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## LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS REÇUE

# A GENERALIZED RELATIONAL SEMANTICS FOR MODAL LOGIC

by

David Kenneth Johnston

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## ABSTRACT

The notion of a relational frame is extended to include structures with relations of any finite arity. For each natural number n, the class of n+1-ary relational frames is shown to determine the logic $G_n$, which is defined using an n-ary modal operator. For each n, a truth condition for the unary modal operator is defined on n-ary frames. Two ways of syntactically defining the resulting unary logic are presented. Several extensions of the $G_n$ logics, using both formulae with n-ary operators and formulae with the unary operator, are presented. Soundness and completeness with respect to classes of n-ary relational frames is proved for each extension. It is proved that the formula $\Box p \rightarrow \Diamond p$ is not determined by any class of n-ary relational frames where n is greater than two.

CONTENTS

INTRODUCTION

In the last two decades the use of binary relational frames to provide a semantics for modal logic has received a great deal of attention, and this has produced a multitude of interesting formal results. This essay will present the basic theory of a generalized relational semantics for modal logic. The generalization is achieved in a very simple way: the notion of a relational frame is extended to include structures with relations of any finite arity. Since this requires only a modest intellectual leap, it is surprising that it has not been done before. A possible explanation of this puzzle is that such a simple generalization does not seem to promise many interesting formal results. It might be thought that it would yield only rather tedious generalizations of results already obtained for binary frames. One of the purposes of this essay is to show that this is not the case.

The generalized notion of a relational frame described in section II is due to R.E. Jennings of Simon Fraser University and P.K. Schotch of Dalhousie University, and was first formulated in 1975. When the search began for the logic determined by these frames it was immediately apparent that this generalization was non-trivial. In fact, several of the completeness results included in section IV were obtained before it was known which logic was being extended. It was not until 1977 that Jennings and Schotch obtained a completeness theorem for the logic described in section III.

The endeavours described above were concerned only with the unary modal operator ◫. But these generalized relational frames also allow for the definition of a truth condition for an n-ary modal operator (where the arity of the frame is n + 1.) The completeness theorem for the n-ary operator was simple by comparison with that for the unary operator, and was made even simpler by the contribution of R.I. Goldblatt. Goldblatt had investigated a binary modal operator with its truth condition defined on ternary frames. This binary operator turned out to be our $\Diamond_2$. Theorem 3 below is essentially a generalization of the completeness theorem that Goldblatt provides in [1].

In section IV the characteristic generalized frame conditions for several traditional formulae are presented. In the binary case, these formulae distinguish themselves by having rather simple frame conditions, such as transitivity or symmetry. In the n-ary case some of these formulae retain this distinction, in that they are characterized by a straightforward n-ary frame condition for each n. But for other formulae this distinction vanishes. For example, in section V it is shown that [D] has no characteristic n-ary frame condition where n is greater than two. Section IV does contain some surprises, despite the fact that it examines only some of the well-known formulae (and none of the more exotic formulae) that are in the literature. This should indicate that more surprises are to be expected as research in this area continues.

Above we contrasted the simplicity of the relational semantics of the n-ary modal operator with the complexities involved with the unary operator. This situation is reversed in the case of neighbourhood semantics. In [4] Segerberg gives a neighbourhood semantics for E, C, and K.

Each of these logics has a correlated logic defined with the n-ary operator. These can be conveniently designated as $E_n$, $C_n$, and $G_n$. (The significance of this last name is explained in section II.) The generalization of the notion of a neighbourhood frame that is required to deal with the n-ary operator is as simple as that required for relational frames: the neighbourhood function maps points onto sets of ordered n-tuples of sets, rather than onto sets of sets. Completeness results are easily obtained for $E_n$ and $C_n$, which indicates that the generalization is correct. However, no completeness result has been forthcoming for $G_n$ (the correlate of K). By contrast, the neighbourhood semantics for the unary operator is quite trivial, whether one uses generalized neighbourhood frames or the standard frames found in [4]. Segerberg also provides a definition of binary relational frames on neighbourhood frames. Attempts to duplicate this achievment for n-ary relational frames have so far failed, both with the generalized and the standard neighbourhood frames. Once again, procedures which are straightforward in the binary case prove to be quite the opposite in the n-ary case.

The problems described above, and the ones described in the sections that follow, should be enough to prove that generalized relational frames are objects worthy of detailed study. It is hoped that this essay will play a part in inspiring such research.

(I) SYNTAX

An <u>n-adic modal language</u> $L_n$ is a triple $\langle At, k, F_n \rangle$ where

$At = \{p_i : i \in Nat\}$

$k = \{\bot, \rightarrow, \Box_n\}$

and where $F_n$ is defined as follows:

(i) $At \subseteq F_n$

(ii) $\bot \in F_n$

(iii) $\forall \alpha, \beta,\ \alpha \in F_n \ \&\ \beta \in F_n \Rightarrow \alpha \rightarrow \beta \in F_n$

(iv) $\forall \alpha_1 \ldots \alpha_n,\ \alpha_1 \in F_n \ \&\ \ldots \ \&\ \alpha_n \in F_n \Rightarrow \Box_n(\alpha_1 \ldots \alpha_n) \in F_n$

Each $\Box_n$, and any operator defined with them, is a <u>modal operator</u>. Here the familiar $\Box$ operator is the modal operator $\Box_1$ of $L_1$.

The abbreviating conventions for PC operators are as usual. The n-ary diamond is defined:

$$\Diamond_n(\alpha_1 \ldots \alpha_n) =_{df} \neg\Box_n(\neg\alpha_1 \ldots \neg\alpha_n)$$

We also define the unary $\Box$ in each $L_n$:

$$\Box\alpha =_{df} \Box_n(\alpha \ldots \alpha)$$

Where the arity of a modal operator is apparent from the context, the subscript 'n' is often omitted. For example, '$\Box_n(\alpha_1 \ldots \alpha_n)$' will often be written as '$\Box(\alpha_1 \ldots \alpha_n)$'.

We maintain the traditional distinction between a system and a logic. Many logics bear the same names as particular formulae. To avoid confusion we enclose the name of a formula in square brackets. For example, [D] denotes the formula $\Box p \rightarrow \Diamond p$. Where [X] is the name of $\alpha$, [X'] is the name of $\Box\alpha$. Where L is a logic and [X] is the name of a formula, LX or L[X] denotes the logic generated by the system L with the addition of [X] as an axiom.

We also make use of the notion of a <u>sector</u>. Where O is an n-ary modal operator, we define the set of formulae $F_O$ as follows:

(i) $At \subseteq F_O$

(ii) $\perp \in F_O$

(iii) $\forall\alpha,\beta,\ \alpha \in F_O \ \&\ \beta \in F_O \Rightarrow \alpha \rightarrow \beta \in F_O$

(iv) $\forall\alpha_1\ldots\alpha_n,\ \alpha_1 \in F_O \ \&\ \ldots\ \&\ \alpha_n \in F_O \Rightarrow O(\alpha_1\ldots\alpha_n) \in F_O$

Where L is a logic, the <u>O-sector</u> <u>of</u> <u>L</u> (L/O) is defined as $L \cap F_O$. Thus $L/\Box = L \cap F_1 = L \cap F_\Box$.

## (II) RELATIONAL FRAMES

A relational frame is a triple $\langle D,f,R\rangle$, where D is a non-empty set, f is a function mapping each element of D onto a natural number, and R is a function from D such that for all x in D, $R(x) \in p(D^{f(x)})$. A relational frame may be said to be first order (for the restricted purposes of this essay) just when f(x) = f(y) for all x, y, in D. Suppose $F$ is a first order relational frame where f(x) = n for all x in D. Then $F$ can be represented as a pair $\langle D,R\rangle$ where R is a subset of $D^{n+1}$. All relational frames which will be considered in this work are first order frames, and so they will be constructed as pairs. Instead of $<x,y_1...y_n> \in R$ we will often write $xRy_1...y_n$. $F$ will be said to be an n+1-ary relational frame if f(x) = n for all x in D, that is, if R is an n+1-ary relation.

A model $M$ on an n+1-ary relational frame $F$ is a pair $\langle F,V\rangle$ where V is a function: At $\rightarrow p(D)$. V is said to be an assignment. The appropriate modal language for an n+1-ary relational model is $L_n$. The truth conditions for PC formulae are as usual. The truth condition for the $\Box_n$ operator is as follows:

$$M\models_x \Box(\alpha_1...\alpha_n) \text{ iff } \forall y_1...y_n, xRy_1...y_n \Rightarrow \exists k(1\leq k\leq n): M\models_{y_k} \alpha_k$$

The notions of truth on a model, validity on a frame, and validity on a class of frames, as well as soundness and completeness with respect to classes of frames, are those which are in common usage.

Both [RR] $\vdash \alpha \rightarrow \beta \Rightarrow \vdash \Box\alpha \rightarrow \Box\beta$ and [RN] $\vdash \alpha \Rightarrow \vdash \Box\alpha$ preserve validity on relational frames of arity greater than one. However, it is easily shown that [K] $\Box p \wedge \Box q \rightarrow \Box(p\wedge q)$ will fail on any class of relational frames with arity greater than one. Thus the generalized notion of a relational frame yields a first order semantics for logics weaker than K. We will now see what these logics are.

