35912

CANADIAN THESES ON MICROFICHE

THÈSES CANADIENNES SUR MICROFICHE

NAME OF AUTHOR/*NOM DE L'AUTEUR* Elena Croitoru Giurcă

TITLE OF THESIS/*TITRE DE LA THÈSE* Simultaneous Extension and Torsion of an Elastic Dielectric Tube

UNIVERSITY/*UNIVERSITÉ* Simon Fraser University

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* M. Sc.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE* 1977

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* Dr. M. Singh

DATED/*DATÉ* 13 December SIGNED/*SIGNÉ* [redacted]

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ* [redacted]

SIMULTANEOUS EXTENSION AND TORSION OF AN ELASTIC DIELECTRIC TUBE

by

Elena Croitoru Giurcá

Dpl. Eng. Electr. Tech. University "Gheorghe Asachi", Jassy, 1961

Drd. Theor. Mech. Tech. University "Gheorghe Asachi", Jassy, 1969

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

December 1977

APPROVAL

Name: Elena Croitoru Giurca

Degree: Master of Science

Title of Thesis: Simultaneous Extension and Torsion of an Elastic Dielectric Tube

Examining Committee:

Chairman: A.H. Lachlan

M. Singh
Senior Supervisor

G.A.C. Graham

D. Sharma

A. Das
External Examiner

Date Approved: December 2, 1977

ABSTRACT

Employing the most general form of the strain energy function in the finite theory of elastic dielectrics, the simultaneous extension and torsion of an incompressible cylindrical tube is considered in the absence of either a body force or a distributed charge. Firstly, the problem is investigated with the dielectric displacement field prescribed in the radial direction. The same simultaneous extension and torsion is then studied with a prescribed axial electric field.

A law is obtained which related the longitudinal force necessary to produce a large simple extension with the torsional modulus for a small torsion superposed on that simple extension. The law depends on the form of the strain energy function which is not the case in finite elasticity theory. In fact, the law is independent of the stored energy function if and only if the electric field is totally absent.

ACKNOWLEDGEMENTS

I would like to thank Dr. Manohar Singh, Department of Mathematics, Simon Fraser University, for suggesting the subject of this thesis and for his assistance in its completion.

I wish also to extend my thanks to Kathy Moore for her precise typing.

TABLE OF CONTENTS

1. INTRODUCTION

The systematic formulation of the theory governing finite deformations of elastic dielectrics was originally put forward by Toupin [1], and then by Eringen [2]. These theories postulated the existence of a local self electric field due to polarization accompanied by the Maxwell field. Later, Singh and Pipkin [3] considered the total stress field and the dielectric displacement field directly as functions of the deformation gradients and the total electric field. Based on these constitutive equations, Singh and Pipkin [3] obtained the complete family of controllable states. A controllable deformation is one in which the deformation and the electric field are prescribed at the outset, and then shown that such a state can be maintained in every homogeneous, isotropic, elastic dielectric without the body force or charge distribution. Since the controllable states do not require the functional form of the stored-energy function, they can therefore be employed in experimental determination of the physical properties of various elastic dielectric materials.

In this presentation, we consider the combined extension and torsion of an incompressible, homogeneous, isotropic, elastic dielectric cylindrical tube first when the radial dielectric displacement field is prescribed and then when an axial electric field is prescribed. Both of these states are shown to be controllable in [3]. Since the purpose in [3] was only to show controllability of these states and not to solve the problem, the precise formsof resultant longitudinal forces and couples required to be applied at the ends of the cylindrical tube in order to maintain these states were not obtained.

In Section 2, we present the basic field equations of continuum electrostatics which are independent of the composition of the material media that may be involved. Section 3 outlines the basic equations governing the theory of finite deformations of elastic dielectrics. The constitutive relations for approximate theories valid for small finite deformations and sufficiently weak electric fields are also reproduced in Section 4.

The stimulation in finite elasticity occurred in 1948 when Rivlin [4] drew attention to the so-called exact solutions. Although the exact deformations did not require beforehand the knowledge of the strain energy function to satisfy equilibrium without body forces, the precise form of surface tractions necessary to maintain such a deformation could not be determined without the complete knowledge of the strain energy function for the material considered. The experimental verification of the effects of finite deformations could not be carried out. However, in 1951, Green and Shield [5] came out with a classic result in which they obtained the ratio of the longitudinal force to the torsional rigidity for a small twist superposed on a large simple extension. This ratio fortunately happened to be completely independent of the strain energy function and hence furnished a vehicle for testing the effects of finite elasticity compared to those of infinitesimal theory. In this paper, we set out to obtain a similar type of result for elastic dielectrics. In Section 6, we obtain the ratio of longitudinal force to the torsional rigidity when the prescribed dielectric displacement field is radial. In Section 7, the same ratio is obtained when there is a prescribed electric field in the axial direction. While the ratio in both these states reduces to that of Green and Shield [5] when the electric field is zero, it is shown that the ratio is independent of the stored energy function if and only if the electric field is totally absent. In Section 8, we have investigated the small but finite simultaneous extension and torsion of the cylindrical tube in the presence of a radial electric field. This state is not controllable if the form of the strain energy function is purely arbitrary [3]. But we show that it is controllable if small finite theory is applied. Once again, we have demonstrated that the ratio of longitudinal force to torsional modulus is independent of the form of stored energy function if and only if electric field vanishes.

In Section 9, we considered the combined extension and torsion of a compressible, homogeneous, isotropic, elastic cylindrical tube. It has been proved by Singh [6] that such a state cannot be in equilibrium by surface tractions alone if the form of the stored energy function is arbitrary. However, special forms of strain energy function can lead to a solution of the problem.

2. CONTINUUM ELECTROSTATICS

The electric field in the Maxwell-Faraday electrostatic theory is determined by the two conditions,

$$\oint_C E_i dx_i = 0 , \tag{1}$$

$$\oint_S D_i n_i dS = Q , \tag{2}$$

and a constitutive equation between the components of the dielectric displacement flux D_i and the electric field E_i. In (1), C is any arbitrary closed space curve while S in (2) is the boundary of an arbitrary closed regular region R, and Q is the total free charge contained in R such that

$$Q = \int_R \sigma dV + \int_S \omega dS ,$$

where σ and ω denote the volume density and surface density of free charge, respectively.

Let V denote the region of space occupied by the dielectric and let B denote the boundary of V. Let V_0 denote the remainder of space. We assume that the electric and flux fields are continuously differentiable functions of position in each of the regions V and V_0. We also assume that the fields E_i and D_i suffer at most a finite discontinuity at the surfaces B. Furthermore, the dielectrics that we are interested in have no net free charge in V and no net surface charge in B so that $\sigma = 0$, $\omega = 0$. With these assumptions, the law

(1) yields that

(a) the electric field $E_i(\underline{x})$ is irrotational in V and V_0, i.e.

$$\underline{\nabla} \times \underline{E} = 0 , \tag{3}$$

and

(b) the tangential component of $E_i(\underline{x})$ is continuous across B, i.e.

$$(\underline{E} - \underline{E}^{(0)}) \times \underline{n} = 0 \quad \text{on} \quad B , \tag{4}$$

where E_i and $E_i^{(0)}$ denote the electric fields inside and outside the dielectric, respectively. The law (2) implies that the displacement field $D_i(\underline{x})$ is solenoidal:

$$\underline{\nabla} \cdot \underline{D} = 0 \quad \text{in} \quad V , \tag{5}$$

and that the normal component of D_i is continuous across the surface B:

$$(\underline{D} - \underline{D}^{(0)}) \cdot \underline{n} = 0 \quad \text{on} \quad B , \tag{6}$$

where D_i and $D_i^{(0)}$, once again, denote the dielectric displacement fields inside and outside the dielectric, respectively.

3. THEORY OF FINITE DEFORMATIONS IN ELASTIC DIELECTRICS

In free space, the dielectric displacement field $D_i^{(0)}(\underline{x})$ is directly proportional to the electric field strength $E_i^{(0)}(\underline{x})$:

$$D_i^{(0)}(\underline{x}) = \varepsilon_0 E_i^{(0)}(\underline{x}) \ , \tag{7}$$

where ε_0 denotes the physical constant for free space. Since the resultant electrostatic force on any region which lies outside the dielectric is zero, we may thus represent the stress force in free space by the Maxwell stress M_{ij} :

$$M_{ij} = \varepsilon_0 \ E_i^{(0)} E_j^{(0)} - \frac{\varepsilon_0}{2} \ E_k^{(0)} E_k^{(0)} \delta_{ij} \ . \tag{8}$$

clearly, $\oint_S M_{ij} n_j ds = 0$ for any closed surface S in free space.

