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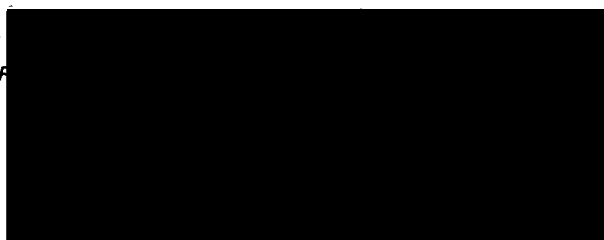
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SECOND ORDER ELASTICITY

by

Gary Robert Nicklason

B.Sc., Simon Fraser University, 1975

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics



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SECOND ORDER ELASTICITY

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ABSTRACT

The second order elasticity theory, formulated by Rivlin (1953), has been applied to the combined extension and torsion of a homogeneous, isotropic, elastic cylindrical tube. With surface tractions given at the plane ends, the displacement field and stress distributions are determined in explicit form. The results so obtained are compared to those of classical elasticity theory.

The problem of a thick spherical shell under uniform internal and external pressures is examined next in the light of Rivlin's second order elasticity theory. The displacement and stress fields are derived explicitly. The following special cases are investigated:

- (i) Infinite medium with spherical cavity under internal pressure only,
- (ii) Solid sphere under external pressure,
- (iii) Stress concentration when the internal radius of a thick hollow sphere approaches zero.

DEDICATION

To my parents, Henry and Anne.

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1. INTRODUCTION

In classical elasticity, the deformation to which the body is subjected is so small that all terms of second degree in the displacement gradients may be neglected in comparison with those of the first degree. This assumption leads to the linear relationship between forces or stress components to the spatial derivatives of the displacement components. Consequently, it becomes unimportant whether the applied surface tractions are specified per unit area of the undeformed or deformed material. Therefore, if applied mechanical forces are given, the displacement and stress fields can usually be found explicitly in a well formulated boundary value problem.

However, this is not the case in finite elasticity. The quantities such as body force, surface traction, and stress components must be stated explicitly whether they are defined with relation to the undeformed body or the deformed body. Moreover, not only the resulting equations governing the theory are nonlinear but the explicit functional form of the strain energy function on the strain invariants is not known in advance.

It is because of these reasons that most of the interest in finite elasticity has centered around the so called exact or controllable deformations. These problems are solved by the inverse method. The deformation is described at the outset, and it is verified that the deformation can be supported without body force, in every homogeneous, isotropic, incompressible, elastic material. These exact deformations were originally discovered by Rivlin [1]. While there is a moderately

large number of exact solutions for incompressible materials [2], Ericksen [3] has shown that if the material considered is compressible, then only pure homogeneous deformations are admissible. Thus, it is apparent that to solve any boundary value problem involving a nonhomogeneous deformation with compressible elastic materials, one must turn to approximate techniques.

One of the approximate techniques often employed is to consider the strain energy function as a polynomial function of the strain invariants and then retain terms in this expansion only to the order desired by the formulation of one's approximation [4]. These approximations are based on the assumption that the principal extensions are sufficiently small, but no restriction is imposed on the magnitudes of the rotations involved. The applications of such techniques, often called complete theories, are rather limited.


The other types of approximations rest on the assumption that when deformations are small but not infinitesimal, then perturbations based on the classical theory solution could be used. The process of such systematic approximations for problems in nonlinear elasticity has been formulated and applied rigorously by Murnaghan [5], Green and Spratt [6], Green and Shield [7], Green and Adkins [8], Grioli [9], and Sheng [10].

In 1953, Rivlin [11] proposed an approximation technique which he calls second order elasticity. It is founded on the assumption that only terms up to third order in displacement gradients be

retained in the polynomial expansion for the strain energy function. In other words, it is assumed the displacement gradients are large enough so that the classical theory is not valid but small enough to neglect terms of degree higher than the second in displacement gradients in stress components. The appropriate constitutive equations, equilibrium equations, and boundary conditions are formulated to within the framework of such a second order theory. Through an elegant mathematical presentation, Rivlin reduces any surface traction boundary value problem in second order theory for homogeneous isotropic, compressible, elastic materials to the solution of two boundary value problems in classical theory. The process is that boundary value problems of classical theory is solved for the given system of forces and the corresponding displacement field determined. This set of displacements is introduced into the equations of the second order theory and the forces required to maintain the given displacements, in addition to those prescribed, are calculated. These additional forces are reversed in direction and the effects which they would produce in the undeformed body are found. Then, by summing the classical displacement field and that obtained from the reversed forces, we can show that the total resulting displacement field is a solution in the second order elasticity theory for the given problem. All quantities in the above are expressed in terms of the material coordinates of the body. This formulation of Rivlin has been reproduced by us in Section 3.

The next section of the paper shows how this second order theory can be applied to a boundary value problem to determine second order effects. Rivlin [11] examined the combined extension and torsion of a cylindrical tube to demonstrate the application of his theory and found that the tube, on twisting, undergoes a fractional simple extension which is proportional to the square of the twist. Besides confirming this result once again, we also present the displacement field and stress distribution, not obtained in Rivlin's paper, complete to second order terms for given surface forces at the plane ends of the undeformed tube. Also investigated are the special cases of pure torsion of a rod, and that of a simple extension. On comparison with Murnaghan [5], where perturbation techniques are used, we show that both theories furnish identical results.

We next consider the second order solution to the problem of a thick spherical shell subjected to uniform internal and external pressures. All of the additional forces, fortunately, turn out to be radial in nature and hence facilitate the determination of second order terms in displacement and stress fields which, are found explicitly. The problems of an infinite medium bounded internally by a spherical hole under internal pressure and the solid sphere under external pressure, which are special cases of the general problem are also solved. The results obtained are then compared with the corresponding ones in classical elasticity theory.



2.1 THEORY OF FINITE ELASTIC DEFORMATIONS

A continuous body occupying a region $D_0 + B_0$ is subjected to body and surface forces. It deforms and occupies the position $D + B$. The deformation is described by the mapping:

$$\xi_i = \xi_i(x_j), \quad (1)$$

where x_i and ξ_i are the coordinates of the same generic particle in B_0 and B , respectively, referred to a fixed rectangular Cartesian system.

We shall consider quasi-static deformations only. The equations of equilibrium are then given by:

$$\frac{\partial t_{ij}}{\partial \xi_j} + \rho X_i = 0, \quad (2)$$

where X_i is the body force per unit mass, ρ the mass density in the deformed state, and t_{ij} the symmetric state of stress in D .

Since our interest lies only in those problems in which the tractions are prescribed all along the surface, the corresponding boundary conditions are:

$$T_i = t_{ij} n_j \quad \text{on } B, \quad (3)$$

where n_i is the unit normal vector to B and T_i represents the surface force per unit area of B .

If the material is homogeneous and isotropic, then the constitutive relations are furnished by:

$$t_{ij} = \frac{2}{\sqrt{I_3}} \left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) g_{ij} - \frac{\partial W}{\partial I_2} g_{ik} g_{kj} + I_3 \frac{\partial W}{\partial I_3} \delta_{ij} \right]. \quad (4)$$

Here, g_{ij} denotes the Finger Strain tensor given by:

$$g_{ij} = \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_k} \quad (5)$$

and W represents the strain energy which is a function of the three invariants:

$$I_1 = g_{ii}, \quad I_2 = \frac{1}{2}(g_{ii}g_{jj} - g_{ij}g_{ij}), \quad I_3 = \det.(g_{ij}). \quad (6)$$

2.2 SOLUTIONS TO PROBLEMS IN FINITE ELASTICITY

The problems which have aroused interest in finite elasticity theory are the so called exact solutions. In these, the deformation is prescribed at the outset, and then it is demonstrated that such a deformation can be supported in every homogeneous, isotropic, elastic material. In other words, the relation (1) being given, the stresses calculated from (4) satisfy the equilibrium equations (2) with body force $X_i = 0$ no matter what the functional form of the strain energy W is in terms of the invariants I_1 , I_2 , and I_3 . The appropriate surface tractions to support such a prescribed deformation are then calculated from (3).

In classical theory, the assumption is made that the displacement gradients $\frac{\partial u_i}{\partial \xi_j} \ll 1$ and, therefore, the stresses are calculated on the basis of neglecting powers higher than one in $\frac{\partial u_i}{\partial \xi_j}$. The equations so obtained governing such a theory are linear and hence techniques to attack boundary value problems for given applied forces can be easily developed. This, however, is not the case in finite elasticity theory

where the equations are obviously nonlinear in the displacements. In the following section, we reproduce the technique developed by Rivlin [11] which makes use of the classical infinitesimal theory as well as the method of inverse calculations to solve boundary value problems in the second order theory of elasticity.

