On Montgomery's Pair Correlation Conjecture to the Zeros
of the Riemann Zeta Function

by

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A Thesis submitted in partial fulfillment
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Abstract

In this thesis, we are interested in Montgomery's pair correlation conjecture which is about the distribution of the spacings between consecutive zeros of the Riemann Zeta function. Our goal is to explain and study Montgomery's pair correlation conjecture and discuss its connection with the random matrix theory.

In Chapter One, we will explain how to define the Riemann Zeta function by using the analytic continuation. After this, several classical properties of the Riemann Zeta function will be discussed.

In Chapter Two, We will explain the proof of Montgomery's main result and discuss the pair correlation conjecture in detail. The main result about the pair correlation functions of the eigenvalues of random matrices will also be proved. These two pair correlation functions turn out amazingly to be the same. Thus the full importance of Montgomery's conjecture is established.
Dedication

For my wife Xiaojuan, who offered me unconditional love and support throughout the course of this thesis.
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Chapter 1

Introduction of the Riemann Zeta function

1.1 Introduction

Georg Friedrich Bernhard Riemann (1826-1866), the great German mathematician, a student of Gauss, in his path-breaking also the only paper on number theory, "On the Number of Primes Less Than a Given Magnitude", published in 1860, derived and proved a number of very important results in the field of analytic number theory, and also proposed several remarkable conjectures which generation after generation of mathematicians were obsessed. Of the conjectures he posed, many of them had been solved, but there is one conjecture which has withstood attack on from many first-class mathematicians for more than one hundred years and maybe will keep attracting mathematicians' attention for another one hundred years. It is known as "The Riemann Hypothesis (RH)", which is now one of the most famous unsolved problem in mathematics.

The Riemann Hypothesis is about the locations of zeros of a very important complex function, known as Riemann Zeta function, denoted by $\zeta(s)$. Riemann conjectured in his paper that "it is very likely" all the "non-trivial" zeros of $\zeta(s)$ lie on the vertical line $\Re(s) = 1/2$ in the complex plane.

In Chapter One, we go through the methods which Riemann used to define his Riemann Zeta function. That is, we first define $\zeta(s)$ on the half plane where $\Re(s) > 1$ by a convergent Dirichlet series and then extend $\zeta(s)$ to the whole complex plane by analytic continuation. After the global definition of $\zeta(s)$ is derived, some of the main properties of $\zeta(s)$ are discussed, such as the functional equation. Then we focus on the zeros of $\zeta(s)$. That includes the locations of zeros, the zero-free region, the density of zeros and the empirical methods to compute the zeros of $\zeta(s)$. All these topics are very important to the research of $\zeta(s)$ and the Riemann Hypothesis.
CHAPTER 1. INTRODUCTION OF THE RIEMANN ZETA FUNCTION

In Chapter Two, we will study the results in Hugh Montgomery's famous paper on his celebrated pair correlation conjecture. We will explain the proof of the main result of Montgomery and discuss the pair correlation conjecture in more detail. Next, we will study random matrix theory which is widely studied and used in theoretical physics. We will prove the main result about the pair correlation functions of the eigenvalues of random matrices. The conjectured distribution function of the pair correlation of the zeros of the Riemann Zeta function and the pair correlation of the eigenvalues of random Hermitian matrices turn out amazingly to be the same. This suggests a close relation between the zeros of the Riemann Zeta function and the eigenvalues of random matrices. Their connection will be discussed at the end and thus the full importance of Montgomery's conjecture is established.

1.2 The Definition of the Riemann Zeta Function

Mathematicians are interested in the Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}$$

for many years. At first they only investigate this series for real numbers s. The simplest case is when \(s = 1\), i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$  

When we look at the partial sum of this series, we have:

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) \\
&= 1 + \frac{3}{2} \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) \\
&= 1 + \frac{3}{2} \\
&\vdots \\
S_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^n} \\
&> 1 + \frac{n}{2}.
\end{align*}
\]
CHAPTER 1. INTRODUCTION OF THE RIEMANN ZETA FUNCTION

This series is called the harmonic series and it is divergent because its partial sums are unbounded. In the eighteenth century, some mathematicians began to look at the series for $s = 2$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$  \hspace{1cm} (1.1)

This is a more challenging question. It was first proved that this series is convergent by James Bernoulli in 1689 but he did not know what value the series converged to. James Stirling gave an approximation of it by 16 decimal places, which is 1.6449340668482264. It was Euler in 1735 who correctly obtained the sum of this series, namely

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Riemann opened a new path for the research on this question. He looked much deeper into this problem than his predecessors. His first big step was studying the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} + \cdots,$$  \hspace{1cm} (1.2)

as a function of a complex variable $s = \sigma + it$, $\sigma > 1$. He gave this function the name "zeta", denoted by $\zeta(s)$. When $\sigma > 1$, we can use the integral test to show that this series is convergent. We have

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \int_{1}^{\infty} \frac{1}{x^\sigma} \, dx + 1 = \left[ \frac{x^{1-\sigma}}{1-\sigma} \right]_1^{\infty} + 1 = \frac{1}{\sigma - 1} + 1 < \infty.$$

This shows that the series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely in the region $\sigma > 1$ and uniformly in the region $\sigma \geq \sigma_0 > 1$, for any $\sigma_0 > 1$. Riemann did not stop at the region $\sigma > 1$. Since for $\sigma \leq 1$ the series (1.2) is divergent, and so its series definition is meaningless, Riemann extended $\zeta(s)$ to be defined over all complex values by analytic continuation. In the following sections we will explain in more detail about how he did this.

1.3 The Riemann Zeta Function in the Region $\sigma > 1$ and the Euler Product Formula

The Riemann Zeta function is defined by the Dirichlet series (1.2) for $\sigma > 1$ and is closely related to prime numbers by the Euler product formula which was discovered by Euler, and it is of fundamental
importance for the research on the Riemann Zeta function. We give a proof of the Euler product formula here.

Let \( s = \sigma + it \). The series (1.2) is absolutely convergent in the region \( \sigma > 1 \) and uniformly convergent in the region \( \sigma \geq \sigma_0 > 1 \). Therefore, it defines an analytic function in the region \( \sigma > 1 \).

We define the Riemann Zeta function

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.
\]  

(1.3)

However, the series (1.2) diverges when \( \sigma < 1 \) and the series (1.3) does not define the Riemann Zeta function \( \zeta(s) \) in the region \( \sigma \leq 1 \). The series (1.2) is closely related to the prime factorization of integers. We have the following Euler product formula.

**Theorem 1.1 (Euler product formula).** For \( s = \sigma + it \) and \( \sigma > 1 \), we have

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},
\]

where the product is over all prime numbers \( p \).

