NAME OF AUTHOR/*NOM DE L'AUTEUR* DAN SONNENSCHEIN

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NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* A.R. FREEDMAN

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A GENERAL THEORY OF ASYMPTOTIC DENSITY

by

Dan Jacob Sonnenschein

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Name: Dan Jacob Sonnenschein

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Examining Committee:

Chairperson: Alistair H. Lachlan

Allen R. Freedman
Senior Supervisor

John J. Sember

Norman R. Reilly

G. A. Cecil Graham
External Examiner

Date Approved: March 30, 1978

ABSTRACT

The usual lower asymptotic density is generalized by defining an abstract density as a set function satisfying certain axioms. A generalized upper density and natural density are defined, and elementary properties of density, upper density and natural density are proved. This abstract density theory is in part analogous to measure theory, except that countable additivity for natural density does not hold.

Various asymptotic densities including the usual one are shown to be associated with summability methods, in particular matrix methods of summability. Necessary and sufficient conditions on a matrix in order for its associated set function to be an asymptotic density are given. An approximation to countable additivity for natural density is shown to hold for the class of densities associated with regular matrix methods of summability.

The order relations among particular densities are discussed, and a natural density which is an extension of the natural ordinary density is presented. Some applications to summability theory and number theory are given, and some unsolved problems are stated.

DEDICATION

to my mother

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TABLE OF CONTENTS

§1 INTRODUCTION

Let I denote the set of positive integers, and let 2^I denote the class of subsets of I. For $A \in 2^I$, define the counting function $A(n)$ as the number of elements in $A \cap \{1, . . . ,n\}$. The usual (or ordinary) lower asymptotic density δ is defined by $\delta(A) = \liminf_n \frac{A(n)}{n}$, and the usual upper asymptotic density $\bar{\delta}$ is defined by $\bar{\delta}(A) = \limsup_n \frac{A(n)}{n}$. If the ordinary lower and upper densities of A are equal, A is said to have natural ordinary density ν_δ, given by $\nu_\delta(A) = \lim_n \frac{A(n)}{n}$.

Among the elementary properties of ordinary density are the asymptotic and translation invariant properties, which are defined in Chapter 2. There two axiom systems are given, defining an abstract asymptotic density and an abstract translation invariant density, both of which are generalizations of the usual lower asymptotic density. It is shown that a translation invariant density is an asymptotic density, but that an asymptotic density is not necessarily translation invariant. The term density is used to refer to both types of abstract density.

An upper density is defined in terms of the density, and a generalized natural density is defined on the class of sets with equal density and upper density. Various elementary properties of density upper density, and natural density are proved in Chapter 2. Of particular importance is that while a natural density is finitely additive, in contrast to a measure it is not countably additive.

Buck [1] has defined in a different way a finitely additive density function, which has other properties of measure as well. He also defines a class of "limit densities," based on the observation that the natural ordinary density of a set, when it exists, is the Cesaro limit of the characteristic sequence of the set. (The usual lower and upper densities of a set are the lim inf and lim sup respectively, of the Cesaro transform of the characteristic sequence of the set.) This observation suggests the possibility of applying other summability methods in the same way to obtain other limit densities. The connection between density and regular matrix methods of summability is studied in Chapter 3. There we give necessary and sufficient conditions on a regular matrix in order for its associated set function to be an asymptotic density. Another main result of this chapter is that an approximation to countable additivity for natural density, called the additivity property, holds for the class of densities associated with regular matrix methods of summability.

The density studied in Chapter 4 is obtained from the Abel summability method, a regular "semi-continuous matrix method" of summability stronger than Cesaro summability. Well-known results are stated which show that natural Abel density is equivalent to natural ordinary density, and thus that Abel density has the additivity property.

The uniform density of Chapter 4 is associated with the summability method of "almost convergence," which is not a regular matrix method. An example is given to show that uniform density does not have the additivity property. Natural ordinary density is shown to be a proper extension of natural uniform density.

The logarithmic density of Chapter 5 is another density obtained from a regular matrix method of summability, and so it has the additivity property. Known results are stated which show that natural logarithmic density is a proper extension of natural ordinary density.

In the conclusion, we indicate some of the uses of the concept of density in summability theory and number theory, and state some of the unsolved problems of this general theory of asymptotic density.

The notation and some conventions that will be used are as follows. References to a set A or "all A" will mean a set $A \in 2^I$, or "all $A \in 2^I$." Unless otherwise indicated, lowercase English letters refer to positive integers, so that for example, "all i" means "all $i \in I$." The set-theoretic difference $\{a \in A \mid a \notin B\}$ is denoted by $A \setminus B$, and the complement $I \setminus A$ is denoted by $\bar{A}$. Proper containment is denoted by $\subset$, and containment by $\subseteq$. Finally, the order of the terms in many infinite series is rearranged without explicitly mentioning the absolute convergence that allows it to be done.

§2 AXIOMS AND ELEMENTARY PROPERTIES

In this chapter, two sets of axioms for an abstract density are given. These axiom systems define an asymptotic density and a translation invariant density, both of which are generalizations of the usual lower asymptotic density. It will be shown that a translation invariant density is an asymptotic density, but that an asymptotic density is not necessarily translation invariant. Unless otherwise indicated, the word "density" is used to refer to both types of density.

An abstract upper density is defined in terms of the density, and a set is said to have natural density if its density and upper density are equal. There is a partial analogy of density, upper density, and natural density to inner measure, outer measure, and measure respectively. The analogy is not complete because countable additivity of a measure does not carry over to countable additivity of a natural density. However, an approximation to countable additivity will be shown to hold for some natural densities.

We begin by giving some definitions which are used in the formulation of the axioms.

<u>Definition 2.1</u>. Given a positive integer n, the <u>translation</u> (by n) of set A, written $A + n$, is $\{x + n \mid x \in A\}$.

<u>Definition 2.2</u>. Set A is <u>asymptotic</u> to set B, written $A \sim B$, means that the symmetric difference $A \Delta B$ is finite.

The relation $\sim$ is easily seen to be reflexive and symmetric.

It is transitive as well, since if both $A \Delta B$ and $B \Delta C$ are finite, then $A \Delta C = (A \cap \overline{C}) \cup (C \cap \overline{A}) \leq [(A \cap \overline{B}) \cup (B \cap \overline{C})] \cup [(C \cap \overline{B}) \cup (B \cap \overline{A})]$, which is finite. Thus $\sim$ is an equivalence relation, and so we may refer to asymptotic sets as equivalent sets. A useful characterization of equivalent sets is given by the following lemma.

<u>Lemma 2.1.</u> $A \sim B$ if and only if there exists an N such that $A \setminus \{1, \ldots, N\} = B \setminus \{1, \ldots, N\}$ or $A \cup \{1, \ldots, N\} = B \cup \{1, \ldots, N\}$.

Proof. If $A \sim B$, so that $A \Delta B$ is finite, there is an initial segment $\{1, \ldots, N\}$ containing the finitely many integers that are in exactly one of A or B. Then an integer greater than N is in neither set or in both sets, so that $A \setminus \{1, \ldots, N\} = B \setminus \{1, \ldots, N\}$ and $A \cup \{1, \ldots, N\} = B \cup \{1, \ldots, N\}$. If either of the latter conditions holds for some N, then there are at most N integers in exactly one of A or B, so that $A \Delta B$ is finite and $A \sim B$.

Two systems of axioms for a function d on 2^I are now given, defining the two abstract densities. The axioms of each set will be shown to be independent. The first set is:

1. for all A, $0 \leq d(A) \leq 1$;

2. $d(I) = 1$;

3. if $A \sim B$, then $d(A) = d(B)$;

4. if $A \cap B = \phi$, then $d(A) + d(B) \leq d(A \cup B)$;

5. for all A, B, $d(A) + d(B) \leq 1 + d(A \cap B)$.

The second set of axioms is obtained from the first by replacing Axiom 3 with:

3'. for all A, $d(A + 1) = d(A)$.

Definition 2.3. A function d on 2^I is called an asymptotic density if it satisfies the first set of axioms.

Definition 2.4. A function d on 2^I is called a translation invariant density if it satisfies the second set of axioms.

Theorem 2.1. The axioms in each of the above systems are independent.

Proof. Define a function d on 2^I by $d(A) = 2\delta(A) - 1$, where δ is the usual lower density. Then $d(I) = 1$ (Axiom 2); $d(A + 1) = 2\delta(A + 1) - 1 = 2\delta(A) - 1 = d(A)$ for all A (Axiom 3'); and if $A \sim B$, then $d(A) = 2\delta(A) - 1 = 2\delta(B) - 1 = d(B)$ (Axiom 3). Also, if $A \cap B = \phi$, then $d(A) + d(B) = 2(\delta(A) + \delta(B)) - 2 \leq 2\delta(A \cup B) - 2 < 2\delta(A \cup B) - 1 = d(A \cup B)$ (Axiom 4); and for all A, B, $d(A) + d(B) = 2(\delta(A) + \delta(B)) - 2 \leq 2(1 + \delta(A \cap B)) - 2 = 2\delta(A \cap B) = 1 + d(A \cap B)$ (Axiom 5). But for $\delta(A) < \frac{1}{2}$, $d(A) < 0$, so Axiom 1 does not hold, and is therefore independent in both sets.

The independence of Axiom 2 is shown by defining d on 2^I by $d(A) = \frac{1}{2}\delta(A)$. It follows easily that all axioms except Axiom 2 hold.

For Axioms 3 and 3', consider the function d on 2^I defined by $d(A) = \sum_{a \in A} \frac{1}{2^a}$. Then clearly Axioms 1 and 2 hold. If $A \cap B = \phi$, then $d(A) + d(B) = \sum_{n \in A} \frac{1}{2^n} + \sum_{n \in B} \frac{1}{2^n} = \sum_{n \in A \cup B} \frac{1}{2^n} = d(A \cup B)$, so that Axiom 4 holds (with equality). Now using this axiom and Axiom 1, we have $d(A) + d(B) = \sum_{n \in A} \frac{1}{2^n} + \sum_{n \in B} \frac{1}{2^n} = \sum_{n \in A \setminus B} \frac{1}{2^n} + \sum_{n \in B \setminus A} \frac{1}{2^n} + 2 \sum_{n \in A \cap B} \frac{1}{2^n} = \sum_{n \in A \Delta B} \frac{1}{2^n} + \sum_{n \in A \cap B} \frac{1}{2^n} + \sum_{n \in A \cap B} \frac{1}{2^n} = \sum_{n \in A \cup B} \frac{1}{2^n} + \sum_{n \in A \cap B} \frac{1}{2^n} \leq 1 + d(A \cap B)$, so Axiom 5 is satisfied. But for $A = \{1\}$, $A + 1 \sim A$ and $d(A + 1) \neq d(A)$, so that Axioms 3 and 3' are not satisfied. Thus these axioms are independent in their respective systems.

To see that Axiom 4 is independent, let the term 3-progression denote an infinite arithmetic progression with common difference 3, and define a function d on 2^I by:

$$d(A) = \begin{cases} 1 & \text{if } A \sim I; \\ \frac{1}{2} & \text{if } A \not\sim I \text{ and contains a 3-progression;} \\ 0 & \text{otherwise.} \end{cases}$$

Axioms 1 and 2 clearly hold, and it is routine to verify that Axioms 3, 3' and 5 are also satisfied. To see that Axiom 4 does not hold, let $A = \{1,4,7, \ldots\}$ and $B = \{2,5,8, \ldots\}$. Then $d(A) = d(B) = d(A \cup B) = \frac{1}{2}$, so that $d(A) + d(B) > d(A \cup B)$.

Finally, to show that Axiom 5 is independent, define a function d on 2^I by:

$$d(A) = \begin{cases} 1 & \text{if } A \text{ contains two disjoint 3-progressions;} \\ 0 & \text{otherwise.} \end{cases}$$

Again, Axioms 1 and 2 clearly hold, and it may be easily verified that Axioms 3, 3' and 4 also hold. But if we let $A = \{1,4,7,...\} \cup \{2,5,8,...\}$, and $B = \{1,4,7,...\} \cup \{3,6,9,...\}$, then $d(A) = d(B) = 1$, while $d(A \cap B) = 0$. Thus $d(A) + d(B) > 1 + d(A \cap B)$, so Axiom 5 does not hold. This concludes the proof.

The next two lemmas are extensions of Axioms 3' and 4, following by induction on i.

