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CERTAIN PERTURBATION METHODS IN NON-LINEAR MECHANICS

by

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ABSTRACT

Certain perturbation methods are used to analyse some problems of non-linear mechanics. A comparison is made of the periodic solutions obtained by the two methods of Poincaré and Krylov-Bogoliubov for systems of ordinary differential equations and it is shown that the solutions obtained by the two methods coincide term by term. The results are then extended to mono-frequent periodic solutions of oscillatory systems governed by weakly non-linear hyperbolic partial differential equations.

The Krylov-Bogoliubov-Mitropolskii (K-B-M) asymptotic method was used to investigate the following problems:

a) The response of a non-linear vibrator under the time-dependent (periodic or non-periodic) external force is investigated. The analysis is extended to non-linear vibrators governed by partial differential equations.

b) The effects of kinematical non-linearities on the vibration frequencies of undamped and damped strings are investigated and it is shown that in the case of undamped string, the natural frequencies are increased by a term which in lowest order is proportional to the square of the amplitude of vibration. The same is true for natural frequencies in the case of a damped string, for times which are small compared with the decay time.

c) The asymptotic solutions are obtained for a class of hyperbolic partial differential equation with slowly varying coefficients. The results are applied to two dimensional vibrations of a damped stretched string.

The nature of plane shock waves in a viscoelastic media displaying cubic elasticity is also investigated, by using the two-time expansion. The non-linearity is taken to occur in the form of terms in the stress-strain relation which are quadratic and cubic in strain and the viscoelasticity is taken as a functional term in the stress-strain relation. Approximate solutions are obtained in the case when the viscoelastic effects are significant only within the shock-layers which develop.

DEDICATED ~~TO~~ MY PARENTS

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INTRODUCTION

With the advance of science and technology every day the study of oscillation theory is becoming more and more important because of its applications in various branches of non-linear mechanics, physics and engineering. The oscillatory processes like vibrations of plants and machinery, electro-magnetic oscillations in radio and electrical engineering, automatic oscillations in control systems, sonic and ultra-sonic oscillations etc. all come under the heading of general oscillation theory.

The origin of the science of oscillations can easily be traced in the classical mechanics of the motion of a pendulum during the times of Galileo and Newton. In the beginning, the theory of oscillations was confined to linear oscillations, because the theory of non-linear differential equations governing the oscillatory system was not developed. Simply these equations were linearized to differential equations with constant coefficients and solved. Though the linearization often leads to quantitative as well as qualitative errors, it was only in a few cases that linearization was not adopted and non-linear oscillations were examined as such. (For example

M.V. Ostrogradskii (1835) studied the equation

$\ddot{x} + \omega^2 x = \alpha x^3)$.

An oscillation is treated as sufficiently close to a linear one, when the governing non-linear differential equation contains a small parameter ε and for $\varepsilon = 0$, this equation reduces to a linear differential equation with constant coefficients. To find the solution of such non-linear systems, one generally seeks approximate solutions. The most commonly used method is the perturbation method. This method was worked by the astronomers for studying planetary motion. The earliest technique to solve a non-linear differential equation was to express the solution sought $u(t,\varepsilon)$ as a power series in ε:

$$u(t,\varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r u_r(t).$$

Substituting $u(t,\varepsilon)$ in the given non-linear differential equation and equating the various powers of ε gives linear differential equations for each u_r which can easily be solved. However such a solution generally involves secular terms of the type $t^m \sin \alpha t$ and $t^m \cos \alpha t$ ($m > 1$, α = constant) which makes the solution valid only for small intervals of time. Such a solution becomes invalid for large times.

In the development of the theory of non-linear oscillations, various methods have been suggested from time to time to overcome the difficulty caused by the appearance of secular terms in the solutions. All these methods have been based on power series expansions.

Although as a rule, the series diverge, the approximate formulae found by taking the first few terms are quite suitable for practical calculations. In fact, these series are asymptotic in the sense that the error in the n-th approximation is proportional to ε^{n+1} and as such can be made as small as we please by taking ε sufficiently small. In other words, the error committed in truncating the series solution after n terms is numerically less than the first neglected term, that is, (n+1)th term. Since it becomes too complicated to calculate higher approximations, in general one resorts to first and second approximations for most practical purposes.

The asymptotic methods were found to be very effective in celestial mechanics by various astronomers such as Lindstedt (1882), Bohlin (1889), Poincaré (1892) and Gyldén (1893). The fundamental idea in Lindstedt - Poincaré approach was based on the observation that the non-linearities in the equation

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x,\dot{x}) \qquad (1)$$

alter the frequency ω_0 of the linear system ($\varepsilon=0$) to $\omega(\varepsilon)$. They made the change in the independent variable t to $\tau = \omega t$ and expanded x and ω in power series of ε as:

$$x = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots \ldots$$
$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \ldots \qquad (2)$$

The choice of ω_r (r=1,2,...) was made in such a way as to avoid the appearance of secular terms. Various forms of this idea have been utilized to obtain approximate solutions to problems in physics and engineering.

In 1926, Vander-Pol suggested a method of finding the approximate solutions of (1) with $f(x,\dot{x}) = (1-x^2)\dot{x}$. The solution was represented by

$$x = a \cos \omega_0 t + b \sin \omega_0 t$$
$$\dot{x} = -\omega_0 a \sin \omega_0 t + \omega_0 b \cos \omega_0 t \qquad (3)$$

where a and b were assumed to be slowly varying functions of time to be determined from the equations

$$\dot{a} = -\frac{\varepsilon}{\omega_0} F(a,b,t) \sin \omega_0 t ,$$
$$\dot{b} = \frac{\varepsilon}{\omega_0} F(a,b,t) \cos \omega_0 t , \qquad (4)$$

where $F(a,b,t) = f(a \cos \omega_0 t + b \sin \omega_0 t,$

$$- a\omega_0 \sin \omega_0 t + b\omega_0 \cos \omega_0 t).$$

The equations for $\dot{a}$ and $\dot{b}$ were further simplified to

$$\dot{a} = - \frac{\varepsilon}{2\omega_0} F_0(a,b) \quad , \qquad \dot{b} = \frac{\varepsilon}{2\omega_0} G_0(a,b)$$

by averaging the equations (4) over the time interval [t, t+T], during which a and b change very little and hence can be taken to be constants on the right-side of equations (4). Here $T = 2\pi/\omega_0$, is the period of the terms on the right side of equations (4). This method gives only the first approximation and is not suitable for higher approximations. A mathematical justification of this method was given by Fatou (1928) and Mendelstan and Papaleksi (1934).

On the basis of the method of Vander-Pol, Krylov, Bogoliubov and Mitropolskii (K-B-M) developed an asymptotic method for solving non-linear problems in mechanics which apart from being suitable for higher approximations, gives idential solutions to those found by Vander-Pol method for first approximations. The K-B-M method was later generalized to the so-called method of averaging. According to K-B-M method, the asymptotic solution of (1) is sought in the form

$$x = a \cos \psi + \sum_{r=1}^{\infty} \varepsilon^r u_r(a,\psi) \qquad (5)$$

where u_r are 2π-periodic in ψ and a,ψ are determined from the differential equations

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots$$

$$\dot{\psi} = \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots \qquad (6)$$

It is assumed that $u_r(a,\psi)$, $r=1,2,\ldots$ are free of first harmonics in ψ. The functions A_r, B_r and u_r are to be so determined that the solution (5)-(6) satisfies (1) to each order of ε. The K-B-M method can also be used to find periodic solutions of (1) if we take $\dot{a} = 0$ or a = constant.

In recent years, apart from other methods, the two-time method, first introduced by Cole and Kevorkian [14] has gained considerable importance. According to this method, the solution of (1) is assumed to be a function of two times τ and η defined by

$$\tau = \varepsilon t \quad , \qquad \eta = (1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \ldots + \varepsilon^{M-1}\omega_M)t$$

where ω_r are constants, and is expressed in the form

$$x(t,\varepsilon) = \tilde{x}(\tau,\eta,\varepsilon) = \sum_{r=0}^{M-1} \varepsilon^r x_r(\tau,\eta) + 0(\varepsilon^M).$$

Each x_r is determined in such a way that the solution is uniformly valid, i.e. x_r/x_{r-1} is of $0(1)$ as $t \to \infty$.

In other words, each x_r is no more singular for large t than the preceeding term x_{r-1}. This requires that the secular terms in each x_r should be eliminated. This requirement of secular terms in each x_r determines the constants ω_r.

This two-time method can be generalized to the method of multiple scales. In this case, we seek the solution as a function of

$$t, \quad \tau_1 = \varepsilon t, \quad \tau_2 = \varepsilon^2 t, \quad \ldots, \quad \tau_n = \varepsilon^n t$$

and express it as

$$x(t,\varepsilon) = \sum_{r=0}^{n} \varepsilon^r x_r(t,\tau_1,\tau_2, \ldots,\tau_n) + 0(\varepsilon^{n+1}).$$

The methods of finding periodic and asymptotic solutions of partial differential equations of the hyperbolic type were recently developed by many authors like Mitropolskii and Mosenkeev [9], Fodchuk [13], Chikwendu and Kevorkian [15], Eckhaus [16], Fink, Hall and Hausrath [17] and Bojadziev and Lardner [10,11,12].

Morrison [19] established the equivalence of the method of averaging and two time method for second order ordinary differential equations of the type (1) where as for partial differential equations, the same has been established by Lardner [20]. These results

are significant because of the rigorous foundation of the average method established by Bogoliubov [4], which therefore indirectly provides a justification of the two time method.

In this thesis, we make use of some of these perturbation methods to analyse certain non-linear problems in mechanics.

In Chapter 1, we have compared the periodic solutions obtained by the methods of Poincaré and Krylov-Bogoliubov for systems of ordinary differential equations. The method has been extended to monofrequent periodic solutions of oscillatory systems governed by weakly non-linear hyperbolic partial differential equations.

In Chapter 2, we have investigated the response of a non-linear vibrator governed by a second order ordinary differential equation under the influence of a time-dependent external force. The method is also extended to partial differential equations in such a case.

In Chapter 3, the effects of kinemetical non-linearities on the vibration frequencies of an undamped and a damped string are investigated.

In Chapter 4, the asymptotic solutions for certain partial differential equations of hyperbolic type with slowly varying coefficients are investigated.

In Chapter 5, two-time expansion is used to investigate the nature of plane shock waves in visco-elastic media displaying cubic elasticity.

CHAPTER 1

ON THE PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OBTAINED BY THE METHODS OF POINCARE AND KRYLOV-BOGOLIUBOV.

1. INTRODUCTION.

As is well known, Poincaré Method [1] in the theory of non-linear vibrations allows us to find the periodic solutions of weakly non-linear systems of ordinary differential equations. The Poincaré method can be extended to find the mono-frequent periodic oscillations of a mechanical system governed by second order non-linear hyperbolic autonomous partial differential equations. The method can equally well be used for the non-autonomous partial differential equations.

The asymptotic method of Krylov-Bogoliubov [3] developed by Bogoliubov and Mitropolskii [4] is used for studying non-stationary vibrations governed by ordinary differential equations with small non-linearities. Later this method was developed by Mitropolskii and Mosenkov [9], Bojadziev and Lardner [10,11,12] and others [13,14,15,16,17] for finding the non-stationary regime of vibrations of mechanical and

electrical systems governed by partial differential equations. In particular the K-B-M method is also applicable to periodic vibrations.

Proskurjakov [6] has compared the periodic solutions by the two methods of the autonomous differential equation

$$\ddot{x} + \omega^2 x = \varepsilon f(x,\dot{x}) , \quad \dot{x} = \frac{dx}{dt} , \qquad (1)$$

where ε is a small positive parameter and $f(x,\dot{x})$ is analytic in x, $\dot{x}$ in some domain. It is concluded after long computation that the first three approximate solutions obtained by both methods coincide entirely. Later on Proskurjakov mentions that in the case of the main resonance, the same conclusion holds for second-order non-autonomous differential equations.

In this chapter, we provide an exposition of the methods of Poincaré and Krylov-Bogoliubov-Mitropolskii as applied to periodic solutions of (1). In particular we shall reconsider some of the results of Proskurjakov [6] on the comparison of the two methods, avoiding long calculations. This we do in section 2. In section 3, we give analogous results for a second-order non-autonomous ordinary differential equation in both the resonance and non-resonance case. We

generalize these results to autonomous and non-autonomous systems of ordinary differential equations. This is done in sections 4 and 5.

In section 6, we extend the investigation to compare monofrequent periodic oscillations governed by autonomous non-linear partial differential equations. In section 7, we give analogous results for non-autonomous partial differential equations in both resonance and non-resonance cases.

In all these sections, the importance of the so-called improved n-th approximation in the K-B-M method is demonstrated when the periodic solutions are being sought.

2.* Second-Order Autonomous Equation.

Let us consider equation (1). According to Poincaré's method we can seek the periodic solutions of (1) in the form

$$x = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots , \qquad (2)$$

where the functions $\{x_s(\tau)\}$ are 2π periodic in τ and

$$t = \frac{\tau}{\omega} (1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) . \qquad (3)$$

The constants $\{h_s\}$ have to be determined. Usually the solution $x(t)$ of (1) is sought under the condition $\dot{x}(0) = 0$, which implies

$$\frac{dx_s(0)}{d\tau} = 0 , \qquad s = 0,1,\ldots . \qquad (4)$$

The function $x_0(\tau)$ is a solution of the generating equation $d^2x_0/d\tau^2 + x_0 = 0$. Taking into account (4) one gets

$$x_0(\tau) = M \cos \tau . \qquad (5)$$

Substituting (2) into (1), making use of (3), and equating the coefficients of $\varepsilon, \varepsilon^2, \ldots$, we have for

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* Sections 2-5 are published in UTILITAS MATHEMATICA Vol.3, (1973) pp.49-64 (with BOJADZIEV and LARDNER).

$x_1, x_2, \ldots$ the equations

$$\frac{d^2x_1}{d\tau^2} + x_1 = \frac{1}{\omega^2} f(M \cos \tau, -\omega M \sin \tau) - 2h_1 M \cos \tau, \tag{6}$$

$$\frac{d^2x_2}{d\tau^2} + x_2 = \frac{1}{\omega^2}\left[2h_1 f(M \cos \tau, -\omega M \sin \tau) + x_1 \frac{\partial f}{\partial x_0} + \omega\left(\frac{dx_1}{d\tau} + h_1 \omega M \sin \tau\right)\frac{\partial f}{\partial \dot{x}_0}\right] - (h_1^2 + 2h_2) M \cos \tau - 2h_1 x_1 . \tag{7}$$

Since $f(M \cos \tau, -\omega M \sin \tau) \equiv f_0(M,\tau)$ is a 2π periodic function in τ, we can expand it as a Fourier series.

$$f_0(M,\tau) = \frac{P_0}{2} + \sum_{n=1}^{\infty} (P_n \cos n\tau + Q_n \sin n\tau) . \tag{8}$$

Substituting (8) into (6) and equating to zero the coefficients of sin τ and cos τ, to get rid of the secular terms, one obtains the amplitude equation

$$Q_1 = 0 \quad \text{or} \quad \int_0^{2\pi} f_0(M,\tau) \sin \tau d\tau = 0 , \tag{9}$$

and an equation for h_1

$$\frac{P_1}{\omega^2} - 2h_1 M = 0 \quad \text{or} \quad h_1 = \frac{1}{2\pi M\omega^2} \int_0^{2\pi} f_0(M,\tau) \cos \tau d\tau \,. \tag{10}$$

Equation (9) determines the amplitude M .

The solution of (6) under condition (4) is

$$x_1(\tau) = M_1 \cos \tau + \frac{P_0}{2\omega^2} + \sum_{n=2}^{\infty} \frac{P_n \cos n\tau + Q_n \sin n\tau - nQ_n \sin \tau}{(1-n^2)\omega^2} \,, \tag{11}$$

where M_1 is a constant to be determined from equation (7). Then the solution (2) of (1) in the first approximation according to (5) and (11) is

$$x = M \cos \tau + \varepsilon\left[M_1 \cos \tau + \frac{P_0}{2\omega^2} + \sum_{n=2}^{\infty} \frac{P_n \cos n\tau + Q_n \sin n\tau - nQ_n \sin \tau}{(1-n^2)\omega^2}\right]. \tag{12}$$

Using (3) and (10) the solution (12) can be written with respect to t .

Now let us apply the (K-B) method to equation (1). The solution is sought in the form

$$x = a \cos \psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots \,, \tag{13}$$

where the functions $\{u_s(a,\psi)\}$ are 2π periodic in ψ and a and ψ are functions of t which satisfy the equations

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots ,$$
$$\dot{\psi} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots . \tag{14}$$

The functions $\{A_s(a), B_s(a)\}$ and $\{u_s(a,\psi)\}$ are to be found from the requirement that x satisfies (1). For example, the formulas for $A_1(a)$ and $B_1(a)$ are [3]

$$A_1(a) = -\frac{1}{2\omega\pi}\int_0^{2\pi} f_0(a,\psi)\sin\psi d\psi ,$$
$$B_1(a) = -\frac{1}{2a\omega\pi}\int_0^{2\pi} f_0(a,\psi)\cos\psi d\psi . \tag{15}$$

To find the periodic solution of (1) one sets

$$\dot{a} = 0, \quad \dot{\psi} = \text{const} = \omega + \varepsilon B_1 + \varepsilon^2 B_2 + \dots , \tag{16}$$

where $\{B_s\}$ are constants to be determined. From (16) we get

$$a = a_0 = \text{const}, \quad \psi = (\omega + \varepsilon B_1 + \varepsilon^2 B_2 + \dots)t, \text{ if } \psi(0) = 0 . \tag{17}$$

From (14) and (15), $\dot{a} = 0$ implies $A_1(a) = 0$ or

$$\int_0^{2\pi} f_0(a,\psi)\sin\psi d\psi = 0 , \tag{18}$$

which determines the amplitude a . The equation (18) is identical with (9) which shows that a = M provided we consider the corresponding roots in the two equations.

The solution (13) can be written in the form (2), where

$$x_0(\psi) = a \cos \psi, \quad x_s(\psi) = u_s(a,\psi), \quad s = 1,2,\ldots .$$

By the uniqueness theorem for periodic solutions, the two solutions (2) and (13) must be the same, provided that the initial conditions (4) are used for the K-B method as well as for the Poincaré method. We observe that in the solution (2), x is 2π periodic in τ, defined in (3), while in the solution (13), x is 2π periodic in ψ defined by (17). It follows that the ratio of ψ and τ must be rational. But ψ and τ are expressible as power series in ε whose zero-order terms are identical, so that they must therefore be equal. The two sets of coefficients in (3) and (18) are related by equations such as

$$B_1 = - \omega h_1, \quad B_2 = \omega (h_1^2 - h_2),\ldots .$$

Since $\tau = \psi$ in the two solutions (2) and (13) we must have term-by-term equality of these power series

in ε , provided that the initial conditions (4) are used for the Poincaré method as well as for the K-B method. Hence we conclude that for periodic solutions the solution according to the K-B method is identical term by term with the solution according to the Poincaré method.

That is why the usual formula for $u_1(a,\psi)$ [3] coincides with (11) if we omit the terms in $\cos\psi$ and $\sin\psi$, for in K-B method it is usually supposed that $u_1(a,\psi)$ does not contain the first harmonics. In the case of periodic solutions we do not make such an assumption, but introduce the initial conditions (4) instead.

Let us note that according to the K-B method, by the n^{th} approximation is usually meant the sum of the first n terms in (13). For example, the solution in the first approximation is $x = a\cos\psi$, where $\dot{a} = \varepsilon A_1(a)$ and $\dot{\psi} = \omega + \varepsilon B_1(a)$. For the same $\dot{a}$ and $\dot{\psi}$ sometimes $x = a\cos\psi + \varepsilon u_1(a,\psi)$ is called the improved first approximation.

In the case of periodic solutions, the improved first approximate solution of (1) is (12). This shows that the first approximation found by Poincaré's method

coincides with the improved first approximation found by the K-B method with respect to both ψ and τ .

In general the n^{th} approximation found by Poincaré's method is the same as the improved n^{th} approximation found by the K-B method, i.e. if we take (n+1) terms and not n terms in (13). This remark is important because in the general case when one applies the K-B method to find nonstationary vibrations, the recommendation given is to take n terms for the n^{th} approximation.

3. Second-Order Non-autonomous Equation.

Consider now the non-autonomous differential equation

$$\ddot{x} + \omega^2 x = \varepsilon f(\nu t, x, \dot{x}) \ , \tag{19}$$

where $\varepsilon > 0$ is a small parameter and $f(\nu t, x, \dot{x})$ is 2π periodic in νt of the form

$$f(\nu t, x, \dot{x}) = \sum_{n=-N}^{N} e^{in\nu t} f_n(x, \dot{x}) \ .$$

The coefficients $f_n(x, \dot{x})$ in this finite sum are required to be polynomial in x and $\dot{x}$.

(a) Nonresonance case. Assume that for any integers m, n,

$$n\nu + m\omega \neq \omega \quad \text{or} \quad \nu \neq \frac{q}{p}\omega \quad (p, q \text{ any integers}).$$

Let us apply Poincaré's method for finding periodic solutions with period $2\pi/\nu$ in t (or with period 2π in νt) of the differential equation (19). We can write the solution as a series

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \ , \tag{20}$$

where the functions $\{x_s(t)\}$ are 2π periodic in νt.

After substituting (20) into (19) we get $x_0(t) = 0$ and $x_1(t)$ is to be found from $\ddot{x}_1 + \omega^2 x_1 = f(\nu t, 0, 0)$. To solve this equation we expand the function $f(\nu t, 0, 0)$ in a Fourier series and get

$$x_1(t) = \frac{1}{\pi} \sum_n \left(\frac{\cos n\theta}{\omega^2 - (n\nu)^2} \int_0^{2\pi} f(\theta, 0, 0) \cos n\theta d\theta \right.$$

$$\left. + \frac{\sin n\theta}{\omega^2 - (n\nu)^2} \int_0^{2\pi} f(\theta, 0, 0) \sin n\theta d\theta \right), \quad \theta = \nu t .$$

Hence the first approximation of the periodic solution of (19) is $x(t) = \varepsilon x_1(t)$. In the same manner one can find the n^{th} approximation.

According to the K-B method [3] the solution of equation (19) in the nonresonance case is sought in the form

$$x = a \cos \psi + \varepsilon u_1(a, \psi, \nu t) + \varepsilon^2 u_2(a, \psi, \nu t) + \ldots \qquad (21)$$

where the functions $\{u_s(a, \psi, \nu t)\}$ are 2π periodic in ψ and νt . The amplitude a and the phase ψ are given by (14) . Here again by the n^{th} approximation is understood the sum of the first n terms in (21).

Substituting (21) into (19), using (14) and equating the coefficients of $\varepsilon, \varepsilon^2, \ldots$ enables us to find $u_s(a,\psi,\theta)$, $A_s(a)$, $B_s(a)$, $s = 1,2,\ldots$ and therefore the solution of (19) in the nonstationary regime.

To find the periodic solution of (19) with period $2\pi/\nu$ in t, we set $a = 0$, $\psi = 0$ in (21). This means the functions $\{u_s\}$ depend on θ only and equations (14) no longer appear. Then (21) becomes

$$x = \varepsilon u_1(0,0,\theta) + \varepsilon^2 u_2(0,0,\theta) + \ldots , \qquad (22)$$

where $\{u_s(0,0,\theta)\}$ are 2π periodic in θ.

If we let $u_s(0,0,\nu t) = x_s(t)$, $s = 1,2,\ldots$, then the sought solution (22) coincides with (20) .

The conclusion is that the periodic solutions found by both methods are the same in every approximation. Of course, we compare the n^{th} approximation in Poincaré's method with the improved n^{th} approximation in the K-B method.

(b) Resonance case. Here we assume that for certain pairs of integers m, n

$$n\nu + m\omega \simeq \omega \quad \text{or} \quad \nu \simeq \frac{q}{p}\,\omega \quad (p,q \text{ integers}) .$$

To simplify the considerations let us consider the case of exact main resonance, $\omega = \nu$, i.e. $p = q = 1$.

According to Poincaré's method we seek the periodic solutions of (19) in the form (20).

The solution of the generating equation

$$\ddot{x}_0 + \nu^2 x_0 = 0 \tag{23}$$

is

$$x_0(t) = M \cos \nu t + N \sin \nu t \ , \tag{24}$$

which is $2\pi/\nu$ periodic in t. The constants M and N are to be determined. For the function $x_1(t)$ we get the equation

$$\ddot{x}_1 + \nu^2 x_1 = f_0(\nu t, M, N) \ , \tag{25}$$

where

$$f_0(\nu t, M, N) = f(\nu t, M \cos \nu t + N \sin \nu t, -M\nu \sin \nu t + N\nu \cos \nu t) \ . \tag{26}$$

To solve (25) we develop (26) in a Fourier series

$$f_0(\nu t, M, N) = \sum_n f_n^{(0)}(M,N) e^{in\nu t} \ ,$$

$$f_n^{(0)}(M,N) = \frac{1}{2\pi} \int_0^{2\pi} f_0(\nu t, M, N) e^{-in\nu t} d(\nu t) \ . \tag{27}$$

To avoid the secular terms in the solution of (25) we set

$$f_1^{(0)}(M,N) = 0, \quad f_{-1}^{(0)}(M,N) = 0 . \tag{28}$$

From the equations (28) , called amplitude equations, we can find M and N .

The periodic solution of (25) then is

$$x_1(t) = M_1 \cos \nu t + N_1 \sin \nu t + \sum_{n\neq\pm 1} \frac{e^{in\nu t}}{2\pi\nu^2(1-n^2)} \int_0^{2\pi} f_0(\nu t,M,N) e^{-in\nu t} d(\nu t) , \tag{29}$$

where M_1 and N_1 are to be determined under the condition that the function $x_2(t)$ from the second approximation should not contain secular terms.

Now let us apply the K-B method. The solution of (19) in the nonstationary regime is sought in the form

$$x = a \cos (\frac{p}{q} \theta + \psi) + \varepsilon u_1(a,\theta,\frac{p}{q} \theta + \psi) + \varepsilon^2 \ldots, \quad \theta = \nu t, \tag{30}$$

where the functions $\{u_s(a,\theta,\frac{p}{q}\theta + \psi)\}$ are 2π periodic in θ and $\frac{p}{q} \theta + \psi$. The quantities a and ψ are given by

the equations

$$\dot{a} = \varepsilon A_1(a,\psi) + \varepsilon^2 \ldots , \quad \dot{\psi} = \omega - \frac{p}{q}\nu + \varepsilon B_1(a,\psi) + \varepsilon^2 \ldots \tag{31}$$

According to K-B method [4] , A_1 and B_1 are to be determined by the equation

$$\left[(\omega - \frac{p}{q}\nu)\frac{\partial A_1}{\partial \psi} - 2a\omega B_1\right]\cos(\frac{p}{q}\theta + \psi)$$

$$- \left[(\omega - \frac{p}{q}\nu)a\frac{\partial B_1}{\partial \psi} + 2\omega A_1\right]\sin(\frac{p}{q}\theta + \psi) \tag{32}$$

$$= \sum_{nq+p(m\mp 1)=0} f_{nm}^{(0)}(a)\, e^{\pm i(\frac{p}{q}\theta+\psi)}\, e^{i(m\pm 1)\psi} ,$$

where

$$f_{nm}^{(0)}(a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f\Big(\theta, a\cos(\frac{p}{q}\theta + \psi), \tag{33}$$

$$- a\nu\sin(\frac{p}{q}\theta + \psi)\Big) e^{-i\{n\theta+m(\frac{p}{q}\theta+\psi)\}} d\theta d(\frac{p}{q}\theta + \psi) .$$

In the case of main resonance $p = q = 1$.