**Proof.** Let \( X \geq 2 \). We define the function \( \zeta_X(s) \) by setting

\[
\zeta_X(s) = \prod_{p \leq X} (1 - \frac{1}{p^s})^{-1}.
\]

(1.4)

Each of the factors on the right in (1.4) is the sum of an infinite geometric progression

\[
\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots = \sum_{m=0}^{\infty} \frac{1}{p^{ms}}.
\]

Since \( p^{-\sigma} < 1 \) for \( \sigma > 1 \), each of these progressions is absolutely convergent. Therefore they can be multiplied term-by-term. Thus, the right side of (1.4) is equal to

\[
\prod_{p \leq X} \sum_{m=0}^{\infty} \frac{1}{p^{ms}} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{1}{(p_1^{m_1} \cdots p_j^{m_j})},
\]

(1.5)

where \( 2 = p_1 < \cdots < p_j \), and \( p_1, \ldots, p_j \) are all of the prime numbers up to \( X \). We note that none of the terms on the right in (1.5) is equal to one another, since if we had

\[
p_1^{m_1} \cdots p_j^{m_j} = p_1^{m'_1} \cdots p_j^{m'_j},
\]

then it would follow from the fundamental theorem of arithmetic that \( m_1 = m'_1, \ldots, m_j = m'_j \). Moreover, again using the fundamental theorem of arithmetic, we see that every number \( n < X \) can be written as

\[
n = p_1^{m_1} \cdots p_j^{m_j},
\]

(1.6)
where the $m_1, \ldots, m_j$ are non-negative integers. Hence the representation in (1.6) is unique. Consequently, the right side of (1.5) takes the form

$$\sum_{n \leq X} \frac{1}{n^s} + \sum' \frac{1}{n^s},$$

(1.7)

where $\sum'$ stands for summation over those natural numbers $n > X$ whose prime divisors are all less than $X$. We give an upper bound for this sum

$$\left| \sum' \frac{1}{n^s} \right| \leq \sum_{n > X} \frac{1}{n^s} \leq \sum_{n > X} \frac{1}{n^\sigma} \leq \int_X^\infty \frac{1}{u^\sigma} \, du = \frac{X^{1-\sigma}}{\sigma-1}.$$ 

From (1.4), (1.5), (1.7) and this upper bound we obtain

$$\zeta_X(s) = \prod_{p \leq X} \frac{1}{1 - \frac{1}{p^s}} = \sum_{n \leq X} \frac{1}{n^s} + O\left(\frac{X^{1-\sigma}}{\sigma-1}\right).$$

Taking $X$ to infinity, since $\sigma > 1$, we have

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

as desired. \qed

The Euler product formula sometimes is called the analytical form of the fundamental theorem of arithmetic.

It is easy to prove that $\zeta(s) \neq 0$ when $\Re(s) > 1$ using the Euler product formula. To see this, since $\zeta(s)$ can be defined as the Euler product when $\sigma > 1$, and this infinite product is convergent, and therefore non-zero, the result follows. So we have

**Theorem 1.2.** For $\Re(s) > 1$,

$$\zeta(s) \neq 0.$$ 

### 1.4 Analytic Continuation of $\zeta(s)$ to $\sigma > 0$

Since the series (1.2) does not converge in the region $\sigma \leq 1$, the definition $\zeta(s)$ in this region needs to be an analytic continuation of the series. Let $f(s)$ be an analytic function in a domain $D$. An analytic function $\tilde{f}$ in a domain $\tilde{D}(\supset D)$ is an analytic continuation of $f$ if $f$ and $\tilde{f}$ agree in the domain $D$. In this section, we obtain a simple analytic continuation of $\zeta(s)$ to the region $\sigma > 0$ and in the next section, we will give the analytic continuation to the whole complex plane $\mathbb{C}$. The series $\sum_{n=1}^\infty 1/n^s$ converges when $\sigma > 1$ but diverges when $\sigma \leq 1$. But through analytic continuation
we find a meromorphic function which is defined on the whole complex plane. When restricted to
$\sigma > 1$, it coincide with the original definition $\sum_{n=1}^{\infty} 1/n^s$ and is analytic at any $s \neq 1$ in $C$. This
meromorphic function only has a simple pole at $s = 1$. The derivation of the well-defined Riemann
Zeta function on the whole complex plane is one of the main contributions of Riemann's eight page
epoch-making memoir.

When $\sigma > 1$, we have

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}) = \sum_{n=1}^{\infty} ns \int_{n}^{n+1} x^{-s-1} dx = s \sum_{n=1}^{\infty} \int_{n}^{n+1} [x] x^{-s-1} dx = s \int_{1}^{\infty} [x] x^{-s-1} dx.
$$

Here $[x]$ is defined to be the integral part of the real number $x$ so that $[x] = n$ if $x$ belongs to the
interval $[n, n+1)$ for some $n \in Z$. Also $\{x\} := x - [x]$ is defined to be the fractional part of $x$. So we have

$$
\zeta(s) = s \int_{1}^{\infty} (x - \{x\}) x^{-s-1} dx = s \int_{1}^{\infty} x^{-s} dx - s \int_{1}^{\infty} \{x\} x^{-s-1} dx = \frac{s}{s-1} - s \int_{1}^{\infty} \{x\} x^{-s-1} dx.
$$

Thus

$$
\zeta(s) = 1 + \frac{1}{s-1} - s \int_{1}^{\infty} \{x\} x^{-s-1} dx. \quad (1.8)
$$

Since the improper integral in (1.8) converges absolutely when $\sigma > 0$ because $0 \leq \{x\} < 1$, the right
hand side of (1.8) defines the analytical continuation of $\zeta(s)$ for the region $\sigma > 0$ and the term $\frac{1}{s-1}$
in (1.8) contributes to the simple pole of $\zeta(s)$ at $s = 1$ with residue 1. Now the domain of $\zeta(s)$ has
been extended to $\sigma > 0$. We see that for $0 < \sigma < 1$, $\zeta(s)$ is still defined despite the fact that the
series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges. For example, in view of (1.8), we have

$$
\zeta\left(\frac{1}{2}\right) = 1 + \frac{1}{\frac{1}{2}-1} - \frac{1}{2} \int_{1}^{\infty} \{x\} x^{-\frac{1}{2}-1} dx = -1 - \frac{1}{2} \int_{1}^{\infty} \{x\} x^{-\frac{3}{2}} dx.
$$
Since $0 \leq \{x\} < 1$, it follows that

$$\left| \zeta \left( \frac{1}{2} \right) + 1 + \frac{1}{2} \int_{1}^{M} \{x\} x^{-\frac{3}{2}} dx \right| \leq \frac{1}{2} \int_{M}^{\infty} x^{-\frac{3}{2}} dx = M^{-1/2},$$

where we split the integral into two parts. The main part

$$-1 - \frac{1}{2} \int_{1}^{M} \{x\} x^{-\frac{3}{2}} dx$$

is the approximation of $\zeta(1/2)$ and the remaining part

$$\frac{1}{2} \int_{M}^{\infty} x^{-\frac{3}{2}} dx = -x^{-1/2} \bigg|_{M}^{\infty} = 1/\sqrt{M}$$

is the error term of this approximation. As an example, take $M = 10,000$. We get $\zeta(\frac{1}{2}) = -1.4553557871\ldots$. The error is within 0.01 as the accurate value of $\zeta(1/2)$ is $-1.4603545\ldots$.

