35892



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TITLE OF THESIS/*TITRE DE LA THÈSE* THE KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD APPLIED TO MODELS OF INTERACTING POPULATION WITH RETARDATION

UNIVERSITY/*UNIVERSITÉ* SIMON FRASER UNIVERSITY

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* M. Sc.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE* 1977

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* Dr. George Bojadziev



DATED/*DATÉ* 17/Aug./77 SIGNED/*SIGNÉ*

PERMANENT ADDRESS/*RÉSIDENCE FIX*[redacted]

# THE KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD APPLIED TO MODELS OF INTERACTING POPULATIONS WITH RETARDATION

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B.Sc., Simon Fraser University, 1974


A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics



SIMON FRASER UNIVERSITY

August 1977

APPROVAL

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DEGREE: Master of Science

TITLE OF THESIS: The Krylov-Bogoliubov-Mitropolskii method applied to models of interacting populations with retardation

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ABSTRACT

One of the most interesting problems in population dynamics is concerned with a population of predators feeding on a population of prey. Lotka and Volterra (L-V) were the first to propose a set of rate equations with quadratic nonlinearities to describe such anecosystem. Later, by incorporating a saturation level term for the prey species, Gause and Witt modified the L-V model into what is known as the Volterra-Gause-Witt (V-G-W) model. Further, the V-G-W model involving constant time lag terms was studied by Wangersky and Cunningham (W-C).

In the literature on population dunamics there are many investigations, mainly dealing with stability problems. However, it is important in some cases that an approximate solution to the modelling equations be obtained. A useful approach to achieve this goal is the small parameter expansion on which the perturbation theory is based. A widely used method in this theory is the asymptotic method of Krylov-Bogoliubov-Mitropolskii (K-B-M). This method is used to study the following V-G-W modified models:

a) V-G-W model with two saturation levels

b) V-G-W model with two saturation levels and small time lag terms in the nonlinear part

c) V-G-W model with two saturation levels and small time lag terms in the linear part

d) W-C model with small time lag

e) W-C model with large time lag.

When the time lag involved is small, the system may be reduced to one without deviating argument. Then investigations can be done with the K-B-M method. For nonlinear oscillatory problems, it is generally known that the frequency of oscillation and the amplitude become interdependent. This behaviour is exhibited in the first order solution in most cases. The fluctuations are shown to be not only frequency dependent but also exponentially damped owing to the presence of saturation levels. For the model with significant time lag, the situation is more complicated. In this case an extension of the K-B-M method is used.

Finally, comparisons of the influences of time lag terms, small and significant, on the amplitude and the phase of these different models are made based on their first order approximate solutions.

# ACKNOWLEDGEMENT

The author would like to express his appreciation to Dr. G. N. Bojadziev for his encouragement and guidance throughout the entire period while this work was done. Also, many thanks to the staff and faculty members in the Mathematics Department of Simon Fraser University who made this thesis possible.

TABLE OF CONTENTS

FOREWARD

Many of the problems faced today by physicists, engineers and applied mathematicians involve difficulties due to nonlinear governing equations involving retarded arguments, small or significant, which preclude exact analytical solutions. To solve these problems we shall resort to a form of approximate solution. Among those systematic methods of approximation techniques, one which is widely used is the perturbation method of Krylov-Bogoliubov-Mitropolskii (K-B-M) which involves asymptotic expansions in powers of a small parameter. Application of this method to biological sciences was not known until recently.

Here the K-B-M method is used to study nonlinear models, mainly with retardation, in problems of population dynamics. Extension of the Volterra, Gause and Witt (V-G-W) model to include time lag, small or significant, will promote the closeness of describing a certain process by a system of nonlinear differential equations with deviating arguments in which past history is also accounted for. Clearly, investigations and derivations of approximate solutions of nonlinear systems with retarded arguments will be more interesting and at the same time involve more complications.

In Chapter 1, the K-B-M method for a second order ordinary differential equation with small nonlinearity is reviewed. A brief introduction of delay differential equations is given. Then an extension of the K-B-M method used for differential equations involving deviating arguments with large delay is outlined. The Lotka-Volterra model and its modification to V-G-W model is briefly explained. Applications of the K-B-M method to models of the V-G-W type by other authors are also mentioned.

In Chapter 2, the V-G-W model with two levels of saturation is studied. The results obtained may then be used for comparison with those involving time lag, small and significant. Two such models of the same type but with small time lag terms are investigated. Terms with deviating arguments in the describing system of differential equations occur in the nonlinear part of the first model and the linear part of the second model. For all three models, approximate solutions ~~up to~~ the second order are obtained. Then the Wangersky and Cunningham (W-C) model with small time lag is also studied. In this case, approximate solutions for both the prey and the predator species are given.

In Chapter 3, the W-C model with large time lag for only the predator species is considered by means of the extended K-B-M method.

As a conclusion, the first order solutions of all the models being investigated are compared in order to exhibit the effect of time lag on the approximate solutions.

CHAPTER 1

## § 1 THE KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD

The study of oscillatory processes is of basic importance in widely diverse branches of mechanics, physics and engineering. Investigators in the theory of linear oscillations attempted to fit the oscillatory processes studied by them into linear schemes, neglecting the nonlinear terms without proper justification. And such a linearization may lead to real errors not only of a quantitative but also of a qualitative nature. In the last century there already existed a mathematical apparatus which, if developed to the necessary extent and generalized, might have served as the tool for investigating nonlinear oscillations, at least those oscillations sufficiently close to the linear ones. An oscillation is treated as sufficiently close to a linear one when, though the corresponding differential equation be nonlinear, there is a parameter $\varepsilon$ in the equation such that for a zero value of $\varepsilon$ the equation degenerates into a linear differential equation with constant coefficients. In this case it is assumed that the parameter $\varepsilon$ is 'small', i.e. it may be taken as sufficiently small in absolute magnitude. However, there immediately arose the difficulty that it was impossible to use the usual method of expansions in powers of a small parameter for arriving at results which would be suited to the study of motion over sufficiently prolonged intervals of time. In fact, the usual expansions in powers of a small parameter lead to approximate formulae for the unknown quantities, characterising the motion, in cases which contain, besides terms depending harmonically on time, also secular terms like $t^m \sin \alpha t$ and $t^m \cos \alpha t$ ($m \geq 1$, $\alpha$ = constant), with time 't' appearing without the sine or cosine symbol. Consequently, the intensity of the secular terms increases rapidly with the increasing values of t. Thus it is clear that the range of application of the approximation formulae is limited to very small intervals of time.

The method of Poisson [1] for solving problems of the pendulum oscillations also revealed the above mentioned difficulty of the ordinary expansions in powers of a small parameter. Suppose we have to find a solution of a nonlinear equation containing a small parameter $\varepsilon$ of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x, \frac{dx}{dt}). \tag{1.1}$$

Then using the method of Poisson we seek a solution, which satisfies (1.1) to an accuracy of order $\varepsilon^{n+1}$ of the form

$$x = x_o + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots + \varepsilon^n x_n. \tag{1.2}$$

By substituting expression (1.2) into equation (1.1) and equating the coefficients of like powers of the parameter $\varepsilon$, we get the following system of equations:

$$\frac{d^2x_o}{dt^2} + \omega^2 x_o = 0,$$

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = f(x_o, \frac{dx_o}{dt}),$$

$$\frac{d^2X_2}{dt^2} + \omega^2 X_2 = f_x'(X_o, \frac{dX_o}{dt})X_1 + f_{\dot{x}}'(X_o, \frac{dX_o}{dt})\dot{X}_1 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

where $f_x' = \frac{\partial f}{\partial X}$ and $\dot{X} = \frac{dX}{dt}$ .

It can be easily seen that the above method entails appearance of secular terms. For instance, we obtain the first order solution

$$X_o = a \cos(\omega t + \theta)$$

and suppose that $f(X_o, \frac{dX_o}{dt}) = -X_o^3$ . Then we get the second order approximation,

$$X_1 = -(3/8\omega)a^3 t \sin(\omega t + \theta) + (1/32\omega^2)a^3 \cos 3(\omega t + \theta) ,$$

which contains a secular term. Hence this method is suitable only for very small intervals of time. Because of the presence of secular terms on the right-hand-side of expression (1.2), it is difficult to establish the periodicity of the solution.

It is natural that the oscillating systems most accessible for investigations are those with small nonlinearity. At present a number of sufficiently general methods are available for treating weakly non-linear systems. In this thesis, we will confine ourselves to the asymptotic method known as the K-B-M method [1, 2]. This method came into existence under the work of N. M. Krylov and N. N. Bogoliubov and Yu. A. Mitropolskii. We shall now construct the asymptotic approximations for the case of oscillations defined by equation (1.1). When perturbation is absent, i.e., when $\varepsilon = 0$ , the oscillations will be purely harmonic with

$$X(t) = a \cos \Psi ,$$

where the amplitude $a$ and the phase $\Psi$ are defined by

$$\frac{da}{dt} = 0 , \qquad \frac{d\Psi}{dt} = \omega .$$

The existence of nonlinear perturbation ($\varepsilon \neq 0$) results in the appearance of overtones in the solution of equation (1.1), a factor that establishes dependence between the instantaneous frequency $\frac{d\Psi}{dt}$ and the amplitude. We shall seek a general solution of equation (1.1) in the form

$$X(t) = a \cos \Psi + \varepsilon U_1(a,\Psi) + \varepsilon^2 U_2(a,\Psi) + \ldots \qquad (1.3)$$

Here $U_1(a,\Psi)$, $U_2(a,\Psi)$, ... , are periodic functions of the phase $\Psi$ with a period $2\pi$ and the quantities $a$, $\Psi$, are functions of time defined by the differential equations

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots ,$$
$$\frac{d\Psi}{dt} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots \qquad (1.4)$$

Although power series expansions are divergent, the approximate formulae obtained by taking a limited number of terms, for $m = 1, 2, \ldots$ , are found to be extremely suitable for practical calculations. In fact, these series are asymptotic in the sense that the error of the m-th approximation is proportional to the (m+1)-th power of the small parameter $\varepsilon$ . Hence for fixed values of $m = 1, 2, \ldots$ the error can be made as small as desired when the value of $\varepsilon$ is sufficiently small. Of course, by increasing m indefinitely we will not get convergence. However, the absence of such a convergence is not of essential importance in practice, because practically the determination of the coefficients of successive powers of $\varepsilon$ becomes so complicated that actually approximations of only the first or the second order, or of not very high order, may be used; but their usefulness is completely conditioned by the property of asymptoticity.

In practice, due to the rapidly growing complexity of the formulae, only the first two or three terms may be effectively derived. However, they are enough for application purposes. Therefore we confine ourselves to a finite number of terms in the expansions (1.3) and (1.4), i.e.,

$$X(t) = a\cos\Psi + \varepsilon U_1(a,\Psi) + \ldots + \varepsilon^m U_m(a,\Psi) , \qquad (1.5)$$

$$\frac{da}{dt} = \varepsilon A_1(a) + \ldots + \varepsilon^m A_m(a) ,$$

$$\frac{d\Psi}{dt} = \omega + \varepsilon B_1(a) + \ldots + \varepsilon^m B_m(a) , \qquad m = 1,2,\ldots \qquad (1.6)$$

Hence the practical applicability of the method is not determined by the convergence of the series (1.5) and (1.6) when $m \to \infty$ but by their asymptotic properties for a given fixed value of m when $\varepsilon \to 0$.

It is only required that when $\varepsilon$ is small, expression (1.5) should give a sufficiently accurate form of the solution of the nonlinear equation (1.1) for a sufficiently long interval of time. The first problem is to deduce suitable expressions for the functions $U_1(a,\Psi)$, $U_2(a,\Psi)$, . . ., $A_1(a)$, $A_2(a)$, . . ., $B_1(a)$ $B_2(a)$, . . . But a certain amount of arbitrariness is involved in defining the above expressions. Hence for the uniqueness of the definition of these coefficients we have to impose additional restrictions on them. Thus we require that the first harmonics are missing in the expressions $U_1(a,\Psi)$, $U_2(a,\Psi)$, . . ., i.e., the conditions

$$\int_0^{2\pi} U_i(a,\Psi)\cos\Psi\, d\Psi = 0 ,$$

$$\int_0^{2\pi} U_i(a,\Psi)\sin\Psi\, d\Psi = 0 , \qquad i = 1,2,\ldots, \tag{1.8}$$

have to be satisfied. Physically, the imposition of these conditions is equivalent to selecting $a$ as the full amplitude of the first fundamental harmonic of oscillation.