For each natural number n, we define the logic $G_n$ as the set of formulae including PC and each of the n instances of the schema:

$$[G_n]\ \Box(p_1 \ldots p_n) \wedge \Box(p_1 \ldots p_{k-1}, q, p_{k+1} \ldots p_n) \rightarrow \Box(p_1 \ldots p_{k-1}, p_k \wedge q, p_{k+1} \ldots p_n)$$

and closed under modus ponens, uniform substitution, and each of the n instances of the following two schemata:

$$[RR_n]\ \vdash \alpha \rightarrow \beta \Rightarrow \vdash \Box(\gamma_1 \ldots \gamma_{k-1}, \alpha, \gamma_{k+1} \ldots \gamma_n) \rightarrow \Box(\gamma_1 \ldots \gamma_{k-1}, \beta, \gamma_{k+1} \ldots \gamma_n)$$

$$[RN_n]\ \vdash \alpha \Rightarrow \vdash \Box(\beta_1 \ldots \beta_{k-1}, \alpha, \beta_{k+1} \ldots \beta_n)$$

(The 'G' is used in recognition of Rob Goldblatt, who provides in [1] what amounts to a completeness theorem for $G_2$.)

We will first prove a theorem which reveals some of the syntactic properties of the $G_n$ logics. Consider the following rule:

$$[RR'_n]\ \vdash \alpha_1 \rightarrow \beta_1\ \&\ \ldots\ \&\ \vdash \alpha_n \rightarrow \beta_n \Rightarrow \vdash \Box(\alpha_1 \ldots \alpha_n) \rightarrow \Box(\beta_1 \ldots \beta_n)$$

<u>Lemma 1.1</u>: If PC $\subseteq$ L, then L is closed under $[RR_n]$ only if L is closed under $[RR'_n]$.

| | | |
|---|---|---|
| Proof: (1) | $\vdash \alpha_1 \rightarrow \beta_1$ | Hypothesis |
| ⋮ | | ⋮ |
| (n) | $\vdash \alpha_n \rightarrow \beta_n$ | Hypothesis |
| (n+1) | $\vdash \Box(\alpha_1 \ldots \alpha_n) \rightarrow \Box(\beta_1, \alpha_2 \ldots \alpha_n)$ | from (1) by $[RR_n]$ |
| (n+2) | $\vdash \Box(\beta_1, \alpha_2 \ldots \alpha_n) \rightarrow \Box(\beta_1, \beta_2, \alpha_3 \ldots \alpha_n)$ | from (2) by $[RR_n]$ |

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots$

(2n) $\vdash \Box(\beta_1 \ldots \beta_{n-1}, \alpha_n) \to \Box(\beta_1 \ldots \beta_n)$ from (n) by [$RR_n$]

(2n+1) $\vdash \Box(\alpha_1 \ldots \alpha_n) \to \Box(\beta_1 \ldots \beta_n)$ from (n+1) ... (2n) by transitivity of $\to$

<u>Lemma 1.2</u>: If PC $\subseteq$ L then L is closed under [$RR'_n$] only if L is closed under [$RR_n$].

Proof: (1) $\vdash \alpha \to \beta$ Hypothesis

(2) $\vdash \gamma_1 \to \gamma_1$ PC

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots$

(k-1) $\vdash \gamma_{k-1} \to \gamma_{k-1}$ PC

(k) $\vdash \gamma_{k+1} \to \gamma_{k+1}$ PC

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots$

(n) $\vdash \gamma_n \to \gamma_n$ PC

(n+1) $\vdash \Box(\gamma_1 \ldots \gamma_{k-1}, \alpha, \gamma_{k+1} \ldots \gamma_n) \to \Box(\gamma_1 \ldots \gamma_{k-1}, \beta, \gamma_{k+1} \ldots \gamma_n)$ from (1) ... (n) by [$RR'_n$]

<u>Lemma 1.3</u>: If PC $\subseteq$ L then L is closed under [$RR_n$] only if L is closed under [RR].

Proof: (1) $\vdash \alpha \to \beta$ Hypothesis

(2) $\vdash \Box_n(\alpha \ldots \alpha) \to \Box_n(\beta, \alpha \ldots \alpha)$ from (1) by [$RR_n$]

(3) $\vdash \Box_n(\beta, \alpha \ldots \alpha) \to \Box_n(\beta, \beta, \alpha \ldots \alpha)$ from (1) by [$RR_n$]

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots$

(n+1) $\vdash \Box_n(\beta \ldots \beta, \alpha) \to \Box_n(\beta \ldots \beta)$ from (1) by [$RR_n$]

(n+2) $\vdash \Box_n(\alpha \ldots \alpha) \to \Box_n(\beta \ldots \beta)$ from (2) ... (n+1) by transitivity of $\to$

(n+3) $\vdash \Box\alpha \to \Box\beta$ from (n+2) by definition of $\Box$ in $L_n$

Lemma 1.4: L is closed under $[RN_n]$ only if L is closed under [RN].

Proof: (1) $\vdash \alpha$ Hypothesis

(2) $\vdash \Box_n(\alpha\ldots\alpha)$ from (1) by $[RN_n]$

(3) $\vdash \Box\alpha$ from (2) by definition of $\Box$ in $L_n$

Lemma 1.5: If L includes PC and is closed under $[RR_n']$, then

(i) $\vdash_L \Box_n(\alpha_1\ldots\alpha_n) \to \Box(\alpha_1\vee\ldots\vee\alpha_n)$

(ii) $\vdash_L \Box(\alpha_1\wedge\ldots\wedge\alpha_n) \to \Box_n(\alpha_1\ldots\alpha_n)$

Proof of (i):

(1) $\vdash \alpha_1 \to \alpha_1\vee\ldots\vee\alpha_n$ PC

$\vdots$ $\vdots$

(n) $\vdash \alpha_n \to \alpha_1\vee\ldots\vee\alpha_n$ PC

(n+1) $\vdash \Box_n(\alpha_1\ldots\alpha_n) \to \Box_n((\alpha_1\vee\ldots\vee\alpha_n)\ldots(\alpha_1\vee\ldots\vee\alpha_n))$ from (1) ... (n) by $[RR_n']$

(n+2) $\vdash \Box_n(\alpha_1\ldots\alpha_n) \to \Box(\alpha_1\vee\ldots\vee\alpha_n)$ from (n+1) by definition of $\Box$ in $L_n$

Proof of (ii):

(1) $\vdash \alpha_1\wedge\ldots\wedge\alpha_n \to \alpha_1$ PC

$\vdots$ $\vdots$

(n) $\vdash \alpha_1\wedge\ldots\wedge\alpha_n \to \alpha_n$ PC

(n+1) $\vdash \Box_n((\alpha_1\wedge\ldots\wedge\alpha_n)\ldots(\alpha_1\wedge\ldots\wedge\alpha_n)) \to \Box_n(\alpha_1\ldots\alpha_n)$ from (1) ... (n) by $[RR_n']$

(n+2) $\vdash \Box(\alpha_1\wedge\ldots\wedge\alpha_n) \rightarrow \Box_n(\alpha_1\ldots\alpha_n)$ from (n+1) by definition of $\Box$ in $L_n$

Thus we have the following theorem:

THEOREM 1: If PC $\subseteq$ L, then:

(i) L is closed under $[RR_n]$ iff L is closed under $[RR'_n]$

(ii) L is closed under [RR] if L is closed under $[RR_n]$

(iii) L is closed under [RN] if L is closed under $[RN_n]$

(iv) $\vdash_L \Box_n(\alpha_1\ldots\alpha_n) \rightarrow \Box(\alpha_1\vee\ldots\vee\alpha_n)$ if L is closed under $[RR'_n]$

(v) $\vdash_L \Box(\alpha_1\wedge\ldots\wedge\alpha_n) \rightarrow \Box_n(\alpha_1\ldots\alpha_n)$ if L is closed under $[RR'_n]$

It follows from Theorem 1 that the $\Box$-sector of $G_n$ is closed under [RR] and [RN]. The following theorem shows that [K] is not a member of the $\Box$-sector of $G_n$ where n > 1.

THEOREM 2: $G_n$ is sound with respect to the class of n+1-ary relational frames.

Proof: trivial

Thus where n > 1, [K] $\notin G_n/\Box$ since [K] will fail on some n+1-ary relational frame.

The canonical domain $D_L$ and the canonical assignment $V_L$ are defined as usual. The canonical relation $R_L$ is defined:

$$xR_L y_1\ldots y_n \text{ iff } \forall\alpha_1\ldots\alpha_n,\ \Box(\alpha_1\ldots\alpha_n) \in x \Rightarrow \exists k(1\leq k\leq n): \alpha_k \in y_k$$

Where L = $G_n$, the following theorem shows that all and only $G_n$ theorems are true on the canonical model $\mathcal{M}_L = \langle D_L, R_L, V_L\rangle$, and therefore constitutes the Fundamental Theorem for relational semantics.

