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ANTI-RAMSEY NUMBERS

An Introduction

by

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ABSTRACT

Let Y be a graph with p edges and let k be a positive integer. The (Y,k)-anti-ramsey number, denoted by ar(Y,k), is the least m such that every complete graph on m vertices whose edges are coloured using no colour more than k times contains a subgraph isomorphic to Y whose edges are coloured with p distinct colours. The existence of ar(Y,k) is proven along with upper and lower bounds for $ar(K_n,k)$ and some exact values of $ar(K_n,k)$ are given (Chapter IV). The restricted problem of finding $ar(K_{1,n},k)$ is investigated and exact values given for n or k equal to 1, 2 or 3 together with upper and lower bounds for $ar(K_{1,n},k)$ (Chapter III). Related problems and conjectures are mentioned in Chapter V.

A viděli jsme

Co?

Právě, my jsme viděli

Ale co?

Viděli jsme to mizet

A už se nikdy nedovíme

Thanks are due to all those friends who suffered through the preparation of this thesis; especially to

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

I. INTRODUCTION

In the beginning there was the Advanced Problem 6034 proposed by Fred Galvin in the May 1975 issue of the American Mathematical Monthly:

> Suppose the edges of the complete graph on n vertices are coloured so that no colour is used more than k times.
>
> (1) If $n \geq k + 2$, show that there is a triangle no two of whose edges have the same colour.
>
> (2) Show that this is not necessarily so if $n = k + 1$.

After a solution[(1)] was found some generalizations began to emerge. In view of (1) it is clear that $n = k + 2$ is the smallest such that the conclusion of (2) holds. Is there such a minimum if instead of triangles we consider cycles of length greater than three? What if one looks at complete graphs on m points, $m > 3$? Can a minimum n be found so that it is impossible to colour the edges of K_n using each colour at most k times and guarantee that every m-star (i.e., $K_{1,m}$) has two edges of the same colour? What are the minima, if they exist? Some of these questions are relatively easy to answer, others seem to be rather difficult. A few were answered by B. Alspach, M. Gerson, G. Hahn and P. Hell and reported in [H3]. These and some more are the subject of the present work.

II. DEFINITIONS AND EASY OBSERVATIONS

Let A be a set. We will denote by $A^{[2]}$ the set of unordered pairs of elements of A and by $|A|$ the cardinality of A.

A graph X is a finite(2) non-empty set $V(X)$ of points(3) together with a subset $E(X)$ of $V(X)^{[2]}$ of edges(4). We write $X = (V(X),E(X))$. The vertex set $V(X)$ of a bipartite graph X is the union of two disjoint sets V_1 and V_2 while $E(X) \cap (V_1^{[2]} \cup V_2^{[2]}) = \emptyset$. A graph Y is a subgraph of the graph X (or is contained in X) if $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$. We write $Y \subseteq X$.

When $|V(X)| = m$ and $E(X) = V(X)^{[2]}$ we say that X is the complete graph on m points and denote it by K_m. Similarly, we denote by $K_{m,n}$ the complete bipartite graph on $m + n$ vertices--here $|V_1| = m$, $|V_2| = n$ and $E(K_{m,n}) = (V_1 \cup V_2)^{[2]} - V_1^{[2]} - V_2^{[2]}$. In particular, we call $K_{1,m}$ an m-star. If $e \in E(X)$ consists of the points u and v we write $e = uv$. It is sometimes convenient to consider edges as sets and say, for example, that $u \in uv$ and that $e \cap f = \emptyset$ if $e,f \in E(X)$.

Intuitively, a graph is a set of points joined by edges. Hence, we say that the vertices u and v of X are adjacent if $uv \in E(X)$ and that the edges e and f of X are adjacent if $e \cap f \neq \emptyset$ and $e \neq f$. We also say that the edge uv is incident with the points u and v. The degree $d(v)$ of a vertex v is the number of edges incident with it. If there is a $d \in N$(5) such that $d(v) = d$ for each $v \in V(X)$ we say that X is regular of degree d; K_m, then, is regular of degree $m - 1$.

A path in a graph X is a sequence of distinct points $u_1, u_2, \ldots, u_n$ (usually written just $u_1u_2\ldots u_n$) with every pair of consecutive vertices joined by an edge. A cycle is a path with the additional edge u_nu_1. A $u - v$ path is one whose first and last points are u and v respectively. A graph X is connected if it contains a $u - v$ path for every pair of points u and v. An equivalent and sometimes useful way of defining a connected graph is to say that for any nontrivial partition of $V(X)$ into V_1 and V_2 there are vertices $v_1 \in V_1$, $v_2 \in V_2$ with $v_1v_2 \in E(X)$. Each maximal connected subgraph of X is a component of it. A subgraph Y of X obtained by taking a subset $V(Y)$ of $V(X)$ (or a subset $E(Y)$ of $E(X)$) together with all the edges of X incident only with vertices in $V(Y)$ (or all the vertices incident with the edges in $E(Y)$) is said to be induced by $V(Y)$ (or $E(Y)$).

If, given graphs X and Y, there is a bijective map $f : V(X) \to V(Y)$ such that $f(u)f(v) \in E(Y)$ exactly when $uv \in E(X)$ for any $u,v \in V(X)$, we say that X and Y are isomorphic. It is trivial to verify that isomorphism is an equivalence relation on the set of graphs and we will henceforth use the name X to denote any element of the equivalence class containing X.

Let $C = \{c_i \mid i < \omega\}$ be a set of colours and for each $k \in N$ and each set A let $F_k(A)$ be the set of functions $\{f : A \to C \mid \forall i < \omega(|f^{-1}(c_i)| \leq k)\}$ where $|f^{-1}(c_i)|$ is the cardinality of the pre-image of c_i. When no confusion might arise we will write simply F_k for $F_k(A)$; in particular, for a graph X we write F_k for $F_k(E(X))$. An edge-colouring of a graph X is a function f from

$E(X)$ into C, an edge-k-colouring is a function $f \in F_k(E(X))$. Given a graph X and an $f \in F_k$ we say that the subgraph Y of X is monochromatic if $|f(E(Y))| = 1$ and that it is a rainbow[6] if $f \restriction E(Y)$ is one-to-one. For a given graph Y a function $f \in F_k(E(X))$ is a (Y,k)-colouring of X if no $Y \subseteq X$ is a rainbow. A graph X admits a (Y,k)-colouring (is (Y,k)-colourable) if there is a (Y,k)-colouring of it. We will think of an edge-colouring of X as just that and, hence, will talk about an edge coloured c_i or having a colour c_i, an X being (Y,k)-coloured, etc.

Let X be coloured by some $f \in F_k$. We say that the colour c_i and the vertex $v \in V(X)$ are incident (with each other) if there is a $u \in V(X)$ with $f(u,v) = c_i$. The colour degree $c(v)$ of $v \in V(X)$ is the number of colours incident with it. Denote by C^* the image of $E(X)$ under f and, without loss of generality, assume $C^* = \{c_1, \ldots, c_p\}$ for some p. The set of edges coloured c_i incident with v will be denoted by $E_i(v)$. The colour structure $s(v)$ of v is a vector of length p with $s_i(v) = |E_i(v)|$. The condensed colour structure $\bar{s}(v)$ is obtained from $s(v)$ by omitting all zero entries; it has, therefore, length $c(v)$.

An edge uv coloured c_i is single at v if $E_i(v) = \{uv\}$, is single if it is single at either u or v and is totally single if it is single at both u and v. The single degree $t(v)$ of v is the number of single edges incident with it and the single in-degree $r(v)$ is the number of edges single at v. The single out-degree $o(v)$ is given by

$$o(v) = t(v) - r(v) + \sigma(v)$$

where $\sigma(v)$ is the number of totally single edges incident with v. The single in-degree and out-degree of X are, respectively,

$$r(X) = \sum_{v \in V(X)} r(v)$$

and

$$o(X) = \sum_{v \in V(X)} o(v)$$

and we observe that $o(X) = r(X)$.

Since we are dealing with discrete structures it is useful to introduce the function symbols $\lfloor\ \rfloor$ and $\lceil\ \rceil$. Let y be a real number. Then $\lfloor y \rfloor$, called the <u>floor</u> of y, is the integer a such that $a \leq y < a + 1$ and $\lceil y \rceil$, called the <u>ceiling</u> of y, is the integer b such that $b - 1 < y \leq b$. We can now make the main definition.

<u>DEFINITION 2.1</u>: Let Y be a graph and k a positive integer. The (Y,k)-anti-ramsey number $ar(Y,k)$ is the least m such that K_m does not admit a (Y,k)-colouring.

It will be shown in Chapter IV that $ar(K_n,k)$ exists for all n and k. To see that the definition makes sense for all Y we note that if $Y \subseteq K_n$ then any (Y,k)-colouring is also a (K_n,k)-colouring and, hence,

$ar(Y,k) \leq ar(K_n,k)$. It is a trivial observation that for any Y there is an n such that $Y \subseteq K_n$. One might have expected the definition of $ar(Y,k)$ to be

<u>DEFINITION 2.2</u>: Let Y be a graph and k a positive integer. The (Y,k)-anti-ramsey number $ar(Y,k)$ is the least m' such that there is a graph X with $|V(X)| = m'$ which does not admit a (Y,k)-colouring.

A little thought, however, shows the two definitions to be equivalent. Let m and m' be $ar(Y,k)$ as defined in Definitions 2.1 and 2.2, respectively. Clearly, $m' \leq m$. If X is a graph on m' vertices which is not (Y,k)-colourable, then, since $X \subseteq K_{m'}$, $K_{m'}$ does not admit a (Y,k)-colouring either. Thus, $m' \geq m$. In view of this, we shall adopt Definition 2.1.

The problem, of course, is that of finding $ar(Y,k)$, given Y and k. In general, this seems rather difficult although some easy observations are at hand. We now list a few.

<u>OBSERVATION 2.1</u>: $$ar(Y,k) \geq |V(Y)|$$

since K_n contains no Y for $n < |V(Y)|$.

<u>OBSERVATION 2.2</u>: $$ar(Y,k) > \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor$$

since K_n can be coloured with only one colour when $\binom{n}{2} \leq k$, which yields $n \leq \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor$.

OBSERVATION 2.3: $ar(Y,1) = |V(Y)|$

which follows from Observation 2.1 and the fact that each colour can be used only once.

OBSERVATION 2.4: $ar(Y,k) \leq ar(K_n,k)$ for $n \geq |V(Y)|$

as was already mentioned.

Considering the difficulty of the general case, it is natural to concentrate on some special cases. The first to come to mind is, as might be expected, the case of $ar(K_n,k)$. At present it seems that even this is not easy and apart from the sparse results of Chapter IV not much is known. The most approachable special case is that of n-stars which is the subject of the following chapter.