From equations (1.3) and (1.4) with some rearrangements in powers of $\varepsilon$, we get

$$\frac{dX}{dt} = -a\omega\sin\Psi + \varepsilon(A_1\cos\Psi - aB_1\sin\Psi + \omega\frac{\partial U_1}{\partial\Psi})$$
$$+ \varepsilon^2(A_2\cos\Psi - aB_2\sin\Psi + A_1\frac{\partial U_1}{\partial a} + B_1\frac{\partial U_1}{\partial\Psi} + \omega\frac{\partial U_2}{\partial\Psi}) + \varepsilon^3 \ldots$$

$$\frac{d^2X}{dt^2} = -a\omega^2\cos\Psi + \varepsilon(-2\omega A_1\sin\Psi - 2\omega aB_1\cos\Psi + \omega^2\frac{\partial^2 U_1}{\partial\Psi^2})$$
$$+ \varepsilon^2\{(A_1\frac{dA_1}{da} - aB_1^2 - 2\omega aB_2)\cos\Psi$$
$$- (2\omega A_2 + 2A_1B_1 + A_1 a\frac{dB_1}{da})\sin\Psi$$
$$+ 2\omega A_1\frac{\partial^2 U_1}{\partial a\partial\Psi} + 2\omega B_1\frac{\partial^2 U_1}{\partial\Psi^2} + \omega^2\frac{\partial^2 U_2}{\partial\Psi^2}\} + \varepsilon^3 \ldots$$

Hence the left-hand-side of equation (1.1) may be written as

$$\frac{d^2X}{dt^2} + \omega^2 X = \varepsilon\{-2\omega A_1 \sin\Psi - 2\omega a B_1 \cos\Psi + \omega^2 \frac{\partial^2 U_1}{\partial \Psi^2} + \omega^2 U_1\}$$

$$+ \varepsilon^2\{(A_1 \frac{dA_1}{da} - aB_1^2 - 2\omega a B_2)\cos\Psi$$

$$- (2\omega A_2 + 2A_1 B_1 + A_1 a \frac{dB_1}{da})\sin\Psi$$

$$+ 2\omega A_1 \frac{\partial^2 U_1}{\partial\alpha\partial\Psi} + 2\omega B_1 \frac{\partial^2 U_1}{\partial\Psi^2} + \omega^2 \frac{\partial^2 U_2}{\partial\Psi^2} + \omega^2 U_2\} + \varepsilon^3 \ldots \quad (1.9)$$

Also, the right-hand-side of equation (1.1) with the expression for $X(t)$ given by (1.3) may be written in the form

$$\varepsilon f(X, \frac{dX}{dt}) = \varepsilon f(a\cos\Psi, -a\omega\sin\Psi)$$

$$+ \varepsilon^2\{U_1 f'_x(a\cos\Psi, -a\omega\sin\Psi)$$

$$+ (A_1\cos\Psi - aB_1\sin\Psi + \omega\frac{\partial U_1}{\partial\Psi}) f'_{\dot{x}}(a\cos\Psi, -a\omega\sin\Psi)\} + \varepsilon^3 \ldots \quad (1.10)$$

In order that the expression (1.3) may satisfy the original equation with an accuracy of the order $\varepsilon^{m+1}$, it is necessary to equate coefficients of equal powers of $\varepsilon$ in the right-hand-side of equations (1.9) and (1.10) up to terms of the m-th order inclusively. Thus we get a system of m equations for the functions $U_i$, for $i = 1, 2, \ldots, m$. Using the Fourier series expansion for the functions $f(a\cos\Psi, -a\omega\sin\Psi)$ and $U_1(a,\Psi)$ in the first equation of the system, we may determine $A_1(a)$, $B_1(a)$ and $U_1(a,\Psi)$. The two Fourier series are

$$f(a\cos\Psi, -a\omega\sin\Psi) = g_o(a) + \sum_{n=1}^{\infty} g_n(a)\cos n\psi + h_n(a)\sin n\psi,$$

$$U_1(a,\Psi) = v_o(a) + \sum_{n=2}^{\infty} v_n(a)\cos n\Psi + w_n(a)\sin n\Psi, \quad (1.11)$$

where

$$g_n(a) = 1/2\pi \int_0^{2\pi} f(a \cos \Psi, -a\omega \sin \Psi) \cos n\Psi \, d\Psi,$$

$$h_n(a) = 1/2\pi \int_0^{2\pi} f(a \cos \Psi, -a\omega \sin \Psi) \sin n\Psi \, d\Psi.$$

Then by equating coefficients of identical harmonics in the first equation of the system we get

$$g_1(a) + 2\omega a B_1 = 0, \qquad h_1(a) + 2\omega A_1 = 0,$$

$$v_o(a) = g_o(a)/\omega^2, \tag{1.12}$$

$$v_n(a) = g_n(a)/\omega^2(1-n^2), \qquad w_n(a) = h_n(a)/\omega^2(1-n^2),$$

for n = 2, 3, ... Thus we have uniquely determined $A_1(a)$ and $B_1(a)$, as well as all the harmonic components of the function $U_1(a,\Psi)$ except the first $v_1(a)$ and $w_1(a)$. However, by virtue of the additional conditions (1.8), these functions do not contain the first harmonic; therefore

$$v_1(a) = 0, \qquad w_1(a) = 0.$$

Also, we have from equations (1.11) and (1.12),

$$A_1(a) = -1/2\pi\omega \int_0^{2\pi} f(a \cos \Psi, -a\omega \sin \Psi) \sin \Psi \, d\Psi,$$

$$B_1(a) = -1/2\pi\omega a \int_0^{2\pi} f(a \cos \Psi, -a\omega \sin \Psi) \cos \Psi \, d\Psi,$$

$$U_1(a,\Psi) = g_o(a)/\omega^2 + 1/\omega^2 \sum_{n=2}^{\infty} \{g_n(a) \cos n\Psi + h_n(a) \sin n\Psi\}/(1-n^2).$$

By completely determining $U_1(a,\Psi)$, $A_1(a)$ and $B_1(a)$, we can apply similar techniques to the other equations in the foregoing system to obtain higher order approximations successively, i.e., the quantities (1.7). By means of the additional conditions (1.8) the functions $A_n(a)$ and $B_n(a)$ ($n = 1, 2, \ldots$) are uniquely determined. As a result, the expressions for $A_n(a)$ and $B_n(a)$ thus obtained ensure the absence of terms with first harmonics in the systems of m equations, which in turn enables us to avoid the appearance of secular terms in the solution.

In particular, using the second equation in the system of m equations, we obtain expressions for the second approximation

$$A_2(a) = -\frac{1}{2\omega}\{2A_1B_1 + A_1 a\frac{dB_1}{da}\} - \frac{1}{2\pi\omega}\int_0^{2\pi} [U_1(a,\Psi)f_x'(a\cos\Psi, -a\omega\sin\Psi)$$
$$+ (A_1\cos\Psi - aB_1\sin\Psi + \omega\frac{\partial U_1}{\partial\Psi})f_{\dot{x}}'(a\cos\Psi, -a\omega\sin\Psi)]\sin\Psi\, d\Psi,$$

$$B_2(a) = -\frac{1}{2\omega}\{B_1^2 - (A_1/a)\frac{dA_1}{da}\} - \frac{1}{2\pi\omega a}\int_0^{2\pi} [U_1(a,\Psi)f_x'(a\cos\Psi, -a\omega\sin\Psi)$$
$$+ (A_1\cos\Psi - aB_1\sin\Psi + \omega\frac{\partial U_1}{\partial\Psi})f_{\dot{x}}'(a\cos\Psi, -a\omega\sin\Psi)]\cos\Psi\, d\Psi.$$

## § 2 DELAY DIFFERENTIAL EQUATIONS

Differential equations with a deviating argument are those in which the unknown function and its derivatives enter, generally speaking, under different values of the argument; e.g.,

$$\dot{X}(t) = f[t, X(t), X(t-\tau)], \qquad \tau > 0 . \tag{1.13}$$

Equations with a deviating argument describe many processes with time delay. They have many applications in the theory of automatic control, the theory of self-oscillating systems, the study of problems connected with combustion in rocket motion, the problem of long-range planning in economics, a series of biological problems, and in many other areas of science and technology. At present, this theory is one of the most rapidly developing branches of mathematical analysis.

Consider the differential equation of n-th order with $r$ deviating arguments

$$X^{(m_o)}(t) = f[t, X(t), \ldots, X^{(m_o-1)}(t), X(t - \tau_1(t)), \ldots,$$
$$X^{(m_1)}(t - \tau_1(t)), \ldots, X(t - \tau_r(t)),$$
$$\ldots, X^{(m_r)}(t - \tau_r(t))], \tag{1.14}$$

where the deviations $\tau_i(t) > 0$, $\max\limits_{0\le i\le r} m_i = n$. Here $X^{(k)}(t - \tau_i(t))$ denotes the k-th derivative of the function $X(Z)$, taken at the point $Z = t - \tau_i(t)$. In equation (1.14), let $\mu = \max\limits_{1\le i\le r} m_i$, and designate

$\lambda = m_o - \mu$. The equations for which $\lambda > 0$ are called equations with a retarded argument; $\lambda = 0$ , equations of neutral type; and $\lambda < 0$ , equations of advanced type. This thesis will involve differential equations with a retarded argument only. Equation (1.13) with delay $\tau$ being a constant is a simple example.

Consider an equation of the form

$$L[X(t-\tau)] = \sum_{p=0}^{n} \sum_{j=0}^{m} a_{pj} X^{(p)}(t - \tau_j) = f(t) \tag{1.15}$$

with coefficients $a_{pj}$ and deviating arguments $\tau_j$'s being constants. Then equation (1.15) is called a linear differential equation with constant coefficients and constant deviating arguments. With no loss of generality, we assume $0 = \tau_o < \tau_1 < . . . < \tau_m$ . In equation (1.15), if $a_{no} \neq 0$ , while the other $a_{nj} = 0$ , then it is an equation with retarded argument.

To begin with we consider the homogeneous equation of (1.15),

$$\sum_{p=0}^{n} \sum_{j=0}^{m} a_{pj} X^{(p)}(t - \tau_j) = 0 , \tag{1.16}$$

called a stationary linear homogeneous equation with a deviating argument. We seek a solution of equation (1.16) which has the form

$$X(t) = e^{kt} , \tag{1.17}$$

where $k$ is a constant.

Substituting equation (1.17) into equation (1.16) and cancelling $e^{kt}$ , one obtains the so called characteristic equation,

$$\sum_{p=0}^{n} \sum_{j=0}^{m} a_{pj} k^p e^{-k\tau_j} = 0 , \tag{1.18}$$

for the determination of $k$ . The left-hand-side of equation (1.16),

$$\Phi(k) = \sum_{p=0}^{n} \sum_{j=0}^{m} a_{pj} k^p e^{-k\tau_j} ,$$

is called the characteristic quasi-polynomial. This quasi-polynomial is an analytic function everywhere. Equation (1.18) has an infinite set of roots; the unique limit point is infinity. Each root corresponds to a particular solution of the form $e^{k_i t}$. Then the solution of the homogeneous equation (1.16), $X_\phi(t)$ with $\phi$ as a given initial function, can be expanded into a series of these basic solutions. For differential equations with a retarded argument, all roots $k_i$ of the quasi-polynomial $\Phi(k)$ lie in a left half plane $Re[k_i] \leq N$ . Here $Re[Z]$ denotes the real part of Z. All the solutions of equation (1.16) are asymptotically stable if all the roots $k_i$ of the quasi-polynomial satisfy the condition $Re[k_i] < 0$ and are asymptotically unstable if at least one root has a positive real part [7]. The solution $X_\phi(t)$ of equation (1.13) is called stable , if for any $\varepsilon > 0$ , there exists a $\delta(\varepsilon) > 0$ , such that from the inequality $|\phi(t) - \psi(t)| < \delta(\varepsilon)$ on the initial set, there follows $|X_\phi(t) - X_\psi(t)| < \varepsilon$ for all $t \geq t_o$ , where $\psi(t)$ is any continuous initial function. A stable solution $X_\phi(t)$ is called asymptotically stable , if $\lim_{t\to\infty} |X_\phi(t) - X_\psi(t)| = 0$ for any continuous initial function $\psi(t)$ , satisfying for sufficiently small $\delta_1 > 0$ the condition $|\phi(t) - \psi(t)| < \delta_1$ . All solutions of the linear equation with a deviating argument, $L[X(t-\tau)] = f(t)$ , and fixed initial point $t_o$ (as for a linear equation without deviating argument) are stable, or unstable simultaneously. For investigations of the stability of some solution $X_\phi(t)$ of equation (1.15), it is possible, by the change of variable $Y(t) = X(t) - X_\phi(t)$, to transform the discussion to stability of the solution $X_\phi(t)$ into that of the trivial solution $Y(t)$ of the corresponding homogeneous equation (1.16).

## § 3 K-B-M METHOD FOR DELAY DIFFERENTIAL EQUATIONS

Since we will consider mainly systems of differential equations with deviating arguments, straight forward application of the K-B-M method will not be valid. If the deviating arguments involved have a small retardation, we can apply the method of expansion in powers of the retardation and the system will then be reduced to one in which there is no deviating argument. The nonlinear system may then be solved by the K-B-M method. For the case when the time lag is large, the situation is a bit more complicated. Any given system of two first order nonlinear differential equations, even with deviating arguments, can always be reduced to a second order differential equation of the following form:

$$\ddot{X}(t) + \underline{\alpha}\dot{X}(t) + \underline{\beta}\dot{X}(t-\Delta) + \underline{\gamma}X(t) + \underline{\eta}X(t-\Delta) = \varepsilon F[X(t), X(t-\Delta), \dot{X}(t), \dot{X}(t-\Delta)], \tag{1.19}$$

where $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, $\underline{\eta}$ are constants and $\Delta = O(1)$. Hence we will extend the K-B-M method in accordance with equation (1.19). This method was given for the first time by Bojadziev and Lardner [4].