3.1 RIVLIN'S SECOND ORDER THEORY

For compressible, isotropic, elastic materials, Rivlin [11] proposed a procedure to solve surface traction problems when the deformations involved are small but finite. The theory rests on retaining terms up to second powers in the displacement gradients. The solution to a boundary value problem is reduced to solving two problems in classical theory plus an inverse calculation.

3.2 STRESS-STRAIN RELATIONS OF SECOND ORDER THEORY

We rewrite equation (4) as:

$$t_{ij} = \frac{2}{\tau} \left[g_{ij} \frac{\partial W}{\partial I_1} - G_{ij} \frac{\partial W}{\partial I_2} + \left(I_3 \frac{\partial W}{\partial I_3} + I_2 \frac{\partial W}{\partial I_2} \right) \delta_{ij} \right], \quad (7)$$

where $\tau = (I_3)^{1/2}$, G_{ij} the cofactor of g_{ij} in $\det.g_{ij}$, and the strain energy W a function of the scalar invariants:

$$I_1 = g_{ii}, \quad I_2 = G_{ii}, \quad I_3 = \text{Det}.g_{ij}. \quad (8)$$

For convenience, we define a set of alternative invariants:

$$J_1 = I_1 - 3, \quad J_2 = I_2 - 2I_1 + 3, \quad J_3 = I_3 - I_2 + I_1 - 1. \quad (9)$$

The invariants J_1 , J_2 , and J_3 are of the first, second, and third order of smallness in $\frac{\partial u_i}{\partial x_j}$. Assuming the strain energy function W to be a polynomial in its arguments, we may write:

$$W = \sum_{\alpha, \beta, \gamma} A_{\alpha\beta\gamma} J_1^\alpha J_2^\beta J_3^\gamma. \quad (10)$$

We may take the medium to be such that, $W = 0$ and that it is unstressed, when undeformed. Since we are interested in formulating a theory in which deformations are so small that powers higher than third in displacement gradients $\frac{\partial u_i}{\partial x_j}$ can be neglected, equation (10) reduces to the form:

$$W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3, \quad (11)$$

where $a_1, a_2, a_3, a_4,$ and a_5 are material constants. The form (11) for W was first obtained by Murnaghan [5].

We introduce the notation:

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad \alpha_{ij} = \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}, \quad (12)$$

$$e_{ii} = 2\Delta, \quad \alpha_{ii} = \alpha, \quad E_{ij} = \text{co-factor of } e_{ij} \text{ in Det. } e_{ij},$$

$$\text{and } E_{ii} = E.$$

With use of (11) and (12), and neglecting terms of degree higher than second in $\frac{\partial u_i}{\partial x_j}$, equation (7) assumes the form:

$$\begin{aligned} t_{ij} = & 2\{[-a_1 e_{ij} + 2(a_1 + 2a_2)\Delta\delta_{ij}]\} + \{(4a_2 - 2a_3 + a_1)\Delta e_{ij} \\ & - a_1 \alpha_{ij} - (a_1 - a_5)E_{ij}\} + \{(a_1 + 2a_2)\alpha + (a_1 + a_3)E \\ & + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta^2\}\delta_{ij}. \end{aligned} \quad (13)$$

It may be noted here that in the expression (13) for t_{ij} , if we neglect terms of degree higher than first in $\frac{\partial u_i}{\partial x_j}$, we recover the

stress-strain relations of classical theory:

$$t_{ij} = 2[-a_1 e_{ij} + 2(a_1 + 2a_2) \Delta \delta_{ij}] \quad (14)$$

From (14), we get:

$$\lambda = 4(a_1 + 2a_2), \quad \mu = -2a_1, \quad (15)$$

where λ and μ are the Lamé constants of classical theory.

Furthermore, (15) allows us to infer:

$$a_1 < 0, \quad a_2 > 0, \quad \text{and} \quad a_1 + 2a_2 > 0. \quad (16)$$

3.3 EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

Suppose the deformation occurs under the body forces X_i per unit mass and surface tractions X_{vi} measured per unit area of the undeformed boundary B_0 . We write here once again the equilibrium equation (2):

$$\frac{\partial t_{ik}}{\partial x_k} + \rho X_i = 0 \quad (17)$$

The boundary condition (3) can be expressed as:

$$X_{vi} \frac{dS}{dS'} = t_{ij} l'_j, \quad (18)$$

where dS and dS' denote elements of surface area in the undeformed

and deformed states, respectively, and l'_i is the unit normal vector to the deformed boundary B.

We shall now like to express relations (17) and (18) in terms of the coordinates x_i which describe the undeformed body. If ρ_0 stands for the density in the undeformed state, then:

$$\rho = \frac{\rho_0}{\tau} \quad (19)$$

With $\xi_i = x_i + u_i$, $\frac{\partial}{\partial \xi_k} = \frac{\partial x_i}{\partial \xi_k} \frac{\partial}{\partial x_i}$, we get:

$$\frac{\partial}{\partial \xi_k} = \frac{1}{\tau} \frac{\partial \tau}{\partial u_k} \frac{\partial}{\partial (\frac{x_i}{\partial x_i})} \quad (20)$$

With (19) and (20), equilibrium equation (17) becomes:

$$\frac{\partial \tau}{\partial (\frac{\partial u_k}{\partial x_j})} \frac{\partial t_{ik}}{\partial x_j} + \rho_0 x_i = 0 \quad (21)$$

If l_i denotes the unit normal to the undeformed boundary B_0 , then:

$$l'_i = \frac{dS}{dS'} \left[\frac{\partial \tau}{\partial u_i} l_j \right] \frac{\partial}{\partial (\frac{x_i}{\partial x_j})} \quad (22)$$

and the boundary condition (18) therefore becomes:

$$x_{vi} = \frac{\partial \tau}{\partial (\frac{\partial u_k}{\partial x_j})} l_j t_{ik} \quad (23)$$

If the first degree terms in $\frac{\partial u_i}{\partial x_j}$ on the right hand side of equation (13) are denoted by t'_{ij} and those of second degree in $\frac{\partial u_i}{\partial x_j}$ by t''_{ij} , then the equilibrium equations (21) and boundary conditions (22) can

be represented as:

$$[(1+\Delta)\delta_{jk} - \frac{\partial u_k}{\partial x_j}] \frac{\partial t'_{ij}}{\partial x_k} + \frac{\partial t''_{ij}}{\partial x_j} + \rho_0 X_i = 0, \quad (24)$$

and

$$X_{vi} = [(1+\Delta)\delta_{jk} - \frac{\partial u_k}{\partial x_j}] \ell_k t'_{ij} + \ell_j t''_{ij}. \quad (25)$$

3.4 A METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS

A systematic procedure to determine the displacement field u_i when X_i and X_{vi} in (24) and (25) are given will be presented.

Suppose v_i denotes the displacement field corresponding to body force X_i and surface traction X_{vi} according to the classical theory. In other words, v_i is given by:

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho_0 X_i = 0, \quad (26)$$

and

$$X_{vi} = \tau_{ij} \ell_j, \quad (27)$$

where

$$\tau_{ij} = 2[-a_1 e'_{ij} + 2(a_1 + 2a_2)\Delta' \delta_{ij}], \quad (28)$$

and

$$e'_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \quad \Delta' = \frac{1}{2} e'_{ii}. \quad (29)$$

We shall now determine the forces which would be required by the second order theory to maintain the displacement field v_i . That is, v_i found from (26) to (29) is substituted in (24) and (25) to find forces X'_i and X'_{vi} :

$$[(1+\Delta')\delta_{jk} - \frac{\partial v_j}{\partial x_k}] \frac{\partial \tau_{ik}}{\partial x_j} + \frac{\partial \tau'_{ij}}{\partial x_j} + \rho_0 X'_i = 0, \quad (30)$$

and

$$X'_{vi} = [(1+\Delta')\delta_{jk} - \frac{\partial v_j}{\partial x_k}] \ell_j \tau_{ik} + \ell_k \tau'_{ik}. \quad (31)$$

Here, τ_{ik} is given by (28) and:

$$\begin{aligned} \tau'_{ik} = & 2\{[(4a_2 - 2a_3 + a_1)\Delta' e'_{ik} - a_1 \alpha'_{ik} - (a_1 - a_5)E'_{ik}] \\ & + \{(a_1 + 2a_2)\alpha' + (a_1 + a_3)E' + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta'^2\}\delta_{ik}\}, \quad (32) \end{aligned}$$

where

$$\begin{aligned} \alpha'_{ik} = & \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_j}, \quad \alpha' = \alpha'_{ii}, \quad E'_{ik} = \text{co-factor of } e'_{ik} \text{ in} \\ & \text{Det. } e'_{ik}, \quad \text{and } E'_{ii} = E'. \quad (33) \end{aligned}$$