A word about the name "anti-ramsey number." In 1930 F. P. Ramsey proved in [R2] a theorem now bearing his name. In the language of graphs theorem states the following: Every infinite complete graph whose edges are coloured with k colours contains a complete monochromatic subgraph on $\aleph_0$ vertices. It follows from this that for any pair of positive integers m and n there is a least $r \in N$ such that every complete graph on r vertices whose edges are coloured with two colours contains either a monochromatic K_m or a monochromatic K_n. This r is denoted by $r(m,n)$ and is called a ramsey number. Thus, with ramsey numbers one is looking for the minimum r such that in any edge-colouring of K_r (with two colours) there is a monochromatic subgraph while with anti-ramsey numbers the minimum m sought is such that in any edge-colouring of K_m (using each colour no more than k times) there is a <u>rainbow</u> subgraph.

It is interesting to note, however, that despite the similarity between $r(m,n)$ and $ar(Y,k)$ the latter problem is more related to one of Turán. We will say more about this in Chapter V.

III. STAR-NUMBERS

We have already defined $K_{1,n}$. An n-star consists of a centre u of degree n and n points of degree one all adjacent to u. If such a graph is not a rainbow under an edge-colouring then there are two edges of the same colour which are both incident with the centre. This is a very useful piece of information. With any other[7] so coloured connected graph one has no idea just where the two edges of the same colour might lie. This is due to the fact that there are at least two vertices of degree at least two in a connected non-star. Thus, with n-stars one can make use of the pigeon-hole principle[8] which makes $ar(K_{1,n},k)$ a natural choice for consideration. This is not to say, however, that the "star numbers" are easy to find.

We will begin with a few observations.

OBSERVATION 3.1: $$ar(K_{1,n},k) \geq n + 1$$

which follows directly from Observation 2.1.

OBSERVATION 3.2: A graph X is $(K_{1,n},k)$-coloured by f if and only if $f \in F_k(E(X))$ and for each $v \in V(X)$, $1 \leq c(v) \leq n - 1$.

OBSERVATION 3.3: If K_m is $(K_{1,n},k)$-coloured then

(a) for each $v \in V(K_m)$, $c(v) \geq \left\lceil \frac{m-1}{k} \right\rceil \geq \frac{m-1}{k}$

(b) $n \geq 1 + \max_{v \in V(K_m)} c(v) \geq 1 + \frac{m-1}{k}$

(c) $ar(K_{1,n},k) \leq (n-1)k + 2$

and we note that (c) is a consequence of (b).

We can now state our first two propositions.

PROPOSITION 3.1: (a) $ar(K_{1,1},k) = 2$

(b) $ar(K_{1,n},1) = n + 1$.

Proof: From Observations 3.1 and 3.3(c).

***

PROPOSITION 3.2: $ar(K_{1,2},k) = \left\lfloor \frac{3 + \sqrt{1 + 8k}}{2} \right\rfloor$

Proof: From Observation 3.2 we have $c(v) = 1$ for each $v \in V(K_m)$ whenever K_m is $(K_{1,2},k)$-coloured. Hence, only one colour is used in such a colouring and, thus, $\binom{m}{2} \leq k$. The result then follows by solving the inequality for m.

***

Our next task will be to try for an improvement on the upper bound for $ar(K_{1,n},k)$. We assume, until the end of proof of Proposition 3.3 $k > 1$.

LEMMA 3.1: Let K_m be $(K_{1,n},k)$-coloured by f and let $f(E(K_m)) = \{c_1, c_2, \ldots, c_p\}$. Let a be a non-negative integer and let $v \in V(K_m)$. For each $1 \leq i \leq p$, if $s_i(v) \geq k - a$ then at least $k - 3a$ edges in $E_i(v)$ are single.

Proof: Consider the component Y containing v of the subgraph of K_m induced by $f^{-1}(c_i)$. Y has a subgraph $K_{1,s_i(v)}$ and $E = E(Y) - E(K_{1,s_i(v)})$ has at most a elements. Since each edge in E is incident with no more than two vertices of

$K_{1,s_i(v)}$ (and is not incident with v) it follows that no more than 2a vertices other than v of $K_{1,s_i(v)}$ have degrees greater than one. That is, at least $k - a - 2a$ edges are single. Note that $k - 3a \geq 1$ if $a \leq \frac{k-1}{2}$.

***

LEMMA 3.2: Let a, b and a_i, $i = 1,2, \ldots, a$ be integers such that

$$\frac{1}{a} \sum_{i=1}^{a} a_i \geq b.$$

Then

$$\sum_{i=1}^{a} (a_i - b) \geq 0.$$

Proof: Easy manipulation.

***

LEMMA 3.3: If, in an edge- k-coloured graph X, a vertex v has condensed colour structure such that

$$\frac{1}{c(v)} \sum_{i=1}^{c(v)} \bar{s}_i(v) \geq k - \left\lfloor \frac{k-1}{3} \right\rfloor$$

then $r(v) < o(v)$.

Proof: Let $b = k - \left\lfloor \frac{k-1}{3} \right\rfloor$ and for $i = 1, \ldots, c(v)$ let $a_i = \bar{s}_i(v) - b$. Using Lemma 3.2 we then have

$$\sum_{a_i \geq 0} a_i - \sum_{a_i < 0} |a_i| = \sum_{i=1}^{c(v)} a_i \geq 0.$$

Now $a_i \geq 0$ means $\overline{s}_i(v) \geq k - \left\lfloor \frac{k-1}{3} \right\rfloor$ and so, by Lemma 3.1, $E_i(v)$ contains $k - 3b_i$ single edges, where $b_i = k - \overline{s}_i(v)$. Since

$$a_i < 3a_i + 1 \leq 3(\overline{s}_i(v) - k + \left\lfloor \frac{k-1}{3} \right\rfloor) + 1 = 3(\left\lfloor \frac{k-1}{3} \right\rfloor - b_i) + 1 \leq k - 3b_i$$

we have

$$o(v) \geqq \sum_{a_i \geq 0} (k - 3b_i) > \sum_{a_i \geq 0} a_i .$$

Also, $r(v) \leqq \sum_{a_i<0} 1 \leq \sum_{a_i<0} |a_i|$ and, hence,

$$o(v) > \sum_{a_i \geq 0} a_i \geq \sum_{a_i<0} |a_i| \geq r(v).$$

***

PROPOSITION 3.3: $ar(K_{1,n},k) \leq \frac{2}{3}k(n-1) + n$ for $n > 1$, $k > 1$.

Proof: If $m = \frac{2}{3}k(n-1) + n$ and K_m is $(K_{1,n},k)$-coloured then for each $v \in V(K_m)$ we have

$$\frac{1}{c(v)} \sum_{i=1}^{c(v)} \overline{s}_i(v) \geq \frac{1}{n-1}(m-1) = \frac{1}{n-1}(\frac{2}{3}k + 1)(n-1)$$
$$\geq k - \left\lfloor \frac{k-1}{3} \right\rfloor .$$

Hence, by Lemma 3.3, $o(v) > r(v)$. Thus,

$$o(K_m) > r(K_m),$$

a contradiction.

***

Comparing this new upper bound with that of Observation 3.3(c) we see that

$$(n - 1)k + 2 \geq \frac{2}{3}(n - 1)k + n$$

whenever $n > 1$ and $k \geq 3$ and thus Proposition 3.3 does, indeed, provide an improvement.

We now turn to more specific results.

PROPOSITION 3.4: $ar(K_{1,n},2) = n + \left\lfloor\frac{n+2}{3}\right\rfloor$.

Proof: A. For $n < 4$ note Obs. 3.1 and Propositions 3.1(a) and 3.2. Otherwise let $m = n + \frac{n-1}{3}$ and $V(K_m) = \{v_0, \ldots, v_{m-1}\}$. Also, let $C' = \{c_{ij} \mid i = 0, \ldots, m-1;\ j = 1, \ldots, \left\lfloor\frac{n-1}{3}\right\rfloor\} \subset C$. We define $f : E(K_m) \to C$ as follows: for each $i = 0, \ldots, m-1$ and $j = 1, \ldots, \left\lfloor\frac{n-1}{3}\right\rfloor$ put $f(v_i v_{i+2j-1}) = f(v_i v_{i+2j}) = c_{ij}$ with $i + 2j - 1$ and $i + 2j$ taken modulo m. This assigns, for each i, colours to edges $v_i v_{i+s}$ for $s = 1, \ldots, 2\left\lfloor\frac{n-1}{3}\right\rfloor$. Since $s, s' \leq 2\left\lfloor\frac{n-1}{3}\right\rfloor$ implies $s + s' < n + \left\lfloor\frac{n-1}{3}\right\rfloor$, no edge is coloured twice. The remaining edges can be coloured with arbitrary colours distinct from one another and from those in C'. For every v we then have $c(v) = \left\lfloor\frac{n-1}{3}\right\rfloor + (m - 1 - 2\left\lfloor\frac{n-1}{3}\right\rfloor) = n - 1$ since $\left\lfloor\frac{n-1}{3}\right\rfloor$ colours from C' and $m - 1 - 2\left\lfloor\frac{n-1}{3}\right\rfloor$ other colours were used. Thus, f is a $(K_{1,n},2)$-colouring of K_m and, hence,

$$ar(K_{1,n},2) \geq n + \left\lfloor \frac{n+2}{3} \right\rfloor.$$

B. Suppose now K_m is $(K_{1,n},2)$-coloured with $m \geq n + \left\lfloor \frac{n+2}{3} \right\rfloor$. Let $v \in V(K_m)$. Then

$$n - 1 \geq c(v) \geq \frac{1}{2}(n + \left\lfloor \frac{n-1}{3} \right\rfloor)$$

by Observations 3.2 and 3.3(a). Consider the condensed colour structure $\bar{s}(v)$. It is a vector consisting of n_1 1's and n_2 2's and we have

$$2n_2 + n_1 \geq n + \left\lfloor \frac{n-1}{3} \right\rfloor$$

and

$$c(v) = n_1 + n_2 \leq n - 1.$$

Hence,

$$n + \left\lfloor \frac{n-1}{3} \right\rfloor \leq n - 1 + n_2 ,$$

that is

$$n_2 \geq \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$

Thus,

$$n_1 \leq n - \left\lfloor \frac{n-1}{3} \right\rfloor - 2.$$

But then

$$2n_2 \geq 2\left\lfloor \frac{n-1}{3} \right\rfloor + 2 > n - \left\lfloor \frac{n-1}{3} \right\rfloor - 2 \geq n_1 .$$

Now

$$o(v) \geq 2n_2 > n_1 = r(v)$$

for an arbitrary $v \in V(K_m)$ and so

$$o(K_m) > r(K_m)$$

an impossible situation. Hence,

$$ar(K_{1,n},2) \leq n + \left\lfloor \frac{n + 2}{3} \right\rfloor .$$

***

Figures 1 and 2 show $(K_{1,n},2)$-colourings of $K_{n+\left\lfloor \frac{n-1}{3} \right\rfloor}$ for $n = 4$ and 9 respectively.