Consider the generating equation of (1.19), when $\varepsilon = 0$,

$$\ddot{X}(t) + \underline{\alpha}\dot{X}(t) + \underline{\beta}\dot{X}(t-\Delta) + \underline{\gamma}X(t) + \underline{\eta}X(t-\Delta) = 0. \tag{1.20}$$

This equation has solutions of the form $Ce^{Zt}$, where C is an arbitrary constant and Z is a root of the characteristic equation

$$\Omega(Z) = Z^2 + \underline{\alpha}Z + \underline{\beta}Ze^{-Z\Delta} + \underline{\gamma} + \underline{\eta}e^{-Z\Delta} = 0. \tag{1.21}$$

The above expression for $\Omega(\dot{Z})$ is known as the quasipolynomial of the homogeneous differential equation (1.20). If $\Delta$ is different from zero, then equation (1.21) has an infinite number of roots in the complex plane. Let $Z = -\xi + i\omega$ denote a particular root of (1.21). Then the corresponding real solution of (1.20) takes the following form:

$$X^{(0)}(t) = \mathrm{Re}[Ce^{(-\xi+i\omega)t}]$$

$$= C_o e^{-\xi t}\cos(\omega t+\omega_o),$$

where $C_o$ and $\omega_o$ are real constants; $\mathrm{Re}[Z]$ is used to denote the real part of Z. We are usually interested in the decaying solutions of equation (1.20), i.e., $\xi > 0$ .

Since the system being considered will involve strong damping and large time delay, a modified form of the standard solution in the K-B-M method for equation (1.19) is used,

$$X(t) = \mathrm{Re}[e^{-\xi\alpha+i\Psi}] + \varepsilon X^{(1)}(\alpha,\Psi) + \varepsilon^2 X^{(2)}(\alpha,\Psi) + \ldots, \tag{1.22}$$

where $\alpha$ and $\Psi$ satisfy the equations

$$\frac{d\alpha}{dt} = 1 + \varepsilon P(\alpha) + \varepsilon^2 \ldots,$$

$$\frac{d\Psi}{dt} = \omega + \varepsilon Q(\alpha) + \varepsilon^2 \ldots \tag{1.23}$$

Then differentiating equation (1.22) and making use of equations (1.23), we obtain

$$\frac{dX}{dt} = \mathrm{Re}\,[e^{-\xi\alpha+i\Psi}(Z + \varepsilon R)] + \varepsilon\{\frac{\partial X^{(1)}}{\partial\alpha} + \omega\frac{\partial X^{(1)}}{\partial\Psi}\} + \varepsilon^2 \ldots, \tag{1.24}$$

$$\frac{d^2X}{dt^2} = \mathrm{Re}[e^{-\xi\alpha+i\Psi}\{Z^2 + \varepsilon(R' + 2ZR\}]$$

$$+ \varepsilon(\frac{\partial^2 X^{(1)}}{\partial\alpha^2} + 2\omega\frac{\partial^2 X^{(1)}}{\partial\alpha\partial\Psi} + \omega^2\frac{\partial^2 X^{(1)}}{\partial\Psi^2}) + \varepsilon^2 \ldots, \qquad (1.25)$$

where $Z = -\xi+i\omega$ and $R(\alpha) = -\xi P(\alpha) + iQ(\alpha)$ and $R' = \frac{dR}{d\alpha}$.

From the expression (1.22), we get

$$X(t-\Delta) = \mathrm{Re}[e^{-\xi\alpha_\Delta+i\Psi_\Delta}] + \varepsilon X^{(1)}(\alpha_\Delta,\Psi_\Delta) + \varepsilon^2 \ldots,$$

$$\frac{dX(t-\Delta)}{dt} = \mathrm{Re}[e^{-\xi\alpha_\Delta+i\Psi_\Delta}\{Z + \varepsilon R(\alpha_\Delta)\}]$$

$$+ \varepsilon\{\frac{\partial X^{(1)}}{\partial\alpha}(\alpha_\Delta,\Psi_\Delta) + \omega\frac{\partial X^{(1)}}{\partial\Psi}(\alpha_\Delta,\Psi_\Delta)\} + \varepsilon^2 \ldots,$$

where $\alpha_\Delta = \alpha(t-\Delta)$ and $\Psi_\Delta = \Psi(t-\Delta)$.

By integrating the first equation of (1.23) from $t-\Delta$ to $t$, we get

$$\Delta = \int_{\alpha(t-\Delta)}^{\alpha(t)} \frac{d\alpha}{1 + \varepsilon P(\alpha) + \varepsilon^2 \ldots}$$

$$= \int_{\alpha(t-\Delta)}^{\alpha(t)} \{1 - \varepsilon P(\alpha) + \varepsilon^2 \ldots\}d\alpha$$

$$= \alpha(t) - \alpha(t-\Delta) - \varepsilon\int_{\alpha(t-\Delta)}^{\alpha(t)} P(\alpha)d\alpha + \varepsilon^2 \ldots$$

Therefore we have

$$\alpha_\Delta = \alpha(t-\Delta) = \alpha(t) - \Delta - \varepsilon P^\Delta(\alpha) + \varepsilon^2 \ldots,$$

where $P^\Delta(\alpha) = \int_{\alpha-\Delta}^{\alpha} P(\tau)\,d\tau$. Similarly, one can show that

$$\Psi_\Delta = \Psi(t-\Delta) = \Psi(t) - \omega\Delta - \varepsilon Q^\Delta(\alpha) + \varepsilon^2 \ldots,$$

where $Q^{\Delta}(\alpha) = \int_{\alpha-\Delta}^{\alpha} Q(\tau)\, d\tau.$

Hence we have

$$X(t-\Delta) = \mathrm{Re}[(e^{-\xi\alpha+i\Psi}e^{-Z\Delta})(1 - \varepsilon R^{\Delta})] + \varepsilon X^{(1)}(\alpha-\Delta,\Psi-\omega\Delta) + \varepsilon^2 \ldots, \quad (1.26)$$

$$\frac{dX(t-\Delta)}{dt} = \mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}\{Z + \varepsilon R(\alpha-\Delta) - \varepsilon Z R^{\Delta}(\alpha)\}]$$
$$+ \varepsilon\{\frac{\partial X^{(1)}}{\partial\alpha}(\alpha-\Delta,\Psi-\omega\Delta) + \omega\frac{\partial X^{(1)}}{\partial\Psi}(\alpha-\Delta,\Psi-\omega\Delta)\} + \varepsilon^2 \ldots, \quad (1.27)$$

where $R^{\Delta}(\alpha) = -\xi P^{\Delta}(\alpha) + iQ^{\Delta}(\alpha) = \int_{\alpha-\Delta}^{\alpha} R(\tau)\, d\tau.$

Then with the expressions given in (1.22), (1.24), (1.25), (1.26) and (1.27), we can substitute into equation (1.19) and compare terms of different orders in $\varepsilon$. Thus we have reduced the differential equation (1.19) with time lag to a system of differential equations in orders of $\varepsilon$ now with no time lag. The zero order terms cancelled identically and the first order $\varepsilon$ equation is as follows:

$$\frac{\partial^2 X^{(1)}}{\partial\alpha^2} + 2\omega\frac{\partial^2 X^{(1)}}{\partial\alpha\partial\Psi} + \omega^2\frac{\partial^2 X^{(1)}}{\partial\Psi^2} + \underline{\alpha}(\frac{\partial X^{(1)}}{\partial\alpha} + \omega\frac{\partial X^{(1)}}{\partial\Psi})$$
$$+ \underline{\beta}\{\frac{\partial X^{(1)}}{\partial\alpha}(\alpha-\Delta,\Psi-\omega\Delta) + \omega\frac{\partial X^{(1)}}{\partial\Psi}(\alpha-\Delta,\Psi-\omega\Delta)\} + \underline{\gamma}X^{(1)}$$
$$+ \underline{\eta}X^{(1)}(\alpha-\Delta,\Psi-\omega\Delta) + \mathrm{Re}[e^{-\xi\alpha+i\Psi}(R' + 2ZR)]$$
$$+ \underline{\alpha}\mathrm{Re}[e^{-\xi\alpha+i\Psi}R] + \underline{\beta}\mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}\{R(\alpha-\Delta) - ZR^{\Delta}(\alpha)\}] = F^{(1)}(\alpha,\Psi), \quad (1.28)$$

where

$$F^{(1)}(\alpha,\Psi) = F\{\mathrm{Re}[e^{-\xi\alpha+i\Psi}], \mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}], \mathrm{Re}[e^{-\xi\alpha+i\Psi}Z], \mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}Z]\}. \quad (1.29)$$

We shall seek a solution of equation (1.28) in Fourier series form,

$$X^{(1)}(\alpha,\Psi) = \sum_{m=-\infty}^{m=+\infty} X_m(\alpha)e^{im\Psi}, \tag{1.30}$$

and also expand $F^{(1)}(\alpha,\Psi)$ in Fourier series,

$$F^{(1)}(\alpha,\Psi) = \sum_{m=-\infty}^{m=+\infty} F_m(\alpha)e^{im\Psi}.$$

From the expression (1.29) for $F^{(1)}(\alpha,\Psi)$, the nonzero coefficients $F_m(\alpha)$ can be determined. Then by substituting (1.30) into equation (1.28) and comparing the coefficients of $e^{im\Psi}$, for the case $m \neq \pm 1$, we have

$$X_m''(\alpha) + (\underline{\alpha} + 2im\omega)X_m'(\alpha) + (\underline{\alpha}im\omega - m^2\omega^2 + \underline{\gamma})X_m(\alpha)$$

$$+ \underline{\beta}X_m'(\alpha-\Delta)e^{-im\omega\Delta} + (\underline{\beta}im\omega + \underline{\eta})X_m(\alpha-\Delta)e^{-im\omega\Delta} = F_m(\alpha).$$

From this equation the coefficients $X_m(\alpha)$ may be determined readily.

For $m = \pm 1$, as usual in the K-B-M method, we assume that the first harmonics are not present in $X_1(\alpha,\Psi)$. We thus have

$$R' + (2Z + \underline{\alpha})R + \underline{\beta}e^{-Z\Delta}R(\alpha-\Delta) - \underline{\beta}Ze^{-Z\Delta}\int_{\alpha-\Delta}^{\alpha} R(s)\,ds = 2e^{\xi\alpha}F_1(\alpha). \tag{1.31}$$

Once $R(\alpha)$ is determined from equation (1.31), then $P(\alpha)$ and $Q(\alpha)$ are known and together with $X^{(1)}(\alpha,\Psi)$ being computed, the first improved approximation of the solution to equation (1.19) is obtained.

## § 4 SOME BASIC MODELS IN POPULATION DYNAMICS

The mathematical study of an ecosystem concerning a population of predators feeding on a population of prey was initiated independently by Lotka and Volterra [5, 6], who proposed a set of rate equations with quadratic nonlinearities to describe the interaction of a prey species with its predator when both populations coexist in an ecological environment with finite resources. In the Lotka and Volterra (L-V) model, it is postulated that the population of species 1 (prey) would grow exponentially in the absence of species 2 (predator), while the population of the predator would extinguish exponentially in the absence of its prey. The interaction between the species is introduced through binary collisions, so that the loss rate of species 1 due to the interaction with species 2 is proportional to the product of their population sizes, as is the growth rate of species 2. The L-V rate equations for a prey-predator system are given by

$$\frac{dN_1}{dt} = \alpha_1 N_1 - \beta_1 N_1 N_2 \,,$$
$$\frac{dN_2}{dt} = -\alpha_2 N_2 + \beta_2 N_1 N_2 \,, \qquad (1.31)$$

where $N_i(t)$, for $i = 1,2$, is the number of individuals of species $i$ at a given time; $\alpha_i$ is the intraspecific coefficient (innate capacity for increase per individual) and $\beta_i$ is the interspecific coefficient; they are all positive constants.

A deficiency of the L-V model is the non-existence of a saturation level for the population of the prey species alone which, in general, does not grow indefinitely in a given environment with limited space and resource. In order to incorporate the saturation effect, Gause and Witt [7] in 1935 introduced a self-interaction term for the population of the prey $N_1$ . The modified equations for the so called V-G-W model are

$$\frac{dN_1}{dt} = \alpha_1 N_1 (1 - N_1/\Theta) - \beta_1 N_1 N_2 \ ,$$

$$\frac{dN_2}{dt} = -\alpha_2 N_2 + \beta_2 N_1 N_2 \ , \qquad (1.32)$$

where $\Theta$ is the carrying capacity (self-saturation level) of $N_1$ . For $\Theta \to \infty$, the system (1.32) reduces to (1.31).