### 1.5 Analytic Continuation of $\zeta(s)$ to $\mathbb{C}$ and Functional Equation

In this section, we will give a functional equation for $\zeta(s)$ that defines the analytic continuation of $\zeta(s)$ to the whole complex plane. Let us recall the Gamma function first. For $\sigma > 0$, we define

$$\Gamma(s) = \int_{0}^{\infty} e^{-t} t^{\sigma-1} dt. \quad (1.9)$$

The improper integral in (1.9) converges when $\sigma > 0$ and defines an analytic function in this region. For $\sigma > 0$, we have

$$\Gamma(s + 1) = \int_{0}^{\infty} e^{-t} t^{s} dt = -t^{s} e^{-t} \bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-t} t^{s-1} dt = s \int_{0}^{\infty} e^{-t} t^{s-1} dt = s \Gamma(s). \quad (1.10)$$

Thus we can continue $\Gamma(s)$ analytically to a meromorphic function in $\mathbb{C}$ by the functional equation (1.10). Clearly $\Gamma(1) = \int_{0}^{\infty} e^{-t} dt = 1$ and hence by (1.10), we have $\Gamma(n + 1) = n!$ for any positive integer $n$. 
CHAPTER 1. INTRODUCTION OF THE RIEMANN ZETA FUNCTION

By Weierstrass’ formula (e.g. [8], page 73), we have for any \( s \in \mathbb{C} \),

\[
\frac{1}{\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right) e^{-s/n},
\]

(1.11)

where \( \gamma \) is Euler’s constant and defined by

\[
\gamma := \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.57721566 \cdots.
\]

It follows from this infinite product (1.11) that \( \Gamma(s) \) has no zero in \( \mathbb{C} \). However, in view of the functional equation (1.10), \( \Gamma(s) \) has simple poles at \( s = 0, -1, -2, \cdots \) with residue 1. This is because

\[
\lim_{s \to 0^+} \frac{1}{\Gamma(s)} = \lim_{s \to 0^+} \frac{s}{\Gamma(s + 1)} = 0,
\]

(1.12)

where \( \Gamma(1) = 1 \). In view of Stirling’s asymptotic formula (e.g. page 73, [8]), we have

\[
\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}).
\]

Furthermore,

\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1})
\]

as \( |s| \to \infty \) and \( -\pi + \delta < \arg s < \pi - \delta \), for any fixed \( \delta > 0 \).

To obtain the analytic continuation of \( \zeta(s) \), we start from \( \Gamma(s) \)

\[
\Gamma(s) = \int_{0}^{\infty} e^{-t} t^{s-1} dt, \quad \sigma > 0.
\]

By substituting \( t = n^2 \pi x \), we get

\[
\Gamma\left( \frac{s}{2} \right) = (n^2 \pi)^{s/2} \int_{0}^{\infty} e^{-n^2 \pi x} x^{s/2 - 1} dx.
\]

Next, dividing the term \( (n^2 \pi)^{s/2} \) on both sides and then summing over \( n \) from 1, 2, 3, \cdots, we get for \( \sigma > 1 \),

\[
\pi^{-s/2} \Gamma\left( \frac{s}{2} \right) \zeta(s) = \pi^{-s/2} \Gamma\left( \frac{s}{2} \right) \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-n^2 \pi x} x^{s/2 - 1} dx.
\]

Since the series \( \sum_{n=1}^{\infty} e^{-n^2 \pi x} \) is uniformly convergent on \([0, \infty)\), interchanging the order of the summation and integration is legitimate and then we obtain

\[
\pi^{-s/2} \Gamma\left( \frac{s}{2} \right) \zeta(s) = \int_{0}^{\infty} \left( \sum_{n=1}^{\infty} e^{-n^2 \pi x} \right) x^{s/2 - 1} dx.
\]

Letting

\[
\omega(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x},
\]

with
we have
\[ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \int_0^\infty \omega(x)x^{\frac{s}{2}-1}dx = \int_0^1 \omega(x)x^{\frac{s}{2}-1}dx + \int_1^\infty \omega(x)x^{\frac{s}{2}-1}dx. \]

By substituting \( \frac{1}{2} \) for \( x \) for the first integral above, we get
\[ \pi^{-\frac{1}{2}} \Gamma \left( \frac{1}{2} \right) \zeta(s) = \int_1^\infty \omega \left( \frac{1}{x} \right) x^{-\frac{s}{2}-1}dx + \int_1^\infty \omega(x)x^{\frac{s}{2}-1}dx. \]

We let \( \theta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x} \). Hence
\[ 2\omega(x) = \theta(x) - 1. \]

The function \( \theta(x) \) is the famous \( \Theta \) function of Jacobi and satisfies the functional equation (e.g. (2.2.6) of [3])
\[ x^{\frac{1}{2}} \theta(x) = \theta(x^{-1}), \quad x > 0. \]

It then follows that
\[ \omega(x^{-1}) = -\frac{1}{2} + \frac{1}{2} x^{\frac{1}{2}} + x^{\frac{1}{2}} \omega(x). \]

So
\[ \int_1^\infty x^{-\frac{1}{2}s-1} \omega(x^{-1})dx = \int_1^\infty x^{-\frac{1}{2}s-1} \left( -\frac{1}{2} + \frac{1}{2} x^{\frac{1}{2}} + x^{\frac{1}{2}} \omega(x) \right)dx = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty x^{-\frac{1}{2}s-1} \omega(x)dx. \]

Therefore, for any \( s \in \mathbb{C}, s \neq 0,1 \), we have
\[ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{3}{2}}) \omega(x)dx. \quad (1.13) \]

Let us examine the convergence of the above improper integral. Since
\[ \omega(x) \leq e^{-\pi x} + e^{-2\pi x} + e^{-3\pi x} + \ldots = \frac{e^{-\pi x}}{1 - e^{-\pi x}}, \]
as \( x \) approaches infinity, the integral converges absolutely for any \( s \) and converges uniformly with respect to \( s \) in any bounded region of the complex plane. Hence the integral represents an entire function of \( s \), and gives the analytic continuation of \( \zeta(s) \) to \( \mathbb{C} \). Now the global definition of \( \zeta(s) \) is
\[ \zeta(s) := \frac{\pi^s}{\Gamma \left( \frac{s}{2} \right)} \left\{ \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{3}{2}}) \omega(x)dx \right\}. \quad (1.14) \]

Also under the map of sending \( s \) to \( 1 - s \), we have
\[ \int_1^\infty (x^{\frac{1-s}{2}} + x^{\frac{1}{2}-\frac{3}{2}s-\frac{1}{2}}) \omega(x)dx = \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{3}{2}}) \omega(x)dx. \]
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and

\[ \frac{1}{(1-s)((1-s)-1)} = \frac{1}{s(s-1)}. \]

Hence the right hand side of (1.14) is invariant under the mapping of substituting \( s \) by \( 1 - s \). Therefore we obtain the functional equation

\[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \] (1.15)

In view of this, we define

\[ \xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s). \] (1.16)

This function is entire, since \( \frac{1}{2} s \Gamma\left(\frac{s}{2}\right) \) has no zeros, and the only possible pole of \( \zeta(s) \) is at \( s = 1 \), which is cancelled out by the factor \( s - 1 \). From (1.15) we immediately have a functional equation for \( \xi(s) \) in a very simple form

\[ \xi(s) = \xi(1-s). \] (1.17)

From (1.17) we deduce that \( \xi(s) \) is symmetric to \( s = \frac{1}{2} \).