Before proceeding to the next case $(ar(K_{1,n},3))$ we digress a little to recall a few facts of combinatorics. A Steiner triple system (STS) of order v is a set of three-element subsets of a set X with $|X| = v$ such that every pair of elements of X appears in exactly one triple. It can be shown[9] that an STS of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$. Consider a complete graph X on $v \equiv 1$ or $3 \pmod 6$ vertices. A Steiner triple system on $V(X)$ is a partition of $E(X)$ into triangles--a pair of elements of $V(X)$ corresponds to an edge and a triple corresponds to a cycle of length three, or a triangle. With this we state

PROPOSITION 3.5:

$$ar(K_{1,n},3) = \begin{cases} 2n & \text{if } n \not\equiv 0 \pmod 3 \\ 2n - 1 & \text{if } n \equiv 0 \pmod 3 . \end{cases}$$

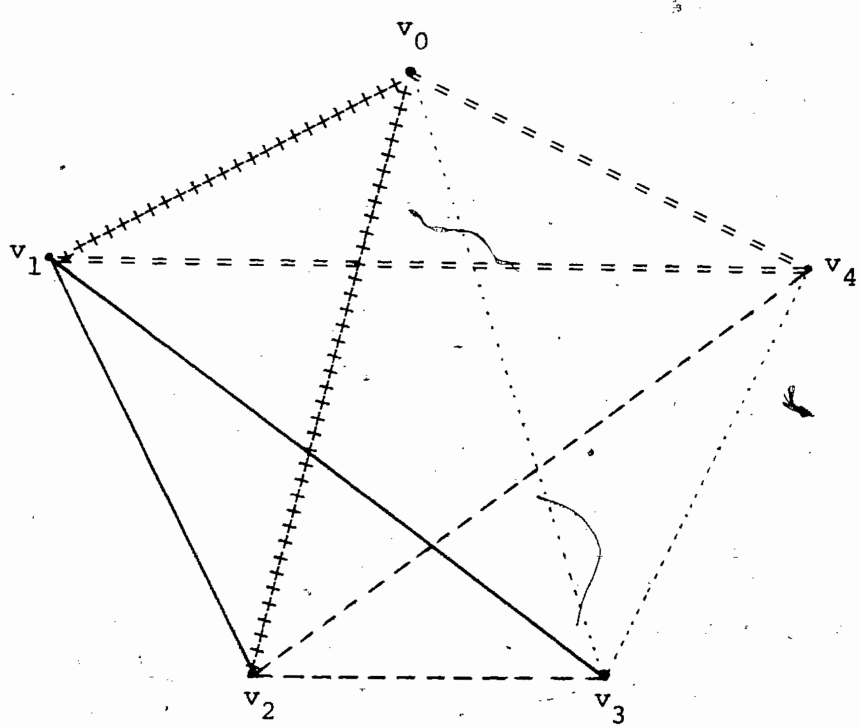

Figure 1. A $(K_{1,4},2)$-colouring of K_5 .

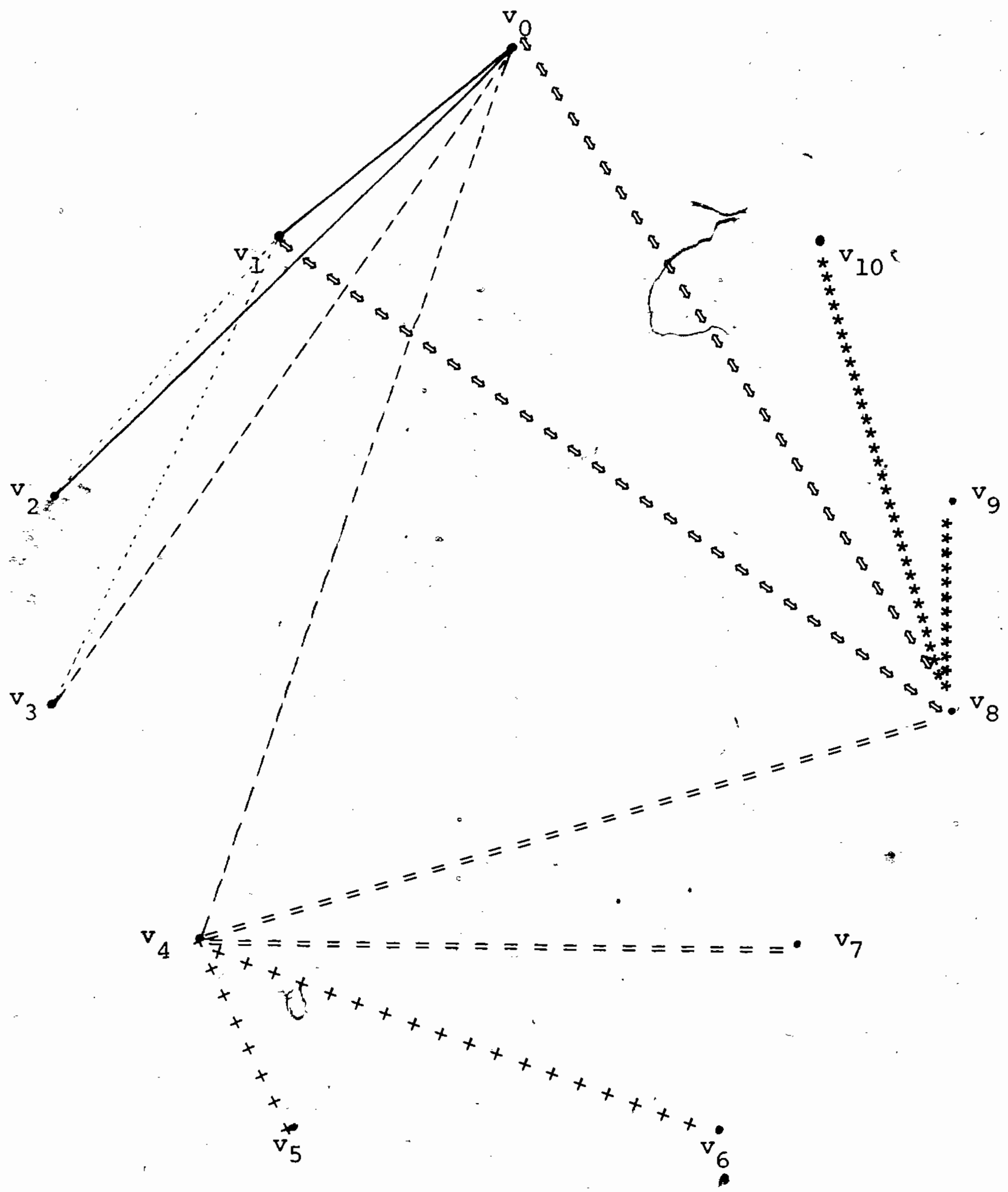

Figure 2. Part of a $(K_{1,9},2)$-colouring of K_{11}.

Proof: A. We have $n \not\equiv 0 \pmod 3$ if and only if $2n - 1 \equiv 1$ or $3 \pmod 6$, that is, exactly when an STS of order $2n - 1$ exists. Thus, $E(K_{2n-1})$ can be partitioned into triangles. Colouring each triangle a distinct colour we obtain an edge-3-colouring of K_{2n-1} with $c(v) = \frac{2n-2}{2} = n - 1$ for each $v \in V(K_{2n-1})$. Hence, this is a $(K_{1,n},3)$-colouring and

$$ar(K_{1,n},3) \geq 2n$$

when $n \not\equiv 0 \pmod 3$. Also, $n \equiv 0 \pmod 3$ if and only if $2n - 2 \equiv 4 \pmod 6$, in which case an STS of order $2n - 3$ exists. Hence, there is an edge-3-colouring of K_{2n-3} as in the previous case, with $c(v) = n - 2$. Since $2n - 3$ is divisible by three when $n \equiv 0 \pmod 3$, the vertex set of K_{2n-3} can be partitioned into triples each of which can be joined by edges of the same colour to a new vertex u. If the edges from distinct triples have distinct colours this will result in

$$c(v) = n - 1 \quad \text{for} \quad v \in V(K_{2n-3})$$

$$c(u) = \frac{2n-3}{3} < n - 1.$$

Therefore,

$$ar(K_{1,n},3) \geq 2n - 1$$

when $n \equiv 0 \pmod 3$.

B. Now let $m < ar(K_{1,n},3)$ and suppose K_m is $(K_{1,n},3)$-coloured. Let $v \in V(K_m)$ and consider the condensed colour structure $\bar{s}(v)$. This vector consists of $n_1(v)$ 1's, $n_2(v)$ 2's and $n_3(v)$ 3's and we observe that $r(v) = n_1(v)$ and $o(v) \geq 3n_3(v)$. We then obtain, writing n_i for $n_i(v)$, $i = 1,2,3$,

$$3n_3 + 2n_2 + n_1 = m - 1$$

and

$$n - 1 \geq c(v) = n_1 + n_2 + n_3$$

and, thus,

$$n_2 = c(v) - n_1 - n_3 .$$

Hence,

$$m - 1 = n_3 - n_1 + 2c(v) \leq n_3 - n_1 + 2(n - 1).$$

Now, if $m = 2n$ then

$$2n - 1 \leq n_3 - n_1 + 2(n - 1)$$

or,

$$n_1 + 1 \leq n_3 \leq 3n_3$$

and from this

$$o(v) \geq 3n_3 > n_1 = r(v).$$

Therefore,

$$o(K_{2n}) > r(K_{2n}),$$

a contradiction. If $m = 2n - 1$ and $n \equiv 0 \pmod 3$ then $m \equiv 5 \pmod 6$ and we get

$$2n - 2 \leq n_3 - n_1 + 2(n - 1)$$

or,

$$n_1 \leq n_3 \leq 3n_3 .$$

Thus

$$o(v) \geq 3n_3 \geq n_1 = r(v).$$

Now $o(K_m) = r(K_m)$ and so the inequalities in

$$o(K_m) \geq 3 \sum_{v \in V(K_m)} n_3(v) \geq \sum_{v \in V(K_m)} n_3(v) \geq \sum_{v \in V(K_m)} n_1(v)$$

$$= r(K_m)$$

are equalities. Thus, $n_3(v) = n_1(v) = 0$. Then $c(v) = n_2(v)$ and $\overline{s}(v)$ consists entirely of 2's. Since no colour appears more than three times this implies the existence of an STS of order $2n - 1 \equiv 5 \pmod 6$, an impossibility.