THEOREM 3: $\mathcal{M}_L \models_x \alpha$ iff $\alpha \in x$ (L = $G_n$)

Proof: The proof is by induction on the length of $\alpha$. The proof is trivial for PC formulae, and the "if" direction for the induction step where $\alpha = \Box(\beta_1 \ldots \beta_n)$ follows from the definition of $R_L$.

($\Rightarrow$) assume that $\Box(\beta_1 \ldots \beta_n) \notin x$. Let $A_0 \ldots A_k \ldots$ be an enumeration of $F_n$. Construct $y_1 \ldots y_n$ as follows:

Let $y_{1_0} = \{\neg\beta_1\}$

$\vdots$

$$y_{1_{k+1}} = y_{1_k} \cup \{A_k\} \text{ iff } \forall\gamma,\ y_{1_k} \vdash_L A_k \rightarrow \neg\gamma \Rightarrow \Box(\gamma,\beta_2 \ldots \beta_n) \notin x$$
$$= y_{1_k} \text{ otherwise}$$

Let $y_1 = \cup\{y_{1_0} \ldots y_{1_k} \ldots\}$

Let $y_{i_0} = \{\neg\beta_i\}$

$\vdots$

$$y_{i_{k+1}} = y_{i_k} \cup \{A_k\} \text{ iff } \forall\gamma,\delta_1 \ldots \delta_{i-1},\ y_{i_k} \vdash_L A_k \rightarrow \neg\gamma \ \& \ \Box(\delta_1 \ldots \delta_{i-1},\gamma,\beta_{i+1} \ldots \beta_n) \in x \Rightarrow \exists j(1 \le j \le i-1): \delta_j \in y_j$$
$$= y_{i_k} \text{ otherwise}$$

Let $y_i = \cup\{y_{i_0} \ldots y_{i_k} \ldots\}$

<u>Lemma 3.1</u>: $\forall i(1 \le i \le n)$, $y_i \nvdash_L \bot$

Proof: The proof is by induction on i.

(i=1) assume $y_1 \vdash_L \bot \ \therefore \exists y_{1_k} \subseteq y_1 : y_{1_k} \vdash_L \bot$

But $y_{1_k} = y_{1_{k-1}} \cup \{A_j\}$ for some $j \le k-1$ or $y_{1_k} = \{\neg\beta_1\}$

If $y_{1_k} = \{\neg\beta_1\}$ then $\vdash_L \neg\beta_1 \rightarrow \bot \ \therefore \ \vdash_L \top \rightarrow \beta_1$

$\therefore \ \vdash_L \Box(\top,\beta_2 \ldots \beta_n) \rightarrow \Box(\beta_1 \ldots \beta_n)$ But $\Box(\top,\beta_2 \ldots \beta_n) \in x$ ($[RN_n]$)

$\therefore \Box(\beta_1 \ldots \beta_n) \in x$, contrary to hypothesis

$\therefore y_{1_k} = y_{1_{k-1}} \cup \{A_j\}$ for some $j \le k-1$

$\therefore y_{1_{k-1}} \vdash_L A_j \rightarrow \bot$

$\therefore \Box(\top,\beta_2\ldots\beta_n) \notin x$ by construction of $y_1$, contrary to $[RN_n]$

(induction step) Assume that $y_i \vdash_L \bot \quad \therefore \exists y_{i_k} \subseteq y_i : y_{i_k} \vdash_L \bot$

But $y_{i_k} = y_{i_{k-1}} \cup \{A_j\}$ for some $j \leq k-1$

(for $y_{i_k} = \{\neg\beta_i\}$ we argue as above)

$\therefore y_{i_{k-1}} \vdash_L A_j \rightarrow \bot$

$\therefore \forall\delta_1\ldots\delta_{i-1}, \Box(\delta_1\ldots\delta_{i-1}, \top,\beta_{i+1}\ldots\beta_n) \in x \Rightarrow$

$\delta_1 \in y_1$ or ... or $\delta_{i-1} \in y_{i-1}$ (by construction of $y_i$)

But $\Box(\beta_1\ldots\beta_{i-1},\top,\beta_{i+1}\ldots\beta_n) \in x$ ($[RN_n]$)

$\therefore \exists j(1 \leq j \leq i-1): \beta_j \wedge \neg\beta_j \in y_j$

$\therefore y_j$ is inconsistent, contrary to the induction hypothesis

<u>Lemma 3.2</u>: $\forall\gamma,\delta_1\ldots\delta_{i-1}, y_i \vdash_L \neg\gamma \;\&\; \neg\delta_1 \in y_1 \;\&\; \ldots \;\&\; \neg\delta_{i-1} \in y_{i-1} \Rightarrow$

$\Box(\delta_1\ldots\delta_{i-1},\gamma,\beta_{i+1}\ldots\beta_n) \notin x$

Proof: The proof is by induction on i.

(i=1) Assume that $y_1 \vdash_L \neg\gamma \quad \therefore \exists y_{1_k} \subseteq y_1 : y_{1_k} \vdash_L \neg\gamma$

But $y_{1_k} = y_{1_{k-1}} \cup \{A_j\}$ for some $j \leq k-1$ or $y_{1_k} = \{\neg\beta_1\}$

If $y_{1_k} = y_{1_{k-1}} \cup \{A_j\}$ then $y_{1_{k-1}} \vdash_L A_j \rightarrow \neg\gamma$

$\therefore \Box(\gamma,\beta_2\ldots\beta_n) \notin x$ by construction of $y_1$

Assume $y_{1_k} = \{\neg\beta_1\} \quad \therefore \vdash_L \neg\beta_1 \rightarrow \neg\gamma \quad \therefore \vdash_L \gamma \rightarrow \beta_1$

$\therefore \vdash_L \Box(\gamma,\beta_2\ldots\beta_n) \rightarrow \Box(\beta_1\ldots\beta_n)$ But $\Box(\beta_1\ldots\beta_n) \notin x$

$\therefore \Box(\gamma,\beta_2\ldots\beta_n) \notin x$

(induction step) Assume that $y_i \vdash_L \neg\gamma \;\&\; \neg\delta_1 \in y_1 \;\&\; \ldots \;\&\; \neg\delta_{i-1} \in y_{i-1}$

$\therefore \exists y_{i_k} \subseteq y_i : y_{i_k} \vdash_L \neg\gamma$ Assume $y_{i_k} = y_{i_{k-1}} \cup \{A_j\}$ for some $j \leq k-1$

But $\delta_1 \notin y_1 \;\&\; \ldots \;\&\; \delta_{i-1} \notin y_{i-1}$ (Lemma 3.1)

$\therefore \Box(\delta_1\ldots\delta_{i-1},\gamma,\beta_{i+1}\ldots\beta_n) \notin x$ by construction of $y_i$

Assume that $y_{i_k} = \{\neg\beta_i\} \therefore \vdash_L \neg\beta_i \rightarrow \neg\gamma \quad \therefore \vdash_L \gamma \rightarrow \beta_i$

$\therefore \vdash_L \Box(\delta_1 \ldots \delta_{i-1}, \gamma, \beta_{i+1} \ldots \beta_n) \rightarrow \Box(\delta_1 \ldots \delta_{i-1}, \beta_i \ldots \beta_n)$ ([$RR_n$])

But $y_{i-1} \vdash_L \neg\delta_{i-1}$ & $\neg\delta_1 \in y_1$ & ... & $\neg\delta_{i-2} \in y_{i-2}$

$\therefore \Box(\delta_1 \ldots \delta_{i-1}, \beta_i \ldots \beta_n) \notin x$ (induction hypothesis)

$\therefore \Box(\delta_1 \ldots \delta_{i-1}, \gamma, \beta_{i+1} \ldots \beta_n) \notin x$

<u>Lemma 3.3</u>: $\forall\gamma$, $\gamma \in y_i$ or $\neg\gamma \in y_i$

Proof: The proof is by induction on i.