<u>Lemma 2.2</u>. If d is a translation invariant density, then for all A and all $i \geq 1$, $d(A + i) = d(A)$.

<u>Lemma 2.3</u>. If d is a density, and $\{A_i\}_{i=1}^{n}$ is a finite collection of disjoint sets, then $\sum_{i=1}^{n} d(A_i) \leq d\left(\bigcup_{i=1}^{n} A_i\right)$.

To prove countable superadditivity of a density, we first require the following monotone property.

<u>Theorem 2.2</u>. If d is a density and $A \subseteq B$, then $d(A) \leq d(B)$.

Proof. By Axioms 1 and 4, $d(A) \leq d(A) + d(B \setminus A) \leq d(A \cup (B \setminus A)) = d(B)$.

Theorem 2.3. If d is a density, and $\{A_i\}_{i=1}^{\infty}$ is a countable collection of disjoint sets, then $\sum_{i=1}^{\infty} d(A_i) \leq d(\bigcup_{i=1}^{\infty} A_i)$.

Proof. Suppose on the contrary that there exists a countable collection of disjoint sets $\{A_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} d(A_i) > d(\bigcup_{i=1}^{\infty} A_i)$. Note that since for all n, $\sum_{i=1}^{n} d(A_i) \leq d(\bigcup_{i=1}^{n} A_i) \leq 1$, we have $\sum_{i=1}^{\infty} d(A_i) \leq 1$. So we may let $\sum_{i=1}^{\infty} d(A_i) = d(\bigcup_{i=1}^{\infty} A_i) + \delta$, where $0 < \delta \leq 1$. Then there exists an N such that $\sum_{i=1}^{N} d(A_i) > d(\bigcup_{i=1}^{\infty} A_i) + \frac{\delta}{2}$, and by monotonicity, $d(\bigcup_{i=1}^{\infty} A_i) \geq d(\bigcup_{i=1}^{N} A_i)$. This gives $\sum_{i=1}^{N} d(A_i) > d(\bigcup_{i=1}^{N} A_i)$, contradicting finite superadditivity. Therefore, for every collection $\{A_i\}_{i=1}^{\infty}$ of disjoint sets, $\sum_{i=1}^{\infty} d(A_i) \leq d(\bigcup_{i=1}^{\infty} A_i)$.

We now introduce the abstract upper density. Given a function d on 2^I, define a function $\bar{d}$ on 2^I by $\bar{d}(A) = 1 - d(\bar{A})$.

Definition 2.5. If d is a density, then the function $\bar{d}$ is called the associated upper density.

Note that the upper density is defined in terms of the density alone. The order relation of density to upper density is as follows.

Theorem 2.4. If d is a density, then for all A, $d(A) \leq \bar{d}(A)$.

Proof. Using Axioms 2 and 4, we have $d(A) = d(A) + d(\bar{A}) - d(\bar{A}) \leq d(A \cup \bar{A}) - d(\bar{A}) = d(I) - d(\bar{A}) = 1 - d(\bar{A}) = \bar{d}(A)$.

The monotone property also holds for upper density.

Theorem 2.5. If $\overline{d}$ is an upper density, and $A \subseteq B$, then $\overline{d}(A) \le \overline{d}(B)$.

Proof. If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$, so by the monotonicity of density, $d(\overline{B}) \le d(\overline{A})$. Then $\overline{d}(A) = 1 - d(\overline{A}) \le 1 - d(\overline{B}) = \overline{d}(B)$.

The following is a "mixed density" result.

Theorem 2.6. If $A \cap B = \phi$, then $d(A \cup B) \le d(A) + \overline{d}(B)$.

Proof. Let $A' = A \cup B$ and $B' = \overline{B}$, so that $A' \cap B' = A$. By Axiom 5, $d(A \cup B) + d(\overline{B}) = d(A') + d(B') \le 1 + d(A' \cap B') = 1 + d(A)$. Thus $d(A) + \overline{d}(B) = d(A) + 1 - d(\overline{B}) \ge d(A \cup B)$.

The next theorem shows that adding (deleting) a set of upper density zero to (from) any given set does not alter the density of the given set.

Theorem 2.7. If d is a density and B is a set such that $\overline{d}(B) = 0$, then for all A, $d(A \cup B) = d(A \setminus B) = d(A)$.

Proof. Let $C = B \setminus A$, so that $C \subseteq B$, $A \cup B = A \cup C$, and $A \cap C = \phi$. Then by monotonicity and Theorem 2.6, $d(A) \le d(A \cup B) = d(A \cup C) \le d(A) + \overline{d}(C) \le d(A) + \overline{d}(B) = d(A)$. So $d(A \cup B) = d(A)$. Furthermore, since $A \setminus B \subseteq A$ and $A \cap B \subseteq B$, we have $d(A \setminus B) \le d(A) = d((A \setminus B) \cup (A \cap B)) \le d(A \setminus B) + \overline{d}(A \cap B) = d(A \setminus B)$. Thus $d(A \setminus B) = d(A)$.

We now show that the upper density of a finite set is zero.

Theorem 2.8. If d is a density and F is a finite set, then $\overline{d}(F) = d(F) = 0$.

Proof. If F is a finite set, then $\overline{F} \sim I$. So if d is an asymptotic density, then by Axioms 2 and 3, $\overline{d}(F) = 1 - d(\overline{F}) = 1 - d(I) = 0$. We also have for a finite set F that for some k, $I + k \subseteq \overline{F}$. So if d is a translation invariant density, then by Axiom 2, Lemma 1.2 and the monotone property, $\overline{d}(F) = 1 - d(\overline{F}) \leq 1 - d(I + k) = 1 - d(I) = 0$. With Axiom 1, this gives $\overline{d}(F) = 0$.

In both cases, since $\overline{d}(F) \geq d(F) \geq 0$, we have $d(F) = 0$.

Now we demonstrate the relation between asymptotic density and translation invariant density that was mentioned earlier.

Theorem 2.9. A translation invariant density is an asymptotic density.

Proof. Let d be a translation invariant density. If $A \sim B$, then by Lemma 1.1 there exists an N such that $A \cup \{1, \ldots, N\} = B \cup \{1, \ldots, N\}$. Then by Theorems 2.7 and 2.8, $d(A) = d(A \cup \{1, \ldots, N\}) = d(B \cup \{1, \ldots, N\}) = d(B)$. Thus Axiom 3 holds, and so d is an asymptotic density.

The converse of the above theorem is not true. An example of an asymptotic density that is not translation invariant will be given in Chapter 3.

In the following theorem we give properties of upper density corresponding to the axioms. Note that Properties $\overline{4}$ and $\overline{5}$ are not exactly analogous to Axioms 4 and 5.

Theorem 2.10. If $\bar{d}$ is an upper density, then:

$\bar{1}$. for all A, $0 \le \bar{d}(A) \le 1$;

$\bar{2}$. $\bar{d}(I) = 1$;

$\bar{3}$. if $A \sim B$, then $\bar{d}(A) = \bar{d}(B)$;

$\bar{4}$. for all A, B, $\bar{d}(A) + \bar{d}(B) \ge \bar{d}(A \cup B)$;

$\bar{5}$. if $A \cup B = I$, then $\bar{d}(A) + \bar{d}(B) \ge 1 + \bar{d}(A \cap B)$.

If $\bar{d}$ is an upper translation invariant density, then:

$\bar{3}'$. for all A, $\bar{d}(A + 1) = \bar{d}(A)$.

Proof.

$\bar{1}$. For any A, $\bar{d}(A) = 1 - d(\bar{A})$ and $0 \le d(\bar{A}) \le 1$, so that $0 \le \bar{d}(A) \le 1$.

$\bar{2}$. Since the density of a finite set is zero, $\bar{d}(I) = 1 - d(\phi) = 1$.

$\bar{3}$. First observe that $A \Delta B = \bar{A} \Delta \bar{B}$. Then $A \sim B$ implies that $\bar{A} \Delta \bar{B}$ is finite, from which $\bar{A} \sim \bar{B}$. So using Axiom 3, we have $\bar{d}(A) = 1 - d(\bar{A}) = 1 - d(\bar{B}) = \bar{d}(B)$.

$\bar{4}$. By definition, $\bar{d}(A) + \bar{d}(B) = 1 - d(\bar{A}) + 1 - d(\bar{B}) = 2 - (d(\bar{A}) + d(\bar{B}))$. By Axiom 5, for all A, B, $d(\bar{A}) + d(\bar{B}) \le 1 + d(\bar{A} \cap \bar{B})$.

Then $\overline{d}(A) + \overline{d}(B) = 2 - (d(\overline{A}) + d(\overline{B})) \geq 2 - (1 + d(\overline{A} \cap \overline{B})) =$ $1 - d(\overline{A} \cap \overline{B}) = 1 - d(\overline{A \cup B}) = \overline{d}(A \cup B)$.

$\overline{5}$. By definition, $\overline{d}(A \cap B) = 1 - d(\overline{A \cap B}) = 1 - d(\overline{A} \cup \overline{B})$. Now $A \cup B = I$ implies that $\overline{A} \cap \overline{B} = \overline{A \cup B} = \phi$, so by Axiom 4 we have $d(\overline{A} \cup \overline{B}) \geq d(\overline{A}) + d(\overline{B})$. Thus $\overline{d}(A \cap B) = 1 - d(\overline{A} \cup \overline{B}) \leq 1 -$ $(d(\overline{A}) + d(\overline{B}))$, and $1 + \overline{d}(A \cap B) \leq 2 - (d(\overline{A}) + d(\overline{B})) = \overline{d}(A) + \overline{d}(B)$.

$\overline{3}'$. First observe that $\overline{A + 1} = (\overline{A} + 1) \cup \{1\}$. For $x \in \overline{A + 1}$ iff $x \neq a + 1$ for any $a \in A$, or $x = 1$ iff $x - 1 \notin A$, or $x = 1$ iff $x - 1 \in \overline{A}$, or $x = 1$ iff $x \in (\overline{A} + 1) \cup \{1\}$. Then by Theorems 2.7 and 2.8, $d(\overline{A + 1}) = d(\overline{A} + 1)$. Therefore, using Axiom 3', $\overline{d}(A + 1) = 1 - d(\overline{A + 1}) = 1 - d(\overline{A} + 1) = 1 - d(\overline{A}) = \overline{d}(A)$.

The next two lemmas are extensions of Properties $\overline{3}'$ and $\overline{4}$, and follow by induction on i.

Lemma 2.4. If $\overline{d}$ is an upper translation invariant density, then for all A and all $i \geq 1$, $\overline{d}(A + i) = \overline{d}(A)$.

Lemma 2.5. If $\overline{d}$ is an upper density, and $\{A_i\}_{i=1}^{n}$ is any finite collection of sets, then $\sum_{i=1}^{n} \overline{d}(A_i) \geq \overline{d}\left(\bigcup_{i=1}^{n} A_i\right)$.

It follows from Theorem 2.8 (zero upper density of a finite set) that the above lemma cannot be extended to countable collections of sets. For $\sum_{i=1}^{\infty} \overline{d}(\{i\}) = 0 < 1 = \overline{d}(I) = \overline{d}\left(\bigcup_{i=1}^{\infty} \{i\}\right)$. It is of interest to note that while for density we have countable superadditivity of collections of disjoint sets, for upper density we have finite but not countable subadditivity of collections of arbitrary sets.

We can use finite subadditivity to show that the addition (deletion) of a set of zero upper density to (from) a given set does not alter the upper density of the given set.

<u>Theorem 2.11</u>. If $\overline{d}$ is an upper density, and B is a set such that $\overline{d}(B) = 0$, then for all A, $\overline{d}(A \cup B) = \overline{d}(A \setminus B) = \overline{d}(A)$.

Proof. By monotonicity and Property $\overline{4}$, $\overline{d}(A) \leq \overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B) = \overline{d}(A)$, so that $\overline{d}(A \cup B) = \overline{d}(A)$. Also, $\overline{d}(A \setminus B) \leq \overline{d}(A) = \overline{d}((A \setminus B) \cup (A \cap B)) \leq \overline{d}(A \setminus B) + \overline{d}(A \cap B) = \overline{d}(A \setminus B)$. Thus $\overline{d}(A \setminus B) = \overline{d}(A)$.

The next three theorems include stronger forms of Axiom 4, Theorem 2.6 and Property $\overline{4}$.