***

Figures 3 and 4 show $(K_{1,3},3)$- and $(K_{1,4},3)$-colourings of K_4 and K_7 respectively. We now have the values of $ar(K_{1,n},k)$ whenever n or

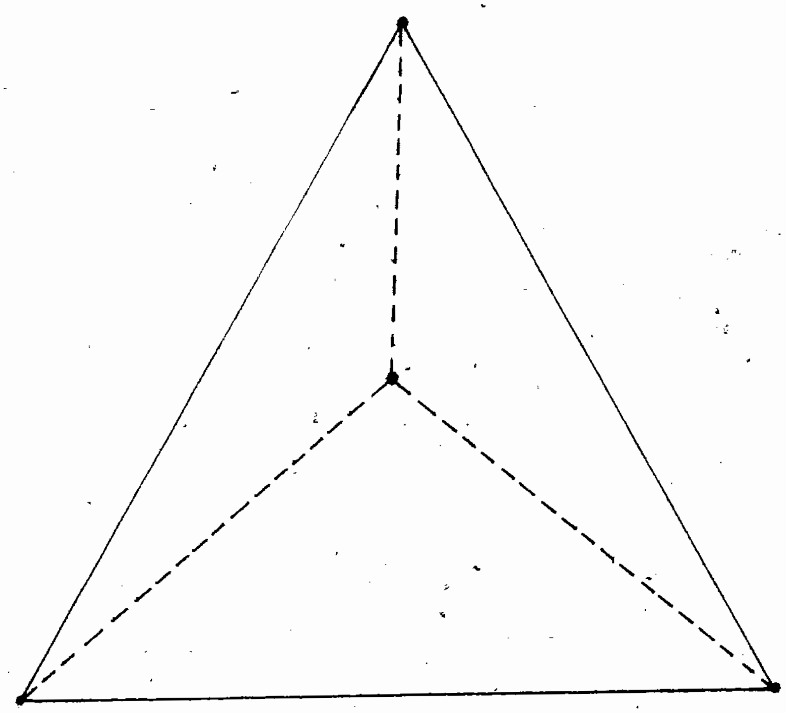

Figure 3. A $(K_{1,3},3)$-colouring of K_4 .

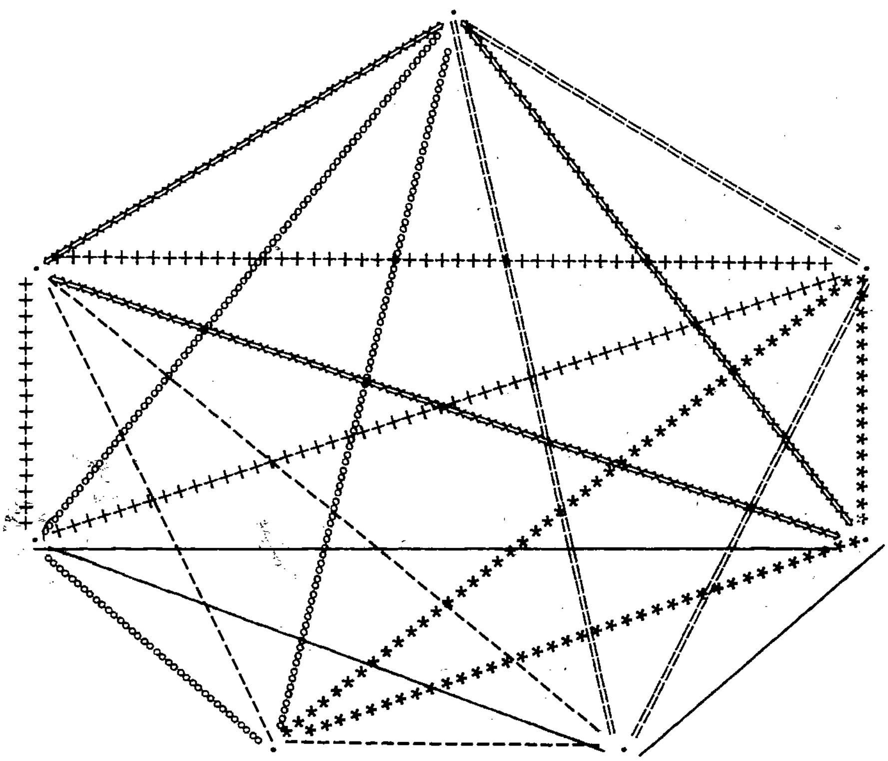

Figure 4. A $(K_{1,4},3)$ - colouring of K_7 .

k is equal to 1 or 2 and for $k = 3$. The results have not been complicated but the proofs, though elementary, have been getting longer. The value of $ar(K_{1,3},k)$ is simple and in keeping with the progression so far established the proof will be the sole objective of the remainder of this chapter. Before embarking on the sequence of le,,as leading to the main result let us make a few remarks. First, recall Observation 3.2. In the case of a $(K_{1,3},k)$-coloured graph it says that the colour degree of any vertex is at most two. Second, for the purposes of proving Lemma 3.4, we will say that in an edge-coloured graph X the colours c_i and c_j meet if there is a vertex in $V(X)$ incident with both of them.

We can now begin the sequence with the surprising Lemma 3.4.

LEMMA 3.4: If $m < ar(K_{1,3},k)$ then there is a $(K_{1,3},k)$-colouring f of K_m with $|f(E(K_m))| \leq 3$.

Proof: Let p be the least integer such that K_m can be $(K_{1,3},k)$-coloured by some g with $g(E(K_m)) = \{c_0, \ldots, c_p\}$. For $i = 0, \ldots, p$ let X_i be the subgraph induced by the edges coloured c_i and let $n_i = |V(X_i)|$. We will show

(A) If $p > 2$ then, without loss of generality, for $i \neq j$, $i,j = 1, \ldots, p$, $V(X_i) \cap V(X_j) = \emptyset$ and all the edges between X_i and X_j are coloured c_0.

(B) If the conclusion of (A) holds then for any partition of $\{1, \ldots, p\}$ into $I \cup J$ we have

$$\min\{\sum_{i \in I} n_i, \sum_{j \in J} n_j\} < \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor .$$

(C) $p \leqq 2$.

We begin with the last claim, assuming (A) and (B). Recall that if $n \leqq \left\lfloor \frac{1+\sqrt{1+8k}}{2} \right\rfloor$ then K_n can be coloured with only one colour. Now, if $p > 2$ then it follows from (A) and (B) that

$$\min \{ \sum_{i=1}^{2} n_i , \sum_{i=3}^{p} n_i \} < \left\lfloor \frac{1+\sqrt{1+8k}}{2} \right\rfloor$$

and, hence, at least one colour can be eliminated.

Proof of (A) : If every two colours meet then there are vertices u and v , u incident with c_0 and c_1 , v with c_2 and c_3 . The edge uv cannot, then, be coloured with any colour without increasing the colour degree of at least one of u and v to three. We may, therefore, assume that, without loss of generality, c_1 and c_2 do not meet. It is clear that $V(X_1) \cap V(X_2) = \emptyset$. Let $u_i \in V(X_i)$, $i = 1,2$. Since $g(u_1u_2) \neq c_1,c_2$ we may also assume, without loss of generality, that u_1u_2 is coloured c_0. Let v be any point of X_2. Then $g(u_1v) = c_0$ since since u_1v is coloured neither c_1 nor c_2 , u_1 is incident with c_0 and c_1 and $c(u_1) = 2$. Similarly, $g(u_2w) = c_0$ for any $w \in V(X_1)$. Thus, all the edges between X_1 and X_2 are coloured c_0. In fact, all vertices of K_m are incident with c_0 since for any point v the edges between v and X_1 or those between v and X_2 are coloured c_0. It is now clear that for all $i \neq j$, $i,j = 1, \ldots, p$, $V(X_i) \cap V(X_j) = \emptyset$ and all the edges between X_i and X_j are coloured with c_0.

Proof of (B) : Assume that the conclusion of (A) holds and that for some I and J

$$\min \{ \sum_{i\in I} n_i , \sum_{j\in J} n_j \} \geq \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor .$$

Clearly, the number of edges coloured c_0 is at most k and at least

$$(\sum_{i\in I} n_i) \cdot (\sum_{j\in J} n_j)$$

which, with the above assumption, implies that

$$k \geq \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor^2 .$$

Let us write now

$$s = \frac{1 + \sqrt{1 + 8k}}{2}$$

and $\lfloor s \rfloor = s - r$. Since $0 \leq r < 1$ and $s \geq 1$ we obtain

$$k \geq (s - r)^2 > (s - 1)^2 \geq \frac{s}{2}(s - 1) = k$$

which is impossible.

This completes the proof.

***

An obvious consequence of Lemma 3.4 is the following :

LEMMA 3.5: $$ar(K_{1,3},k) \le \left\lfloor \frac{3 + \sqrt{1 + 24k}}{2} \right\rfloor$$

Proof: All we need to do is solve the inequality implied by Lemma 3.4, namely $\binom{m}{2} \le 3k$ whenever $m < ar(K_{1,3},k)$.

***

The conjecture at hand, that is, $ar(K_{1,3},k) = \left\lfloor \frac{3 + \sqrt{1 + 24k}}{2} \right\rfloor$, will turn out to be true except when $k = 2$ or 7. In order to prove it we need two more lemmas.

LEMMA 3.6: For every non-negative integer k there are unique non-negative integers n and i such that $i \le 3n$ and $k = \frac{3n^2 - n}{2} + i$.

Proof: It is easy to verify that $\frac{3n^2 + n}{2}$ is an integer for each $n \in N$. Let $k \in N$. The set $S_k = \{n \mid \frac{3n^2 - n}{2} \le k\}$ has a largest element, say n_k. We set $i_k = k - \frac{3n_k^2 - n_k}{2}$ and notice that

$$\frac{3n_k^2 - n_k}{2} + 3n_k + 1 = \frac{3(n_k + 1)^2 - (n_k + 1)}{2},$$

i.e., $i_k \le 3n_k$ otherwise n_k was not largest. If $k = \frac{3n^2 - n}{2} + i$ with $i \le 3n$ for $n < n_k$ then

$$\frac{3n^2 - n}{2} + i \le \frac{3n^2 - n}{2} + 3n = \frac{3(n + 1)^2 - (n + 1)}{2} - 1 < \frac{3n_k^2 - n_k}{2},$$

that is $k < k$. Thus, the pair (n_k, i_k) is unique.

***

LEMMA 3.7: For $k = \frac{3n^2 - n}{2} + i$, $i \leq 3n$, we have

$$\left\lfloor \frac{1 + \sqrt{1 + 24k}}{2} \right\rfloor = \begin{cases} 3n & \text{if } 0 \leq i < n \\ 3n + 1 & \text{if } n \leq i \leq 2n \\ 3n + 2 & \text{if } 2n < i \leq 3n. \end{cases}$$

Proof: We have

$$\sqrt{1 + 24k} = \sqrt{36n^2 - 12n + 24i + 1}$$

and we consider the three cases.

(a) $0 \leq i < n$

Then

$$6n - 1 \leq \sqrt{36n^2 - 12n + 24i + 1} < \sqrt{36n^2 + 12n + 1} = 6n + 1$$

and

$$3n \leq \frac{1 + \sqrt{1 + 24k}}{2} < 3n + 1.$$

(b) $n \leq i \leq 2n$

Then

$$6n + 1 \leq \sqrt{36n^2 - 12n + 24i + 1} < \sqrt{36n^2 + 36n + 9} = 6n + 3$$

and

$$3n + 1 \leq \frac{1 + \sqrt{1 + 24k}}{2} < 3n + 2.$$

(c) $2n < i \leq 3n$.