Some authors have studied system (1.31), or similar systems, by qualitative investigations on the phase plane; for instance, Goel, Maitra and Montroll [8] in 1971. Since an exact solution of the proposed problem has not been obtained so far, then one could go about seeking an approximate solution. The standard procedure is to linearize the nonlinear equation in the neighborhood of the equilibrium point in the phase plane, and assume the effect of the nonlinear terms to be small. Under this assumption, the elegant asymptotic method of K-B-M could be applied to determine the weak nonlinear effects. An attempt to find an approximate solution of system (1.32) using the K-B-M method was made by Dutt, Ghosh and Karmakar [9] , but mistakes of principle character were made due to incorrect consideration of those different $\varepsilon$ order terms generated from the nonlinearity of system (1.32).

A more general V-G-W model which is described by the nonlinear system of first order differential equations

$$\frac{dN_1}{dt} = \alpha_1 N_1 [k_1 - N_1/\Theta - f_1(N_2)],$$

$$\frac{dN_2}{dt} = \alpha_2 N_2 [k_2 - f_2(N_1)], \tag{1.33}$$

where $\alpha_1$ , $\alpha_2$ , $k_1$ , $k_2$ are constants, $\Theta$ is the saturation level of $N_1$ , and $f_1$ , $f_2$ are analytic functions of their arguments. This system was recently studied in a paper by Bojadziev [10] and an approximate solution was obtained by means of the K-B-M method. The system (1.32), which is a particular case of (1.33), is considered in detail also in that paper. The approximate solutions for system (1.32), up to the second order, are given.

# CHAPTER 2

## § 1 V-G-W MODEL WITH TWO LEVELS OF SATURATION

Let us consider a further modification of the Volterra, Gause and Witt (V-G-W) model by incorporating another saturation level for the second species. This includes the effect of competition for food of the second species among themselves when its population size gets large which will lead to a decrease in the population eventually. The system of nonlinear differential equations describing this model is

$$\frac{dN_1}{dt} = N_1(t)\,[\kappa_1 - \alpha_{11}N_1(t)/\Theta_1 - \alpha_{12}N_2(t)]\ ,$$
$$\frac{dN_2}{dt} = N_2(t)[-\kappa_2 + \alpha_{21}N_1(t) - \alpha_{22}N_2(t)/\Theta_2]\ , \qquad (2.1)$$

where $\Theta_i$ is the carrying capacity (self-saturation level) of $N_i$ and $\kappa_i$'s, $\alpha_{ij}$'s are positive constants, for $i,j = 1,2$ . The non-zero equilibrium positions can be obtained from (2.1) by setting

$\frac{dN_i}{dt} = 0$ , $i = 1,2$ ; then we get

$$\kappa_1 - \alpha_{11}q_1/\Theta_1 - \alpha_{12}q_2 = 0\ ,$$
$$-\kappa_2 + \alpha_{21}q_1 - \alpha_{22}q_2/\Theta_2 = 0\ .$$

The solution of this system is

$$q_1 = \frac{\kappa_1\alpha_{22}/\Theta_2 + \kappa_2\alpha_{12}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}}\ ,$$

$$q_2 = \frac{-\kappa_2\alpha_{11}/\Theta_1 + \kappa_1\alpha_{21}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}}\ .$$

We are interested in the small vibrations occurring in the vicinity of the equilibrium position $(q_1, q_2)$. In order to accomplish this, we use the tranformations

$$N_i(t) = q_i + \varepsilon X_i(t) , \qquad i = 1,2 , \tag{2.2}$$

where $\varepsilon$ is a small positive parameter $(\varepsilon << 1)$ ; $X_1(t)$ and $X_2(t)$ are the two new variables. Substituting (2.2) into the system (2.1), dividing by $\varepsilon$ and keeping terms up to order of $\varepsilon^2$, gives

$$\begin{aligned} \frac{dX_1}{dt} &= -\alpha_{12}q_1X_2 - \varepsilon(2b_1X_1 + \alpha_{12}X_1X_2) - \varepsilon^2\, 2b_1\, X_1^2/q_1 , \\ \frac{dX_2}{dt} &= \alpha_{21}q_2X_1 - \varepsilon(2b_2X_2 - \alpha_{21}X_1X_2) - \varepsilon^2\, 2b_2\, X_2^2/q_2 , \end{aligned} \tag{2.3}$$

where the terms $\alpha_{ii}q_i/\Theta_i$ , i = 1,2 , are small when compared to the frequency of the linearized system of (2.1). Hence we let $\alpha_{ii}q_i/\Theta_i = \varepsilon 2b_i$ . Also, the terms $-\varepsilon(\alpha_{ii}/\Theta_i)\, X_i^2$ can be written as $-\varepsilon^2(2b_i/q_i)\, X_1^2$ and therefore are not considered as first order terms in $\varepsilon$ .

The linear system of (2.3), for $\varepsilon = 0$ , can be reduced to the following form:

$$\begin{aligned} \ddot{X}_1 + k^2X_1 &= 0 , \\ \ddot{X}_2 + k^2X_2 &= 0 , \end{aligned} \tag{2.4}$$

where $\ddot{X}_i = \dfrac{d^2X_i}{dt^2}$ , i = 1,2 , and $k^2 = \alpha_{12}\alpha_{21}q_1q_2$ . Note that for the linear case we have $q_1 = \kappa_2/\alpha_{21}$ and $q_2 = \kappa_1/\alpha_{12}$ .

The solutions of system (2.4) are

$$X_1(t) = P_1\cos(kt+n) ,$$
$$X_2(t) = P_2\sin(kt+n) ,$$

where $n$ is an arbitrary constant and the amplitudes are related as

$$\frac{P_1}{P_2} = \frac{\alpha_{12}}{\alpha_{21}}[\kappa_2/\kappa_1]^{\frac{1}{2}} .$$

Since $X_1$ and $X_2$ are related by the first equation in (2.3), the solution for $X_2$ can be derived from the knowledge of $X_1$ and $\dot{X}_1$. Henceforth we will consider the equation for $X_1$ only; and from the nonlinear system (2.4), we get

$$\ddot{X}_1 + k^2X_1 = \varepsilon(-2b_1\dot{X}_1 + 2b_2\alpha_{12}q_1X_2 - k^2X_1^2/q_1 - \alpha_{12}\dot{X}_1X_2 - \alpha_{12}\alpha_{21}q_1X_1X_2)$$
$$+ \varepsilon^2(-4b_1X_1\dot{X}_1/q_1 + 2b_2\alpha_{12}q_1X_2^2/q_2 + 2b_2\alpha_{12}X_1X_2 - \alpha_{12}\alpha_{21}X_1^2X_2) \quad (2.5)$$

In order to apply the K-B-M perturbation method, equation (2.5) is reduced to the following canonical form:

$$\ddot{X}_1 + k^2X_1 = \varepsilon f(X_1,\dot{X}_1) , \quad (2.6)$$

where

$$f(X_1,\dot{X}_1) = -2(b_1+b_2)\dot{X}_1 - k^2X_1^2/q_1 + \alpha_{21}X_1\dot{X}_1 + \dot{X}_1^2/q_1$$
$$+ \varepsilon(-4b_1b_2X_1 + 2b_1\alpha_{21}X_1^2 - 2b_1X_1\dot{X}_1/q_1 + 2b_2\alpha_{21}\dot{X}_1^2/k^2 - X_1\dot{X}_1^2/q_1^2) + \varepsilon^2 \ldots$$

We shall seek a general solution of equation (2.6) in the form

$$X_1(t) = a \cos \Psi + \varepsilon U_1(a,\Psi) + \varepsilon^2U_2(a,\Psi) + \varepsilon^3 . . .$$

with $a(t)$ and $\Psi(t)$ defined by

$$\frac{da}{dt} = A_1(a) + \varepsilon^2A_2(a) + \varepsilon^3 . . .,$$
$$\frac{d\Psi}{dt} = k + \varepsilon B_1(a) + \varepsilon^2B_2(a) + \varepsilon^3 . . . \quad (2.7)$$

We are interested in the first order approximate solution of equation (2.6) and $f(X_1,\dot{X}_1)$ in (2.6) takes the following form:

$$f(X_1,\dot{X}_1) = M_0(X_1,\dot{X}_1) + \varepsilon M_1(X_1,\dot{X}_1) \ .$$

Then using Taylor's formula for each of the $M_i$'s, for $i = 0,1$, the above expression becomes

$$\begin{aligned} f(X_1,\dot{X}_1) &= M_0^0(a,\Psi) + \varepsilon M_0^1(a,\Psi) + \varepsilon^2\ldots \\ &\quad + \varepsilon M_1^0(a,\Psi) + \varepsilon^2 M_1^1(a,\Psi) + \varepsilon^3\ldots \\ &= M_0^0(a,\Psi) + \varepsilon\{M_0^1(a,\Psi) + M_1^0(a,\Psi)\} + \varepsilon^2\ldots, \end{aligned} \tag{2.8}$$

where

$$M_0^0(a,\Psi) = M_0(a\cos\Psi, -ak\sin\Psi) \ ,$$

$$M_1^0(a,\Psi) = M_1(a\cos\Psi, -ak\sin\Psi) \ ,$$

$$M_0^1(a,\Psi) = U_1 \frac{\partial M_0}{\partial X_1}(a\cos\Psi, -ak\sin\Psi)$$

$$+ \left[A_1\cos\Psi - aB_1\sin\Psi + k\frac{\partial U_1}{\partial\Psi}\right]\frac{\partial M_0}{\partial\dot{X}_1}(a\cos\Psi, -ak\sin\Psi) \ .$$

Thus, according to the K-B-M method, we have

$$\begin{aligned} f_0(a,\Psi) &= M_0(a\cos\Psi, -ak\sin\Psi) \ , \\ f_1(a,\Psi) &= M_0^1(a,\Psi) + M_1^0(a,\Psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\Psi + (2A_1B_1 + aA_1\frac{dB_1}{da})\sin\Psi, \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} A_1(a) &= -1/2\pi k \int_0^{2\pi} f_0(a,\Psi)\sin\Psi\, d\Psi \ , \\ B_1(a) &= -1/2\pi ak \int_0^{2\pi} f_0(a,\Psi)\cos\Psi\, d\Psi \ . \end{aligned} \tag{2.10}$$

From the expressions (2.6), (2.8) and (2.9) we obtain

$$f_0(a,\Psi) = 2(b_1+b_2)ak\sin\Psi - (a^2k^2/q_1)\cos 2\Psi - \tfrac{1}{2}\alpha_{21}a^2k\sin 2\Psi \ .$$

Hence the non-zero Fourier coefficients of $M_0(a\cos\Psi, -ak\sin\Psi)$ are

$$g_2 = -a^2k^2/q_1 ,$$

$$h_1 = 2(b_1+b_2)ak ,$$

$$h_2 = -\frac{1}{2}\alpha_{21}a^2k ;$$

so formulae (2.10) give

$$A_1 = -h_1/2k = -(b_1+b_2)a ,$$

$$B_1 = -g_1/2ak = 0 .$$

Thus, up to the first order, we get

$$\frac{da}{dt} = -\varepsilon(b_1+b_2)a = -\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)a ,$$

$$\frac{d\Psi}{dt} = k ,$$

which have solutions

$$a(t) = P_1 e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)t} , \qquad (2.11)$$

$$\Psi(t) = kt + k_0 ,$$

where $P_1$ and $k_0$ are constants of integration. Therefore the first approximation of equation (2.6) is

$$X_1(t) = P_1 e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)t} \cos(kt + k_0) .$$

In order to derive the first improved approximation, $U_1(a,\Psi)$ is expressed in Fourier series with the form

$$U_1(a,\Psi) = v_o(a) + \sum_{n=1}^{\infty} v_n(a)\cos n\Psi + w_n(a)\sin n\Psi ,$$

where

$$v_0(a) = g_0(a)/k^2 ,$$

$$v_n(a) = g_n(a)/k^2(1-n^2) ,$$

$$w_n(a) = h_n(a)/k^2(1-n^2) , \qquad n = 2,3,\ldots .$$

Therefore we have

$$v_0 = v_1 = 0 \, ,$$

$$v_2 = a^2/3q_1 \, ,$$

$$w_2 = \alpha_{21} a^2/6k \, ,$$

and otherwise zeroes. Thus the first improved approximation is given by

$$X_1(t) = a \cos \Psi + \varepsilon (a^2/3)\{(1/q_1)\cos 2\Psi + (\alpha_{21}/2k)\sin 2\Psi\}. \qquad (2.12)$$

Since $B_1 = 0$, in order to obtain dependence of the phase on the amplitude, we have to calculate the second approximation. From expressions (2.7), (2.9), (2.10) and (2.12) with a series of elaborations we get

$$f_1(a,\Psi) = \frac{1}{2}a^2 (b_1+b_2)\alpha_{21}$$

$$+ \{-4b_1 b_2 a + a^3\alpha_{21}^2/12 + a^3k^2/12q_1^2 + a(b_1+b_2)^2\}\cos\Psi$$

$$+ (0)\sin\Psi + \text{other terms of higher harmonics;}$$

then

$$A_2(a) = -1/2\pi k \int_0^{2\pi} f_1(a,\Psi)\sin\Psi \, d\Psi = 0 \, ,$$

$$B_2(a) = -1/2\pi ak \int_0^{2\pi} f_1(a,\Psi)\cos\Psi \, d\Psi \qquad (2.13)$$

$$= -\frac{1}{2k}\{(b_1-b_2)^2 + a^2(\alpha_{21}^2 + k^2/q_1^2)/12\} \, .$$

Also from expressions (2.8), (2.11) and (2.13) we have

$$a(t) = P_1 e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)t} \, ,$$

$$\Psi(t) = kt\{1 - (1/8k^2)(\alpha_{11}q_1/\Theta_1 - \alpha_{22}q_2/\Theta_2)^2\} \qquad (2.14)$$

$$+ \varepsilon^2 \frac{a^2(\alpha_{21}^2 + k^2/q_1^2)}{24k(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)} + k_0 \, ,$$

together with (2.12) as the second approximate solution for the nonlinear differential equation (2.6).