From equations (26) to (33), we can now calculate the forces $X'_i - X_i$ and $X'_{vi} - X_{vi}$, which are required to support v_i in addition to the given forces X_i and X_{vi} :

$$-\rho_0 (X'_i - X_i) = [\Delta' \delta_{jk} - \frac{\partial v_j}{\partial x_k}] \frac{\partial \tau_{ik}}{\partial x_j} + \frac{\partial \tau'_{ij}}{\partial x_j}, \quad (34)$$

$$X'_{vi} - X_{vi} = [\Delta' \delta_{jk} - \frac{\partial v_j}{\partial x_k}] \ell_j \tau_{ik} + \ell_j \tau'_{ij}. \quad (35)$$

The forces given by equations (34) and (35) do not actually act. In order to negate their effect, we now calculate the displacement field w_i which their negatives would produce in the undeformed body according to classical theory. That is, we determine w_i from:

$$\frac{\partial \tau''_{ij}}{\partial x_j} + \rho_0 (x_i - x'_i) = 0, \quad (36)$$

and

$$x_{\nu i} - x'_{\nu i} = \tau''_{ik} k, \quad (37)$$

where

$$\tau''_{ij} = 2[-a_1 e''_{ij} + 2(a_1 + 2a_2) \Delta'' \delta_{ij}], \quad (38)$$

and

$$e''_{ij} = \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}, \quad \Delta'' = \frac{1}{2} e''_{ii}.$$

We now set:

$$u_i = v_i + w_i. \quad (39)$$

With u_i calculated as in (39), it is easily verified that the equilibrium equation (24) and boundary condition (25) are identically satisfied up to terms of the second degree in the displacement gradients.

Thus, we see that by solving two appropriate classical theory problems defined by (26) to (29) and (36) to (38) along with an inverse calculation defined by (30) to (35), we are able to obtain a solution u_i of second order theory equations (24) and (25).

4.1 SIMULTANEOUS EXTENSION AND TORSION OF A CYLINDRICAL TUBE

Our purpose here is to employ Rivlin's theory presented in the preceding section to investigate the simultaneous extension and torsion of a cylindrical tube. The tube has internal radius b and external radius a and is subjected to a couple M and a longitudinal force N at its plane ends. The body forces shall be assumed to be zero.

4.2 THE CLASSICAL SOLUTION

The classical solution for this problem can be found by superposing the classical solutions of the simple extension and pure torsion problems. This well known solution is given by:

$$\begin{aligned} v_1 &= -\sigma \epsilon x_1 - \psi x_2 x_3, \\ v_2 &= -\sigma \epsilon x_2 + \psi x_1 x_3, \\ v_3 &= \epsilon x_3, \end{aligned} \tag{40}$$

where ϵ denotes the extension parallel to the axis of the tube and ψ the amount of torsion about the axis of the tube. Here, Poisson's ratio σ can be expressed as:

$$\sigma = \frac{a_1 + 2a_2}{a_1 + 4a_2}, \tag{41}$$

when use is made of (15).

As to what should be the values of ϵ and ψ corresponding to the couple M and the longitudinal force N given at the plane ends, we can calculate these by introducing the solution (40) into the equations of classical theory as follows.

Introducing (40) into (28) and (29), we get:

$$\begin{aligned} e'_{11} = e'_{22} = -2\sigma\epsilon, \quad e'_{12} = 0, \quad e'_{23} = \psi x_1, \quad e'_{13} = -\psi x_2, \\ e'_{33} = 2\epsilon, \quad \Delta' = \epsilon(1-2\sigma), \end{aligned} \quad (42)$$

and

$$\begin{aligned} \tau_{11} = \tau_{12} = \tau_{22} = 0, \quad \tau_{33} = 8\epsilon[a_2 - \sigma(a_1 + 2a_2)], \\ \tau_{23} = -2a_1\psi x_1, \quad \tau_{31} = 2a_1\psi x_2. \end{aligned} \quad (43)$$

With (43), equation (26) gives $x_i = 0$ which, of course, is to be expected.

On the lateral surfaces of the tube, $l_1 = x_1/a$, $l_2 = x_2/a$, and $l_3 = 0$. With (43) in (27), we obtain:

$$X_{vi} = 0. \quad (44)$$

On a plane end of the tube, $l_1 = l_2 = 0$, $l_3 = 1$. Equation (27) then furnishes:

$$X_{v1} = 2a_1\psi x_2, \quad X_{v2} = -2a_1\psi x_1, \quad X_{v3} = 8\epsilon[a_2 - \sigma(a_1 + 2a_2)]. \quad (45)$$

The system of surface tractions X_{v1} and X_{v2} in (45) is equivalent to an azimuthal surface traction $\bar{\Theta}_v$ given by:

$$\bar{\Theta}_v = -2a_1\psi r, \quad (46)$$

where $r = \sqrt{x_1^2 + x_2^2}$.

, Thus, the quantities ϵ and ψ in the displacement field (40)

which is produced by the given resultant couple M and the given longitudinal force N are furnished by:

$$M = \int_0^a \int_b^{\infty} \theta_{\nu} r^2 dr d\theta = -\pi a_1 \psi (a^4 - b^4), \quad (47)$$

and

$$N = \int_0^a \int_b^{\infty} X_{\nu 3} r dr d\theta = 8\pi \epsilon [a_2 - \sigma(a_1 + 2a_2)] (a^2 - b^2). \quad (48)$$

4.3 SECOND ORDER FORCES REQUIRED TO MAINTAIN THE DEFORMATION v_i OF CLASSICAL THEORY

We seek to determine here the body forces $X'_i - X_i$ and surface forces $X'_{\nu i} - X_{\nu i}$ which would have to be applied, in addition to the forces given, to maintain the deformation (40) within the framework of the second order theory. These can be found from (34) and (35) after we have calculated the expressions for α'_{ij} , E'_{ij} , and τ'_{ij} from (32) and (33) which we do now.

Introducing (40) into (33), we obtain:

$$\begin{aligned} \alpha'_{11} &= \sigma^2 \epsilon^2 + \psi^2 x_2^2 + \psi^2 x_3^2, & \alpha'_{22} &= \sigma^2 \epsilon^2 + \psi^2 x_1^2 + \psi^2 x_3^2, \\ \alpha'_{33} &= \epsilon^2, & \alpha'_{12} &= -\psi^2 x_1 x_2, & \alpha'_{23} &= \epsilon \psi x_1, \\ \alpha'_{31} &= -\epsilon \psi x_2, & \text{and } \alpha' &= 2\sigma^2 \epsilon^2 + \epsilon^2 + \psi^2 r^2 + 2\psi^2 x_3^2, \end{aligned} \quad (49)$$

where $r^2 = x_1^2 + x_2^2$.

From (42) and (33), we have:

$$\begin{aligned} E'_{11} &= -4\sigma\epsilon^2 - \psi^2 x_1^2, & E'_{22} &= -4\sigma\epsilon^2 - \psi^2 x_2^2, \\ E'_{33} &= 4\sigma^2\epsilon^2, & E'_{12} &= -\psi^2 x_1 x_2, & E'_{23} &= 2\sigma\psi x_1, \\ E'_{31} &= -2\sigma\psi x_2, & \text{and} & & E' &= -8\sigma\epsilon^2 - \psi^2 r^2 + 4\sigma^2\epsilon^2. \end{aligned} \quad (50)$$

Substitution of (49) and (50) into (32) gives:

$$\begin{aligned} \tau'_{11} &= 2\psi^2 [-(2a_1 - a_5)x_2^2 + (a_1 + 4a_2)x_3^2 + (a_1 + 2a_2 - a_3 - a_5)r^2] + \gamma, \\ \tau'_{22} &= 2\psi^2 [-(2a_1 - a_5)x_1^2 + (a_1 + 4a_2)x_3^2 + (a_1 + 2a_2 - a_3 - a_5)r^2] + \gamma, \\ \tau'_{33} &= 2\psi^2 [2(a_1 + 2a_2)x_3^2 + (2a_2 - a_3)r^2] + \gamma', \\ \tau'_{12} &= 2\psi^2 (2a_1 - a_5)x_1 x_2, \\ \tau'_{23} &= 2\psi\epsilon x_1 [(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_5)], \end{aligned}$$

and

$$\tau'_{31} = -2\psi\epsilon x_2 [(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_5)], \quad (51)$$

where

$$\begin{aligned} \gamma &= 2\epsilon^2 [\sigma^2 (a_1 + 4a_2 + 12a_3 + 48a_4) - 2\sigma(-a_1 - 4a_2 + 10a_3 + 24a_4 + 2a_5) \\ &\quad + (-a_1 - 2a_2 + 4a_3 + 12a_4)], \end{aligned}$$

and

$$\gamma' = 4\epsilon^2 [\sigma^2 (-3a_1 - 6a_2 + 10a_3 + 24a_4 + 2a_5) - 2\sigma(a_1 + 4a_3 + 12a_4) + 3(a_2 + 2a_4)]. \quad (52)$$