(i=1) Assume $\exists\gamma$: $\gamma \notin y_1$ & $\neg\gamma \notin y_1$

Let $\gamma = A_j$ and $\neg\gamma = A_k$ in the ordering of $F_n$

$\therefore \exists\eta$: $y_{1_j} \vdash_L \gamma \rightarrow \neg\eta$ & $\Box(\eta, \beta_2 \ldots \beta_n) \in x$ and

$\exists\vartheta$: $y_{1_k} \vdash_L \neg\gamma \rightarrow \neg\vartheta$ & $\Box(\vartheta, \beta_2 \ldots \beta_n) \in x$ (construction of $y_1$)

But $y_{1_j} \subseteq y_1$ & $y_{1_k} \subseteq y_1$ $\therefore y_1 \vdash_L \eta\wedge\vartheta \rightarrow \gamma\wedge\neg\gamma$

$\therefore y_1 \vdash_L \neg(\eta\wedge\vartheta)$ $\therefore \Box(\eta\wedge\vartheta, \beta_2 \ldots \beta_n) \notin x$ (Lemma 3.2)

But $\Box(\eta, \beta_2 \ldots \beta_n) \wedge \Box(\vartheta, \beta_2 \ldots \beta_n) \in x$

$\therefore \Box(\eta\wedge\vartheta, \beta_2 \ldots \beta_n) \in x$ by [$G_n$], which is absurd

$\therefore \gamma \in y_1$ or $\neg\gamma \in y_1$

(induction step) Assume $\exists\gamma$: $\gamma \notin y_i$ & $\neg\gamma \notin y_i$

Let $\gamma = A_j$ and $\neg\gamma = A_k$ in the ordering of $F_n$

$\therefore \exists\eta, \varphi_1 \ldots \varphi_{i-1}$: $y_{i_j} \vdash_L \gamma \rightarrow \neg\eta$ & $\Box(\varphi_1 \ldots \varphi_{i-1}, \eta, \beta_{i+1} \ldots \beta_n) \in x$ &

$\forall g(1 \le g \le i-1)$, $\varphi_g \notin y_g$, and

$\exists\vartheta, \psi_1 \ldots \psi_{i-1}$: $y_{i_k} \vdash_L \neg\gamma \rightarrow \neg\vartheta$ & $\Box(\psi_1 \ldots \psi_{i-1}, \vartheta, \beta_{i+1} \ldots \beta_n) \in x$ &

$\forall g(1 \le g \le i-1)$, $\psi_g \notin y_g$ (by construction of $y_i$)

$\therefore (\varphi_1 \vee \psi_1) \notin y_1$ & ... & $(\varphi_{i-1} \vee \psi_{i-1}) \notin y_{i-1}$ (induction hypothesis and Lemma 3.1)

But $y_{i_j} \subseteq y_i$ and $y_{i_k} \subseteq y_i$

$\therefore y_i \vdash_L \eta\wedge\vartheta \rightarrow \gamma\wedge\neg\gamma$ $\therefore y_i \vdash_L \neg(\eta\wedge\vartheta)$

$\therefore\ \Box(\varphi_1 \vee \psi_1 \ldots \varphi_{i-1} \vee \psi_{i-1}, \eta \wedge \vartheta, \beta_{i+1} \ldots \beta_n) \notin x$ (construction of $y_i$)

But $\forall g(1 \le g \le i-1)$, $\vdash_L \varphi_g \rightarrow \varphi_g \vee \psi_g$ & $\vdash_L \psi_g \rightarrow \varphi_g \vee \psi_g$

$\therefore\ \Box(\varphi_1 \vee \psi_1 \ldots \varphi_{i-1} \vee \psi_{i-1}, \eta, \beta_{i+1} \ldots \beta_n) \wedge \Box(\varphi_1 \vee \psi_1 \ldots \varphi_{i-1} \vee \psi_{i-1}, \vartheta,$
$\beta_{i+1} \ldots \beta_n) \in x$ ($i-1$ applications of $[RR_n]$)

$\therefore\ \Box(\varphi_1 \vee \psi_1 \ldots \varphi_{i-1} \vee \psi_{i-1}, \eta \wedge \vartheta, \beta_{i+1} \ldots \beta_n) \in x$ by $[G_n]$, which is absurd

$\therefore\ \gamma \in y_i$ or $\neg\gamma \in y_i$

Lemma 3.4: $xR_L y_1 \ldots y_n$

Proof: By Lemmas 3.1 and 3.3, $y_1 \ldots y_n$ are L-maximal consistent

$\therefore\ y_1 \in D_L$ & ... & $y_n \in D_L$

Assume $\Box(\gamma_1 \ldots \gamma_n) \in x$ and $\gamma_n \notin y_n$ $\therefore\ \neg\gamma_n \in y_n$

$\therefore\ y_n \vdash_L \neg\gamma_n$ $\therefore\ \neg\gamma_1 \notin y_1$ or ... or $\neg\gamma_{n-1} \notin y_{n-1}$ (Lemma 3.2)

$\therefore\ \gamma_1 \in y_1$ or ... or $\gamma_{n-1} \in y_{n-1}$

$\therefore\ \Box(\gamma_1 \ldots \gamma_n) \in x \Rightarrow \gamma_1 \in y_1$ or ... or $\gamma_n \in y_n$

$\therefore\ xR_L y_1 \ldots y_n$

Thus we have shown $\exists y_1 \ldots y_n$: $xR_L y_1 \ldots y_n$ & $\beta_1 \notin y_1$ & ... & $\beta_n \notin y_n$.

$\therefore\ M_L \not\models_x \Box(\beta_1 \ldots \beta_n)$

$\therefore\ M_L \models_x \Box(\beta_1 \ldots \beta_n) \Rightarrow \Box(\beta_1 \ldots \beta_n) \in x$

Corollary 3.1: $G_n$ is complete with respect to the class of n+1-ary relational frames.

Proof: The proof is from Theorem 3 by the usual argument.

## (III) THE $\Box$-SECTOR OF $G_n$

By the definition of $\Box_1$ in $L_n$, a truth condition for the unary operator on n+1-ary relational models is easily derived:

$$M \models_x \Box\alpha \text{ iff } \forall y_1 \ldots y_n,\ xRy_1 \ldots y_n \Rightarrow \exists k (1 \leq k \leq n): M \models_{y_k} \alpha$$

Thus each class of first order relational frames of a particular arity will determine a logic which is a subset of $F_1$. Where the arity of the frames is n+1, this logic will be identical with $G_n/\Box$.

The logic $K_n$ is defined as the smallest set including PC and each instance of the following schema:

$$[K_n]_m\ \Box p_1 \wedge \ldots \wedge \Box p_m \rightarrow \Box((p_1 \wedge \ldots \wedge p_i) \vee \ldots \vee (p_j \wedge \ldots \wedge p_m))$$

(where $i = \frac{m+n-1}{n}$ is the number of conjuncts in each disjunct) and closed under [RR] and [RN]. This logic was at first thought to be complete with respect to the class of n+1-ary relational frames. The strategy of the completeness proof was to prove the Fundamental Theorem by showing, on the assumption that $\Box\alpha \notin x$, that the set $\{\beta: \Box\beta \in x\}$ can be partitioned into n sets, each consistent with $\neg\alpha$. When apparent counter-examples were found, Jennings and Schotch produced an expanded axiom set by defining several infinite series of axioms, all similar to the $[K_n]_m$ series given above. This axiom set is given in [3] and will not be reproduced here.

These axioms have been called <u>aggregation principles</u>. The strongest aggregation principle is [K], where $\Box$ collects all of its propositional letters into a single conjunction. As n increases, these aggregation principles become weaker. Where $n > 1$, $K_n$ is said to be <u>quasi-</u>

aggregative.

Jennings and Schotch have succeeded in showing that their axiomatization is complete. However, it is still not known whether $G_n/\Box$ is finitely axiomatizable. Some attempts have been made to prove completeness by methods similar to those used in Theorem 3, but so far these have not been successful.

We now present another way of characterizing $G_n/\Box$ syntactically using only the monadic modal language $L_1$. The function $T: F_1 \to F_1$ is defined as follows:

(i) $T(\alpha) = \alpha$ if $\alpha \in At$

(ii) $T(\bot) = \bot$

(iii) $T(\alpha \to \beta) = T(\alpha) \to T(\beta)$

(iv) $T(\Box\alpha) = \Box\Diamond T(\alpha)$

The range and domain of T is extended to subsets of $F_1$ in the obvious way:

$$T(L) = \{\beta \in F_1: \exists\alpha \in L: T(\alpha) = \beta\} \quad (\text{for } L \subseteq F_1)$$

In Segerberg [4] we find the following formula:

$$[Alt_n]\ \Box p_1 \vee \Box(p_1 \to p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_n \to p_{n+1})$$

The desired syntactic characterization of $G_n/\Box$ is provided by the following fact:

THEOREM 4: $T(G_n/\Box) = KD'Alt_n'/\Box\Diamond$.

Proof: Let $C$ be the class of binary relational frames such that:

$$\forall x,y,\ xRy \Rightarrow 1 \leq cardR(y) \leq n$$

It can be shown that $C$ determines $KD'Alt_n'$.

Let $C^*$ be the class of all n+1-ary relational frames.

Lemma 4.1: $\forall\alpha \in F_1$, $C^* \models \alpha \Leftrightarrow C \models T(\alpha)$.

Proof: For the "only if" direction, assume that $C \not\models T(\alpha)$

$\therefore \exists F \in C: F \not\models T(\alpha)$

$\therefore$ there is an $M$ on $F$, x in D of $M$ such that $M \not\models_x T(\alpha)$

Let $R' \subseteq D^{n+1} = \{<x,y_1 \ldots y_n>: \exists w: xRw \;\&\; wRy_i \text{ for each } i \leq n\}$

Let $F' = \langle D,R'\rangle$ and $M' = \langle F',V\rangle$ Obviously $F' \in C^*$

Then: $\forall\beta$, $M' \models_x \beta \Rightarrow M \models_x T(\beta)$

Proof: The proof is by induction on the length of $\beta$. We give only the induction step for $\beta = \Box\gamma$.