To find the $2\pi/\nu$ periodic solutions in t of (19) we set

$$\dot{a} = 0;\ \dot{\psi} \triangleq 0,\ \text{i.e. } a = a_0 = \text{const},\ \psi = \psi_0 = \text{const}. \quad (34)$$

This implies $A_s(a_0,\psi_0) = 0$, $B_s(a_0,\psi_0) = 0$, $s = 1,2,\ldots$ and we do not need equations (31) . Then (30) becomes

$$x = a_0 \cos(\theta + \psi_0) + \varepsilon u_1(a_0,\theta,\theta + \psi_0) + \varepsilon^2 \ldots . \quad (35)$$

The term $u_0 = a_0 \cos(\theta + \psi_0)$ is a solution of

$$\nu^2 \frac{d^2u_0}{d\theta^2} + \nu^2 u_0 = 0 . \quad (36)$$

The equations (23) and (36) are the same, hence their solutions x_0 and u_0 coincide; i.e.

$$M \cos \nu t + N \sin \nu t = a_0 \cos(\theta + \psi_0) , \quad \theta = \nu t,$$

or

$$M = a_0 \cos \psi_0, \quad N = - a_0 \sin \psi_0 . \quad (37)$$

According to the uniqueness theorem for periodic solutions, the two solutions (20) and (35) coincide term by term in the power series in ε .

One can show that the amplitude equations (28) coincide with the corresponding equations in the K-B method. From (32), taking into account that $p = q = 1$, $A_1 = B_1 = 0$, we get

$$\sum_{n=\pm 1} f_n^{(0)}(a_0,\psi_0)e^{in\theta} = 0 ,$$

$$f_n^{(0)}(a_0,\psi_0) = \frac{1}{2\pi}\int_0^{2\pi} f[\theta,\ a_0\cos(\theta+\psi_0),\ -\nu a_0\sin(\theta+\psi_0)]e^{-in\theta}d\theta, \tag{38}$$

or

$$f_1^{(0)}(a_0,\psi_0) = 0, \quad f_{-1}^{(0)}(a_0,\psi_0) = 0 . \tag{39}$$

From (26), (27), (37), and (38) it is seen that the amplitude equations (28) coincide with the equations (39). Hence we find again the same result as in sections 2 and 3a.

4. Systems of Autonomous Differential Equations.

Consider the non-linear system

$$\ddot{x} + \omega^2 x = \varepsilon f(x,\dot{x}) \ , \tag{40}$$

where $\varepsilon > 0$ is a small parameter, $x = (x_1,\ldots,x_n)$, $f = (f_1,\ldots,f_n)$ are n vectors and $\omega = \mathrm{diag}(\omega_1,\ldots,\omega_n)$ is a diagonal matrix. We assume that f is an analytic function of $x,\dot{x}$, in a domain containing the solution of the generating system ($\varepsilon = 0$) . We are interested in the periodic solutions of (40) with period $2\pi/\omega_1 + 0(\varepsilon)$ which corresponds to the frequency ω_1. We also assume that $\omega_i \neq \kappa\omega_1$, κ is any integer, $i = 2,\ldots,n$.

According to Poincaré's method we seek the solution in the form

$$x(\tau) = x^0(\tau) + \varepsilon x^{(1)}(\tau) + \varepsilon^2 x^{(2)}(\tau) + \ldots \ , \tag{41}$$

where $x^{(s)} = (x_1^{(s)}, \ldots, x_n^{(s)})$ are 2π periodic in τ and

$$t = \frac{\tau}{\omega_1}(1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) \ . \tag{42}$$

Using (41) and (42) in (40) and equating the various powers of ε , we have equations for $x^{(s)}(\tau)$, $s = 0,1,\ldots$.

We seek the solutions of these equations under the conditions

$$\frac{dx_1^{(s)}(0)}{d\tau} = 0 , \qquad s = 0,1,\ldots . \tag{43}$$

The 2π-periodic solutions of the generating system $\omega_1^2 \frac{d^2x^{(0)}}{d\tau^2} + \omega^2 x^0 = 0$ using (43) are

$$x_1^{(0)} = M \cos \tau , \quad x_k^{(0)} = 0 , \quad k = 2,\ldots,n . \tag{44}$$

The amplitude M and the constant h_1 are determined by (9) and (10) but instead of f_0 we have f_{10}, the first component of the vector function $f_0 = f(x^{(0)}, \dot{x}^{(0)})$. Analogous formulas to (11) and (12) but in vector form are found for $x^{(1)}(\tau)$ and x.

Let us now apply the K-B method to find the periodic solutions of (40) which correspond to the frequency ω_1. Here we seek the solution in the form

$$x = u^{(0)}(a,\psi) + \varepsilon u^{(1)}(a,\psi) + \varepsilon^2 u^{(2)}(a,\psi) + \ldots , \tag{45}$$

where $u^{(s)}(a,\psi) = (u_1^{(s)},\ldots,u_n^{(s)})$, $s = 0,1,\ldots$, are 2π periodic in ψ, $u_1^{(0)}(a,\psi) = a \cos \psi$, $u_k^{(0)}(a,\psi) = 0$, $k = 2,\ldots,n$,

$$\dot{a} = 0 \ , \quad \dot{\psi} = \omega_1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots \ . \tag{46}$$

From (46) we have

$$a = a_0 = \text{const.} \ ,$$

$$\psi = [\omega_1 + \varepsilon B_1(a_0) + \varepsilon^2 B_2(a_0) + \ldots] \, t \ , \quad \text{if } \psi(0) = 0 \ , \tag{47}$$

where a_0 and $\{B_s\}$ are to be found.

The solution (45) can be written in the form (41), where $a_0 = M$. As in section 2 one can see that τ given by (42) and ψ by (47) are the same; i.e. $\tau = \psi$, and we have the same relations between $\{B_s\}$ and $\{h_s\}$, except that ω is replaced by ω_1 .

Hence, in the case of periodic solutions, the solution (45) for the K-B method reduces to the form (41) and the results obtained are the same.

5. Systems of Non-autonomous Differential Equations.

Consider the system of non-autonomous differential equations

$$\ddot{x} + \omega^2 x = \varepsilon f(\theta, x, \dot{x}) , \qquad \theta = \nu t , \qquad (48)$$

where $x = (x_1, \ldots, x_n)$ and $f = (f_1, \ldots, f_n)$ are n-vectors, $\omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$. The function f is 2π periodic in θ of the form $f = \sum_{n=-N}^{N} e^{in\theta} \Phi_n$ where Φ_n is polynomial in $x, \dot{x}$.

Let us apply Poincaré's method to find 2π periodic solution in θ of the equation (48) in the non-resonance case. We can write the solution in the form

$$x(t) = x^{0}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t) + \ldots , \qquad (49)$$

where $x^{(s)}(t)$, $s = 1,2,\ldots$ are n-vectors.

As in section 3 we get $x^{(0)}(t) = 0$ and $x^{(1)}(t)$ is to be determined from the equation
$\ddot{x}^{(1)} + \omega^2 x^{(1)} = f(\theta, 0, 0)$.

According to K-B method [4], where more general systems are considered, it is supposed that: (a) the generating system has vibrations with frequency ω_1 of

the form $x_1 = a_0 \cos(\omega_1 t + \psi_0)$, $x_k = 0$, $k = 2,\ldots,n$, which depend on two arbitrary constants a_0 and ψ_0 .

(b) The set of values $\{\kappa\omega_1 , \kappa = 1,2\ldots\}$ never equal any of the frequencies $\omega_2,\ldots\omega_n$.

Consider the nonresonance case $\nu \neq \frac{p}{q}\omega_1$ (p,q are integers). Under these assumptions the system (48) has an asymptotic solution of the form

$$x = u^{(0)}(a,\psi,\theta) + \varepsilon u^{(1)}(a,\psi,\theta) + \varepsilon^2 \ldots , \qquad (50)$$

$$u_1^{(0)}(a,\psi,\theta) = a\cos\psi, \quad u_k^{(0)}(a,\psi,\theta) = 0, \quad k = 2,\ldots,n ,$$

where $u^{(s)}(a,\psi,\theta) = (u_1^{(s)},\ldots,u_n^{(s)})$ are 2π periodic in ψ and θ , $s = 1,2,\ldots$, and a and ψ are given by (14) . By the integration of the equations (14) one introduces two constants. Hence (50) is not the general solution of (48) but only a two-parameter solution which corresponds to the frequency ω_1 .

To find the 2π periodic solution in θ we set, as in section 3, $a = 0$, $\psi = 0$. Then $u^{(0)} = 0$ and (50) reduces to

$$x = \varepsilon u^{(1)}(0,0,\theta) + \varepsilon^2 u^{(2)}(0,0,\theta) + \ldots ,$$

which coincides with the solution (49) .

Here again we have the same result as in sections 2, 3 and 4.

A comparison in the main resonance case $\nu = \omega_1$ has also been made and the result obtained is the same.

An identical result holds for the more general system

$$A\ddot{x} + Bx = \varepsilon f(\theta, x, \dot{x}, \varepsilon) , \quad \theta = \nu t ,$$

where x and f are vectors, A and B are n×n constant matrices.

6. SECOND ORDER AUTONOMOUS PARTIAL DEFFERENTIAL EQUATION.

Consider the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}[K(x)\frac{\partial u}{\partial x}] - u + \varepsilon F(x,u,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \varepsilon) \qquad (51)$$

where ε is a small parameter. We suppose that $u(x,t)$ must satisfy a pair of linear boundary conditions $B_j(u) = 0$, $(j = 1,2)$ which involve the values of u and u_x at the end points $x = 0$ and $x = \ell$.

First of all consider the generating equation

$$\rho(x)\frac{\partial^2 u_0}{\partial t^2} = \frac{\partial}{\partial x}[K(x)\frac{\partial u_0}{\partial x}] - u_0 \quad , \quad B_j(u_0) = 0, \ j = 1,2 \qquad (52)$$

This has a complete set of separable solutions of the form,

$$a_n\phi_n(x)\cos(\omega_n t + \alpha_n) \quad , \quad n = 1,2,... \qquad (53)$$

where a_n and α_n are constants.

The set of functions $\{\phi_n(x)\}$ satisfy the differential equations

$$\frac{d}{dx}[K(x)\phi_n'(x)] + \omega_n^2\phi_n(x)\rho(x) - \phi_n(x) = 0 \qquad (54)$$

and the boundary conditions $B_j(\phi_n) = 0$, $j = 1,2$.

These boundary conditions determine the allowed set of eigen-frequencies $\{\omega_n\}$. If the boundary conditions satisfy the usual self-adjointness condition of S-L theory, then the set of functions $\{\phi_n(x)\}$ are complete and orthogonal w.r.t. the weight function $\rho(x)$. By suitably normalizing the $\phi_n(x)$, we can thus achieve the result

$$\int_0^{\ell} \rho(x)\phi_n(x)\phi_m(x)dx = \delta_{nm} \tag{55}$$

where δ_{nm} is Kroneker delta.

The solution (53) of the generating system (52) is periodic with period $2\pi/\omega_n$ for $n = 1,2,3,\ldots$.

We shall find the periodic solution of (51) corresponding to the frequency ω_1 of the generating system, i.e. corresponding to

$$u_0(x,t) = A \cos(\omega_1 t + \alpha)\, \phi_1(x)$$

where $A = a_1$, $\alpha = \alpha_1 = \text{constant}$.

We require this solution to satisfy

$$u(x,\ t + 2\pi/\omega_1) = u(x,t) \tag{56}$$

We shall also assume that ω_n is not an integral multiple of ω_1 for $n \neq 1$.

According to Poincaré's method, we make the following transformation

$$t = \frac{\tau}{\omega_1} (1 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots.) \tag{57}$$

and let

$$u(x,\tau) = u_0(x,\tau) + \sum_{r=1}^{\infty} \varepsilon^r u_r(x,\tau) \tag{58}$$

where now in view of (56) and (57), $u(x,\tau)$ or $u_r(x,\tau)$, $r = 0,1,2,\ldots$ are 2π periodic in τ.

Because of the time translation arbitrariness in the autonomous system, we select the origin of τ such that

$$\int_0^{\ell} \rho(x) \frac{\partial u}{\partial \tau} (x,0) \phi_1(x) dx = 0$$

or

$$\int_0^{\ell} \rho(x) \frac{\partial u_r}{\partial \tau} (x,0) \phi_1(x) dx = 0, \quad r = 0,1,2,\ldots \tag{59}$$

This is the corresponding condition to (4) in case of ordinary autonomous differential equation.

The function $u_0(x,\tau)$ is the solution of the generating equation

$$\rho(x)\omega_1^2 u_{0\tau\tau} = \frac{\partial}{\partial x}[K(x)u_{0x}] - u_0$$

Taking into account (59), the solution is

$$u_0(x,\tau) = M\,\phi_1(x)\cos\tau \tag{60}$$

where $\phi_1(x)$ satisfies (54) for $n = 1$.

Substituting (58) into (51), making use of (57) and equating coefficients of $\varepsilon, \varepsilon^2, \ldots$, we have for $u_1, u_2, \ldots$ the equations:

$$\rho(x)\omega_1^2 u_{1\tau\tau} - \frac{\partial}{\partial x}[K(x)u_{1x}] + u_1$$

$$= 2h_1 M\left[\frac{\partial}{\partial x}(K(x)\phi_1'(x)) - \phi_1(x)\right]\cos\tau$$

$$+ F_0(x,M,\tau) \tag{61}$$

$$\rho(x)\omega_1^2 u_{2\tau\tau} - \frac{\partial}{\partial x}\,[K(x)u_{2x}] + u_2$$

$$= 2h_1\,[\frac{\partial}{\partial x}\,(K(x)u_{1x}) - u_1] + (h_1^2 + 2h_1)[\frac{\partial}{\partial x}(K(x)u_{0x}) - u_0]$$

$$+ 2h_1 F_0(x,M,\tau) + u_1 \frac{\partial F_0}{\partial u_0}\,(x,M,\tau) + u_{1x}\frac{\partial F_0}{\partial u_{0x}}$$

$$+ (u_{1\tau} - h_1 u_{0\tau})\frac{\partial F_0}{\partial u_{0\tau}} \qquad (62)$$

where

$$F_0(x,M,\tau) = F(x,M\phi_1(x)\cos\tau,\, M\phi_1'(x)\cos\tau,\, -\omega_1 M\phi_1(x)\sin\tau)$$

To solve (61), we expand $u_1(x,\tau)$ in terms of known eigenfunctions $\{\phi_n(x)\}$ given by (54):

$$u_1(x,\tau) = \sum_{m=1}^{\infty} \phi_m(x)\, V_m(\tau) \qquad (63)$$

In view of (55) and (59), (63) implies

$$V_1'(0) = 0\,. \qquad (64)$$

Using (63) in (61), multiplying both sides by $\phi_n(x)$, integrating from 0 to ℓ and using (55), we get

$$\omega_1^2 V_n''(\tau) + \omega_n^2 V_n(\tau) = -2h_1 M\delta_{1n}\cos\tau + F_n(M,\tau) \qquad (65)$$

where

$$F_n(M,\tau) = \int_0^{\ell} F_0(x,M,\tau)\phi_n(x)dx \ . \tag{66}$$

Since $F_n(M,\tau)$ are 2π periodic in τ, we can expand them as Fourier series

$$F_n(M,\tau) = \frac{P_0^{(n)}(M)}{2} + \sum_{r=1}^{\infty} \{P_r^{(n)}(M) \cos r\tau + Q_r^{(n)}(M) \sin r\tau\} \tag{67}$$

Substituting (67) in (65) and taking n = 1, equating the coefficients of sin τ and cos τ to zero to get rid of the secular terms, we obtain the amplitude equation:

$$Q_1^{(1)}(M) = 0 \quad \text{or} \quad \int_0^{2\pi} F_1(M,\tau) \sin \tau \, d\tau = 0 \tag{68}$$

and an equation for h_1 :

$$-2Mh_1 + P_1^{(1)}(M) = 0 \quad \text{or} \quad h_1 = \frac{1}{4\pi M} \int_0^{2\pi} F_1(M,\tau) \cos \tau \, d\tau \tag{69}$$

Equation (68) determines the amplitude M. The solution for $V_1(\tau)$ under the condition (64) is

$$V_1(\tau) = M_1 \cos \tau + \frac{P_0^{(1)}}{2\omega_1^2} + \sum_{r=2}^{\infty} \frac{P_r^{(1)} \cos r\tau + Q_r^{(1)} \sin r\tau - rQ_r^{(1)} \sin \tau}{\omega_1^2(1 - r^2)}$$

and for $n \neq 1$,

$$V_n(\tau) = \frac{P_0^{(n)}}{2\omega_n^2} + \sum_{r=1}^{\infty} \frac{P_r^{(n)} \cos r\tau + Q_r^{(n)} \sin r\tau}{\omega_n^2 - r^2\omega_1^2}$$

Thus from (63), the solution for $u_1(x,\tau)$ is

$$u_1(x,\tau) = \phi_1(x) \left[M_1 \cos \tau + \frac{P_0}{2\omega_1^2} + \sum_{r=2}^{\infty} \frac{P_r^{(1)} \cos r\tau + Q_r^{(1)} \sin r\tau - rQ_r^{(1)} \sin \tau}{\omega_1^2(1 - r^2)} \right]$$

$$+ \sum_{n=2}^{\infty} \phi_n(x) \left[\frac{P_0^{(n)}}{2\omega_n^2} + \sum_{r=1}^{\infty} \frac{P_r^{(n)} \cos r\tau + Q_r^{(n)} \sin r\tau}{\omega_n^2 - r^2\omega_1^2} \right] \tag{70}$$

This solution has one unknown constant M_1 which is determined from the solution of u_2.

Now let us apply K-B-M method to find a mono-frequent solution of equation (51) corresponding to the frequency ω_1. The solution is sought in the form

$$u(x,t) = a\phi_1(x) \cos \psi + \sum_{r=1}^{\infty} \varepsilon^r v_r(x,a,\psi) \tag{71}$$

where $\{u_r(x,a,\psi)\}$ are 2π periodic in ψ and a,ψ are given by the differential equations

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots$$

$$\dot{\psi} = \omega_1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots\ldots \qquad (72)$$

The functions $\{v_r(x,a,\psi), A_r(a), B_r(a)\}$ are to be found from the requirement that u satisfies (51) to each order of ε. For example, the formulas for $A_1(a)$ and $B_1(a)$ are [10]

$$A_1(a) = -\frac{1}{4\pi\omega_1}\int_0^{2\pi} F_1(a,\psi)\sin\psi\, d\psi$$

$$B_1(a) = -\frac{1}{4\pi a\omega_1}\int_0^{2\pi} F_1(a,\psi)\cos\psi\, d\psi \qquad (73)$$

where

$$F_n(a,\psi) = \int_0^{\ell} F_0(x,a,\psi)\phi_n(x)\, dx$$

and

$$F_0(x,a,\psi) = F(x, a\phi_1(x)\cos\psi, a\phi_1'(x)\cos\psi, - \omega_1 a\phi_1'(x)\sin\psi)$$

To find the periodic solution of (51), one sets

$\dot{a} = 0$, $\dot{\psi}$ = constant $= \omega_1 + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots$

where $\{B_r\}$ are the constants to be determined.

These equations imply that

$$a = a_0 = \text{constant},$$

$$\psi = (\omega_1 + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots.)t \quad \text{if} \quad \psi(0) = 0 \ldots \tag{74}$$

From (72) and (73), $\dot{a} = 0$ implies $A_1(a) = 0$ or

$$\int_0^{2\pi} F_1(a,\psi) \sin \psi \, d\psi = 0 \, . \tag{75}$$

which determines the amplitude a. The equation (75) is identical with (68) which shows that a = M, provided we consider the corresponding roots in the two equations. The second equation (73) shows that $B_1 = - h_1/\omega_1$ and hence to first order, $\psi = \tau$.

The solution (71) can be written in the form (58), where

$$u_0(a,\psi,x) = a\phi_1(x) \cos \psi, \quad u_r(x,a,\psi) = v_r(x,a,\psi), \quad r = 1,2,\ldots$$

As in section 2, we have $\tau = \psi$ and

$$B_1 = - h_1/\omega_1 \, , \quad B_2 = \frac{1}{\omega_1} (h_1^2 - h_2)$$

If the periodic solution is assumed to be unique, the two solutions (58) and (71) must be same term by term, provided that the same initial conditions (59) are used for both methods. Of course we compare the n^{th} approximation of Poincaré's method with the improved n^{th} approximation of the K-B-M method.

7. SECOND ORDER NON-AUTONOMOUS PARTIAL DIFFERENTIAL EQUATION.

Consider the non-autonomous partial differential equation

$$\rho(x)u_{tt} = \frac{\partial}{\partial x}[K(x)u_x] - u + \varepsilon F(\theta,x,u,u_x,u_t,\varepsilon), \quad \theta = \nu t \tag{76}$$

under the same boundary conditions (52). F is supposed to have period $2\pi/\nu$ in t, i.e. 2π periodic in θ and to be of the form

$$F(\theta,x,u,u_x,u_t,\varepsilon) = \sum_{n=-N}^{N} e^{in\theta} F_n(x,u,u_x,u_t,\varepsilon)$$

The coefficients $F_n(x,u,u_x,u_t,\varepsilon)$ in this finite sum are required to be polynomial in their arguments.

Here two cases arise.

(a) NON-RESONANCE CASE.

Assume that for all integers m and n,

$$n\nu + m\omega_r \neq \omega_r \quad \text{or} \quad \omega_r \neq (p/q)\nu \quad (p,q \text{ any integers})$$

for any integer r.

The generating equation of (76) is (52) and the formulas (53)-(55) still hold.

Let us apply Poincaré's method for finding the periodic solutions with period $2\pi/\nu$ in t (or 2π periodic in $\theta = \nu t$) of the differential equation (76). We can write the solution as a series

$$u(x,t) = \sum_{r=0}^{\infty} \varepsilon^r u_r(x,t) \tag{77}$$

where the functions $\{u_r(x,t)\}$ are 2π periodic in νt.

After substituting (77) into (76), we get $u_0(x,t) = 0$. $u_1(x,t)$ is to be found from

$$\rho(x)u_{1tt} = \frac{\partial}{\partial x}[K(x)u_{1x}] - u_1 + F(\theta,x,0,0,0,0) \tag{78}$$

To solve this, we expand $u_1(x,t)$ as

$$u_1(x,t) = \sum_{r=1}^{\infty} \phi_r(x)V_r(\theta) ,$$

substitute in (78), multiply by $\phi_n(x)$, integrate w.r.t. x from 0 to ℓ, use (54) and (55) to get

$$\nu^2 V_n''(\theta) + \omega_n^2 V_n(\theta) = F_n(\theta)$$

where

$$F_n(\theta) = \int_0^{\ell} F(\theta,x,0,0,0,0)\phi_n(x)dx .$$

Since $F_n(\theta)$ is 2π periodic in θ, we can expand it in Fourier series and get a 2π periodic solution in θ for V_n and thus for u_1 as

$$u_1(x,t) = \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \left[\frac{\phi_n(x) \cos r\theta}{\omega_n^2 - \nu^2 r^2} \int_0^{2\pi} F_n(\theta) \cos r\theta \, d\theta \right.$$

$$\left. + \frac{\sin r\theta \, \phi_n(x)}{\omega_n^2 - \nu^2 r^2} \int_0^{2\pi} F_n(\theta) \sin r\theta \, d\theta \right] \quad (79)$$

According to the K-B-M method [10] the solution of (76) is sought in the form

$$u(x,t) = \sum_n a_n \phi_n(x) \cos \psi_n + \varepsilon v_1(x,\{a_n\},\{\psi_n\},\theta)$$

$$+ \varepsilon^2 v_2(x,\{a_n\},\{\psi_n\},\theta) + \ldots. \quad (80)$$

where $\{v_r\}$ are 2π periodic in ψ_n $(n=1,2,\ldots)$ and also 2π periodic in θ .

$\{a_n\}$ and $\{\psi_n\}$ satisfy the differential equations

$$\dot{a}_n = \sum_{r=1}^{\infty} \varepsilon^r A_r^{(n)}(\{a_p\})$$

$$\dot{\psi}_n = \omega_n + \sum_{r=1}^{\infty} \varepsilon^r B_r^{(n)}(\{a_p\}) \quad (81)$$

Substituting (80) into (76), using (81) and equating coefficients of ε, ε^2, ... enables us to find

$$v_r(x,\{a_n\},\{\psi_n\},\theta),\ A_r(\{a_n\}),\ B_r(\{a_n\}),\quad r = 1,2,...$$

and therefore the solution of (76) in the non-stationary regime.

To find the periodic solution of (76) with period $2\pi/\nu$ in t, we set $a_n = 0$, $n = 1,2,...$ in (80) and assume that $\{v_r\}$ are independent of $\{\psi_n\}$. This means that $\{v_r\}$ depend on x and θ only and equations (81) no longer appear. Then (80) becomes

$$u(x,t) = \varepsilon v_1(x,0,0,\theta) + \varepsilon^2 v_2(x,0,0,\theta) + ... \qquad (82)$$

If we let $v_r(x,0,0,\theta) = u_r(x,t)$, $r = 1,2,....$

then the sought solution (82) coincides with (77).

The conclusion is that the periodic solution found by both methods are the same in every approximation. Of course, we compare the n^{th} approximation of Poincaré's method with the improved n^{th} approximation in the K-B-M method.

(b) RESONANCE CASE.

Let us now assume that there are some pairs of integers m and n such that there is one frequency, call it ω_1 for which

$$n\nu + m\omega_1 \simeq \omega_1 \quad \text{or} \quad \omega_1 \simeq \frac{p}{q}\nu \quad (p,q \text{ integers}).$$

To simplify the considerations, let us consider the case of exact main resonance, $\omega_1 = \nu$, i.e. $p = q = 1$.

According to Poincaré' method, we seek the periodic solutions of (76) in the form (77). The solution of the generating equation

$$\rho(x)u_{0tt} = \frac{\partial}{\partial x}[K(x)u_{0x}] - u_0 \qquad (82')$$

which is 2π periodic in νt or $\omega_1 t$ is

$$u_0 = a\phi_1(x)\cos(\omega_1 t + \alpha)$$

or $$u_0 = (M\cos\nu t + N\sin\nu t)\phi_1(x) \qquad (83)$$

where M and N are constants to be determined and $\phi_1(x)$ satisfies (54).

For the function $u_1(x,t)$, we get the equation

$$\rho(x)u_{1tt} - \frac{\partial}{\partial x}[K(x)u_{1x}] + u_1$$

$$= F(\theta, x, \phi_1'(x)(M \cos \nu t + N \sin \nu t),$$

$$\phi_1(x)(- M\nu \sin \nu t + N\nu \cos \nu t), 0). \tag{84}$$

To solve this, we expand u_1 in terms of $\{\phi_n(x)\}$ as

$$u_1(x,t) = \sum_{r=1}^{\infty} \phi_r(x) V_r(\theta) \tag{85}$$

Substituting (85) in (84), using (54), multiplying by $\phi_n(x)$, integrating from 0 to ℓ and using (55), we have

$$\nu^2 V_n''(\theta) + \omega_n^2 V_n(\theta) = F_n(\theta, M, N) \tag{86}$$

where

$$F_n(\theta, M, N) = \int_0^{\ell} F(\theta, x, u_{0x}, u_{0t}, 0)\phi_n(x)dx$$

To solve (86), we expand $F_n(\theta, M, N)$ in Fourier series:

$$F_n(\theta, M, N) = \sum_r F_r^{(n)}(M, N)e^{ir\theta} \tag{87}$$

where

$$F_r^{(n)}(M, N) = \frac{1}{2\pi} \int_0^{2\pi} F_n(\theta, M, N)e^{-ir\theta}d\theta$$

To avoid secular terms in the solution of (86) (when n=1), we set

$$F_1^{(1)}(M,N) = 0 , \quad F_{-1}^{(1)}(M,N) = 0 \tag{88}$$

From equations (88), called the amplitude equations, we determine M and N. The periodic solution of (86) then is

$$V_1(\theta) = M^{(1)} \cos \nu t + N^{(1)} \sin \nu t + \sum_{r \neq \pm 1} \frac{e^{ir\theta}}{2\pi\nu^2(1-r^2)} \int_0^{2\pi} F_1(\theta,M,N) e^{-ir\theta} d\theta$$

and

$$V_n(\theta) = M^{(n)} \cos \nu t + N^{(n)} \sin \nu t + \sum_r \frac{e^{ir\theta}}{2\pi(\omega_n^2 - \nu^2 r^2)} \int_0^{2\pi} F_n(\theta,M,N) e^{-ir\theta} \quad (n \neq 1) \tag{89}$$

where $M^{(n)}$, $N^{(n)}$, $n=1,2,\ldots$ are to be determined under the condition that the function $u_2(x,t)$ from the second approximation should not contain secular terms.