1.6 Zeros of the Riemann Zeta Function

In this section we summarize what we have obtained about zeros of the Riemann Zeta function and discuss the following questions:

1. Where are the zeros located?

2. How many types of zeros does the Riemann Zeta function have?

3. What are the basic properties of the zeros?

From (1.14) we can define the Riemann Zeta function to be

\[ \zeta(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \int_{1}^{\infty} \left( x^{\frac{1}{2} s - 1} + x^{-\frac{1}{2} s - \frac{1}{2}} \right) \omega(x) dx \right\}, \quad s \in \mathbb{C}, s \neq 1. \] (1.18)

Since the Gamma function \( \Gamma\left(\frac{s}{2}\right) \) in (1.9) has simple poles at \( s = 0, -2, -4, \ldots \), its reciprocal has simple zeros at \( s = 0, -2, -4, \ldots \). Therefore \( \zeta(s) \) has simple zeros at \( s = 0, -2, -4, \ldots \). These zeros, coming from the Gamma function, are called trivial zeros of the Riemann Zeta function which lie on the negative real axis of the complex plane. We also know from Theorem 1.2 that \( \zeta(s) \neq 0 \) in the region \( \sigma > 1 \). However, it is not trivial to show that \( \zeta(s) \neq 0 \) on the vertical line \( \sigma = 1 \). In fact, it was first proved by Hadamard and De la Vallée Poussin and it is equivalent to the prime number theorem. Furthermore, by the functional equation (1.15), the zeros of \( \zeta(s) \) are symmetric to the vertical line \( \sigma = \frac{1}{2} \) in the region \( 0 \leq \sigma \leq 1 \). Hence \( \zeta(s) \neq 0 \) on the vertical line \( \sigma = 0 \). Also
in view of (1.14), the zeros are symmetric to the real axis because the conjugates of zeros are also zeros. Hence if the Riemann Zeta function has any zeros except the trivial ones, they can only lie in the region $0 < \sigma < 1$. This region is called the critical strip, the vertical line $\sigma = \frac{1}{2}$ is called the critical line, and the zeros in the critical strip are called the non-trivial zeros. Note that in view of (1.16), the zeros of $\xi(s)$ are lying in the region $0 \leq \sigma \leq 1$ and they are precisely the non-trivial zeros of $\zeta(s)$.

So where are these non-trivial zeros? Maybe after computing the first few zeros and also inspired by the functional equation, Riemann made a celebrated conjecture that all the non-trivial zeros lie on the critical line $\Re(s) = \frac{1}{2}$. The reason for Riemann to make this conjecture is still unknown. The proof of the celebrated "Riemann Hypothesis (RH)" has been eluding many mathematicians for more than one hundred years.

We state the Riemann Hypothesis as follows.

**Conjecture 1.3. (Riemann Hypothesis)** All the non-trivial zeros of $\zeta(s)$ lie on the vertical line $\sigma = 1/2$.

In 1900 the mathematician Hilbert listed the RH to be the eighth problem of his famous twenty-three problems. Some of these problems have been solved, but the RH remains the hardest one resisting any attempt. People still do not know when the RH will be solved but one can use a computer to verify the conjecture up to millions of zeros. No counter example of a non-trivial zeros off the critical line has been found yet. The Clay Mathematical Institute put the RH to be one of the seven millennium problems. Interestingly enough, people who just find a counter example to disprove the RH, without involving any deep theory, do not qualify for the prize.

We would also like to discuss a little about Riemann's Memoir, in which the RH was stated. This eight page memoir has become a legendary paper in mathematical history. Not only did it propose the RH, but it also laid down the groundwork for the development of analytic number theory. Two main results were proved, which are the derivation of $\zeta(s)$ and the discovery of the functional equation for $\zeta(s)$. More than that, in this paper Riemann introduced important new methods and tools which later on turned out to be very useful in further development of these topics. In addition to the RH, Riemann conjectured the asymptotic formula for the number $N(T)$ of zeros of $\zeta(s)$ in the critical strip with $0 \leq \gamma < T$ to be

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

which was proved by Mangold in 1905. The product representation of $\xi(s)$ as

$$\xi(s) = e^{A+Bs} \prod_{\rho} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}}$$

where $A$, $B$ are constants and $\rho$ represents all non-trivial zeros of $\zeta(s)$ was proved by Hadamard in 1893.
1.7 Zero-free Region

In this section, we prove a zero-free region inside the critical strip $0 < \sigma < 1$. We know that if the RH is correct, then all non-trivial zeros of Riemann Zeta function would lie on the critical line $\sigma = \frac{1}{2}$. So a zero-free region inside the critical strip could be the whole region but a single line. Even though we don’t know RH to be true or false, we might attempt to prove a zero-free region which is a vertical strip, even with very slim width, in the critical region. We will see in this section that we are unable to prove such a vertical strip exists and that it is quite a big step towards solving the RH if one can prove that $\zeta(s) \neq 0$ for $1 - c \leq \sigma \leq c$ for some $1/2 < c < 1$.

First let us prove that there is no zero on the vertical line $\sigma = 1$.

Recall the definition of $\xi(s)$,

$$\xi(s) = \frac{1}{2}s(s - 1)\pi^{-rac{1}{4}}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$$

and the functional equation of $\xi(s)$

$$\xi(s) = \xi(1 - s).$$

When $\sigma > 1$, the Euler product formula tells us that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where $s = \sigma + it$. Now we take logarithms on each side of (1.21) and get

$$\log \zeta(s) = -\sum_p \log \left(1 - \frac{1}{p^s}\right).$$

Using the Taylor expansion of $\log(1 - x)$ at $x = 0$, we have

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}$$

$$= \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}p^{-imt}$$

$$= \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}e^{-imt\log p}. \quad (1.22)$$

It follows that the real part of $\log \zeta(s)$ is

$$\Re \log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(mt \log p). \quad (1.23)$$