<u>Theorem 2.12</u>. If d is a density and $\overline{d}(A \cap B) = 0$, then $d(A) + d(B) \leq d(A \cup B)$.

Proof. Since $\overline{d}(A \cap B) = 0$, we have $d(A \cap B) = 0$. Then using Axiom 4 and Theorem 2.7, $d(A \cup B) = d((A \Delta B) \cup (A \cap B)) \geq d(A \Delta B) + d(A \cap B) = d(A \Delta B) = d((A \setminus B) \cup (B \setminus A)) \geq d(A \setminus B) + d(B \setminus A) = d(A \setminus (A \cap B)) + d(B \setminus (A \cap B)) = d(A) + d(B)$.

<u>Corollary 2.1</u>. If d is a density, and $\{A_i\}_{i=1}^{\infty}$ is a

countable collection of sets such that for $i \neq j$, $\overline{d}(A_i \cap A_j) = 0$, then $\sum_{i=1}^{\infty} d(A_i) \leq d\left(\bigcup_{i=1}^{\infty} A_i\right)$.

Proof. This follows from the above theorem like the proofs of Lemma 2.3 and Theorem 2.3.

Theorem 2.13. If d is a density and $\overline{d}(A \cap B) = 0$, then $d(A \cup B) \leq d(A) + \overline{d}(B) \leq \overline{d}(A \cup B)$.

Proof. Let $A' = A \cup B$ and $B' = \overline{B}$. Then $A' \cap B' = A \setminus B = A \setminus (A \cap B)$, so by Theorem 2.7, $d(A' \cap B') = d(A)$. The proof of $d(A \cup B) \leq d(A) + \overline{d}(B)$ then follows like that of Theorem 2.6.

For the second inequality, since $\overline{A} \cup (A \cup B) = I$, we have by Property $\overline{5}$ and Theorem 2.11 that $\overline{d}(\overline{A}) + \overline{d}(A \cup B) \geq 1 + \overline{d}(\overline{A} \cap (A \cup B)) = 1 + \overline{d}(B \setminus A) = 1 + \overline{d}(B \setminus (A \cap B)) = 1 + \overline{d}(B)$. Therefore $\overline{d}(A \cup B) \geq 1 - \overline{d}(\overline{A}) + \overline{d}(B) = d(A) + \overline{d}(B)$.

Theorem 2.14. If d is a density and $d(A \cup B) = 1$, then $\overline{d}(A) + \overline{d}(B) \geq 1 + \overline{d}(A \cap B)$.

Proof. We have $\overline{d}(\overline{A} \cap \overline{B}) = 1 - d(A \cup B) = 0$, so by Theorem 2.12, $d(\overline{A}) + d(\overline{B}) \leq d(\overline{A} \cup \overline{B})$. The rest of the proof follows like that of Property $\overline{5}$.

We can calculate or give inequalities for the abstract density of certain classes of sets, as the next two theorems show.

Theorem 2.15. Let A be such that for $i \neq j$, $(A + i) \cap (A + j) \sim \phi$, and let d be a translation invariant density. Then

$d(A) = 0$.

Proof. For each i, let $B_i = A + i$, so that for $i \neq j$, $B_i \cap B_j \sim \phi$. By Axiom 3', $d(B_i) = d(A)$ for all i. Then by Corollary 2.1, $\sum_{i=1}^{\infty} d(A) = \sum_{i=1}^{\infty} d(B_i) \leq d\left(\bigcup_{i=1}^{\infty} B_i\right) \leq 1$. This implies that $d(A) = 0$.

Examples of "sparse" sets meeting the requirements of the above theorem are $\{2^n\}$ and $\{n^2\}$.

Theorem 2.16. Let $A = \{kn\}_{n=1}^{\infty}$, for some $k \geq 1$, and let d be a translation invariant density. Then $d(A) \leq \frac{1}{k}$ and $\bar{d}(A) \geq \frac{1}{k}$.

Proof. Let $A_0 = A$, and for $i = 1, \ldots, k-1$, let $A_i = \{kn + i\}_{n=1}^{\infty} = A_0 + i$. Then by Axiom 3', $d(A_i) = d(A)$ for $i = 1, \ldots, k-1$. Also, the A_i are disjoint and $\bigcup_{i=0}^{k-1} A_i = I$. Thus by Lemma 2.3, $kd(A) = \sum_{i=0}^{k-1} d(A_i) \leq d(\bigcup_{i=0}^{k-1} A_i) = 1$, from which $d(A) \leq \frac{1}{k}$. Similarly, using Lemma 2.5, we get $k\bar{d}(A) \geq 1$ and $\bar{d}(A) \geq \frac{1}{k}$.

Definition 2.7. Let a density d be called a normal density if for every arithmetic progression $A = \{kn\}_{n=1}^{\infty}$ with $k \geq 1$, $d(A) = \bar{d}(A) = \frac{1}{k}$.

An example of a normal density is the usual (lower) asymptotic density. For an example of a non-normal density, consider the set function β defined on 2^I by:

$$\beta(A) = \begin{cases} 1 & \text{if } A \sim I; \\ 0 & \text{otherwise.} \end{cases}$$

It may be easily verified that β is a translation invariant density.

Definition 2.8. The function β is called the discrete density.

It follows that the upper discrete density is given by

$$\overline{\beta}(A) = \begin{cases} 1 & \text{if } A \text{ is infinite;} \\ 0 & \text{if } A \text{ is finite.} \end{cases}$$

Now given $A = \{kn\}_{n=1}^{\infty}$ for some $k \geq 2$, we have $\beta(A) = 0 < \frac{1}{k} < 1 = \overline{\beta}(A)$, so that β is a non-normal density.

The example of discrete density also shows that the conclusion of Theorem 2.15 need not hold for upper density. This can be seen by letting A be any infinite set satisfying the hypothesis of the theorem, say $A = \{n^2\}_{n=1}^{\infty}$. Then $\overline{\beta}(A) = 1$.

Furthermore, the example of β shows that Theorem 2.7 is not true if the hypothesis $\overline{d}(B) = 0$ is replaced by $d(B) = 0$. For if

$A = \{1,3,5, \ldots\}$ and $\overline{B} = \{2,4,6, \ldots\}$, then $\beta(B) = 0$, but $\beta(A \cup B) = \beta(I) = 1 > 0 = \beta(A)$.

We conclude this discussion of the discrete density by giving its order relation to the ordinary asymptotic density. First the usual notation is defined.

<u>Definition 2.6.</u> If d_1 and d_2 are functions on 2^I, then $d_1 \leq d_2$ means that for all A, $d_1(A) \leq d_2(A)$; and $d_1 = d_2$ means that for all A, $d_1(A) = d_2(A)$.

<u>Lemma 2.6.</u> If d_1 and d_2 are densities such that $d_1 \leq d_2$, then $\overline{d}_2 \leq \overline{d}_1$.

Proof. For all A, $\overline{d}_2(A) = 1 - d_2(\overline{A}) \leq 1 - d_1(\overline{A}) = \overline{d}_1(A)$.

<u>Theorem 2.17</u> If β is the discrete density, and δ is the usual asymptotic density, then $\beta \leq \delta \leq \overline{\delta} \leq \overline{\beta}$.

Proof. If $\beta(A) = 0$, then by Axiom 1, $\beta(A) \leq \delta(A)$. If $\beta(A) = 1$, then $A \sim I$ and $\delta(A) = \delta(I) = 1$. Thus $\beta \leq \delta$, and the conclusion follows from Theorem 2.4 and the preceding lemma.

We now define the natural density and give some of its elementary properties.

<u>Definition 2.8.</u> Given a density d, let $N_d = \{A \mid d(A) = \overline{d}(A)\}$. Then the <u>natural density</u> $\nu_d : N_d \to [0,1]$ is is defined by $\nu_d(A) = d(A) = \overline{d}(A)$. We may write ν for ν_d, when d is understood from the context.

Theorem 2.18. If d is a density and $A \in N_d$, then $\overline{A} \in N_d$ with $\nu(\overline{A}) = 1 - \nu(A)$.

Proof. We have $\overline{d}(\overline{A}) = 1 - d(\overline{\overline{A}}) = 1 - d(A) = 1 - \overline{d}(A) = d(\overline{A})$, so $\overline{A} \in N_d$. Also, $\nu(\overline{A}) = \overline{d}(\overline{A}) = 1 - d(A) = 1 - \nu(A)$.

Theorem 2.19. If d is a translation invariant density and $A \in N_d$, then $A + 1 \in N_d$, with $\nu(A + 1) = \nu(A)$.

Proof. We have $\overline{d}(A + 1) = 1 - d(\overline{A + 1}) = 1 - d(\overline{A} + 1) = 1 - d(\overline{A}) = 1 - \overline{d}(\overline{A}) = 1 - \overline{d}(\overline{A} + 1) = 1 - \overline{d}(\overline{A + 1}) = d(A + 1)$, so $A + 1 \in N_d$. Also, $\nu(A + 1) = d(A + 1) = d(A) = \nu(A)$.

The following is a mixed density and natural density result.

Theorem 2.20. Let d be a density, and let $B \in N_d$. If $\nu_d(A \cap B) = 0$, then $d(A \cup B) = d(A) + \nu_d(B)$, and $\overline{d}(A \cup B) = \overline{d}(A) + \nu_d(B)$.

Proof. Note that $\nu_d(A \cap B) = 0$ is equivalent to $\overline{d}(A \cap B) = 0$. Then by Axiom 4 and the first part of Theorem 2.12, $d(A) + \nu_d(B) = d(A) + d(B) \leq d(A \cup B) \leq d(A) + \overline{d}(B) = d(A) + \nu_d(B)$. So $d(A \cup B) = d(A) + \nu_d(B)$.

Furthermore, using Property $\overline{4}$ and the second part of Theorem 2.12, we have $\overline{d}(A) + \nu_d(B) = \overline{d}(A) + d(B) \leq \overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B) = \overline{d}(A) + \nu_d(B)$. Thus $\overline{d}(A \cup B) = \overline{d}(A) + \nu_d(B)$.

We now show that natural density is finitely additive.

Theorem 2.21. If d is a density, and $A, B, A \cap B \in N_d$ with $\nu_d(A \cap B) = 0$, then $A \cup B \in N_d$ and $\nu_d(A \cup B) = \nu_d(A) + \nu_d(B)$.

Proof. Using Theorems 2.12 and 2.4, and Property $\overline{4}$, we have $\nu(A) + \nu(B) = \underline{d}(A) + \underline{d}(B) \leq \underline{d}(A \cup B) \leq \overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B) = \nu(A) + \nu(B)$, which implies that $\underline{d}(A \cup B) = \overline{d}(A \cup B)$ so that $A \cup B \in N_d$. We also have $\nu(A \cup B) = d(A \cup B) = \nu(A) + \nu(B)$.

Definition 2.6. Sets A and B are called almost disjoint (with respect to density d) if $\nu_d(A \cap B) = 0$.

Note that by Theorem 2.8, disjoint sets are almost disjoint with respect to any density.

Now it follows from Theorem 2.16 by the usual induction that natural density is finitely additive for collections of almost disjoint sets. Natural density is not countably additive, however, since if d is any density, then $\nu_d(\bigcup_{i=1}^{\infty}\{i\}) = 1 \neq 0 = \sum_{i=1}^{\infty} \nu_d(\{i\})$. There are some natural densities though, for which an approximation to countable additivity called the additivity property (AP) holds.

Additivity Property. Let $\{A_i\}_{i=1}^{\infty}$ be a collection of disjoint sets in N_d. Then there exists a collection $\{B_i\}_{i=1}^{\infty}$ with each $B_i \sim A_i$, such that $\bigcup_{i=1}^{\infty} B_i \in N_d$ and $\nu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \nu(B_i)$.

We may also formulate a weak additivity property (WAP) by replacing the requirement that each $B_i \sim A_i$ by the condition that for each i, $\overline{d}(B_i \triangle A_i) = 0$. Since the upper density of a finite set is zero, it follows that the additivity property implies the weak additivity property. It is not known if the converse is true.

In the next section, a class of natural densities is shown to have the additivity property.

§3 MATRIX METHODS AND THE ADDITIVITY PROPERTY

Let $x = (x_k)$ be a sequence, and let $M = (a_{nk})$ be an infinite matrix. Then multiplication of the sequence by the matrix results in a new sequence $Mx = y = (y_n)$, where $y_n = (Mx)_n = \sum_k a_{nk}x_k$. M is said to define a regular matrix method of summability if the convergence of x to s implies the convergence of Mx to s.