Inside the continuous dielectric medium, we assume that the surface forces are described by a system of stress vectors distributed over the surface of any arbitrary region v. Let T_i denote the field of stress vectors. We also assume that the resultant force F_i and moment G_i exerted on the material, not including gravitational or inertial forces, can be described completely by the field of stress vectors. That is,

$$\underline{F} = \oint_S \underline{T} ds \ , \quad \underline{G} = \oint_S \underline{x} \times \underline{T} ds \ ,$$

where S is the surface containing any arbitrary region v. The requirement of static equilibrium then leads to the field equations

$$T_i = \sigma_{ij} n_j \ , \tag{9}$$

$$\sigma_{ij} = \sigma_{ji} \ , \tag{10}$$

and

$$\sigma_{ij,j} + \rho f_i = 0 \ , \tag{11}$$

where σ_{ij} denotes the stress tensor, ρ the mass density, and f_i the body force per unit mass. At the bounding surface B of the dielectric then, we get

$$T_i = (\sigma_{ij} - M_{ij}) n_j \quad \text{on} \quad B \ , \tag{12}$$

where T_i now represents the applied mechanical force at the boundary B, and n_i the unit outward normal to B. It is easy to observe that the Maxwell tensor M_{ij} satisfies equations (9) to (12) in free space identically.

To obtain constitutive equations inside the dielectric, we assume that the stress tensor σ_{ij} and the dielectric displacement field D_i have the forms:

$$\sigma_{ij} = \sigma_{ij}\left(\frac{\partial x_p}{\partial X_q} , E_r \right) \ , \tag{13}$$

$$D_i = D_i\left(\frac{\partial x_p}{\partial X_q} , E_r \right) \ , \tag{14}$$

where X_i and x_i denote the coordinates of the same generic particle referred to a fixed Cartesian frame in the undeformed and deformed positions, respectively.

We shall confine our attention to homogeneous and isotropic dielectrics only. The invariance of constitutive equations (13) and (14) to rigid rotation or translation and the symmetry imposed by the isotropy of the material leads to the following forms [7]:

$$\sigma_{ij} = \frac{2}{(I_3)^{\frac{1}{2}}} [(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2})g_{ij} - \frac{\partial W}{\partial I_2} g^2_{ij} + I_3 \frac{\partial W}{\partial I_3} \delta_{ij}$$

$$+ \frac{\partial W}{\partial I_4} E_i E_j + \frac{\partial W}{\partial I_5} (g_{jk} E_i E_k + g_{ik} E_k E_j)$$

$$+ \frac{\partial W}{\partial I_6} (g^2_{jk} E_i E_k + g^2_{ik} E_j E_k) + \frac{\partial W}{\partial I_6} g_{ik} g_{jl} E_k E_l] \ , \tag{15}$$

and

$$D_i = 2(\frac{\partial W}{\partial I_4} \delta_{ij} + \frac{\partial W}{\partial I_5} g_{ij} + \frac{\partial W}{\partial I_6} g^2_{ij})E_j \ . \tag{16}$$

Here g^2_{ij} denotes the ij element of the square of the matrix g_{ij} which is the Finger strain tensor defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \ , \tag{17}$$

and W stands for the stored energy function whose arguments are the following six scalar invariants:

$$
\begin{aligned}
I_1 &= g_{ii} , \\
I_2 &= \frac{1}{2}[g_{ii}g_{jj} - g_{ij}g_{ij}] , \\
I_3 &= \text{Det.}|g_{ij}| , \\
I_4 &= E_iE_i , \\
I_5 &= g_{ij}E_iE_j , \\
I_6 &= g_{ij}^2E_iE_j .
\end{aligned} \tag{18}
$$

If, in addition to being homogeneous and isotropic, the elastic dielectric considered in incompressible also, then the strain invariant I_3 is unity in all deformations so that W is a function of the invariants I_1, I_2, I_4, I_5, and I_6. An arbitrary pressure p arises as a reaction to the constraint of no volume change. The constitutive relation (15) then assumes the form:

$$
\begin{aligned}
\sigma_{ij} = -p\delta_{ij} + 2[(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2})g_{ij} - \frac{\partial W}{\partial I_2} g_{ij}^2 + \frac{\partial W}{\partial I_4} E_iE_j \\
+ \frac{\partial W}{\partial I_5} (g_{ik}E_kE_j + g_{jk}E_kE_i) + \frac{\partial W}{\partial I_6} (g_{ik}^2E_kE_j + g_{jk}^2E_kE_i) \\
+ \frac{\partial W}{\partial I_6} g_{ik}g_{jl}E_kE_l] .
\end{aligned} \tag{19}
$$

In some cases, it is more convenient to consider the displacement field D_i rather than electric field E_i as the independent variable in the formulation of constitutive equations (13) and (14). If that is done, then for homogeneous, isotropic, elastic dielectrics, the relations (15) and (16) are respectively, replaced by:

$$\sigma_{ij} = \frac{2}{(I_3^*)^{\frac{1}{2}}} [(\frac{\partial W}{\partial I_1^*} + I_1^* \frac{\partial W}{\partial I_2^*}) g_{ij} - \frac{\partial W}{\partial I_2^*} g_{ij}^2 + I_3^* \frac{\partial W}{\partial I_3^*} \delta_{ij}$$

$$+ \frac{\partial W}{\partial I_4^*} D_i D_j + \frac{\partial W}{\partial I_5^*} (g_{ik} D_k D_j + g_{jk} D_k D_i)$$

$$+ \frac{\partial W}{\partial I_6^*} (g_{ik}^2 D_k D_j + g_{jk}^2 D_k D_i + g_{ik} g_{jl} D_k D_l)] , \qquad (20)$$

and

$$E_i = 2 (\frac{\partial W}{\partial I_4^*} \delta_{ij} + \frac{\partial W}{\partial I_5^*} g_{ij} + \frac{\partial W}{\partial I_6^*} g_{ij}^2) D_j , \qquad (21)$$

where the stored energy function W in now the function of the six scalar invariants

$$\begin{aligned} I_1^* &= g_{ii} , \\ I_2^* &= \frac{1}{2} [g_{ii} g_{jj} - g_{ij} g_{ij}] , \\ I_3^* &= \text{Det.} |g_{ij}| , \\ I_4^* &= D_i D_i , \\ I_5^* &= g_{ij} D_i D_j , \\ I_6^* &= g_{ij}^2 D_i D_j . \end{aligned} \qquad (22)$$

For an incompressible dielectric, $I_3^* = 1$ and relation (19) then is replaced by

$$\sigma_{ij} = -p\delta_{ij} + 2[(\frac{\partial W}{\partial I_1^*} + I_1^* \frac{\partial W}{\partial I_2^*})g_{ij} - \frac{\partial W}{\partial I_2^*} g_{ij}^2 + \frac{\partial W}{\partial I_4^*} D_i D_j$$

$$+ \frac{\partial W}{\partial I_5^*} (g_{ik} D_k D_j + g_{jk} D_i D_k) \tag{23}$$

$$+ \frac{\partial W}{\partial I_6^*} (g_{ik}^2 D_k D_j + g_{jk}^2 D_k D_i + g_{ik} g_{jl} D_k D_l)].$$

4. SMALL FINITE DEFORMATIONS

If the deformation is small and the electric field sufficiently weak, then we can assume the stored energy function $W(I_1, I_2, \ldots, I_6)$ to be a polynomial in its arguments and obtain an approximation to any desired order in the principal extensions and powers of the electric field by neglecting terms above an appropriate degree in the polynomial expansion for W. A first approximation can be arrived at by retaining in the polynomial expansion for W all terms involving principal extensions to a lower degree than third and electric field components to a degree lower than fourth. That is

$$W = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 , \qquad (24)$$

where $a_0, a_1, \ldots, a_6$ are material constants, and $J_1, J_2, \ldots, J_5$ are defined as:

$$\begin{aligned} J_1 &= I_1 - 3 \\ J_2 &= (I_2 - 3) - 2(I_1 - 3) , \\ J_3 &= (I_3 - 1) - (I_2 - 3) + (I_1 - 3) , \\ J_4 &= I_4 , \\ J_5 &= I_5 - I_4 . \end{aligned} \qquad (25)$$

Since we can take the stored energy to be zero in the undeformed and unstressed state, $a_0 = a_1 = 0$, so that

$$W = a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 \, . \tag{26}$$

Substituting for W from (26) into (15) and (16), we obtain the constitutive equation of the first order finite theory:

$$\begin{aligned}\sigma_{ij} = 2\{[a_2 + (a_2+2a_3)J_1 + a_6 J_4]g_{ij} - a_2 g_{ij}^2 \\ + (a_4-a_5)E_i E_j + a_5(g_{ik}E_k E_j + g_{jk}E_k E_i)\} \, ,\end{aligned} \tag{27}$$

$$D_i = 2[(a_4-a_5)\delta_{ij} + a_5 g_{ij}]E_j \, . \tag{28}$$

When the dielectric is incompressible, $I_3 = 1$, and the stored energy function W for the first order approximation assumes the form:

$$W = b_1 J_1 + b_2 J_2 + b_3 J_5 \, , \tag{29}$$

where b_1, b_2, b_3 are constants of the material. With W from (29), constitutive relations become

$$\sigma_{ij} = -p\delta_{ij} + C_1 g_{ij} + C_2 E_i E_j + C_3(g_{ik}E_k E_j + g_{jk}E_k E_i) \, , \tag{30}$$

$$D_i = C_2 E_i + C_3 g_{ij} E_j \, , \tag{31}$$

where p is an arbitrary pressure and C's are physical constants of the dielectric medium.