We need the following elementary inequality

$$3 + 4 \cos \theta + \cos(2\theta) \geq 0, \quad (1.24)$$
for any $\theta$. This inequality can be proved easily by observing that the left hand side of (1.24) is $2(1 + \cos \theta)^2$, which is non-negative for any $\theta$. We apply this inequality to the following

$$3\Re \log \zeta(\sigma) + 4\Re \log \zeta(\sigma + it) + i \log \zeta(\sigma + 2it)$$

$$= 3 \sum_p \sum_{m=1}^{\infty} \frac{1}{p^{\sigma}} m^{-\sigma m} + 4 \sum_p \sum_{m=1}^{\infty} \frac{1}{p^{\sigma}} m^{-\sigma m} \cos(mt \log p)$$

$$+ \sum_p \sum_{m=1}^{\infty} \frac{1}{p^{\sigma}} m^{-\sigma m} \cos(2mt \log p)$$

$$= \sum_p \sum_{m=1}^{\infty} \frac{1}{m \sigma m} \left( 3 + 4 \cos(mt \log p) + \cos(2mt \log p) \right),$$

by (1.23). Since $\log W = \log |W| + i \arg(W)$, so

$$3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| \geq 0,$$

or equivalently,

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1. \tag{1.25}$$

Since $\zeta(\sigma)$ has a simple pole at $s = 1$ with residue 1, the Laurent series of $\zeta(\sigma)$ at $\sigma = 1$ is

$$\zeta(\sigma) = \frac{1}{1 - \sigma} + a_0 + a_1(\sigma - 1) + a_2(\sigma - 1)^2 + \cdots = \frac{1}{1 - \sigma} + g(\sigma)$$

where $g(\sigma)$ is analytic at $\sigma = 1$ and hence for $1 < \sigma \leq 2$, we have $g(\sigma) = O(1)$ and

$$\zeta(\sigma) = \frac{1}{1 - \sigma} + O(1).$$

Now we will show that $\zeta(1 + it) \neq 0$ by using (1.25). Suppose to the contrary that there is a zero on the line $\sigma = 1$, say, $\zeta(1 + it) = 0$ and $t \neq 0$. Then for any $\sigma > 1$,

$$|\zeta(\sigma + it)| = |\zeta(\sigma + it) - \zeta(1 + it)|$$

$$= |\sigma - 1||\zeta'(\sigma_0 + it)|, \quad 1 < \sigma_0 < \sigma$$

$$\leq A_1(\sigma - 1),$$

by the mean-value theorem and $A_1$ is a constant depending only on $t$. Also when $\sigma$ approaches 1, we have $|\zeta(\sigma + 2it)| < A_2$, where $A_2$ depends only on $t$. Since the degree of the term $\sigma - 1$, which is four, is greater than that of the term $\frac{1}{\sigma - 1}$, which is three, for fixed $t$, as $\sigma$ tend to $1^+$, we have

$$\lim_{\sigma \to 1^+} |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)|$$

$$\leq \lim_{\sigma \to 1^+} \left( \frac{1}{\sigma - 1} + O(1) \right)^3 A_1^4(\sigma - 1)^4 A_2$$

$$= 0.$$

This contradicts (1.25). Hence we conclude that $\zeta(1 + it) \neq 0$ for any $t$. 
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Up to now, all our work is outside critical strip, or at most on the boundary of it. Next we will extend our argument inside the critical strip and show that there exists a thin region to the left of $\sigma < 1$ which is zero free and the width of this region is approaching zero when the height, $t$, goes to infinity. More precisely, the width is proportional to $c(\log t)^{-1}$ at height $t$. The original work is due to De La Vallée Poussin in 1899.

Before we begin our work, let us first introduce three important arithmetic functions $\pi(x)$, $\psi(x)$, and $\Lambda(x)$. First $\pi(x)$ counts the number of primes less than $x$. The celebrated prime number theory, proved by de Vallée Poisson and Hadamard in 1896, gives us a nice asymptotic formula of $\pi(x)$

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}.$$ 

The function $\psi(x)$ is defined as

$$\psi(x) = \sum_{p \leq x} (\log p).$$

It puts weight on each prime number and the asymptotic formula of $\psi(x)$ is $\psi(x) \sim x$. The function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} 0 & \text{if } n \neq p^m \\ \log p & \text{if } n = p^m. \end{cases}$$

(1.26)

So if a number $n$ is not a prime power, then $\Lambda(n) = 0$. When we sum them up, we have

$$\sum_{n \leq x} \Lambda(n) = \sum_{\ell \leq x} (\log p),$$

which is slightly different from $\psi(x)$.

In view of (1.22), we have, for $\sigma > 1$,

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-sm}.$$ 

After differentiating both sides with respect to $s$, it becomes

$$-\frac{\zeta'}{\zeta}(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-sm} (-m \log p)$$

$$= \sum_p \sum_{m=1}^{\infty} (\log p) p^{-ms}.$$ 

Since $\Lambda(p^m) = (\log p)$, so

$$\sum_{m=1}^{\infty} (\log p) p^{(-ms)} = \frac{1}{p^s} \log p + \frac{1}{p^{2s}} \log p + \frac{1}{p^{3s}} \log p + \cdots$$

$$= \frac{1}{p^s} \Lambda(p) + \frac{1}{p^{2s}} \Lambda(p^2) + \frac{1}{p^{3s}} \Lambda(p^3) + \cdots$$

$$= \sum_{m=1}^{\infty} \Lambda(p^m)(p^m)^{(-s)}.$$
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Hence for \( \sigma > 1 \),

\[
-\frac{\zeta'}{\zeta}(s) = \sum_{p} \sum_{m=1}^{\infty} (\log p)p^{-ms} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}.
\]

Therefore if \( s = \sigma + it \), then

\[
-\Re\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma} \cos(t \log n).
\]

By using the inequality (1.24) again, we have

\[
-\Re\frac{\zeta'}{\zeta}(\sigma) - 4\Re\frac{\zeta'}{\zeta}(\sigma + it) - \Re\frac{\zeta'}{\zeta}(\sigma + 2it) = \sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma} \{3 + 4 \cos(t \log n) + \cos(2t \log n)\} \geq 0.
\]

Next we will give bounds for these three terms \( \Re\frac{\zeta'}{\zeta}(\sigma) \), \( \Re\frac{\zeta'}{\zeta}(\sigma + it) \), and \( \Re\frac{\zeta'}{\zeta}(\sigma + 2it) \) respectively.

First let us look at the term \( \Re\frac{\zeta'}{\zeta}(\sigma) \). Since \( \zeta(s) \) has a simple pole at \( s = 1 \) with residue 1, so does \( -\frac{\zeta'}{\zeta}(s) \). So the Laurent series of \( \Re\frac{\zeta'}{\zeta}(s) \) at \( s = 1 \) is

\[
-\frac{\zeta'}{\zeta}(s) = \frac{1}{s - 1} + a_0 + a_1(s - 1) + \cdots.
\]

Therefore,

\[
-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma - 1} + A_3, \quad 1 < \sigma \leq 2,
\]

where \( A_3 \) is an absolute constant.