By taking the limit as $\Theta_1$ , $\Theta_2 \to \infty$ in the expressions (2.11), we obtain the first improved approximation for the L-V model (1.31).

$$X_1(t) = a_o \cos \Psi + (a_o^2\beta_2/3)\{(1/\alpha_2)\cos 2\Psi + (1/2k)\sin 2\Psi\}$$

with $\Psi(t) = kt + k_o$ and $a(t) = a_o$ , where $k = (\alpha_1\alpha_2)^{\frac{1}{2}}$ .

In using the K-B-M perturbation technique, the calculation was based on the assumption of small fluctuations of the populations of species $N_1$ and $N_2$ about their respective equilibrium levels. Essentially, this means that both $a/q_1$ and $a/q_2$ are small quantities. Hence the qualitative features shown by the first order results may not be true for large nonlinear fluctuations.

From the linearized system (2.4) of the V-G-W model we have an expression for the linear frequency which is dependent on the intraspecific coefficients $\kappa_1$, $\kappa_2$ and the other coefficients as well, in contrast to the L-V model, in which case the linear frequency depends only on the intraspecific coefficients, $\alpha_1$ and $\alpha_2$ . The extra dependence occurs merely due to the presence of two saturation level terms.

For the first approximation, the frequency of oscillation is found to be the same as in the linear case, since $B_1 = 0$ , and the amplitude is exponentially damped. This damping effect appears due to the two saturation levels, since $\alpha_{ii}$ and $q_i$ are positive constants, and the power of the damping factor can be zero only if $\Theta_i \to \infty$ , for $i = 1,2$ . Consequently, the amplitude in the V-G-W model is shown to be decaying exponentially with a small damping constant, $-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2) = -\varepsilon(b_1 + b_2)$ , while in the L-V model it is just a constant. Nonlinearity is then exhibited through the dependence on $a(t)$ in the expression (2.12) for $U_1(a,\Psi)$. Further, the dependence of $\Psi(t)$ on $a(t)$ is revealed in the second approximation through the term $B_2(a)$ which is given in equation (2.13). This would mean that the period of the harmonic terms stays constant, $2\pi/k$, in the first order solution, with a correction factor occurring in the second order solution. In fact, we can see from the second equation in expressions (2.14) that the period,

$$2\pi/[k\{1 - (1/8k^2)(\alpha_{11}q_1/\Theta_1 - \alpha_{22}q_2/\Theta_2)^2\}] ,$$

is actually increased.

## § 2 V-G-W MODEL WITH SMALL TIME LAG IN THE NONLINEAR PART AND TWO LEVELS OF SATURATION

It is known in ecology that the degree of pattern regularity in population oscillations depends on the population history. Perturbations applied to systems may have aftereffects which do not show up until a certain time after they have been applied. This is the effect due to time delay mechanisms. The previous V-G-W model (3.1) is generalized to include deviating arguments in the nonlinear part as follows:

$$\frac{dN_1}{dt} = \kappa_1 N_1(t) - \alpha_{11}N_1(t)N_1(t-\Delta)/\Theta_1 - \alpha_{12}N_1(t)N_2(t-\Delta) ,$$
$$\frac{dN_2}{dt} = -\kappa_2 N_2(t) + \alpha_{21}N_2(t)N_1(t-\Delta) - \alpha_{22}N_2(t)N_2(t-\Delta)/\Theta_2 , \tag{2.15}$$

where $\Theta_i$ is the saturation level for the species $N_i$, $\kappa_i$, and $\alpha_{ij} > 0$, for $i, j = 1, 2$.

Since the deviation is small, we can use the method of expansion in powers of the deviation and thus reduce the system to one without deviating argument. Then the method of K-B-M may be applied directly to solve the system. We let the time lag $\Delta = \varepsilon\tau$, where $\varepsilon << 1$ ; therefore one can expand $N_i(t-\varepsilon\tau)$ in powers of the retardation $\varepsilon\tau$ using the Taylor series expansion as suggested in El'sgol'ts [3] and get

$$N_i(t-\varepsilon\tau) = N_i(t) - \varepsilon\tau N_i(t) + \varepsilon^2\tau^2 N_i(t)/2! - \dots + (-1)^m\varepsilon^m\tau^m N_i^{(m)}(t)/m! , \quad i = 1,2. \tag{2.16}$$

With $\varepsilon = 0$ and setting $dN_i/dt = 0$ in system (2.15), we obtain the nonzero equilibrium position $(q_1, q_2)$ which is given by

$$q_1 = \frac{\kappa_1\alpha_{22}/\Theta_2 + \kappa_2\alpha_{12}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}} \ ,$$

$$q_2 = \frac{-\kappa_2\alpha_{11}/\Theta_1 + \kappa_1\alpha_{21}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}} \ .$$

As in the previous case, the substitutions

$$N_i(t) = q_1 + \varepsilon X_i(t) \ , \qquad i = 1,2 \ , \tag{2.17}$$

are used and expression (2.16) becomes

$$N_i(t-\varepsilon\tau) = q_i + \varepsilon X_i(t) - \varepsilon^2\tau\dot{X}_i(t) + \varepsilon^3\tau^2\ddot{X}_i(t)/2 - \dots \ . \tag{2.18}$$

By substituting the expressions (2.17) and (2.18) into the system (2.15) we then have

$$\begin{aligned}\frac{dX_1}{dt} &= -\alpha_{12}q_1X_2 + \varepsilon(-2b_1X_1 - \alpha_{12}X_1X_2 + \alpha_{12}q_1\tau\dot{X}_2) \\ &\quad + \varepsilon^2(-2b_1X_1^2/q_1 + 2b_1\tau\dot{X}_1 - \tfrac{1}{2}\alpha_{12}q_1\tau^2\ddot{X}_2) + \varepsilon^3 \dots \ , \\ \frac{dX_2}{dt} &= \alpha_{21}q_2X_1 + \varepsilon(-2b_2X_2 + \alpha_{21}X_1X_2 - \alpha_{21}q_2\tau\dot{X}_1) \\ &\quad + \varepsilon^2(-2b_2X_2^2/q_2 + 2b_2\tau\dot{X}_2 + \tfrac{1}{2}\alpha_{21}q_2\tau^2\ddot{X}_1) + \varepsilon^3 \dots \ ,\end{aligned} \tag{2.19}$$

where $\alpha_{ii}q_i/\Theta_i = \varepsilon 2b_i$ are assumed to be small quantities, for $i = 1,2$; and therefore $\varepsilon\alpha_{ii}(X_i^2 - q_i\tau\dot{X}_i)/\Theta_i$ are considered as terms of order $\varepsilon^2$.

Hence for $\varepsilon = 0$, the linear system of (2.19) is

$$\ddot{X}_1 + k^2X_1 = 0 \ ,$$

$$\ddot{X}_2 + k^2X_2 = 0 \ ,$$

where $k^2 = \alpha_{12}\alpha_{21}q_1q_2$. This system possesses a solution of the form

$$X_1 = P_1 \cos(kt + n) ,$$

$$X_2 = P_2 \sin(kt + n) ,$$

where $P_1$ , $P_2$ and $n$ are constants.

The canonical form is obtained by differentiating the first equation once in the system (2.19) and using the second one to eliminate $X_2$ . Then system (2.19) is reduced to the following differential equation, up to terms of order $\varepsilon^2$ ,

$$\ddot{X}_1 + k^2 X_1 = \varepsilon\{-2(b_1+b_2-k^2\tau)\dot{X}_1 - k^2 X_1^2/q_1 + \dot{X}_1^2/q_1 + \alpha_{21} X_1 \dot{X}_1\}$$
$$+ \varepsilon^2\{-4b_1 b_2 X_1 + 2\alpha_{21} b_1 X_1^2 - 2b_1 X_1 \dot{X}_1/q_1 + \alpha_{21}[(2b_2/k^2)-\tau]\dot{X}_1^2 - X_1 \dot{X}_1^2/q_1^2\}. \quad (2.20)$$

And similarly we have

$$\ddot{X}_2 + k^2 X_2 = \varepsilon\{-2(b_1+b_2-k^2\tau)X_2 - k^2 X_2^2/q_2 + \dot{X}_2^2/q_2 - \alpha_{12} X_2 \dot{X}_2\}$$
$$+ \varepsilon^2\{-4b_1 b_2 X_2 - 2\alpha_{12} b_2 X_2^2 - 2b_2 X_2 \dot{X}_2/q_2 - \alpha_{12}[(2b_1/k^2)-\tau]\dot{X}_2^2 - X_2 \dot{X}_2^2/q_2^2\} .$$

Now we shall consider equation (2.20) which in standard form is

$$\ddot{X}_1 + k^2 X_1 = \varepsilon f(X_1,\dot{X}_1)$$
$$= \varepsilon M_0(X_1,\dot{X}_1) + \varepsilon^2 M_1(X_1,\dot{X}_1) .$$

This is similar to the case in the previous section, hence we have

$$f_0(a,\Psi) = M_0(a \cos \Psi, -ak \sin \Psi) ,$$
$$f_1(a,\Psi) = M_0^1(a,\Psi) + M_1^0(a,\Psi) \quad (2.21)$$
$$+ (aB_1^2 - A_1 \frac{dA_1}{da}) \cos \Psi + (2A_1 B_1 + aA_1 \frac{dB_1}{da}) \sin \Psi .$$

From equations (2.20) and (2.21) we get

$$f_0(a,\Psi) = 2(b_1+b_2-k^2\tau) ak \sin \Psi - \tfrac{1}{2}\alpha_{21} a^2 k \sin 2\Psi - (a^2 k^2/q_1) \cos 2\Psi .$$

Thus the nonzero Fourier coefficients for $f_0(a,\Psi)$ are

$$g_2 = -a^2k^2q_1 ,$$

$$h_1 = 2(b_1+b_2-k^2\tau)ak ,$$

$$h_2 = -\frac{1}{2}a^2k\alpha_{21} .$$

Therefore we have

$$A_1(a) = -(b_1+b_2-k^2\tau)a ,$$

$$B_1(a) = 0 .$$

Then the amplitude is given by

$$a(t) = P_1e^{-\varepsilon(b_1+b_2-k^2\tau)t}$$

$$= P_1e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2 - 2k^2\Delta)t} ,$$

and the phase is

$$\Psi(t) = kt + \Psi_0 ,$$

where $P_1$ and $\Psi_0$ are integration constants. Further, we have for the first improved approximation,

$$X_1(t) = a\cos\Psi + \varepsilon U_1^{(1)}(a,\Psi) ,$$

where

$$U_1^{(1)}(a,\Psi) = \frac{1}{3}a^2\{(1/q_1)\cos 2\Psi + (\alpha_{21}/2k)\sin 2\Psi\} . \tag{2.21}$$

Calculation of the second approximation is necessary in order to obtain dependence of the phase on the amplitude. From expressions (2.19), (2.20) and (2.21) we get

$$f_1(a,\Psi) = \frac{1}{2}a^2(b_1+b_2)\alpha_{21}$$

$$+ \{-4b_1b_2a+\alpha_{21}^2/12 + a^3k^2/12q_1^2 + a(b_1+b_2-k^2\tau)^2\}\cos\Psi$$

$$+ (0)\sin\Psi + \text{other terms of higher harmonics,}$$

so we have

$$A_2(a) = -1/2\pi k \int_0^{2\pi} f_1(a,\Psi)\sin\Psi \, d\Psi = 0 ,$$

$$B_2(a) = -1/2\pi ak \int_0^{2\pi} f_1(a,\Psi)\cos\Psi \, d\Psi = 0$$

$$= -(1/2k)\{(b_1-b_2)^2 - 2(b_1+b_2)k^2\tau + k^4\tau^2 + a^2(\alpha_{21}^2+k^2/q_1^2)/12\} .$$

As a second approximation we have

$$a(t) = P_1 e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2 - 2k^2\Delta)t}$$

$$\Psi(t) = kt\{1 + \tfrac{1}{2}[(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)\Delta - k^2\Delta^2 - \frac{1}{4k^2}(\alpha_{11}q_1/\Theta_1-\alpha_{22}q_2/\Theta_2)^2]\}$$

$$+ \varepsilon^2(1/24k)a^2(\alpha_{21}^2 + k^2/q_1^2)(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2 - 2k^2\Delta)^{-1} + \Psi_o .$$

In the first approximation we can see that the amplitude is damped and the damping constant depends on the saturation levels $\Theta_1$, $\Theta_2$ and also the time lag $\Delta$ . But the phase remains the same as in the linear case. Complicated dependence of the phase $\Psi$ on the amplitude is obtained in the second approximation. One can see that the improvement of the phase consists of two main parts: (1) a small correction to the linear term $kt$ due to the saturation levels and the time delay $\Delta$ , (2) an exponential term $a^2$ with a fractional factor involving all the parameters of the system including $\Theta_i$'s and $\Delta$ .