Substituting (89) into (85), $u_1(x,t)$ is known.

Now let us apply the K-B-M method. The solution of (76) in the non-stationary regime is sought in the form

$$u(x,t) = \phi_1(x)a\cos\psi + \varepsilon v_1(x,a,\psi,\theta) + \varepsilon^2 v_2(x,a,\psi,\theta) + \ldots. \tag{90}$$

where $v_r(x,a,\psi,\theta)$ are 2π periodic in ψ and θ. The functions a and ψ are now given by the equations

$$\dot{a} = \sum_{r=1}^{\infty} \varepsilon^r A_r(a,\phi)$$

$$\dot{\phi} = \omega_1 - p/q\ \nu + \varepsilon B_1(a,\phi) + \varepsilon^2 \ldots. \quad ; \quad \psi = \frac{p}{q}\theta + \phi \tag{91}$$

where $\{A_r(a,\phi),\ B_r(a,\phi)\}$ are 2π periodic in ϕ .

According to the K-B-M method [10], $A_1(a,\phi)$ and $B_1(a,\phi)$ are determined from the equation

$$[(\omega_1 - p/q\ \nu)\frac{\partial A_1}{\partial \phi} - 2a\omega_1 B_1]\cos\psi - [a(\omega_1 - p/q\ \nu)\frac{\partial B_1}{\partial \phi} + 2\omega_1 A_1]\sin\psi$$

$$= \sum_{(m\pm 1)p+nq=0} F_{nm}^{(1)}(a)\, e^{i(n\theta+m\psi)} \tag{92}$$

where

$$F_{nm}^{(r)}(a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_r(a,\psi,\theta)e^{-i(n\theta+m\psi)}d\theta d\psi \tag{93}$$

and

$$F_r(a,\psi,\theta) = \int_0^{\ell} F(\theta, x, \phi_1(x)a\cos\psi, \phi_1'(x)a\cos\psi, - \phi_1(x)a\omega_1\sin\psi, 0)\phi_r(x)dx \tag{94}$$

In the case of the main resonance, p=q=1 .

To find $2\pi/\nu$ periodic solutions in t of (76), we set

$\dot{a} = 0$, $\dot{\phi} = 0$, i.e. $a = a_0 = \text{constant}$, $\phi = \phi_0 = \text{const.}$

This implies $A_r(a_0,\phi_0) = 0$, $B_r(a_0,\phi_0) = 0$, r=1,2,.... and we do not require equations (91). Then (90) becomes

$$u(x,t) = \phi_1(x)a_0\cos(\theta+\phi_0) + \varepsilon v_1(x,a_0,\theta+\phi_0,\theta) + \varepsilon^2... \tag{95}$$

The term $v_0 = \phi_1(x)a_0\cos(\theta+\phi_0)$ is the solution of

$$\rho(x)\nu^2 v_{0\theta\theta} = \frac{\partial}{\partial x}[K(x)v_{0x}] - v_0 \tag{96}$$

The equations (96) and (82') are the same, hence their solutions v_0 and u_0 coincide, i.e.

$$M\cos\nu t + N\sin\nu t = a_0\cos(\theta+\phi_0) , \qquad \theta = \nu t$$

or $$M = a_0\cos\phi_0 , \qquad N = -a_0\sin\phi_0 \tag{97}$$

Assuming the periodic solution to be unique, the two solutions (85) and (95) coincide term by term in the power series in ε.

It is easy to show that the amplitude equations (88) coincide with the corresponding equations in the K-B-M method. From (92), taking into account $p = q = 1$, $A_1 = B_1 = 0$, we have:

$$\sum_{(m+n)\pm 1=0} F_{nm}^{(1)}(a_0) e^{i(n\theta+m\psi)} = 0$$

or $$\sum_{(m+n)=\pm 1} F_{nm}^{(1)}(a_0) e^{i(m+n)\theta} e^{im\phi_0} = 0$$

or $$\sum_{r=\pm 1} F_r(a_0,\phi_0) e^{ir\theta} = 0 \qquad (98)$$

where

$$F_r(a_0,\phi_0) = \frac{1}{2\pi} \int_0^{2\pi} F_1(a_0,\theta+\phi_0,\theta) e^{-ir\theta} d\theta \qquad (99)$$

The equation (98) when expanded gives

$$F_1^{(1)}(a_0,\phi_0) e^{i\theta} + F_{-1}^{(1)}(a_0,\phi_0) e^{-i\theta} = 0$$

Since $F_1^{(1)}(a_0,\phi_0) = F_{-1}^{(1)}(a_0,\phi_0) = 0$.

These equations coincide with equations (88) in view of (97), (86'), (94) and (99).

Hence the periodic solution found by K-B-M method coincides with the one found by Poincaré's method, provided we consider the improved approximation in the K-B-M method.

CHAPTER 2

RESPONSE OF A NON-LINEAR VIBRATOR UNDER THE INFLUENCE OF A TIME-DEPENDENT EXTERNAL FORCE.

INTRODUCTION.

The asymptotic solutions of the autonomous differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x, dx/dt) \tag{1}$$

where ε is a small parameter have been investigated in detail by Krylov, Bogoliubov and Mitropolskii [3,4].

In this chapter, we study the influence of an external excitation force (periodic or non-periodic) on the oscillatory system. For systems governed by ordinary differential equations, this process can be modeled by an equation of the type

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}) + \varepsilon \Phi(t) \tag{2}$$

where $\Phi(t)$ may or may not be periodic function of time. The case when $\Phi(t)$ is 2π periodic in t has been investigated by Bogoliubov and Mitropolskii [4] in great detail. In fact in [4], a more general case is studied when the

right hand side of equation (2) being a 2π periodic in t is a finite sum of terms of the type $\epsilon F(t,x,\dot{x}) = \epsilon \sum F_n(x,\dot{x})e^{int}$. Here F_n are polynomials in their arguments and clearly $F(t,x,\dot{x})$ is 2π periodic in t.

We place no such restriction on $\Phi(t)$, but assume that it is a well-behaved function for $t > 0$ and bounded as $t \to \infty$.

In section I, we deal with non-linear non-resonant vibrators governed by ordinary differential equations of type (2) and the results are extended when a significant damping force is present.

In section II, the method is extended to vibratory systems governed by partial differential equations of the hyperbolic type.

SECTION I* (SYSTEM GOVERNED BY ORDINARY DIFF. EQUATION)

THE ASYMPTOTIC SOLUTION

When perturbing forces are completely absent ($\varepsilon = 0$) oscillations will evidently be purely harmonic, $x_0 = a\cos\psi$, $dx_0/dt = a\omega\sin\psi$, with a constant amplitude and a uniformly increasing phase angle, i.e. $da/dt = 0$, $d\psi/dt = \omega$.

The solution of (2) according to the K-B-M method is sought in the form

$$x = a\cos\psi + \varepsilon u_1(a,\psi,t) + \varepsilon^2 u_2(a,\psi,t) + \ldots . \qquad (3)$$

Here the functions u_k, $k = 1,2,\ldots$ are supposed 2π periodic in variable ψ (in [4] they are 2π periodic in ψ and t) and a and ψ are determined by the differential equations

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots ,$$

$$\frac{d\psi}{dt} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots . \qquad (4)$$

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*This work has been published in INTERNATIONAL JOURNAL OF CONTROL, APRIL (1975), pp.233-34 (with Bojadziev and Farooqui).

Substituting (3) and (4) in equation (2) and equating the coefficients of various powers of ε, we get partial differential equations for $u_1, u_2, \ldots$ as follows

$$\omega^2\frac{\partial^2 u_1}{\partial\psi^2} + 2\omega\frac{\partial^2 u_1}{\partial\psi\partial t} + \frac{\partial^2 u_1}{\partial t^2} + \omega^2 u_1 = \Phi(t) + f_0(a,\psi)$$

$$+ 2\omega A_1 \sin\psi + 2a\omega B_1 \cos\psi \ , \ f_0(a,\psi) = f(a\cos\psi, a\omega\sin\psi) \ , \tag{5}$$

$$\omega^2\frac{\partial^2 u_2}{\partial\psi^2} + 2\omega\frac{\partial^2 u_2}{\partial\psi\partial t} + \frac{\partial^2 u_2}{\partial t^2} + \omega^2 u_2 = u_1 \frac{\partial f_0}{\partial x_0}$$

$$+ (A_1\cos\psi - aB_1 \sin\psi + \omega\frac{\partial u_1}{\partial\psi} + \frac{\partial u_1}{\partial t})\frac{\partial f_0}{\partial \dot{x}_0}$$

$$+ 2\omega(A_2 \sin\psi + aB_2 \cos\psi)$$

$$+ (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi + (aA_1\frac{dB_1}{da} + 2A_1B_1)\sin\psi$$

$$-2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2} - 2A_1\frac{\partial^2 u_1}{\partial a\partial t} - 2\omega A_1\frac{\partial^2 u_1}{\partial a\partial\psi} - 2B_1\frac{\partial^2 u_1}{\partial\psi\partial t} \ , \tag{6}$$

.....

The equation (5) is a linear partial differential equation with the right hand side as sum of a function of

t alone and a function of ψ alone. This motivates us to seek the solution of (5) in the form

$$u_1(a,\psi,t) = v_1(a,\psi) + w_1(t) \ . \tag{7}$$

Substituting (7) in (5), we find that v_1 and w_1 satisfy the following differential equations

$$\omega^2 \frac{\partial^2 v_1}{\partial \psi^2} + \omega^2 v_1 = f_0(a,\psi) + 2\omega A_1 \sin\psi + 2a\omega B_1 \cos\psi, \tag{8}$$

$$\frac{d^2 w_1}{dt^2} + \omega^2 w_1 = \Phi(t) \ . \tag{9}$$

The equation (8) is exactly the same as in the autonomous case (1), which leads to the solution [3]

$$v_1(a,\psi) = \frac{g_0(a)}{\omega^2} + \frac{1}{\omega^2} \sum_{n=2}^{\infty} \frac{g_n(a)\cos n\psi + h_n(a)\sin n\psi}{1 - n^2} \ , \tag{10}$$

where

$$g_n(a) = \frac{1}{2\pi} \int_0^{2\pi} f_0(a,\psi)\cos n\psi d\psi, \quad h_n(a) = \frac{1}{2\pi} \int_0^{2\pi} f_0(a,\psi)\sin n\psi d\psi$$

and to the following values for A_1 and B_1

$$A_1 = -\frac{1}{2\pi\omega} \int_0^{2\pi} f_0(a,\psi)\sin\psi d\psi \ ,$$

$$B_1 = -\frac{1}{2\pi a\omega} \int_0^{2\pi} f_0(a,\psi)\cos\psi d\psi \ . \tag{11}$$

The solution of equation (9) can be found straight forward.

Thus the solution of equation (2) up to first improved approximation is

$$x = a\cos\psi + \varepsilon[v_1(a,\psi) + w_1(t)] , \qquad (12)$$

where $da/dt = \varepsilon A_1$ and $d\psi/dt = \omega + \varepsilon B_1$.

Consider now the equation (6). Its right hand side consists of a function of ψ alone, a function of t alone and also the terms which are products of trigonometric functions of ψ and functions of t. This leads us to assume a solution of equation (6) of the form

$$u_2(a,\psi,t) = v_2(a,\psi) + w_2(t) + \sum_n [P_n(a)\phi_n(t)\cos n\psi + Q_n(a)\psi_n(t)\sin n\psi], \qquad (13)$$

where the nature of $\phi_n(t)$ and $\psi_n(t)$ will depend on the nature of the given function $\Phi(t)$, and the number of terms in the summation are finite.

Substituting of (13) into (6) leads to differential equations for $v_2(a,\psi)$ and $w_2(t)$ and algebraic equations for $P_n(a)$ and $Q_n(a)$. The solution $v_2(a,\psi)$ coincides with

the solution of the autonomous case [4] and the same is true for $A_2(a)$ and $B_2(a)$.

Knowing $u_2(a,\psi,t)$, A_2 and B_2, the solution of (2) up to the second improved approximation is

$$x = a\cos\psi + \varepsilon u_1(a,\psi,t) + \varepsilon^2 u_2(a,\psi,t), \tag{14}$$

where

$$da/dt = \varepsilon A_1 + \varepsilon^2 A_2 , \quad d\psi/dt = \omega + \varepsilon B_1 + \varepsilon^2 B_2 . \tag{15}$$

This is illustrated by the following example.

VAN DER POL'S EQUATION

Consider the equation of Van der Pol's type in the case of an external acting nonperiodic force

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon(1-x^2)\frac{dx}{dt} + \varepsilon E e^{-pt}\sin\nu t . \tag{16}$$

Making use of formulas (11), (10) and solving equations (9) we obtain

$$A_1 = \frac{a}{2}\left(1-\frac{a^2}{4}\right) , \quad B_1 = 0 ,$$

$$v_1(a,\psi) = -\frac{a^3}{32\omega}\sin 3\psi , \quad w_1(t) = (\alpha\sin\nu t + \beta\cos\nu t)e^{-pt},$$

where

$$\alpha = \frac{E(p^2+\omega^2-\nu^2)}{(p^2+\omega^2-\nu^2)^2 + 4p^2\nu^2}, \quad \beta = \frac{2p\nu E}{(p^2+\omega^2-\nu^2)^2 + 4p^2\nu^2}. \tag{17}$$

Thus to the first order of ε the solution of (16) is

$$x = a\cos\psi + \varepsilon[-\frac{a^3}{32\omega}\sin 3\psi + (\alpha\sin\nu t + \beta\cos\nu t)e^{-pt}], \tag{18}$$

where a and ψ are determined by

$$\frac{da}{dt} = \varepsilon\frac{a}{2}(1-\frac{a^2}{4}), \quad \frac{d\psi}{dt} = \omega$$

and α and β are given by (17).

Let us find the second approximate solution of (16). The partial differential equation (6) now reduces to

$$\begin{aligned}
&\omega^2\frac{\partial^2 u_2}{\partial\psi^2} + 2\omega\frac{\partial^2 u_2}{\partial\psi\partial t} + \frac{\partial^2 u_2}{\partial t^2} + \omega^2 u_2 \\
&(2a\omega B_2 + \frac{a}{4} - \frac{a^3}{4} + \frac{7a^5}{128})\cos\psi \\
&+ \frac{a^2}{32}(-3 + 5a + \frac{3a^2}{2} - \frac{5a^3}{4})\cos 3\psi \\
&+ \frac{5a^5}{128}\cos 5\psi + 2\omega A_2\sin\psi \\
&- \frac{a^2}{2}\{(\alpha\nu-\beta p)\cos\nu t - (\beta\nu + \alpha p)\sin\nu t\}e^{-pt}\cos 2\psi \\
&+ a^2\omega(\alpha\sin\nu t + \beta\cos\nu t)e^{-pt}\sin 2\psi \\
&+ (1-\frac{a^2}{2})[(\alpha\nu-\beta p)\cos\nu t - (\beta\nu + \alpha p)\sin\nu t]e^{-pt}.
\end{aligned} \tag{19}$$

According to (13) we seek a solution of (19) of the form

$$u_2(a,\psi,t) = v_2(a,\psi) + w_2(t) + (k_1\sin\nu t + k_2\cos\nu t)e^{-pt}\sin 2\psi + (k_3\sin\nu t + k_4\cos\nu t)e^{-pt}\cos 2\psi \ . \qquad (20)$$

Substituting (20) into equation (19) and equating the terms involving ψ alone and t alone and the coefficients of $e^{-pt}\cos\nu t$ and $e^{-pt}\sin\nu t$ gives two differential equations for v_2 and w_2 and a system of four linear algebraic equations for k_i, $i = 1, \ldots, 4$.

Assuming as usual in K-B-M method that $u_2(a,\psi,t)$ does not contain the first harmonics $\sin\psi$ and $\cos\psi$ and solving the equations for v_2 and w_2 we get

$$A_2 = 0 \ , \quad B_2 = \frac{1}{8\omega}\left(-1 + a^2 - \frac{7a^4}{32}\right) \ ,$$

$$v_2(a,\psi) = -\frac{a^3}{256}\left(2-\frac{a^2}{4}\right)\cos 3\psi - \frac{5a^5}{3072\omega^2}\cos 5\psi, \qquad (21)$$

$$w_2(t) = \left(1-\frac{a^2}{2}\right)(P\cos\nu t - Q\sin\nu t)e^{-pt} \ , \qquad (22)$$

where

$$P = \frac{1}{E}[(\alpha^2-\beta^2)\nu - 2\alpha\beta p], \quad Q = \frac{1}{E}[(\alpha^2-\beta^2)p + 2\alpha\beta\nu] \ .$$

From the algebraic system one can obtain easily the constants k_1, k_2, k_3 and k_4 because the determinant of the system is $16[(p^2-\nu^2-3\omega^2)^2 - (4\nu\omega)^2]^2$, hence different from zero.

Thus the second improved approximate solution of (16) is given by (14), where the first two terms present the first approximation (18) and u_2 is given by (20). The functions v_2 and w_2 in (20) are found by (21) and (22). The amplitude a and the phase ψ in (14) are determined by (15), where A_1, B_1, A_2 and B_2 are already obtained.

SIGNIFICANT DAMPING FORCE

The K-B-M method was developed by Popov [7] for the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + c^2x = \varepsilon f(x, \frac{dx}{dt}), \quad c^2 > b^2 , \qquad (23)$$

which models the motion of a material system with one degree of freedom with large damping. Equations of that type are important in the theory of automatic control.

Here we extend the results discussed above for the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + c^2x = \varepsilon f(x,\frac{dx}{dt}) + \varepsilon\Phi(t) \ , \qquad (24)$$

where $\Phi(t)$ is the same function as introduced in equation (2) and $b > 0$.

For $\varepsilon = 0$ the unperturbed equation of (24) has decaying solution of the type $x = a\cos\psi$, where $da/dt = -ba$ and $d\psi/dt = \omega$.

According to [7] the solution of equation (23) is sought in the form (3), but in this case a and ψ as functions of t are given by the differential equations

$$\frac{da}{dt} = -ba + \varepsilon A_1(a) + \varepsilon^2 \ldots . \qquad (25)$$

$$\frac{d\psi}{dt} = \omega + \varepsilon B_1(a) + \varepsilon^2 \ldots . \qquad (25)$$

The equation for u_1 now is

$$a^2b^2\frac{\partial^2 u_1}{\partial a^2} + \omega^2\frac{\partial^2 u_1}{\partial \psi 2} + \frac{\partial^2 u_1}{\partial t 2} + 2\omega\frac{\partial^2 u_1}{\partial\psi\partial t} - 2ab\omega\frac{\partial^2 u_1}{\partial\psi\partial a} -$$

$$- 2ab\frac{\partial^2 u_1}{\partial t\partial a} - ab^2\frac{\partial u_1}{\partial a} + 2b\omega\frac{\partial u_1}{\partial\psi} + 2b\frac{\partial u_1}{\partial t} + c^2u_1$$

$$= (ab\frac{\partial A_1}{\partial a} - bA_1 + 2a\omega B_1)\cos\psi + (2\omega A_1 - a^2b\frac{\partial B_1}{\partial a})\sin\psi$$

$$+ \bar{f}_0(a,\psi) + \Phi(t) \ , \qquad (26)$$

where $\bar{f}_0(a,\psi) = f(a\cos\psi, -ab\cos\psi - a\omega\sin\psi)$.

We seek a solution of (26) of the type (7). For $v_1(a,\psi)$ and $w_1(t)$ we obtain

$$a^2b^2\frac{\partial^2 v_1}{\partial a^2} + \omega^2\frac{\partial^2 v_1}{\partial \psi^2} - 2ab\omega\frac{\partial^2 v_1}{\partial \psi \partial a} - ab^2\frac{\partial v_1}{\partial a} + 2b\omega\frac{\partial v_1}{\partial \psi} + c^2 v_1 \tag{27}$$

$$= (ab\frac{\partial A_1}{\partial a} - bA_1 + 2a\omega B_1)\cos\psi + (2\omega A_1 - a^2 b\frac{\partial B_1}{\partial a})\sin\psi + \bar{f}_0(a,\psi)$$

$$\frac{d^2 w_1}{dt^2} + 2b\frac{dw_1}{dt} + c^2 w_1 = \Phi(t) . \tag{28}$$

To solve (27), we develop the functions $v_1(a,\psi)$ and $\bar{f}_0(a,\psi)$ in Fourier series in ψ. As usually we require that $v_1(a,\psi)$ does not contain the first harmonics which imply the following differential equations for A_1 and B_1

$$-ab\frac{dA_1}{da} + bA_1 - 2a\omega B_1 = \frac{1}{\pi}\int_0^{2\pi} \bar{f}_0(a,\psi)\cos\psi d\psi$$

$$a^2 b\frac{dB_1}{da} - 2\omega A_1 = \frac{1}{\pi}\int_0^{2\pi} \bar{f}_0(a,\psi)\sin\psi d\psi . \tag{29}$$

Eliminating A_1 from (29) gives for B_1 a second order Euler's differential equation which can be solved

easily. For the Fourier's coefficients of the function $v_1(a,\psi)$ we get second order coupled differential equations also of Euler's type.

The differential equation (28) is linear and can be solved straight forward. Thus up to the first improved approximation the solution of (24) is found. In the process of finding the second approximation a lot of difficulties of numerical character are involved. However usually in applications the first approximation is enough.

Let us illustrate in brief the case of significant damping by the equation of Van der Pol's type

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + c^2x = \varepsilon(1-x^2)\frac{dx}{dt} + \varepsilon E e^{-pt}\sin\nu t , \qquad (30)$$

For the functions $\bar{f}_0(a,\psi)$ and $\Phi(t)$ we have now

$$\bar{f}_0(a,\psi) = -a(1-\frac{a^2}{2})(b\cos\psi + \omega\sin\psi)$$

$$+ \frac{a^3}{4}\,[b(\cos 3\psi + \cos\psi) + \omega(\sin 3\psi - \sin\psi)] , \qquad (31)$$

$$\Phi(t) = Ee^{-pt}\sin\nu t . \qquad (32)$$

Then the system (29) has the following solution

$$A_1(a) = \frac{a}{2}[1 - \frac{a^2}{4}(1 + \frac{4b^2}{b^2+\omega^2})] ,$$

$$B_1(a) = \frac{b}{2\omega} - \frac{\omega b}{4(b^2+\omega^2)}a^2 , \qquad (33)$$

Substituting (33) in the differential equations

$$\frac{da}{dt} = -ba + \varepsilon A_1(a), \quad \frac{d\psi}{dt} = \omega + \varepsilon B_1(a)$$

we can find a and b. To find the first improved solution $x = a\cos\psi + \varepsilon[v_1(a,\psi) + w_1(t)]$ we need to determine v_1 and w_1 by solving equations of the type (27) and (28) with $\bar{f}_0(a,\psi)$ and $\Phi(t)$ given by (31) and (32).

SECTION II (SYSTEM GOVERNED BY PARTIAL DIFF. EQUATION)

In this case, the vibrating process can be modeled by an equation of the type

$$\rho(x)u_{tt} - \frac{\partial}{\partial x}[k(x)u_x] + u = \varepsilon F(x,u,u_x,u_t) + \varepsilon\Phi(t) , \qquad (34)$$

where ε is a small parameter, F is analytic in its arguments and $\Phi(t)$ as in section I is assumed to be a well behaved function for $t > 0$ and bounded as $t \to \infty$. $\Phi(t)$ may or may not be periodic in t. A more general case of the above equation, when the right side is of the form $F(x,u,u_x,u_t,t)$ where F is 2π periodic in t has been studied by Keller [21], and Bojadziev and Lardner [10].

We also suppose that $u(x,t)$ satisfies a pair of boundary conditions $B_j(u) = 0$, $j = 1,2$, where B_j involves the values of u and u_x at the ends $x = 0$, $x = \ell$.

ASYMPTOTIC SOLUTIONS

For $\varepsilon = 0$, the generating equation of (34) is

$$\rho(x)u_{tt} - \frac{\partial}{\partial x}[k(x)u_x] + u = 0 , \quad B_j(u) = 0 , \quad j = 1,2. \qquad (35)$$

This equation has a complete set of separable solutions of the form $a_n\phi_n(x)\cos(\omega_n t + \alpha_n)$, $n = 1,2,\ldots$, where

a_n and α_n are arbitrary constants. The set of eigen functions $\{\phi_n(x)\}$ satisfy the differential equation

$$\frac{d}{dx}[k(x)\phi_n'(x)] + \omega_n^2\rho(x)\phi_n(x) - \phi_n(x) = 0 \qquad (36)$$

and the boundary conditions $B_j(\phi_n) = 0$, which determine the allowed set of eigen frequencies $\{\omega_n\}$. By suitably normalizing the $\{\phi_n(x)\}$, we can write

$$\int_0^\ell \rho(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}, \qquad (37)$$

where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$.

We wish to find the mono-frequent solution of (34) which corresponds to the frequency ω_1.

According to K-B-M method and [10], we seek the asymptotic solutions of (34) in the form

$$u(x,t) = a\phi_1(x)\cos\psi + \varepsilon u_1(x,a,\psi,t) + \varepsilon^2(x,a,\psi,t) + \ldots \qquad (38)$$

where u_r $(r=1,2,\ldots)$ are 2π periodic in ψ (in [10], they are 2π periodic in ψ and t). The functions a and ψ satisfy the differential equation

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots.$$

$$\dot{\psi} = \omega_1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots. \tag{39}$$

Here A_r , B_r , u_r are to be determined from the requirement that u satisfies (34) to each order of ε.

Substituting (38) into (34), making use of (39) and comparing coefficients of ε, ε^2 on both sides, we have:

$$\rho(x)\left[\omega_1^2 \frac{\partial^2 u_1}{\partial \psi^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial \psi \partial t} + \frac{\partial^2 u_1}{\partial t^2}\right] - \frac{\partial}{\partial x}\left[k(x)\frac{\partial u_1}{\partial x}\right] + u_1$$

$$= F_0(a,\psi,x) + 2\omega_1\phi_1(A_1\sin\psi + aB_1\cos\psi) + \Phi(t) \tag{40}$$

$$\rho(x)\left[\omega_1^2 \frac{\partial^2 u_2}{\partial \psi^2} + 2\omega_1 \frac{\partial^2 u_2}{\partial \psi \partial t} + \frac{\partial^2 u_2}{\partial t^2}\right] - \frac{\partial}{\partial x}\left[k(x)\frac{\partial u_2}{\partial x}\right] + u_2$$

$$= u_1 \frac{\partial F_0}{\partial u_0} + \frac{\partial u_1}{\partial x}\frac{\partial F_0}{\partial u_{0,x}} + \left[\omega_1 \frac{\partial u_1}{\partial \psi} + \frac{\partial u_1}{\partial t} + \phi_1(A_1\cos\psi - aB_1\sin\psi)\right]\frac{\partial F_0}{\partial u_{0,t}} + \phi_1\Big((2\omega_1 A_2 + 2A_1B_1 + aA_1B_1')\sin\psi$$

$$+ (2a\omega_1 B_2 + aB_1^2 - A_1A_1')\cos\psi\Big) - 2A_1\left(\omega_1 \frac{\partial^2 u_1}{\partial a \partial \psi} + \frac{\partial^2 u_1}{\partial a \partial t}\right)$$

$$- 2B_1\left(\omega_1 \frac{\partial^2 u_1}{\partial \psi^2} + \frac{\partial^2 u_1}{\partial t \partial \psi}\right) \tag{41}$$

Here $F_0(a,\psi,x) = F(x, a\phi_1\cos\psi, a\phi_1'\cos\psi, -a\omega_1\phi_1\sin\psi)$.