The following well-known Silverman/Toeplitz conditions are necessary and sufficient for a matrix (a_{nk}) to be regular.

i) $\forall k,\ a_{nk} \to 0$ as $n \to \infty$;

ii) $\sup_n \sum_k |a_{nk}| < \infty$;

iii) $\sum_k a_{nk} \to 1$ as $n \to \infty$.

One of the most familiar examples of a regular matrix method is the Cesaro method, which is defined by the matrix

$$C = \begin{pmatrix} 1 & 0 & \cdots & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & & & & \end{pmatrix},$$

where $c_{nk} = \begin{cases} 1 & \text{if } k \le n; \\ 0 & \text{if } k > n. \end{cases}$

The usual (lower) asymptotic density is related to the Cesaro method as follows. Given a set A, let $\chi_A = (\chi_A(k))$ be the characteristic sequence of A defined by

$$\chi_A(k) = \begin{cases} 1 & \text{if } k \in A; \\ 0 & \text{if } k \notin A. \end{cases}$$

Then $A(n) = \sum_{k=1}^{n} \chi_A(k)$, and $\frac{A(n)}{n} = \frac{\sum_{k=1}^{n} \chi_A(k)}{n} = \sum_{k=1}^{n} \frac{1}{n} \chi_A(k) =$ $\sum_{k=1}^{\infty} c_{nk}\chi_A(k)$. So $\delta(A) = \liminf_n \frac{A(n)}{n} = \liminf_n (C\chi_A)_n$.

Furthermore, the discrete density is related to the infinite identity matrix, denoted by $J = (j_{nk})$. Given a set A, $(J\chi_A)_n = \sum_k j_{nk}\chi_A(k) = \chi_A(n)$, so $\beta(A) = \liminf_n \chi_A(n) = \liminf_n (J\chi_A)_n$.

Now for an arbitrary infinite matrix $M = (a_{nk})$, define $d_M : 2^I \to R$ by $d_M(A) = \liminf_n (M\chi_A)_n$. A major aim of this chapter is to characterize those matrices M for which d_M is a density.

Firstly, d_M is well defined if $(M\chi_A)_n$ exists for all A and all n; i.e., if $\sum_k a_{nk}\chi_A(k)$ converges for all A and all n. This condition is equivalent to $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, for all n. The following definition is required for the proof.

<u>Definition 3.1</u>. Let $a \in R$. Then a^+, the positive part of a, is defined by $a^+ = \max\{0,a\}$, and a^-, the negative part of a, is defined by $a^- = \max\{0,-a\}$.

Note that $a = a^+ - a^-$, $|a| = a^+ + a^-$, and $a^+ = a$ iff $a \geq 0$. Also, if $\sum a_n$ is a series with (b_n) and (c_n) the sequences of its non-negative and negative terms respectively, then $\sum b_n = \sum a_n^+$ and $\sum c_n = - \sum a_n^-$.

<u>Theorem 3.1</u>. In order for $\sum_k a_{nk}\chi_A(k)$ to converge for all A and all n, it is necessary and sufficient that for all n, $\sum_k |a_{nk}| < \infty$.

Proof. The condition is sufficient, since if $\sum_k |a_{nk}| < \infty$, for all n, then for all n and all A, $\left|\sum_k a_{nk}\chi_A(k)\right| \le \sum_k |a_{nk}\chi_A(k)| = \sum_k |a_{nk}|\chi_A(k) \le \sum_k |a_{nk}| < \infty$.

To see that it is necessary, suppose there exists an n such that $\sum_k |a_{nk}|$ diverges. Then by a well-known theorem on absolute convergence, $\sum_k a_{nk}^+$ and $\sum_k a_{nk}^-$ cannot both converge. If $\sum_k a_{nk}^+$ diverges, define A by $k \in A$ iff $a_{nk} > 0$, so that $\sum_k a_{nk}\chi_A(k) = \sum_k a_{nk}^+$ diverges. If only $\sum_k a_{nk}^-$ diverges, define A by $k \in A$ iff $a_{nk} < 0$. Then $\sum_k a_{nk}\chi_A(k) = -\sum_k a_{nk}^-$ diverges. In either case, there exists an n and an A such that $\sum_k a_{nk}\chi_A(k)$ diverges.

The conclusion follows by the contrapositive.

The condition $\lim_n \sum_k a_{nk} = 1$ is not necessary in order for d_M to be a density. For example, consider the following matrix constructed from the Cesaro matrix: $M = \begin{pmatrix} 1 & 0 & \cdots & & & \\ 1 & 0 & \cdots & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & \\ 1 & 1 & 0 & \cdots & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & \\ 1 & 1 & 1 & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & \end{pmatrix}$.

Then for all A, $d_M(A) = \lim_n \inf (M\chi_A)_n = \lim_n \inf (C\chi_A)_n = \delta(A)$, so that d_M is a density.

A conjecture here is that if d_M is a density, then there exists a matrix $M' = (a_{nk}')$ with $\lim_n \sum_k a_{nk}' = 1$, such that for all

A, $d_{M'}(A) = d_M(A)$.

Another consideration in regard to the condition $\lim_n \sum_k a_{nk} = 1$ is given by the following theorem [4].

Theorem 3.2. Let $M = (a_{nk})$ be a matrix such that d_M is a density. Then in order that $\overline{d}_M(A) = \limsup_n (M\chi_A)_n$ for all A, it is necessary and sufficient that $\lim_n \sum_k a_{nk} = 1$.

Proof. To see that it is necessary, first note that $(M\chi_I)_n = \sum_k a_{nk}$. Now if $\overline{d}_M(A) = \limsup_n (M\chi_A)_n$ for all A, then $\limsup_n \sum_k a_{nk} = \limsup_n (M\chi_I)_n = \overline{d}(I) = 1$. Since $\liminf_n \sum_k a_{nk} = d_M(I) = 1$ also, it follows that $\lim_n \sum_k a_{nk} = 1$.

For the converse, let $\underline{1}$ denote the unit sequence $\{1, 1, 1, \ldots\}$, and observe that $\chi_{\overline{A}} = \underline{1} - \chi_A$. Then using the fact that multiplication by M is linear, and the hypothesis $\lim_n \sum_k a_{nk} = 1$, we have $\overline{d}_M(A) = 1 - d_M(\overline{A}) = 1 - \liminf_n (M\chi_{\overline{A}})_n = 1 - \liminf_n (M(\underline{1} - \chi_A))_n = 1 - \liminf_n (M\underline{1} - M\chi_A)_n = 1 + \limsup_n (M\chi_A - M\underline{1})_n = 1 + \limsup_n (M\chi_A)_n - \lim_n (M\underline{1})_n = \limsup_n (M\chi_A)_n$.

In view of the above considerations, the convergence of the sequence of row sums to 1 is assumed in the following theorem. Some definitions are required first.

Definition 3.2. Given $M = (a_{nk})$ with $\lim_n \sum_k a_{nk} = 1$, such that d_M is a density, let $N_{d_M} = \{A \mid d_M(A) = \overline{d}_M(A)\}$. For $A \in N_{d_M}$, define $\nu_{d_M}(A) = \lim_n (M\chi_A)_n$. As before, we may write only ν when d_M is understood.

Definition 3.3. Let $M = (a_{nk})$ be an infinite matrix. Then M is essentially non-negative means that $\lim_n \sum_k a_{nk}^- = 0$.

Theorem 3.3. If $M = (a_{nk})$ is a matrix with $\lim_n \sum_k a_{nk} = 1$ such that d_M is a density, then M is regular and essentially non-negative.

Proof. Since d_M is a well-defined density, by Theorem 2.1 we have for all n that $\sum_k |a_{nk}| < \infty$. After the proof that M is essentially non-negative, it will be shown that the absolute row sums are uniformly bounded. To see that the column limits are zero, for any given k, let $A = \{k\}$. Then $(M\chi_A)_n = \sum_i a_{ni}\chi_A(i) = a_{nk}$, and since $A \in N_{d_M}$ with $\nu(A) = 0$, it follows that $\lim_n a_{nk} = \lim_n (M\chi_A)_n = \nu(A) = 0$.

To see that M is essentially non-negative, suppose on the contrary that there exists $\delta > 0$ such that $\sum_k a_{nk}^- > \delta$ for infinitely many n. Let n_1 be such an n, and choose finitely many k, say $k_1 < k_2 < \ldots < k_{e_1}$, such that $\sum_{j=1}^{e_1} a_{n_1k_j}^- > \frac{\delta}{2}$ and $a_{n_1k_j} < 0$ for $j = 1, \ldots, e_1$. Since the absolute row sums are bounded, there exists $M_1 > k_{e_1}$ such that $\sum_{k=M_1}^{\infty} |a_{n_1k}| < \frac{\delta}{8}$. And since the column limits are zero, there exists N_1 such that for all $n \geq N_1$, $\sum_{k=1}^{M_1} |a_{nk}| < \frac{\delta}{8}$. Choose an $n_2 \geq N_1$ so that $\sum_k a_{n_2k}^- > \delta$ (and $\sum_{k=1}^{M_1} |a_{n_2k}| < \frac{\delta}{8}$).

Then continuing the procedure, choose finitely many k, say $k_{e_1+1} < k_{e_1+2} < \ldots < k_{e_2}$, with $M_1 < k_{e_1+1}$ such that $\sum_{j=e_1+1}^{e_2} a_{n_2k_j}^- > \frac{\delta}{2}$, and $a_{n_2k_j} < 0$ for $j = e_1 + 1, \ldots, e_2$. Also, find $M_2 > k_{e_2}$ such that $\sum_{k=M_2}^{\infty} |a_{n_2k}| < \frac{\delta}{8}$, and N_2 such that for all $n \geq N_2$, $\sum_{k=1}^{M_2} |a_{nk}| < \frac{\delta}{8}$. Then choose an $n_3 \geq N_2$ such that

$\sum_k a_{n_3 k}^- > \delta$, and so on.

In general, we have integers $k_{e_{i-1}+1} < k_{e_{i-1}+2} < \cdots < k_{e_i}$ $(e_0 = 0)$ with $M_{i-1} < k_{e_{i-1}+1}$ $(M_0 = 0)$, and an integer n_i such that $\sum_{j=e_{i-1}+1}^{e_i} a_{n_i k_j}^- > \frac{\delta}{2}$, and $a_{n_i k_j} < 0$ for $j = e_{i-1}+1, \ldots, e_i$. Also, there exists $M_i > k_{e_i}$ such that $\sum_{k=M_i}^{\infty} |a_{n_i k}| < \frac{\delta}{8}$, and N_i such that for all $n \geq N_i$, $\sum_{k=1}^{M_i} |a_{n_i k}| < \frac{\delta}{8}$.

Now let $A = \{k_1, k_2, \ldots, k_{e_1}, k_{e_1+1}, \ldots, k_{e_2}, \ldots\}$.

Then for $i > 1$,

$$(M\chi_A)_{n_i} = \sum_k a_{n_i k}\chi_A(k)$$

$$= \sum_{k=1}^{M_{i-1}} a_{n_i k}\chi_A(k) + \sum_{k=M_{i-1}+1}^{M_i-1} a_{n_i k}\chi_A(k) + \sum_{k=M_i}^{\infty} a_{n_i k}\chi_A(k)$$

$$\leq \sum_{k=1}^{M_{i-1}} |a_{n_i k}| + \sum_{k=M_{i-1}+1}^{M_i-1} a_{n_i k}\chi_A(k) + \sum_{k=M_i}^{\infty} |a_{n_i k}|.$$

Now we have $M_{i-1} + 1 \leq k_{e_{i-1}+1} < \cdots < k_{e_i} \leq M_i - 1$, and so by the definition of A, $\sum_{k=M_{i-1}+1}^{M_i-1} a_{n_i k}\chi_A(k) = \sum_{j=e_{i-1}+1}^{e_i} a_{n_i k_j}$. Since $a_{n_i k_j} < 0$ for $j = e_{i-1}+1, \ldots, e_i$, and $\sum_{j=e_{i-1}+1}^{e_i} a_{n_i k_j}^- > \frac{\delta}{2}$, it follows that $\sum_{j=e_{i-1}+1}^{e_i} a_{n_i k_j} < -\frac{\delta}{2}$.