Then

$$6n + 3 < \sqrt{36n^2 - 12n + 24i + 1} < \sqrt{36n^2 + 60n + 25} = 6n + 5$$

and

$$3n + 2 < \frac{1 + \sqrt{1 + 24k}}{2} < 3(n + 1).$$

***

We are now almost ready for a proof of our conjecture. The missing link is Lemma 3.8 which we shall assume for the moment.

PROPOSITION 3.6: $ar(K_{1,3},k) = \left\lfloor \frac{3 + \sqrt{1 + 24k}}{2} \right\rfloor$

except that

$$ar(K_{1,3},2) = 4$$

and

$$ar(K_{1,3},7) = 7.$$

Proof: Let us dispose of the exceptions first. From Proposition 3.4 we have $ar(K_{1,3},2) = 4$. From Lemma 3.8 we will see that $ar(K_{1,3},7) \leq 7$ and Figure 5 shows $ar(K_{1,3},7) \geq 7$. In the general case--$k \neq 2,7$--write $k = \frac{3n^2 - n}{2} + i$ as in Lemma 3.6 and consider the three cases of Lemma 3.7. In each one we will show that the complete graph on $\left\lfloor \frac{1 + \sqrt{1 + 24k}}{2} \right\rfloor$ vertices is $(K_{1,3},k)$-colourable by exhibiting such a colouring.

(a) $0 \leq i < n$

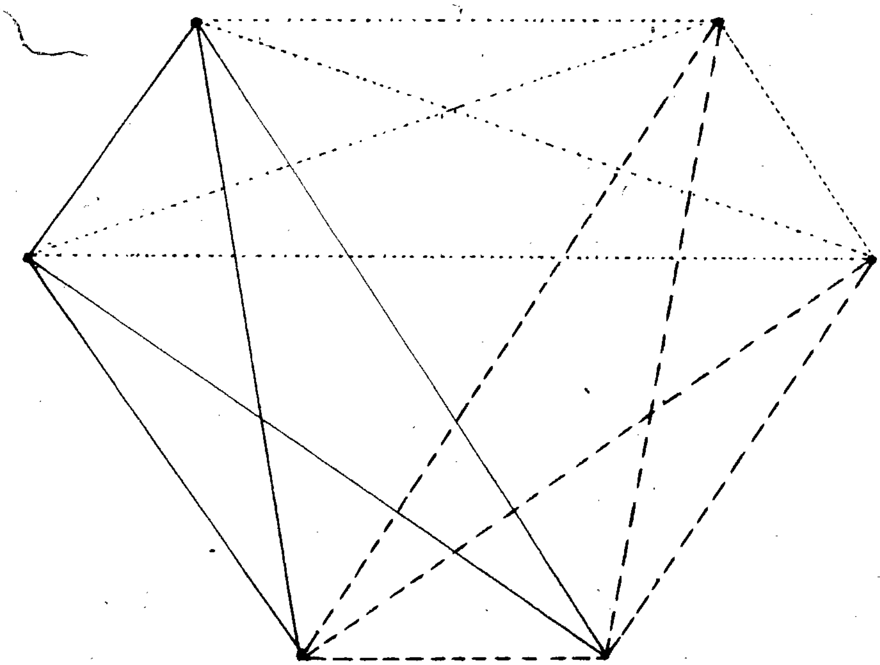

Figure 5. A $(K_{1,3},7)$-colouring of K_6 .

Then $\left\lfloor \frac{1 + \sqrt{1 + 24k}}{2} \right\rfloor = 3n$ and we let $V(K_{3n}) =$ $V_0 \cup V_1 \cup V_2 \cup \{v\}$ where $|V_0| = |V_2| = |V_1| + 1 = n$. For $i = 0,1,2$ we colour with c_i the edges of the complete graph on V_i and the edges between V_i and V_{i+1}, taking $i + 1$ modulo 3. The edges from v to V_0 will be coloured with c_0 and those from v to V_1 and V_2 with c_1. This colours all the edges of K_{3n} and each colour is used exactly $\frac{3n^2 - n}{2} \le k$ times as is easily checked. Figure 6(a) shows a schema of the colouring.

(b) $n \le i \le 2n$

Then $\left\lfloor \frac{1 + \sqrt{1 + 24k}}{2} \right\rfloor = 3n + 1$. In view of the special cases $(k = 2,7)$ we assume that $n \ge 3$.

Let $V(K_{3n+1}) = V_0 \cup V_1 \cup V_2 \cup V \cup \{u\} \cup \{v\}$ with $|V_0| = |V_2| = |V_1| + 3 = n$, $|V| = 2$. For $i = 0,1,2$, colour with c_i the edges of the complete graph on V_i and the edges from V_i to V_{i+1}, taking $i + 1$ modulo 3. Colour with c_0 the edges between V and V_0 and between u and $V_0 \cup V_1 \cup V$ plus the edge in the graph on V. Colour with c_2 the edges from v to V_0 and with c_1 all the remaining edges. It is not difficult to check that all the edges of K_{3n+1} are coloured and that each colour is used $\frac{3n^2 + n}{2} \le k$ times. Figure 6(b) shows the schema of the colouring.

(c) $2n < i \le 3n$

Then $\left\lfloor \frac{1 + \sqrt{1 + 24k}}{2} \right\rfloor = 3n + 2$. We let $V(K_{3n+2}) =$ $V_0 \cup V_1 \cup V_2 \cup \{u\} \cup \{v\}$ with $|V_i| = n$ for $i = 0,1,2$ and colour c_i the edges of the complete graph on V_i plus the edges between V_i and V_{i+1}, taking $i + 1$ modulo 3. Also, colour with c_0 the edges from u to $V_0 \cup V_1$ and with c_2 those from v to $V_0 \cup V_2$. The remaining edges will be coloured with c_1. Then each colour appears at most $\frac{3n^2 - n}{2} + 2n + 1 \leq k$ times. Figure 6(c) shows the schema. Clearly, in each case each vertex has colour degree two and so we have constructed $(K_{1,3},k)$-colourings.

***

We now provide the missing link. Since $ar(K_{1,n},k)$ exists for all n and k there is a maximum number of colours that are necessary to $(K_{1,n},k)$-colour any K_m when $m < ar(K_{1,n},k)$. Let this maximum be $p(n,k)$.

<u>LEMMA 3.8</u>: If $m < ar(K_{1,n},k)$ and $\binom{m}{2} = k \cdot p(n,k)$ then, letting

$$q = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil,$$

$$m \leq \left\lfloor \frac{2k(n - 1)}{q} \right\rfloor + 1.$$

Proof: For simplicity we let $p = p(n,k)$. Let m be as in the statement of the lemma. Then any $(K_{1,n},k)$-colouring of K_m must use p colours. Let f be such a colouring. For each $i = 1, \ldots, p$ let X_i be the monochromatic subgraph of K_m induced by the edges coloured c_i. Since f is a $(K_{1,n},k)$-colouring no point of K_m lies in more than $n - 1$ of the X_i's. Letting $a = \min_i |V(X_i)|$ we observe that

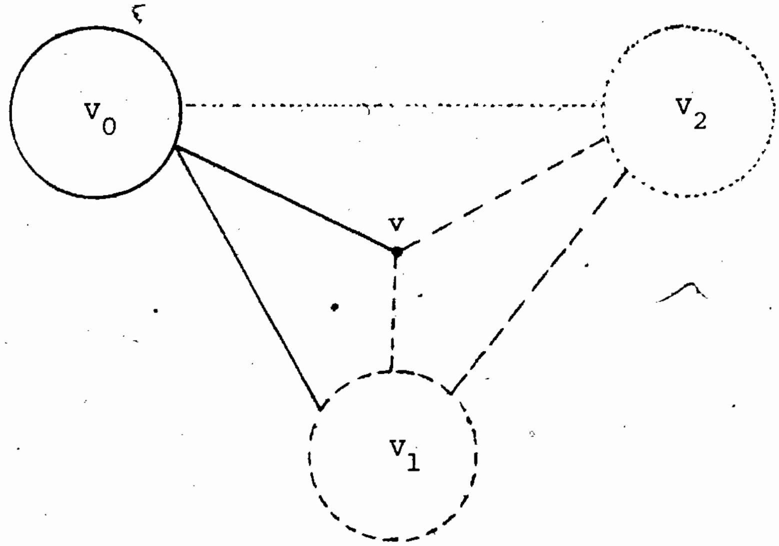

Figure 6 (a). The colouring scheme for a $(K_{1,3},k)$-colouring of K_{3n} .

Counting the edges by colour:

c_0 : $\binom{n}{2} + n + n(n - 1) = \frac{3n^2 - n}{2}$

c_1 : $[\binom{n-1}{2} + (n - 1) + n] + n(n - 1) = \frac{3n^2 - n}{2}$

c_2 : $\binom{n}{2} + n^2 = \frac{3n^2 - n}{2}$

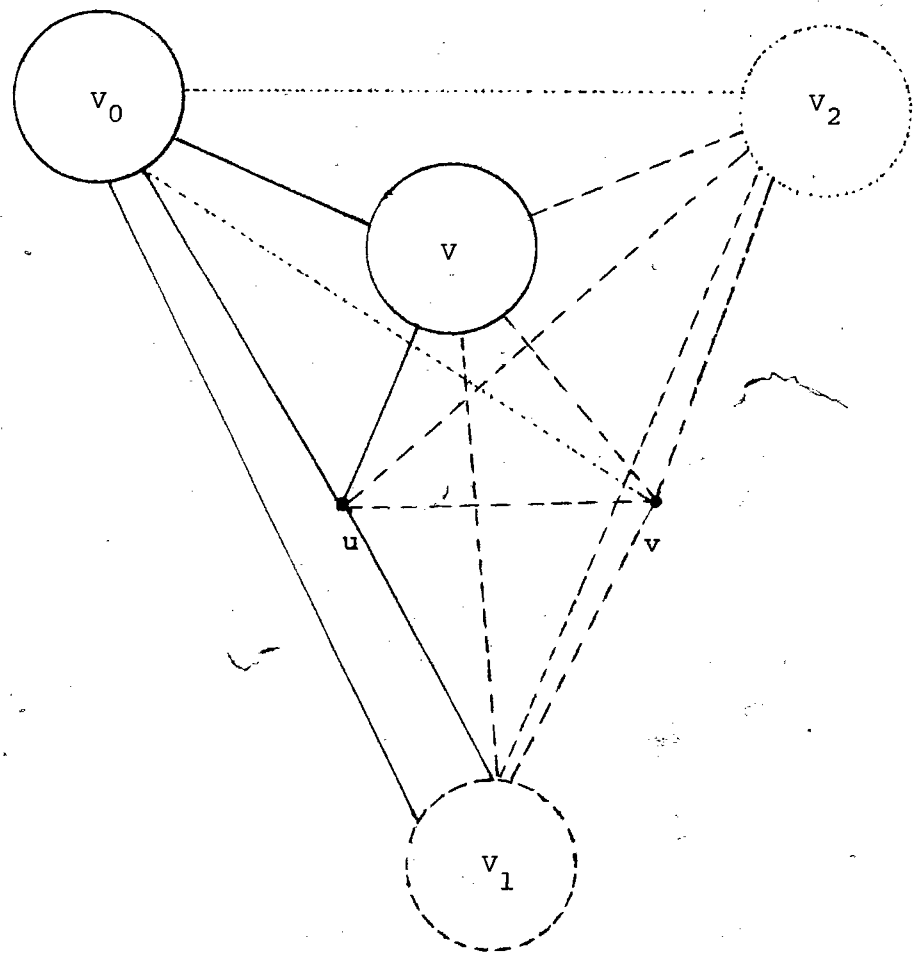

Figure 6 (b). The colouring scheme for a $(K_{1,3},k)$-colouring of K_{3n+1}.