Let us first consider equation (40). The function $u_1(x,a,\psi,t)$ may be expanded in terms of the eigenfunctions $\{\phi_n(x)\}$ as

$$u_1(x,a,\psi,t) = \sum_{k=1}^{\infty} V_k(a,\psi,t)\phi_k(x) \qquad (42)$$

Substituting (42) into (40), multiplying both sides by $\phi_r(x)$, integrating from 0 to ℓ and using (36) and (37), gives

$$\omega_1^2 \frac{\partial^2 V_r}{\partial\psi^2} + 2\omega_1 \frac{\partial^2 V_r}{\partial\psi\partial t} + \frac{\partial^2 V_r}{\partial t^2} + \omega_1^2 V_r = F_r + c_r\Phi(t)$$

$$+ 2\omega_1\delta_{1r}(A_1\sin\psi + aB_1\cos\psi) \qquad (43)$$

where

$$c_r = \int_0^{\ell} \phi_r(x)dx$$

$$F_r(a,\psi) = \int_0^{\ell} F_0(x,a,\psi)\phi_r(x)dx$$

The equation (43) is a linear partial differential equation with the right hand side as a sum of a function of (a,ψ) and a function of t alone. This motivates us

to seek the solution of (43) in the form

$$V_r(a,\psi,t) = P_r(a,\psi) + Q_r(t) \tag{44}$$

Using this in (43), we obtain

$$\omega_1^2 \frac{\partial^2 P_r}{\partial \psi^2} + \omega_r^2 P_r = F_r(a,\psi) + 2\omega_1 \delta_{1r}(A_1 \sin\psi + aB_1 \cos\psi) \tag{45}$$

and

$$Q_r'' + \omega_r^2 Q_r = c_r \Phi(t) \ . \tag{46}$$

The solution of (45) can be found by expanding P_r and F_r in Fourier series:

$$\begin{aligned} P_r(a,\psi) &= \tfrac{1}{2}A_{0r}(a) + \sum_{n=1}^{\infty} [A_{nr}(a)\cos n\psi + B_{nr}(a)\sin n\psi] \\ F_r(a,\psi) &= \tfrac{1}{2}P_{0r}(a) + \sum_{n=1}^{\infty} [P_{nr}(a)\cos n\psi + B_{nr}(a)\sin n\psi] \end{aligned} \tag{47}$$

Substituting (47) in (45), we get

$$\tfrac{1}{2}\omega_r^2 A_{0r} + \sum_{n=1}^{\infty} (\omega_r^2 - n^2\omega_1^2)(A_{nr}\cos n\psi + B_{nr}\sin n\psi)$$

$$= \tfrac{1}{2} P_{0r} + \sum_{n=1}^{\infty} (P_{nr}\cos n\psi + Q_{nr}\sin n\psi)$$

$$+ \ 2\omega_1 \delta_{1r}(A_1 \sin\psi + aB_1 \cos\psi)$$

This gives for $r \neq 1$,

$$A_{nr} = \frac{P_{nr}}{\omega_r^2 - n^2\omega_1^2} \quad , \quad B_{nr} = \frac{Q_{nr}}{\omega_r^2 - n^2\omega_1^2} \quad \text{and} \quad A_{0r} = \frac{P_{0r}}{\omega_r^2} \quad .$$

The coefficients A_{11} and B_{11} remain undetermined. Thus we assume, as it is customary in the K-B-M method, that $P_r(a,\psi)$ does not contain the first harmonics in ψ. Hence $P_r(a,\psi)$ are completely determined.

For $r = n = 1$, one finds that

$$A_1 = -\frac{1}{2\omega_1} Q_{11} = -\frac{1}{4\pi\omega_1} \int_0^{2\pi} F_1(a,\psi)\sin\psi d\psi \quad ,$$

$$B_1 = -\frac{1}{2a\omega_1} P_{11} = -\frac{1}{4\pi a\omega_1} \int_0^{2\pi} F_1(a,\psi)\cos\psi d\psi \quad . \tag{48}$$

The equation (46) can be easily solved by elementary methods.

Thus the solution of (34) up to the first improved approximation is

$$u(x,t) = a\phi_1(x)\cos\psi + \varepsilon \sum_{r=1}^{\infty} [P_r(a,\psi) + Q_r(t)]\phi_r(x) \tag{49}$$

where $\dot{a} = \varepsilon A_1$, $\dot{\psi} = \omega_1 + \varepsilon B_1$.

As an illustration, let us consider again an equation of the Vander Pol type.

$$u_{tt} - u_{xx} + u = \varepsilon(1 - u^2)u_t + \varepsilon E e^{-pt}\sin\nu t \ , \qquad (50)$$

where E, p,ν are positive constants. Let the boundary conditions be

$$u(0,t) = u(\ell,t) = 0 \ . \qquad (51)$$

In this case, the orthonormal eigen functions $\{\phi_n(x)\}$ and the eigenfrequencies $\{\omega_n\}$ are given by

$$\phi_n = \sqrt{2}\ \sin(\frac{n\pi}{\ell}x) \quad , \qquad \omega_n^2 = 1 + \frac{n^2\pi^2}{\ell^2} \ . \qquad (52)$$

whereas, for $F_0(x,a,\psi)$ and $\Phi(t)$, we have:

$$F_0(x,a,\psi) = - a\omega_1[1 - 2a^2\sin^2(\frac{\pi x}{\ell})\cos^2\psi] \cdot \sqrt{2}\ \sin(\frac{\pi x}{\ell})\sin\psi$$

$$\Phi(t) = Ee^{-pt}\sin\nu t$$

Then the system (48) has the solution

$$A_1 = \frac{a\ell}{16}\ (8 - 3a^2) \ , \qquad B_1 = 0 \ . \qquad (53)$$

The equations (45) lead to the solutions

$$P_1(a,\psi) = -\,3\,\frac{a^3\ell}{64\omega_1}\sin 3\psi \quad , \quad P_r(a,\psi) = 0, \quad \text{for } r \neq 1,3$$

$$\text{and } P_3(a,\psi) = -\,\frac{a^3\omega_1\ell}{8}\left[\frac{1}{\omega_3^2 - \omega_1^2}\sin\psi + \frac{1}{\omega_3^2 - 9\omega_1^2}\sin 3\psi\right] .$$

The equations (48) lead to the solutions

$$Q_r(t) = \frac{2\sqrt{2}\,\ell E}{r\pi}\, e^{-pt}\left[\frac{(p^r + \omega_r^2 - \nu^2)\sin\nu t + 2p\nu\cos\nu t}{(p^2 + \omega_r^2 - \nu^2)^2 + 4p^2\nu^2}\right]$$

if r is even

$$= 0 \quad \text{if r is odd.}$$

Substituting these values in (49), we get the first improved approximation for the equation (50).

CHAPTER 3

THE EFFECT OF KINEMATICAL NONLINEARITIES ON THE VIBRATION FREQUENCIES OF A STRETCHED STRING.

INTRODUCTION.

In the usual elementary discussion of the transverse vibrations of a stretched string, it is assumed that the motion of the particles is entirely transverse and that the gradient of the transverse displacement remains small. With these approximations, the governing equation is shown to reduce to the wave equation. In a more complete treatment, there is a non-linear coupling between the transverse and longitudinal modes of vibration arising from purely kinematical sources. So, in this chapter we examine the effect of this nonlinearity on the natural frequencies of transverse vibration (i) when no damping is present (ii) when the transverse damping is there. In section I, we investigate the case when there is no damping and in section II we investigate the case when damping is present.

SECTION I*

WHEN NO DAMPING IS PRESENT

Let x be a coordinate along the string and t be time and let u(x,t) and v(x,t) be respectively the longitudinal and transverse displacements of the string (see Fig. 1). Consider an element (x, x+dx) of the string, which at time t has length $d\ell$ given by

$$d\ell = dx[(1 + u_x)^2 + v_x^2]^{1/2} .$$

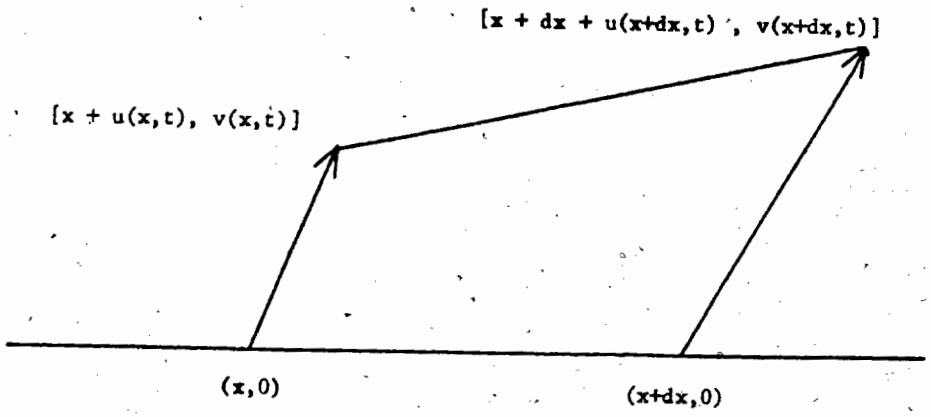

Figure 1

(Here and in the future we use suffices to indicate partial differentiation). If dx_0 is the unstretched

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*This work has been published in UTILITAS MATHEMATICA Vol.6(1974) pp.307-320 (with R. LARDNER).

length of the element dx, then the constant initial strain e_0 is given by $1 + e_0 = dx/dx_0$, and the strain at time t, $e(x,t)$, is given by

$$e(x,t) = \frac{d\ell - dx_0}{dx_0} = (1 + e_0)[(1 + u_x)^2 + v_x^2]^{1/2} - 1 .$$

The potential and kinetic energy densities of the string are then respectively

$$W = \frac{1}{2} \lambda e^2 , \quad T = \frac{1}{2} \rho (u_t^2 + v_t^2) ,$$

where λ is the elastic modulus and ρ the mass per unit length of the string. It is apparent from the first of these equations that we are assuming the string to obey Hooke's Law - we are including no nonlinearities of material behaviour. Using these energies, the Lagrange equations of motion become

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial u_t} \right) = \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial u_x} \right) , \quad \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial v_t} \right) = \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial v_x} \right) .$$

After expanding W and keeping only terms up to fourth order in u_x and v_x, these equations of motion become

$$\begin{aligned} u_{tt} - c_1^2 u_{xx} &= c^2 (v_x v_{xx} - u_{xx} v_x^2 - 2 u_x v_x v_{xx}) \\ v_{tt} - c_2^2 v_{xx} &= c^2 (u_{xx} v_x + u_x v_{xx} - 2 u_x v_x u_{xx} - u_x^2 v_{xx} + \tfrac{3}{2} v_x^2 v_{xx}) . \end{aligned} \tag{1}$$

Here we have introduced the definitions

$$\lambda(1 + e_0)^2\rho^{-1} = c_1^2 , \quad \lambda e_0(1 + e_0)\rho^{-1} = c_2^2 , \quad c^2 = c_1^2 - c_2^2 .$$

c_1 and c_2 are the usual velocities of longitudinal and transverse wave propogation on the string.

We intend to use a perturbation method for the solution of equations (1), and to make it easier to keep track of the various orders, we shall replace u and v by εu and εv , where ε is a small parameter. Equations (1) then take the form.

$$\begin{aligned} u_{tt} - c_1^2 u_{xx} &= \varepsilon F(u_x, v_x, u_{xx}, v_{xx}, \varepsilon) \\ v_{tt} - c_2^2 v_{xx} &= \varepsilon G(u_x, v_x, u_{xx}, v_{xx}, \varepsilon) \end{aligned} \tag{2}$$

where

$$F = c^2(v_x v_{xx} - \varepsilon u_{xx} v_x^2 - 2\varepsilon u_x v_x v_{xx})$$

$$G = c^2(u_{xx} v_x + u_x v_{xx} - 2\varepsilon u_x v_x u_{xx} - \varepsilon u_x^2 v_{xx} + \frac{3}{2}\varepsilon v_x^2 v_{xx}) .$$

Of course we have already omitted some terms of order ε^2 and higher from F and G , but since we shall be finding the solution only to order ε^2 these terms do not matter.

In addition to the partial differential equations (2), u and v each satisfy a pair of boundary conditions involving their values and the values of their derivatives with respect to x at the two ends $x = 0$ and $x = \ell$ of the string. We shall write these conditions in the symbolic form $\mathcal{B}_i^{(1)}(u) = 0$, $\mathcal{B}_i^{(2)}(v) = 0$ $(i = 1,2)$.

If we set $\varepsilon = 0$ in equations (2) we obtain the corresponding generating equations, $u_{tt} - c_1^2 u_{xx} = 0$, $v_{tt} - x_2^2 v_{xx} = 0$. These equations possess sets of separable solutions of the form

$$u(x,t) = A_n \Phi_n(x) \cos(\Omega_n t + \Psi_n),$$

$$v(x,t) = a_n \phi_n(x) \cos(\omega_n t + \psi_n) \qquad (3)$$

where A_n, a_n, Ψ_n and ψ_n are constants and where the eigen functions satisfy the differential equations

$$\Omega_n^2 \Phi_n'' + c_1^2 \Phi_n = 0, \quad \omega_n^2 \phi_n'' + c_2^2 \phi_n = 0. \qquad (4)$$

The boundary conditions $\mathcal{B}_i^{(1)}(\Phi_n) = 0$, $\mathcal{B}_i^{(2)}(\phi_n) = 0$, $(i = 1;2)$, enable the set of allowed eigenfunctions and eigenfrequencies $\{\Omega_n, \omega_n\}$ to be determined. It is a well-known result that if the boundary conditions are unmixed,

the eigenfunctions form complete orthogonal sets, and by suitable normalisation we can arrange that

$$\int_0^{\ell} \phi_n(x)\phi_m(x)\,dx = \int_0^{\ell} \Phi_n(x)\Phi_m(x)\,dx = \delta_{mn} . \qquad (5)$$

We wish to find solutions of the nonlinear system (2) which are close to the normal mode solutions (3) of the generating system. In particular, we are interested in the solution which is close to the basic mode of transverse vibration,

$$u(x,t) = 0 , \quad v(x,t) = a\phi_1(x)\cos(\omega_1 t + \psi) . \qquad (6)$$

We shall seek the solution using an extension of the Krylov-Bogoliubov-Mitropolskii asymptotic method.

The asymptotic solution

The asymptotic method of Krylov, Bogoliubov and Mitropolskii [3,4] has been extended to partial differential equations by Mitropolskii and Moseenkov [9] and Bojadziev and Lardner [10,11,12]. The method may readily be extended further to systems of partial differential equations such as equations (2). In accordance with this method, we seek the solution in the form

$$u(x,t) = \sum_{s=1}^{\infty} \varepsilon^s u_s(x,a,\psi) \tag{7}$$

$$v(x,t) = a\phi_1(x) \cos\psi + \sum_{s=1}^{\infty} \varepsilon^s v_s(x,a,\psi) , \tag{8}$$

where the functions $\{u_s , v_s\}$ are assumed to be 2π periodic functions of ψ and where a and ψ are functions of t assumed to satisfy differential equations of the form

$$\frac{da}{dt} = \sum_{s=1}^{\infty} \varepsilon^s A_s(a) , \quad \frac{d\psi}{dt} = \omega_1 + \sum_{s=1}^{\infty} \varepsilon^s B_s(a) . \tag{9}$$

The functions $\{A_s(a), B_s(a), u_s(x,a,\psi), v_s(x,a,\psi)\}$ are to be determined from the requirement that the solutions (7) and (8) should satisfy equations (2).

It should be noted that if we set $\varepsilon = 0$ in equations (7) - (9), the assumed solution reduces to the basic mode (6) of transverse vibration.

Substituting equations (7) - (9) into equations (2) and comparing the coefficients of successive powers of ε, we obtain a hierarchy of equations for the functions $\{u_s, v_s\}$. The terms of zero order in ε cancel identically, while the terms of orders ε and ε^2 give the

following four equations.

$$\omega_1^2 \frac{\partial^2 u_1}{\partial \psi^2} - c_1^2 \frac{\partial^2 u_1}{\partial x^2} = c^2 \phi_1'(x)\phi_1''(x)\, a^2 \cos^2\psi \tag{10}$$

$$\omega_1^2 \frac{\partial^2 u_2}{\partial \psi^2} - c_1^2 \frac{\partial^2 u_2}{\partial x^2} = c^2 a \cos\psi \left[\phi_1'(x) \frac{\partial^2 v_1}{\partial x^2} + \phi_1''(x) \frac{\partial v_1}{\partial x}\right]$$

$$- 2\omega_1 A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} - 2\omega_1 B_1 \frac{\partial^2 u_1}{\partial \psi^2} \tag{11}$$

$$\omega_1^2 \frac{\partial^2 v_1}{\partial \psi^2} - c_2^2 \frac{\partial^2 v_1}{\partial x^2} = 2\omega_1 \phi_1(x)\,(A_1 \sin\psi + aB_1 \cos\psi) \tag{12}$$

$$\omega_1^2 \frac{\partial^2 v_2}{\partial \psi^2} - c_2^2 \frac{\partial^2 v_2}{\partial x^2} = \phi_1(x)(2\omega_1 A_2 \sin\psi + 2a\omega_1 B_2 \cos\psi$$

$$- A_1 \frac{dA_1}{da} \cos\psi + 2A_1 B_1 \sin\psi$$

$$+ aA_1 \frac{dB_1}{da} \sin\psi + aB_1^2 \cos\psi)$$

$$+ c^2 a \cos\psi \left[\phi_1'(x) \frac{\partial^2 u_1}{\partial x^2} + \phi_1''(x) \frac{\partial u_1}{\partial x}\right]$$

$$+ \frac{3}{2} c^2 a^3 \cos^3\psi\, \phi_1'(x)^2 \phi_1''(x)$$

$$- 2\omega_1 \left(A_1 \frac{\partial^2 v_1}{\partial a \partial \psi} + B_1 \frac{\partial^2 v_1}{\partial \psi^2}\right). \tag{13}$$

In solving equations (10) - (13) we expand each of the four functions u_1, u_2, v_1, v_2 in double series with respect to x and ψ in the forms

$$u_s(x,a,\psi) = \sum_{k=1}^{\infty}\left[\frac{1}{2}A_{0k}^{(s)}(a) + \sum_{n=1}^{\infty}\left(A_{nk}^{(s)}(a)\cos n\psi + B_{nk}^{(s)}(a)\sin n\psi\right)\right]\Phi_k(x) \tag{14}$$

$$v_s(x,a,\psi) = \sum_{k=1}^{\infty}\left[\frac{1}{2}C_{0k}^{(s)}(a) + \sum_{n=1}^{\infty}\left(C_{nk}^{(s)}(a)\cos n\psi + D_{nk}^{(s)}(a)\sin n\psi\right)\right]\phi_k(x) \tag{15}$$

(s=1,2). Substituting these expansions into equations (10) - (13) and using the conditions (5) to compare the coefficients of $\Phi_k(x)$ or $\phi_k(x)$ in these equations we obtain a series of algebraic equations for the quantities $\{A_{nk}^{(s)}, B_{nk}^{(s)}, C_{nk}^{(s)}, D_{nk}^{(s)}\}$. Writing down first of all the results derived from equations (10) and (12) for u_1 and v_1 we have

$$\frac{1}{2}\Omega_k^2 A_{0k}^{(1)} + \sum_{n=1}^{\infty}(\Omega_k^2 - n^2\omega_1^2)(A_{nk}^{(1)}\cos n\psi + B_{nk}^{(1)}\sin n\psi) = \frac{1}{2}a^2c^2K_k(1+\cos 2\psi) \tag{16}$$

$$\frac{1}{2}\omega_k^2 C_{0k}^{(1)} + \sum_{n=1}^{\infty}(\omega_k^2 - n^2\omega_1^2)(C_{nk}^{(1)}\cos n\psi + D_{nk}^{(1)}\sin n\psi)$$

$$= 2\omega_1\delta_{k1}(A_1\sin\psi + aB_1\cos\psi) \qquad (17)$$

where δ_{kj} is the Kronecker delta and where the constants K_k are defined by

$$K_k = \int_0^{\ell} \phi_1'(x)\,\phi_1''(x)\,\Phi_k(x)\,dx \ . \qquad (18)$$

From equations (17), considering the case k=1 , comparing the coefficients of cos ψ and sin ψ shows that $A_1 = B_1 = 0$. For all other pairs of values of k and n , we see that $C_{nk}^{(1)} = D_{nk}^{(1)} = 0$. We may also assume that $C_{11}^{(1)} = D_{11}^{(1)} = 0$, since any terms such as $C_{11}^{(1)}\phi_k(x)\cos\psi$ and $D_{11}^{(1)}\phi_k(x)\sin\psi$ in v_1 could be included in the first term in the solution (8). From equation (16), comparing the different coefficients of cos $n\psi$ and sin $n\psi$, we see that all the coefficients $B_{nk}^{(1)}$ are zero, and all the $A_{nk}^{(1)}$ are zero except

$$A_{0k}^{(1)} = \frac{a^2c^2K_k}{\Omega_k^2} \ , \quad A_{2k}^{(1)} = \frac{a^2c^2K_k}{2(\Omega_k^2 - 4\omega_1^2)} \ . \qquad (19)$$

Since $A_1 = B_1 = v_1 = 0$, equation (11) for u_2 has a zero right hand side and we readily see after making use

of the expansion (14) that u_2 is also identically zero. The right hand side of equation (13) is also considerably simplified, and after substituting the expansion (15) we obtain that

$$\frac{1}{2}\omega_k^2 C_{0k}^{(2)} + \sum_{n=1}^{\infty}(\omega_k^2 - n^2\omega_1^2)(C_{nk}^{(2)}\cos n\psi + D_{nk}^{(2)}\sin n\psi)$$

$$= 2\omega_1\delta_{k1}(A_2\sin\psi + aB_2\cos\psi) + \frac{3}{8}c^2a^3\alpha_k(3\cos\psi + \cos 3\psi)$$

$$+ ac^2\sum_{\ell}\beta_{k\ell}[\frac{1}{2}A_{0\ell}^{(1)}\cos\psi + \frac{1}{2}A_{2\ell}^{(1)}(\cos\psi + \cos 3\psi)] \qquad (20)$$

where

$$\alpha_k = \int_0^{\ell} \phi_1'(x)^2\phi_1''(x)\,\phi_k(x)\,dx\,,$$

$$\beta_{k\ell} = \int_0^{\ell} [\phi_1'(x)\,\Phi_\ell''(x) + \phi_1''(x)\,\Phi_\ell'(x)]\phi_k(x)dx\,. \qquad (21)$$

Again comparing the terms for which $k = n = 1$ we obtain $A_2 = 0$ and

$$B_2 = -\frac{9}{16}\frac{a^2c^2}{\omega_1}\alpha_1 - \frac{1}{4}\frac{c^2}{\omega_1}\sum_{\ell}(A_{0\ell}^{(1)} + A_{2\ell}^{(1)})\beta_{1\ell} \qquad (22)$$

As before we may assume that $C_{11}^{(2)} = D_{11}^{(2)} = 0$. Comparing the other terms in (20) then shows that all the $D_{nk}^{(2)}$ are

zero and all $C_{nk}^{(2)}$ are zero except for

$$C_{1k}^{(2)} = \frac{1}{\omega_k^2 - \omega_1^2}\left[\frac{9}{8}c^2a^3\alpha_k + \frac{1}{2}ac^2\sum_{\ell}(A_{0\ell}^{(1)} + A_{2\ell}^{(1)})\beta_{k\ell}\right] \quad (k\neq 1)$$

and (23)

$$C_{3k}^{(2)} = \frac{1}{\omega_k^2 - 9\omega_1^2}\left[\frac{3}{8}c^2a^3\alpha_k + \frac{1}{2}ac^2\sum_{\ell}A_{2\ell}^{(1)}\beta_{k\ell}\right] \quad (\text{all } k) .$$

Substituting the results (19) into (22) we obtain that $B_2 = Ea^2$ where

$$E = -\frac{c^2}{16\,\omega_1^2}\left[9\alpha_1 + 4c^2\sum_k \frac{3\Omega_k^2 - 8\omega_1^2}{2\Omega_k^2(\Omega_k^2 - 4\omega_1^2)}K_k\beta_{1k}\right]. \qquad (24)$$

Since A_1 and A_2 have both been shown to vanish, it follows from the first of equations (9) that a is constant to order ε^2, which is an expected result in a system which involves no damping. The second of equations (9) may therefore be integrated, giving, to order ε^2, $\psi = (\omega_1 + \varepsilon^2\omega_1 Ea^2)t + \psi_0$.

Thus to order ε^2, the solution becomes

$$u(x;t) = \varepsilon u_1(x,t) ,$$

$$v(x,t) = a\phi_1(x)\cos[\omega_1 t(1 + \varepsilon^2 Ea^2) + \psi_0] + \varepsilon^2 v_2(x,t)$$

with u_1 and $\tilde{v}_2$ being given by equations (14) and (15) in which the non-zero coefficients are given by equations (19) and (23). Bearing in mind that the physical transverse displacement is $\varepsilon v(x,t)$, we see that one effect of the kinematical nonlinearities is to increase the natural frequency of the basic mode from its linear value of ω_1 by a factor which is one plus E times the square of the amplitude of the vibration.

Finally, we note that our analysis of equations such as (16), (17), and (20) is valid only if $(\Omega_k^2 - n^2\omega_1^2)$ and $(\omega_k^2 - n^2\omega_1^2)$ never vanish for any pairs of values of n and k (except of course for k = n = 1 for the second of these quantities). This condition is generally found to be necessary when the KBM method is applied to continuous systems [9,10] . When it is violated, the system is said to display internal resonance, and the extension of the KBM method to such systems [11] involves considerably more complexities than the method as used here.

A particular example

Let us choose units of length in such a way that the ends of the string are at x = 0 and x = 1 . We consider the case when the boundary conditions on u and v

are of the form

$$u(0,t) = u(1,t) = 0$$

$$v(0,t) = 0 \ , \quad hv_x(1,t) + v(1,t) = 0 \ ,$$

where h is some constant. This corresponds to having the ends of the string fixed except for the end x = 1 which is held by some elastic fixture in the transverse direction. The normalised eigenfunctions and eigenvalues satisfying equations (4), (5) and the boundary conditions are therefore

$$\Phi_n(x) = \sqrt{2} \sin n\pi x \ , \quad \Omega_n = n\pi c_1 \ ;$$

$$\phi_n(x) = H_n \sin q_n x \ , \quad \omega_n = q_n c_2$$

where

$$H_n^2 = 2[1 + h(1 + h^2 q_n^2)^{-1}]^{-1} \tag{25}$$

and q_n (n = 1,2,...) are the roots of the equation

$$\tan q + hq = 0 \ . \tag{26}$$

Using these eigenfunctions in the definitions (18) and (21) we obtain that

$$K_n = \frac{(-1)^n n\pi H_1^2 q_1^3}{\sqrt{2}(n^2\pi^2 - 4q_1^2)} \sin 2q_1$$

$$\alpha_1 = -\frac{H_1^4 q_1^4}{8}\left(1 - \frac{\sin 4q_1}{4q_1}\right)$$

$$\beta_{1n} = \frac{(-1)^n n\pi H_1^2 q_1 (n^2\pi^2 - 2q_1^2)}{\sqrt{2}(n^2\pi^2 - 4q_1^2)} \sin 2q_1 .$$

Substituting these into equation (24), the sum over k may be performed using the result that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha}\cot \pi\alpha ,$$

and after some simplification we obtain that

$$E = \frac{H_1^4 q_1}{256r}\{4q_1(3+r)(1-r) - r(4-3r)\sin 4q_1$$

$$- 2\sin^2 2q_1[q_1^{-1}(1-r)^2 + \sqrt{r}(1-2r)\cot(2q_1\sqrt{r})]\} .$$

where $r = c_2^2/c_1^2 = e_0/(1+e_0)$.