So we have $(M\chi_A)_{n_i} \leq \frac{\delta}{8} - \frac{\delta}{2} + \frac{\delta}{8} = -\frac{\delta}{4}$, and $d_M(A) = \liminf_n (M\chi_A)_n \leq -\frac{\delta}{4} < 0$, contradicting Axiom 1. This proves that we must have $\lim_n \sum_k a_{nk}^- = 0$; i.e., that M is essentially non-negative.

Now $\sum_k |a_{nk}| = \sum_k a_{nk}^+ + a_{nk}^- = \sum_k a_{nk}^+ + \sum_k a_{nk}^- \to \sum_k a_{nk}^+$ as $n \to \infty$. Also, $\sum_k a_{nk} = \sum_k a_{nk}^+ - a_{nk}^- = \sum_k a_{nk}^+ - \sum_k a_{nk}^- \to \sum_k a_{nk}^+$ as $n \to \infty$. Thus $\lim_n \sum_k |a_{nk}| = \lim_n \sum_k a_{nk} = 1$ (by hypothesis), so that $\sup_n \sum_k |a_{nk}| < \infty$. Since all three Silverman/Toeplitz conditions are now fulfilled, we conclude that M is regular.

A partial converse to the above theorem will now be proved. We begin with some definitions and lemmas.

Definition 3.3. $M = (a_{nk})$ is a non-negative matrix, written $M \geq 0$, if for all n and all k, $a_{nk} \geq 0$.

A non-negative matrix M^+ can be associated with an arbitrary matrix $M = (a_{nk})$ by defining $M^+ = (a_{nk}^+)$.

Definition 3.4. Two matrices M and M' are equivalent, written $M \equiv M'$, if for all A, $\lim_n ((M\chi_A)_n - (M'\chi_A)_n) = 0$.

Lemma 3.1. If $M \equiv M'$, then $d_M = d_{M'}$.

Proof. Using well-known properties of lim inf, we have for sequences x_n and y_n that $\liminf_n (x_n - y_n) \leq \liminf_n x_n - \liminf_n y_n \leq \limsup_n (x_n - y_n)$. Now given A, $d_M(A) - d_{M'}(A) = \liminf_n (M\chi_A)_n - \liminf_n (M'\chi_A)_n$, and since for all A, $\lim_n ((M\chi_A)_n - (M'\chi_A)_n) = 0$, it follows that $d_M(A) - d_{M'}(A) = 0$, and so $d_M(A) = d_{M'}(A)$.

Lemma 3.2. If M is an essentially non-negative matrix, then $M \equiv M^+$.

Proof. Let $M = (a_{nk})$. For all A, $(M\chi_A)_n - (M^+\chi_A)_n =$ $\sum_k a_{nk}\chi_A(k) - \sum_k a_{nk}^+\chi_A(k) = \sum_k (a_{nk}^+ - a_{nk}^-)\chi_A(k) - \sum_k a_{nk}^+\chi_A(k) =$ $-\sum_k a_{nk}^-\chi_A(k)$. Since $0 \le \sum_k a_{nk}^-\chi_A(k) \le \sum_k a_{nk}^-$, and $\lim_n \sum_k a_{nk}^- = 0$, we have $\lim_n((M\chi_A)_n - (M^+\chi_A)_n) = 0$. Thus $M \equiv M^+$.

<u>Theorem 3.4</u> [4]. If M is regular and essentially non-negative, then d_M is an asymptotic density, but not necessarily a translation invariant density.

Proof. From Lemma 3.2 we have $d_M = d_{M^+}$, so we may simply assume that M is a non-negative matrix. Then for all A, $d_M(A) = \liminf_n (M\chi_A)_n = \liminf_n \sum_k a_{nk}\chi_A(k) \ge 0$, and $d_M(A) = \liminf_n \sum_k a_{nk}\chi_A(k) \le \liminf_n \sum_k a_{nk}\chi_I(k) = \liminf_n \sum_k a_{nk} = 1$, since M is regular. So Axioms 1 and 2 hold.

Now suppose that $A \sim B$. Then there exists an N such that for all $k \ge N$, $\chi_A(k) = \chi_B(k)$. Let $x_n = \sum_k a_{nk}\chi_A(k)$ and $y_n =$ $\sum_k a_{nk}\chi_B(k)$. Then $|x_n - y_n| = |\sum_k a_{nk}\chi_A(k) - \sum_k a_{nk}\chi_B(k)| =$ $|\sum_k a_{nk}(\chi_A(k) - \chi_B(k))| = |\sum_{k=1}^{N} a_{nk}(\chi_A(k) - \chi_B(k))| \le \sum_{k=1}^{N} a_{nk}|\chi_A(k) - \chi_B(k)| \le$ $\sum_{k=1}^{N} a_{nk}$. Since the column limits are zero, it follows that $\lim_n \sum_{k=1}^{N} a_{nk} = 0$. Now it may be assumed that $d_M(A) \ge d_M(B)$, so $d_M(A) - d_M(B) = \liminf_n x_n - \liminf_n y_n \le \limsup_n (x_n - y_n) = 0$. That is, $d_M(A) = d_M(B)$, and Axiom 3 is proved.

For Axiom 4, let $A \cap B = \phi$, and observe that in this case, $\chi_{A\cup B} = \chi_A + \chi_B$. Then $d_M(A \cup B) = \liminf_n (M\chi_{A\cup B})_n =$ $\liminf_n \sum_k a_{nk}(\chi_A(k) + \chi_B(k)) \ge \liminf_n \sum_k a_{nk}\chi_A(k) + \liminf_n \sum_k a_{nk}\chi_B(k) =$

$d_M(A) + d_M(B)$.

Axiom 5 is proved via Property $\overline{4}$ for upper density. It is sufficient to prove the latter, since if $\overline{d}(A) + \overline{d}(B) \geq \overline{d}(A \cup B)$ for all A, B, then for all A, B, $1 - d(A) + 1 - d(B) = \overline{d}(\overline{A}) + \overline{d}(\overline{B}) \geq \overline{d}(\overline{A} \cup \overline{B}) = \overline{d}(\overline{A \cap B}) = 1 - d(A \cap B)$; i.e., $d(A) + d(B) \leq 1 + d(A \cap B)$. Note that in general, $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$. Observe also that in the proof of Theorem 3.2, it is only required that $d_M = \liminf_n (M\chi_A)_n$ satisfies Axiom 2. Since this has already been shown, we may apply the result here. Then for all A, B, $\overline{d}_M(A \cup B) = \limsup_n (M\chi_{A \cup B})_n =$
$\limsup_n \sum_k a_{nk}(\chi_A(k) + \chi_B(k) - \chi_A(k)\chi_B(k)) \leq \limsup_n \sum_k a_{nk}\chi_A(k) +$
$\limsup_n \sum_k a_{nk}\chi_B(k) + \limsup_n \sum_k - a_{nk}\chi_A(k)\chi_B(k)$. Since M is non-negative
$\limsup_n \sum_k - a_{nk}\chi_A(k)\chi_B(k) \leq 0$, so we have $\overline{d}_M(A \cup B) \leq$
$\limsup_n \sum_k a_{nk}\chi_A(k) + \limsup_n \sum_k a_{nk}\chi_B(k) = \overline{d}_M(A) + \overline{d}_M(B)$, which was to be shown.

This proves that d_M is an asymptotic density. To see that an asymptotic density obtained from an essentially non-negative regular matrix may not be translation invariant, consider the density d_M associated with

$$M = \begin{pmatrix} 1 & 0 & \cdots & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & & \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots \\ \vdots & & & & & & \end{pmatrix}.$$

If A is the set of odd numbers, then for all n, $(M\chi_A)_n = 1$ so that $d_M(A) = 1$. But for all n, $(M\chi_{A+1})_n = 0$ so that $d_M(A + 1) = 0$.

The above theorem is only a partial converse to Theorem 3.3, since the question of necessary and sufficient conditions for translation invariance remains open.

We now show that the additivity property holds for the following class of densities.

Definition 3.5 [4]. A density d is a regular matrix method density if there exists a regular matrix M such that $d_M = d$.

We restate a conjecture made earlier as follows. If M is a matrix such that d_M is a density, then there exists a regular matrix M' such that $d_{M'} = d_M$.

Theorem 3.5. All regular matrix method densities have the additivity property.

Proof. Let d_M be a given matrix method density. Then by Theorem 3.3, $M = (a_{nk})$ is essentially non-negative and regular. By Lemmas 3.1 and 3.2, we may assume without loss of generality that M is a non-negative regular matrix.

Thus for all n, $\sum_k |a_{nk}| = \sum_k a_{nk} < \infty$. So for all n we can find $s(n)$ such that

$$\sum_{k=s(n)+1}^{\infty} a_{nk} < \frac{1}{n}, \quad \text{and} \quad s(n+1) > s(n). \tag{3.1}$$

Now given a collection $\{A_i\}_{i=1}^{\infty}$ of disjoint sets in N_{d_M}, let

$\nu_{d_M}(A_i) = \alpha_i$. By the finite additivity of natural density, we have for each j that $\bigcup_{i=1}^{j} A_i \in N_{d_M}$, and that

$$\limsup_n \sum_k a_{nk}(\chi_{A_1}(k) + \ldots + \chi_{A_j}(k))$$

$$= \limsup_n (M \cdot \chi_{A_1 \cup \ldots \cup A_j})_n$$

$$= \nu_{d_M}\left(\bigcup_{i=1}^{j} A_i\right)$$

$$= \sum_{i=1}^{j} \nu_{d_M}(A_i)$$

$$= \sum_{i=1}^{j} \alpha_i .$$

So for each j, there exists an $N(j)$ such that if $n \geq N(j)$, then

$$\sum_k a_{nk}(\chi_{A_1}(k) + \ldots + \chi_{A_j}(k)) \leq \sum_{i=1}^{j} \alpha_i + \frac{1}{j} , \qquad 3.2$$

and we may arrange that $N(j + 1) > N(j)$.

For $n \geq N(1)$, define $p(n)$ to be such that $N(p(n)) \leq n$ $N(p(n) + 1)$. Given any integer M, let $n_0 = N(M + 1)$. Then by definition, for all $n \geq n_0$, $p(n) \geq M + 1 > M$. Thus $\lim_n p(n) = \infty$.

Now for each i, define B_i by $B_i =$

$A_i \setminus \{1, \ldots, s(N(i+1))\}$, so that for each i, $B_i \sim A_i$ and $\nu d_M(B_i) = \nu d_M(A_i)$. Given n, if $i > p(n)$, then

$$i + 1 > p(n) + 1 \Rightarrow N(i+1) > N(p(n)+1) > n$$

$$\Rightarrow s(N(i+1)) > s(n)$$

$$\Rightarrow B_i \cap \{1, \ldots, s(n)\} = \phi,$$

so that for $k \le s(n)$, $\chi_{B_i}(k) = 0$. So for all n,

$$\sum_{k=1}^{s(n)} a_{nk} \sum_{i=1}^{\infty} \chi_{B_i}(k) = \sum_{k=1}^{s(n)} a_{nk} \sum_{i=1}^{p(n)} \chi_{B_i}(k). \qquad 3.3$$

Let $B = \bigcup_{i=1}^{\infty} B_i$. Then

$$\overline{d}(B) = \limsup_n (M \cdot \chi_B)_n$$

$$= \limsup_n \sum_{k=1}^{\infty} a_{nk}\chi_B(k)$$

$$\le \limsup_n \left(\frac{1}{n} + \sum_{k=1}^{s(n)} a_{nk}\chi_B(k)\right) \qquad 3.1$$

$$= \limsup_n \sum_{k=1}^{s(n)} a_{nk}\chi_B(k)$$

$$= \limsup_n \sum_{k=1}^{s(n)} a_{nk} \sum_{i=1}^{\infty} \chi_{B_i}(k)$$

$$= \limsup_n \sum_{k=1}^{s(n)} a_{nk} \sum_{i=1}^{p(n)} \chi_{B_i}(k) \qquad 3.3$$

$$\leq \limsup_n \sum_{k=1}^{s(n)} a_{nk} \sum_{i=1}^{p(n)} \chi_{A_i}(k)$$

$$\leq \limsup_n \sum_{k=1}^{\infty} a_{nk} \sum_{i=1}^{p(n)} \chi_{A_i}(k)$$

$$\leq \limsup_n \left(\sum_{i=1}^{p(n)} \alpha_i + \frac{1}{p(n)} \right) \qquad 3.2$$

$$\leq \limsup_n \sum_{i=1}^{p(n)} \alpha_i + \limsup_n \frac{1}{p(n)}$$

$$= \limsup_n \sum_{i=1}^{p(n)} \alpha_i$$

$$= \sum_{i=1}^{\infty} \alpha_i \,.$$

The last step follows since $0 \leq \sum_{i=1}^{\infty} \alpha_i = \sum_{i=1}^{\infty} d(A_i) \leq d(\bigcup_{i=1}^{\infty} A_i) \leq 1$ implies that $\sum_{i=1}^{\infty} \alpha_i$ is a convergent series. Now using countable superadditivity of density, we have $\sum_{i=1}^{\infty} \nu(A_i) = \sum_{i=1}^{\infty} \nu(B_i) = \sum_{i=1}^{\infty} d(B_i) \leq d(\bigcup_{i=1}^{\infty} B_i) \leq \bar{d}(B) \leq \sum_{i=1}^{\infty} \nu(A_i)$. From this it follows that

$\bigcup_{i=1}^{\infty} B_i \in N_{d_M}$ and that $\nu_{d_M}(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \nu_{d_M}(B_i)$. Thus the arbitrary regular matrix method density d_M has the additivity property.