## § 3 V-G-W MODEL WITH SMALL TIME LAG IN THE LINEAR PART AND TWO LEVELS OF SATURATION

This model is described by a nonlinear system involving small time lag in the linear part of the describing equations:

$$\frac{dN_1}{dt} = N_1(t-\Delta)[\kappa_1 - \alpha_{11}N_1(t)/\Theta_1 - \alpha_{12}N_2(t)] \ ,$$

$$\frac{dN_2}{dt} = N_2(t-\Delta)[-\kappa_2 + \alpha_{21}N_1(t) - \alpha_{22}N_2(t)/\Theta_2] \ , \qquad (2.22)$$

where the time lag $\Delta$ is assumed to be small and we let $\Delta = \varepsilon\tau$ .

At equilibrium, with $\frac{dN_i}{dt} = 0$ , $i = 1,2$ ; we get

$$q_1 = \frac{\kappa_1\alpha_{22}/\Theta_2 + \kappa_2\alpha_{12}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}} \ ,$$

$$q_2 = \frac{-\kappa_2\alpha_{11}/\Theta_1 + \kappa_1\alpha_{21}}{\alpha_{11}\alpha_{22}/\Theta_1\Theta_2 + \alpha_{12}\alpha_{21}} \ .$$

Again, we use the substitutions

$$N_i(t) = q_i + \varepsilon X_i(t) \ , \qquad i = 1,2 \ . \qquad (2.23)$$

Also expanding $N_i(t-\varepsilon\tau)$ up to order $\varepsilon^2$ , we have

$$N_i(t-\varepsilon\tau) = q_i + \varepsilon X_i - \varepsilon^2\tau\dot{X}_i \ . \qquad (2.24)$$

Substituting expressions (2.23) and (2.24) into the system (2.22) we obtain

$$\dot{X}_1 = -q_1\alpha_{12}X_2 - \varepsilon(2b_1X_1 + \alpha_{12}X_1X_2) + \varepsilon^2(\tau\alpha_{12}\dot{X}_1X_2 - 2b_1X_1^2/q_1) \ ,$$

$$\dot{X}_2 = q_2\alpha_{21}X_1 - \varepsilon(2b_2X_2 - \alpha_{21}X_1X_2) + \varepsilon^2(-\tau\alpha_{21}X_1\dot{X}_2 - 2b_2X_2^2/q_2) \ , \qquad (2.25)$$

where $\alpha_{ii}q_i/\Theta_i = \varepsilon 2b_i$ . Note that terms involving $\tau$ occur in the order of $\varepsilon^2$ only. System (2.25) is reduced to canonical form by differentiating the first equation and eliminating $X_2$ by means of the second equation, we then have

$$\ddot{X}_1 + k^2 X_1 = \varepsilon f(X_1, \dot{X}_1) \,, \tag{2.26}$$

where $f(X_1,\dot{X}_1) = [-2(b_1+b_2)\dot{X}_1 + (\dot{X}_1^2 - k^2X_1^2)/q_1 + \alpha_{21}X_1\dot{X}_1]$

$$+ \varepsilon\{-4b_1b_2X_1 + 2b_1\alpha_{21}X_1^2 - [(2b_1 + k^2\tau)/q_1]X_1\dot{X}_1 + (2b_2\alpha_{21}/k^2)\dot{X}_1^2 - X_1\dot{X}_1^2/q_1^2\} \,,$$

and $k^2 = q_1q_2\alpha_{12}\alpha_{21}$ .

Applying the K-B-M method, we observe that the first order and the first improved approximations coincide with the results obtained in Chapter 2 §1, i.e.,

$$A_1(a) = -(b_1+b_2)a \,,$$

$$B_1(a) = 0 \,, \tag{2.27}$$

$$u_1(a,\Psi) = \frac{1}{3}a^2\{(1/q_1)\cos 2\Psi + \frac{1}{2}(\alpha_{21}/k)\sin 2\Psi\} \,.$$

Dependence of the phase on the amplitude may be obtained by deriving the second approximation. From expressions (2.26) and (2.27) we get

$$f_1(a,\Psi) = a^2(b_1+b_2)\alpha_{21}$$

$$+ \{-4b_1b_2a + a^3\alpha_{21}^2/12 + a^3k^2/12q_1^2 + a(b_1+b_2)^2\}\cos\Psi$$

$$+ (0)\sin\Psi + \{(2b_1 + k^2\tau)a^2k/2q_1\}\sin 2\Psi + \ldots$$

One can easily compare with the results obtained in §1 and see that $A_2(a)$ and $B_2(a)$ are also the same, with

$$A_2(a) = 0 \,,$$

$$B_2(a) = -(1/2k)\{(b_1-b_2)^2 + a^2(\alpha_{21}^2 + k^2/q_1^2)/12\} \,.$$

Hence, up to the second order approximation, the small time lag terms occurring in the linear parts have no effect on the small vibrations in the vicinity of the equilibrium position. Difference may be anticipated in higher order solutions but it is of no interest to go any further from here.

## § 4 THE W-C MODEL WITH SMALL TIME LAG

In this section we shall consider a special case of the non-linear Wangersky and Cunningham (W-C) [11] model in which the growth rate of the predator species may depend on the interaction between the populations of the two species at a small time prior to a given moment. This model is governed by the following system of equations:

$$\frac{dN_1}{dt} = \alpha_1 N_1(t)[1 - N_1(t)/\Theta] - \beta_1 N_1(t) N_2(t) ,$$

$$\frac{dN_2}{dt} = -\alpha_2 N_2(t) + \beta_2 N_1(t-\Delta) N_2(t-\Delta) , \tag{2.28}$$

where $\alpha_i, \beta_i > 0$ and $\Delta$ is the small time lag, $\Theta$ is the carrying capacity for $N_1$ .

As in previous cases, with $\varepsilon = 0$ and setting $\frac{dN_{1,2}}{dt} = 0$ , the non-zero equilibrium populations then obtained are

$$q_1 = \alpha_2/\beta_2 , \quad q_2 = \alpha_1(1-\alpha_2/\beta_2\Theta)/\beta_1 .$$

By using the transformations

$$N_i(t) = q_1 + \varepsilon X_i(t), \qquad i = 1,2 ,$$

in the vicinity of the equilibrium position $(q_1,q_2)$, and the expansions for $N_i(t-\Delta)$ , with $\Delta = \varepsilon\tau$ ,

$$N_i(t-\varepsilon\tau) = q_i + \varepsilon X_i - \varepsilon^2 \tau \dot{X}_i + \varepsilon^3 (\tau^2/2) \ddot{X}_i - \ldots$$

System (2.28) then becomes

$$\frac{dX_1}{dt} = -\beta_1 q_1 X_2 + \varepsilon(-2bX_1 - \beta_1 X_1 X_2) + \varepsilon^2 \ldots ,$$

$$\frac{dX_2}{dt} = \beta_2 q_2 X_1 + \varepsilon\beta_2 (X_1 X_2 - \tau q_2 \dot{X}_1 - \tau q_1 \dot{X}_2) + \varepsilon^2 \ldots , \tag{2.29}$$

where $\alpha_1 q_1/\Theta = \varepsilon 2b$ is a small quantity.

The linear system of (2.29), with $\varepsilon = 0$, can be reduced as

$$\ddot{X}_i + k^2 X_i = 0 , \tag{2.30}$$

for $i = 1,2$ ; where $k^2 = \alpha_1\alpha_2$ .

For the nonlinear system, with $\varepsilon \neq 0$, after some manipulations of the two equations in (2.29), we obtain

$$\ddot{X}_1 + k^2 X_1 = \varepsilon\{\alpha_2 k^2 \tau X_1 - (2b - k^2\tau)\dot{X}_1 + \beta_2 X_1 \dot{X}_1 - (k^2 X_1^2 - \dot{X}_1^2)/q_1\} + O(\varepsilon^2) , \tag{2.31}$$

where $k^2 = \beta_1\beta_2 q_1 q_2$ . Then expanding the right-hand-side of equation (2.31), by Taylor's formula we get

$$\ddot{X}_1 + k^2 X_1 = \varepsilon f^o(a \cos \Psi, -ak \sin \Psi) + O(\varepsilon^2)$$

where $f^o(a \cos \Psi, -ak \sin \Psi) = (2b - k^2\tau) ak \sin \Psi + \alpha_2 k^2 \tau a \cos \Psi$

$$- \tfrac{1}{2}\beta_2 a^2 k \sin 2\Psi + (a^2 k^2/q_1) \cos 2\Psi .$$

Hence the non-zero Fourier coefficients for $f^o(a \cos \Psi, -ak \sin \Psi)$ are

$$g_o = 0 ,$$

$$g_1 = \alpha_2 k^2 \tau a , \qquad h_1 = (2b - k^2\tau) ak ,$$

$$g_2 = a^2 k^2/q_1 , \qquad h_2 = - \tfrac{1}{2}\alpha_2 a^2 k .$$

Thus we have, according to the K-B-M method,

$$A_1 = - \frac{h_1}{2k} = -a(b - \tfrac{1}{2}k^2\tau) ,$$

$$B_1 = \frac{g_1}{2ak} = -\tfrac{1}{2}\alpha_2 k\tau .$$

Since $\dot{a} = \varepsilon A_1$ ,

$$a(t) = a_o e^{-\varepsilon(b-\frac{1}{2}k^2\tau)t} = a_o e^{-\frac{1}{2}(\alpha_1 q_1/\Theta - k^2\Delta)t} .$$

Also, $\dot{\Psi} = k + \varepsilon B_1$ , and

$$\Psi(t) = kt - \varepsilon\tfrac{1}{2}\alpha_2 k\tau t + \Psi_o = kt(1 - \tfrac{1}{2}\alpha_2\Delta) + \Psi_o ,$$

where $a_o$ and $\Psi_o$ are constants of integration.

With

$$U_1(a,\Psi) = g_o/k^2 - (1/3k^2)\{g_2 \cos 2\Psi + h_2 \sin 2\Psi\} ,$$

we thus get

$$X_1(t) = a \cos \Psi + \varepsilon(a^2/3)\{(1/q_1)\cos 2\Psi + (\beta_2/2k)\sin 2\Psi\}. \quad (2.32)$$

Now $X_1(t)$ is determined as a first improved approximation. For equation (2.29) with $X_1(t)$ known, $X_2(t)$ can be solved and is given by

$$X_2(t) = (a/\beta_1)\{(k/q_1)\sin \Psi - (\alpha_1/2\Theta)\cos \Psi\} - \tfrac{1}{2}a\Delta\beta_2\{(k/\beta_1)\sin \Psi + q_2\cos \Psi\}$$
$$+ \varepsilon(a^2/3\beta_1 q_1)\{(k/2q_1)\sin 2\Psi + \beta_2 \cos 2\Psi\} .$$

The linear frequency remains the same as in the case without time lag. In the first order correction terms, there are contributions from the time lag for both the amplitude and frequency. The amplitude is damped exponentially with a small coefficient of decay or growth which takes the form $G = -\varepsilon(b - \frac{1}{2}k^2\tau) = -\frac{1}{2}(\alpha_1 q_1/\Theta - k^2\Delta)$ . If $\alpha_1 q_1/\Theta > k^2\Delta$ , then $G < 0$ and the amplitude is decaying; if $\alpha_1 q_1/\Theta < k^2\Delta$ , then $G > 0$ and $a$ is increasing. The coefficient $G$ is zero when $\alpha_1 q_1/\Theta = k^2$ , which implies that the fluctuations $X_1(t)$ and $X_2(t)$ are periodic. As a particular case when there is no saturation level $(\Theta \to \infty)$ and no delay $(\Delta = 0)$ , $G$ is zero. The result degenerates to that for the L-V model. The introduction of a saturation level term causes some sort of decaying damping effect. But the introduction of a small time lag tends to diminish the damping effect, and to cause growth of the amplitude. In this case, $B_1$ is different from zero, and a small correction $-\frac{1}{2}\alpha_2 k\Delta$ to the frequency is determined. The phase $\Psi$, in the first approximation, is independent of the amplitude but depends on the time lag. Due to this dependence, the period of the harmonic terms in the solution for $X_1(t)$, expression (2.32), is influenced. For instance, the period of $\cos\Psi$ is $\frac{2\pi}{k}(1 + \frac{1}{2}\alpha_2\Delta)$, up to the order of $\varepsilon$. Hence the introduction of a small time lag tends to increase the period, $\frac{2\pi}{k}$, of the solution of the linear system (2.30).