Observing that q_1 is the smallest root of equation (26) and that H_1 is given by equation (25) we see that E depends only on the two parameters r and h - that is on

the initial strain in the string and on the elasticity of the end fixture at x=1. Curves of E against h for different values of r are shown in Fig. 2.

It can be seen that E becomes infinite when $2q_1\sqrt{r} = k\pi$ for any integer k . In Fig. 2, this is seen to occur on the curve for $r = 0.4$ at approximately $h = 0.3$. This condition is equivalent to $\Omega_k = 2\omega_1$, and so corresponds to an internal resonance between the basic transverse mode and one of the longitudinal modes. As remarked on page 89, the above derivation becomes invalid for internal resonances, and therefore the results for E are not meaningful in the neighbourhoods of such asymptotes.

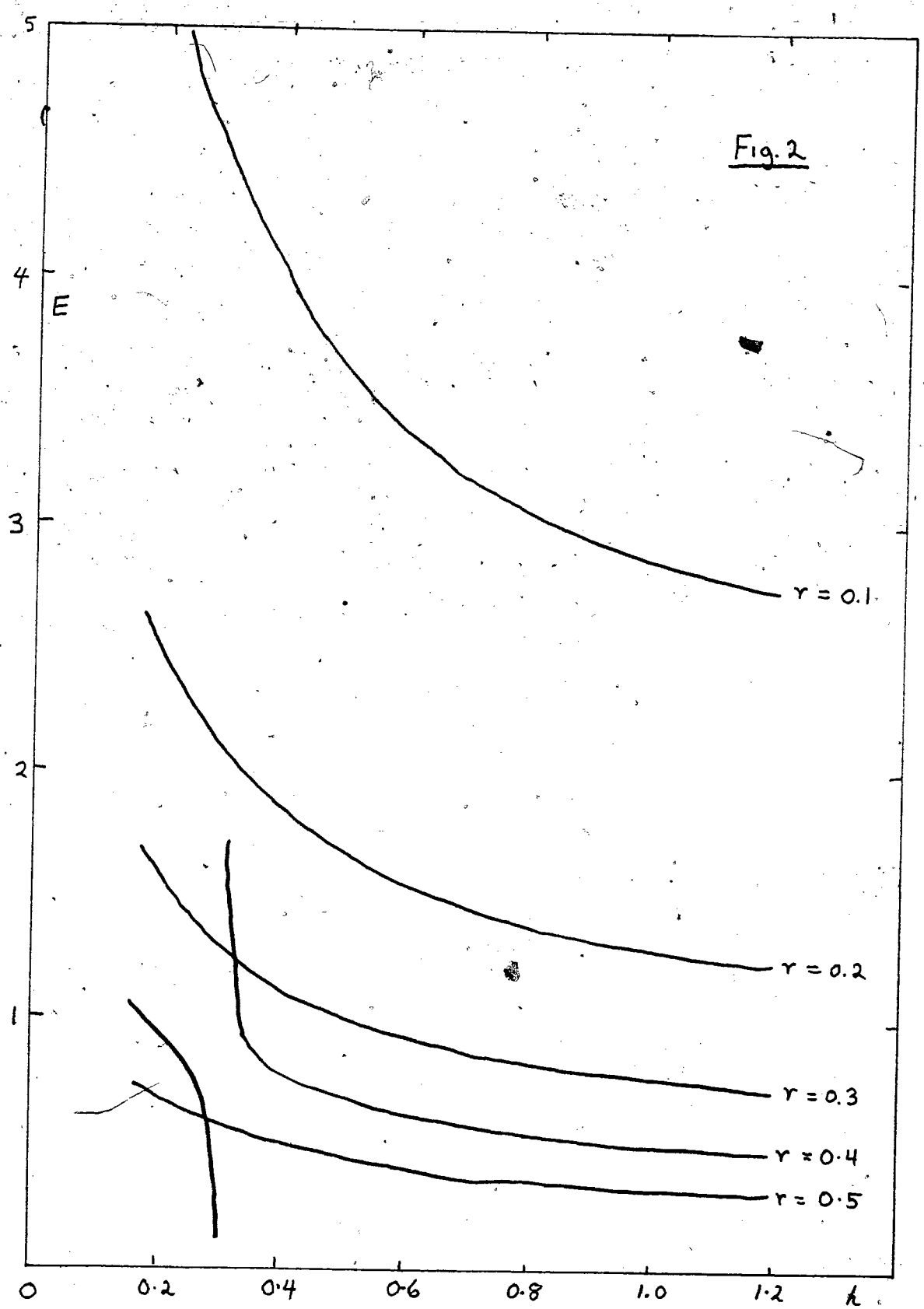

Fig. 2

SECTION II*

DAMPING PRESENT

If γ denotes the coefficient of transverse damping, then as in (2) the equations of motion of the string in this case are

$$u_{tt} - c_1^2 u_{xx} = \varepsilon F(u_x, v_x, u_{xx}, v_{xx}, \varepsilon)$$
$$v_{tt} + 2\gamma v_t - c_2^2 v_{xx} = \varepsilon G(u_x, v_x, u_{xx}, v_{xx}, \varepsilon) \tag{27}$$

If we set $\varepsilon = 0$ in (27), we obtain the corresponding generating equations

$$u_{tt}^{(0)} - c_1^{2(0)} u_{xx} = 0 \ , \quad \beta_i^{(1)}(u^{(0)}) = 0$$
$$(i = 1,2)$$
$$v_{tt}^{(0)} + 2\gamma v_t^{(0)} - c_2^2 v_{xx}^{(0)} = 0 \ , \quad \beta_i^{(2)}(v^{(0)}) = 0 \tag{28}$$

The problem (28) possesses sets of separable solutions for $u^{(0)}$ and $v^{(0)}$ which can be written as

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* This work has been accepted for publication in ACTA MECHANICA.

$$u^{(0)}(x,t) = \alpha_n \Phi_n(x) \cos(\Omega_n t + \Psi_n)$$

$$v^{(0)}(x,t) = a_n \phi_n(x) e^{-\gamma t} \cos(\omega_n t + \psi_n) \qquad n = 1,2,3,... \tag{29}$$

where α_n, a_n, Ψ_n, ψ_n are arbitrary constants and where the eigenfunctions satisfy the differential equations

$$c_1^2 \Phi_n''(x) + \Omega_n^2 \Phi_n(x) = 0, \quad c_2^2 \phi_n''(x) + \lambda_n^2 \phi_n(x) = 0 \tag{30}$$

$$\text{where } \lambda_n^2 = \omega_n^2 + \gamma^2 \quad (n = 1,2,...) .$$

We shall assume that the damping is less than critical; i.e., $\gamma < \lambda_n$ for all n, which means that all ω_n^2 are positive.

The boundary conditions $\mathcal{B}_i^{(1)}(\Phi_n) = 0$ and $\mathcal{B}_i^{(2)}(\phi_n) = 0$ $(i = 1,2)$, enable the set of allowed eigenfunctions and eigenfrequencies $\{\Omega_n, \omega_n\}$ to be determined. If the self-adjointness criteria are satisfied by the boundary conditions, then the sets of functions $\{\Phi_n(x)\}$ and $\{\phi_n(x)\}$ form a Sturm-Liouville system and they have the property that besides being complete, they are orthogonal.

By suitably normalizing $\Phi_n(x)$ and $\phi_n(x)$, these functions can be made to satisfy equations (5).

The general solution of the generating system (28) can be found as the sum of these separable solutions.

We are interested in finding the solutions of the nonlinear system (27) which are close to the normal mode solutions (29) of the generating system. In particular, we are interested in the solution which is close to the basic mode of transverse vibration:

$$u^{(0)}(x,t) = 0, \quad v^{(0)}(x,t) = a_0 e^{-\gamma t}\phi_1(x)\cos(\omega_1 t + \psi_1) \tag{31}$$

General asymptotic solution.

As in [12], we seek the solution of (27) in the form

$$u(x,t) = 0 + \sum_{s=1}^{\infty} \varepsilon^s u_s(x,\alpha,\psi) \tag{32}$$

$$v(x,t) = e^{-\gamma\alpha}\phi_1(x)\cos\psi + \sum_{s=1}^{\infty} \varepsilon^s v_s(x,\alpha,\psi) \tag{33}$$

where α and ψ are functions of t satisfying the differential equations

$$\frac{d\alpha}{dt} = 1 + \varepsilon A_1(\alpha) + \varepsilon^2 A_2(\alpha) + \ldots$$

$$\frac{d\psi}{dt} = \omega_1 + \varepsilon B_1(\alpha) + \varepsilon^2 B_2(\alpha) + \ldots \tag{34}$$

This approach is a modification of Popov's method [7].

The functions $\{u_s, v_s\}$ are assumed to be 2π periodic in ψ. The functions $\{A_n(\alpha), B_n(\alpha), u_n(x,\alpha,\psi), v_n(x,\alpha,\psi)\}$ are to be determined from the requirement that $u(x,t)$, $v(x,t)$ satisfy (27) to each order of ε.

In addition, we also assume that ω_n $(n \neq 1)$ is not an integral multiple of ω_1 ; i.e., $\omega_n \neq p\omega_1$, for p any integer and $n \geq 2$.

It should be noted that for $\varepsilon = 0$, the solutions (32), (33) of equations (27) reduce to the basic mode (31) of the transverse vibration.

Substituting (32)-(34) into equations (27) and comparing the coefficients of successive powers of ε, we obtain a hierarchy of differential equations for $\{u_s, v_s\}$. The terms of zero order in ε cancel identically, while the terms of order ε and ε^2 give the following four equations.

$$\frac{\partial^2 u_1}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 u_1}{\partial \psi^2} - c_1^2 \frac{\partial^2 u_1}{\partial x^2}$$

$$= c^2 \phi_1'(x)\, \phi_1''(x)\, e^{-2\gamma\alpha} \cos^2 \psi \qquad (35)$$

$$\frac{\partial^2 u_2}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 u_2}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 u_2}{\partial \psi^2} - c_1^2 \frac{\partial^2 u_2}{\partial x^2}$$

$$= c^2 e^{-\gamma\alpha} \cos\psi \left[\phi_1'(x) \frac{\partial^2 v_1}{\partial x^2} + \phi_1''(x) \frac{\partial v_1}{\partial x} \right]$$

$$- 2 \left(A_1 \frac{\partial^2 u_1}{\partial \alpha^2} + B_1 \frac{\partial^2 u_1}{\partial \alpha \partial \psi} + A_1 \omega_1 \frac{\partial^2 u_1}{\partial \alpha \partial \psi} + \omega_1 B_1 \frac{\partial^2 u_1}{\partial \psi^2} \right)$$

$$- \frac{dA_1}{d\alpha} \frac{\partial u_1}{\partial \alpha} - \frac{dB_1}{d\alpha} \frac{\partial u_1}{\partial \psi} \qquad (36)$$

$$\frac{\partial^2 v_1}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 v_1}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 v_1}{\partial \psi^2} - c_2^2 \frac{\partial^2 v_1}{\partial x^2} + 2\gamma \left(\frac{\partial v_1}{\partial \alpha} + \omega_1 \frac{\partial v_1}{\partial \psi} \right)$$

$$= \phi_1(x) e^{-\gamma\alpha} \left[\left(\gamma \frac{dA_1}{d\alpha} + 2\omega_1 B_1 \right) \cos\psi + \left(\frac{dB_1}{d\alpha} - 2\gamma\omega_1 A_1 \right) \sin\psi \right] \qquad (37)$$

and

$$\frac{\partial^2 v_2}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 v_2}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 v_2}{\partial \psi^2} - c_2^2 \frac{\partial^2 v_2}{\partial x^2} + 2\gamma \left(\frac{\partial v_2}{\partial \alpha} + \omega_1 \frac{\partial v_2}{\partial \psi} \right)$$

$$= \phi_1(x) e^{-\gamma\alpha} \left[\gamma \frac{dA_2}{d\alpha} \cos\psi + \frac{dB_2}{d\alpha} \sin\psi + 2\omega_1 (-\gamma A_2 \sin\psi + B_2 \cos\psi) \right.$$

$$- \gamma^2 A_1^2 \cos\psi - 2\gamma A_1 B_1 \sin\psi + \gamma A_1 \frac{dA_1}{d\alpha} \cos\psi$$

$$\left. + A_1 \frac{dB_1}{d\alpha} \sin\psi + B_1^2 \cos\psi \right]$$

$$+ c^2 e^{-\gamma\alpha}\left[\frac{\partial u_1}{\partial x}\,\phi_1''(x) + \frac{\partial^2 u_1}{\partial x^2}\phi_1'(x)\right]\cos\psi - \frac{3}{2}\,c^2 e^{-3\gamma\alpha}(\phi_1'(x))^2$$

$$\cdot\phi_1''(x)\cos^3\psi - \frac{dA_1}{d\alpha}\frac{\partial v_1}{\partial\alpha} - \frac{dB_1}{d\alpha}\frac{\partial v_1}{\partial\psi} - 2A_1\frac{\partial^2 v_1}{\partial\alpha^2} - 2A_1\omega_1\frac{\partial^2 v_1}{\partial\alpha\partial\psi}$$

$$- 2B_1\frac{\partial^2 v_1}{\partial\alpha\partial\psi} - 2\omega_1 B_1\frac{\partial^2 v_1}{\partial\psi^2} - 2\gamma\left(A_1\frac{\partial v_1}{\partial\alpha} + B_1\frac{\partial v_1}{\partial\psi}\right) \qquad (38)$$

In solving equations (35)-(38) we expand each of the four functions u_1, u_2, v_1, v_2 as double Fourier series in ψ and x using the harmonic basis for ψ and the bases $\{\Phi_n(x)\}$, $\{\phi_n(x)\}$ for x:

$$u_k(x,\alpha,\psi) = \sum_{r=1}^{\infty}\{1/2\,A_{0r}^{(k)}(\alpha) + \sum_{n=1}^{\infty}[A_{nr}^{(k)}(\alpha)\cos n\psi + B_{nr}^{(k)}(\alpha)\sin n\psi]\}\Phi_r(x) \qquad (39)$$

$$v_k(x,\alpha,\psi) = \sum_{r=1}^{\infty}\{1/2\,C_{0r}^{(k)}(\alpha) + \sum_{n=1}^{\infty}[C_{nr}^{(k)}(\alpha)\cos n\psi + D_{nr}^{(k)}(\alpha)\sin n\psi]\}\phi_r(x) \qquad (40)$$

$(k = 1,2)$.

Substituting these expansions into equations (35)-(38) and using the conditions (30) to compare the coefficients of $\Phi_k(x)$ or $\phi_k(x)$ in these equations, we obtain a series of differential equations for $A_{nr}^{(k)}$, $B_{nr}^{(k)}$,

$C_{nr}^{(k)}$ and $D_{nr}^{(k)}$. We write down first of all the results derived from equations (35) and (37) for u_1 and v_1 , which give:

$$\frac{1}{2}\left[\frac{d^2A_{0r}^{(1)}}{d\alpha^2} + \Omega_r^2 A_{0r}^{(1)}\right] + \sum_{n=1}^{\infty}\left[\frac{d^2A_{nr}^{(1)}}{d\alpha^2} + 2n\omega_1 \frac{dB_{nr}^{(1)}}{d\alpha}\right.$$

$$\left. + (\Omega_r^2 - n^2\omega_1^2) A_{nr}^{(1)}\right] \cos n\psi + \sum_{n=1}^{\infty}\left[\frac{d^2B_{nr}^{(1)}}{d\alpha^2} - 2n\omega_1 \frac{dA_{nr}^{(1)}}{d\alpha}\right.$$

$$\left. + (\Omega_r^2 - n^2\omega_1^2) B_{nr}^{(1)}\right] \sin n\psi = \frac{1}{2} c^2 K_r e^{-\gamma\alpha}(1 + \cos 2\psi) \tag{41}$$

$$\frac{1}{2}\left[\frac{d^2C_{0r}^{(1)}}{d\alpha^2} + 2\gamma \frac{dC_{0r}^{(1)}}{d\alpha} + \lambda_r^2 C_{0r}^{(1)}\right]$$

$$+ \sum_{n=1}^{\infty}\left[\frac{d^2C_{nr}^{(1)}}{d\alpha^2} + 2\left(\gamma \frac{dC_{nr}^{(1)}}{d\alpha} + n\omega_1 \frac{dD_{nr}^{(1)}}{d\alpha}\right) + (\lambda_r^2 - n^2\omega_1^2) C_{nr}^{(1)}\right.$$

$$\left. + 2n\omega_1\gamma D_{nr}^{(1)}\right] \cos n\psi$$

$$+ \sum_{n=1}^{\infty}\left[\frac{d^2D_{nr}^{(1)}}{d\alpha^2} + 2\left(\gamma \frac{dD_{nr}^{(1)}}{d\alpha} - n\omega_1 \frac{dC_{nr}^{(1)}}{d\alpha}\right)\right.$$

$$\left. + (\lambda_r^2 - n^2\omega_1^2) D_{nr}^{(1)} - 2n\omega_1\gamma C_{nr}^{(1)}\right] \sin \psi$$

$$= \delta_{1r} e^{-\gamma\alpha}\left[\left(\gamma \frac{dA_1}{d\alpha} + 2\omega_1 B_1\right) \cos \psi + \left(\frac{dB_1}{d\alpha} - 2\gamma\omega_1 A_1\right) \sin \psi\right] \tag{42}$$

where the constants K_r are defined by

$$K_r = \int_0^{\ell} \phi_1'(x)\, \phi_1''(x)\, \Phi_r(x)\, dx . \tag{43}$$

From equation (41), considering the case r = 1, comparing the coefficients of $\cos\psi$ and $\sin\psi$, we find that $A_1 = B_1 = 0$. For all other pairs of values of n and r (except n = r = 1), we have $C_{nr}^{(1)} = D_{nr}^{(1)} = 0$.

The constants $C_{11}^{(1)}$ and $D_{11}^{(1)}$ remain undetermined. Returning to (33), we see that we may take $C_{11}^{(1)} = D_{11}^{(1)} = 0$ since any terms in $v_1(x,t)$ proportional to $\phi_1(x)\cos\psi$ and $\phi_1(x)\sin\psi$ can be included with the first term in the expansion.

From equation (42), comparing the different coefficients of $\cos n\psi$ and $\sin n\psi$, we see that all $A_{nr}^{(1)}$ and $B_{nr}^{(1)}$ are zero except

$$\begin{aligned} A_{0r}^{(1)} &= c^2 K_r e^{-\gamma\alpha} / (\Omega_r^2 + 4\gamma^2) \\ A_{2r}^{(1)} &= -B_{2r}^{(1)} / (8\gamma\omega_1) = c^2 P_r K_r / [2(P_r^2 + 64\gamma^2\omega_1^2)] \end{aligned} \tag{44}$$

where $P_r = \Omega_r^2 + 4\gamma^2 - 4\omega_1^2$.

Thus from (39) and (40), we have

$$u_1(x,\alpha,\psi) = \sum_r \left(1/2\, A_{0r}^{(1)} + A_{2r}^{(1)} \cos 2\psi + B_{2r}^{(1)} \sin 2\psi\right) \Phi_r(x)$$

$$v_1(x,\alpha,\psi) = 0 \tag{45}$$

where $A_{0r}^{(1)}$, $A_{2r}^{(1)}$ and $B_{2r}^{(1)}$ are given by (44).

Since $A_1 = B_1 = v_1 = 0$, equation (36) for u_2 has a zero right-hand side and after making use of expansion (39), we readily find that u_2 is also identically zero. The equation (38) for v_2 is also considerably simplified to:

$$\frac{\partial^2 v_2}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 v_2}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 v_2}{\partial \psi^2} - c_2^2 \frac{\partial^2 v_2}{\partial x^2} + 2\gamma\left(\frac{\partial v_2}{\partial \alpha} + \omega_1 \frac{\partial v_2}{\partial \psi}\right)$$

$$= \phi_1(x) e^{-\gamma\alpha}\left[\gamma \frac{dA_2}{d\alpha} \cos\psi + \frac{dB_2}{d\alpha} \sin\psi + 2\gamma(-A_2 \sin\psi + B_2 \cos\psi)\right]$$

$$+ c^2 e^{-\gamma\alpha}\left[\frac{\partial u_1}{\partial x} \phi_1''(x) + \frac{\partial^2 u_1}{\partial x^2} \phi_1'(x)\right] \cos\psi - \frac{3}{2} c^2 e^{-3\gamma\alpha} (\phi_1'(x))^2$$

$$\phi_1''(x) \cos^3 \psi \,.$$

After substituting the expansion (40) for v_2, this gives:

$$\frac{1}{2}\left[\frac{d^2C_{0r}^{(2)}}{d\alpha^2} + 2\gamma\frac{dC_{0r}^{(2)}}{d\alpha} + \lambda_r^2 C_{0r}^{(2)}\right]$$

$$+ \sum_{n=1}^{\infty}\left[\frac{d^2C_{nr}^{(2)}}{d\alpha^2} + 2\gamma\frac{dC_{nr}^{(2)}}{d\alpha} + 2n\omega_1\frac{dD_{nr}^{(2)}}{d\alpha}\right.$$

$$\left. + (\lambda_r^2 - n^2\omega_1^2)\, C_{nr}^{(2)} + 2n\gamma\omega_1 D_{nr}^{(2)}\right]\cos n\psi$$

$$+ \sum_{n=1}^{\infty}\left[\frac{d^2D_{nr}^{(2)}}{d\alpha^2} + 2\gamma\frac{dD_{nr}^{(2)}}{d\alpha} - 2n\omega_1\frac{dC_{nr}^{(2)}}{d\alpha}\right.$$

$$\left. + (\lambda_r^2 - n^2\omega_1^2)\, D_{nr}^{(2)} - 2n\omega_1\gamma C_{nr}^{(2)}\right]\sin n\psi$$

$$= \delta_{1r}e^{-\gamma\alpha}\left[\left(\gamma\frac{dA_2}{d\alpha} + 2\omega_1 B_2\right)\cos\psi + \left(\frac{dB_2}{d\alpha} - 2\omega_1\gamma A_2\right)\sin\psi\right]$$

$$+ c^2 e^{-\gamma\alpha}\sum_k\left[\frac{1}{2}A_{0k}^{(1)}\cos\psi + \frac{1}{2}A_{2k}^{(1)}(\cos 3\psi + \cos\psi)\right.$$

$$\left. + \frac{1}{2}B_{2k}^{(1)}(\sin 3\psi + \sin\psi)\right]\beta_{kr} + \frac{3}{8}c^2 e^{-3\gamma\alpha}\alpha_r(\cos 3\psi + 3\cos\psi) \tag{46}$$

where

$$\alpha_r = \int_0^{\ell}(\phi_1'(x))^2\,\phi_1''(x)\,\phi_r(x)\,dx \tag{47}$$

$$\beta_{kr} = \int_0^{\ell}[\Phi_k''(x)\,\phi_1'(x) + \Phi_k'(x)\,\phi_1''(x)]\phi_r(x)\,dx$$

As before we may assume $C_{11}^{(2)} = D_{11}^{(2)} = 0$.

Comparing the terms for which $n = r = 1$, we obtain the following differential equations for A_2 and B_2:

$$\gamma \frac{dA_2}{d\alpha} + 2\omega_1 B_2 = -\frac{c^2}{2} \sum_k (A_{0k}^{(1)} + A_{2k}^{(1)}) \beta_{k1} - \frac{9}{8} c^2 \alpha_1 e^{-2\gamma\alpha} ;$$

$$\frac{dB_2}{d\alpha} - 2\omega_1 \gamma A_2 = -\frac{1}{2} c^2 \sum_k B_{2k}^{(1)} \beta_{k1} .$$

These simultaneous differential equations on solving yield:

$$A_2 = Re^{-2\gamma\alpha} , \quad B_2 = Qe^{-2\gamma\alpha}$$

where

$$Q = -\frac{c^2}{16}\left[\frac{9\omega_1\alpha_1}{(\gamma^2 + \omega_1^2)} + \frac{4c^2}{\omega_1} \sum_k \left(\frac{1}{\Omega_k^2 + 4\gamma^2} + \frac{P_k}{2(P_k^2 + 64\gamma^2\omega_1^2)}\right) K_k \beta_{k1}\right] \tag{48}$$

$$R = -\frac{Q}{\omega_1} - c^4 \sum_k \frac{K_k \beta_{k1}}{(P_k^2 + 64\gamma^2\omega_1^2)}$$

Comparing the other terms in (46), we find that all $C_{nr}^{(2)}$ and $D_{nr}^{(2)}$ are zero except

$$C_{1r}^{(2)} = \frac{c^2 e^{-\gamma\alpha}}{2(\omega_r^2 - \omega_1^2)} \sum_k (A_{0k}^{(1)} + A_{2k}^{(1)}) \beta_{kr} + \frac{9c^2 S_r \alpha_r e^{-3\gamma\alpha}}{8(S_r^2 + 16\gamma^2\omega_1^2)}$$

$$(r \neq 1)$$

$$D_{1r}^{(2)} = \frac{c^2 e^{-\gamma\alpha}}{2(\omega_r^2 - \omega_1^2)} \sum_k B_{2k}^{(1)} \beta_{kr} - \frac{9c^2\gamma\omega_1\alpha_r e^{-3\gamma\alpha}}{2(S_r^2 + 16\omega_1^2\gamma^2)} \quad (r \neq 1)$$

$$C_{3r}^{(2)} = \frac{c^2 e^{-\gamma\alpha}}{2(\omega_r^2 - 9\omega_1^2)} \sum_k A_{2k}^{(1)} \beta_{kr} + \frac{3c^2 \alpha_r Q_r e^{-3\gamma\alpha}}{8(Q_r^2 + 144\gamma^2\omega_1^2)} \quad (\text{all } r)$$

$$D_{3r}^{(2)} = \frac{c^2 e^{-\gamma\alpha}}{2(\omega_r^2 - 9\omega_1^2)} \sum_k B_{2k}^{(1)} \beta_{kr} - \frac{9c^2\gamma\omega_1\alpha_r e^{-3\gamma\alpha}}{2(Q_r^2 + 144\gamma^2\omega_1^2)} \quad (\text{all } r) \tag{49}$$

where

$$S_r = \omega_r^2 - \omega_1^2 + 4\gamma^2$$

$$Q_r = \omega_r^2 - 9\omega_1^2 + 4\gamma^2 .$$

Thus from (39) and (40) we have

$$u_2(x,\alpha,\psi) = 0$$

$$v_2(x,\alpha,\psi) = \sum_{r=2}^{\infty} (C_{1r}^{(2)} \cos\psi + D_{1r}^{(2)} \sin\psi)\, \phi_r(x) + \sum_{r=1}^{\infty} (C_{3r}^{(2)} \cos 3\psi + D_{3r}^{(2)} \sin 3\psi)\, \phi_r(x) \tag{50}$$

where the constants $C_{1r}^{(2)}$, $D_{1r}^{(2)}$, $C_{3r}^{(2)}$. and $D_{3r}^{(2)}$ are determined by (49).

Hence to the order of ε^2, the solution becomes

$$u(x,t) = \varepsilon u_1(x,\alpha,\psi)$$
$$v(x,t) = e^{-\gamma\alpha}\phi_1(x)\cos\psi + \varepsilon^2 v_2(x,\alpha,\psi) \tag{51}$$

where $u_1(x,\alpha,\psi)$ and $v_2(x,\alpha,\psi)$ are given by (45) and (50) and α, ψ by

$$\frac{d\alpha}{dt} = 1 + \varepsilon^2 R e^{-2\gamma\alpha}$$
$$\frac{d\psi}{dt} = \omega_1 + \varepsilon^2 Q e^{-2\gamma\alpha} \tag{52}$$

and Q, R by equations (48).