Assume that $M \not\models_x T(\beta)$ $\therefore M \not\models_x \Box\Diamond T(\gamma)$

$\therefore \exists y: xRy \;\&\; M \not\models_y \Diamond T(\gamma)$

$\therefore \forall z$, $yRz \Rightarrow M \not\models_z T(\gamma)$ But $\text{card}R(y) \geq 1$ since $F \in C$

$\therefore \exists z: yRz \;\&\; M \not\models_z T(\gamma)$

$\therefore xR'z \ldots z \;\&\; M' \not\models_z \gamma$ by the induction hypothesis and definition of R'

$\therefore M' \not\models_x \Box\gamma$ $\therefore \forall\beta$, $M' \models_x \beta \Rightarrow M \models_x T(\beta)$

$\therefore F' \not\models \alpha$ But $F' \in C^*$ $\therefore C^* \not\models \alpha$

$\therefore C^* \models \alpha \Rightarrow C \models T(\alpha)$

For the "if" direction, assume that $C^* \not\models \alpha$

$\therefore \exists F \in C^*: F \not\models \alpha$

$\therefore$ there is an $M$ on $F$, an x in D of $M$ such that $M \not\models_x \alpha$

Let $D' = D \cup R$

Define $R' \subseteq D' X D'$ as follows:

(i) $<x,y_1 \ldots y_n> \in R \Rightarrow xR'<x,y_1 \ldots y_n>R'y_i$ for each $i \leq n$

(ii) $<x,y_1 \ldots y_n>R'z \Rightarrow zR'z$

Then $F' = \langle D',R'\rangle \in C$. Let $M' = \langle F',V\rangle$

Then: $\forall\beta$, $M' \models_x T(\beta) \Rightarrow M \models_x \beta$.

Proof: The proof is by induction on the length of $\beta$. We give only the induction step for $\beta = \Box\gamma$.

Assume $M \not\models_x \beta$ $\therefore$ $M \not\models_x \Box\gamma$

$\therefore$ $\exists y_1 \ldots y_n$: $xRy_1 \ldots y_n$ & $M \not\models_{y_1} \gamma$ & ... & $M \not\models_{y_n} \gamma$

$\therefore$ $xR'\langle x, y_1 \ldots y_n\rangle R'y_i$ & $M' \not\models_{y_i} T(\gamma)$ (for each $i \leq n$) by the induction hypothesis and definition of R'

But $F' \in C$ $\therefore$ $\text{card}R'(\langle x, y_1 \ldots y_n\rangle) \leq n$

$\therefore$ $M' \models_{\langle x, y_1 \ldots y_n\rangle} \Box\neg T(\gamma)$ $\therefore$ $M' \models_x \Diamond\Box\neg T(\gamma)$

$\therefore$ $M' \not\models_x \Box\Diamond T(\gamma)$ $\therefore$ $M' \not\models_x T(\Box\gamma)$

$\therefore$ $\forall\beta$, $M' \models_x T(\beta) \Rightarrow M \models_x \beta$

$\therefore$ $F' \not\models T(\alpha)$ But $F' \in C$ $\therefore$ $C \not\models T(\alpha)$

$\therefore$ $C \models T(\alpha) \Rightarrow C^* \models \alpha$

This proves the lemma. To prove the theorem, first suppose that $\alpha \in T(G_n/\Box)$.

$\therefore$ $\alpha = T(\beta)$ for some $\beta \in G_n/\Box$ .

$\therefore$ $C^* \models \beta$ since $G_n$ is sound with respect to $C^*$.

$\therefore$ $C \models T(\beta)$ by the lemma. But $T(\beta) = \alpha$

$\therefore$ $\alpha \in KD'Alt'_n$ since $KD'Alt'_n$ is complete with respect to $C$.

$\therefore$ $T(G_n/\Box) \subseteq KD'Alt'_n/\Box\Diamond$

Next suppose that $\alpha \in KD'Alt'_n/\Box\Diamond$ $\therefore C \models \alpha$ since $KD'Alt'_n$ is sound with respect to $C$.

But $\alpha = T(\beta)$ for some $\beta \in F_1$. $\therefore$ $C^* \models \beta$ by the lemma.

$\therefore$ $\beta \in G_n/\Box$ since $G_n$ is complete with respect to $C^*$.

$\therefore$ $\alpha \in T(G_n/\Box)$ $\therefore$ $KD'Alt'_n/\Box\Diamond \subseteq T(G_n/\Box)$

Thus $T(G_n/\Box) = KD'Alt'_n/\Box\Diamond$

## (IV) SOME EXTENSIONS OF THE $G_n$ AND $K_n$ LOGICS

In section III it was mentioned that Jennings and Schotch had succeeded in axiomatizing the □-sector of $G_n$ by producing a completeness theorem for $K_n$. But it is obvious that not every formula valid on the class of n+l-ary relational frames is a theorem of $K_n$, since some members of $F_n$ will be valid on this class, and $K_n \subseteq F_1$. In this section we will need to make explicit the weaker notion of completeness used implicitly in the remarks mentioned above. Where $C$ is a class of relational frames and L is a logic, we say that L is complete mod $F_n$ with respect to $C$ iff every member of $F_n$ which is valid on $C$ is a theorem of L. Thus in [3] it is shown that $K_n$ is complete mod $F_1$ with respect to the class of n+l-ary relational frames. Where L and $C$ are as above, we say that $C$ determines L mod $F_n$ iff L is sound and complete mod $F_n$ with respect to $C$.

The classes of frames to be examined in this section are defined by the following conditions on n+l-ary relations:

R is reflexive iff $\forall x$, $xRx\ldots x$ .

R is symmetric iff $\forall x,y_1\ldots y_n$, $xRy_1\ldots y_n \Rightarrow \exists k(1\leq k\leq n)$: $y_kRx\ldots x$

R is quasi-transitive iff $\forall x,y_1\ldots y_n,z_1^1\ldots z_n^1\ldots z_1^n\ldots z_n^n$,
$xRy_1\ldots y_n$ & $\forall k(1\leq k\leq n)$, $y_kRz_1^k\ldots z_n^k \Rightarrow \exists j(1\leq j\leq n)$: $xRz_1^j\ldots z_n^j$

R is euclidian iff $\forall x,y_1\ldots y_n,z_1\ldots z_n$, $xRy_1\ldots y_n$ & $xRz_1\ldots z_n \Rightarrow$
$\exists k(1\leq k\leq n)$: $y_kRz_1\ldots z_n$

Classes of n+l-ary frames satisfying these conditions determine extensions of the $G_n$ logics which are given by the following formulae:

$[T_n]$ $\Box(p_1\ldots p_n) \rightarrow p_1\vee\ldots\vee p_n$

$[B_n]$ $\Diamond(\Box(p_{1_1}\ldots p_{1_n})\ldots\Box(p_{n_1}\ldots p_{n_n})) \rightarrow$
$p_{1_1}\vee\ldots\vee p_{1_n}\vee\ldots\vee p_{n_1}\vee\ldots\vee p_{n_n}$

$[4_n]$ $\Box(p_{1_1}\ldots p_{1_n}) \wedge \ldots \wedge \Box(p_{n_1}\ldots p_{n_n}) \rightarrow$
$\Box(\Box(p_{1_1}\ldots p_{1_n})\ldots (p_{n_1}\ldots p_{n_n}))$

$[5_n]$ $\Diamond((p_{1_1}\wedge\ldots\wedge p_{1_n})\ldots(p_{n_1}\wedge\ldots\wedge p_{n_n})) \rightarrow$
$\Box(\Diamond(p_{1_1}\ldots p_{n_1})\ldots\Diamond(p_{1_n}\ldots p_{n_n}))$

<u>THEOREM 5</u>: $G_nT_n$ is determined by the class of reflexive n+1-ary relational frames.

Proof: The proof is trivial and is omitted.

<u>THEOREM 6</u>: $G_nB_n$ is determined by the class of symmetric n+1-ary relational frames.

Proof: Soundness is trivial. For completeness, we show that the canonical relation $R_L$ ($L = G_nB_n$) is symmetric.

Assume $\exists x,y_1\ldots y_n$: $xR_Ly_1\ldots y_n$ & $\forall k(1\leq k\leq n)$, $\sim y_kR_Lx\ldots x$

$\therefore$ $\forall k(1\leq k\leq n)$, $\exists\alpha_{k_1}\ldots\alpha_{k_n}$ : $\Box(\alpha_{k_1}\ldots\alpha_{k_n}) \in y_k$ & $\alpha_{k_1} \notin x$ &...& $\alpha_{k_n} \notin x$

But $xR_Ly_1\ldots y_n$ $\therefore$ $\Diamond(\Box(\alpha_{1_1}\ldots\alpha_{1_n})\ldots\Box(\alpha_{n_1}\ldots\alpha_{n_n})) \in x$

$\therefore$ $\alpha_{1_1}\vee\ldots\vee\alpha_{1_n}\vee\ldots\vee\alpha_{n_1}\vee\ldots\vee\alpha_{n_n} \in x$ ($[B_n]$)

$\therefore$ x is inconsistent, which is absurd

$\therefore$ $R_L$ is symmetric

<u>THEOREM 7</u>: $G_n4_n$ is determined by the class of quasi-transitive n+1-ary relational frames.