Counting the edges by colour:

c_0 : $[\binom{n}{2} + n(n-3) + n + 2n] + (n-3) + 2 + 1 = \frac{3n^2+n}{2}$

c_1 : $[\binom{n-3}{2} + (n-3)n + (n-3) + 2(n-3)] + [2n + n + n] +$
$+ 2 + 1 = \frac{3n^2+n}{2}$.

c_2 : $[\binom{n}{2} + n^2] + n = \frac{3n^2+n}{2}$

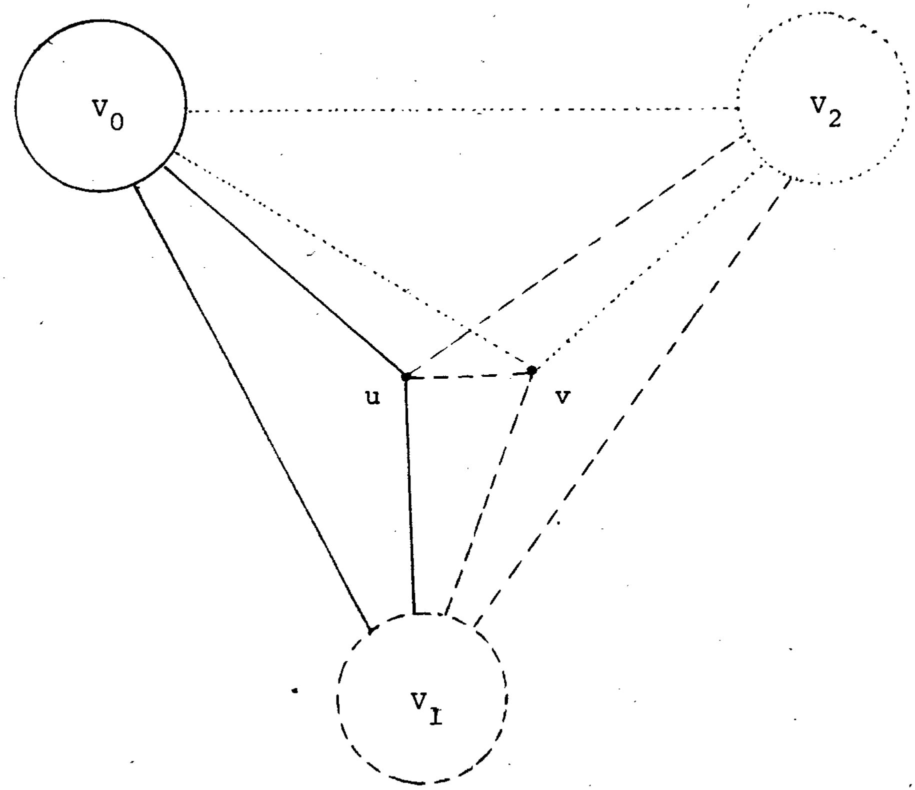

Figure 6 (c). The colouring scheme for a $(K_{1,3},k)$-colouring of K_{3n+2}.

Counting the edges by colour:

c_0 : $[\binom{n}{2} + n^2 + n] + n = \frac{3n^2 - n}{2} + 2n$

c_1 : $[\binom{n}{2} + n^2 + n] + [n + 1] = \frac{3n^2 - n}{2} + 2n + 1$

c_2 : $[\binom{n}{2} + n^2 + n] + n = \frac{3n^2 - n}{2} + 2n$

$$m(n-1) \geq pa = \binom{m}{2}\frac{a}{k}$$

or,

$$m \leq \frac{2k(n-1)}{a} + 1.$$

Since each colour appears k times, $\binom{a}{2} \geq k$, that is,

$$a \geq \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil = q.$$

Thus,

$$m \leq \left\lfloor \frac{2k(n-1)}{q} \right\rfloor + 1$$

since m is an integer.

***

We close this chapter with two conjectures.

CONJECTURE 3.1: $\operatorname{ar}(K_{1,n},k) \leq \frac{n-1}{2}(-1+\sqrt{1+8k}) + 2.$

To justify this, let $a = \sqrt{1+8k}$. Then $2k = \frac{a^2-1}{4}$ and if $q = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$ then

$$\frac{n-1}{2}(a-1) = \frac{(a^2-1)(n-1)}{2(a+1)} = \frac{(a^2-1)(n-1)}{4(\frac{a+1}{2})} \geq$$

$$\geq \left\lfloor \frac{\frac{a^2-1}{4}(n-1)}{\left\lceil \frac{a+1}{2} \right\rceil} \right\rfloor = \left\lfloor \frac{2k(n-1)}{q} \right\rfloor$$

with equality when a is an (odd) integer. Consider now the known star numbers.

(a) For $n = 1$ we have

$$ar(K_{1,n},k) = 2 = \frac{n-1}{2}(-1 + \sqrt{1 + 8k}) + 2.$$

(b) For $n = 2$

$$ar(K_{1,n},k) = \left\lfloor \frac{3 + \sqrt{1 + 8k}}{2} \right\rfloor \leq \frac{3 + \sqrt{1 + 8k}}{2} =$$

$$= \frac{n-1}{2}(-1 + \sqrt{+ 8k}) + 2$$

with equality when $\sqrt{1 + 8k}$ is an integer.

(c) For $n = 3$

$$ar(K_{1,n},k) = \left\lfloor \frac{3 + \sqrt{1 + 24k}}{2} \right\rfloor \leq \frac{2 + \sqrt{4 + 32k}}{2} =$$

$$= \frac{n-1}{2}(-1 + \sqrt{1 + 8k}) + 2.$$

(d) For $k = 1$

$$ar(K_{1,n},k) = n + 1 = 2\,\frac{n-1}{2} + 2 =$$

$$= \frac{n-1}{2}(-1 + \sqrt{1 + 8k}) + 2.$$

(e) For $k = 2$

$$ar(K_{1,n},k) = n + \left\lfloor \frac{n+2}{3} \right\rfloor \leq \frac{4n + 2}{3} \leq \frac{3(n-1)}{2} + 2 \leq$$

$$\leq \frac{n-1}{2}(-1 + \sqrt{1 + 8k}) + 2.$$

(f) For $k = 3$

$$ar(K_{1,n},k) \leq 2n = \frac{n-1}{2}(-1 + \sqrt{1 + 8k}) + 2.$$

So the upper bound is achievable and is good for all the known star numbers. Conjecture 3.1 is an improvement on a previous one:

CONJECTURE 3.2: $ar(K_{1,n},k) \leq \left\lfloor \frac{2k(n-1)}{q'} \right\rfloor$

where $q' = \left\lfloor \frac{1 + \sqrt{1 + 8k}}{2} \right\rfloor$.

For clearly, $\frac{2k(n-1)}{q'} \geq \frac{2k(n-1)}{\frac{1+\sqrt{1+8k}}{2}} = \frac{n-1}{2}(-1 + \sqrt{1+8k})$. But, of course, both conjectures are open.

IV. COMPLETE NUMBERS[10]

In this chapter we turn our attention to $ar(K_n,k)$. We will prove the existence of $ar(K_n,k)$ for all n, k and, thus, justify the statements made previously about the existence of anti-ramsey numbers. Furthermore, we will state and prove most of the sparse results known at this point. First, a lemma.

LEMMA 4.1: If $0 \le m_1 \le \ldots \le m_p \le k$ are integers and $\sum_{i=1}^{p} m_i = ak + b,\ 0 \le b < k$, then

(a) $\sum_{i=1}^{p} m_i^2 \le ak^2 + b^2$

(b) $\sum_{i=1}^{p} \binom{m_i}{2} \le a\binom{k}{2} + \binom{b}{2} < \frac{k-1}{2} \sum_{i=1}^{p} m_i$.

Proof: (a) Since for $i > j$

$$(m_i + 1)^2 + (m_j - 1)^2 = m_i^2 + m_j^2 + 2(m_i - m_j + 1)$$

$$> m_i^2 + m_j^2 ,$$

the sum $\sum_{i=1}^{p} m_i^2$ is maximized when $m_1 = m_2 = \ldots = m_{p-a-1} = 0$, $m_{p-a} = b$ and $m_{p-a+1} = \ldots = m_p = k$. Thus, (a) holds.

(b) The first inequality follows directly from (a). For the second inequality we have

$$a\binom{k}{2} + \binom{b}{2} = \frac{ak(k-1) + b(b-1)}{2} < \frac{(k-1)(ak+b)}{2} =$$

$$= \frac{k-1}{2} \sum_{i=1}^{p} m_i$$

***

OBSERVATION 4.1: $ar(K_2,k) = 2$

which is a trivial re-statement of Proposition 3.1(a).

OBSERVATION 4.2: $ar(K_1,k) = 1$,

vacuously.

The second part of the following proposition together with Observations 2.3, 4.1 and 4.2 and Proposition 4.2 shows that $ar(K_n,k)$ always exists and, hence, so does $ar(Y,k)$.

PROPOSITION 4.1: If $n \geq 4$ and $k \geq 2$ then

(a) $ar(K_n,k) \geq k(n-1) + 1$

(b) $ar(K_n,k) \leq \frac{1}{4} n(n-1)(n-2)(k-1) + 2$.

Proof: (a) Consider the complete graph on the vertex set $L \times M$ where $L = \{1,2, . . . ,k\}$ and $M = \{1,2, . . . ,n-1\}$. Let $D = \{0, . . . , \lfloor \frac{n-1}{2} \rfloor\}$ and let $C^* = \{c_{i,j,r} \mid i \in L, j \in M, r \in D\}$. We can assume $C^* \subset C$. Let e be the edge between (i,j) and (i',j') and suppose, without loss of generality, that $j' - j \equiv r \pmod{n-1}$ with $r \in D$.

If $r = 0$, colour e with $c_{\min(i,i'),j,r}$.

If $0 < r < \lfloor \frac{n-1}{2} \rfloor$, colour e with $c_{i,j,r}$.