§4 ABEL DENSITY

In the last chapter, the usual asymptotic (and translation invariant) density was shown to be associated with the Cesaro summability method. In this chapter, we study a density associated in a similar way with the Abel summability method.

Definition 4.1. A series $\sum_k a_k$ is Abel summable to the value L if $\lim_{x\to 1^-} \sum_k a_k x^k = L$. This may be written $\sum_k a_k = L \quad (A)$. Equivalently, a sequence (s_k) is Abel limitable to the value L if $\lim_{x\to 1^-} (1 - x) \sum_k s_k x^k = L$. This may be written $\lim(s_k) = L \quad (A)$.

Although Abel summability is not a matrix method, it can be related to a "semi-continuous matrix" of the form

$$M = \begin{pmatrix} f_1(x) & f_2(x) & \cdots \\ \downarrow & \downarrow & \end{pmatrix},$$ where x increases over some given interval (which may be changed by a transformation of variable). For a given sequence $s = (s_k)$, define Ms by $Ms(x) = \sum_k s_k f_k(x)$. Let

$$A = \begin{pmatrix} x - x^2 & x^2 - x^3 & \cdots \\ \downarrow & \downarrow & \end{pmatrix}, \quad 0 \le x < 1.$$ Then the Abel limit of a sequence $s = (s_k)$, when it exists, is given by $\lim_{x\to 1^-} \sum_k s_k (x^k - x^{k+1}) = \lim_{x\to 1^-} As(x)$.

Now let a function α be defined on 2^I by $\alpha(A) = \liminf_{x\to 1^-} A\chi_A(x) = \liminf_{x\to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1})$. This function is well defined, since for all A and for $0 \le x < 1$, we have

$$0 \le \sum_k \chi_A(k)(x^k - x^{k+1}) \le \sum_k (1-x)x^k < \infty.$$

Lemma 4.1. If α is as defined above, and $\bar{\alpha}$ is defined as usual by $\bar{\alpha}(A) = 1 - \alpha(\bar{A})$, then $\bar{\alpha}$ is given by $\bar{\alpha}(A) = \limsup_{x\to 1^-} A\chi_A(x)$.

Proof. Note that $\lim_{x\to 1^-} \sum_{k=1}^{\infty} x^k - x^{k+1} = \lim_{x\to 1^-} (1-x)\frac{x}{1-x} = 1.$

Then

$$\begin{aligned}
\bar{\alpha}(A) = 1 - \alpha(\bar{A}) &= 1 - \liminf_{x\to 1^-} \sum_k \chi_{\bar{A}}(k)(x^k - x^{k+1}) \\
&= 1 - \liminf_{x\to 1^-} \sum_k (1 - \chi_A(k))(x^k - x^{k+1}) \\
&= 1 - \liminf_{x\to 1^-} (\sum_k x^k - x^{k+1} - \sum_k \chi_A(k)(x^k - x^{k+1})) \\
&= 1 - (1 + \liminf_{x\to 1^-} - \sum_k \chi_A(k)(x^k - x^{k+1})) \\
&= - \liminf_{x\to 1^-} - \sum_k \chi_A(k)(x^k - x^{k+1}) \\
&= \limsup_{x\to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1}).
\end{aligned}$$

Theorem 4.1. The function α defined above is a translation invariant density.

Proof. For all A and for $0 \le x < 1$, we have

$0 \le \sum_k \chi_A(k)(x^k - x^{k+1}) \le (1-x) \sum_k x^k = (1-x)\frac{x}{1-x} = x < 1.$ Thus for all A, $0 \le \alpha(A) \le 1$, and Axiom 1 holds.

Axiom 2 is satisfied since $\alpha(I) = \liminf_{x\to 1^-} \sum_{k=1}^{\infty} (1-x)x^k = \liminf_{x\to 1^-} (1-x)\frac{x}{1-x} = \liminf_{x\to 1^-} x = 1.$

For Axiom 3', first note that $\chi_{A+1}(k) = \chi_A(k-1)$ (where $\chi_A(0) = 0$). Then

$$\begin{aligned}
\alpha(A+1) &= \liminf_{x\to 1^-} \sum_k \chi_{A+1}(k)(x^k - x^{k+1}) \\
&= \liminf_{x\to 1^-} \sum_k \chi_A(k-1)(x^k - x^{k+1})
\end{aligned}$$

$$= \liminf_{x \to 1^-} x \sum_k \chi_A(k-1)(x^{k-1} - x^k)$$

$$= \liminf_{x \to 1^-} \sum_k \chi_A(k-1)(x^{k-1} - x^k)$$

$$= \alpha(A).$$

For Axiom 4, let $A \cap B = \phi$ so that $\chi_{A \cup B} = \chi_A + \chi_B$. Then

$$\alpha(A \cup B) = \liminf_{x \to 1^-} \sum_k \chi_{A \cup B}(k)(x^k - x^{k+1})$$

$$= \liminf_{x \to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1}) + \chi_B(k)(x^k - x^{k+1})$$

$$= \liminf_{x \to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1}) + \sum_k \chi_B(k)(x^k - x^{k+1})$$

$$\geq \liminf_{x \to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1}) + \liminf_{x \to 1^-} \sum_k \chi_B(k)(x^k - x^{k+1})$$

$$= \alpha(A) + \alpha(B).$$

For Axiom 5, observe that for all A, B, $\chi_{A \cap B} = \chi_A + \chi_B - \chi_{A \cup B}$. Then since $\limsup_{x \to 1^-} \sum_k \chi_{A \cup B}(k)(x^k - x^{k+1}) \leq \limsup_{x \to 1^-} \sum_k (x^k - x^{k+1}) = 1$,

$$\alpha(A \cap B) = \liminf_{x \to 1^-} \sum_k (\chi_A(k) + \chi_B(k) - \chi_{A \cup B}(k))(x^k - x^{k+1})$$

$$\geq \liminf_{x \to 1^-} \sum_k \chi_A(k)(x^k - x^{k+1}) +$$

$$+ \liminf_{x \to 1^-} \sum_k \chi_B(k)(x^k - x^{k+1}) +$$

$$+ \liminf_{x \to 1^-} - \sum_k \chi_{A \cup B}(k)(x^k - x^{k+1})$$

$$= \alpha(A) + \alpha(B) - \limsup_{x \to 1^-} \sum_k \chi_{A \cup B}(k)(x^k - x^{k+1})$$

$$\geq \alpha(A) + \alpha(B) - 1.$$

This concludes the proof.

Definition 4.2. The function α of the above theorem is called Abel density.

We have the following order relation.

Theorem 4.2. If α is the Abel density, and δ is the usual density, then $\underline{\delta} \leq \underline{\alpha} \leq \overline{\alpha} \leq \overline{\delta}$.

Proof. Given A, let $\sigma_n = \frac{1}{n} \sum_{i=0}^{n} \chi_A(i)$ and let $f(x) = (1 - x) \sum_{i=0}^{\infty} \chi_A(i)x^i$ $(\chi_A(0) = 0)$. Then $\frac{f(x)}{(1-x)^2} = \frac{\sum_i \chi_A(i)x^i}{1 - x} = (\sum_i \chi_A(i)x^i)(\sum_i x^i)$. A well-known theorem [8] states that if $\sum a_i$ and $\sum b_i$ are absolutely convergent, and $c_i = a_0 b_i + a_1 b_{i-1} + \ldots + a_i b_0$, then the product series $\sum c_i$ is convergent. Thus $\frac{f(x)}{(1-x)^2} = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} \chi_A(j))x^i = \sum_{i=0}^{\infty} i\sigma_i x^i$.

We have $\overline{\delta}(A) = \limsup_n \sigma_n$, so given $\varepsilon > 0$, we can find N such that for all $n \geq N$, $\sigma_n < \overline{\delta}(A) + \varepsilon$. Then

$$\bar{\alpha}(A) = \limsup_{x\to 1^-} f(x)$$

$$= \limsup_{x\to 1^-}\left((1-x)^2 \sum_{i=1}^{N} i\sigma_i x^i + (1-x)^2 \sum_{i=N+1}^{\infty} i\sigma_i x^i\right)$$

$$= \limsup_{x\to 1^-}(1-x)^2 \sum_{i=N+1}^{\infty} i\sigma_i x^i$$

$$< (\bar{\delta}(A) + \varepsilon) \limsup_{x\to 1^-}(1-x)^2 \sum_{i=N+1}^{\infty} ix^i$$

$$< (\bar{\delta}(A) + \varepsilon) \limsup_{x\to 1^-}(1-x)^2 \sum_{i=0}^{\infty} ix^i$$

$$= \bar{\delta}(A) + \varepsilon.$$

The last step follows since $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$. Since the inequality holds for all A and all $\varepsilon > 0$, we have $\bar{\alpha} \leq \bar{\delta}$. Then by the definition of upper density and Theorem 2.4, $\delta \leq \alpha \leq \bar{\alpha} \leq \bar{\delta}$.

It follows from the above theorem that $N_\delta \subseteq N_\alpha$. We actually have $N_\delta = N_\alpha$, as a consequence of the following well-known result in summability theory [6], which is given without proof. First note that if a sequence (s_n) is Cesaro limitable to the value L, we write $\lim(s_n) = L \ (C)$.

<u>Theorem 4.3</u>. If (s_n) is a bounded sequence with $\lim(s_n) = L \ (A)$, then $\lim(s_n) = L \ (C)$.

The natural Abel density of a set A, if it exists, is the Abel limit of the sequence χ_A. Likewise the natural ordinary density of a set A, if it exists, is the Cesaro limit of χ_A. Since for all A, χ_A is bounded, Theorems 4.3 and 4.2 together give the following.

Theorem 4.3. Natural Abel density and natural ordinary density are equivalent. That is, $N_\alpha = N_\delta$ and $\nu_\alpha = \nu_\delta$.

Since ordinary density has the additivity property and is normal, it follows from the above theorem that the same holds true for Abel density.

§5 UNIFORM DENSITY

We begin the study of uniform density with the following generalization of the counting function $A(n)$.

Definition 5.1. If A is a set and $m \le n$, then $A[m,n]$ denotes the number of integers in $A \cap \{m, m+1, \ldots, n\}$.

Note that $A[1,n] = A(n)$, and that $A[m,n] = (A \cap [m,n])(n) = A(n) - A(m-1)$. We now introduce a sequence in terms of which uniform density is defined.

Definition 5.2. For a set A and $n \ge 1$, let $\alpha_n = \alpha_n(A) = \min\limits_{m \ge 0} \frac{A[m+1, m+n]}{n}$.

Lemma 5.1. Given a set A, let α_n be as defined above. Then $\lim\limits_n \alpha_n$ exists (and equals $\sup\limits_n \alpha_n$).