Proof: Soundness is trivial. For completeness, we show that the canonical relation $R_L$ ($L = G_n4_n$) is quasi-transitive.

Assume $\exists x,y_1\ldots y_n,z_1^1\ldots z_n^1\ldots z_1^n\ldots z_n^n$:

$xR_Ly_1\ldots y_n$ & $\forall k(1\leq k\leq n)$, $y_kR_Lz_1^k\ldots z_n^k$ & $\forall k(1\leq k\leq n)$, $\sim xR_Lz_1^k\ldots z_n^k$

$\therefore \forall k(1 \le k \le n),\ \exists \alpha_{k_1} \ldots \alpha_{k_n} : \Box(\alpha_{k_1} \ldots \alpha_{k_n}) \in x \ \& \ \alpha_{k_1} \notin z_1^k \ \& \ldots \& \ \alpha_{k_n} \notin z_n^k$

$\therefore \Box(\Box(\alpha_{1_1} \ldots \alpha_{1_n}) \ldots \Box(\alpha_{n_1} \ldots \alpha_{n_n})) \in x \quad ([4_n])$

But $xR_L y_1 \ldots y_n \quad \therefore \exists k(1 \le k \le n) : \Box(\alpha_{k_1} \ldots \alpha_{k_n}) \in y_k$

But $y_k R_L z_1^k \ldots z_n^k \quad \therefore \exists j(1 \le j \le n) : \alpha_{k_j} \wedge \neg \alpha_{k_j} \in z_j^k$, which is absurd

$\therefore R_L$ is quasi-transitive

<u>THEOREM-8</u>: $G_n 5_n$ is determined by the class of euclidian n+1-ary relational frames.

Proof: Soundness is trivial. For completeness, we show that the canonical relation $R_L$ ($L = G_n 5_n$) is euclidian.

Assume $\exists x, y_1 \ldots y_n : xR_L y_1 \ldots y_n \ \& \ xR_L z_1 \ldots z_n \ \& \ \forall k(1 \le k \le n), \sim y_k R_L z_1 \ldots z_n$

$\therefore \forall k(1 \le k \le n),\ \exists \alpha_{k_1} \ldots \alpha_{k_n} : \Box(\alpha_{k_1} \ldots \alpha_{k_n}) \in y_k \ \&$

$\alpha_{k_1} \notin z_1 \ \& \ldots \& \ \alpha_{k_n} \notin z_n$

$\therefore \neg \alpha_{1_1} \wedge \ldots \wedge \neg \alpha_{n_1} \in z_1 \ \& \ldots \& \ \neg \alpha_{1_n} \wedge \ldots \wedge \neg \alpha_{n_n} \in z_n$

But $xR_L z_1 \ldots z_n \quad \therefore \ ((\neg \alpha_{1_1} \wedge \ldots \wedge \neg \alpha_{n_1}) \ldots (\neg \alpha_{1_n} \wedge \ldots \wedge \neg \alpha_{n_n})) \in x$

$\therefore \Box(\Diamond(\neg \alpha_{1_1} \ldots \neg \alpha_{1_n}) \ldots (\neg \alpha_{n_1} \ldots \neg \alpha_{n_n})) \in x \quad ([5_n])$

But $xR_L y_1 \ldots y_n \quad \therefore \exists k(1 \le k \le n) : \Diamond(\neg \alpha_{k_1} \ldots \neg \alpha_{k_n}) \in y_k$

$\therefore \neg \Box(\alpha_{k_1} \ldots \alpha_{k_n}) \in y_k \quad \therefore y_k$ is inconsistent, which is absurd

$\therefore R_L$ is euclidian

The following formulae will be familiar:

[Con] $\neg \Box \bot$

[T] $\Box p \rightarrow p$

[B] $\Diamond \Box p \rightarrow p$

[5] $\Diamond p \rightarrow \Box \Diamond p$

[D] $\Box p \rightarrow \Diamond p$

[4] $\Box p \rightarrow \Box \Box p$

The next two lemmas reveal some notable properties of [T] and [B].

Lemma 9.1: $G_nT$ is determined by the class of reflexive n+1-ary relational frames.

Proof: We show that $R_L$ ($L = G_nT$) is reflexive.

Assume that $\square_n(\alpha_1\ldots\alpha_n) \in x \therefore \square(\alpha_1 \vee\ldots\vee \alpha_n) \in x$ (Theorem 1(iv))

$\therefore \alpha_1 \vee\ldots\vee \alpha_n \in x$ ([T]) $\therefore \exists k(1\leq k\leq n): \alpha_k \in x$

$\therefore xR_L x\ldots x$

Lemma 9.2: $G_nB$ is determined by the class of symmetric n+1-ary relational frames.

Proof: We show that $R_L$ ($L = G_nB$) is symmetric.

Assume that $xR_Ly_1\ldots y_n$ & $\forall k(1\leq k\leq n), \sim y_kR_Lx\ldots x$

$\therefore \forall k(1\leq k\leq n), \exists\alpha_{k_1}\ldots\alpha_{k_n}: \square_n(\alpha_{k_1}\ldots\alpha_{k_n}) \in y_k$ & $\neg\alpha_{k_1}\wedge\ldots\wedge\neg\alpha_{k_n} \in x$

$\therefore \neg\alpha_{1_1}\wedge\ldots\wedge\neg\alpha_{1_n}\wedge\ldots\wedge\neg\alpha_{n_1}\wedge\ldots\wedge\neg\alpha_{n_n} \in x$

$\therefore \square\Diamond(\neg\alpha_{1_1}\wedge\ldots\wedge\neg\alpha_{1_n}\wedge\ldots\wedge\neg\alpha_{n_1}\wedge\ldots\wedge\neg\alpha_{n_n}) \in x$ ([B])

But $xR_Ly_1\ldots y_n$

$\therefore \exists k(1\leq k\leq n): \neg\square(\alpha_{1_1}\vee\ldots\vee\alpha_{1_n}\vee\ldots\vee\alpha_{n_1}\vee\ldots\vee\alpha_{n_n}) \in y_k$

But $\square_n(\alpha_{k_1}\ldots\alpha_{k_n}) \in y_k \therefore \square(\alpha_{k_1}\vee\ldots\vee\alpha_{k_n}) \in y_k$ (Theorem 1(iv))

$\therefore \square(\alpha_{1_1}\vee\ldots\vee\alpha_{1_n}\vee\ldots\vee\alpha_{k_1}\vee\ldots\vee\alpha_{k_n}\vee\ldots\vee\alpha_{n_1}\vee\ldots\vee\alpha_{n_n}) \in y_k$ ([RR] and PC)

$\therefore y_k$ is inconsistent, which is absurd

$\therefore R_L$ is symmetric

Thus we have the following result:

THEOREM 9: If $G_n \subseteq L$, then

$\vdash_L$ [T] iff $\vdash_L$ $[T_n]$

$\vdash_L$ [B] iff $\vdash_L$ $[B_n]$

(Here, and in the following, we use the name of a formula to abbreviate the formula; for example, '$\vdash_L$ [T]' means '$\vdash_L \Box p \rightarrow p$'.)

Before moving on to extensions of the $K_n$ systems, we will examine one more extension of the $G_n$ systems, the significance of which will become apparent later. It is obvious how seriality should be defined for an n-ary relation.

THEOREM 10: $G_n$Con is determined by the class of serial n+1-ary relational frames.

Proof: We show that $R_L$ (L = $G_n$Con) is serial.

By definition of $R_L$, $\Box(\alpha_1 \ldots \alpha_n) \in x$ iff $\forall y_1 \ldots y_n$, $xR_L y_1 \ldots y_n \Rightarrow$ $\alpha_1 \in y_1$ or ... or $\alpha_n \in y_n$

$\therefore \Box\bot \notin x$ iff $\exists y_1 \ldots y_n$: $xR_L y_1 \ldots y_n$ & $\bot \notin y_1$ & ... & $\bot \notin y_n$

But $\neg\Box\bot \in x$ $\therefore \exists y_1 \ldots y_n$: $xR_L y_1 \ldots y_n$

$\therefore R_L$ is serial

(This elegant proof is due to B.F. Chellas. It replaces a much longer proof, contained in an earlier draft of this essay, which parallels the proof of Theorem 3.)