From equations (40) to (43), (51), and (34), we find, with

$$x_3 = 0 :$$

$$\begin{aligned} -\rho_0 (X'_1 - X_1) &= 2\psi^2 x_1 (3a_1 + 4a_2 - 2a_3 - 3a_5) , \\ -\rho_0 (X'_2 - X_2) &= 2\psi^2 x_2 (3a_1 + 4a_2 - 2a_3 - 3a_5) , \\ -\rho_0 (X'_3 - X_3) &= 0 . \end{aligned} \quad (53)$$

The system of body forces described by equation (53) is equivalent to a system of radial body forces $-R'$ per unit mass of material such that:

$$-\rho_0 R' = 2\psi^2 r (3a_1 + 4a_2 - 2a_3 - 3a_5) . \quad (54)$$

On the outer curved surface of the tube, we obtain:

$$\begin{aligned} X'_{v1} - X_{v1} &= [2\psi^2 (a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma] \frac{x_1}{a} , \\ X'_{v2} - X_{v2} &= [2\psi^2 (a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma] \frac{x_2}{a} , \\ X'_{v3} - X_{v3} &= 0 . \end{aligned} \quad (55)$$

The force system (55) is equivalent to a radial force R_{v1} acting outwards and measured per unit area of surface in the undeformed position:

$$\begin{aligned} R_{v1} &= 2\psi^2 (a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma \\ &= R'_{v1} + \gamma . \end{aligned} \quad (56)$$

Similarly, the surface force on the inner curved surface can be calculated. It acts radially inwards and has magnitude R_{v2} :

$$\begin{aligned}
 R_{V2} &= 2\psi^2 [a_1 + 2a_2 - a_3 - a_5] b^2 + \gamma \\
 &= R'_{V2} + \gamma .
 \end{aligned} \tag{57}$$

On the plane end of the tube $x_3 = 0$, we obtain:

$$\begin{aligned}
 X'_{V1} - X_{V1} &= 4\psi \epsilon x_2 [(-2a_2 + a_3) + \sigma(a_1 + 4a_2 - 2a_3 - a_5)] , \\
 X'_{V2} - X_{V2} &= -4\psi \epsilon x_1 [(-2a_2 + a_3) + \sigma(a_1 + 4a_2 - 2a_3 - a_5)] , \\
 X'_{V3} - X_{V3} &= 2\psi^2 r^2 (2a_2 - a_3) + \gamma' - 16\sigma \epsilon^2 [a_2 - \sigma(a_1 + 2a_2)] .
 \end{aligned} \tag{58}$$

The first two components in (58) combine into an azimuthal surface traction Θ'_V :

$$\Theta'_V = -4\psi \epsilon r [(-2a_2 + a_3) + \sigma(a_1 + 4a_2 - 2a_3 - a_5)] . \tag{59}$$

The system of forces (58) are then statically equivalent to a resultant couple M' and a longitudinal force $X'_{V3} - X_{V3}$ given by:

$$M' = \int_0^{2\pi} \int_b^a \Theta'_V r^2 dr d\theta = -2\pi\psi \epsilon [(-2a_2 + a_3) + \sigma(a_1 + 4a_2 - 2a_3 - a_5)] (a^4 - b^4) \tag{60}$$

$$\begin{aligned}
 X'_{V3} - X_{V3} &= 2\psi^2 r^2 (2a_2 - a_3) + \gamma' - 16\sigma \epsilon^2 [a_2 - \sigma(a_1 + 2a_2)] \\
 &= Z + \gamma' - 16\sigma \epsilon^2 [a_2 - \sigma(a_1 + 2a_2)] ,
 \end{aligned} \tag{61}$$

where

$$Z = 2\psi^2 r^2 (2a_2 - a_3) . \tag{62}$$

Thus, in addition to the given forces M and N as required by the classical theory, we see that in order to maintain the deformation (40) of the tube according to second order theory, we also need apply the

following system of forces:

- (i) a set of radial body forces R' per unit mass given by (54);
- (ii) on the lateral surface $r = a$, a radially outward force R_{v1} measured per unit area of surface in the undeformed state, given by (56);
- (iii) on the lateral surface $r = b$, a radially inward surface force R_{v2} measured per unit area of the undeformed surface, given by (57);
- (iv) on a plane end $x_3 = 0$, a resultant couple M' and a longitudinal force $X'_{v3} - X_{v3}$ given by (60) and (61) respectively.

4.4 CLASSICAL DISPLACEMENT FIELDS PRODUCED BY THE REVERSED ADDITIONAL FORCES LISTED IN (i) TO (iv) OF SECTION 4.3

As illustrated in our theory in Section 3, we are not interested in actually applying these additional forces. Instead, we can counteract their effects by considering the actions of forces of equal magnitude but opposite direction to those described in (i) to (iv) at the end of the previous Section 4.3. Since the displacement field w_i that we seek here is to be obtained on the basis of classical theory, we may split up the additional force system in any manner convenient for algebraic calculations and then employ the Superposition Principle. We thus make the following groupings of the additional force system:

- (A) A body force $-R' \underline{r}$ per unit mass given by (54), a surface force $-R'_{\nu 1} \underline{r}$ on $r = a$ given by (56), and a surface force $-R'_{\nu 2} \underline{r}$ on $r = b$ given by (57).
- (B) A surface force $-Z$ acting on the plane end $x_3 = 0$ and given by (61).
- (C) A resultant couple $-M'$ acting on the plane end $x_3 = 0$ and given by (60).
- (D) A surface force $-\gamma \underline{r}$ on $r = a$, a surface force $-\gamma \underline{r}$ on $r = b$, and a surface force $-\gamma' + 16\sigma\epsilon^2 [a_2 - \sigma(a_1 + 2a_2)]$ on the plane end $x_3 = 0$.

What we need is to determine the classical displacement fields which correspond to the force systems (A), (B), (C), and (D) separately and then add them to get the required displacement field w_i .

First of all, we shall consider the force system (C). For convenience, we write:

$$-M' = a_1 \pi \psi' (a^4 - b^4), \quad (63)$$

where

$$\psi' = \frac{2\psi\epsilon}{a_1} [(-2a_2 + a_3) + \sigma(a_1 + 4a_2 - 2a_3 - a_5)]. \quad (64)$$

Comparison of (63) with (47) and (40) yields the desired displacement field which we call as $w_i^{(1)}$:

$$w_1^{(1)} = -\psi' x_2 x_3, \quad w_2^{(1)} = \psi' x_1 x_3, \quad w_3^{(1)} = 0. \quad (65)$$

Second, we investigate the force system (A). It should be noted that $R'_{\nu 1}$, $R'_{\nu 2}$, and R' are independent of the amount of

extension ϵ . We first assume that the tube is held at constant length and that a radial displacement, which we call $w_r^{(2)}(r)$ only occurs. Denoting the stresses so ensued by τ_{rr}'' , $\tau_{r\theta}''$, $\tau_{\theta\theta}''$, τ_{zz}'' , $\tau_{\theta z}''$, and τ_{zr}'' in a cylindrical system, we find that:

$$\tau_{rr}'' = 2[-2a_1 \frac{dw_r^{(2)}}{dr} + 2(a_1 + 2a_2) (\frac{dw_r^{(2)}}{dr} + \frac{w_r^{(2)}}{r})],$$

$$\tau_{\theta\theta}'' = 2[-2a_1 \frac{w_r^{(2)}}{r} + 2(a_1 + 2a_2) (\frac{dw_r^{(2)}}{dr} + \frac{w_r^{(2)}}{r})],$$

$$\tau_{zz}'' = 4(a_1 + 2a_2) (\frac{dw_r^{(2)}}{dr} + \frac{w_r^{(2)}}{r}),$$

and

$$\tau_{r\theta}'' = \tau_{\theta z}'' = \tau_{zr}'' = 0. \quad (66)$$

Introducing the above stresses into equilibrium equation (36) when the body force is $-\rho_0 R' \underline{r}$, we obtain the differential equation governing $w_r^{(2)}$:

$$8a_2 \left(\frac{d^2 w_r^{(2)}}{dr^2} + \frac{1}{r} \frac{dw_r^{(2)}}{dr} - \frac{w_r^{(2)}}{r^2} \right) = -Kr, \quad (67)$$

where

$$K = 2\psi^2 (3a_1 + 4a_2 - 2a_3 - 3a_5). \quad (68)$$

Upon integration, (67) yields:

$$w_r^{(2)} = -\frac{K}{64a_2} r^3 + Ar + \frac{B}{r}, \quad (69)$$

where A and B are arbitrary constants. Furthermore, when the stress distribution (66) is inserted in boundary conditions (37), we get:

$$(\tau''_{rr})_{r=a} = -R'_{\sqrt{1}} = -2\psi^2 a^2 (a_1 + 2a_2 - a_3 - a_5), \quad (70)$$

and

$$(\tau''_{rr})_{r=b} = -R'_{\sqrt{2}} = -2\psi^2 b^2 (a_1 + 2a_2 - a_3 - a_5). \quad (71)$$