The first order correction terms to $a\cos\Psi$ in $X_1(t)$ involve the harmonics $\cos 2\Psi$ and $\sin 2\Psi$ with small amplitudes of the type $\varepsilon a^2 C_1$, where $C_1$ is a constant depending on the given parameters of the system (2.28). Correction terms of the same nature appear in the expression for $X_2(t)$ as well. In addition there are correction terms, due to the presence of time lag, involving $\cos\Psi$ and $\sin\Psi$ with small amplitudes of the form $\Delta a C_2$. Also there is a small correction term involving $\cos\Psi$ with the amplitude $a$ in the form $(a/\Theta)C_3$. Here $C_2$ and $C_3$ depend on the given parameters too.

CHAPTER 3

## § 1 A SPECIAL W-C MODEL WITH SIGNIFICANT TIME LAG

A special W-C model is investigated which involves a deviating argument for the predator species only in the nonlinear part of the second equation. This will account for effects on the growth rate of the predator due to its population size at some time prior to a given moment. The model is described by the following system of differential equations:

$$\frac{dN_1}{dt} = \alpha_1 N_1(t)\ 1 - N_1(t)/\Theta \ - \beta_1 N_1(t) N_2(t) \ ,$$

$$\frac{dN_2}{dt} = -\alpha_2 N_2(t) + \beta_2 N_1(t) N_2(t-\Delta) \ , \qquad (3.1)$$

where $\Theta$ is the saturation level for the species $N_1$ , and $\alpha_i$'s, $\beta_i$'s > 0, for $i = 1,2$. Here the significant time lag $\Delta$ is not small and has the same order as the parameters $\alpha_i$'s and $\beta_i$'s . Even the presence of significant time lag for $N_2$ only in the nonlinear part of the second equation introduces a lot of complications to the problem.

The equilibrium position $(q_1,q_2)$ is also given by

$$q_1 = \alpha_2/\beta_2 \ , \qquad q_2 = (\alpha_1/\beta_1)(1 - \alpha_2/\beta_2\Theta) \ .$$

Again we use the substitutions

$$N_i(t) = q_i + \varepsilon X_i(t) \ , \qquad i = 1,2.$$

Then system (3.1) becomes

$$\frac{dX_1}{dt} = -\beta_1 q_1 X_2 - \varepsilon(2bX_1 + \beta_1 X_1 X_2),$$

$$\frac{dX_2}{dt} = -\alpha_2 X_2 + \beta_2 q_2 X_1 + \alpha_2 X_2(t-\Delta) + \varepsilon\beta_2 X_1 X_2(t-\Delta), \tag{3.2}$$

where $\varepsilon 2b = \alpha_1 q_1/\Theta$. Here we cannot expand $X_2(t-\Delta)$ in Taylor series since $\Delta$ is not small. System (3.2) becomes, after eliminating $X_2$ ,

$$\ddot{X}_1 + \alpha_2 \dot{X}_1 - \alpha_2 \dot{X}_1(t-\Delta) + k^2 X_1$$

$$= \varepsilon\{-2bX_1 + \dot{X}_1^2/q_1 - 2\alpha_2 bX_1 + 2\alpha_2 bX_1(t-\Delta) - k^2 X_1^2/q_1 + 2\beta_2 X_1 \dot{X}_1(t-\Delta)$$

$$- \beta_2 X_1(t-\Delta)\dot{X}_1(t-\Delta)\}, \tag{3.3}$$

where $k^2 = \beta_1 \beta_2 q_1 q_2$.

For $\varepsilon = 0$, we obtain the generating equation of (3.3)

$$\ddot{X}_1 + \alpha_2[\dot{X}_1 - \dot{X}_1(t-\Delta)] + k^2 X_1 = 0, \tag{3.4}$$

which possesses solutions of the form $Ce^{Zt}$ , where C is an arbitrary constant and Z is a root of the characteristic equation

$$\Omega(Z) = Z^2 + \alpha_2 Z + k^2 - \alpha_2 Ze^{-Z\Delta} = 0. \tag{3.5}$$

The above expression for $\Omega(Z)$ is called the quasipolynomial of equation (3.4). If $\Delta$ is different from zero and $2n\pi/k$, for $n = 1, 2, \ldots$, equation (3.5) has an infinite number of roots in the complex plane. Let $Z = -\xi + i\omega$ denote a particular root of equation (3.5). Then the corresponding real solution of equation (3.4) to this root takes the form

$$x_I^{(o)}(t) = \mathrm{Re}[Ce^{(-\xi+i\omega)t}] = a_o e^{-\xi t} \cos(\omega t+\omega_o), \tag{3.6}$$

where $a_o$ and $\omega_o$ are real constants and Re[Z] is used to denote the real part of Z. We are usually interested in the decaying solution of equation (3.4), i.e., $\xi > 0$.

In most cases the investigation of the stability of the null solution of equation (3.3) is similar to the investigation of the stability of the null solution of the linear equation (3.4). It is known that (from Bellman and Cook [12], or El'sgol'ts [3]) the null solution of the nonlinear equation (3.3) is asymptotically stable if all the roots of the characteristic equation (3.5) have negative real parts. Referring to the book by Bellman and Cook (Theorem 13.10), one can see that the conditions stated in this theorem are fulfilled by the coefficients of equation (3.5). The necessary and sufficient condition such that all the roots of $\Omega(Z) = 0$ will lie to the left of the imaginary axis is that

$$1 - \cos \sigma_r > 0, \tag{3.7}$$

where $\sigma_n$ $(n \geq 0)$ is the sole root of the equation

$$\tan \sigma = (k^2\Delta^2 - \sigma^2)/\alpha_2\Delta\sigma,$$

which lies on the interval $(n\pi-\frac{1}{2}\pi,\ n\pi+\frac{1}{2}\pi)$. The number r is defined as, since $-\alpha_2\Delta < 0$, the even n for which $\sigma_n$ lies closest to $k\Delta$. Generally, condition (3.7) can be satisfied except when $\sigma_r$ takes on the value of $2n\pi$, for n = 1, 2, ... The case when $\sigma_r = 2n\pi$ would imply that $\Delta = 2n\pi/k$ and this has been excluded as mentioned earlier. Otherwise, purely imaginary roots would result instead. Thus all the roots of equation (3.5) have negative real parts. This would mean that the null solution $X_{1,0} = 0$ of the nonlinear equation (3.3) is asymptotically stable. Similarly, eliminating $X_1$ from the system (3.2) will lead to a nonlinear equation in terms of $X_2$ of the type (3.3) with the same linear part and hence the same characteristic equation (3.5) will be obtained. Therefore the null solution $X_{2,0} = 0$ is also asymptotically stable. Taking into account the substitutions, $N_i(t) = q_i + \varepsilon X_i(t)$ , for $i = 1,2$ , it follows that the populations $N_1$ and $N_2$ will be asymptotically stable in the vicinity of the equilibrium position $(q_1,q_2)$ .

## § 2 ASYMPTOTIC SOLUTION

We attempt now to find an asymptotic solution for equation (3.3) Some nonlinear equations similar to (3.3) have been studied previously by Minorsky (1962), Rubinik (1969), and Mitropolskii and Martinyuk (1969). However, all the models considered involve nonlinear equations either with small time lag, or with significant time lag taking part only as small nonlinear terms. Besides, the characteristic equations corresponding to the models investigated involve purely imaginary roots. The solution associated iwth such an imaginary root describes an almost sinusoidal oscillation. As in equation (3.4), significant time lag is involved in the linear part, and the corresponding characteristic equation will have an infinite number of roots depending on the time lag $\Delta$ . Models exhibiting this feature are considered in the paper by Bojadziev and Chan [13]. In this case, imaginary roots are possible only for particular values of the coefficients of the linear equation. Otherwise, roots of the characteristic equation will all be complex. Hence the solution (3.6) corresponding to a particular root $Z = -\xi + i\omega$ are both damped and oscillatory. It is this feature that prevents the straight forward application of the K-B-M method from working. An extension and modification of the K-B-M method for damped oscillations encountering significant time lag as given in Chapter 1 is used instead.

A solution of the nonlinear equation (3.3), which tends to the solution (3.6) of equation (3.4) as $\varepsilon \to 0$ , is sought in the form

$$X_i(t) = \text{Re}[e^{-\xi\alpha+i\Psi}] + \varepsilon X_i^{(1)}(\alpha,\Psi) + \varepsilon^2 \ldots, \qquad i = 1,2, \tag{3.7}$$

where $\alpha$ and $\Psi$ are determined by the following equations

$$\frac{d\alpha}{dt} = 1 + \varepsilon P(\alpha) + \varepsilon^2 \ldots,$$

$$\frac{d\Psi}{dt} = \omega + \varepsilon Q(\alpha) + \varepsilon^2 \ldots \tag{3.8}$$

Solutions of the type (3.7) are not the most general, but are chosen because usually the real part $-\xi$, of a particular root of equation (3.5), is appreciably smaller than the real parts of all the other roots. For this reason, in the solution of an arbitrary initial value problem, all the other modes will die out relatively quickly, and only the one mode with the smallest $-\xi$ will persist for any length of time.

By substituting the corresponding expressions for $\ddot{X}_1(t)$, $\dot{X}_1(t)$, $\dot{X}_1(t-\Delta)$, into equation (3.3) and comparing those terms of different orders in $\varepsilon$, the zero order terms cancel identically and for the terms of $\varepsilon$ order, we get the following equation for $X_1^{(1)}$,

$$\frac{\partial^2 X_1^{(1)}}{\partial\alpha^2} + 2\omega\frac{\partial^2 X_1^{(1)}}{\partial\alpha\partial\Psi} + \omega^2\frac{\partial^2 X_1^{(1)}}{\partial\Psi^2} + \alpha_2\left\{\frac{\partial X_1^{(1)}}{\partial\alpha} + \omega\frac{\partial X_1^{(1)}}{\partial\Psi}\right\}$$

$$- \alpha_2\left\{\frac{\partial X_1^{(1)}}{\partial\alpha}(\alpha-\Delta,\Psi-\omega\Delta) + \omega\frac{\partial X_1^{(1)}}{\partial\Psi}(\alpha-\Delta,\Psi-\omega\Delta)\right\} + k^2 X_1^{(1)}$$

$$+ \text{Re}[e^{-\xi\alpha+i\Psi}\{R'(\alpha) + (2Z + \alpha_2)R(\alpha) - \alpha_2 R(\alpha-\Delta)e^{-Z\Delta} + \alpha_2 Z R^{\Delta}(\alpha)e^{-Z\Delta}\}]$$

$$= F^{(1)}(\alpha,\Psi), \tag{3.9}$$

where

$$F^{(1)}(\alpha,\Psi) = -2b\mathrm{Re}[e^{-\xi\alpha+i\Psi}Z] + \{\mathrm{Re}[e^{-\xi\alpha+i\Psi}Z]\}^2/q_1$$

$$- 2\alpha_2 b\mathrm{Re}[e^{-\xi\alpha+i\Psi}] + 2\alpha_2 b\mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}]$$

$$- \beta_2\mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}]\mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}Z]$$

$$+ 2\beta_2\mathrm{Re}[e^{-\xi\alpha+i\Psi}]\mathrm{Re}[e^{-\xi\alpha+i\Psi}e^{-Z\Delta}Z]$$

$$- (k^2/q_1)\{\mathrm{Re}[e^{-\xi\alpha+i\Psi}]\}^2 \ . \qquad (3.10)$$

Also, $R' = \frac{dR}{d\alpha}$ , $R(\alpha) = -\xi P(\alpha) + iQ(\alpha)$ and $R^\Delta = -\xi P^\Delta + iQ^\Delta$ , together with

$$P^\Delta = \int_{\alpha-\Delta}^{\alpha} P(\tau)\ d\tau, \qquad \text{and} \qquad Q^\Delta = \int_{\alpha-\Delta}^{\alpha} Q(\tau)\ d\tau.$$

We shall seek a solution of equation (3.9) in Fourier series form

$$X_1^{(1)}(\alpha,\Psi) = \sum_{m=-\infty}^{m=+\infty} X_{1m}(\alpha)e^{im\Psi}, \qquad (3.11)$$

and also expand $F^{(1)}(\alpha,\Psi)$ in Fourier series

$$F^{(1)}(\alpha,\Psi) = \sum_{m=-\infty}^{m=+\infty} F_m(\alpha)e^{im\Psi}.$$

From the expression (3.10), it follows that the only nonzero coefficients $F_m(\alpha)$ arise when m = 0, ±1, ±2, and they are given by

$$F_0(\alpha) = e^{-2\xi\alpha}\{\frac{\xi^2+\omega^2}{2q_1} + e^{2\xi\Delta}[\tfrac{1}{2}\beta_2\xi - (k^2/2q_1 + \beta_2\xi)\cos\omega\Delta + \beta_2\omega\sin\omega\Delta]\},$$

$$F_1(\alpha) = -be^{-\xi\alpha}\{\alpha_2 - \xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta + i(\omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta)\},$$

$$F_2(\alpha) = e^{-2\xi\alpha}(\frac{\omega^2-\xi^2}{4q_1} - \beta_2 e^{2\xi\Delta}(\omega\sin 2\omega\Delta - \xi\cos 2\omega\Delta)$$

$$+ \tfrac{1}{2}e^{\xi\Delta}[-(k^2/q_1 + \beta_2\xi)\cos\omega\Delta + \beta_2 \sin\omega\Delta]$$

$$+ i\{\xi\omega/2q_1 - \beta_2 e^{2\xi\Delta}(\omega\cos 2\omega\Delta + \xi\sin 2\omega\Delta)$$

$$+ \tfrac{1}{2}e^{\xi\Delta}[\beta_2 \cos\omega\Delta + (k^2/2q_1 + \beta_2\xi)\sin\omega\Delta]\}) \ ,$$

and $F_{(-m)}(\alpha) = [F_m(\alpha)]^*$ , for $m = 1,2$; where $[Z]^*$ represents the complex conjugate of $Z$ .