The equations (52) may be integrated to give

$$e^{2\gamma\alpha} = Ce^{2\gamma t} - \varepsilon^2 R$$
$$\psi = (\omega_1 + \varepsilon^2 Q e^{-2\gamma\alpha})\, t + \psi_0 \tag{53}$$

where C and ψ_0 are constants of integration.

Eliminating α from equations in (53), we have

$$\psi = \omega_1 t + \varepsilon^2 Qt/(Ce^{2\gamma t} - \varepsilon^2 R) + \psi_0 . \tag{54}$$

As a particular case, when $\gamma t << 1$, from (51), (53), and (54) we have

$$\text{amplitude of } v \approx e^{-\gamma\alpha} \approx \frac{1}{\sqrt{C}}$$

and

$$\psi \sim (\omega_1 + \frac{\varepsilon^2 Q}{C})\ t + \psi_0 \ .$$

Thus for $\gamma t << 1$, we have

$$\text{angular frequency} \approx \omega_1 + \frac{\varepsilon^2 Q}{C} \approx \omega_1 + \varepsilon^2 Q\ (\text{amplitude})^2 \ .$$

Bearing in mind that the physical transverse displacement is $\varepsilon v(x,t)$, we see that one effect of the kinematical nonlinearities is to increase the initial natural frequency of the basic mode from its linear value ω_1 by an amount equal to Q times the square of the amplitude of vibration. This result is consistent with the one described in the previous section, when the damping force is absent.

Also for large times, i.e., $\gamma t >> 1$, (54) implies $\psi \sim \omega_1 t + \psi_0$ which is as expected, because after a large time interval the amplitude of vibration decays, the nonlinear effects become negligible and the string returns

to its natural frequency of vibration.

A particular example.

Let us choose the units of length in such a way that the ends of the string are at x = 0 and x = 1. We consider the case with fixed-end boundary conditions; that is,

$$u(0,t) = u(1,t) = 0$$

$$v(0,t) = v(1,t) = 0 .$$

In this case, the normalized eigenfunctions and eigenvalues satisfying equations (30) and (5) are

$$\Phi_n(x) = \sqrt{2}\,\sin(n\pi x), \quad \Omega_n = n\pi c_1$$

$$\phi_n(x) = \sqrt{2}\,\sin(n\pi x), \quad \omega_n^2 = n^2\pi^2 c_2^2 - \gamma^2 .$$

Using these eigenfunctions in the definitions (43) and (47), we obtain

$$K_r = -\frac{\pi^3}{\sqrt{2}}\,\delta_{2r}\ , \quad \alpha_1 = -\pi^4/2, \quad \beta_{k1} = -\sqrt{2}\,\pi^3\delta_{k2}$$

where δ_{kp} is the Kronecker delta symbol.

Substituting these into equations (48), summing

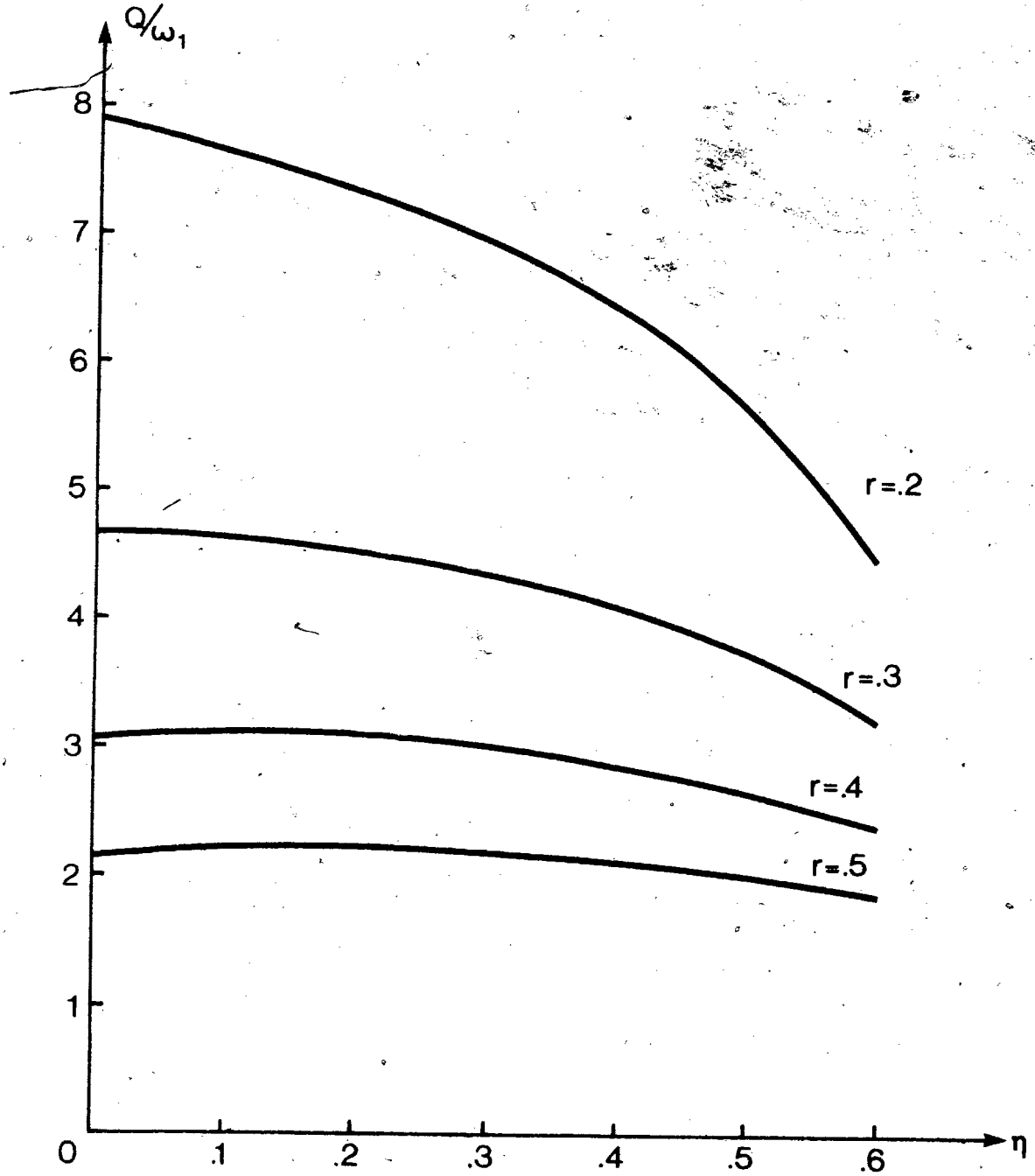
over k and rescaling γ as $\gamma^2 = \pi^2 c_2^2 \eta$, we have

$$\frac{Q}{\omega_1} = \frac{\pi^2(1 - r)}{32r}\left[9 - \frac{1 - r}{1 - \eta}\left(\frac{2}{1 + r\eta} + \frac{1 - r + 2r\eta}{(1 - r)^2 + 4r\eta}\right)\right]$$

$$R = \frac{\pi^2(1 - r)}{32r}\left[-9 + \frac{1 - r}{1 - \eta}\left(\frac{2}{1 + r\eta} + \frac{1 - 3r + 4r\eta}{(1 - r)^2 + 4r\eta}\right)\right]$$

where $r = c_2^2/c_1^2$.

The graph of Q/ω_1 against the damping factor η is shown in the figure for different values of r.


Q/ω1
8
7
6
5
4
3
2
1
0
.1
.2
.3
.4
.5
.6
η
r=.2
r=.3
r=.4
r=.5

CHAPTER 4

ASYMPTOTIC SOLUTION OF A NONLINEAR HYPERBOLIC DIFFERENTIAL EQUATION WITH SLOWLY VARYING COEFFICIENTS.

1. INTRODUCTION.

In recent years, Mitropolskii-Mosenkeev [9], Bojadziev and Lardner [10,11,12] and others have extended the asymptotic method of Krylov-Bogoliubov-Mitropolskii (K-B-M) to solve a number of second order partial differential equations of hyperbolic type. We use this method to find the monofrequent oscillations of a 1-dimensional continuum which is subjected to slowly varying damping and is also elastically aging with time. The vibrations of such a medium are taken to be described by the partial differential equation:

$$\rho(x)[u_{tt} + 2\gamma(\tau)u_t] - c^2(\tau)\frac{\partial}{\partial x}[K(x)u_x] = \varepsilon F(x,u,u_x,u_t,..) \tag{1}$$

where ε is a small parameter, $\tau = \varepsilon t$ ia a slowly varying time and $\rho(x)$, $K(x)$ are given positive functions of x on $0 < x < \ell$. The function F is some given non-linear function defined on the same interval $0 < x < \ell$,

and is assumed to have a sufficient number of derivatives with respect to its arguments. The terms $\gamma(\tau)$ and $c(\tau)$ in the equation appear because of slowly varying damping and slowly varying elastic property of the medium. We further assume that u also satisfies a set of homogeneous boundary conditions $\mathcal{B}_j(u) = 0$, $j = 1,2$ which involve u and u_x at the end points $x = 0$, $x = \ell$.

Setting $\varepsilon = 0$, $\tau = \tau_0$ in (1), the generating eqation

$$\rho(x)[u_{tt}^{(0)} + 2\gamma(\tau_0)u_t^{(0)}] - c^2(\tau_0)\frac{\partial}{\partial x}[K(x)u_x^{(0)}] = 0, \quad \mathcal{B}_j(u^{(0)}) = 0 \tag{2}$$

has an infinite set of separable solutions of the form

$$a_n\phi_n(x)e^{-\gamma(\tau_0)t}\cos(\omega_n(\tau_0)t + \Psi_n), \quad n = 1,2,... \tag{3}$$

where a_n, Ψ_n are constants and $\phi_n(x)$ satisfy the differential equation

$$c^2(\tau_0)\frac{d}{dx}[K(x)\phi_n'(x)] + \lambda_n^2(\tau_0)\rho(x)\phi_n(x) = 0 \tag{4}$$

$$\lambda_n^2(\tau_0) = \omega_n^2(\tau_0) + \gamma^2(\tau_0), \quad \mathcal{B}_j(\phi_n) = 0, \quad j = 1,2$$

We shall assume that the damping is less than critical, that is $\gamma < \lambda_n$ for all n or $\omega_n^2 > 0$.

It is well known that if the boundary conditions satisfy the self-adjointness criteria, the eigenfunctions $\{\phi_n(x)\}$ form a complete set of orthogonal functions. After suitable normalization, we have:

$$\int_0^{\ell} \rho(x)\phi_n(x)\phi_m(x)dx = \delta_{nm} \tag{5}$$

where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$.

We seek the monofrequent solutions of (1) which for $\varepsilon = 0$ correspond to $n = 1$ in (4), that is

$$u^{(0)}(x,t) = a_0 e^{-\gamma(\tau_0)t}\phi_1(x)\cos(\omega_1(\tau_0)t + \Psi_1) \tag{6}$$

2. ASYMPTOTIC SOLUTION.

According to K-B-M method, we seek the solution of nonlinear equation (1) in the form

$$u(x,t) = \phi_1(x)e^{-\gamma\alpha}\cos\psi + \varepsilon u_1(\tau,\alpha,\psi,x) + \varepsilon^2 u_2(\tau,\alpha,\psi,x) + \ldots\ldots \tag{7}$$

where $\gamma = \gamma(\tau)$ and $u_r(\tau,\alpha,\psi,x)$ are 2π periodic in ψ for each $r = 1,2,\ldots$. The parameters α and ψ are functions of time t and are given by the differential equations

$$\frac{d\alpha}{dt} = 1 + \varepsilon A_1(\tau,\alpha) + \varepsilon^2 A_2(\tau,\alpha) + \ldots\ldots$$

$$\frac{d\psi}{dt} = \omega_1(\tau) + \varepsilon B_1(\tau,\alpha) + \varepsilon^2 B_2(\tau,\alpha) + \ldots\ldots \tag{8}$$

We assume the truth of equations (2) and (4) when τ_0 is replaced by τ . The functions $\phi_n(x)$ are independent of τ .

The functions $\{A_r(\tau,\alpha),\ B_r(\tau,\alpha),\ u_r(\tau,\alpha,\psi,x)\}$ are to be determined from the requirement that the solution (7)-(8) should satisfy the equation (1).

It should be noted that for $\varepsilon = 0$, the solution (7)-(8) reduces to the solution (6) of the generating equation. We shall restrict our investigation to the

first approximation only.

Substituting (7)-(8) into (1) and equating the coefficients of ε on both sides, we have:

$$\rho(x)\left[\frac{\partial^2 u_1}{\partial\alpha^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial\alpha\partial\psi} + \omega_1^2 \frac{\partial^2 u_1}{\partial\psi^2} + 2\gamma \frac{\partial u_1}{\partial\alpha} + 2\gamma\omega_1 \frac{\partial u_1}{\partial\psi}\right] - c^2 \frac{\partial}{\partial x}\left[K(x)\frac{\partial u_1}{\partial x}\right]$$

$$= \rho(x)\phi_1(x)e^{-\gamma\alpha}\left[\left(\gamma \frac{\partial A_1}{\partial\alpha} + 2\omega_1 B_1 + 2\gamma'\right)\cos\psi + \left(\frac{\partial B_1}{\partial\alpha} - 2\gamma\omega_1 A_1 - 2\alpha\gamma'\omega_1 + \omega_1'\right)\sin\psi\right] + F_0(x,\alpha,\psi) \tag{9}$$

where $\gamma' = \frac{d\gamma}{d\tau}$, $\omega_1' = \frac{d\omega_1}{d\tau}$ and

$$F_0(x,\alpha,\psi) = F\Big(x, \phi_1(x)e^{-\gamma\alpha}\cos\psi,\ \phi_1'(x)e^{-\gamma\alpha}\cos\psi,\ - \phi_1(x)e^{-\gamma\alpha}(\gamma\cos\psi + \omega_1\sin\psi)\Big)$$

To solve this, we expand $u_1(\tau,\alpha,\psi,x)$ in terms of eigenfunctions $\{\phi_n(x)\}$ as

$$u_1(\tau,\alpha,\psi,x) = \sum_{r=1}^{\infty} V_r(\tau,\alpha,\psi)\phi_r(x) \tag{10}$$

Substituting (10) into (9), multiplying both sides by $\phi_n(x)$, integrating from 0 to ℓ with respect to x, and using (4) and (5), we have:

$$\frac{\partial^2 V_n}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 V_n}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 V_n}{\partial \psi^2} + 2\gamma \frac{\partial V_n}{\partial \alpha} + 2\gamma\omega_1 \frac{\partial V_n}{\partial \psi} + \lambda_n^2 V_n$$

$$= \delta_{1n} e^{-\gamma\alpha}[(\gamma \frac{\partial A_1}{\partial \alpha} + 2\omega_1 B_1 + 2\gamma') \cos \psi$$

$$+ (\frac{\partial B_1}{\partial \alpha} - 2\gamma\omega_1 A_1 - 2\alpha\gamma'\omega_1 + \omega_1') \sin \psi] + F_n(\alpha,\psi) \qquad (11)$$

where

$$F_n(\alpha,\psi) = \int_0^{\ell} F_0(x,\alpha,\psi)\, \phi_n(x)\, dx$$

Since $V_n(\tau,\alpha,\psi)$ and $F_n(\alpha,\psi)$ are both 2π periodic in ψ, we can expand them in Fourier series as

$$V_n(\tau,\alpha,\psi) = \frac{1}{2} A_{n0}(\tau,\alpha) + \sum_{r=1}^{\infty} \{A_{nr}(\tau,\alpha)\cos r\psi + B_{nr}(\tau,\alpha)\sin r\psi\} \qquad (12)$$

$$F_n(\alpha,\psi) = \frac{1}{2} C_{n0}(\alpha) + \sum_{r=1}^{\infty} \{C_{nr}(\alpha) \cos r\psi + D_{nr}(\alpha) \sin r\psi\}$$

Using (12) in (11) implies

$$\frac{1}{2}\left(\frac{\partial^2 A_{n0}}{\partial\alpha^2} + 2\gamma\frac{\partial A_{n0}}{\partial\alpha} + \lambda_n^2 A_{n0}\right)$$

$$+ \sum_{r=1}^{\infty}\Bigg[\Bigg(\frac{\partial^2 A_{nr}}{\partial\alpha^2} - 2r\omega_1\frac{\partial B_{nr}}{\partial\alpha} + 2\gamma\frac{\partial A_{nr}}{\partial\alpha} + (\lambda_n^2 - r^2\omega_1^2)A_{nr} + 2r\omega_1\gamma B_{nr}\Bigg)\cos r\psi$$

$$+ \Bigg(\frac{\partial^2 B_{nr}}{\partial\alpha^2} - 2r\omega_1\frac{\partial A_{nr}}{\partial\alpha} + 2\gamma\frac{\partial B_{nr}}{\partial\alpha} + (\lambda_n^2 - r^2\omega_1^2)B_{nr} - 2r\omega_1\gamma A_{nr}\Bigg)\sin\psi\Bigg]$$

$$= \delta_{1n}e^{-\gamma\alpha}\left[\left(\gamma\frac{\partial A_1}{\partial\alpha} + 2\omega_1 B_1 + 2\gamma'\right)\cos\psi + \left(\frac{\partial B_1}{\partial\alpha} - 2\gamma\omega_1 A_1 - 2\alpha\omega_1\gamma' + \omega_1'\right)\sin\psi\right] + \frac{1}{2}C_{n0} + \sum_{r=1}^{\infty}\left(C_{nr}\cos r\psi + D_{nr}\sin r\psi\right) \quad (13)$$

We also assume that u_1 is independent of first harmonics in ψ which in turn means that V_n are independent of cos ψ and sin ψ. This implies that the summation on the left of (13) starts at r = 2 (and not r = 1) when n = 1.

For n = 1, r = 1, comparing the coefficients of cos ψ and sin ψ in (13), we get

$$\gamma \frac{\partial A_1}{\partial \alpha} + 2\omega_1 B_1 + 2\gamma' = - C_{11} e^{\gamma\alpha}$$

$$\frac{\partial B_1}{\partial \alpha} - 2\gamma\omega_1 A_1 - 2\alpha\omega_1\gamma' + \omega_1' = - D_{11} e^{\gamma\alpha} \qquad (14)$$

These equations determine A_1 and B_1.

Comparing the coefficients of cos $r\psi$ and sin $r\psi$ in (13), we get

$$\frac{\partial^2 A_{n0}}{\partial \alpha^2} + 2\gamma \frac{\partial A_{n0}}{\partial \alpha} + \lambda_n^2 A_{n0} = C_{n0} \quad (n \geq 1)$$

$$\frac{\partial^2 A_{nr}}{\partial \alpha^2} + 2r\omega_1 \frac{\partial B_{nr}}{\partial \alpha} + 2\gamma \frac{\partial A_{nr}}{\partial \alpha} + (\lambda_n^2 - r^2\omega_1^2) A_{nr} + 2n\omega_1\gamma B_{nr} = C_{nr}$$

$$\frac{\partial^2 B_{nr}}{\partial \alpha^2} - 2r\omega_1 \frac{\partial A_{nr}}{\partial \alpha} + 2\gamma \frac{\partial B_{nr}}{\partial \alpha} + (\lambda_n^2 - r^2\omega_1^2) B_{nr} - 2n\omega_1\gamma A_{nr} = D_{nr}$$

The last two equations hold for all n and r except when $n = 1$, $r = 1$. The coefficients A_{nr} and B_{nr} are determined from these equations.

From (12) V_n are determined and then from (10) $u_1(\tau,\alpha,\psi,x)$ is determined. Therefore the solution of (1) upto first improved approximation is

$$u = \phi_1(x)e^{-\gamma\alpha} \cos \psi + \varepsilon u_1(\tau,\alpha,\psi,x)$$

where $\frac{d\alpha}{dt} = 1 + \varepsilon A_1(\tau,\alpha)$, $\frac{d\psi}{dt} = \omega_1(\tau) + \varepsilon B_1(\tau,\alpha)$

and A_1, B_1 are determined by the differential equations (14).

3. TWO-DIMENSIONAL VIBRATIONS OF A STRING.

Although we have investigated the solution of a vibrating medium in one-dimension, the method can be extended to a pair of coupled equations representing the coupled transverse and longitudinal vibrations of a stretched string. As in Chapter 3, section II, the equations of a vibrating string with transverse damping, in this case will be

$$\begin{aligned} u_{tt} - c_1^2(\tau)u_{xx} &= \varepsilon c^2(\tau)v_x v_{xx} \\ v_{tt} + 2\gamma(\tau)v_t - c_2^2(\tau)v_{xx} &= \varepsilon c^2(\tau)(u_{xx}v_x + u_x v_{xx}) \end{aligned} \tag{15}$$

where we have neglected the terms of order ε^2 and higher because we shall be restricting ourselves to find the solution upto order ε. In these equations, u, v represent respectively the longitudinal and transverse displacements, c_1, c_2 are the velocities of longitudinal and transverse wave propagation, and $c^2 = c_1^2 = c_2^2$. c_1, c_2, c are functions of τ because of the elastically aging property of the string and $\gamma = \gamma(\tau)$ because damping slowly varies with time.

For $\varepsilon = 0$, $\tau = \tau_0$, the generating equations are

$$u_{tt}^{(0)} - c_1^2(\tau_0)u_{xx}^{(0)} = 0 , \quad v_{tt}^{(0)} + 2\gamma(\tau_0)v_t^{(0)} - c_2^2(\tau_0)v_{xx}^{(0)} = 0$$

$$B_j(u^{(0)}) = B_j(v^{(0)}) = 0 , \quad j = 1,2. \tag{16}$$

These equations possess a set of separable solutions for $u^{(0)}$ and $v^{(0)}$ which can be written as

$$u^{(0)}(x,t) = \alpha_n \Phi_n(x) \cos(\Omega_n(\tau_0) + \Psi_n)$$

$$v^{(0)}(x,t) = a_n \phi_n(x) e^{-\gamma(\tau_0)t} \cos(\omega_n(\tau_0)t + \psi_n) \tag{17}$$

where α_n, a_n, Ψ_n, ψ_n are constants and $\Phi_n(x)$, $\phi_n(x)$ satisfy the equations

$$c_1^2(\tau_0)\Phi_n''(x) + \Omega_n^2(\tau_0)\Phi_n(x) = 0$$

$$c_2^2(\tau_0)\phi_n''(x) + \lambda_n^2(\tau_0)\phi_n(x) = 0 , \tag{18}$$

$$\lambda_n^2(\tau_0) = \omega_n^2(\tau_0) + \gamma^2(\tau_0)$$

The equations corresponding to (5) in this case are

$$\int_0^\ell \Phi_n(x)\Phi_m(x)dx = \int_0^\ell \phi_n(x)\phi_m(x)dx = \delta_{nm} \tag{19}$$

We seek the particular solution of (15) which is close to the basic mode of transverse vibration:

$$u^{(0)}(x,t) = 0 \ ,$$

$$v^{(0)}(x,t) = a_0 e^{-\gamma(\tau_0)t}\phi_1(x) \cos\left(\omega_1(\tau_0)t + \psi_1\right) \tag{20}$$

According to the KBM method, we seek the solution of (15) in the form

$$u = \varepsilon u_1(\tau,\alpha,\psi,x) + \varepsilon^2 \ldots\ldots$$

$$v = e^{-\gamma(\tau)\alpha}\phi_1(x) \cos\psi + \varepsilon V_1(\tau,\alpha,\psi,x) + \varepsilon^2 \ldots\ldots \tag{21}$$

where α and ψ are functions of t given by

$$\frac{d\alpha}{dt} = 1 + \varepsilon A_1(\alpha,\tau) + \varepsilon^2 \ldots$$

$$\frac{d\psi}{dt} = \omega_1(\tau) + \varepsilon B_1(\alpha,\tau) + \varepsilon^2 \ldots\ldots \tag{22}$$

We also assume that $\omega_n \neq p\omega_1$ for p any integer and $n > 2$ and as before we assume the validity of equations (16) and (18) when c_1, c_2, Ω_n, λ_n are functions of τ, not constants. We also assume that $\phi_n(x)$ and $\Phi_n(x)$ are independent of τ.

Substituting (21)-(22) in (15), and equating the coefficients of ε on both sides, we obtain the following equations for u_1 and V_1:

$$\frac{\partial^2 u_1}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 u_1}{\partial \psi^2} - c_1^2 \frac{\partial^2 u_1}{\partial x^2}$$

$$= c^2 e^{-2\gamma\alpha} \phi_1'(x)\phi_1''(x) \cos^2 \psi \qquad (23)$$

and

$$\frac{\partial^2 V_1}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 V_1}{\partial \alpha \partial \psi} + \omega_1^2 \frac{\partial^2 V_1}{\partial \psi^2} + 2\gamma\omega_1 \frac{\partial V_1}{\partial \psi} + 2\gamma \frac{\partial V_1}{\partial \alpha} - c_2^2 \frac{\partial^2 V_1}{\partial x^2}$$

$$= \phi_1(x) e^{-\gamma\alpha} \left[(2\gamma' + 2\omega_1 B_1 + 2\alpha\gamma\gamma' + \gamma \frac{\partial A_1}{\partial \alpha}) \cos \psi \right.$$

$$\left. + (\omega_1' + \frac{\partial B_1}{\partial \alpha} - 2\gamma\omega_1 A_1 - 2\alpha\omega_1\gamma') \sin \psi \right] \qquad (24)$$

To solve (23)-(24), we expand u_1, V_1 as double Fourier series in ψ and x using the harmonic basis for ψ and the bases $\{\Phi_n(x)\}$, $\{\phi_n(x)\}$ for x :

$$u_1(\tau,\alpha,\psi,x) = \sum_{r=1}^{\infty} [\frac{1}{2} A_{0r}(\tau,\alpha) + \sum_{n=1}^{\infty} \{A_{nr}(\tau,\alpha) \cos n\psi$$

$$+ B_{nr}(\tau,\alpha) \sin n\psi\}]\Phi_n(x) \qquad (25)$$

$$V_1(\tau,\alpha,\psi,x) = \sum_{r=1}^{\infty} [\frac{1}{2} C_{0r}(\tau,\alpha) + \sum_{n=1}^{\infty} \{C_{nr}(\tau,\alpha) \cos n\psi$$

$$+ D_{nr}(\tau,\alpha) \sin n\psi\}]\phi_n(x) \qquad (26)$$

Substituting (25) into (23), making use of conditions (18)-(19) and then comparing coefficients of cos $n\psi$, sin $n\psi$ on both sides, we find that all A_{nr}, B_{nr} are zero except that

$$A_{0r} = c^2 K_r e^{-2\gamma\alpha}/(\Omega_r^2 + 4\gamma^2)$$

$$A_{2r} = -\frac{B_{2r}}{8\gamma\omega_1} = c^2 P_r K_r/\{2(P_r^2 + 64\gamma^2\omega_1^2)\} \tag{27}$$

where $P_r = \Omega_r^2 + 4\gamma^2 - \omega_1^2$

and $K_r = \int_0^{\ell} \phi_1'(x)\phi_1''(x)\Phi_r(x)dx$

Thus,

$$u_1(\tau,\alpha,\psi,x) = \sum_{r=1}^{\infty} (\tfrac{1}{2} A_{0r} + A_{2r} \cos 2\psi + B_{2r} \sin 2\psi)\Phi_r(x) \tag{28}$$

where A_{0r} , A_{2r} , B_{2r} are given by (27).