From (66), (69), (70), and (71), the values of the constants A and B turn out to be:

$$A = \frac{1}{4} \left[\frac{K(a_1 + 8a_2)}{16a_2} - K' \right] \frac{a^2 + b^2}{a_1 + 4a_2},$$

$$B = -\frac{1}{4} \left[\frac{K(a_1 + 8a_2)}{16a_2} - K' \right] \frac{a^2 b^2}{a_1}, \quad (72)$$

where

$$K' = 2\psi^2 (a_1 + 2a_2 - a_3 - a_5). \quad (72')$$

But, it is quite clear from (66) that we have the stress component τ''_{zz} which will certainly contribute to a longitudinal force at the plane ends. We shall combine this force with the force system (B), and calculate the total force:

$$N'_1 = -\int_0^a \int_b^a (\tau''_{zz} + Z)r \, dr \, d\theta$$

$$= -\pi\psi^2 (a^4 - b^4) \frac{a_5(a_1 + 2a_2) - a_1(a_1 - a_3 + 4a_2)}{a_1 + 4a_2}. \quad (73)$$

The force N'_1 gives rise to the fractional extension (or compression)

ϵ' :

$$\epsilon' = \frac{\psi^2}{8} (a^2 + b^2) \frac{a_5(a_1 + 2a_2) - a_1(a_1 - a_3 + 4a_2)}{a_1(a_1 + 3a_2)}. \quad (74)$$

The displacement field, which we denote by $w_i^{(3)}$, due to the extension

ϵ' , i.e. due to the longitudinal force N'_1 , is:

$$w_r^{(3)} = -\sigma\epsilon'r, \quad w_\theta^{(3)} = 0, \quad w_z^{(3)} = \epsilon'z. \quad (74')$$

This field (74') combines with $w_r^{(2)}(r)$ found already to give us the field corresponding to the force systems (A) and (B).

It remains to explore the force system (D), which we do now.

To start with, we assume that the tube is held at constant length and is acted upon by surface tractions $-\gamma$ on each of the lateral surfaces.

We consider then a radial displacement field, call it $w_r(r)$.

Denoting the stresses by σ_{rr} , $\sigma_{\theta\theta}$, ..., we find:

$$\sigma_{rr} = 2[-2a_1 \frac{dw_r}{dr} + 2(a_1 + 2a_2)(\frac{dw_r}{dr} + \frac{w_r}{r})],$$

$$\sigma_{\theta\theta} = 2[-2a_1 \frac{w_r}{r} + 2(a_1 + 2a_2)(\frac{dw_r}{dr} + \frac{w_r}{r})],$$

$$\sigma_{zz} = 4(a_1 + 2a_2)(\frac{dw_r}{dr} + \frac{w_r}{r}),$$

and

$$\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zr} = 0. \quad (75)$$

The equations of equilibrium and boundary conditions become:

$$\frac{d^2 w_r}{dr^2} + \frac{1}{r} \frac{dw_r}{dr} - \frac{w_r}{r^2} = 0, \quad (76)$$

$$(\sigma_{rr})_{r=a} = -\gamma, \quad \text{and} \quad (\sigma_{rr})_{r=b} = -\gamma. \quad (77)$$

On solving (76) and (77), we obtain:

$$w_r = A'r, \quad (78)$$

where $A' = \frac{-\gamma}{4(a_1+4a_2)}$.

The stress distribution (75) has a component σ_{zz} which has to be dealt with. The total longitudinal force to be considered is then:

$$\begin{aligned} N_2' &= -\int_0^a \int_b^a \{\sigma_{zz} + \gamma' - 16\sigma\epsilon^2 [a_2 - \sigma(a_1+2a_2)]\} r dr d\theta \\ &= 2\pi\sigma\gamma(a^2-b^2) - \pi[\gamma' - 16\sigma\epsilon^2\{a_2 - \sigma(a_1+2a_2)\}](a^2-b^2). \end{aligned} \quad (79)$$

The displacement field corresponding to the force system (D) is then the radial field w_r plus the field due to the longitudinal force (79).

It is:

$$\begin{aligned} w_r^{(4)} &= -\sigma\epsilon''r - \frac{\gamma}{4(a_1+4a_2)} r, \\ w_\theta^{(4)} &= 0, \\ w_z^{(4)} &= \epsilon''z, \end{aligned} \quad (80)$$

where

$$\epsilon'' = -\frac{1}{8} \frac{(a_1+4a_2)}{a_1(a_1+3a_2)} [2\sigma\gamma - \gamma' + 16\sigma\epsilon^2\{a_2 - \sigma(a_1+2a_2)\}]. \quad (81)$$

The displacement fields corresponding to the force systems (A), (B), (C), and (D) are determined. By taking the sum of these fields, we should obtain the total field w_i which corresponds to the negative of the additional forces required to maintain the deformation (40) within the formulation of second order theory.

Addition of (65), (69), (74'), and (80) yields:

$$w_1 = -\psi'x_2x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2}\right)x_1 + [A + A' - \sigma(\epsilon' + \epsilon'')]x_1,$$

$$w_2 = \psi' x_1 x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2} \right) x_2 + [A + A' - \sigma(\epsilon' + \epsilon'')] x_2 ,$$

$$w_3 = (\epsilon' + \epsilon'') x_3 , \quad (82)$$

where $r^2 = x_1^2 + x_2^2$.

The displacement field u_i which would therefore occur according to the second order theory if the tube considered is subjected to a given twisting couple M and a given longitudinal force N is given by $v_i + w_i$.

Adding (40) and (82), we obtain:

$$u_1 = v_1 + w_1$$

$$= -(\psi + \psi') x_2 x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2} \right) x_1 + [A + A' - \sigma(\epsilon + \epsilon' + \epsilon'')] x_1 ,$$

$$u_2 = v_2 + w_2$$

$$= (\psi + \psi') x_1 x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2} \right) x_2 + [A + A' - \sigma(\epsilon + \epsilon' + \epsilon'')] x_2 ,$$

$$u_3 = v_3 + w_3$$

$$= (\epsilon + \epsilon' + \epsilon'') x_3 . \quad (83)$$

The stress distribution corresponding to the solution (83) can be easily obtained now by direct substitution of (83) into (13).

5. SPECIAL CASES

We shall examine now the displacement field (83) of the second order theory and compare it with that of the classical theory.

(i) Suppose $\epsilon = 0$. The field (83) becomes:

$$\begin{aligned} u_1 &= -\psi x_2 x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2} \right) x_1 + (A - \sigma\epsilon') x_1, \\ u_2 &= \psi x_1 x_3 + \left(\frac{B}{r^2} - \frac{Kr^2}{64a_2} \right) x_2 + (A - \sigma\epsilon') x_2, \\ u_3 &= \epsilon' x_3. \end{aligned} \quad (84)$$

Comparison of (40) and (84) allows us to infer that whereas the pure torsion of a circular tube without the longitudinal force is possible in the classical theory, that is not the case when second order effects are taken into account. In fact, the extension ϵ' in (84), which is given by (74), depends upon the square of the twist ψ . It was an extension of just this type, proportional to the square of the twist, which was demonstrated by Poynting [12] during his experiments on torsion. However, it should be borne in mind that in a complete finite elasticity theory for incompressible materials [2] where Mooney's strain energy form is sometimes used for small but finite deformations, pure torsion cannot be produced without the longitudinal force.

(ii) Suppose $\psi = 0$. Then (83) takes the form:

$$\begin{aligned} u_1 &= [A' - \sigma(\epsilon + \epsilon'')] x_1, \\ u_2 &= [A' - \sigma(\epsilon + \epsilon'')] x_2, \end{aligned}$$

$$u_3 = (\epsilon + \epsilon'')x_3 \quad (85)$$

As expected, the form of (85) is clearly the same as that of (40) as well as that for simple extension in any complete theory of finite elasticity.

6. COMPARISON WITH MURNAGHAN'S RESULTS

Using a technique which also takes the classical solution as a starting point, Murnaghan [5] has obtained the second order solution for torsion of a circular cylindrical rod. We propose to show now that the two theories produce qualitatively similar results.

Setting the internal radius b of the tube equal to zero, the value of constant A in (72) reduces to:

$$A = \frac{1}{4} \left[\frac{K(a_1 + 8a_2)}{16a_2} - K' \right] \frac{a^2}{a_1 + 4a_2} \quad (86)$$

where K and K' are given by (68) and (72') respectively.