5. INVERSE METHOD OF SOLUTIONS IN ELASTIC DIELECTRICS

According to this method, we prescribe at the outset an appropriate deformation $x_i(X_k)$, and an appropriate electric field E_i which satisfies the field equation (3) as well as boundary condition (4). After obtaining g_{ij} from (17) corresponding to the prescribed deformation, the stress field and the dielectric displacement are derived from (16) and (19), respectively. It is then verified that such a stress satisfies equilibrium equation (11) without any body force, and the electric displacement D_i meets (5) as well as (6). The functional form of the stored energy function remains arbitrary throughout the procedure. Such a combination of the prescribed deformation $x_i(X_k)$ and the electric field E_i is said to constitute a solution or a controllable state. The appropriate mechanical surface tractions to be applied at the boundary of the dielectric which shall then maintain such a state in equilibrium are furnished by (12).

If the deformation is small but finite and the electric field sufficiently weak, then instead of the general arbitrary form of the strain energy function we use the form given by (29). The procedure remains the same as outlined above.

6. SIMULTANEOUS EXTENSION AND TORSION OF AN INCOMPRESSIBLE CYLINDRICAL TUBE IN A RADIAL ELECTRIC DISPLACEMENT FIELD.

Our purpose here is to employ the general theory presented in the previous sections to investigate the simultaneous extension and torsion of a circular cyclindrical tube. The body forces and distributed charges shall be assumed to be zero.

The tube which has the length ℓ_0, internal radius a and external radius b in its initial unstressed and unstrained field free state is elongated uniformly in the axial direction of extension ratio λ along the axis and of extension ratio μ along any direction perpendicular to it. It is also twisted such that planes perpendicular to the axis of the tube are rotated in their own plane through an angle proportional to the distance of the plane from one end, the constant of proportionality being φ. The deformation described is characterized by the mapping:

$$\begin{aligned} r &= \mu R , \\ \theta &= \Theta + \varphi Z , \\ z &= \lambda Z , \end{aligned} \tag{32}$$

where (R,Θ,Z) denote the cylindrical polar coordinates of that generic particle in the undeformed state which occupies the position (r,θ,z) after the deformation.

The tube is subjected to a radial dielectric displacement field $(D_r,0,0)$. Since the divergence of $\underline{D}$ has to vanish, we get $\underline{D}$ of the type:

$$D_r = \frac{Q}{2\pi r}\ , \quad D_\theta = 0\ , \quad D_z = 0\ . \tag{33}$$

We shall take this field to be inside as well as outside the dielectric. The condition of normal continuity across the curved surface of the tube is then satisfied. The field (33) may be produced by placing the tube between the plates of a coaxial cylindrical condenser whose charge per unit length on the interior plate is Q.

The deformation characterized by (32) leads to the following physical components of the strain tensor g_{ij} :

$$\begin{aligned} g_{rr} &= \mu^2\ , \\ g_{\theta\theta} &= \mu^2 + \varphi^2 r^2\ , \\ g_{zz} &= \lambda^2\ , \\ g_{r\theta} &= g_{rz} = 0\ , \\ g_{\theta z} &= \lambda\varphi r\ . \end{aligned} \tag{34}$$

With (33) and (34), the invariants $I_1^*, I_2^*, \ldots, I_6^*$ in (22) are given as:

$$\begin{aligned} I_1^* &= 2\mu^2 + \lambda^2 + \varphi^2 r^2\ , \\ I_2^* &= \mu^4 + 2\mu^2\lambda^2 + \mu^2\varphi^2 r^2\ , \\ I_3^* &= \mu^4\lambda^2\ , \end{aligned}$$

$$I_4^* = \frac{Q^2}{4\pi^2 r^2} ,$$

$$I_5^* = \mu^2 \frac{Q^2}{4\pi^2 r^2} ,$$

$$I_6^* = \mu^4 \frac{Q^2}{4\pi^2 r^2} . \qquad (35)$$

Under the incompressibility condition:

$$\mu^4 \lambda^2 = 1 \quad \text{or} \quad \mu = \frac{1}{\sqrt{\lambda}} \qquad (36)$$

the relations (32), (34) and (35) respectively become:

$$r = \frac{1}{\sqrt{\lambda}} R ,$$

$$\theta = \Theta + \varphi Z , \qquad (37)$$

$$z = \lambda Z ,$$

$$g_{rr} = \frac{1}{\lambda} ,$$

$$g_{\theta\theta} = \frac{1}{\lambda} + \varphi^2 r^2 ,$$

$$g_{zz} = \lambda^2 , \qquad (38)$$

$$g_{r\theta} = g_{rz} = 0 ,$$

$$g_{\theta z} = \lambda \varphi r ,$$

and

$$
\begin{aligned}
I_1^* &= \left(\frac{2}{\lambda} + \lambda^2 \right) + \varphi^2 r^2 , \\
I_2^* &= \left(\frac{1}{\lambda^2} + 2\lambda \right) + \frac{1}{\lambda} \varphi^2 r^2 , \\
I_3^* &= 1 , \\
I_4^* &= \frac{Q^2}{4\pi^2 r^2} , \\
I_5^* &= \frac{1}{\lambda} \frac{Q^2}{4\pi^2 r^2} , \\
I_6^* &= \frac{1}{\lambda^2} \frac{Q^2}{4\pi^2 r^2} .
\end{aligned}
\tag{39}
$$

It may be noted here that the invariants (39) are functions of r only. From (7), the electric field in the medium surrounding the dielectric is given by:

$$
E_r^{(0)} = \frac{Q}{2\pi\varepsilon_Q r} , \quad E_\theta^{(0)} = 0 , \quad E_z^{(0)} = 0 . \tag{40}
$$

The electric field inside the dielectric is furnished by (21):

$$
\begin{aligned}
E_r &= 2\left(\frac{\partial W}{\partial I_4^*} + \frac{1}{\lambda} \frac{\partial W}{\partial I_5^*} + \frac{1}{\lambda^2} \frac{\partial W}{\partial I_6^*} \right) \frac{Q}{2\pi r} , \\
E_\theta &= 0 , \quad E_z = 0 .
\end{aligned}
\tag{41}
$$

It is clearly apparent that the electric field in (40) or (41) is conservative and that the condition of continuity of the tangential component across the boundary of the tube is met identically. With use of (33) and (38) in (23), we find the state of stress within the dielectric:

$$\sigma_{rr} = -p + \frac{1}{\lambda}\Psi_1 + \frac{1}{\lambda^2}\Psi_2 + (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6)\frac{Q^2}{4\pi^2 r^2},$$

$$\sigma_{\theta\theta} = -p + (\frac{1}{\lambda}+\varphi^2 r^2)\Psi_1 + (\frac{1}{\lambda^2} + \frac{2}{\lambda}\varphi^2 r^2 + \varphi^4 r^4 + \lambda^2\varphi^2 r^2)\Psi_2 ,$$

$$\sigma_{zz} = -p + \lambda^2\Psi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Psi_2 , \tag{42}$$

$$\sigma_{\theta z} = \lambda\varphi r[\Psi_1 + (\frac{1}{\lambda} + \varphi^2 r^2 + \lambda^2)\Psi_2] ,$$

$$\sigma_{r\theta} = \sigma_{rz} = 0 ,$$

where, to simplify the writing, we have put

$$\Psi_1 = 2(\frac{\partial W}{\partial I_1^*} + I_1^*\frac{\partial W}{\partial I_2^*}) ,$$

$$\Psi_2 = -2\frac{\partial W}{\partial I_2^*} ,$$

$$\Psi_4 = 2\frac{\partial W}{\partial I_4^*} , \tag{43}$$

$$\Psi_5 = 4\frac{\partial W}{\partial I_5^*} ,$$

$$\Psi_6 = 6\frac{\partial W}{\partial I_6^*} .$$

Note that $\Psi_1,\Psi_2,\ldots,\Psi_6$ are functions of r only. The equations of static equilibrium (11), without any body forces, in cylindrical coordinates can be written as:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rz,z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \ ,$$

$$\sigma_{r\theta,r} + \frac{1}{4}\sigma_{\theta\theta,\theta} + \sigma_{\theta z,z} + \frac{2}{r}\sigma_{r\theta} = 0 \ , \qquad (44)$$

$$\sigma_{rz,r} + \frac{1}{r}\sigma_{\theta z,\theta} + \sigma_{zz,z} + \frac{2}{r}\sigma_{rz} = 0 \ .$$