By substituting the series (3.11) into equation (3.8) and comparing the coefficients of $e^{im\Psi}$ for $m \neq \pm 1$ , we have

$$X''_{1m}(\alpha) + (\alpha_2 + i2m\omega)X'_{1m}(\alpha) + (i\alpha_2 m - m^2\omega^2)X_{1m}(\alpha) - \alpha_2 X'_{1m}(\alpha-\Delta)e^{-im\omega\Delta}$$

$$- i\alpha_2 m\omega X_{1m}(\alpha-\Delta)e^{-im\omega\Delta} + k^2 X_{1m}(\alpha) = F_m(\alpha) \ ; \qquad (3.12)$$

where $X'_{1m} = \frac{dX_{1m}}{d\alpha}$ . From the above equations, the coefficients $X_{1m}(\alpha)$ may readily be determined.

For $m = \pm 1$ , as usual in the K-B-M method we assume that the first harmonics are not present in $X_1^{(1)}(\alpha,\Psi)$ , i.e., $X_{11}(\alpha) = X_{1,-1}(\alpha) = 0$. We obtain

$$R' + (2Z + \alpha_2)R - \alpha_2 R(\alpha-\Delta)e^{-Z\Delta} + \alpha_2 Z \int_{\alpha-\Delta}^{\alpha} R(\tau)\, d\tau \ e^{-Z\Delta}$$

$$= 2e^{\xi\alpha}F_1(\alpha)$$

$$= -2b\{\alpha_2 - \xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta + i(\omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta)\}. \qquad (3.13)$$

Now we will seek the solution, $R(\alpha)$, of equation (3.13) with the form

$$R(\alpha) = W_1\alpha + W_2 , \tag{3.14}$$

where $W_1$ , $W_2$ are complex constants.

By substituting expression (3.14) into equation (3.13) and comparing the coefficients of $\alpha$ and the free term, we get

$$W_1[2Z + \alpha_2(1 - e^{-Z\Delta} + Z\Delta e^{-Z\Delta})] = 0,$$

$$W_2[2Z + \alpha_2(1 - e^{-Z\Delta} + Z\Delta e^{-Z\Delta})] + W_1[1 + \alpha_2\Delta(e^{-Z\Delta} - \tfrac{1}{2}Z\Delta e^{-Z\Delta})]$$

$$= -2b[\alpha_2 - \xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta + i(\omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta)].$$

Solving the above system gives

$$W_1 = 0,$$

$$W_2 = -2b[\alpha_2 - \xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta + i(\omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta)]/\Omega'(Z) ,$$

where $\Omega'(Z) = \dfrac{d\Omega}{dZ} = 2Z + \alpha_2(1 - e^{-Z\Delta} + Z\Delta e^{-Z\Delta})$. It is assumed that $Z = -\xi + i\omega$ is a simple root of $\Omega(Z) = 0$, i.e., $\Omega'(Z) \neq 0$. Then for convenience, we let

$$W_2 = -\xi w_1 + i w_2 \equiv \frac{-2b(G_1 + iG_2)}{H_1 + iH_2} , \tag{3.15}$$

where

$$G_1 = \alpha_2 - \xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta ,$$

$$G_2 = \omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta ,$$

$$H_1 = \alpha_2 - 2\xi - \alpha_2 e^{\xi\Delta}\cos\omega\Delta - \alpha_2 \Delta e^{\xi\Delta}(\xi\cos\omega\Delta - \omega\sin\omega\Delta),$$

$$H_2 = 2\omega + \alpha_2 e^{\xi\Delta}\sin\omega\Delta + \alpha_2 \Delta e^{\xi\Delta}(\omega\cos\omega\Delta + \xi\sin\omega\Delta).$$

Solving equation (3.15) and comparing the real and imaginary parts, we have

$$\begin{aligned} -\xi w_1 H_1 - w_2 H_2 &= -2bG_1 , \\ -\xi w_1 H_2 + w_2 H_1 &= -2bG_2 . \end{aligned} \tag{3.16}$$

Since $W_1 = 0$, $R(\alpha) = -\xi P(\alpha) + iQ(\alpha)$, and $W_2 = -\xi w_1 + iw_2$, equations (3.14) and (3.16) imply

$$\begin{aligned} P(\alpha) &= w_1 = \frac{2b(G_1H_1 + G_2H_2)}{\xi(H_1^2 + H_2^2)} , \\ Q(\alpha) &= w_2 = \frac{2b(H_2G_1 - H_1G_2)}{(H_1^2 + H_2^2)} . \end{aligned} \tag{3.17}$$

Therefore equation (3.8) with expressions (3.17) takes the form

$$\begin{aligned} \frac{d\alpha}{dt} &= 1 + \varepsilon w_1 , \\ \frac{d\Psi}{dt} &= \omega + \varepsilon w_2 , \end{aligned} \tag{3.18}$$

which have solutions of the following form

$$e^{-\xi\alpha} = a_o e^{-\xi(1+\varepsilon w_1)t} , \tag{3.19}$$

$$\Psi(t) = (\omega + \varepsilon w_2)t + \Psi_o . \tag{3.20}$$

Thus the first approximation is then

$$X_1(t) = a \cos \Psi , \tag{3.21}$$

where the amplitude $a = e^{-\xi\alpha}$ is given by equation (3.19) and the phase $\Psi$ by (3.20). If, in addition, equation (3.12) is solved for $X_{1m}$, then $X_1^{(1)}(\alpha,\Psi)$ can be determined making use of the series (3.11) and we obtain the so called first improved approximation

$$X_1(t) = e^{-\xi\alpha}\cos \Psi + \varepsilon X_1^{(1)}(\alpha,\Psi) ,$$

which represents improvement over the first approximation only over time intervals of length O(1), when it is accurate to within an error of order $\varepsilon^2$.

## § 3 DISCUSSION

In this section, we will confine our attention to the first approximate solution (3.21) which has the general nature of an oscillatory part $\cos \Psi$ multiplied by a decaying amplitude $a$, since $\xi > 0$. The decay is of exponential type with decay constant $\xi$, but its form is modified by an additional term $\varepsilon w_1$. Also, there is a small correction term $\varepsilon w_2$ to the linear frequency $\omega$. Both correction terms involve the time lag $\Delta$. To investigate the vibration determined by expression (3.21), we can introduce a logarithmic decrement $\delta_1 = \ln(\frac{a_j}{a_{j+1}})$ to measure the decay of the amplitude. Here $a_j$ and $a_{j+1}$ represent the amplitudes $a = e^{-\xi\alpha}$, given by equation (3.19), for the j-th and (j+1)-th cycles, for times $t_j$ and $t_j+T$, where $T$ is the quasi-period. Then we get

$$\delta_1 = \xi(1 + \varepsilon w_1)T . \tag{3.22}$$

The quasi-period $T$ defined by $\Psi(t_j+T) - \Psi(t_j) = 2\pi$ can be found from equation (3.20) and, as a first approximation, is given by

$$T = \frac{2\pi}{\omega}(1 - \varepsilon w_2/\omega) . \tag{3.23}$$

For $\varepsilon = 0$, the linear case, from expressions (3.22) and (3.23) we get the known results $\delta_1^{(o)} = 2\pi\xi/\omega$ and $T^{(o)} = 2\pi/\omega$.

When the time lag $\Delta$ takes on some special values, $2n\pi/k$, for $n = 0, 1, 2, \ldots$, the characteristic equation (3.5) would give purely imaginary roots $\pm ik$. This corresponds to the case $\xi = 0$ and $\omega = k$.

Then the solution for the nonlinear equation (3.3) is sought according to the standard K-B-M method and we obtain

$$A_1 = -ab/(1 + \tfrac{1}{2}k\Delta) ,$$

$$B_1 = 0 .$$

Thus the first approximation for this special case is

$$X_1(t) = a_o e^{-\varepsilon bt/(1+\frac{1}{2}k\Delta)} \cos(kt + \Psi_o) ,$$

where $\Delta = 2n\pi/k$ , $a_o$ and $\Psi_o$ are constants of integration.

Clearly, the vibrating process is slowly decaying due to a decreasing exponential amplitude. Since the magnitude $|A_1|$ is largest when $\Delta = 0$, the time lag $\Delta$ acts in a way to diminish the damping effect produced due to the presence of a saturation level. The phase stays the same as in the linear case. As $n$ increases, the change in the amplitude decreases. This means that if $n \to \infty$ , the amplitude $a \to a_o$ and the decaying process approaches a periodic one. Hence the time lag $\Delta$ has a destabilizing effect. But this latter case is rather restrictive since $\Delta$ can be taken only as certain values; therefore it is not that interesting for us to pursue any further.

For applied problems modeled by ordinary differential equations, the initial conditions are given at a point $t = t_o$ . But the initial conditions for delay differential equations will be given on an interval $[t_o-\Delta, t_o]$ . Hence the initial value problem for the differential equation (3.3) with time lag can be stated as follows: find a function $X(t)$ , which for $t > t_o$ satisfies equation (3.3) and the conditions

$$X(t) = p(t) \ ,$$

$$\frac{dX(t)}{dt} = \frac{dp(t)}{dt} \ , \qquad t \in [t_o - \Delta,\ t_o] \ .$$

# CONCLUSION

The aim of this thesis is to study the effect of time lag on different modified models of the V-G-W and the W-C types in population dynamics. As a final discussion of the results, we confine our consideration to the first approximate solutions only of those models studied: (i) No time lag, (ii) Small time lag, $\Delta = \varepsilon\tau$, in the nonlinear part, (iii) Small time lag in the linear part, (iv) W-C model with small time lag, (v) Special W-C model with significant time lag. The first order approximations for the above models are all designated by $X(t) = a\cos\Psi$, where $a$ and $\Psi$ are given respectively as follows:

$$a = a_o e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)t}, \qquad \Psi = kt + \Psi_o; \tag{1}$$

$$a = a_o e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2 - 2k^2\Delta)t}, \qquad \Psi = kt + \Psi_o; \tag{2}$$

$$a = a_o e^{-\frac{1}{2}(\alpha_{11}q_1/\Theta_1 + \alpha_{22}q_2/\Theta_2)t}, \qquad \Psi = kt + \Psi_o; \tag{3}$$

$$a = a_o e^{-\frac{1}{2}(\alpha_1 q_1/\Theta - k^2\Delta)t}, \qquad \Psi = kt(1 - \tfrac{1}{2}\alpha_2\Delta) + \Psi_o; \tag{4}$$

$$a = a_o e^{-(\xi + C_1\alpha_1 q_1/\Theta)t} ,$$

$$\Psi = (\omega + C_2\alpha_1 q_1/\Theta)t + \Psi_o . \qquad (5)$$

In the first three models, one can see that the small positive terms $\alpha_{ii}q_i/2\Theta_i = \varepsilon b_i$ in solutions (1), (2) and (3), due to the saturation levels, cause a small damping effect and make the amplitude $a$ vary slowly as the time $t$ increases. The small time lag $\Delta$ in the second model produces a term $k^2\Delta$ which tends to diminish the damping effect, i.e., has a destabilizing influence on the amplitude $a$, while in the third model, the small time lag occurring in the linear part of the system generates no effect on the vibrations of the population as far as the first approximation. This is true even up to the second order approximation. The phase in all three cases involves no contribution from the time lag $\Delta$. Clearly, when $\Delta = 0$, all three solutions coincide. Hence one sees that the introduction of a small deviating argument will result in a small change of the amplitude only.

For model (iv), the small time lag produces the same effect in the amplitude as in the model (ii) and at the same time contributes a correction term to the phase $\Psi$ as well. The frequency is then increased as a consequence.

However, for the case of significant time lag. correction factors due to $\Delta$ occur as product terms with the small quantity $\varepsilon b$, in the form of $C_1\alpha_1 q_1/\Theta$ and $C_2\alpha_1 q_1/\Theta$, for the amplitude and the phase respectively. $C_1$ and $C_2$ are some expressions involving $\Delta$. The amplitude $a$ is strongly decaying due to the decaying coefficient $-\xi$ which dominates the correction term $C_1\alpha_1q_1/\Theta$. Since $\xi$ depends on $\Delta$, the significant time lag produces a strong stabilizing influence on the nonlinear vibrating system. Also, the phase $\Psi$ is slightly affected due to the time lag $\Delta$. Note that in the special case when $\Delta = 2n\pi/k$, for n = 1, 2, ..., this corresponds to $\xi = 0$, and the amplitude $a \to a_o$ as $n \to \infty$. The significant time lag will then cause a destabilizing effect instead.

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