Requiring that the displacement field u_i given by (83) be bounded demands that $B = 0$. The fractional extension ϵ' as obtained in (74) becomes:

$$\epsilon' = \frac{\psi^2}{8} a^2 \frac{a_5(a_1 + 2a_2) - a_1(a_1 - a_3 + 4a_2)}{a_1(a_1 + 3a_2)} \quad (87)$$

The field w_i in (82) then assumes the form:

$$\begin{aligned} w_1 &= -\frac{Kr^2}{64a_2} x_1 + (A - \sigma\epsilon')x_1, \\ w_2 &= -\frac{Kr^2}{64a_2} x_2 + (A - \sigma\epsilon')x_2, \\ w_3 &= \epsilon'x_3. \end{aligned} \quad (88)$$

The above field w_i in cylindrical polar coordinates can be written as:

$$w_r = \left(A - \sigma\epsilon' - \frac{Kr^2}{64a_2} \right) r, \quad w_\theta = 0, \quad w_z = \epsilon'z. \quad (89)$$

Murnaghan shows that if the lateral surface of the cylindrical rod is force free and if the forces acting on the plane ends reduce to a couple whose moment is equal to the moment predicted by the classical theory to maintain a twist ψ per unit length of the cylinder, then in the second order approximation the rod undergoes a radial contraction which is a linear combination of r and r^3 and a longitudinal extension which depends linearly on z . The multiplicative and other constants involved in the expressions depend upon the square of the twist and the material constants. In view of our result (89), the behaviour predicted by the theory presented is similar to that of Murnaghan's. In particular, we have shown that the effects of the second order theory are to decrease the radius of the rod by an amount proportional to $\psi^2 a^3$ and increase the length by an amount proportional to $\psi^2 a^2$.

7.1 COMPRESSION OF A SPHERICAL SHELL

We now use the second order Rivlin's theory presented in Section 3 to examine the second order effects produced in a spherical shell by a state of uniform compression. The internal and external radii of the shell initially are a and b respectively. It is subjected to a uniform pressure p_1 on the inner surface and p_2 on the outer surface. The body forces are taken to be zero. We shall assume, of course, spherical symmetry, and employ spherical polar coordinates (r, θ, ϕ) .

7.2 THE CLASSICAL SOLUTION

The classical solution to this problem is readily obtained. Assuming that any particle which was originally at a distance r from the centre of the shell undergoes a displacement v in the radial direction, we can find the expressions for the components of strain and stress. Denoting these by $e'_{rr}, e'_{\theta\theta}, \dots, e'_{\phi r}$ and $\tau'_{rr}, \tau'_{\theta\theta}, \dots, \tau'_{\phi r}$ respectively, we obtain:

$$e'_{rr} = 2 \frac{dv}{dr}, \quad e'_{\theta\theta} = e'_{\phi\phi} = \frac{2v}{r},$$

$$e'_{r\theta} = e'_{\theta\phi} = e'_{\phi r} = 0, \quad \Delta' = \frac{1}{2} e'_{ii} = \frac{dv}{dr} + \frac{2v}{r}, \quad (90)$$

and

$$\begin{aligned}\tau_{rr} &= 2[-2a_1 \frac{dv}{dr} + 2(a_1+2a_2) (\frac{dv}{dr} + \frac{2v}{r})] , \\ \tau_{\theta\theta} &= \tau_{\phi\phi} = 2[-2a_1 \frac{v}{r} + 2(a_1+2a_2) (\frac{dv}{dr} + \frac{2v}{r})] , \\ \tau_{r\theta} &= \tau_{\theta\phi} = \tau_{\phi r} = 0 .\end{aligned}\quad (91)$$

Introducing (91) into the equations of equilibrium in polar spherical coordinates, we find that they reduce, in the absence of body forces, to the ordinary differential equation:

$$\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} - \frac{2v}{r^2} = 0 . \quad (92)$$

This equation has the general solution:

$$v = c_1 r + \frac{c_2}{r^2} , \quad (93)$$

where c_1 and c_2 are constants of integration to be determined from the boundary conditions:

$$[\tau_{rr}]_{r=a} = -p_1 , \quad [\tau_{rr}]_{r=b} = -p_2 . \quad (94)$$

From (91), (93), and (94) we get:

$$\begin{aligned}c_1 &= \frac{1}{8(a_1+3a_2)} \frac{p_1 a^3 - p_2 b^3}{b^3 - a^3} , \\ c_2 &= -\frac{1}{8a_1} \frac{a^3 b^3 (p_1 - p_2)}{b^3 - a^3} .\end{aligned}\quad (95)$$

The state of stress within the shell is found from (91) and (93) to be:

$$\tau_{rr} = 8(a_1 + 3a_2)c_1 + \frac{8a_1c_2}{r^3},$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = 8(a_1 + 3a_2)c_1 - \frac{4a_1c_2}{r^3},$$

and

$$\tau_{r\theta} = \tau_{\theta\phi} = \tau_{\phi r} = 0. \quad (96)$$

Since the main body of the theory presented in Section 3 is formulated in terms of a rectangular cartesian reference frame, it is desirable to express the preceding results in terms of cartesian coordinates. Denoting the displacements by v_i in the cartesian system, it follows that (93) can be written as:

$$v_i = \left(c_1 + \frac{c_2}{r^3}\right)x_i, \quad (97)$$

and the stresses (96) by:

$$\tau_{ij} = 8(a_1 + 3a_2)c_1 \delta_{ij} - \frac{4a_1c_2}{r^5} (r^2 \delta_{ij} - 3x_i x_j). \quad (98)$$

in which $r^2 = x_1^2 + x_2^2 + x_3^2$.

Differentiating (97) with respect to x_j provides the useful relation:

$$\frac{\partial v_i}{\partial x_j} = c_1 \delta_{ij} + \frac{c_2}{r^5} (r^2 \delta_{ij} - 3x_i x_j). \quad (99)$$

7.3 SECOND ORDER FORCES REQUIRED TO MAINTAIN THE DEFORMATION PREDICTED BY CLASSICAL THEORY

We now apply the displacement field (97) calculated for the classical problem to the constitutive relations of the second order

theory. This will allow us to determine those body forces $X'_i - X_i$ and surface forces $X'_{vi} - X_{vi}$ which are required, in addition to the pressures p_1 and p_2 , to maintain the deformation described by (97) within the framework of the second order theory.

Introducing (99) into (29), we find:

$$e'_{ik} = 2c_1 \delta_{ik} + \frac{2c_2}{r^5} (r^2 \delta_{ik} - 3x_i x_k),$$

$$\Delta' = \frac{1}{2} e'_{ii} = 3c_1. \quad (100)$$

From equations (99) and (33):

$$\alpha'_{ik} = c_1^2 \delta_{ik} + \frac{2c_1 c_2}{r^5} (r^2 \delta_{ik} - 3x_i x_k) + \frac{c_2^2}{r^8} (r^2 \delta_{ik} + 3x_i x_k),$$

$$E'_{ik} = 4 \left[c_1^2 \delta_{ik} - \frac{c_1 c_2}{r^5} (r^2 \delta_{ik} - 3x_i x_k) - \frac{c_2^2}{r^8} (2r^2 \delta_{ik} - 3x_i x_k) \right],$$

$$\alpha' = \alpha'_{ii} = 3 \left[c_1^2 + \frac{2c_2^2}{r^6} \right],$$

and

$$E' = E'_{ii} = 12 \left[c_1^2 - \frac{c_2^2}{r^6} \right]. \quad (101)$$

The components τ'_{ik} are given from (100), (101), and (32) as:

$$\tau'_{ik} = 2 \left[2(-a_1 - 3a_2 + 18a_3 + 54a_4 + 2a_5) c_1^2 \delta_{ik} \right. \\ \left. + 4(2a_1 + 6a_2 - 3a_3 - a_5) (r^2 \delta_{ik} - 3x_i x_k) \frac{c_1 c_2}{r^5} \right. \\ \left. + (a_1 + 12a_2 - 12a_3 - 8a_5) \frac{c_2^2}{r^6} \delta_{ik} - 3(5a_1 - 4a_5) \frac{c_2^2}{r^8} x_i x_k \right]. \quad (102)$$

Differentiating τ'_{ik} with respect to x_j and then contracting on j and k yields:

$$\frac{\partial \tau'_{ij}}{\partial x_j} = 36(3a_1 - 4a_2 + 4a_3) \frac{c_2^2}{r} x_i. \quad (103)$$

Also, from (98):

$$\frac{\partial \tau_{ik}}{\partial x_j} = \frac{12 a_1 c_2}{r^7} [r^2 x_j \delta_{ik} + r^2 x_k \delta_{ij} + r^2 x_i \delta_{kj} - 5x_i x_k x_j]. \quad (104)$$

By contracting on j and k in (104), we may observe directly that the components τ_{ik} do satisfy the equations of equilibrium $\tau_{ij,j} = 0$.