Introducing the stress field (42) in the equilibrium equations (44), we obtain:

$$\frac{\partial}{\partial r}(\sigma_{rr}) + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \ ,$$

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \ . \qquad (45)$$

The pressure p depends, therefore, only on r. By integration it can be expressed as:

$$p(r) = p(r_b) + L(r) \ , \qquad (46)$$

such that

$$L(r) = \int_{r_b}^{r} \{ \frac{1}{\lambda}\Psi_1' + \frac{1}{\lambda^2}\Psi_2' + (\Psi_4' + \frac{1}{\lambda}\Psi_5' + \frac{1}{\lambda^2}\Psi_6')\frac{Q^2}{4\pi^2\xi^2}$$

$$- (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6)\frac{Q^2}{4\pi^2\xi^3} - \varphi^2\xi\Psi_1$$

$$- [(\frac{2}{\lambda} + \lambda^2)\varphi^2\xi + \varphi^4\xi^3]\Psi_2\} d\xi \ , \qquad (47)$$

where "prime" denotes the derivative with respect to r. The state of stress in the charge free medium surrounding the dielectric is described by the Maxwell stress. Then (8) yields:

$$
\begin{aligned}
\sigma_{rr}^{(0)} &= M_{rr} = \frac{Q^2}{8\pi^2\varepsilon_0 r^2}, \\
\sigma_{\theta\theta}^{(0)} &= M_{\theta\theta} = -\frac{Q^2}{8\pi^2\varepsilon_0 r^2}, \\
\sigma_{zz}^{(0)} &= M_{zz} = -\frac{Q^2}{8\pi^2\varepsilon_0 r^2}, \\
\sigma_{r\theta}^{(0)} &= \sigma_{\theta z}^{(0)} = \sigma_{rz}^{(0)} = 0 .
\end{aligned}
\tag{48}
$$

The surface tractions per unit area of the deformed configuration that must be applied to maintain the prescribed state are now furnished by (12):

$$
\begin{aligned}
(T_r)_{r=r_b} &= (\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_b} \\
&= -p(r_b) + \left[\frac{1}{\lambda}\Psi_1 + \frac{1}{\lambda^2}\Psi_2 + \left(\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6\right)\frac{Q^2}{4\pi^2 r_b^2} - \frac{Q^2}{8\pi^2\varepsilon_0 r_b^2}\right]_{r=r_b}, \\
(T_\theta)_{r=r_b} &= (T_z)_{r=r_b} = 0 .
\end{aligned}
\tag{49}
$$

The surface tractions on the inner surface of the tube can be similarly obtained:

$$(T_r)_{r=r_a} = -(\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_a}$$

$$= p(r_a) - [\frac{1}{\lambda}\Psi_1 + \frac{1}{\lambda^2}\Psi_2 + (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6)\frac{Q^2}{4\pi^2 r_a^2} - \frac{Q}{8\pi^2\varepsilon_0 r_a^2}]_{r=r_a},$$

$$(T_\theta)_{r=r_a} = (T_z)_{r=r_a} = 0 \; . \qquad (50)$$

We can choose one of the curved surfaces, say $r = r_b$, as force free. Equation (49) then furnishes:

$$p(r_b) = [\frac{1}{\lambda}\Psi_1 + \frac{1}{\lambda^2}\Psi_2 + (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6)\frac{Q^2}{4\pi^2 r_b^2} - \frac{Q^2}{8\pi^2\varepsilon_0 r_b^2}]_{r=r_b} \; . \qquad (51)$$

To support the given state, we also require normal and azimuthal surface tractions on the plane ends of the tube.

On $z = \ell$, we have

$$(T_r)_{z=\ell} = \sigma_{rz} - \sigma_{rz}^{(0)} = 0 \; ,$$

$$(T_\theta)_{z=\ell} = \sigma_{\theta z} - \sigma_{\theta z}^{(0)} = \lambda\varphi r[\Psi_1 + (\frac{1}{\lambda} + r^2\varphi^2 + \lambda^2)\Psi_2] \; , \qquad (52)$$

$$(T_z)_{z=\ell} = \sigma_{zz} - \sigma_{zz}^{(0)} = \lambda^2\Psi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Psi_2 - p(r_b) - L(r) + \frac{Q^2}{8\pi^2\varepsilon_0 r^2} \; .$$

The above distribution of surface tractions (52) at the plane end $z = \ell$ is statically equivalent to a torsional couple M about the axis of the tube and a longitudinal force N along the axis of the tube:

$$M = \int_{r_a}^{r_b}\int_0^{2\pi} T_\theta r^2 dr d\theta = 2\pi\lambda\varphi \int_{r_a}^{r_b} [\Psi_1 + (\frac{1}{\lambda} + r^2\varphi^2 + \lambda^2)\Psi_2] r^3 dr , \quad (53)$$

and

$$N = \int_{r_a}^{r_b}\int_0^{2\pi} T_z r dr d\theta = 2\pi \int_{r_a}^{r_b} [\lambda^2\Psi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Psi_2 - p(r) + \frac{Q^2}{8\pi^2\varepsilon_0 r^2}] r dr$$

$$= 2\pi \int_{r_a}^{r_b} [\lambda^2\Psi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Psi_2 - p(r_b) + \frac{Q^2}{8\pi^2\varepsilon_0 r^2}] r dr$$

$$+ \pi \int_{r_a}^{r_b} \{\frac{1}{\lambda}\Psi_1' + \frac{1}{\lambda^2}\Psi_2' + (\Psi_4' + \frac{1}{\lambda}\Psi_5' + \frac{1}{\lambda^2}\Psi_6') \frac{Q^2}{4\pi^2 r^2}$$

$$- (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6) \frac{Q^2}{4\pi^2 r^3} - \varphi^2 r \Psi_1$$

$$- [(\frac{2}{\lambda} + \lambda^2)\varphi^2 r + \varphi^4 r^3]\Psi_2\} r^2 dr . \quad (54)$$

SPECIAL CASES.

(i) If $\lambda = 1$, the torsion is unaccompanied by the simple extension. The resultant couple M and the resultant longitudinal force N at the end $z = \ell$ reduce to

$$M = 2\pi\varphi \int_{r_a}^{r_b} [\Psi_1 + (2 + \varphi^2 r^2)\Psi_2] r^3 dr , \quad (55)$$

$$N = 2\pi \int_{r_a}^{r_b} [\Psi_1 + (1 + \varphi^2 r^2)\Psi_2 - p(r_b) + \frac{Q^2}{8\pi^2\varepsilon_0 r^2}] r dr$$

$$+ \pi \int_{r_a}^{r_b} [\Psi_1' + \Psi_2' + (\Psi_4' + \Psi_5' + \Psi_6') \frac{Q^2}{4\pi^2 r^2} - (\Psi_4 + \Psi_5 + \Psi_6) \frac{Q^2}{4\pi^2 r^3}$$

$$- \varphi^2 r \Psi_1 - (3\varphi^2 r + \varphi^4 r^3) \Psi_2] r^2 dr \quad . \tag{56}$$

where

$$p(r_b) = [\Psi_1 + \Psi_2 + (\Psi_4 + \Psi_5 + \Psi_6) \frac{Q^2}{4\pi^2 r_b^2} - \frac{Q^2}{8\pi^2 \varepsilon_0 r_b^2}]_{\substack{r=r_b \\ \lambda=1}} .$$

(ii). Suppose $\varphi = 0$. There is no torsion, and the resultant couple M at $z = \ell$ is zero. The longitudinal force N takes the form:

$$N = 2\pi \int_{r_a}^{r_b} [\lambda^2 \Psi_1 + \lambda^4 \Psi_2 - p(r_b) + \frac{Q^2}{8\pi^2 \varepsilon_0 r^2}] r dr$$

$$+ \pi \int_{r_a}^{r_b} [\frac{1}{\lambda} \Psi_1' + \frac{1}{\lambda^2} \Psi_2' + (\Psi_4' + \frac{1}{\lambda} \Psi_5' + \frac{1}{\lambda^2} \Psi_6') \frac{Q^2}{4\pi^2 r^2}$$

$$- (\Psi_4 + \frac{1}{\lambda} \Psi_5 + \frac{1}{\lambda^2} \Psi_6) \frac{Q^2}{4\pi^2 r^3}] r^2 dr \quad . \tag{57}$$

In (57), the expressions for $\Psi_1, \Psi_2 \ldots$ do not involve the specific twist φ .