For the second and third functions, from (6) of Chapter 12 in [8] we have

\[
\xi(s) = e^{A + Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},
\]

where \( \rho \) is a non-trivial zeros of \( \zeta(s) \) and \( A \) and \( B \) are absolute constants. Taking differentiation on logarithm of \( \xi(s) \), we have

\[
\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \frac{-1}{s - \frac{s}{\rho}} + \sum_{\rho} \frac{1}{\rho} = B + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right).
\]

Since \( \xi(s) \) is also defined as

\[
\xi(s) = \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{1}{2} s\right)\zeta(s),
\]

by taking differentiation of the logarithm of (1.29), we have

\[
\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s - 1} + \frac{1}{2} \log \pi + \frac{1}{2} \Gamma'(\frac{1}{2}s) + \frac{\zeta'(s)}{\zeta(s)}.
\]
Now combining (1.28) and (1.30), we have
\[- \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(1/2s + 1)}{\Gamma(s)} - \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).\]

Since if \( t \geq 2 \) and \( 1 \leq \sigma \leq 2 \), by the Stirling formula for \( \Gamma(s) \),
\[\left| \frac{\Gamma'(1/2s + 1)}{\Gamma(s)} \right| \leq A_4 \log t.\]

So we have
\[- \Re \frac{\zeta'(s)}{\zeta}(s) \leq A_5 \log t - \sum_{\rho} \Re \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right). \tag{1.31}\]

Since
\[\Re \left( \frac{1}{s-\rho} \right) = \frac{\sigma - \beta}{|s-\rho|^2} > 0\]
and
\[\Re \left( \frac{1}{\rho} \right) = \frac{\beta}{|\rho|^2} > 0,\]
so we have by letting \( s = \sigma + it \),
\[- \Re \frac{\zeta'(s)}{\zeta}(\sigma + 2it) < A_6 \log t,\]
which is a bound for the third term.

For the second term, we let \( s = \sigma + it \) and \( \beta + i\gamma \) be any non trivial zero of \( \zeta(s) \) with \( \gamma \geq 2 \),
\[- \Re \frac{\zeta'(s)}{\zeta}(\sigma + it) < A_7 \log t - \Re \frac{1}{s-\rho} \]
\[= A_7 \log t - \frac{\sigma - \beta}{(\sigma - \beta)^2 + (\gamma - t)^2}.\]

Choosing \( t = \gamma \), we get
\[- \Re \frac{\zeta'(s)}{\zeta}(\sigma + it) < A_7 \log \gamma - \frac{1}{\sigma - \beta}.\]

Recall that
\[\left| \frac{\zeta'(s)}{\zeta}(\sigma) \right|, \left| \frac{\zeta'(s)}{\zeta}(\sigma + it) \right|, \left| \frac{\zeta'(s)}{\zeta}(\sigma + 2it) \right| \geq 1,\]
and hence
\[\frac{4}{\sigma - \beta} - \frac{3}{\sigma - 1} \leq A_8 \log \gamma.\]

By choosing \( \sigma = 1 + \frac{\delta}{\log \gamma} \) and substituting upper bounds into this inequality, we have
\[\beta < 1 + \frac{\delta}{\log \gamma} - \frac{4\delta}{(3 + A_8\delta) \log \gamma} = 1 - \frac{c}{\log \gamma},\]
which gives us the desired result.
1.8 Counting the Zeros of $\zeta(s)$ Inside a Rectangle

The argument principle in complex analysis gives a very useful tool to count the zeros or the poles of a meromorphic function inside a specified region.

**Argument Principle.** Let $f$ be meromorphic in a domain interior to a positively oriented simple closed contour $C$ such that $f$ is analytic and non-zero on $C$. Then

$$\frac{1}{2\pi} \Delta_C \arg(f(z)) = Z - P$$

where $Z$ is the number of zeros and $P$ is the number of poles of $f(z)$, counting multiplicities, inside $C$.

**Proof.** This is a well known result in complex analysis, e.g., see §79 of [4].

We apply this principle to count the number of zeros of $\zeta(s)$ within the rectangle $\{\sigma + it \in \mathbb{C} : 0 < \sigma < 1, 0 \leq t < T\}$. Here we denote the number of zeros of $\zeta(s)$ inside this rectangle by

$$N(T) := \#\{\sigma + it : 0 < \sigma < 1, 0 \leq t < T, \zeta(\sigma + it) = 0\}.$$ 

Since $\xi(s)$ has the same zeros as $\zeta(s)$ in the critical strip and $\xi(s)$ is an entire function, we apply the argument principle to $\xi(s)$. Let $R$ be the rectangular contour with vertices $-1, 2, 2 + iT$ and $-1 + iT$ and positively oriented. By the principle argument we have

$$N(T) = \frac{1}{2\pi} \Delta_R \arg(\xi(s))$$

where $\Delta_R \arg(\xi(s))$ denotes the change of argument of $\xi(s)$ along the rectangular contour $R$. We now divide $R$ into three sub-contours. Let $L_1$ be the horizontal line from $-1$ to $2$; $L_2$ be the contour consisting of a vertical line from $2$ to $2 + iT$ and then a horizontal line from $2 + iT$ to $\frac{1}{2} + iT$; $L_3$ be the contour consisting of a horizontal line from $\frac{1}{2} + iT$ to $-1 + iT$ and a vertical line from $-1 + iT$ to $-1$. Therefore we get

$$\Delta_R \arg(\xi(s)) = \Delta_{L_1} \arg(\xi(s)) + \Delta_{L_2} \arg(\xi(s)) + \Delta_{L_3} \arg(\xi(s)). \quad (1.32)$$

Let's trace the argument change of $\xi(s)$ along each contour.

First, there is no argument change along $L_1$ since the values of $\xi(s)$ along $L_1$ are real and hence all arguments of $\xi(s)$ are zero constantly. Thus

$$\Delta_{L_1} \arg(\xi(s)) = 0. \quad (1.33)$$

From the functional equation (1.20), we have

$$\xi(\sigma + it) = \xi(1 - \sigma - it) = \xi(1 - \sigma + it).$$
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So the argument change of $\xi(s)$ when $s$ moves along $L_3$ is the same as the argument change of $\xi(s)$ when $s$ moves along $L_2$. Hence from this and (1.33), (1.32) becomes

$$N(T) = \frac{1}{2\pi} 2\Delta_{L_2} \arg(\xi(s)) = \frac{1}{\pi} \Delta_{L_2} \arg(\xi(s))$$  \hspace{1cm} (1.34)

Recall the definition of $\xi(s)$,

$$\xi(s) = (s - 1)\pi^{-\frac{1}{2}s} \Gamma \left( \frac{1}{2}s + 1 \right) \zeta(s).$$ \hspace{1cm} (1.35)

We next work out the argument changes along $L_2$ on each of the four factors on the right hand side of (1.35).

First,

$$\Delta_{L_2} \arg(s - 1) = \arg \left( \frac{1}{2} + iT \right) - \arg(1)$$

$$= \arg \left( \frac{1}{2} + iT \right)$$

$$= \pi \over 2 + \arctan \left( \frac{1}{2T} \right)$$

$$= \pi \over 2 + O(T^{-1})$$

because $\arctan(1/T) = O(T^{-1})$.