We are now in a position to find the additional body forces which must be applied to the shell to maintain the deformation (97) according to the second order theory. Substituting from equations (99), (100), (103), and (104) into (34) and taking $X_i = 0$ gives:

$$-\rho_0 X'_i = 36(a_1 - 4a_2 + 4a_3) \frac{c_2^2}{r} x_i. \quad (105)$$

On the outer surface of the shell $r = b$, so that:

$$[\tau_{ik}]_{r=b} = 4[2(a_1 + 3a_2)c_1 - \frac{a_1 c_2}{b^3}] \delta_{ik} + \frac{12a_1 c_2}{5b} x_i x_k, \quad (106)$$

and

$$\begin{aligned} [\tau'_{ik}]_{r=b} &= 2[2(-a_1 - 3a_2 + 18a_3 + 54a_4 + 2a_5)c_1^2 \delta_{ik} \\ &+ 4(2a_1 + 6a_2 - 3a_3 - a_5)(b^2 \delta_{ik} - 3x_i x_k) \frac{c_1 c_2}{b^5} \\ &+ (a_1 + 12a_2 - 12a_3 - 8a_5) \frac{c_2^2}{b^6} \delta_{ik} - 3(5a_1 - 4a_5) \frac{c_2^2}{b^8} x_i x_k]. \end{aligned} \quad (107)$$

The components l_i of the outward unit normal vector to the external surface are:

$$l_i = \frac{x_i}{b}. \quad (108)$$

We are now able to determine the additional second order forces which must be applied to the external surface of the shell to maintain the displacement field (97). Introducing (99), (100), and (106) to (108) into (35), we find that on the surface $r = b$:

$$\begin{aligned} X'_{vi} - X_{vi} &= 4[(3a_1 + 9a_2 + 18a_3 + 54a_4 + 2a_5)c_1^2 \\ &- 4(3a_2 - 3a_3 - a_5) \frac{c_1 c_2}{b^3} - (3a_1 - 6a_2 + 6a_3 - 2a_5) \frac{c_2^2}{b^6}] \left(\frac{x_i}{b} \right). \end{aligned} \quad (109)$$

In a similar manner, on the surface $r = a$:

$$\begin{aligned} X'_{vi} - X_{vi} &= 4[(3a_1 + 9a_2 + 18a_3 + 54a_4 + 2a_5)c_1^2 \\ &- 4(3a_2 - 3a_3 - a_5) \frac{c_1 c_2}{a^3} - (3a_1 - 6a_2 + 6a_3 - 2a_5) \frac{c_2^2}{a^6}] \left(-\frac{x_i}{a} \right). \end{aligned} \quad (110)$$

The system of body forces described by equation (105) is equivalent to a radial body force R' , per unit mass of the material, given by:

$$-\rho_0 R' = \beta \frac{1}{r}, \quad (111)$$

where

$$\beta = 36(a_1 - 4a_2 + 4a_3)c_2^2. \quad (112)$$

The force system (109) is equivalent to a radial distribution R_{v1} , acting outward from the surface of the shell and measured per unit area of surface in the undeformed state:

$$R_{v1} = \beta_1 c_1^2 + \beta_2 \frac{c_1 c_2}{b^3} + \beta_3 \frac{c_2^2}{b^6}. \quad (113)$$

Similarly, the surface tractions on the inner surface of the shell are equivalent to a radial distribution of amount R_{v2} :

$$R_{v2} = \beta_1 c_1^2 + \beta_2 \frac{c_1 c_2}{a^3} + \beta_3 \frac{c_2^2}{a^6}. \quad (114)$$

In equations (113) and (114), we have set:

$$\beta_1 = 4(3a_1 + 9a_2 + 18a_3 + 54a_4 + 2a_5) ,$$

$$\beta_2 = -16(3a_2 - 3a_3 - a_5) ,$$

and

$$\beta_3 = -4(3a_1 - 6a_2 + 6a_3 - 2a_5) . \quad (115)$$

In summary, we see that in order to maintain the deformation of the shell described by equation (97), the second order theory requires that the following additional system of forces be imposed:

- i) a radial body force R' per unit mass of material, given by (111);
- (ii) on the surface $r = b$, radial surface tractions of amount R_{v1} , measured per unit area of surface in the undeformed state, given by (113);
- (iii) on the surface $r = a$, radial surface tractions of amount R_{v2} , measured per unit area of surface in the undeformed state, given by (114).

7.4 DEFORMATION OF THE SHELL PRODUCED BY THE NEGATIVE OF THE ADDITIONAL FORCES DETERMINED IN SECTION 7.3

We can now consider the effects which would be produced in the undeformed shell if the additional forces do not act. These are determined by calculating, according to classical theory, the

displacements which would occur in the undeformed body by the action of forces which are equal in magnitude and opposite in direction to the body force $\rho_0 R'$ and surface tractions $R_{\nu 1}$ and $R_{\nu 2}$.

The system of body forces $-\rho_0 R'$ and surface tractions $-R_{\nu 1}$ and $-R_{\nu 2}$ is assumed to produce a displacement field which is radial in nature and depends only on r . Denoting this displacement by w , the components of stress, τ''_{rr} , $\tau''_{r\theta}$, ..., $\tau''_{\phi r}$, referred to a polar, spherical coordinate system are:

$$\tau''_{rr} = 2[-2a_1 \frac{dw}{dr} + 2(a_1 + 2a_2) (\frac{dw}{dr} + \frac{2w}{r})] ,$$

$$\tau''_{\theta\theta} = \tau''_{\phi\phi} = 2[-2a_1 \frac{w}{r} + 2(a_1 + 2a_2) (\frac{dw}{dr} + \frac{2w}{r})] ,$$

and

$$\tau''_{r\theta} = \tau''_{\theta\phi} = \tau''_{\phi r} = 0 . \quad (116)$$

The equations of equilibrium with body force $-\rho_0 R'$ become:

$$8a_2 \left[\frac{d^2 w}{dr^2} + \frac{2}{r} \frac{dw}{dr} - \frac{2w}{r^2} \right] + \frac{\beta}{r^7} = 0 , \quad (117)$$

and the associated boundary conditions reduce to:

$$[\tau''_{rr}]_{r=b} = -R_{\nu 1} = -[\beta_1 c_1^2 + \beta_2 \frac{c_1 c_2}{b^3} + \beta_3 \frac{c_2^2}{b^6}] ,$$

$$[\tau''_{rr}]_{r=a} = -R_{\nu 2} = -[\beta_1 c_1^2 + \beta_2 \frac{c_1 c_2}{a^3} + \beta_3 \frac{c_2^2}{a^6}] . \quad (118)$$

Integration of equation (117) yields :

$$w = c_3 r + \frac{c_4}{r^2} - \frac{\beta}{144a_2} \frac{1}{r^5} \quad (119)$$

where c_3 and c_4 are arbitrary constants. From equations (116)

and (119), it follows that :

$$\tau_{rr}'' = 8(a_1 + 3a_2)c_3 + \frac{8a_1c_4}{r^3} - \frac{a_1 - 3a_2}{18a_2} \frac{\beta}{r^6} \quad (120)$$

With (120), conditions (118) furnish :

$$\begin{cases} c_3 = \frac{1}{8(a_1 + 3a_2)} \left[-\beta_1 c_1^2 + \left\{ \beta_3 c_2^2 - \frac{\beta(a_1 - 3a_2)}{18a_2} \right\} \frac{1}{a^3 b^3} \right], \\ c_4 = \frac{1}{8a_1} \left[-\beta_2 c_1 c_2 - \left\{ \beta_3 c_2^2 - \frac{\beta(a_1 - 3a_2)}{18a_2} \right\} \frac{a^3 + b^3}{a^3 b^3} \right]. \end{cases} \quad (121)$$

By virtue of equation (39), we can now write down the total displacement field (u_r, u_θ, u_ϕ) . From (93) and (119):

$$\begin{aligned} u_r = u = v + w &= (c_1 + c_3)r + (c_2 + c_4) \frac{1}{r^2} - \frac{\beta}{144a_2} \frac{1}{r^5}, \\ u_\theta = u_\phi &= 0. \end{aligned} \quad (122)$$

This displacement field (122) corresponds to the problem of a spherical shell subjected to internal pressure p_1 and external pressure p_2 within the framework of second order theory formulated in Section 3.

The stress distribution corresponding to the solution (122), of course, can be readily obtained by direct substitution of (122) into (13).