(iii). Suppose φ is small. That is, the deformation consists of a small twist superposed on a very large simple extension so that terms containing φ^2 can be neglected. In such a case, $I_1^*, I_2^*, \ldots, I_6^*$ and hence $\Psi_1, \Psi_2, \ldots$ are independent of φ. The resultant couple M and the longitudinal force N are given by:

$$M = 2\pi\lambda\varphi \int_{r_a}^{r_b} [\Psi_1 + (\frac{1}{\lambda} + \lambda^2)\Psi_2] r^3 dr \tag{58}$$

$$N = 2\pi \int_{r_a}^{r_b} [\lambda^2\Psi_1 + \lambda^4\Psi_2 - p(r_b) + \frac{Q}{8\pi^2\varepsilon_0 r^2}]\, rdr$$
$$+ \pi \int_{r_a}^{r_b} [\frac{1}{\lambda}\Psi_1' + \frac{1}{\lambda^2}\Psi_2' + (\Psi_4' + \frac{1}{\lambda}\Psi_5' + \frac{1}{\lambda^2}\Psi_6')\frac{Q^2}{4\pi^2 r^2}$$
$$- (\Psi_4 + \frac{1}{\lambda}\Psi_5 + \frac{1}{\lambda^2}\Psi_6)\frac{Q^2}{4\pi^2 r^3}] r^2 dr \ . \tag{59}$$

From (58) and (59), it is easily observed that the ratio of longitudinal force N to the torsional modulus M/φ is not independent of the stored energy function which is represented through $\Psi_1, \Psi_2, \ldots, \Psi_6$. On the other hand, if the electrical effects are totally absent, then

$$N = 2\pi \int_{r_a}^{r_b} [\lambda^2\Psi_1 + \lambda^4\Psi_2 - p(r_b)] rdr \ , \tag{60}$$

where

$$p(r_b) = (\frac{1}{\lambda}\Psi_1 + \frac{1}{\lambda^2}\Psi_2)$$

and

$$M = 2\pi\lambda\varphi \int_{r_a}^{r_b} [\Psi_1 + (\frac{1}{\lambda} + \lambda^2)\Psi_2] r^3 dr \ . \tag{61}$$

Since Ψ_1 and Ψ_2 are constant in this case, we obtain

$$\frac{N}{M/\varphi} = 2(\lambda - \frac{1}{\lambda^2}) \frac{r_b^2 - r_a^2}{r_b^4 - r_a^4} \tag{62}$$

The ratio (62), which is independent of the strain energy function, was first obtained by Green and Shield [5]. We have shown that this ratio is not independent of the stored energy function if the electrical effects are taken into account.

7. SIMULTANEOUS EXTENSION AND TORSION OF AN INCOMPRESSIBLE TUBE IN AN AXIAL ELECTRIC FIELD.

The deformation (38) described in Section 6 can also be supported without body forces or distributed charges in the presence of a uniform axial electric field given both within the dielectric and in the medium surrounding it by:

$$E_r = 0 \; , \quad E_\theta = 0 \; , \quad E_z = H \; . \tag{63}$$

where H is a constant. The electric field given in (63) meets the required condition of being conservative and the tangential component across the cylindrical surfaces of the tube is continuous. With the strain components given by (38) and the electric field by (63), it follows that the invariants defined in (18) are functions of r only:

$$\begin{aligned}
I_1 &= \frac{2}{\lambda} + \lambda^2 + \varphi^2 r^2 \; , \\
I_2 &= \frac{1}{\lambda^2} + 2\lambda + \frac{1}{\lambda}\varphi^2 r^2 \; , \\
I_3 &= 1 \; , \\
I_4 &= H^2 \; , \\
I_5 &= \lambda^2 H^2 \; , \\
I_6 &= (\lambda^4 + \lambda^2 \varphi^2 r^2) H^2 \; .
\end{aligned} \tag{64}$$

The dielectric displacement field components for the inside of elastic dielectric are found with the aid of (16) to be:

$$D_r = 0 ,$$

$$D_\theta = 2\{\lambda\varphi r \frac{\partial W}{\partial I_5} + [(\frac{1}{\lambda} + \varphi^2 r^2)\lambda\varphi r + \lambda^3\varphi r] \frac{\partial W}{\partial I_6}\}H ,$$

$$D_z = 2[\frac{\partial W}{\partial I_4} + \lambda^2 \frac{\partial W}{\partial I_5} + (\lambda^4 + \lambda^2\varphi^2 r^2) \frac{\partial W}{\partial I_6}] H . \tag{65}$$

From (7), the dielectric displacement field in the medium surrounding the tube can be expressed as:

$$D_r^{(0)} = 0 , \quad D_\theta^{(0)} = 0 , \quad D_z^{(0)} = \varepsilon_0 H . \tag{66}$$

Thus, the dielectric displacement field $\underline{D}$ given in (65) and (66) satisfies the condition $\underline{\nabla}\cdot\underline{D} = 0$ inside and outside of the dielectric and the normal component of $\underline{D}$ is continuous across the boundary surfaces $r = r_a$ and $r = r_b$. Substitution of strain components (38) and electric field (63) in (19) gives us the state of stress throughout the dielectric:

$$\sigma_{rr} = -p + \frac{1}{\lambda}\Phi_1 + \frac{1}{\lambda^2}\Phi_2 ,$$

$$\sigma_{\theta\theta} = -p + (\frac{1}{\lambda} + \varphi^2 r^2)\Phi_1 + (\frac{1}{\lambda^2} + \frac{2}{\lambda}\varphi^2 r^2 + \varphi^4 r^4 + \lambda^2\varphi^2 r^2)\Phi_2$$

$$+ \lambda^2\varphi^2 r^2 H^2 \Phi_6 ,$$

$$\sigma_{zz} = -p + \lambda^2\Phi_1 + (\lambda^4+\lambda^2\varphi^2r^2)\Phi_2 + H^2[\Phi_4 + 2\lambda^2\Phi_5 + (3\lambda^4 + 2\lambda^2\varphi^2r^2)\Phi_6],$$

$$\sigma_{\theta z} = \lambda\varphi r\{\Phi_1 + (\frac{1}{\lambda} + \varphi^2r^2 + \lambda^2)\Phi_2 + H^2[\Phi_5 + (\frac{1}{\lambda} + \varphi^2r^2 + 2\lambda^2)\Phi_6]\},$$

$$\sigma_{r\theta} = \sigma_{rz} = 0, \qquad (64)$$

where

$$\Phi_1 = 2(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}),$$

$$\Phi_2 = -2\frac{\partial W}{\partial I_2},$$

$$\Phi_4 = 2\frac{\partial W}{\partial I_4}, \qquad (65)$$

$$\Phi_5 = 2\frac{\partial W}{\partial I_5},$$

$$\Phi_6 = 2\frac{\partial W}{\partial I_6}.$$

The stress field (64) implies that the equilibrium equations(44) are satisfied provided p is such that

$$\frac{\partial}{\partial r}(\sigma_{rr}) - \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0,$$

and

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \qquad (66)$$

Hence,

$$p(r) = p(r_b) + L(r), \qquad (67)$$

where

$$L(r) = \int_{r_b}^{r} [\, \frac{1}{\lambda} \Phi_1' + \frac{1}{\lambda^2} \Phi_2' - \varphi^2 \xi \Phi_1 - (\frac{2}{\lambda} \varphi^2 \xi + \lambda^2 \varphi^2 \xi + \varphi^4 \xi^3) \Phi_2$$

$$- \lambda^2 \varphi^2 \xi H^2 \Phi_6] d\xi \; . \tag{68}$$

Introducing the above relations (67) and (68) into (64) we obtain the stress distribution within the dielectric. The stress components evaluated outside the dielectric, described by Maxwell stress (8) are as follows:

$$\sigma_{rr}^{(0)} = - \frac{\varepsilon_0}{2} H^2 \, ,$$

$$\sigma_{\theta\theta}^{(0)} = - \frac{\varepsilon_0}{2} H^2 \, ,$$

$$\sigma_{zz}^{(0)} = \frac{\varepsilon_0}{2} H^2 \, , \tag{69}$$

$$\sigma_{r\theta}^{(0)} = \sigma_{rz}^{(0)} = \sigma_{\theta z}^{(0)} = 0 \; .$$