If $r = \frac{n-1}{2}$, colour e with $c_{i,j,r}$ whenever n is even or n is odd and $j \leq \frac{n-1}{2}$..

This colours all the edges and no colour appears more than k times. Suppose now that this edge-coloured graph contains a rainbow K_n. Then $V(K_n)$ intersects each $L \times \{j\}$ in at most two vertices. Further, there is a $j \in M$ such that $L \times \{j\}$ contains two vertices of K_n and, hence, $V(K_n) \cap L \times \{j - 1\} = \emptyset$ (taking $L \times \{n - 1\}$ when $j = 1$). Thus there is a $j' \neq j$ with $L \times \{j'\}$ containing two points of K_n. Without loss of generality, $j' - j \equiv r \pmod{n - 1}$, $r \in D$ and, hence, the edges between either of the two points in $L \times \{j\}$ and the two points in $L \times \{j'\}$ have the same colour. This is a contradiction.

(b) Let K_m be (K_n,k)-coloured by f and let $f(E(K_m)) = \{c_1, \ldots, c_p\}$. Let m_i be the number of edges coloured c_i. There are $\sum_{i=1}^{p} \binom{m_i}{2}$ pairs of edges of the same colour in this edge-coloured graph, each appearing in at most $\binom{m-3}{n-3}$ sets of n vertices (since $\binom{m-3}{n-3} \geq \binom{m-4}{n-4}$ and each such pair appears in $\binom{m-3}{n-3}$ sets if the edges are adjacent and in $\binom{m-4}{n-4}$ sets if they are disjoint). Now K_m contains $\binom{m}{n}$ complete graphs on n points each of which must contain a pair of edges of the same colour. Thus,

$$\binom{m-3}{n-3} \sum_{i=1}^{p} \binom{m_i}{2} \geq \binom{m}{n} .$$

Also, for each $i = 1, \ldots, p$ $m_i \leq k$ and

$$\sum_{i=1}^{p} m_i = \binom{m}{2} . \qquad (1)$$

Writing $\binom{m}{2} = ak + b$ with $0 \leq b < k$ we get, by Lemma 4.1,

$$\sum_{i=1}^{p} \binom{m_i}{2} < \frac{k-1}{2}\binom{m}{2} .$$

From this and (1),

$$\binom{m-3}{n-3}\binom{m}{2}\frac{k-1}{2} > \binom{m}{n}$$

or,

$$\frac{k-1}{4} > \frac{m-2}{n(n-1)(n-2)} .$$

Hence,

$$m < \frac{n(n-1)(n-2)(k-1)}{4} + 2$$

which proves the result.

***

For the next two results put $N(v) = \{u \in V(X) \mid uv \in E(X)\}$.

PROPOSITION 4.2: $ar(K_3,k) = k + 2$.

Proof: Consider the complete graph on the vertex set $V = \{1, . . . ,k + 1\}$. Colour the edge ij with colour $c_{min(i,j)}$. Clearly, no colour is used more than k times and if $V(K_3) = \{m,n,p\} \subseteq V$ with $m < n < p$ then the edges mn and mp have the same colour. Thus,

$$ar(K_3,k) \geq k + 2.$$

Suppose now that $m \geq k + 2$ and that K_m is (K_3,k)-coloured by f. Let d be the maximum number of edges of the same colour incident with the same vertex and let v be such a vertex with the d edges coloured c. Let $U = \{u \in V(K_m) \mid f(uv) = c\}$ and $V = N(v) - U$. Clearly, neither is empty and we can find $w \in V$ with $f(vw) = c' \neq c$. Now for each $u \in U$, $f(uw) = c$ or c' and since d is maximal there is a $u' \in U$ with $f(wu') = c$. Thus, there are at least $d + (m - d - 1) = m - 1 > k$ edges coloured c, a contradiction.

***

The following observation is due to Pavol Hell.

<u>LEMMA 4.2</u>: If K_{k+1} is (K_3,k)-coloured and $k > 3$ then there is a vertex which is the centre of a monochromatic k-star.

Proof: The argument of the second part of the proof of Proposition 4.2 will be used twice.

Let $k > 3$ and let f be a (K_3,k)-colouring of K_{k+1}. Let d, v, U, V, w, c, c' be as in part (b) of the proof of Proposition 4.2. If $d < k$, neither U nor V is empty and U contains at least two points. Again, for each $w' \in V$ there is a $u' \in U$ with $f(u'w') = c$. In fact, since no colour appears more than k times, the u' is unique with respect to w' and no edges within U or V are coloured c. Thus, w is incident with d edges coloured c' and we can apply the same argument to the sets

$$U' = \{z \in V(K_{k+1}) \mid f(wz) = c'\},\ V' = N(w) - U'.$$

In particular, if $u \in U$ is the unique vertex joined to w by an edge coloured c, then $u \in V'$ and all but one edge from u to U' are coloured c. This is a contradiction unless U has only two elements. Then $|V| \geq 2$. Let $U = \{u,u'\}$. Then, since $f(uw) = c$ and $f(u'w) = c'$, the edge uu' is coloured c'. By the maximality of d we have $f(ww') \neq c'$ for all $w' \in V$ and since f is a (K_3,k)-colouring, $f(vw') = c' = f(uw')$. Using the maximality of d again, we see that $|V| = 2$. Thus $k = 4$. But we have already the colour c' on the edges uu', vw, vw', wu' and uw', a contradiction.

***

Figure 7 may help in understanding the above proof.

Let us now consider two particular cases. We shall show that $ar(K_4,2) = 7$ and $ar(K_5,2) \geq 10$. The value of $ar(K_4,2)$ shows that the upper bound of Proposition 4.1 is sharp. This, of course, stems from the fact that in the proof of the proposition we considered each subgraph K_n as containing exactly one pair of edges of the same colour. Clearly, this is hardly the case as n grows large.

<u>LEMMA 4.3:</u> $ar(K_4,2) = 7$.

Proof: Consider K_6 with vertices $0, \ldots, 5$. For $i = 0,1,2$ colour the edges from $2i$ to $2i+1$ and $2i+2$ with c_{2i} and the

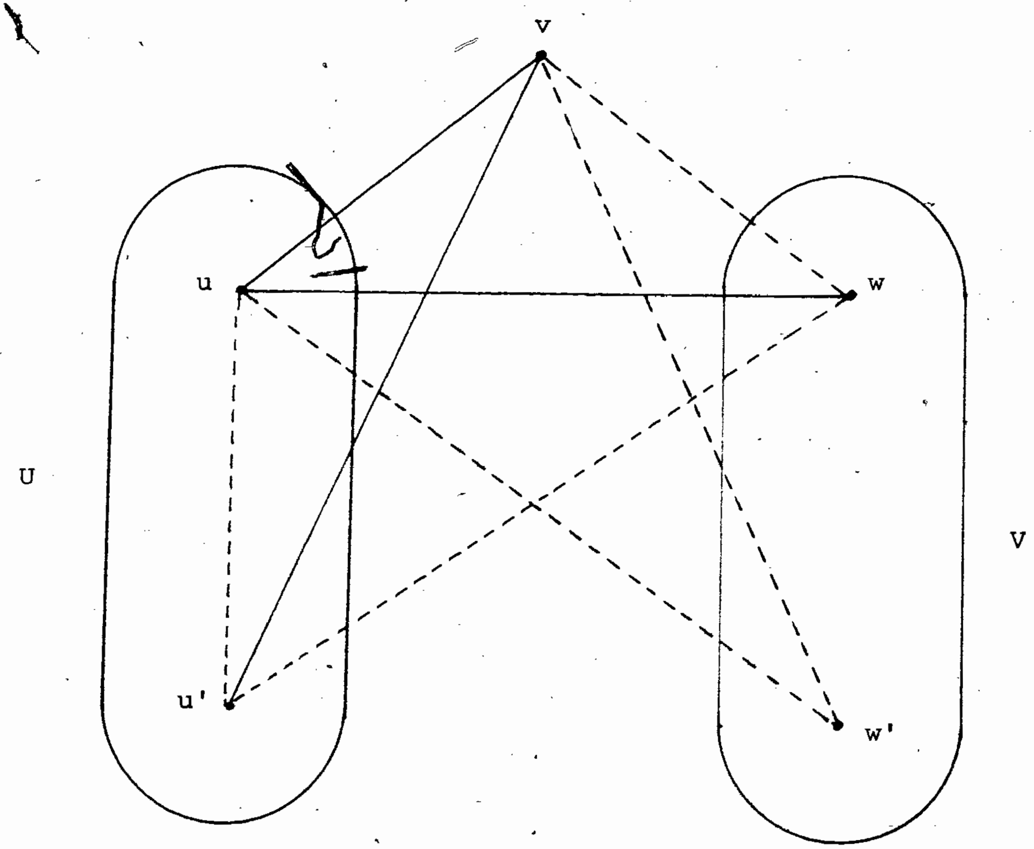

Figure 7. An aid to the proof of Lemma 4.2.

——— c   ------ c'

edges from $2i + 1$ to $2i + 3$ and $2i + 4$ with c_{2i+1}, addition taken modulo 6. It is routine to verify that any completion of this colouring to an edge-2-colouring will result in a $(K_4,2)$-coloured graph. Thus, $ar(K_4,2) \geq 7$.

Suppose now that K_7 is $(K_4,2)$-coloured by f. Since K_7 has 21 edges, no more than ten colours appear twice and no generality is lost by assuming exactly ten colours occur twice each. Each pair of adjacent edges lies in four K_4's and each pair of non-adjacent edges in one K_4. Letting p and q be the numbers of pairs of adjacent and non-adjacent edges of the same colour, respectively, and observing that there are 35 K_4's in K_7 we obtain

$$4p + q \geq 35$$

$$p + q = 10.$$

Since p and q are integers it follows that $q \leq 1$.

Let us call an occurrence of a pair of edges of the same colour in a K_4 already containing one such pair a <u>waste</u>. Also, let us say that an edge is <u>redundant</u> if it does not lie in a monochromatic $K_{1,2}$. Now, consider a pair of edges uv, uw of the same colour such that vw is not redundant. Then there is a unique z such that--without loss of generality--the edges vw and vz have the same colour and the complete graph on $\{u,v,w,z\}$ contains a waste. In fact, any three such pairs determine three wastes unless they form a <u>triple</u>, that is, a

K_4 on $\{u,v,w,z\}$ with each of the pairs (uv,uw), (vw,vz), (wz,uz) coloured one colour (see Figures 8). If the number of triples is r, then the number of wastes is at least

$$(10 - q) - (2q + 1) - r = 9 - 3q - r.$$

This is because there are $10 - q$ monochromatic 2-stars, $2q + 1$ redundant edges and every triple contains two wastes. Furthermore, $r \leq 2$ since two triples have at most one point in common and, hence, three triples require at least nine vertices. Thus, the number of wastes is at least $7 - 2q$. When $q = 0$, no more than forty K_4's contain pairs of edges of the same colour and, hence, no more than five wastes are possible. Similarly, when $q = 1$ only two wastes are allowed. Each case yields a contradiction. Thus, $ar(K_4,2) \leq 7$.