Substituting (26) in (24), making use of conditions (18)-(19) and then comparing the coefficients of cos $n\psi$ and sin $n\psi$, we get a series of following differential equations for C_{nr} , D_{nr} :

$$\frac{\partial^2 C_{0r}}{\partial \alpha^2} + 2\gamma \frac{\partial C_{0r}}{\partial \alpha} + \lambda_r^2 C_{0r} = 0$$

$$\frac{\partial^2 C_{nr}}{\partial \alpha^2} + 2\left(\gamma \frac{\partial C_{nr}}{\partial \alpha} + n\omega_1 \frac{\partial D_{nr}}{\partial \alpha}\right) + (\lambda_r^2 - n^2\omega_1^2) C_{nr} + 2n\gamma\omega_1 D_{nr}$$

$$= \delta_{1r}\delta_{1n} e^{-\gamma\alpha}\left(2\omega_1 B_1 + \gamma \frac{\partial A_1}{\partial \alpha} + 2\alpha\gamma\gamma' + 2\gamma'\right)$$

(29)

$$\frac{\partial^2 D_{nr}}{\partial \alpha^2} + 2\left(\gamma \frac{\partial D_{nr}}{\partial \alpha} - n\omega_1 \frac{\partial C_{nr}}{\partial \alpha}\right) + (\lambda_r^2 - n^2\omega_1^2) D_{nr} - 2n\gamma\omega_1 C_{nr}$$

$$= -\delta_{1r}\delta_{1n} e^{-\gamma\alpha}\left(2\omega_1\gamma A_1 - \frac{\partial B_1}{\partial \alpha} + 2\alpha\omega_1\gamma' - \omega_1'\right)$$

We seek particular solutions of the system (29). Now (29)(i) implies that $C_{0r} = 0$ and (29)(ii) and (iii) imply that

$$C_{nr} = D_{nr} = 0 \quad \text{for } n \geq 2$$

and $$C_{1r} = D_{1r} = 0 \quad \text{for } r \neq 1.$$

Also for $n = 1$, $r = 1$, equations (29)(ii),(iii) lead to differential equations for A_1 , B_1 :

$$2\omega_1 B_1 + \gamma \frac{\partial A_1}{\partial \alpha} + 2\gamma'(1 + \alpha\gamma) = 0$$

$$2\gamma\omega_1 A_1 - \frac{\partial B_1}{\partial \alpha} + 2\gamma'\alpha\omega_1 - \omega_1' = 0$$

These equations for A_1 , B_1 lead to the solutions

$$A_1 = \frac{1}{2\gamma\omega_1^2}(\omega_1\omega_1' + \gamma\gamma' - 2\alpha\gamma'\omega_1^2) \ , \quad B_1 = \frac{\gamma'}{2\omega_1}(1 + 2\alpha\gamma) \qquad (30)$$

In (29), C_{11} and D_{11} remain undetermined and we assume that $C_{11} = D_{11} = 0$.

Thus from (26), $V_1 \equiv 0$.

Therefore upto first improved approximation, the solution of (15) is

$$u = \varepsilon u_1(\tau, \alpha, \psi, x) \ , \qquad V = \phi_1(x) e^{-\gamma\alpha} \cos\psi \qquad (31)$$

where $\frac{d\alpha}{dt} = 1 + \varepsilon A_1(\tau, \alpha)$, $\frac{d\psi}{dt} = \omega_1(\tau) + \varepsilon B_1(\tau, \alpha)$

and A_1 , B_1 and u_1 are given by (30) and (28).

The differential equations for α and ψ in (31) can be integrated in particular cases.

Substituting (30) into (31), we have:

$$\frac{d\alpha}{dt} = 1 + \frac{\varepsilon}{2\gamma\omega_1^2}\,(\omega_1\omega_1' + \gamma\gamma' - 2\alpha\gamma'\omega_1^2)$$

$$\text{or}\quad \varepsilon\frac{d\alpha}{d\tau} = 1 + \frac{\varepsilon}{2\gamma\omega_1^2}\,(\omega_1\omega_1' + \gamma\gamma') - \varepsilon\frac{\gamma'}{\gamma}\,\alpha$$

Therefore,

$$\varepsilon\left(\gamma\frac{d\alpha}{d\tau} + \alpha\frac{d\gamma}{d\tau}\right) = \gamma + \frac{\varepsilon}{4\omega_1^2}\frac{d}{d\tau}\,(\omega_1^2 + \gamma^2)$$

$$\text{or}\quad \varepsilon\frac{d}{d\tau}\,(\alpha\gamma) = \gamma + \frac{\varepsilon}{4\omega_1^2}\frac{d}{d\tau}\,(\omega_1^2 + \gamma^2) \qquad (32)$$

$$\text{and}\quad \frac{d\psi}{dt} = \omega_1 + \varepsilon\frac{\gamma'}{2\omega_1}\,(1 + 2\alpha\gamma)$$

Let us consider the particular case in which γ is constant and $\lambda_n(\tau) = \lambda_n c_2(\tau)$ where λ_n are constants.

Then using (18), the equations (32)(iii) reduce to

$$\varepsilon\gamma\frac{d\alpha}{d\tau} = \gamma + \frac{\varepsilon}{2(\lambda_1^2 c_2^2 - \gamma^2)}\cdot\lambda_1^2 c_2 c_2'$$

$$\text{and}\quad \frac{d\psi}{dt} = \omega_1 .$$

Integrating the first equation between 0 and τ and the second equation with respect to t , we have:

$$\varepsilon[\gamma\alpha|_{\tau} - \gamma\alpha|_{0}] = \gamma\tau + \frac{\varepsilon}{4}\{\ell n(\lambda_1^2 c_2^2 - \gamma_1^2)\}_0^{\tau}$$

$$\text{or} \quad \gamma\alpha|_{\tau} = \gamma\alpha|_{0} + \gamma t + \frac{1}{4}\ln\left[\frac{\lambda_1^2 c_2^2(\tau) - \gamma^2}{\lambda_1^2 c_2^2(0) - \gamma^2}\right] \tag{33}$$

$$\text{and} \quad \psi = \int^{t} \omega_1 dt \ .$$

CHAPTER 5

PLANE SHOCK WAVES IN VISCOELASTIC MEDIA DISPLAYING CUBIC ELASTICITY.

1. INTRODUCTION.

In this chapter we consider the plane vibrations of a medium which is predominantly linearly elastic but which displays in addition small nonlinear elastic and linear viscoelastic behaviour. The nonlinearity in elasticity is supposed to occur in the form of terms in the stress-strain relation which are quadratic and cubic in strain and the viscoelasticity is taken as a functional term in the stress-strain relation. The case when the stress-strain relation contains only nonlinear quadratic term in strain in addition to linear term has been investigated by Lardner [22] and it has been shown that the solution of any non-trivial initial value problem for such a medium develops a singularity in the form of a shock-wave after a finite interval of time. In [22], the structure of the shock layer has also been investigated.

The consideration of the case when the stress-strain relation contains quadratic as well as cubic terms in strain is important since for many materials it can be expected that the stress is an odd function of strain measured from its natural state. In such a case the departure from linearity can be expected to be cubic and not quadratic in strain.

Similar problems have been investigated earlier by Mortell and Varley [23], Mortel and Seymour [24,25], Kruskal and Zabusky [26], Keller and Kogelman [27] and Chukwendu and Kevorkian [15]. The methods of approximation used were the averaging method or closely related Krylov-Bogoliubov-Mitropolskii (K-B-M) method, method of strained co-ordinates or the two time method. In [24-27], the two-time method was used to find the solution in the form of an expansion in spatial eigenfunctions. The system of equations for the time-dependent amplitudes obtained by the two-time method is the same as obtained by the averaging method. In fact a comparison of two-time and averaging methods [20] has shown that in their lowest approximations, these two methods are formally equivalent for a wide class of hyperbolic partial

differential equations with small nonlinearities. However, it has been pointed out by Nayfeh [28], that for certain nonlinear wave equations, a direct use of the two-time method without an eigenfunction expansion allows the solution to be generated more directly than through the averaging method. The problem to be investigated in this chapter falls under this category.

The equations describing the physical model are written down in section 2, where we show that they lead to a certain nonlinear wave equation for longitudinal displacement of the column.

In section 3, the solution is obtained by using two-time expansion. The solution to the lowest order consists of the superposition of two modulated and dispersed travelling waves, one travelling to the right and the other to the left. Unlike the case of quadratic elasticity, the two waves in this case are not quite uncoupled and a term proportional to the total energy contained in one wave appears in the differential equation which describes the propagation of the other wave and vice versa.

In section 4, we examine the case of a purely elastic regime when dissipation can be ignored and the energy in each of the two waves is constant. When shocks form in the solution, the dissipation becomes significant and the coupling between the two waves becomes less trivial.

In the remaining sections, we examine the structure of shock-wave when the predominant dissipative mechanism is viscoelastic in nature. In section 5, we consider the Voigt model, when the viscoelasticity occurs as a small term in the stress-strain relation proportional to the strain rate. The complete solution involving the shock-wave is found by means of a matching technique.

In section 6, we examine the case of general viscoelastic relaxation function. In this case we find an integral equation for the inner solution within the shock-layer and obtain the condition for the shock-velocity.

In section 7, we investigate a particular case when the relaxation function is of exponential nature and derive the explicit solution for the integral equation of section 6.

2. THE PHYSICAL MODEL.

We shall consider the plane vibrations of a viscoelastic slab. Let x by a coordinate through the slab and let u(x,t) denote the displacement in the x-direction at time t of the cross-section of the slab which occupies the position x in the natural state. Then the longitudinal strain e(x,t) and particle velocity v(x,t) are defined by

$$e = u_x \, , \quad v = u_t \, , \tag{1}$$

subscripts denoting partial differentiation. The equation of motion takes the form

$$\rho u_{tt} = \sigma_x \tag{2}$$

where $\sigma(x,t)$ is the longitudinal stress and ρ the density.

We shall suppose that the constitutive law relating stress and strain for the material of which the slab is composed takes the form

$$\sigma(x,t) = E\{e(x,t) + \mu_1[e(x,t)]^2 + \tfrac{1}{3}\nu[e(x,t)]^3 + 0(e^4)\}$$
$$- \int_0^\infty G_1(s)[e(x,t-s) - e(x,t)]ds. \tag{3}$$

Here E represents the long-time plane extensional modulus and the μ_1- and ν-terms provide contributions to the stress respectively quadratic and cubic in the strain. We shall assume that these two terms which produce the deviation from linearity of material behaviour are small compared to the leading term in eqn. (3). However, we shall not assume that the cubic term is necessarily of smaller order than the quadratic term.

The final term in eqn. (3) is the viscoelastic contribution to the stress, $G_1(s)$ being the relaxation modulus. We shall assume that this term in never more significant than the nonlinear elastic terms. Later on, in deriving explicit solutions, we shall make the further restriction that the viscoelastic term is significant only within the shock layers which develop in the solution- that is, only within regions in which the strain is varying rapidly.

It is useful in keeping note of the different orders of magnitude to introduce a re-scaled strain $e^*(x,t)$ which is of order unity and to write $e(x,t) = \sqrt{\varepsilon}\, e^*(x,t)$ ($\varepsilon << 1$). The other variables are

similarly re-scaled: $u = \sqrt{\varepsilon}\, u^*$, $v = \sqrt{\varepsilon}\, v^*$, $\sigma = \sqrt{\varepsilon}\, \sigma_*$. We also write $\mu_1 = \sqrt{\varepsilon}\, \mu$ and $G_1(s) = \varepsilon E G(s)$, so that the stress-strain relation in terms of the new variables takes the form

$$\sigma^*(x,t)/E = e^*(x,t) + \varepsilon\{\mu[e^*(x,t)]^2 + \tfrac{2}{3}\nu[e^*(x,t)]^3 - \int_0^\infty G(s)[e^*(x,t-s) - e^*(x,t)]ds\} + O(\varepsilon^2). \tag{4}$$

Equations (1) and (2) remain unchanged in terms of the re-scaled variables.

From now on, we shall drop the *'s, writing simply u for the re-scaled displacement u*, and so on.

Equations (1), (2) and (4) can be combined into a single equation for u(x,t) which takes the form

$$u_{tt} - u_{xx} = \varepsilon\{2\mu u_x u_{xx} + 2\nu u_x^2 u_{xx} - \int_0^\infty G(s)[u_{xx}(x,t-s) - u_{xx}(x,t)]ds\} \tag{5}$$

where units have been chosen in such a way that $E/\rho = 1$.

The viscoelastic relaxation function G(s) is assumed to decay to zero as $s \to \infty$ with a characteristic

decay time which is denoted by δ. If it happens that the strain changes slowly relative to the time scale δ, the strain $e(x,t-s)$ in the relaxation term in eqn. (4) may be expanded in a Taylor expansion about $s = 0$, and we obtain the approximate result that

$$\sigma(x,t)/E = e(x,t) + \varepsilon\{\mu[e(x,t)]^2 + \frac{2}{3}\nu[e(x,t)]^3 + ke_t(x,t)\} + O(\varepsilon^2) \qquad (6)$$

where $k = \int_0^\infty sG(s)ds$. We shall refer to this expression as the Voigt approximation. Within this approximation, eqn. (5) is replaced by the equation

$$u_{tt} - u_{xx} = \varepsilon(2\mu u_x u_{xx} + 2\nu u_x^2 u_{xx} + k u_{xxt}). \qquad (7)$$

In addition to these field equations it is necessary to impose initial and boundary conditions on the system. We shall suppose that the slab occupies the region $0 < x < \ell$ and that the faces $x = 0$ and $x = \ell$ are held fixed. Then the boundary conditions take the form

$$u(0,t) = u(\ell,t) = 0.$$

As initial conditions we shall assume that the initial displacement and velocity, $u(x,0)$ and $v(x,0) = u_t(x,0)$,

are both prescribed. It follows therefore that the initial strain $e(x,0) = u_x(x,0)$ is also known.

3. APPROXIMATE SOLUTION USING TWO-TIME METHOD.

Both eqn's. (5) and (7) are special examples of the general nonlinear wave equation

$$u_{tt} - u_{xx} = \varepsilon\Phi\{u\} \tag{8}$$

where Φ is some nonlinear functional of u. We shall study the solution of equations of this type using a two-time method. According to this method, we introduce the slow time variable $\tau = \varepsilon t$ and seek the solution in the form of an expansion

$$u(x,t) = u_0(x,t,\tau) + \varepsilon u_1(x,t,\tau) + 0(\varepsilon^2). \tag{9}$$

In order that the lowest order solution $u = u_0(x,t,\tau)$ remain a valid approximation for times of order ε^{-1}, we must impose the requirement that u_1 should not contain secular terms growing in proportion to t.

After substituting the expansion (9) into eqn. (8) and comparing terms of different orders of ε we obtain the following equations for u_0 and u_1:

$$u_{0,tt} - u_{0,xx} = 0$$

$$u_{1,tt} - u_{1,xx} = -2u_{0,t\tau} + \Phi\{u_0\}.$$

Here the subscripts after the comma indicate partial differentiation.

The solution for u_0 may be written down immediately, and is of the form

$$u_0(x,t,\tau) = F(\alpha,\tau) + G(\beta,\tau) \tag{10}$$

where $\alpha = t + x$, $\beta = t - x$, and F and G are two as yet arbitrary functions. The equation for u_1 thus becomes

$$4u_{1,\alpha\beta} = -2\left[F_{\alpha\tau}(\alpha,\tau) + G_{\beta\tau}(\beta,\tau)\right] + \Phi\left(F(\alpha,\tau) + G(\beta,\tau)\right). \tag{11}$$

The requirement that u_1 should not contain secular terms will turn out to impose conditions on F and G. Although it is possible to write down these conditions for a general functional Φ, it is more convenient in the present case to examine the particular forms for Φ in eqns. (5) and (7) individually.

In the case of the Voigt approximation, eqn. (11) takes the form

$$\begin{aligned} 4u_{1,\alpha\beta} = &-2(F_{\alpha\tau} + G_{\beta\tau}) + 2\mu(F_\alpha - G_\beta)(F_{\alpha\alpha} + G_{\beta\beta}) \\ &+ 2\nu(F_\alpha - G_\beta)^2(F_{\alpha\alpha} + G_{\beta\beta}) + k(F_{\alpha\alpha\alpha} + G_{\beta\beta\beta}). \end{aligned} \tag{12}$$

The general solution of this equation is

$$u_1 = \frac{1}{4}\beta\left(-2F_\tau + \mu F_\alpha^2 + \frac{2}{3}\nu F_\alpha^3 + kF_{\alpha\alpha}\right)$$

$$+ \frac{1}{4}\alpha\left(-2G_\tau - \mu G_\beta^2 + \frac{2}{3}\nu G_\beta^3 + kG_{\beta\beta}\right)$$

$$+ \frac{1}{2}\mu\left(FG_\beta - F_\alpha G\right) + \frac{1}{2}\nu\left[G_\beta \int F_\alpha^2 d\alpha + F_\alpha \int G_\beta^2 d\beta - F_\alpha^2 G - FG_\beta^2\right]$$

$$+ F_1(\alpha,\tau) + G_1(\beta,\tau) \qquad (13)$$

where F_1 and G_1 are arbitrary functions of the indicated variables.

The secular terms in u_1 arise from the terms proportional to α and β and also from the two terms involving integrals. If we use the notation

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\theta,\tau)\, d\theta \qquad (14)$$

to denote the long-time average of a general function f (where θ is to be identified with either α or β, as appropriate), then we can write

$$G_\beta \int F_\alpha^2 d\alpha = \alpha \langle F_\alpha^2 \rangle G_\beta + \text{non-secular term}$$

$$F_\alpha \int G_\beta^2 d\beta = \beta \langle G_\beta^2 \rangle F_\alpha + \text{non-secular term.}$$

The secular terms in eqn. (13) will vanish therefore if

we impose the following conditions on F and G:

$$2F_\tau = \mu F_\alpha^2 + \frac{2}{3}\nu F_\alpha^3 + kF_{\alpha\alpha} + 2\nu\langle G_\beta^2\rangle F_\alpha$$

$$2G_\tau = -\mu G_\beta^2 + \frac{2}{3}\nu G_\beta^3 + kG_{\beta\beta} + 2\nu\langle F_\alpha^2\rangle G_\beta. \qquad (15)$$

Let us now examine the case where the full functional form for the viscoelastic term is taken, eqn. (5). In this case the differential equation for u_1 is the same as eqn. (12) except that the k-term is replaced by the expression

$$-\int_0^\infty G(s)\Big[F_{\alpha\alpha}(\alpha - s, \tau - \varepsilon s) - F_{\alpha\alpha}(\alpha,\tau) + G_{\beta\beta}(\beta - s, \tau - \varepsilon s) - G_{\beta\beta}(\beta,\tau)\Big]ds.$$

$G(s)$ decays to zero within an interval δ which can be assumed to be small compared to ε^{-1}, so that to a good approximation we may replace the arguments $\tau - \varepsilon s$ by τ in this integral. Consequently in the solution (13) for u_1, the k-term is replaced by

$$-\frac{1}{4}\beta\int_0^\infty G(s)\Big[F_\alpha(\alpha - s,\tau) - F_\alpha(\alpha,\tau)\Big]ds$$

$$-\frac{1}{4}\alpha\int_0^\infty G(s)\Big[G_\beta(\beta - s,\tau) - G_\beta(\beta,\tau)\Big]ds.$$

Both of these terms are secular, so that the final equations obtained for F and G, in place of eqn. (15), are

$$2F_\tau = \mu F_\alpha^2 + \frac{2}{3}\nu F_\alpha^3 + 2\nu\langle G_\beta^2\rangle F_\alpha - \int_0^\infty G(s)\left[F_\alpha(\alpha - s,\tau) - F_\alpha(\alpha,\tau)\right]ds$$

$$2G_\tau = -\mu G_\beta^2 + \frac{2}{3}\nu G_\beta^3 + 2\nu\langle F_\alpha^2\rangle G_\beta - \int_0^\infty G(s)\left[G_\beta(\beta - s,\tau) - G_\beta(\beta,\tau)\right]ds \tag{16}$$

The conclusion is therefore that the solution is given in lowest order by

$$u(x,t) = F(\alpha,\tau) + G(\beta,\tau) + O(\varepsilon) \tag{17}$$

where F and G satisfy eqns. (15) in the Voigt approximation and eqns. (16) when the full functional viscoelasticity is used. It is expected that this solution will remain valid to within an error which is $O(\varepsilon)$ up to values of time which are of order ε^{-1}.

Before proceeding to solve these equations for F and G, let us observe the effects on these functions of the fixed-end boundary conditions. Putting $x = 0$ and $x = \ell$ in

eqn. (17) we obtain the conditions that

$$F(t,\tau) + G(t,\tau) = 0, \quad F(t + \ell,\tau) + G(t - \ell,\tau) = 0 \quad (\tau = \varepsilon t). \tag{18}$$

The first of these conditions is satisfied if we take $G(\theta,\tau) = - F(\theta,\tau)$ for any pair of arguments (θ,τ). Under this identification, the two equations (15) become identical, as also do the two eqns. (16). Thus consistency is achieved by taking the solution in the form

$$u(x,t) = F(\alpha,\tau) - F(\beta,\tau) + O(\varepsilon) \tag{19}$$

in this case, where $F(\theta,\tau)$ satisfies the equation

$$2F_\tau = \mu F_\theta^2 + \frac{2}{3}\nu F_\theta^3 + 2\nu\langle F_\theta^2\rangle F_\theta + kF_{\theta\theta} \tag{20}$$

in the case of the Voigt approximation, and more generally satisfies

$$2F_\tau = \mu F_\theta^2 + \frac{2}{3}\nu F_\theta^3 + 2\nu\langle F_\theta^2\rangle F_\theta - \int_0^\infty G(s)\left[F_\theta(\theta - s,\tau) - F_\theta(\theta,\tau)\right]ds. \tag{21}$$

Similarly, the second of the boundary conditions (18) is met by assuming that F is periodic in its first argument with period 2ℓ : $F(\theta + 2\ell,\tau) = F(\theta,\tau)$. This

condition is consistent with the differential equations (20) or (21).

It is convenient to introduce $f(\theta,\tau) = F_\theta(\theta,\tau)$. After differentiation with respect to θ, eqns. (20) and (21) then take the form

$$f_\tau = \left[\mu f + \nu f^2 + \nu C(\tau)\right] f_\theta + \frac{1}{2} k f_{\theta\theta} \tag{20'}$$

$$f_\tau = \left[\mu f + \nu f^2 + \nu C(\tau)\right] f_\theta - \frac{1}{2}\int_0^\infty G(s)\left[f_\theta(\theta - s,\tau) - f_\theta(\theta,\tau)\right]ds, \tag{21'}$$

where

$$C(\tau) = \langle f^2 \rangle = \frac{1}{2\ell}\int_0^{2\ell} [f(\theta,\tau)]^2 d\theta. \tag{22}$$

In this last equation we have made use of the periodicity of $f(\theta,\tau)$ in θ in changing from the infinite average in the original definition (14) to the average over one period in eqn. (22).

The strain and velocity are expressible, to lowest order, in terms of the function $f(\theta,\tau)$, for, upon differentiating eqn. (19) with respect to x and t in turn, we obtain that

$$e(x,t) = f(\alpha,\tau) + f(\beta,\tau) + 0(\varepsilon)$$

$$v(x,t) = f(\alpha.\tau) - f(\beta,\tau) + 0(\varepsilon). \qquad (23)$$

From the initial values of e and v, the following initial values of $f(\theta,\tau)$ are obtained:

$$f(\theta,0) = \frac{1}{2}\left[e(\theta,0) + v(\theta,0)\right] \ , \ f(-\theta,0) = \frac{1}{2}\left[e(\theta,0) - v(\theta,0)\right],$$

$$(0 < \theta < \ell). \qquad (24)$$

Consequently the values $f(\theta,0)$ are known for $-\ell < \theta < \ell$. In view of the periodicity of $f(\theta,\tau)$ in θ, it follows therefore that the values $f(\theta,0)$ are known for all values of θ.

For values of t which are $o(\varepsilon^{-1})$, τ is small and may be replaced by zero in the solution (17). The solution then may be interpreted as consisting of two waves, an α-wave with amplitude F travelling to the left and a β-wave with amplitude G travelling to the right. For values of τ significantly different from zero this interpretation breaks down because of the τ-dependence of F and G (although, as we shall see in the next section, an interpretation in terms of simple waves is still possible in the purely

elastic case). Over time-scales of order unity, the F and G parts of the solution still appear approximately as waves propagating in each direction, but the τ-dependence of these functions introduces a slow dispersion and modulation of the two waves.

When $\nu = 0$, the two eqns. (15) become entirely decoupled, indicating that the two parts F and G of the solution propagate independently of one another. This feature of quadratic elastic materials has been noted previously in slightly different contexts by Mortell and Varley [23] and Mortell and Seymour [24,25]. For the case of cubic elasticity ($\nu \neq 0$) the two waves do not propagate independently, although the coupling between them only arises through the average values $\langle F_\beta^2 \rangle$ and $\langle G_\beta^2 \rangle$.

When $\nu = 0$, eqn. (20') reduces to Burgers' equation, as has been shown elsewhere [23].

4. THE ELASTIC SOLUTION.

In this section we shall examine the solution of eqn. (20') or (21') in the case when the viscoelastic terms are absent. In that case the equation for $f(\theta,\tau)$ is the quasi-linear first order equation

$$f_\tau = \left[\mu f + \nu f^2 + \nu C(\tau)\right] f_\theta \ , \tag{25}$$

which may readily be solved by Lagrange's method. The solution may be written in the form

$$f(\theta,\tau) = H(\theta_1) \tag{26}$$

$$\theta_1 = \theta + \left[\mu H(\theta_1) + \nu H(\theta_1)^2\right]\tau + \nu C_1(\tau) \tag{27}$$

$$C_1(\tau) = \int_0^\tau C(\tau')d\tau'.$$

The variable θ_1 represents a characteristic variable for the differential equation (25).

The strain and velocity are obtained from eqn. (23) to be

$$e(x,t) = H(\alpha_1) + H(\beta_1), \quad v(x,t) = H(\alpha_1) - H(\beta_1) \tag{28}$$

where α_1 and β_1 are defined by eqn. (27) in which θ is

replaced respectively by α and β and θ_1 respectively by α_1 and β_1.

In the xt-plane there are two families of characteristic curves corresponding to α_1 = constant and to β_1 = constant. Along any curve of the α-family, the "signal" $H(\alpha_1)$ is constant while along any curve of the β-family, the signal $H(\beta_1)$ is constant. If through a point (x,t) there passes just one curve from each family, carrying the signals $H(\alpha_1)$ and $H(\beta_1)$ respectively, then the strain and velocity at (x,t) are obtained by combining those two signals in accordance with eqns. (28).

Thus we may speak of the signal $H(\alpha_1)$ propagating along the characteristic α_1 = constant. Replacing θ by $\alpha = t + x$ in eqn. (27), we obtain the velocity of propagation to be

$$\left.\frac{dx}{dt}\right|_{\alpha_1} = -1 - \varepsilon\left[\mu H(\alpha_1) + \nu H(\alpha_1)^2 + \nu C(\tau)\right]. \qquad (29)$$

A similar expression is obtained for the velocity of propagation along a β-characteristic. We see that the velocity of propagation on any characteristic depends on

the amplitude of the signal carried by that characteristic.

The amplitudes $H(\theta_1)$ are completely determined by the initial strain and velocity in the medium. Setting $t = 0$ ($\tau = 0$) in eqn. (27) we see that $\theta_1 = \theta$ initially, and hence from eqns. (26) and (24),

$$H(\theta) = \frac{1}{2}\left[e(\theta,0) + v(\theta,0)\right], \quad H(-\theta) = \frac{1}{2}\left[e(\theta,0) - v(\theta,0)\right]$$

$$(0 < \theta < \ell). \qquad (30)$$

(Note that, from eqns. (26) and (27), $H(\theta_1)$ is periodic in θ_1 with period 2ℓ, so that eqns. (30) determine $H(\theta)$ for all values of θ.)

Let us now determine $C(\tau)$. In the definition (22) we substitute the solution (26) and change to θ_1 as integration variable using eqn. (27) thus obtaining

$$C(\tau) = \frac{1}{2\ell}\int_0^{2\ell} H(\theta_1)^2\left[1 - \left(\mu + 2\nu H(\theta_1)\right)\tau H'(\theta_1)\right] d\theta_1 . \qquad (31)$$

Since $H(\theta_1)$ is periodic in θ_1 with period 2ℓ, the terms in eqn. (31) involving the factor $H'(\theta_1)$ vanish, leaving

$$C(\tau) = \frac{1}{2\ell}\int_0^{2\ell} H(\theta_1)^2 d\theta_1 . \qquad (32)$$

It is clear therefore that $C(\tau)$ is in fact a constant, independent of τ. This fact can actually be verified directly from the differential equation (25), but we have kept $C(\tau)$ variable up to this point for future use.