In particular, the radial and tangential stresses are:

$$\begin{aligned} \tau_{rr} &= 8(a_1 + 3a_2)(c_1 + c_3) + 8a_1(c_2 + c_4) \frac{1}{r^3} - \frac{a_1 - 3a_2}{18a_2} \frac{\beta}{r^6} \\ &+ 4[(-a_1 - 3a_2 + 18a_3 + 54a_4 + 2a_5)c_1^2 - 4(2a_1 + 6a_2 - 3a_3 - a_5) \frac{c_1 c_2}{r^3} \\ &+ (-7a_1 + 6a_2 - 6a_3 + 2a_5) \frac{c_2^2}{r^6}] , \end{aligned}$$

$$\begin{aligned}
t_{\theta\theta} = & 8(a_1+3a_2)(c_1+c_3) - 4a_1(c_2+c_4) \frac{1}{r^3} + \frac{2a_1+3a_2}{18a_2} \frac{\beta}{r^6} \\
& + 2[2(-a_1-3a_2+18a_3+54a_4+2a_5)c_1^2 + 4(2a_1+6a_2-3a_3-a_5) \frac{c_1c_2}{r^3} \\
& + (a_1+12a_2-12a_3-8a_5) \frac{c_2^2}{r^6}] . \quad (123)
\end{aligned}$$

We wish to investigate here the nature of these stresses.

In the classical shell problem, the tangential stresses are found to be monotone functions of r which attain their maximum and minimum values on the surfaces of the shell. To see if $t_{\theta\theta}$ exhibits similar behaviour, we differentiate with respect to r to obtain:

$$\begin{aligned}
\frac{dt_{\theta\theta}}{dr} = & [12a_1(c_2+c_4) - 24(2a_1+6a_2-3a_3-a_5)c_1c_2] \frac{1}{r^4} \\
& - \left[\frac{2a_1+3a_2}{3a_2} \beta + 6(a_1+12a_2-12a_3-8a_5)c_2^2 \right] \frac{1}{r^7} . \quad (124)
\end{aligned}$$

If $t_{\theta\theta}$ has a relative extremum, then $\frac{dt_{\theta\theta}}{dr} = 0$ for some value of r . This condition will be satisfied provided:

$$\begin{aligned}
& [12a_1(c_2+c_4) - 24(2a_1+6a_2-3a_3-a_5)c_1c_2]r^3 \\
& - \left[\frac{2a_1+3a_2}{3a_2} \beta + 6(a_1+12a_2-12a_3-8a_5)c_2^2 \right] = 0 . \quad (125)
\end{aligned}$$

Since (125) is a cubic equation, there does exist at least one real value of r for which the first derivative of $t_{\theta\theta}$ vanishes. Thus, unlike the classical theory, the second order theory admits the possibility that the tangential stress attains an extreme value in the interior of the shell. It is clear that the value of r obtained

from (125) will depend both on the deformation and the type of material being considered, and until more is known about these, we cannot say if this value lies in the interval $a \leq r \leq b$.

8. SOLID SPHERE UNDER PRESSURE

This problem happens to be a special case of the preceding problem when $a = 0$ and $p_1 = 0$. We find:

$$c_1 = \frac{-P_2}{8(a_1+3a_2)}, \quad c_2 = 0, \quad c_3 = \frac{-\beta_1 P_2^2}{512(a_1+3a_2)^3}$$

$$c_4 = 0, \quad \text{and} \quad \beta = 0. \quad (126)$$

Introducing (126) into (122) yields:

$$u_r = (c_1 + c_3)r,$$

$$u_\theta = u_\phi = 0. \quad (127)$$

The corresponding state of stress is found from (123):

$$t_{rr} = t_{\theta\theta} = 8(a_1+3a_2)(c_1+c_3) + 4(-a_1-3a_2+18a_3+54a_4+2a_5)c_1^2. \quad (128)$$

9. ALMOST SOLID SPHERE UNDER PRESSURE

Here, we set $p_1 = 0$ and let $a \rightarrow 0$ instead of being identically zero. From equation (122):

$$[u_r]_{r=a} = (c_1 + c_3)a + (c_2 + c_4) \frac{1}{a^2} - \frac{\beta}{144a_2} \frac{1}{a^5}. \quad (129)$$

Noting the forms of the constants c_2 , c_4 , and β , we see that each term on the right hand side of (129) depends on at least the first power of a , so that:

$$\lim_{a \rightarrow 0} [u_r]_{r=a} = 0. \quad (130)$$

Thus, the displacement field remains bounded as a tends to zero and its limit value agrees with the value obtained from (127) with $r = 0$.

In the classical solution the tangential stress on the inner surface of the shell experiences a stress concentration of amount $-\frac{3}{2} p_2$. To determine what happens in the second order solution, we consider:

$$\begin{aligned} [t_{\theta\theta}]_{r=a} = & 8(a_1 + 3a_2)(c_1 + c_3) - 4a_1(c_2 + c_4) \frac{1}{a^3} + \frac{2a_1 + 3a_2}{18a_2} \frac{\beta}{a^6} \\ & + 4[(-a_1 - 3a_2 + 18a_3 + 54a_4 + 2a_5)c_1^2 - 4(2a_1 + 6a_2 - 3a_3 - a_5) \frac{c_1 c_2}{a^3} \\ & + (-7a_1 + 6a_2 - 6a_3 + 2a_5) \frac{c_2^2}{a^6}] , \end{aligned} \quad (131)$$

where

$$c_1 = \frac{1}{8(a_1 + 3a_2)} \frac{-p_2 b^3}{b^3 - a^3}, \quad c_2 = \frac{1}{8a_1} \frac{a^3 b^3 p_2}{b^3 - a^3}, \quad (132)$$

and c_3 , c_4 , and β are respectively defined by (121) and (112).

On taking the limit of t_{00} as a tends to zero, we obtain a complicated expression which involves terms which are not found in the stress field (128). Thus, the second order solution exhibits a tangential stress concentration beyond that which is found in the classical solution.

10. INFINITE MEDIUM WITH A SPHERICAL CAVITY SUBJECTED TO UNIFORM INTERNAL PRESSURE.

This problem, too, is a special case of the general problem discussed in Section 7. When we put $p_2 = 0$ and allow $b \rightarrow \infty$, we find:

$$c_1 = 0, \quad c_2 = -\frac{1}{8a_1} p_1 a^3, \quad c_3 = 0,$$

$$c_4 = \frac{1}{8a_1} \left[\frac{\beta}{18a_2} (a_1 - 3a_2) - \beta_3 c_2^2 \right] \frac{1}{a^3}. \quad (133)$$

The displacement field (122) becomes:

$$u_r = (c_2 + c_4) \frac{1}{r^2} - \frac{1}{144a_2} \frac{\beta}{r^5},$$

$$u_\theta = u_\phi = 0. \quad (134)$$

From (134), it is evident that the displacements become negligible for arbitrarily large values of r . The corresponding radial and tangential stresses are:

$$t_{rr} = 8a_1 (c_2 + c_4) \frac{1}{r^3} - \frac{a_1 - 3a_2}{18a_2} \frac{\beta}{r^6} + 4(-7a_1 + 6a_2 - 6a_3 + 2a_5) \frac{c_2^2}{r^6},$$

$$t_{\theta\theta} = -4a_1 (c_2 + c_4) \frac{1}{r^3} + \frac{2a_1 + 3a_2}{18a_2} \frac{\beta}{r^6} + 2(a_1 + 12a_2 - 12a_3 - 8a_5) \frac{c_2^2}{r^6}. \quad (135)$$

It is clear from (135) that the stresses t_{rr} and $t_{\theta\theta}$ both vanish at infinity, as required.

BIBLIOGRAPHY

1. Rivlin, R.S. Large elastic deformations of isotropic materials. Trans. Roy. Soc. London A 241, 379 (1948).
2. Green, A.E. & W. Zerna. *Theoretical Elasticity.* New York: Oxford University Press, 1968.
3. Ericksen, J.L. Deformations possible in every isotropic compressible, perfectly elastic material. J. Math. Phys. 34, 126 (1955).
4. Rivlin, R.S. Some topics in finite elasticity. Proceedings of the First Symposium on Naval Structural Mechanics, Pergamon Press, New York, 1960.
5. Murnaghan, F.D. *Finite deformation of an elastic solid.* New York: John Wiley and Sons, 1951.
6. Green, A.E. and E.B. Spratt. Second-order effects in the deformation of elastic bodies. Proc. Roy. Soc. London. A 224, 347 (1954).
7. Green, A.E. & R.T. Shield. Finite extension and torsion of cylinders. Phil. Trans. A 244, 47 (1951).
8. Green A.E. & J.E. Adkins. *Large Elastic Deformations and Nonlinear Continuum Mechanics.* Oxford at the Clarendon Press, 1960.
9. Grioli, S. *Mathematical Theory of Elastic Equilibrium.* Ergeli. Angew. Math. No. 7. Berlin-Göttingen-Heidelberg: Springer, 1962.
10. Sheng, P.L. *Secondary Elasticity.* Taipei: Chinese Ass. for the Advancement of Science, Monographs, no. 1, 1955.
11. Rivlin, R.S. The solution of problems in second order elasticity theory. J. Rat. Mech. Anal. 2, 53 (1953).
12. Poynting, J.H. On pressure perpendicular to the shear-planes in finite pure shears, and on the lengthening of loaded wires when twisted. Proc. Roy. Soc. London. A 82, 546 (1909).