The surface forces per unit area which must be applied on the boundary of the deformed body in order to maintain the required state are given by (12). We obtain that the surface tractions acting on the curved boundary are in the radial direction only. On outer surface $r = r_b$ we have:

$$(T_r)_{r=r_b} = (\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_b} = -p(r_b) + (\frac{1}{\lambda} \Phi_1 + \frac{1}{\lambda^2} \Phi_2 + \frac{\varepsilon_0}{2} H^2)_{r=r_b} \, ,$$

$$(T_\theta)_{r=r_b} = (T_z)_{r=r_b} = 0 \; . \tag{70}$$

The distribution of surface tractions which must be applied on the inner surface $r = r_a$ are:

$$(T_r)_{r=r_a} = -(\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_a} = p(r_a) - (\frac{1}{\lambda}\Phi_1 + \frac{1}{\lambda^2}\Phi_2 + \frac{\varepsilon_0}{2}H^2)_{r=r_a} ,$$

$$(T_\theta)_{r=r_a} = (T_z)_{r=r_a} = 0 . \qquad (71)$$

If we impose that the boundary surface $r = r_b$ is force free, then we obtain:

$$p(r_b) = (\frac{1}{\lambda}\Phi_I + \frac{1}{\lambda^2}\Phi_2 + \frac{\varepsilon_0}{2}H^2)_{r=r_b} \qquad (72)$$

In addition, the following surface forces must act on the plane ends of the tube:

$$(T_r)_{z=\ell} = 0 ,$$

$$(T_\theta)_{z=\ell} = (\sigma_{\theta z} - \sigma_{\theta z}^{(0)})_{z=\ell}$$

$$= \lambda\varphi r[\Phi_1 + (\frac{1}{\lambda} + \varphi^2 r^2 + \lambda^2)\Phi_2] + \lambda\varphi r H^2[\Phi_5 + (\frac{1}{\lambda} + \varphi^2 r^2 + 2\lambda^2)\Phi_6],$$

$$(T_z)_{z=\ell} = (\sigma_{zz} - \sigma_{zz}^{(0)})_{z=\ell}$$

$$= -p(r) + \lambda^2\Phi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Phi_2 + H^2[\Phi_4 + 2\lambda^2\Phi_5$$

$$+ (3\lambda^4 + 2\lambda^2\varphi^2 r^2)\Phi_6] - \frac{\varepsilon_0}{2}H^2 . \qquad (73)$$

The system of forces (73) are statically equivalent to a resultant couple M and a longitudinal force N given by:

$$M = \int_{r_a}^{r_b} \int_0^{2\pi} T_\theta r^2 dr d\theta$$

$$= 2\pi\lambda\varphi \int_{r_a}^{r_b} \{\Phi_1 + (\frac{1}{\lambda} + \varphi^2 r^2 + \lambda^2)\Phi_2$$

$$+ H^2[\Phi_5 + (\frac{1}{\lambda} + \varphi^2 r^2 + 2\lambda^2)\Phi_6]\} r^3 dr \qquad (74)$$

$$N = \int_{r_a}^{r_b} \int_0^{2\pi} T_z r dr d\theta$$

$$= 2\pi \int_{r_a}^{r_b} \{\lambda^2\Phi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Phi_2 - p(r_b)$$

$$+ H^2[\Phi_4 + 2\lambda^2\Phi_5 + (3\lambda^4 + 2\lambda^2\varphi^2 r^2)\Phi_6] - \frac{\varepsilon_0}{2} H^2\} r dr$$

$$+ \pi \int_{r_a}^{r_b} [\frac{1}{\lambda}\Phi_1' + \frac{1}{\lambda^2}\Phi_2' - \varphi^2 r\Phi_1 - (\frac{2}{\lambda}\varphi^2 r + \lambda^2\varphi^2 r + \varphi^4 r^3)\Phi_2$$

$$- \varphi^2\lambda^2 r H^2 \Phi_6] r^2 dr . \qquad (75)$$

SPECIAL CASES

(i) For a pure torsion, i.e. $\lambda = 1$, the total couple M and the resultant force N take the form:

$$M = 2\pi\varphi \int_{r_a}^{r_b} \{\Phi_1 + (2 + \varphi^2 r^2)\Phi_2 + H^2[\Phi_5 + (3 + \varphi^2 r^2)\Phi_6]\} r^3 dr .$$

$$N = 2\pi \int_{r_a}^{r_b} \{\Phi_1 + (1 + \varphi^2 r^2)\Phi_2 - p(r_b) + H^2[\Phi_4 + 2\Phi_5 + (3 + 2\varphi^2 r^2)\Phi_6] - \frac{\varepsilon_0}{2} H^2\} r dr$$

$$+ \pi \int_{r_a}^{r_b} [\Phi_1' + \Phi_2' - \varphi^2 r \Phi_1 - (3\varphi^2 r + \varphi^4 r)\Phi_2 - \varphi^2 r H^2 \Phi_6] r^2 dr . \qquad (77)$$

where

$$p(r_b) = (\Phi_1 + \Phi_2 + \frac{\varepsilon_0}{2} H^2)_{\substack{r=r_b \\ \lambda=1}}$$

(ii). If $\varphi = 0$, the deformation (38) restricts to just an extension, and the total couple M required on plane ends is zero. Also, for $\varphi = 0$, the coefficients Φ are no longer dependent on r. With these assumptions, the total force N becomes:

$$N = \pi[(\lambda^2 - \frac{1}{\lambda})\Phi_1 + (\lambda^4 - \frac{1}{\lambda^2})\Phi_2 + H^2(\Phi_4 + 2\lambda^2\Phi_5 + 3\lambda^4\Phi_6) - \varepsilon^0 H^2](r_b^2 - r_a^2) . \qquad (78)$$

(iii). Suppose φ is small. In the case of a small twist superposed on a large simple extension we obtain the following total couple M and the total longitudinal force N:

$$M = \pi\lambda\varphi\{\Phi_1 + (\frac{1}{\lambda} + \lambda^2)\Phi_2 + H^2[\Phi_5 + (\frac{1}{\lambda} + 2\lambda^2)\Phi_6]\} \frac{r_b^4 - r_a^4}{2} ,$$

$$N = \pi[(\lambda^2 - \frac{1}{\lambda})\Phi_1 + (\lambda^4 - \frac{1}{\lambda^2})\Phi_2 + H^2(\Phi_4 + 2\lambda^2\Phi_5 + 3\lambda^4\Phi_6) - \varepsilon_0 H^2](r_b^2 - r_a^2) .$$

The ratio $N/\frac{M}{\varphi}$ provides

$$\frac{N}{M/\varphi} = 2\,\frac{(\lambda^2 - \frac{1}{\lambda})\Phi_1 + (\lambda^4 - \frac{1}{\lambda^2})\Phi_2 + H^2(\Phi_4 + 2\lambda^2\Phi_5 + 3\lambda^4\Phi_6) - \varepsilon_0 H^2}{\lambda\{\Phi_1 + (\frac{1}{\lambda} + \lambda^2)\Phi_2 + H^2[\Phi_6 + (\frac{1}{\lambda} + 2\lambda^2)\Phi_6]\}}\,\frac{r_b^2 - r_a^2}{r_b^4 - r_a^4} \tag{79}$$

For zero electrical effects, we obtain the relation:

$$\frac{N}{M/\varphi} = 2(\lambda - \frac{1}{\lambda^2})\,\frac{r_b^2 - r_a^2}{r_b^4 - r_a^4}\,, \tag{80}$$

which is the same as obtained by Green and Shield [5] when the electric field is not taken into account. It is apparent that (79) is independent of the stored evergy function if and only if there are no electrical effects.

8. SMALL FINITE EXTENSION AND TORSION OF AN INCOMPRESSIBLE TUBE IN A RADIAL ELECTRIC FIELD.

Here we subject the tube to a small finite deformation in a weak electric field. The following mapping is prescribed initially:

$$r = \frac{1}{\sqrt{\lambda}} R ,$$

$$\theta = \Theta + \varphi Z , \tag{81}$$

$$z = \lambda Z .$$

As we have shown in Section 6, the above deformation is isochoric and represents a simultaneous simple extension of extension ratio λ along the axis of the tube and a torsion of φ per unit length.