***

The following construction is due to Brian Alspach.

<u>LEMMA 4.4:</u> $ar(K_5,2) \geq 10.$

Proof: Consider the complete graph on the vertex set $V = I \times I$ with $I = \{0,1,2\}$. Let $C^* = \{c_{i,j,r} \mid i,j,r \in I\}$ and colour the edge $(i,j)(i',j')$--with labels taken modulo 3--by

(a) $c_{i,j,0}$ if $i = i'$ and $j < j'$;

(b) $c_{i,j,1}$ if $i' = i + 1$ and $j \neq j'$;

(c) $c_{i,j,2}$ if $i' = i + 1$ and $j = j' = i$;

(d) any other colour not already used if the edge is not

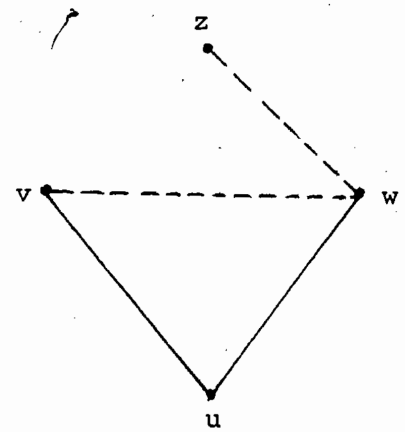

Figure 8(a). A waste determined (uniquely) by (uv,uw).

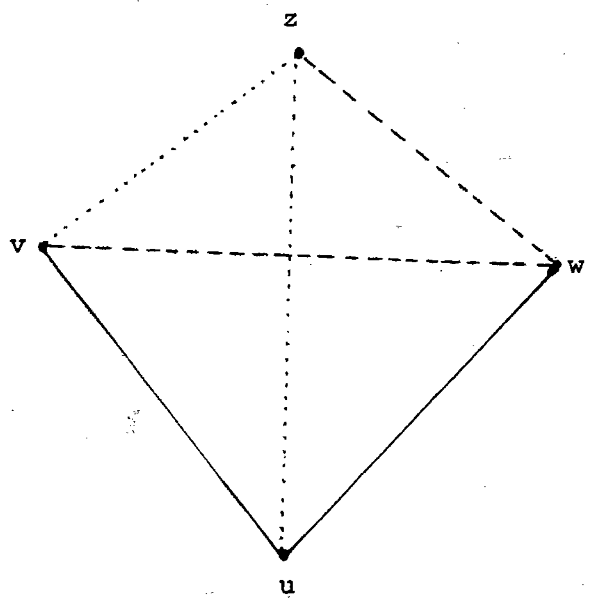

Figure 8(b). A triple.

coloured by any of (a), (b), (c).

It remains to show that any K_5 contained in this K_9 contains two edges of the same colour. Consider any K_5. If three of its vertices lie in $i \times I$ for some i, we are done by (a). If not then there are distinct i, i' such that $i \times I, i' \times I$ contain two vertices of the K_5 each. Without loss of generality assume that $i' = i + 1$ and that the four points are (i,j_1), (i,j_2), (i',j_1'), (i',j_2'). If $j_1 \neq j_1',j_2'$ or $j_2 \neq j_1',j_2'$ we are done by (b). No generality is lost by assuming now that $j_1 = j_1'$ and $j_2 = j_2'$. The fifth vertex must be $(i + 2,j)$ and if $j \neq j_1,j_2$ we are done by (b) again. If, without loss of generality, $j = j_1$ then two of the edges of the triangle on $\{(i,j_1),(i + 1,j_1),(i + 2,j_2)\}$ have the same colour by (c).

***

This exhausts the present results on $ar(K_n,k)$.

V. SOME RELATED PROBLEMS

As was already mentioned in the introduction, in the beginning was a problem proposed by Fred Galvin. A more general version of it was communicated in [Gl]. He begins with a modification of the partition symbol(11) $a \rightarrow (b)^r_k$.

DEFINITION 5.1: $a \overset{*}{\rightarrow} (b)^r_k$ if and only if for any colouring of the r-element subsets of an a-element set such that no colour is used more than k times there is a b-element subset in which all the r-element subsets have different colours.

Denote by $*(b)^r_k$ the least a such that $a \overset{*}{\rightarrow} (b)^r_k$. If we define, similarly, $(b)^r_k$ to be the least a such that $a \rightarrow (b)^r_k$ we see that

$$*(b)^r_k \leq (b)^r_k$$

and the question, of course, is what are these numbers given b and k. Clearly, $ar(K_n,k) = *(n)^2_k$.

DEFINITION 5.2: Let M be a set of cardinality m and define the Turán number $T^r_n(m)$ as

$$T^r_m(n) = \min \left| \{ S \subseteq \tbinom{M}{r} \mid \forall X \in \tbinom{M}{n} \exists Y \in S \; Y \subseteq X \} \right|$$

where $\binom{M}{r} = \{A \subseteq M \mid |A| = r\}$ and $m \geq n \geq r$.

Galvin's results include:

(1) $*(m+1)^1_k = mk + 1 = (m+1)^1_k$

(2) $mk + 1 \overset{*}{\not\rightarrow} (m+2)^2_k$

(3) $*(3)^2_k = k + 2$

(4) $*(4)^2_2 = 7$

(5) $10 \leq *(5)^2_2 \leq 12$

(6) $r \geq 3 \Rightarrow r + 2 < *(r+1)^r_2$

(7) $T^{r+1}_n(m) > \frac{k-1}{2}\binom{m}{r} \Rightarrow m \overset{*}{\rightarrow} (n)^r_k$

(8) $*(n)^r_k \leq \left\lfloor \frac{(k-1)(r+1)}{2}\binom{n}{r+1} \right\rfloor + r + 1$.

Some of these are recognizable as results also derived in this work. None have been published yet nor has the improvement of (8) by P. Erdös to

(9) $*(n)^r_k \leq cn^r$ for n sufficiently large, with c a constant depending on r and k.

A result of Richard Wilson ([W1]) is worth mentioning. It states that for a given graph Y, K_m can be partitioned into edge disjoint subgraphs all isomorphic to Y, if and only if m is sufficiently large and

(a) $m(m-1)$ is divisible by $|E(Y)|$;

(b) $m - 1$ is divisible by the greatest common divisor of the degrees of the vertices of Y.

CONJECTURE 5.1: If, given n and k, K_m can be decomposed into edge-disjoint K_p's with $\binom{p}{2} = k$ and $m - 1 = (p - 1)(n - 1)$ then $ar(K_{1,n},k) = m + 1$.

QUESTION 5.1: There is, for each n and k, a number $p(K_{1,n},k)$ such that no $(K_{1,n},k)$-colouring uses more than $p(K_{1,n},k)$ colours. What is it? What is the corresponding $p(K_n,k)$? And what are $p(Y,k)$, given Y and k?

QUESTION 5.2: What is the relationship between $ar(Y,k)$ and $p(Y,k)$?

QUESTION 5.3: We saw that any $(K_{1,3},k)$-colourable graph can be so coloured using at most three colours. For a fixed n, is there a number $p(K_{1,n})$ such that any $(K_{1,n},k)$-colourable graph can be so coloured using at most $p(K_{1,n})$ colours? What about $p(Y)$ for a fixed Y?

Note that for a fixed k the number of colours necessary to (Y,k)-colour a graph may be arbitrarily large.

Let $d = \langle a_1, \ldots, a_n \rangle$ be a sequence of non-negative integers. We say that d is graphical if there is a graph Y with d as its degree sequence, that is, $V(Y) = \{v_1, \ldots, v_n\}$ and $d(v_i) = a_i$, $i = 1, \ldots, n$. Let X be a graph with an edge-colouring f. Let, as usual, $f(E(X)) = C^* = \{c_1, \ldots, c_p\}$. A colour matrix $A(X,f)$ of X under f is a $|V(X)| \times |C^*|$ matrix with $A_{v,c_i} = |E_i(v)|$. It is easy to see that a v^{th} row of A is the colour structure of v and that a c_i^{th} column of A is the degree sequence of a monochromatic subgraph of X

induced by the edges coloured c_i. Conversely, we say that an $m \times p$ matrix is graphical if it is a colour matrix of K_m under some f with $|f(E(K_m))| = p$. Clearly, any product of a graphical matrix with a permutation matrix is again graphical. Also, if $f \in F_k(E(K_m))$ then $\sum_{v \in V(K_m)} A_{v,c_i} \leq 2k$. We can ask several related questions:

QUESTION 5.4: When is a matrix graphical?

QUESTION 5.5: When is a graphical matrix a colour matrix of a (Y,k)-coloured graph, for a given Y?

QUESTION 5.6: If a matrix is graphical, can we reconstruct the colouring of the graph?

These lead to the following:

QUESTION 5.7: Is a graph determined by a set of its edge-disjoint subgraphs? If so, when is it determined uniquely?

QUESTION 5.8: Given two graphical sequences d_1 and d_2 such that $d_1 - d_2$ is again graphical (if $d_1 = \langle a_{11}, \ldots, a_{1,n} \rangle$ and $d_2 = \langle a_{21}, \ldots, a_{2n} \rangle$ then $d_1 - d_2 = \langle a_{11} - a_{21}, \ldots, a_{1n} - a_{2n} \rangle$, when is it true that some graph with degree sequence d_1 contains edge-disjoint subgraphs with respective degree sequences d_2 and $d_1 - d_2$?

This last question is related to Question 5.4 in the sense that a necessary condition for a matrix to be graphical is that the sum of any i of its columns be graphical, $0 \leq i \leq p$. A partial answer can be found in some papers by S. Kundu ([K1],[K2],[K3]).

For some results on infinite anti-ramsey theory and related problems see [E1] and [E2].

With this we end the present work.

NOTES

(1) Galvin knew the answer. The proof given in Proposition 4.2 is due to Pavol Hell.

(2) Although the definition can be extended to infinite sets, we only consider (finite) graphs here. See [H1], pp. 8, 16.

(3) Also called vertices or nodes. See, e.g., [H1], p. 9.

(4) Also called lines or arcs. See, e.g., [H1], p. 9.

(5) N denotes the set of non-negative integers, as usual.

(6) In [E1] a graph whose edges are coloured with distinct colours is called totally multicoloured.

(7) "other" means "not n-star for any n."

(8) If n objects are distributed among $m < n$ pigeon-holes at least one hole contains at least two objects.

(9) See, e.g., [H2], pp. 237-241.

(10) This chapter consists mostly of the contents of [H3].

(11) The symbol $a \rightarrow (b)^r_k$ means that for any partition into k equivalence classes of the r-element subsets of an a-element set there is a b-element subset all of whose r-element subsets belong to one equivalence class. For example, Ramsey's theorem states that $\alpha \rightarrow (\omega)^r_k$ for any infinite ordinal α; in our special case $r = 2$.

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Note: Entries marked * indicate books.