Proof. We first show that given $N > 0$ and $\varepsilon > 0$, there exists an M such that for all $n \ge M$, $\alpha_n > \alpha_N - \varepsilon$. To see this, let $M > N$ be such that for all $n \ge M$,

$$\frac{1}{([\frac{n}{N}] + 1)} < \varepsilon. \qquad 5.1$$

We have for all n that

$$[\tfrac{n}{N}]N \le n < ([\tfrac{n}{N}] + 1)N. \qquad 5.2$$

Also, by definition we have $\frac{A[m+1,m+N]}{N} \ge \alpha_N$ for all $m \ge 0$, so that for all $n \ge N$,

$$\frac{A[m+1,m+[\frac{n}{N}]N]}{N} = \frac{A[m+1,m+N]}{N} + \frac{A[m+N+1,m+2N]}{N} + \cdots + \frac{A[m+([\frac{n}{N}]-1)N+1,m+[\frac{n}{N}]N]}{N}$$

$$\ge [\tfrac{n}{N}]\alpha_N . \qquad 5.3$$

Now for all $n \ge M$ and for all m,

$$\frac{A[m+1,m+n]}{n} > \frac{A[m+1,m+[\frac{n}{N}]N]}{([\frac{n}{N}]+1)N} \qquad 5.2$$

$$= \frac{A[m+1,m+[\frac{n}{N}]N]}{[\frac{n}{N}]N} - \frac{NA[m+1,m+[\frac{n}{N}]N]}{[\frac{n}{N}]N([\frac{n}{N}]+1)N}$$

$$\ge \frac{A[m+1,m+[\frac{n}{N}]N]}{[\frac{n}{N}]N} - \frac{N[\frac{n}{N}]N}{[\frac{n}{N}]N([\frac{n}{N}]+1)N}$$

$$\geq \frac{A[m + 1, m + [\frac{n}{N}]N]}{[\frac{n}{N}]N} - \varepsilon \qquad 5.1$$

$$\geq \alpha_N - \varepsilon. \qquad 5.3$$

Thus for all $n \geq M$, $\alpha_n = \min_m \frac{A[m + 1, m + n]}{n} \geq \alpha_N - \varepsilon$.

Now given $\varepsilon > 0$, there exists n_0 such that $\alpha_{n_0} > \sup_N \alpha_N - \frac{\varepsilon}{2}$, and by the result already proved, there exists an $M > n_0$ such that for all $n \geq M$, $\alpha_n > \alpha_{n_0} - \frac{\varepsilon}{2} > \sup_N \alpha_N - \varepsilon$. So $\liminf_n \alpha_n \geq \sup_N \alpha_N - \varepsilon$. Thus we have $\sup_N \alpha_N - \varepsilon \leq \liminf_n \alpha_n \leq \limsup_n \alpha_n \leq \sup_N \alpha_N$. Since ε was arbitrary, it follows that $\lim_n \alpha_n = \sup_N \alpha_N$.

Theorem 5.1. Let α_n be defined as above. Then υ defined on 2^I by $\upsilon(A) = \lim_n \alpha_n$ is a translation invariant density.

Proof. For all $m \geq 0$, $n \geq 1$, and all A, $0 \leq A[m + 1, m + n] \leq n$, so for all $n \geq 1$, $0 \leq \alpha_n \leq 1$. Thus for all A, $0 \leq \upsilon(A) \leq 1$, satisfying Axiom 1.

Since for all $m \geq 0$ and $n \geq 1$, $I[m + 1, m + n] = n$, it follows that for all $n \geq 1$, $\alpha_n = 1$. Thus $\upsilon(I) = 1$, satisfying Axiom 2.

For Axiom 3', observe that $(A + 1)[m + 1, m + n] = A[(m - 1) + 1, (m - 1) + n]$, and that $A[0, n - 1] = A[1, n - 1] = A(n - 1)$. Since for all A, $\lim_n \frac{A(n - 1)}{n} - \frac{A(n)}{n} = 0$, for all A,

$$\lim_{n} \min_{m\geq 0} \frac{A[(m-1)+1,(m-1)+n]}{n} = \lim_{n} \min_{m\geq 0} \frac{A[m+1,m+n]}{n} = \upsilon(A).$$

Therefore, $\upsilon(A+1) = \lim_{n} \min_{m\geq 0} \frac{(A+1)[m+1,m+n]}{n} =$

$$\lim_{n} \min_{m\geq 0} \frac{A[(m-1)+1,(m-1)+n]}{n} = \upsilon(A).$$

For Axiom 4, let $A \cap B = \phi$. Then

$$\upsilon(A \cup B) = \lim_{n} \min_{m} \frac{(A \cup B)[m+1,m+n]}{n}$$

$$= \lim_{n} \min_{m} \frac{A[m+1,m+n] + B[m+1,m+n]}{n}$$

$$\geq \lim_{n} \left(\min_{m} \frac{A[m+1,m+n]}{n} + \min_{m} \frac{B[m+1,m+n]}{n} \right)$$

$$= \upsilon(A) + \upsilon(B).$$

Now given any A and B, we have

$$\upsilon(A) + \upsilon(B) = \lim_{n} \min_{m} \frac{A[m+1,m+n]}{n} + \lim_{n} \min_{m} \frac{B[m+1,m+n]}{n}$$

$$\leq \lim_{n} \min_{m} \frac{A[m+1,m+n] + B[m+1,m+n]}{n}$$

$$= \lim_{n} \min_{m} \frac{(A \cup B)[m+1,m+n] + (A \cap B)[m+1,m+n]}{n}$$

$$\leq \lim_n \min_m \left(1 + \frac{(A \cap B)[m+1, m+n]}{n}\right)$$

$$= 1 + \lim_n \min_m \frac{(A \cap B)[m+1, m+n]}{n}$$

$$= 1 + \upsilon(A \cap B).$$

So Axiom 5 holds, and the proof is complete.

Definition 5.3. Let α_n be defined as above. Then the function υ defined on 2^I by $\upsilon(A) = \lim_n \alpha_n(A)$ is called uniform density.

We now give an explicit form of upper uniform density.

Theorem 5.2. For a set A, let $\overline{\alpha}_n = \overline{\alpha}_n(A) = \max_{m \geq 0} \frac{A[m+1, m+n]}{n}$. If υ is the uniform density, then $\overline{\upsilon}$ is given by $\overline{\upsilon}(A) = \lim_n \overline{\alpha}_n$.

Proof. For all A,

$$\overline{\upsilon}(A) = 1 - \upsilon(\overline{A})$$

$$= 1 - \lim_n \min_m \frac{\overline{A}[m+1, m+n]}{n}$$

$$= 1 - \lim_n \min_m \frac{n - A[m+1, m+n]}{n}$$

$$= 1 - \lim_n (1 - \max_m \frac{A[m+1,m+n]}{n})$$

$$= 1 - \lim_n (1 - \overline{\alpha}_n).$$

Since the limit exists, it follows that $\lim_n \overline{\alpha}_n$ exists, and $\overline{\upsilon}(A) = 1 - 1 + \lim_n \overline{\alpha}_n = \lim_n \overline{\alpha}_n$.

A sequence (s_k) is said to be almost convergent to L if $\lim_n \frac{1}{n} \sum_{k=m+1}^{m+n} s_k = L$, uniformly in m. Note that $\sum_{k=m+1}^{m+n} \chi_A(k) = A[m+1,m+n]$. Thus a set A has natural uniform density α if its characteristic sequence χ_A is almost convergent to α.

Uniform density is our first example of a density without the additivity property. Before proving this we give the following definition and lemma.

Definition 5.4. A block in set A is a set $\{k, k+1, \ldots, k+n\} \subseteq A$, and a gap in A is a block in $\overline{A}$. The length of a block (gap) is the number of integers in the block (gap).

Lemma 5.2. If a set A has arbitrarily long gaps, then $\upsilon(A) = 0$. If A has arbitrarily long blocks, then $\overline{\upsilon}(A) = 1$.

Proof. If A has arbitrarily long gaps, then for all n, there exists an m such that $A[m+1,m+n] = 0$. Thus for all n, $\alpha_n = 0$, so $\upsilon(A) = 0$. If A has arbitrarily long blocks, then $\overline{A}$ has arbitrarily long gaps, so $\overline{\upsilon}(A) = 1 - \upsilon(\overline{A}) = 1$.

Theorem 5.3. Uniform density does not have the additivity property.

Proof. Define a collection of sets $\{A_i\}_{i=1}^{\infty}$ by $\{A_1\} = \{2^n\}$, and for $i > 1$, $A_i = (A_1 + (i-1)) \setminus \bigcup_{j=1}^{i-1} A_j$. Then the A_i are disjoint. If $m + 1 \le 2^k \le m + n$, then $\log_2(m+1) \le k \le \log_2(m+n)$, so for all m and n, $A_1[m+1, m+n] = [\log_2(m+n) - \log_2(m+1)] = [\log_2 \frac{m+n}{m+1}] \le [\log_2 n]$. Thus for all n, $\max_m \frac{A_1[m+1,m+n]}{n} = \frac{A_1(n)}{n} \le \frac{[\log_2 n]}{n}$, so $\overline{\upsilon}(A_1) = 0$. Then using the monotone property, and the fact that υ is a translation invariant density, we have for all i that $\overline{\upsilon}(A_i) = 0$. Thus each of the A_i has natural uniform density zero.

Now let $\{B_i\}_{i=1}^{\infty}$ be any collection of sets such that $B_i \sim A_i$ for each i. Since υ is an asymptotic density, $\overline{\upsilon}(B_i) = 0$ for each i.

Given any m, $\bigcup_{i=1}^{m} B_i$ has blocks of length m. To see this, observe that by Lemma 2.1, for $i = 1, \ldots, m$, there exists $N(i)$ such that $B_i \setminus \{1, \ldots, N(i)\} = A_i \setminus \{1, \ldots, N(i)\}$. So if $N = \max_{i=1,\ldots,m} N(i)$, then for $i = 1, \ldots, m$, $B_i \setminus \{1, \ldots, N\} = A_i \setminus \{1, \ldots, N\}$. Thus for n such that $2^n > N$, we have $2^n + i - 1 \in B_i$, $i = 1, \ldots, m$.

Now since $\bigcup_{i=1}^{\infty} B_i$ has arbitrarily long blocks, by Lemma 5.2, $\overline{\upsilon}(\bigcup_{i=1}^{\infty} B_i) = 1$. Thus if $\bigcup_{i=1}^{\infty} B_i \in N_\upsilon$, we must have $\nu_\upsilon(\bigcup_{i=1}^{\infty} B_i) = 1 \ne 0 = \sum_{i=1}^{\infty} \nu_\upsilon(B_i)$. This proves that uniform density does not have the additivity property.

We now give the order relation of uniform density to ordinary density.

Theorem 5.4. If υ is the uniform density, and δ is the ordinary density, then $\upsilon \le \delta \le \overline{\delta} \le \overline{\upsilon}$, and $N_\upsilon \subset N_\delta$ with $\nu_\delta = \nu_\upsilon$ on N_υ.

Proof. For all n, $\min_{m\ge0} \frac{A[m+1,m+n]}{n} \le \frac{A(n)}{n}$, so for all A, $\upsilon(A) \le \delta(A)$. Then by Lemma 2.6, $\upsilon \le \delta \le \overline{\delta} \le \overline{\upsilon}$. With these inequalities and the definition of natural density, it is clear that if $A \in N_\upsilon$, then $A \in N_\delta$, and $\nu_\delta(A) = \nu_\upsilon(A)$.

Now let A be defined by $\chi_A = \{1,0,0,1,1,0,0,0,0,1,1,1,0, . . .\}$, where at the n^{th} step, n consecutive ones are followed by 2^n consecutive zeros. Maximum values of $\frac{A(n)}{n}$ occur when n is the final integer of a block of A. The formula for these integers is $\frac{n}{2}(1+n) + 2^n - 2$. Then $\overline{\delta}(A) = \limsup_n \frac{A(n)}{n} =$

$$\limsup_n \frac{A[\frac{n}{2}(1+n) + 2^n - 2]}{\frac{n}{2}(1+n) + 2^n - 2} = \limsup_n \frac{\frac{n}{2}(1+n)}{\frac{n}{2}(1+n) + 2^n - 2} = 0,$$ so

$A \in N_\delta$. But since A has arbitrarily long gaps and arbitrarily long blocks, by Lemma 5.2, $\upsilon(A) = 0$ and $\overline{\upsilon}(A) = 1$, so $A \notin N_\upsilon$. Thus $N_\upsilon \subset N_\delta$.