We subject the tube to a radial electric field $(E_r, 0, 0)$. In view of conditions (3) and (4), the radial field has to be of the form:

$$E_r = E_r(r) \ , \quad E_\theta = E_z = 0 \quad \text{inside the dielectric} \tag{82}$$

and

$$E_r^{(0)} = E_r^{(0)}(r) \ , \quad E_\theta^{(0)} = E_z^{(0)} = 0 \quad \text{in the surrounding medium} \tag{83}$$

The problem proposed here is to determine within the first order approximation formulated in Section 4, the electric fields $E_r(r)$ and $E_r^{(0)}(r)$ and the surface tractions necessary to maintain the deformation without mechanical body forces or distributed change. The physical components of Finger strain are furnished by (38):

$$g_{rr} = \frac{1}{\lambda} ,$$

$$g_{\theta\theta} = \frac{1}{\lambda} + \varphi^2 r^2 ,$$

$$g_{zz} = \lambda^2 , \qquad (84)$$

$$g_{\theta z} = \lambda\varphi r ,$$

$$g_{r\theta} = g_{rz} = 0 .$$

To obtain the components of dielectric displacement field inside the dielectric body, we substitute (82) and (84) into (31). Thus,

$$D_r = (C_2 + \frac{1}{\lambda} C_3) E_r ,$$

$$D_\theta = 0 , \quad D_z = 0 . \qquad (85)$$

Outside the elastic dielectric tube, the dielectric displacement field follows from (7):

$$D_r^{(0)} = \varepsilon_0 E_r^{(0)} ,$$

$$D_\theta^{(0)} = 0 , \quad D_z^{(0)} = 0 \qquad (86)$$

The conditions (5) provide:

$$E_r = \frac{K}{r} , \quad E_r^{(0)} = \frac{L}{r} . \qquad (87)$$

where K and L are any constants.

Since the normal component of dielectric displacement field is continuous across the boundary surface of dielectric, the condition (6) implies:

$$L = (C_2 + \frac{1}{\lambda} C_3) \frac{K}{\varepsilon_0} \tag{88}$$

In place of (82) and (83), therefore, we now have:

$$E_r = \frac{K}{r}, \quad E_\theta = E_z = 0 \tag{89}$$

and

$$E_r^{(0)} = (C_2 + \frac{1}{\lambda} C_3) \frac{K}{\varepsilon_0 r}, \quad E^{(0)} = E_z^{(0)} = 0 \tag{90}$$

The corresponding stresses inside the dielectric follow using (84) and (89) in (30), and appear as:

$$\begin{aligned}
\sigma_{rr} &= -p + \frac{1}{\lambda} C_1 + (C_2 + \frac{2}{\lambda} C_3) \frac{K^2}{r^2}, \\
\sigma_{\theta\theta} &= -p + (\frac{1}{\lambda} + \varphi^2 r^2) C_1, \\
\sigma_{zz} &= -p + \lambda^2 C_1, \\
\sigma_{\theta z} &= \lambda\varphi r C_1, \\
\sigma_{r\theta} &= \sigma_{rz} = 0.
\end{aligned} \tag{91}$$

With the stress field (91), the equations of equilibrium (44) become:

$$\frac{\partial}{\partial r}(\sigma_{rr}) + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 ,$$
$$\frac{\partial p}{\partial \theta} = 0 , \quad \frac{\partial p}{\partial z} = 0 \tag{92}$$

The last two equations in (92) give us the information that the pressure p depends only on r. Hence, from the first equilibrium equation p(r) can be written as follows:

$$p(r) = \tfrac{1}{2}(C_2 + \frac{2}{\lambda} C_3) \frac{K^2}{r^2} - \tfrac{1}{2}C_1\varphi^2 r^2 + A \tag{93}$$

where A is a constant of integration.

In the medium surrounding the dielectric the stress field represented by the Maxwell stress (8) is given by:

$$\sigma^{(0)}_{rr} = (C_2 + \frac{1}{\lambda} C_3) \frac{K^2}{2\varepsilon_0 r^2} ,$$
$$\sigma^{(0)}_{\theta\theta} = -(C_2 + \frac{1}{\lambda} C_3) \frac{K^2}{2\varepsilon_0 r^2} ,$$
$$\sigma^{(0)}_{zz} = -(C_2 + \frac{1}{\lambda} C_3) \frac{K^2}{2\varepsilon_0 r^2} , \tag{94}$$
$$\sigma^{(0)}_{r\theta} = \sigma^{(0)}_{rz} = \sigma^{(0)}_{\theta z} = 0 .$$

Now we determine the surface tractions which should be applied on the boundaries of the tube to maintain the prescribed state.

On the surface $r = r_b$, the following forces are required:

$$(T_r)_{r=r_b} = (\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_b}$$

$$= (\frac{1}{\lambda} + \tfrac{1}{2}\varphi^2 r_b^2)C_1 + (C_2 + \frac{2}{\lambda}C_3)\frac{K^2}{2r_b^2} - (C_2 + \frac{1}{\lambda}C_3)\frac{K^2}{2\varepsilon_0 r_b^2} - A ,$$

$$(T_\theta)_{r=r_b} = (T_z)_{r=r_b} = 0 . \tag{95}$$

Assuming the outer surface $r = r_b$ force free, the arbitrary constant A can be evaluated as:

$$A = (\frac{1}{\lambda} + \tfrac{1}{2}\varphi^2 r_b^2)C_1 + (C_2 + \frac{2}{\lambda}C_3)\frac{K^2}{2r_b^2} - (C_2 + \frac{1}{\lambda}C_3)\frac{K^2}{2\varepsilon_0 r_b^2} \tag{96}$$

The following surface forces then must act on the inner surface $r = r_a$:

$$(T_r)_{r=r_b} = \tfrac{1}{2}\varphi^2(r_b^2 - r_a^2)C_1 + [\frac{1}{\varepsilon_0}(C_2 + \frac{1}{\lambda}C_3) - (C_2 + \frac{2}{\lambda}C_3)]\frac{r_b^2 - r_a^2}{2r_a^2 r_b^2}K^2 ,$$

$$(T_\theta)_{r=r_a} = (T_z)_{r=r_a} = 0 . \tag{97}$$

Besides the forces given by expression (97), normal and azimuthal surface tractions are required on the plane ends:

$$(T_r)_{z=\ell} = 0 ,$$

$$(T_\theta)_{z=\ell} = \lambda\varphi r C_1 , \tag{98}$$

$$(T_z)_{z=\ell} = (\lambda^2 - \frac{1}{\lambda})C_1 - \tfrac{1}{2}[(C_2 + \frac{2}{\lambda}C_3)K^2 - (C_2 + \frac{1}{\lambda}C_3)\frac{K^2}{\varepsilon_0}](\frac{1}{r^2} + \frac{1}{r_b^2})$$

These last two distributions of surface tractions, $(T_\theta)_{z=\ell}$ and $(T_z)_{z=\ell}$, yield the resultant couple M about the axis of the tube and the resultant longitudinal force N:

$$M = \int_{r_a}^{r_b} \int_0^{2\pi} T_\theta r^2 dr d\theta = \tfrac{1}{2}\pi\lambda\varphi(r_b^4 - r_a^4) \ , \tag{100}$$

$$\begin{aligned} N &= \int_{r_a}^{r_b} \int_0^{2\pi} T_z r dr d\theta \ , \\ &= \pi\{(\lambda^2 - \frac{1}{\lambda})(r_b^2 - r_a^2)C_1 \\ &\quad - [(C_2 + \frac{2}{\lambda} C_3)K^2 - (C_2 + \frac{1}{\lambda} C_3) \frac{K^2}{\varepsilon_0}](\frac{r_b^2 - r_a^2}{2r_b^2} + \ln \frac{r_b}{r_a}) \ . \end{aligned} \tag{101}$$

Once again, we observe that the ratio $\frac{N}{M/\varphi}$ is dependent on the stored energy function. However, if electrical effects are absent, then $K = 0$ and we find

$$\frac{N}{M/\varphi} = 2(\lambda - \frac{1}{\lambda^2}) \frac{r_b^2 - r_a^2}{r_b^4 - r_a^4} \ ,$$

which reestablishes the result of Green and Shield [5].

9. EXTENSION, INFLATION AND TORSION OF A COMPRESSIBLE TUBE IN A RADIAL DISPLACEMENT FIELD

Here we consider the extension, inflation and torsion of a compressible dielectric tube. The deformation is described by the mapping:

$$r = r(R) \, , \quad \theta = \Theta + \varphi Z \, , \quad z = \lambda Z \, . \tag{102}$$

The deformable dielectric is subjected to a known radial dielectric displacement field:

$$D_r = D_r(r) \, , \quad D_\theta = 0 \, , \quad D_z = 0 \, , \tag{103}$$

where

$$D_r = \frac{Q}{2\pi r} \, ,$$

in view of the conditions (5) and (6).

As has been mentioned in the introduction, the state described by (102) and (103) is not controllable. However, our purpose here is to indicate the procedure of finding the surface tractions in case some special functional form of the stored energy is given.