Secondly,

$$\Delta_{L_2} \arg(s - \frac{1}{2}) = \Delta_{L_2} \arg \left( \exp \left( \left( -\frac{1}{2}s \right) \log \pi \right) \right)$$

$$= \arg \left( \exp \left( \left( -\frac{1}{2} + iT \right) \log \pi \right) \right)$$

$$= \arg \left( \exp \left( \left( \frac{1}{4} - iT \right) \log \pi \right) \right)$$

$$= -\frac{T}{2} \log \pi.$$  

Thirdly, we need to use Stirling’s formula of $\Gamma(s)$ to facilitate our calculation. Stirling’s formula gives us an asymptotic estimate for $\Gamma(s)$, (see (5) of Chapter 10 in [8])

$$\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}),$$

which is valid when $|s|$ tends to $\infty$ and the argument satisfies $-\pi + \delta < \arg s < \pi - \delta$ for any fixed $\delta > 0$. So we get

$$\Delta_{L_2} \arg \left( \Gamma \left( \frac{1}{2}s + 1 \right) \right) = \Im \log \Gamma \left( \frac{5}{4} + iT \right)$$

$$= \Im \left\{ \left( \frac{3}{4} + iT \right) \log \left( \frac{5}{4} + iT \right) - \frac{5}{4} - iT + \frac{1}{2} \log 2\pi + O(T^{-1}) \right\}$$

$$= \frac{1}{2} T \log \frac{T}{2} - \frac{1}{2} T + \frac{3}{8} \pi + O(T^{-1}).$$
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Putting all these together, we obtain from (1.34)

\[ N(T) = \frac{1}{\pi} \Delta_{L_1} \arg \zeta(s) \]
\[ = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}) \]  \hspace{1cm} (1.36)

where

\[ S(T) = \frac{1}{\pi} \Delta_{L_2} \arg \zeta(s) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right). \]  \hspace{1cm} (1.37)

It remains to estimate \( S(T) \). From (1.31), if \( s = \sigma + it, 2 \leq t \) and \( 1 < \sigma \leq 2 \), then we have

\[-\Re\zeta'(s) < A_3 \log t - \sum_{\rho} \Re \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right). \]  \hspace{1cm} (1.38)

Since \( \zeta'(s) \) is analytic at \( s = 2 + iT \), so

\[-\Re\zeta'(2 + iT) \leq A_9 \]  \hspace{1cm} (1.39)

where \( A_9 \) is some positive absolute constant. If \( \rho = \beta + i\gamma \) and \( s = 2 + iT \), then

\[ \Re \left( \frac{1}{s - \rho} \right) = \Re \left( \frac{1}{2 - \beta + i(T - \gamma)} \right) \]
\[ = \frac{2 - \beta}{(2 - \beta)^2 + (T - \gamma)^2} \]
\[ \geq \frac{1}{4 + (T - \gamma)^2} \]
\[ \gg \frac{1}{1 + (T - \gamma)^2} \]

because \( 0 < \beta < 1 \). Also we have \( \Re \left( \frac{1}{\rho} \right) = \frac{\beta}{2\beta + i\gamma} \geq 0 \). So by (1.38) and (1.39)

\[ \sum_{\rho} \frac{1}{1 + (T - \gamma)^2} \ll \sum_{\rho} \Re \left( \frac{1}{s - \rho} \right) \ll \log T. \]

Therefore we have proved that for \( T \geq 1 \),

\[ \sum_{\rho} \frac{1}{1 + (T - \gamma)^2} = O(\log T). \]

It immediately follows that

\[ \# \left\{ \rho = \beta + i\gamma : 0 < \beta < 1, T \leq \gamma \leq T + 1, \zeta(\rho) = 0 \right\} \]
\[ \leq 2 \sum_{T \leq \gamma \leq T+1} \frac{1}{1 + (T - \gamma)^2} \]
\[ \leq 2 \sum_{\rho} \frac{1}{1 + (T - \gamma)^2} \ll \log T. \]  \hspace{1cm} (1.40)
CHAPTER 1. INTRODUCTION OF THE RIEMANN ZETA FUNCTION

For large $t$ and $-1 \leq \sigma \leq 2$,
\[
\frac{\zeta'}{\zeta}(s) = O(\log t) + \sum_{\rho} \left\{ \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right\}.\]

When $|\gamma - t| > 1$, we have
\[
\left| \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \leq \frac{3}{(\gamma-t)^2}.
\]
So
\[
\left| \sum_{|\gamma-t|>1} \left( \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) \right| \leq \sum_{|\gamma-t|>1} \frac{3}{(\gamma-t)^2} \ll \sum_{\rho} \frac{1}{1+(\gamma-t)^2} \ll \log t.
\]
Now we get
\[
\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|\leq 1} \left( \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log t).
\]

It follows that
\[
\pi S(T) = \arg \zeta\left(\frac{1}{2} + iT\right) = \int_{\frac{1}{2} + iT}^{2 + iT} \frac{\zeta'}{\zeta}(s)ds = \log \zeta(s)\bigg|_{\frac{1}{2} + iT}^{2 + iT}.
\]
Since $\log \omega = \log |\omega| + i \arg \omega$, so
\[
-\int_{\frac{1}{2} + iT}^{2 + iT} \Im \frac{\zeta'}{\zeta}(s)ds = -\arg(\zeta(s))\bigg|_{\frac{1}{2} + iT}^{2 + iT}
\]
\[
= -\arg(\zeta(2 + iT)) + \arg \left( \zeta\left(\frac{1}{2} + iT\right) \right).
\]
Therefore we get
\[
-\int_{\frac{1}{2} + iT}^{2 + iT} \Im \frac{\zeta'}{\zeta}(s)ds = O(1) + \pi S(T).
\]
Hence
\[
S(T) \ll \sum_{|\gamma-t|<1} \int_{\frac{1}{2} + iT}^{2 + iT} \Im \left( \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) ds + \log T.
\]
and we have
\[
\int_{\frac{1}{2} + iT}^{2 + iT} \Im \left( \frac{1}{s-\rho} \right)ds = \Im \log(s-\rho)\bigg|_{\frac{1}{2} + iT}^{2 + iT} = \arg(s-\rho)\bigg|_{\frac{1}{2} + iT}^{2 + iT} \ll 1.
\]
So we obtain that
\[
S(T) \ll \sum_{|\gamma-T|<1} 1 + \log T \ll \log T
\]
by (1.40). We have therefore showed that
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]

We order the zeros of $\zeta(s)$ with $\gamma > 0$ by their heights first and then by their real parts and let $\{\gamma_n\}$ be the zeros above the real axis under the ordering. For example, if $\Im(\gamma_n) < \Im(\gamma_m)$, then $n < m$ and if $\Im(\gamma_n) = \Im(\gamma_m)$ and $\Re(\gamma_n) < \Re(\gamma_m)$, then $n < m$. Then we have
\[
N(\gamma_n) = n \sim \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi}.
\] (1.41)
Taking the logarithm on each side, we have
\[
\log n \sim \log \gamma_n + \log \log \gamma_n - \log 2\pi \\
\sim \log \gamma_n.
\]
So by replacing \(\log \gamma_n\) by \(\log n\) in (1.41) we get
\[
n \sim \frac{\gamma_n}{2\pi} \log n.
\]
Hence
\[
\gamma_n \sim \frac{2\pi n}{\log n}
\]
as \(n \to \infty\). This formula gives us a way to estimate the height of the \(n\)th zero. For example, what is the height of \(\gamma_{1000}\)? By this formula, it can be easily calculated that
\[
\gamma_{1000} \sim \frac{2\pi \times 1000}{\log 1000} \sim 909.6 \ldots.
\]