Since C is constant, it follows that the velocity of propagation of the characteristics are constant (c.f. eqn. (29)). In the xt-plane, the α- and β-characteristics form two families of straight lines.

Substituting eqn. (30) into eqn. (32) we obtain the result that

$$C = \frac{1}{4\ell} \int_0^{\ell} \left[e(\theta,0)^2 + v(\theta,0)^2 \right] d\theta,$$

showing that, apart from a constant factor, C is equal to the energy in the initial state of strain and motion of the slab (up to terms of order ε).

Equations (26) and (27) provide a solution of eqn. (25) only if eqn. (27) can be solved for θ_1 for given (θ,τ). This amounts to requiring that only one characteristic from each family passes through each point in the (x,t) plane. Now in fact, the family of lines defined by eqn. (27) does have an envelope which there-

fore must provide a boundary beyond which the solution we have found becomes invalid.

If we write $P(\theta_1) = \mu H(\theta_1) + \nu[H(\theta_1)^2 + C]$, eqn. (27) takes the form $\theta_1 = \theta + \tau P(\theta_1)$. The envelope is obtained by differentiating with respect to θ_1, which gives $\tau P'(\theta_1) = 1$, and so is given parametrically by the equations

$$\tau = [P'(\theta_1)]^{-1} , \quad \theta = \theta_1 - P(\theta_1)/P'(\theta_1) .$$

If we denote by θ_m the value of θ_1 at which $P'(\theta_1)$ is a maximum, then the smallest value of τ on the envelope is given by

$$\tau_f = [P'(\theta_m)]^{-1} .$$

Thus for $\tau < \tau_f$, the solution (26), (27) is valid for all values of θ, while for $\tau \geqslant \tau_f$ it becomes invalid for at least some set of values of θ. Since $P(\theta_1)$ is periodic, provided that it is not identically constant, there is always some value of θ_1 at which $P'(\theta_1) > 0$, so that a finite value of τ_f always exists.

For $\tau \geq \tau_f$, it is necessary to include a shock wave in the solution, and in the following sections we shall investigate the nature of the shock layer in the case when the material displays a small degree of viscoelasticity.

5. SOLUTION IN THE VOIGT APPROXIMATION.

In this section we shall examine the solution of eqn. (20') in the case when the constant k is small. This amounts to assuming that the viscoelastic contribution to the stress in eqn. (6) is small compared to the contribution of the nonlinear elastic terms except in regions where the strain is rapidly varying. In particular, the viscoelastic terms are significant within the neighbourhood of any shock wave which may develop in the solution.

Under these conditions, as long as the initial values of the strain and velocity gradients are not large, the elastic solution found in Section 4 will provide a good approximation, within $0(k)$ of the correct solution of eqn. (20'), up to times at which a shock forms. Furthermore, even for $\tau > \tau_f$, the last term in eqn. (20') can be ignored for points which are outside the shock layer, and at such points the solution is given approximately by eqn. (26) and (27). We shall refer to this approximation as the outer solution. We note however that there is one distinction between the outer solution for $\tau > \tau_f$ and the elastic approximation for $\tau < \tau_f$, namely that in the former $C(\tau)$ is no longer constant. Physically we may regard this as a result of absorption of energy by

the viscoelastic mechanisms which are brought into play by the rapid changes occurring in the shock layer.

We shall investigate the inner solution, that is the approximate solution within the shock layer. Let $\theta = \theta_s(\tau)$ represent the path of the shock wave in the $\theta\tau$-plane, and let us introduce the inner variable $\xi = k^{-1}[\theta - \theta_s(\tau)]$. Then eqn. (20') becomes

$$\frac{1}{2} f_{\xi\xi} + \left[\mu f + \nu f^2 + \bar{\theta}_s'(\tau)\right] f_\xi = k f_\tau \tag{33}$$

where we have introduced the abbreviation

$$\bar{\theta}_s(\tau) = \theta_s(\tau) + \nu C_1(\tau). \tag{34}$$

In the lowest approximation, the right-hand side of eqn. (33) may be replaced by zero. The resulting equation may then be integrated, and after the first integration we obtain

$$f_\xi = -\frac{2\nu}{3}(f - f_1)(f - f_2)(f - f_3) \tag{35}$$

where f_1, f_2, f_3 are the three roots of the equation

$$f^3 + (3\mu/2\nu)f^2 + (3\bar{\theta}_s'/\nu)f + a = 0,$$

$a(\tau)$ being a "constant" of integration. In particular, $\{f_i\}$ satisfy the conditions

$$f_1 + f_2 + f_3 = -(3\mu/2\nu),\ f_1f_2 + f_2f_3 + f_3f_1 = 3\bar{\theta}_s'(\tau)/\nu. \tag{36}$$

As θ varies from values below $\theta_s(\tau)$ to values above $\theta_s(\tau)$, ξ varies from large negative to large positive values. Any solution of eqn. (35) tends to one of the values f_1, f_2, f_3 as $\xi \to \pm\infty$, and we shall arbitrarily denote by f_2 and f_3 the roots which give the limiting values of f as $\xi \to -\infty$ and $\xi \to +\infty$ respectively. For any finite value of ξ, the solution of eqn. (35) then lies between f_2 and f_3, and is given explicitly by

$$\xi + b(\tau) = -a_1 \ell n|f - f_1| + a_2 \ell n|f - f_2| - a_3 \ell n|f - f_3| \tag{37}$$

where $b(\tau)$ is a constant of integration and a_1, a_2, a_3 are three constants given by

$$a_1 = 3/[2\nu(f_2 - f_1)(f_3 - f_1)],\ a_2 = 3/[2\nu\Delta(f_2 - f_1)],$$

$$a_3 = 3/[2\nu\Delta(f_3 - f_1)]$$

where $\Delta = f_3 - f_2$ measures the jump across the shock.

Since, according to the definition of f_2 and f_3, a_2 and a_3 must be positive, it follows that the third root f_1 is less than both f_2 and f_3 when $\nu\Delta > 0$ and is greater than both f_2 and f_3 when $\nu\Delta < 0$. The sign of a_1 is then seen to be the same as that of ν.

The constants a_1 , a_2 , a_3 can be written more succinctly if we introduce the notation

$$\Sigma = f_2 + f_3 \ , \ E_{\pm} = \mu + \nu\Sigma \pm \frac{1}{3}\nu\Delta. \tag{38}$$

Then after using the first of conditions (36) it is found that

$$a_1 = (\Delta E_-)^{-1} - (\Delta E_+)^{-1}, \ a_2 = (\Delta E_-)^{-1}, \ a_3 = (\Delta E_+)^{-1}. \tag{39}$$

We shall choose $b(\tau)$ in such a way that the centre of the shock layer $f = \Sigma/2$ occurs when $\xi = 0$. From the solution (37) therefore it follows that

$$b(\tau) = a_1 \ell n \left| \nu\Delta/3(\mu + \nu\Sigma) \right|.$$

It can be expected that this value of b would have to be modified in higher approximations, but at the present level it suffices.

In expressing the form of the inner solution, it is convenient to introduce the dimensionless variable $r(\theta)$

by means of the definition

$$f(\theta) = \frac{1}{2}[\Sigma + \Delta r(\theta)]. \tag{40}$$

As θ varies from below to above the shock layer, f varies from f_2 to f_3, whence r changes from -1 to +1. It is given by the relation

$$\frac{\theta - \theta_s}{k} = a_2 \ell n(1 + r) - a_3 \ell n(1 - r) - a_1 \ell n\left|1 + \tfrac{1}{3}\nu\Delta r/(\mu+\nu\Sigma)\right|. \tag{41}$$

On crossing the shock layer, the inner solution (37) for f changes from the value f_2 at points well below the layer to f_3 at points above the layer. These limiting values must be determined by matching with the outer solution at points just below and just above the shock.

At any point $\theta = \theta_s(\tau)$ on the shock, there are two values of θ_1 for which eqn. (27) is satisfied, which we denote by $\theta_1^{\pm}$. This means that the two characteristics with parameters $\theta_1 = \theta_1^-$ and $\theta_1 = \theta_1^+$ both pass through the point $[\theta_s(\tau),\tau]$ (see Fig.1), the first arriving at

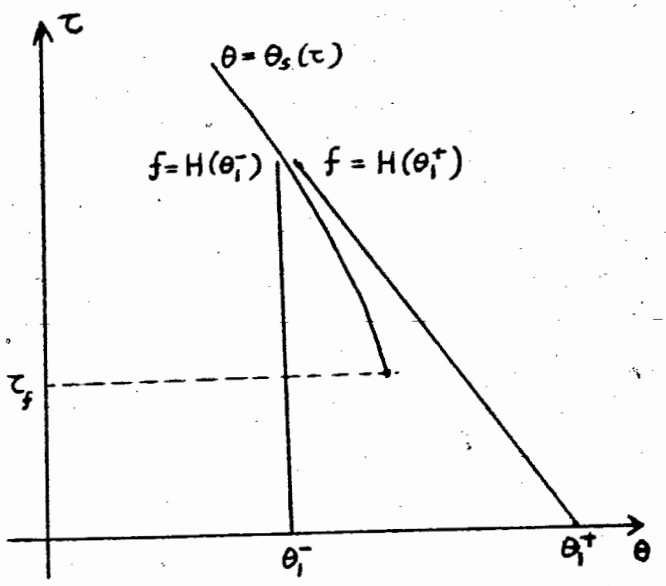


Fig. 1

the lower side of the shock, the second at the upper side. The limiting values f_2 and f_3 of the inner solution must therefore coincide with the signals carried on these two characteristics. Consequently, we arrive at the matching conditions.

$$f_2 = H(\theta_1^-), \quad f_3 = H(\theta_1^+). \tag{42}$$

Equation (27) for the two characteristics takes the forms

$$\theta_1^{\pm} = \bar{\theta}_s(\tau) + \left[\mu H(\theta_1^{\pm}) + \nu H(\theta_1^{\pm})^2\right]\tau. \tag{43}$$

In addition, eqns. (36) must be satisfied, and after eliminating f_1 the following condition relating f_2 and f_3 is obtained:

$$\bar{\theta}_s' = -\frac{1}{2}\mu(f_2 + f_3) - \frac{1}{3}\nu(f_2^2 + f_2 f_3 + f_3^2). \tag{44}$$

After substituting for f_2 and f_3 from eqns. (42), eqn. (44) together with eqns. (43) provide three equations for the three quantities θ_1^-, θ_1^+ and $\bar{\theta}_s$.

In the case when $\nu = 0$, these three equations can be shown [22] to lead to what is called the equal areas rule [29,30,31] by means of which the shock path can be

determined through a certain geometrical construction. A similar constructive method does not appear to be feasible when $\nu \neq 0$.

We observe that, in order for the situation illustrated in Figure 1 to be realized, the slopes of the two characteristics must satisfy the conditions $(d\theta/d\tau)_+ < (d\theta/d\tau)_-$. But from eqn. (27),

$$(d\theta/d\tau)_\pm = -\left[\mu H(\theta_1^\pm) + \nu H(\theta_1^\pm)^2 + \nu C(\tau)\right],$$

so that the following condition is obtained after using eqns. (42):

$$\Delta(\mu + \nu\Sigma) > 0. \tag{45}$$

This condition relates the sign of the jump across the shock to the quadratic and cubic elastic constants and to the average value of f at the centre of the shock $(\Sigma/2)$.

The three equations (43), (44) determine $\bar{\theta}_s(\tau)$, but the shock path $\theta_s(\tau)$ can be obtained from eqn. (34) only when $C(\tau)$ has been found. With the presence of a shock, $C(\tau)$ is no longer constant. In order to obtain an expression for $C(\tau)$, let us multiply eqn. (20') by

$f(\theta,\tau)$ and integrate over θ from 0 to 2ℓ. In view of the periodicity of f, several of the integrals vanish, and the end result is

$$\int_0^{2\ell} ff_\tau d\theta = \frac{1}{2} k \int_0^{2\ell} ff_{\theta\theta} d\theta.$$

From the definition (22) of $C(\tau)$ therefore,

$$\frac{dC}{d\tau} = \frac{k}{2\ell} \int_0^{2\ell} ff_{\theta\theta} d\theta.$$

In the absence of a shock wave, the right-hand side here is small, of order k. However, when the shock is present, a contribution of order unity to this right-hand side is made by the shock layer itself. To evaluate the contribution of the layer, the inner variable ξ may be used, in which case we can write

$$\frac{dC}{d\tau} = \frac{1}{2\ell} \int_{-\infty}^{\infty} ff_{\xi\xi} d\xi.$$

Now the value of $f_{\xi\xi}$ in the inner solution is given by eqn. (33) in which k is set equal to zero. After substituting and performing the integration over ξ, we obtain therefore

$$\frac{dC}{d\tau} = -\frac{1}{\ell}\left[\frac{1}{3}\mu f^3 + \frac{1}{4}\nu f^4 + \frac{1}{2}\bar{\theta}'_s f^2\right]_{-\infty}^{\infty}$$

$$= -\frac{1}{\ell}\left[\frac{1}{3}\mu(f_3^3 - f_2^3) + \frac{1}{4}\nu(f_3^4 - f_2^4) + \frac{1}{2}\bar{\theta}'_s(f_3^2 - f_2^2)\right].$$

After substitution of $\bar{\theta}'_s$ from eqn. (44), this takes the form

$$\frac{dC}{d\tau} = -\frac{1}{12\ell}\Delta^3(\mu + \nu\Sigma). \tag{46}$$

It follows from the condition (45) that $C(\tau)$ is a decreasing function of τ.

After solving eqns. (43) and (44) for $\theta_1^{\pm}$, f_2 and f_3 can be found from eqns. (42), so that eqn. (46) can be integrated, in principle, to get $C(\tau)$.

By virtue of eqns. (23), the shock at $\theta = \theta_s(\tau)$ in $f(\theta,\tau)$ leads to shocks at positions $x = \pm[t - \theta_s(\tau)]$ in $e(x,t)$ and $v(x,t)$. The shock velocities in the xt-plane are $\pm[1 - \varepsilon\theta'_s(\tau)]$. At a fixed value of t, the limit $\xi \to -\infty$ corresponds to a value of x ahead of each shock and $\xi \to +\infty$ corresponds to a value of x behind the shock. Thus the value $f = f_2$ measures the state of strain at the leading side of the shock layer and $f = f_3$ the state of strain at the trailing side (apart from the contribution of the other wave to the strain). When

$\Delta = f_3 - f_2 > 0$ we therefore have a rarefaction wave in which the medium dilates as the shock passes while when $\Delta < 0$ we have a compression wave. Which of these two types of shock is possible is determined by the condition (45).

6. FUNCTIONAL VISCOELASTICITY.

In Section 2, the Voigt approximation was derived on the assumption that the viscoelastic decay time δ is short compared to the time scale for changes in strain. When shocks occur in the medium, this requirement becomes a very restrictive one, since within the shock layer the strain rate is large- of order k^{-1}. To avoid this weakness of the Voigt model, it is necessary to return to the functional form for the viscoelasticity, that is, to eqn. (5) and the resulting lowest order approximation given by eqns. (21') and (23). We shall examine the solution of eqn. (21') under the condition that the viscoelastic term is only significant within shock layers, which corresponds to the assumption $k \ll 1$ which we made in the last section. We shall also continue to assume that the decay time δ is short compared to the time scale for changes in the outer solution, so that the Voigt approximation is valid outside the shock layers.

Under these assumptions, the solution for $\tau < \tau_f$ and outside the shock layer for $\tau > \tau_f$ continues to be given by the purely elastic solution (26) and (27). For $\tau < \tau_f$, $C(\tau) = \text{const.}$

For $\tau \gtrsim \tau_f$, we must obtain an inner solution of eqn. (21') holding within the shock layer. For $\theta - \theta_s(\tau)$ small, we may make the approximate replacement $\partial f/\partial\tau \approx -\theta_s'(\tau)\, \partial f/\partial\theta$ since, within the shock layer, rates of change across the layer dominate rates of change parallel to the path of the layer. Consequently, eqn. (21') takes on the approximate form

$$\left[\mu f + \nu f^2 + \bar{\theta}_s'\right] f_\theta - \frac{1}{2}\int_0^\infty G(s)\left[f_\theta(\theta - s) - f_\theta(\theta)\right]ds = 0 \tag{47}$$

where $\bar{\theta}_s$ is defined in eqn. (34). The τ-dependence has been suppressed throughout this equation, since τ appears in it only as a parameter.

Let us integrate eqn. (47) from θ^- to θ^+:

$$\left[\frac{1}{2}\mu f^2 + \frac{1}{3}\nu f^3 + \bar{\theta}_s' f\right]_{\theta^-}^{\theta^+} - \frac{1}{2}\int_0^\infty G(s)\left[f(\theta^+ - s) - f(\theta^+)\right]ds +$$

$$+ \frac{1}{2}\int_0^\infty G(s)\left[f(\theta^- - s) - f(\theta^-)\right]ds = 0. \tag{48}$$

Now first of all let us take θ^- and θ^+ to be points below and above the shock layer. Then the Voigt approximation may be used for each of the integrals, for example

$$\int_0 G(s)\left[f(\theta^- - s) - f(\theta^-)\right]ds = -kf_\theta(\theta^-), \tag{49}$$

Since $k << 1$, these integrals are therefore both small. Consequently, with such a choice for θ^- and θ^+,

$$\left[\frac{1}{2}\mu f^2 + \frac{1}{3}\nu f^3 + \bar{\theta}'_s f\right]_{\theta^-}^{\theta^+} = 0 \tag{50}$$

The inner solution satisfying eqn. (47) must be matched to the outer solution (26) and (27), and the appropriate matching conditions are that $f(\theta^+) = H(\theta_1^+)$, $f(\theta^-) = H(\theta_1^-)$, where $\theta_1^\pm$ are the characteristic parameters for the two characteristics reaching the shock at time τ, as illustrated in Figure 1. Consequently, eqn. (50) gives the result

$$\bar{\theta}'_s(\tau) = -\frac{1}{2}\mu\Sigma - \frac{1}{12}\nu(3\Sigma^2 + \Delta^2) \tag{51}$$

$$\Sigma = H(\theta_1^+) + H(\theta_1^-), \quad \Delta = H(\theta_1^+) - H(\theta_1^-).$$

This is identical with eqn. (44). Equations (43) continue to apply, since they simply refer to the two characteristics reaching the point $[\theta_s(\tau),\tau]$, so that the solution for $\bar{\theta}'_s(\tau)$ is the same for the general functional viscoelasticity as in the Voigt approximation.

The structure of the inner solution can, however, be entirely different from that given by eqn. (41) in the Voigt approximation. In eqn. (48) let us again take θ^- to be well below the shock layer, so that the second integral is again very small according to eqn. (49), but let us take $\theta^+ = \theta$ to be a general point within the layer. Then after substituting for $\bar{\theta}'_s$ from eqn. (51), the following equation for $f(\theta)$ within the layer is obtained

$$\left[f - H(\theta_1^-)\right]\left[H(\theta_1^+) - f\right]\left[\tfrac{1}{2}\mu + \tfrac{1}{3}\nu(f + \Sigma)\right]$$

$$+ \tfrac{1}{2}\int_0^\infty G(s)\left[f(\theta - s) - f(\theta)\right]ds = 0 \qquad (52)$$

This equation becomes a little simpler if we introduce the dimensionless quantity $r(\theta)$ defined in eqn. (40). We obtain

$$\left[1 - r(\theta)^2\right]\left[\mu + \nu\Sigma + \tfrac{1}{3}\nu\Delta r(\theta)\right]$$

$$+ 2\Delta^{-1}\int_0^\infty G(s)\left[r(\theta - s) - r(\theta)\right]ds = 0. \qquad (53)$$

For a general relaxation function, this non-linear integral equation cannot be solved in closed form,

although numerical solutions could be obtained in any particular case. In the next section we shall examine the solution for the special case of an exponential relaxation function, for which an explicit solution can be found.

Equation (46) continues to hold for a general relaxation function, provided that viscoelastic dissipation is significant only within the shock layer. For, from eqns. (21') and (22) it follows that

$$\frac{dC}{d\tau} = -\frac{1}{2\ell}\int_0^{2\ell} f(\theta)\int_0^{\infty} G(s)\left[f_\theta(\theta - s) - f_\theta(\theta)\right]ds\,d\theta.$$

If the θ integral is restricted to the shock layer, eqn. (47) may be used with the result that

$$\frac{dC}{d\tau} = -\frac{1}{\ell}\int_{\theta^-}^{\theta^+} f(\theta)\left[\mu f(\theta) + \nu f(\theta)^2 + \bar{\theta}_s'(\tau)\right]f_\theta(\theta)\,d\theta.$$

After evaluating this integral, eqn. (46) is obtained.

7. EXPONENTIAL RELAXATION FUNCTION.

We shall now examine the case when the relaxation function has the form $G(s) = de^{-s/\delta}$, characteristic of a Maxwell-Voigt material. In this case $k = \int_0^\infty sG(s)ds = d\delta^2$, and eqn. (53) takes the form

$$\int_0^\infty e^{-s/\delta} r(\theta - s)ds$$

$$= \delta r(\theta) - \frac{\Delta\delta^2}{2k}\left[1 - r(\theta)^2\right]\left[\mu + \nu\Sigma + \frac{1}{3}\nu\Delta r(\theta)\right]. \qquad (54)$$

Differentiating with respect to θ, integrating by parts and then using eqn. (54) to eliminate the integral term which results, we obtain the following differential equation for $r(\theta)$:

$$\frac{dr}{d\theta}\left[\frac{2k}{\Delta\delta} - \frac{1}{3}\nu\Delta + 2(\mu + \nu\Sigma)r + \nu\Delta r^2\right]$$

$$= \delta^{-1}(1 - r^2)(\mu + \nu\Sigma + \frac{1}{3}\nu\Delta r) .$$

This equation is readily integrated, to give

$$\frac{\theta - \theta_s}{k} = d_2 \ell n(1 + r) - d_3 \ell n(1 - r)$$

$$- d_1 \ell n\left(1 + \frac{1}{3}\nu\Delta r/(\mu + \nu\Sigma)\right) \qquad (55)$$

where the constant of integration has been chosen in such a way that the centre of the shock ($r = 0$) occurs at $\theta = \theta_s$, and

$$d_1 = (\Delta E_-)^{-1} - (\Delta E_+)^{-1} + \delta/k, \quad d_2 = (\Delta E_-)^{-1} - \delta/k, \quad d_3 = (\Delta E_+)^{-1} + \delta/k, \tag{56}$$

$E_\pm'$ being defined in eqns. (38)

It is interesting to note that the solution (55) for a general exponential relaxation function has the same qualitative form as that found in eqn. (41) under the more restrictive assumptions of the Voigt approximation. This contrasts with the result found for the quadratic elastic case, $\nu = 0$. In that case, the solution for the Voigt model becomes

$$(\theta - \theta_s)/k = (2/\mu\Delta)\tanh^{-1} r$$

which means that r is antisymmetric about the centre $\theta = \theta_s$ of the shock layer; the solution for the exponential relaxation function on the other hand does not show this symmetry about its centre- the leading

side of the layer is compressed relative to the trailing side. When cubic elasticity is significant, the structure of the shock layer loses its symmetry about the centre even for the Voigt approximation.

An estimate of the width of the leading side of the shock layer (i.e., the side of which $f \to f_2$) can be obtained by setting $r = -1 + 1/N$ with N some suitably large number. We then obtain that $\theta - \theta_s \approx - ka_2 \ell nN$ for the voigt approximation and $\theta - \theta_s \approx - kd_2 \ell nN$ for the exponential relaxation function. For the trailing side of the layer, we can set $r = 1 - 1/N$, obtaining $\theta - \theta_s \approx ka_3 \ell nN$ for the Voigt approximation and $\theta - \theta_s \approx kd_3 \ell nN$ from eqn. (55). The total width of the shock is obtained by adding these two half-widths together, giving the values $k(a_2 + a_3)\ell nN$ and $k(d_2 + d_3)\ell nN$ for the two solutions. But comparing eqns. (39) and (56) we see that these two expressions are identical, so that the total width of the shock layer is given correctly by the Voigt approximation.

Since $a_2 > a_3$ for $\nu > 0$, it follows that in the Voigt approximation the trailing side of the layer is compressed relative to the leading side when the cubic

elasticity is of a "hardening" nature. Conversely when the cubic elasticity is "softening" (i.e., $\nu < 0$), in the Voigt approximation the leading side of the shock is compressed in comparison to the trailing side.

In this latter case ($\nu < 0$) it follows from eqns. (56) that $d_2 < d_3$ so that with exponential relaxation the leading side of the shock is compressed relative to the trailing side, just as in the Voigt approximation. In the case $\nu > 0$ however the two effects compete, the cubic elasticity tending to compress the trailing side while the effect of the non-zero relaxation time is to compress the leading side of the shock. The leading side is the shorter if

$$2\delta\Delta/k > E_-^{-1} - E_+^{-1} .$$

A comparison of the two solutions (41) and (55) allows us to derive the precise conditions under which the Voigt model may be applied. We see that $d_i \approx a_i$ ($i = 1,2,3$) when the two conditions

$$\delta/k << (\Delta E_\pm)^{-1} \qquad (57)$$

are met, and in this eventuality, the solutions (41) and

(55) become approximately the same. These conditions could be derived in a qualitative sense from the solution (41) alone, since an intuitive condition for the validity of the Voigt approximation is that the decay time δ should be much smaller than the widths of the two sides of the shock layer. Thus we are led to the conditions $\delta \ll ka_2$ and $\delta \ll ka_3$ which are identical with conditions (57).

The solution (41) always satisfies the boundary conditions $r \to \pm 1$ as $(\theta - \theta_s)/k \to \pm\infty$ since a_2 and a_3 are positive. However while the condition $r \to +1$ as $(\theta - \theta_s)/k \to +\infty$ is always met by solution (55), the second boundary condition is only satisfied provided that $d_2 > 0$. This condition may be rewritten in the form

$$\Delta(\mu + \nu\Sigma - \tfrac{1}{3}\nu\Delta) < k/\delta. \qquad (58)$$

Unless this condition is met, eqn. (54) has no solution satisfying the required boundary condition.

When $\nu = 0$, the inequality (58) provides simply an upper bound on Δ, the jump in $f(\theta, r)$ across the shock. For stronger shocks than satisfy this bound, no smooth solution of eqn. (54) exists. However, it is interesting that when $\nu \neq 0$, condition (58) does not provide merely an

upper bound on Δ. If we set $\Delta_m = 3(\mu + \nu\Sigma)/\nu$ and $\kappa = 1 - 12k/\nu\delta\Delta_m^2$ then the conditions on Δ arising from the inequality (58) can be summarised as follows. (In deriving the conditions which follow we have made use of the restriction (45) and also of the condition $|\Delta/\Delta_m| < 1$ which is necessary in order that the third logarithm in eqn. (55) should not be singular for $|r| \leqslant 1$.) For $\nu < 0$, it is necessary that

$$0 < -(\Delta/\Delta_m) < \min\left(1, \tfrac{1}{2}(\sqrt{\kappa} - 1)\right).$$

For $\nu > 0$ and $\kappa > 0$, it is necessary that

$$\text{either } 0 < (\Delta/\Delta_m) < \tfrac{1}{2}(1 - \sqrt{\kappa}) \text{ or } \tfrac{1}{2}(1 + \sqrt{\kappa}) < (\Delta/\Delta_m) < 1.$$

For $\nu > 0$ and $\kappa < 0$, the only condition is $0 < (\Delta/\Delta_m) < 1$. Thus when ν and κ are both positive, besides a band of weak shocks there also exists a band of stronger shocks for which a smooth solution of eqn. (54) exists.

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