The physical components of strain g_{ij} are given by the following expressions:

$$
\begin{aligned}
g_{rr} &= r'^2 , \\
g_{\theta\theta} &= \left(\frac{r}{R}\right)^2 + \varphi^2 r^2 , \\
g_{zz} &= \lambda^2 , \\
g_{\theta z} &= \lambda\varphi r , \\
g_{r\theta} &= g_{rz} = 0 ,
\end{aligned}
\tag{104}
$$

where the prime denotes differentiation with respect to R.

The six invariants defined in (22) are found to be given by:

$$
\begin{aligned}
I_1^* &= r'^2 + \left(\frac{r}{R}\right)^2 + \varphi^2 r^2 + \lambda^2 , \\
I_2^* &= r'^2\left[\lambda^2 + \left(\frac{r}{R}\right)^2 + \varphi^2 r^2\right] + \lambda^2\left(\frac{r}{R}\right)^2 , \\
I_3^* &= \lambda^2 r'^2 \left(\frac{r}{R}\right)^2 , \\
I_4^* &= \frac{Q^2}{4\pi^2 r^2} , \\
I_5^* &= r'^2 \frac{Q^2}{4\pi^2 r^2} , \\
I_6^* &= r'^4 \frac{Q^2}{4\pi^2 r^2} .
\end{aligned}
\tag{105}
$$

The invariants $I_1^*, I_2^*, \ldots, I_6^*$ are functions of r (or R) only.

The electric field throughout the dielectric is furnished by relation (21). We obtain

$$E_r = \frac{2}{\lambda^2 r'^2 \left(\frac{r}{R}\right)^2} \left[\frac{\partial W}{\partial I_4^*} + r'^2 \frac{\partial W}{\partial I_5^*} + r'^4 \frac{\partial W}{\partial I_6^*} \right] \frac{Q}{2\pi r} ,$$

(106)

$$E_\theta = E_z = 0 .$$

Outside the dielectric, the electric field components follow from (7):

$$E_r^{(0)} = \frac{Q}{2\pi\varepsilon_0 r} ,$$

(107)

$$E_\theta^{(0)} = E_z^{(0)} = 0 .$$

The tangential component of electric field is trivially continuous across the cylindrical surfaces of the tube. Also, $\underline{\nabla} \cdot \underline{E} = 0$ within the dielectric and outside it. Thus, the electric field given as in (106) and (107) meets the conditions (3) and (4).

Introducing (104), (105) and (106) in (23) we find the stress field inside the dielectric:

$$\sigma_{rr} = r'^2\Psi_1 + r'^4\Psi_2 + \Psi_3 + [\Psi_4 + 2r'^2\Psi_5 + 3r'^4\Psi_6] \frac{Q^2}{4\pi^2 r^2} ,$$

$$\sigma_{\theta\theta} = \left[\left(\frac{r}{R}\right)^2 + \varphi^2 r^2\right]\Psi_1 + \left[\left(\frac{r}{R}\right)^2 + 2\varphi^2 r^2 \left(\frac{r}{R}\right)^2 + \varphi^4 r^4 + \lambda^2\varphi^2 r^2\right]\Psi_2 + \Psi_3 ,$$

$$\sigma_{zz} = \lambda^2\Psi_1 + (\lambda^4 + \lambda^2\varphi^2 r^2)\Psi_2 + \Psi_3 ,$$

$$\sigma_{\theta z} = \lambda\varphi r\Psi_1 + \lambda\varphi r[\lambda^2 + \varphi^2 r^2 + (\frac{r}{R})^2]\Psi_2 ,$$

$$\sigma_{r\theta} = \sigma_{rz} = 0 , \tag{108}$$

where

$$\Psi_1 = \frac{2}{I_3^{*\frac{1}{2}}} (\frac{\partial W}{\partial I_1^*} + I_1^* \frac{\partial W}{\partial I_3^*}) ,$$

$$\Psi_2 = - \frac{2}{I_3^{*\frac{1}{2}}} \frac{\partial W}{\partial I_3^*} ,$$

$$\Psi_3 = \frac{2}{I_3^{*\frac{1}{2}}} I_3^* \frac{\partial W}{\partial I_3^*} ,$$

$$\Psi_4 = \frac{2}{I_3^{*\frac{1}{2}}} \frac{\partial W}{\partial I_4^*} , \tag{109}$$

$$\Psi_5 = \frac{2}{I_5^{*\frac{1}{2}}} \frac{\partial W}{\partial I_5} ,$$

$$\Psi_6 = \frac{2}{I_6^{*\frac{1}{2}}} \frac{\partial W}{\partial I_6} .$$

Since the stress components are independent of θ and z, the equations of equilibrium reduce to the single equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{110}$$

With stress field given by relations (108), the equation (110) can be rewritten in the form:

$$r'^2\Psi_1' + r'^4\Psi_2' + \Psi_3' + [\Psi_4' + 2r'^2\Psi_5' + 3r'^4\Psi_6] \frac{Q^2}{4\pi^2 r^2}$$

$$+ 2\Psi_1 rr' + 4\Psi_2 r'^3 r'' + [4r'r''\Psi_5 + 12r'^3 r''\Psi_6] \frac{Q^2}{4\pi^2 r^2}$$

$$- [\Psi_4 + 2r'^2\Psi_5 + 3r'^4\Psi_6] \frac{Q^2}{4\pi^2 r^3} + \frac{1}{r} \{[r'^2 - (\frac{r}{R})^2 - \varphi^2 r^2]\Psi_1$$

$$+ [r'^4 - (\frac{r}{R})^2 - 2\varphi^2 r^2 (\frac{r}{R})^2 - \varphi^4 r^4 - \lambda^2\varphi^2 r^2]\Psi_2\} = 0 . \quad (111)$$

If we know the dependence of $\Psi_1, \Psi_2, \ldots \Psi_6$ on r, i.e. the stored energy W for the compressible body is known explicitly, then the solution, if it exists, of the ordinary non-linear equation (111) will give us the dependence of r on R i.e. $r = r(R)$.

The stress components in the medium surrounding the dielectric follow from (7):

$$\sigma^{(0)}_{rr} = \frac{Q^2}{8\pi^2\varepsilon_0 r^2} ,$$

$$\sigma^{(0)}_{\theta\theta} = -\frac{Q^2}{8\pi^2\varepsilon_0 r^2} , \quad (112)$$

$$\sigma^{(0)}_{zz} = -\frac{Q^2}{8\pi^2\varepsilon_0 r^2} ,$$

$$\sigma^{(0)}_{r\theta} = \sigma^{(0)}_{rz} = \sigma^{(0)}_{\theta z} = 0 .$$

In order to support the deformation, the following forces per unit area of the deformed configuration must be applied on the boundary:

(i) radial surface tractions on the inner surface $r = r_b$ and outer surface $r = r_b$:

$$(T_r)_{r=r_a} = -(\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_a} \; , \qquad (113)$$

$$(T_r)_{r=r_b} = (\sigma_{rr} - \sigma_{rr}^{(0)})_{r=r_b} \; , \qquad (114)$$

(ii) azimuthal and normal surface tractions on plane ends:

$$(T_\theta)_{z=\ell} = (\sigma_{\theta z} - \sigma_{\theta z}^{(0)})_{z=\ell} \; , \qquad (115)$$

$$(T_z)_{z=\ell} = (\sigma_{zz} - \sigma_{zz}^{(0)})_{z=\ell} \; , \qquad (116)$$

where, assuming that $r = r(R)$ is known, $\sigma_{rr}, \sigma_{\theta z} \ldots$ are given by expressions (108) and (112).

REFERENCES

1. Toupin, R.A., The Elastic Dielectric. Journal of Rational Mechanics and Analysis, 5 849 (1956).

2. Eringen, A.C., On the foundations of electro-elastostatics. Intl. J. Engr. Sci. 1, 127 (1963).

3. Singh, M. and A.C. Pipkin, Controllable States of Elastic Dielectrics. Arch. Rational Mech. Anal. 21, 169 (1966).

4. Rivlin, R.S., Large Elastic Deformations of Isotropic Materials. Phil. Trans. Roy. Soc. Lond. A. 240 459 (1948).

5. Green, A.E. and R.T. Shield, Finite Extension and Torsion of Cylinders. Phil. Trans. Roy. Soc. Lond. A 244, 47 (1951).

6. Singh, M., Controllable States in Compressible Elastic Dielectrics. ZAMP. 17, 449 (1966).

7. Singh, M., Small Finite Deformations of Elastic Dielectrics. Quarterly of Applied Mathematics. 25, 275 (1967).