Our completeness results for the $K_n$ extensions are obtained in the usual manner; it is shown that the canonical frame is a member of the class of frames in question. Since these $K_n$ extensions are included in $F_1$, canonical frames different from those used for the $G_n$ extensions are available. Let $M'_L = \langle D_L, R'_L, V_L \rangle$. As usual, $D_L$ is the set of L-maximal consistent sets and $V_L$ is an assignment such that $x \in V_L(p)$ iff $p \in x$. $R'_L$ is defined as follows:

$$xR'_L y_1 \ldots y_n \text{ iff } \forall\alpha, \Box\alpha \in x \Rightarrow \alpha \in y_1 \text{ or } \ldots \text{ or } \alpha \in y_n$$

Where $K_n \subseteq L$, it can be shown that

$$M'_L \models_x \alpha \text{ iff } \alpha \in x$$

for all $\alpha \in F_1$. The proof of this theorem is included in [3] and will not be reproduced here.

We can now give completeness results for some extensions of the $K_n$ logics.

THEOREM 11: $K_nT$ is determined (mod $F_1$) by the class of reflexive n+1-ary relational frames.

Proof: The proof is trivial.

THEOREM 12: $K_nB$ is determined (mod $F_1$) by the class of symmetric n+1-ary relational frames.

Proof: We show that the canonical relation $R_L'$ ($L = K_nB$) is symmetric.

Assume that $\exists x, y_1 \ldots y_n$: $xR_L'y_1 \ldots y_n$ & $\forall k(1 \le k \le n)$, $\sim y_k R_L' x \ldots x$

$\therefore \exists \alpha_1 \ldots \alpha_n$: $\forall k(1 \le k \le n)$, $\Box\alpha_k \in y_k$ & $\alpha_k \notin x$

$\therefore \neg(\alpha_1 \vee \ldots \vee \alpha_n) \in x$ $\therefore \Box\Diamond\neg(\alpha_1 \vee \ldots \vee \alpha_n) \in x$ ([B])

But $xR_L'y_1 \ldots y_n$ $\therefore \exists k(1 \le k \le n)$: $\neg\Box(\alpha_1 \vee \ldots \vee \alpha_n) \in y_k$

But $\Box\alpha_k \in y_k$ $\therefore \Box(\alpha_1 \vee \ldots \vee \alpha_k \vee \ldots \vee \alpha_n) \in y_k$ (PC and [RR])

$\therefore y_k$ is inconsistent, which is absurd

$\therefore R_L'$ is symmetric

THEOREM 13: $K_n5$ is determined (mod $F_1$) by the class of euclidian n+1-ary relational frames.

Proof: We show that the canonical relation $R_L'$ ($L = K_n5$) is euclidian.

Assume that $\exists x, y_1 \ldots y_n, z_1 \ldots z_n$: $xR_L'y_1 \ldots y_n$ & $xR_L'z_1 \ldots z_n$ and $\forall k(1 \le k \le n)$, $\sim y_k R_L' z_1 \ldots z_n$

$\therefore \exists \alpha_1 \ldots \alpha_n$: $\forall k(1 \le k \le n)$, $\Box\alpha_k \in y_k$ & $\alpha_k \notin z_1$ & ... & $\alpha_k \notin z_n$

$\therefore \neg(\alpha_1 \vee \ldots \vee \alpha_n) \in z_1$ & ... & $\neg(\alpha_1 \vee \ldots \vee \alpha_n) \in z_n$

But $xR_L'z_1 \ldots z_n$ $\therefore \Diamond\neg(\alpha_1 \vee \ldots \vee \alpha_n) \in x$

$\therefore \Box\Diamond\neg(\alpha_1 \vee \ldots \vee \alpha_n) \in x$ ([5]) But $xR'_L y_1 \ldots y_n$

$\therefore \exists k (1 \leq k \leq n): \neg\Box(\alpha_1 \vee \ldots \vee \alpha_n) \in y_k$ But $\Box\alpha_k \in y_k$

$\therefore \Box(\alpha_1 \vee \ldots \vee \alpha_k \vee \ldots \vee \alpha_n) \in y_k$ (PC and [RR])

$\therefore y_k$ is inconsistent, which is absurd

$\therefore R'_L$ is euclidian

The straightforward generalizations of relational frame conditions that can be made for [T], [B], and [5] are not so easily obtained for [D] and [4]. In fact, it can be shown that [D] is not determined by any class of first-order n-ary relational frames where $n > 2$. (See section V.) Thus, although [D] and [Con] are equivalent in K (i.e. $G_1$ or $K_1$) Theorem 10 shows that they are not equivalent in $G_n$ where $n > 1$, and hence that they are not equivalent in $K_n$ where $n > 1$. One should suspect, then, that [Con] ought to be regarded as the syntactic representative of seriality in relational frame theory. Such a view is supported further by the next result:

<u>THEOREM 14</u>: $K_n$Con is determined (mod $F_1$) by the class of serial n+1-ary relational frames.

Proof: This is easily shown by a simple adaptation of the proof of Theorem 10.

It is still not known whether $K_n 4$ is determined by a class of n-ary relational frames where $n > 2$. However, an interesting result is available. Where m = 1, the following schema yields the traditional [4] axiom:

$$[4]_m \; \Box p_1 \wedge \ldots \wedge \Box p_m \rightarrow \Box(\Box p_1 \wedge \ldots \wedge \Box p_m)$$

The result is this:

<u>THEOREM 15</u>: $K_n[4]_n$ is determined (mod $F_1$) by the class of quasi-transitive n+1-ary relational frames.

Proof: (Soundness)

Assume that $M$ is a model on a quasi-transitive n+1-ary frame and that $M \nvDash_x \Box(\Box p_1 \wedge \ldots \wedge \Box p_n)$

$\therefore \exists y_1 \ldots y_n : M \nvDash_{y_1} \Box p_1 \wedge \ldots \wedge \Box p_n$ & ... & $M \nvDash_{y_n} \Box p_1 \wedge \ldots \wedge \Box p_n$ & $xRy_1 \ldots y_n$

$\therefore \forall k, \exists j: M \nvDash_{y_k} \Box p_{j_k} \quad (1 \le k \le n)$

$\therefore \forall k(1 \le k \le n), \exists z_1^k \ldots z_n^k : y_k R z_1^k \ldots z_n^k$ & $M \nvDash_{z_1^k} p_{j_k}$ & ... & $M \nvDash_{z_n^k} p_{j_k}$

But $\exists k(1 \le k \le n): xRz_1^k \ldots z_n^k$ (frame condition)

$\therefore M \nvDash_x \Box p_{j_k} \quad \therefore M \nvDash_x \Box p_1 \wedge \ldots \wedge \Box p_n$

(Completeness)

We show that the canonical relation $R_L'$ ($L = K_n[4]_m$) is quasi-transitive.

Assume $\exists x, y_1 \ldots y_n, z_1^1 \ldots z_n^1 \ldots z_1^n \ldots z_n^n$:

$xR_L' y_1 \ldots y_n$ & $\forall k(1 \le k \le n), y_k R_L' z_1^k \ldots z_n^k$ & $\forall k(1 \le k \le n), \sim xR_L' z_1^k \ldots z_n^k$

$\therefore \exists \alpha_1 \ldots \alpha_n : \forall k, \Box\alpha_k \in x$ & $\alpha_k \notin z_1^k$ & ... & $\alpha_k \notin z_n^k$

$\therefore \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \in x \quad \therefore \Box(\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \in x \quad ([4]_n)$

But $xR_L' y_1 \ldots y_n \quad \therefore \exists k(1 \le k \le n): \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \in y_k$

$\therefore \Box\alpha_k \in y_k$ But $y_k R_L' z_1^k \ldots z_n^k$

$\therefore \exists j(1 \le j \le n): \alpha_k \wedge \neg\alpha_k \in z_j^k$, which is absurd

$\therefore R_L'$ is quasi-transitive

It is easily seen that

$$\vdash_L [4]_m \Rightarrow \vdash_L [4]_n \quad (PC \subseteq L)$$

for any n, m such that m > n; one merely substitutes $p_n$ for $p_{n+1} \ldots p_m$. It can also be shown that

$$\vdash_L [4]_m \text{ iff } \vdash_L [4]_j \quad (K_n \subseteq L)$$

for any j, m ≥ n by the following result:

Corollary 15.1: If $m \geq n$, then $K_n[4]_m$ is determined (mod $F_1$) by the class of quasi-transitive n+1-ary relational frames.

Proof: This is easily shown by an adaptation of the proof of Theorem 15.

Thus [4] is equivalent to each $[4]_n$ in K. However, our next result shows that this is not true for any weaker $K_n$ logic.

THEOREM 16: $\nvdash_{K_n 4} [4]_n$ where $n > 1$.

Proof: First we note that any n-ary frame is equivalent mod $F_1$ to an n+1-ary frame. Let $F = \langle D,R\rangle$ be an n-ary frame. Define R* as follows:

$$R^* = \{<x_1 \ldots x_n, x_n>: <x_1 \ldots x_n> \in R\}$$

Let $F^* = \langle D,R^*\rangle$. It is easily shown that $F$ and $F^*$ are equivalent mod $F_1$.