The existence of natural uniform density could be taken as a definition of a set being "uniformly distributed." Then we would expect arithmetic progressions to have natural uniform density, as the next result shows is the case.

Theorem 5.5. Uniform density is a normal density.

Proof. Let $A = \{kn\}_{n=1}^{\infty}$ for some $k \geq 1$. For all m, $\frac{(n-1)}{k} \leq A[m+1, m+n] \leq \frac{(n+1)}{k}$, so $\nu_{\upsilon}(A) = \lim_n \alpha_n = \lim_n \overline{\alpha}_n = \frac{1}{k}$.

To conclude this chapter, we calculate the natural uniform density of another class of sets.

Definition 5.5. Let a set A be given as an increasing sequence (a_n). Then A is a lacunary set if $\lim_n a_{n+1} - a_n = \infty$.

Theorem 5.6. The natural uniform density of a lacunary set is zero.

Proof. Let $A = (a_n)$ be any lacunary set. It is sufficient to prove that $\overline{\upsilon}(A) = 0$. Given $\varepsilon > 0$, let K be such that $\frac{1}{K} < \varepsilon$, and let N be such that for all $n \geq N$, $a_{n+1} - a_n > K$. Let $n_i = a_N + iK$. For all m, $A[m+1, m+n_i] \leq N + \frac{a_N + iK}{K}$, so $\overline{\alpha}_{n_i}(A) = \max_m \frac{A[m+1, m+n_i]}{n_i} \leq \frac{N}{a_N + iK} + \frac{1}{K}$. Then $\overline{\upsilon}(A) = \lim_n \overline{\alpha}_n(A) = \lim_i \overline{\alpha}_{n_i}(A) \leq \frac{1}{K} < \varepsilon$. Since ε was arbitrary, $\overline{\upsilon}(A) = 0$.

§6 LOGARITHMIC DENSITY

Our final density is a regular matrix method density. It is derived from the logarithmic summability method, a special case of a Riesz typical mean [6]. Let

$$L = (a_{nk}), \quad \text{where} \quad a_{nk} = \begin{cases} \dfrac{1}{k \ln n} & \text{if } k \le n; \\ 0 & \text{if } k > n. \end{cases}$$

Define λ on 2^I by $\lambda(A) = d_L(A) = \liminf_n (L\chi_A)_n$. Note that $\lambda(A) = \liminf_n \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k} = \liminf_n \frac{1}{\ln n} \sum_{\substack{a \in A \\ a \le n}} \frac{1}{a}$.

We now give a sufficient condition on a matrix M in order for d_M to be a translation invariant density.

<u>Theorem 6.1</u>. If $M = (a_{nk})$ is an essentially non-negative regular matrix with $\lim_n \sum_k |a_{n,k} - a_{n,k+1}| = 0$, then d_M is a translation invariant density.

Proof. By Theorem 3.4, d_M is an asymptotic density. To see that d_M is translation invariant, first note that $\chi_{A+1}(k) = \chi_A(k-1)$. Then

$$|(M\chi_A)_n - (M\chi_{A+1})_n| = |\sum_k a_{nk}\chi_A(k) - \sum_k a_{nk}\chi_A(k-1)|$$

$$= |\sum_k (a_{nk} - a_{n,k+1})\chi_A(k)|$$

$$\leq \sum_k |a_{nk} - a_{n,k+1}|,$$

so $\lim_n (M\chi_A)_n - (M\chi_{A+1})_n = 0$. Thus $|d_M(A) - d_M(A+1)| = |\liminf_n (M\chi_A)_n - \liminf_n (M\chi_{A+1})_n| \leq |\limsup (M\chi_A)_n - (M\chi_{A+1})_n| = 0$, and $d_M(A+1) = d_M(A)$.

Corollary 6.2. The function $\lambda = d_L$ defined above is a translation invariant density.

Proof. The matrix $L = (a_{nk})$ defined above is non-negative. It is regular, since for all k, $\lim_n a_{nk} = 0$, and $\lim_n \sum_k a_{nk} = \lim_n \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} = 1$. Finally, $\lim_n \sum_k |a_{n,k} - a_{n,k+1}| = \lim_n \frac{1}{\ln n}\left(\sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{k+1} + \frac{1}{n}\right) = \lim_n \frac{1}{\ln n} = 0$. So by the above theorem, $\lambda = d_L$ is a translation invariant density.

Definition 6.1. The function λ defined above is called the logarithmic density.

We note that by Theorem 3.2, the upper logarithmic density $\overline{\lambda}$ is given by $\overline{\lambda}(A) = \limsup_n \frac{1}{\ln n} \sum_{\substack{a \in A \\ a \leq n}} \frac{1}{a}$; and by Theorem 3.5, logarithmic density has the additivity property. Since the usual density is normal, the following result shows that logarithmic density is also normal.

Theorem 6.2. If δ is the usual density, and λ is the logarithmic density, then $\underline{\delta} \le \underline{\lambda} \le \overline{\lambda} \le \overline{\delta}$.

Proof. Given sequences (x_i) and (y_i), with $s_n = \sum_{i=1}^{n} x_i$, the Abel summation formula is $\sum_{i=n+1}^{n+k} x_i y_i = \sum_{i=n+1}^{n+k} s_i(y_i - y_{i+1}) - s_n y_{n+1} + s_{n+k} y_{n+k+1}$. Given A, let $x_i = \chi_A(i)$ and $y_i = \frac{1}{i}$, so that $s_n = A(n)$. Then given $N > 0$,

$$\sum_{i=N+1}^{n} \frac{\chi_A(i)}{i} = \sum_{i=N+1}^{n-1} \frac{A(i)}{i(i+1)} + \frac{A(n)}{n(n+1)} - \frac{A(N)}{N+1} + \frac{A(n)}{n+1}$$

$$= \sum_{i=N+1}^{n-1} \frac{A(i)}{i} \cdot \frac{1}{i+1} + \frac{A(n)}{n} - \frac{A(N)}{N+1} .$$

Since $\frac{A(n)}{n} - \frac{A(N)}{N+1}$ is bounded, we have

$$\underline{\lambda}(A) = \liminf_{n} \frac{1}{\ln n}\left(\sum_{i=1}^{N} \frac{\chi_A(i)}{i} + \sum_{i=N+1}^{n} \frac{\chi_A(i)}{i}\right)$$

$$= \liminf_{n} \frac{1}{\ln n}\left(\sum_{i=1}^{N} \frac{\chi_A(i)}{i} + \sum_{i=N+1}^{n-1} \frac{A(i)}{i} \cdot \frac{1}{i+1} + O(1)\right)$$

$$= \liminf_{n} \frac{1}{\ln n} \sum_{i=N+1}^{n-1} \frac{A(i)}{i} \cdot \frac{1}{i+1}$$

$$\geq \inf_{i>N} \frac{A(i)}{i} \liminf_{n} \frac{1}{\ln n} \sum_{i=N+1}^{n-1} \frac{1}{i+1}$$

$$= \inf_{i>N} \frac{A(i)}{i} .$$

Then letting $N \to \infty$, we get $\lambda(A) \geq \liminf_{n} \frac{A(n)}{n} = \delta(A)$. It follows by Theorem 2.5 and Lemma 2.6 that $\delta \leq \lambda \leq \overline{\lambda} \leq \overline{\delta}$.

By the above theorem, $N_\delta \subseteq N_\lambda$. We actually have $N_\delta \subset N_\lambda$, since sets of the following type have natural logarithmic density but may not have natural ordinary density.

<u>Definition 6.2</u>. Let $A = (a_n)$. A is a <u>primitive</u> <u>set</u> if for $i \neq j$, $a_i \nmid a_j$.

We state the following results from Sequences [5] without proof: If A is a primitive set, then $\delta(A) = \lambda(A) = \overline{\lambda}(A) = 0$, and $0 \leq \overline{\delta}(A) < \frac{1}{2}$. Moreover, given any $\varepsilon > 0$, there is a primitive set A such that $\overline{\delta}(A) > \frac{1}{2} - \varepsilon$. Together with Theorem 6.2, this gives the following.

<u>Theorem 6.3</u>. If δ is the usual density, and λ is the logarithmic density, then $N_\delta \subset N_\lambda$, with $\nu_\lambda = \nu_\delta$ on N_δ .

§7 CONCLUSION

In this chapter we mention some other generalizations of asymptotic density, and give some applications of the density concept in summability theory and number theory. We restate the unsolved problems that have been mentioned, and give some new ones as well.

In this paper, a general asymptotic density has been defined axiomatically, using as axioms some of the elementary properties of ordinary asymptotic density. Asymptotic density for sets of integers, and for arbitrary sets, has been generalized along more measure-theoretic lines by Buck [1,2]. If sets A and B have natural density according to his definitions, then so do $A \cup B$ and $A \cap B$; this does not hold for natural ordinary density [1]. Asymptotic density has also been generalized to sets of n-tuples of integers by Freedman [3].

The density concept has been applied in summability theory by Freedman and Sember [4]. Given an asymptotic density d, a sequence x is said to be (d)-nearly convergent to the real number L, if there exists an A with $\overline{d}(A) = 0$ such that, when the terms of x indexed by elements of A are deleted, the resulting sequence converges to L in the ordinary sense.

The set ω_d of (d)-nearly convergent sequences is a linear space, and can be used to describe the convergence fields and strong convergence fields of summability methods related to d. Whether or not d has the additivity property for sets of natural density zero affects the results here. Some applications of the general theorems are that if $|\sigma_1|$ is the space of strongly Cesaro-summable sequences, $|AC|$ is the space of strongly almost-convergent sequences, and m is the space of bounded sequences, then $|\sigma_1| \cap m = \omega_\delta \cap m$, and $|AC| = |AC| \cap m \supset \omega_\upsilon \cap m$, with $|AC| = \overline{\omega_\upsilon \cap m}$.

One of the uses of asymptotic density in number theory is in the concept of "almost all," similar to the use of "almost all" in the sense of measure. A property is said to hold for almost all integers if the set of integers for which the property does not hold is of natural ordinary density zero. An example of a result stated in this form is that almost all integers are the sum of a k^{th} power and a prime [7]. Clearly, the definition of "almost all" may be modified to "(d)-almost all" for a general density d. If N_d^0 denotes the class of sets A with $\nu_d(A) = 0$, then $N_\upsilon^0 \subset N_\delta^0$, so for example, the statement that P holds for (υ)-almost all integers is stronger than the statement that P holds for (δ)-almost all integers.

An example of a general theorem in additive number theory involving ordinary asymptotic density is Ostmann's result that if $\delta(A) + \delta(B) > 1$, then $A + B \sim I$ [7]. There is also an important theorem by Kneser, which states that under certain conditions, $\delta(A + B) \geq \delta(A) + \delta(B)$. [5]. It may be asked if these results hold for the generalized density d.

Other unsolved problems in the theory of asymptotic density presented here are:

(i) Is every natural density associated with a summability method? That is, given a density d, is there a method M such that $\lim \chi_A = \alpha$ (M) if and only if $A \in N_d$ with $\nu_d(A) = \alpha$.

(ii) Is every matrix-method density a regular matrix-method density?

(iii) Is the weak additivity property equivalent to the additivity property, or do there exist densities which have the WAP but not the AP? In particular, does uniform density have the WAP?

(iv) Find a necessary and sufficient condition on a matrix $M = (a_{nk})$ in order for d_M to be a translation invariant density. We note again that a sufficient condition is $\lim_n \sum_k |a_{n,k} - a_{n,k+1}| = 0$.

(v) Is it true that for all A and B, $d(A) + d(B) \leq \bar{d}(A \cup B) + d(A \cap B)$?

(vi) Find a set A for which $\delta(A) < \alpha(A)$. Since $\delta \leq \alpha \leq \bar{\alpha} \leq \bar{\delta}$, it is necessary that $A \notin N_\delta$.

(vii) Is $\alpha \leq \lambda$? We cannot have $\lambda \leq \alpha \leq \bar{\alpha} \leq \lambda$, since then the existence of natural logarithmic density would imply the existence of natural ordinary density. The known ordering of asymptotic densities that have been studied here is

$$\beta \leq \upsilon \leq \delta \leq \lambda \leq \bar{\lambda} \leq \bar{\delta} \leq \bar{\upsilon} \leq \bar{\beta} .$$

Furthermore, ν_υ, ν_δ, and ν_λ are proper extensions of ν_β, ν_υ, and ν_δ respectively.

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