Lemma 16.1: $\nvdash_{K_2 4} [4]_2$

Proof: Let $F = \langle D,R\rangle$ be a ternary frame where

$$D = \{x, y_1, y_2, z_1, z_2, z_3\}$$

$$R = \{<x,y_1,y_2>, <y_1,z_1,z_2>, <y_2,z_2,z_3>, <x,z_1,z_3>\}$$

The structure of $F$ can be illustrated as in Figure 1.

Since $\Box\alpha$ will be true at each $z_i$ for any $\alpha$, $\Box\Box\alpha$ cannot fail at any $z_i$ or $y_i$. Thue [4] holds at each $y_i$ and $z_i$. Suppose that $\Box\Box p$ fails at x. Then $\Box p$ fails at $y_1$ and $y_2$, and thus p fails at $z_1$ and $z_3$. But $xRz_1,z_3$. Therefore $\Box p$ fails at x. But on a model where $V(p) = \{y_1, z_1\}$ and $V(q) = \{y_2, z_3\}$, $[4]_2$ will fail at x. Thus $\nvdash_{K_2 4} [4]_2$

It follows from Lemma 16.1 and the preceding remarks that $\nvdash_{K_n 4} [4]_2$ where $n > 1$. But $\vdash_L [4]_n \Rightarrow \vdash_L [4]_2$ for any $n > 2$. Thus $\nvdash_{K_n 4} [4]_n$. This proves the theorem.

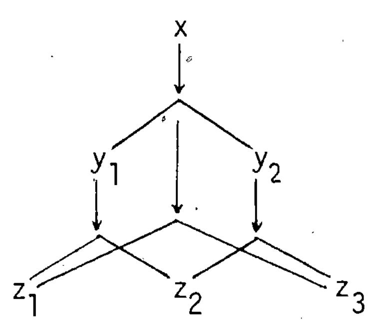



FIGURE 1

## (V) FIRST ORDER DEFINABILITY

In Goldblatt [2] we find the following definition of first order definability:

> A modal sentence $\alpha$ is first order definable iff there is a first order sentence $\alpha^*$ such that, for any frame $F$, $F \models \alpha$ iff $F$ is a model for $\alpha^*$ in the first order sense.

Here $\alpha^*$ is a sentence of a first order language containing a single dyadic predicate, and $F$ is a binary relational frame. Our generalized notion of a relational frame requires a more general notion of first order definability:

> $\alpha$ is n-adically first order definable (f.o.d) iff there is a first order sentence $\alpha^*$ such that for any n-ary frame $F$, $F \models \alpha$ iff $F$ is a model for $\alpha^*$ in the first order sense

where $\alpha^*$ is a sentence of a first order language containing a single n-adic predicate.

> $\alpha$ is universally f.o.d. iff $\alpha$ is n-adically f.o.d. for each n.

We will now show that [D] is not triadically f.o.d. For each $i \in$ Nat, we define the ternary frame $F_i = \langle D_i, R_i \rangle$ as follows:

$D_i = \{x, y_1 \ldots y_{2i+1}\}$

$xR_i y_j, y_k$ where $j = k-1$

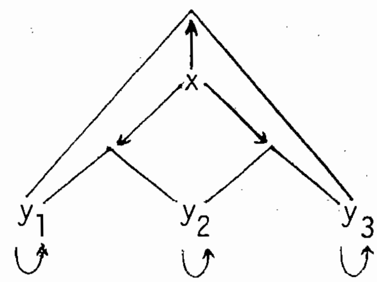
$xR_i y_{2i+1}, y_1$

$y_j R_i y_j, y_j$ for each $y_j$

The first two frames are illustrated in Figure 2.

To have [D] fail on one of these frames, we must have $\Box p$ and $\Box\neg p$

$F_1$




$F_2$

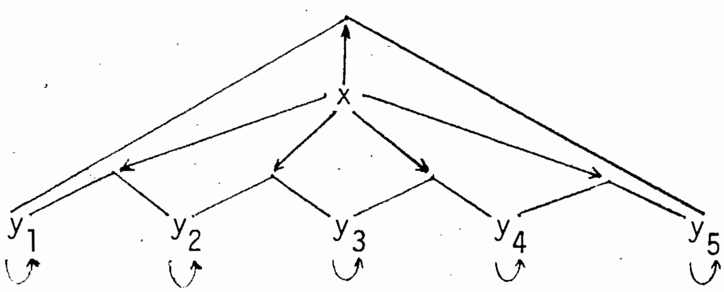



FIGURE 2

holding at some point. Obviously this cannot occur at any of the $y_i$'s. Suppose that $\Box p$ and $\Box\neg p$ hold at x in $F_1$. Then p must hold at $y_1$ or $y_2$. Suppose p holds at $y_1$. Then $\neg p$ must hold at $y_2$. Therefore p must hold at $y_3$ since $xR_1y_2,y_3$. But $xR_1y_3,y_1$, so we have a related pair where $\neg p$ fails at both coordinates. Thus $\Box\neg p$ fails at x. In general, if we make p true at $y_1$, we must make it at all odd-numbered $y_i$'s if we want $\Box\neg p$ to hold at x. But $y_{2i+1}$ will always have an odd index, and x will always be related to $y_{2i+1}$, $y_1$. Thus $\Box p$ and $\Box\neg p$ cannot both hold at x in any of the $F_i$'s. It is clear that the same considerations arise when $\neg p$ is true at $y_1$.

Now let G be a non-principal ultrafilter on Nat. The <u>ultraproduct</u> of the $F_i$'s over G ($F_G$) is defined as in Goldblatt [2] except for the relation $R_G$:

$$\hat{f}R_G\hat{g}_1,\hat{g}_2 \text{ iff } \{i: f(i)R_ig_1(i),g_2(i)\} \in G$$

The structure of $F_G$ is illustrated in Figure 3. Since $D_G$ will be non-denumerable this diagram does not fully illustrate the structure. But this is not important. What is important is that we can define a valuation where [D] will fail at $\hat{f}$:

$$\hat{g}_j \in V(p) \text{ if } j \text{ is odd}$$

$$\hat{g}_{j'} \in V(p) \text{ if } j' \text{ is even}$$

Thus each pair will have a coordinate where p holds and a coordinate where $\neg p$ holds, and so $\Box p \wedge \Box\neg p$ will hold at $\hat{f}$.

It is easily seen why $F_G$ has the structure illustrated. The existence of particular sets in the ultrafilter guarantees the existence of particular points in the ultraproduct domain. To get $\hat{f}$, one chooses a

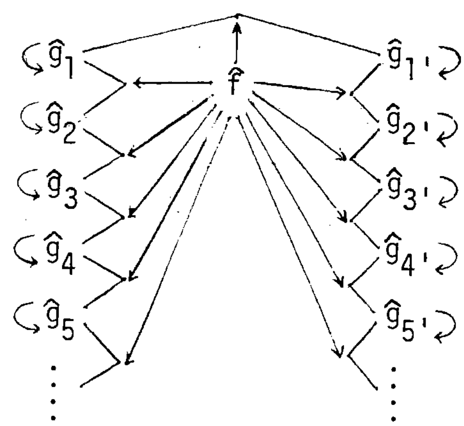



FIGURE 3

function f in $\Pi_{i \ \mathrm{Nat}} D_i$ such that f(i) = x for all i. Since Nat $\in$ G, $\hat{f}$ is just the set containing this function. Nat also guarantees the existence of the points "shared" by all the $F_i$'s; that is, $\hat{g}_1$, $\hat{g}_2$, $\hat{g}_3$, $\hat{g}_{1'}$, $\hat{g}_{2'}$, $\hat{g}_{3'}$ (the first, second, third, last, second-to-last, and third-to-last points). By examining the definition of $R_G$ it can be seen that the relation diagrammed does hold between these points.

To get the points not "shared" by all the $F_i$'s (e.g. $\hat{g}_4$ and $\hat{g}_{4'}$) we appeal to the fact that G is a non-principal ultrafilter. Since G will contain all cofinite sets, it will contain Nat - {1}. To get $\hat{g}_4$, we choose those functions which map i onto $y_4$ for i > 1. There will be four distinct functions of this type, namely those which map 1 onto x, $y_1$, $y_2$, and $y_3$. $\hat{g}_{4'}$ is formed in a similar way, as are $\hat{g}_5$ and $\hat{g}_{5'}$, these being points which are "shared" by all the $F_i$'s except for $F_1$.

We know from Los' Theorem that every class of first order models is closed under ultraproducts. Thus every first order sentence true on all of the $F_i$'s will be true on $F_G$. Now suppose that [D] is f.o.d. by a triadic first order sentence $\alpha^*$. Then by the definition $\alpha^*$ holds on all of the $F_i$'s since [D] holds on all of them. But then $\alpha^*$ holds on $F_G$ by Los' Theorem, and so [D] must hold on $F_G$, contrary to what we have shown. Thus we have proved the following:

<u>Lemma 17.1</u>: [D] is not triadically f.o.d.

<u>THEOREM 17</u>: [D] is not n-adically f.o.d. if n > 2.

Proof: This follows from Lemma 17.1 and the fact that every n-ary relational frame has an equivalent n+1-ary relational frame.

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