1.9 How to Find a Zero of Riemann Zeta Function

In view of the RH we know that the zeros lying on the critical line are very important. How can we compute these zeros? In the seventeenth century, when Euler tried to find out the value of \(\zeta(2)\), he used a way now called the Euler-Maclaurin summation formula to compute \(\zeta(s)\). Later people found that this formula actually can be used to compute any value of \(\zeta(s)\). When people use this formula to compute \(\zeta(\frac{1}{2} + it)\), points on the critical line, they found that this formula is not computationally efficient. If one wants to compute \(\zeta(\frac{1}{2} + it)\) for a large \(t\) using the Euler-Maclaurin summation formula, it takes \(O(t)\) steps to get the result. When Riemann studied \(\zeta(s)\), he developed a very efficient way to evaluate \(\zeta(s)\), which is only of order \(O(\sqrt{t})\) steps to compute \(\zeta(\frac{1}{2} + it)\) when \(t\) is very large. Unfortunately, Riemann’s efficient method to compute \(\zeta(s)\) remained unknown to the mathematics community until Siegel, after carefully studying Riemann’s unpublished notes, rediscovered it and made it public. Now this method is called the Riemann-Siegel formula.

We know that \(\xi(s)\) and \(\zeta(s)\) have same zeros within the critical strip \(0 < \sigma < 1\). So in order to find non-trivial zeros of \(\zeta(s)\), we only need to find zeros of \(\xi(s)\). Recall that
\[
\xi(s) = \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),
\]
and so if we can compute the value of \(\xi(s)\), then we can obtain the value of \(\zeta(s)\). By the functional equation of \(\xi(s)\)
\[
\xi(s) = \xi(1 - s),
\]
we have
\[
\xi(1/2 + it) = \xi(1 - (1/2 + it)) = \xi(1/2 - it) = \xi(1/2 + it) = \xi(1/2 + it),
\] (1.42)
which tells us in fact $\xi(1/2 + it)$ is real. Since
\[
\xi(1/2 + it) = \frac{1}{2}(1/2 + it)(1/2 + it - 1)\pi^{-1/4(1/2+it)}\Gamma\left(\frac{1/2 + it}{2}\right)\zeta(1/2 + it)
\]
and $-\frac{1}{2}(1/4 + t^2)\pi^{-1/4}$ is real, in order to have $\xi(1/2 + it)$ real, the argument of the complex number $\pi^{-it/2}\Gamma(1/4 + it/2)\zeta(1/2 + it)$ must be a multiple of $2\pi$. If we look at $\pi^{-it/2}\Gamma(1/4 + it/2)\zeta(1/2 + it)$ as a product of two complex numbers $\pi^{-it/2}\Gamma(1/4 + it/2)$ and $\zeta(1/2 + it)$, then the sum of the arguments of $\pi^{-it/2}\Gamma(1/4 + it/2)$ and $\zeta(1/2 + it)$ must be a multiple of $2\pi$. Now let us define
\[
\theta(t) := \text{arg}(\pi^{-it/2}\Gamma(1/4 + it/2))
\]
and
\[
Z(t) := \exp(i\theta(t))\zeta(1/2 + it).
\]
Note that $\text{arg}(\zeta(1/2 + it)) = -\theta(t)$. We claim that $Z(t)$ is a real valued function. This is because
\[
Z(t) = \exp(i\theta(t))\zeta(1/2 + it) = \exp(i\theta(t))\exp(-i\theta(t))|\zeta(1/2 + it)| = |\zeta(1/2 + it)|.
\]
Since $\zeta(1/2 + it)$ is analytic, it follows that $Z(t)$ is continuous. Thus if $Z(t)$ changes sign within an interval, that means there is at least one zero of $\zeta(1/2 + it)$ inside this interval. By evaluating $Z(t)$ numerically, we can locate the positions of the zeros of $\zeta(1/2 + it)$. For example, using step $t = 0.01$ and evaluating $Z(t)$ from 0 to 100 along critical line, we can easily locate the intervals where $Z(t)$ has sign changes and then we know there is a zero of $\zeta(s)$ inside this interval.

When we evaluate $Z(t)$, we don’t use its definition directly because it involves $\zeta(1/2 + it)$. Instead we use an asymptotic formula to approximate $Z(t)$. This formula is the famous Riemann-Siegel formula. For the proof of this formula, we refer to Chapter 7 of [4]. Here we state the formula and demonstrate how to use this formula to locate a zero of $\zeta(s)$.

**Theorem 1.4 (Riemann-Siegel Formula).** With the definitions above, the following identity holds,
\[
Z(t) = 2 \sum_{k=1}^{\nu(t)} \frac{1}{\sqrt{k}} \cos[\theta(t) - t \ln k] + R(t)
\]
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where
\[ \nu(t) = \left\lfloor \frac{\sqrt{t}}{2\pi} \right\rfloor, \]
\[ p = \sqrt{\frac{t}{2\pi}} - \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor, \]
\[ R(t) = (-1)^{\nu(t)-1} \left( \frac{t}{2\pi} \right)^{-1/4} \sum_{k=0}^{\infty} c_k \left( \sqrt{\frac{t}{2\pi}} - \nu(t) \right) \left( \frac{t}{2\pi} \right)^{-k/2}. \]

Here \( \lfloor x \rfloor \) is the floor function. The first five \( c_k(p) \) are given by

\[ c_0(p) = \psi(p), \]
\[ c_1(p) = -\frac{\psi^{(3)}(p)}{96\pi^2}, \]
\[ c_2(p) = \frac{\psi^{(2)}(p)}{64\pi^2} + \frac{\psi^{(6)}(p)}{18432\pi^4}, \]
\[ c_3(p) = \frac{\psi^{(1)}(p)}{64\pi^2} - \frac{\psi^{(5)}(p)}{3840\pi^2} - \frac{\psi^{(9)}(p)}{5308416\pi^6}, \]
\[ c_4(p) = \frac{\psi(p)}{128\pi^2} + \frac{19\psi^{(4)}(p)}{24576\pi^4} + \frac{11\psi^{(8)}(p)}{5898240\pi^6} + \frac{\psi^{(12)}(p)}{2038431744\pi^8}, \]

where \( \psi(p) \) is given by
\[ \psi(p) = \frac{\cos(2\pi(p^2 - p - 1/16))}{\cos(2\pi p)}. \]

Also \( \vartheta(t) \) can be approximated by using Stirling’s formula for \( \Gamma(s) \)
\[ \vartheta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} \cdots. \]

In our demonstration, we use the Riemann-Siegel formula to compute \( Z(30) \) and \( Z(31) \). For
simplicity, we only use first approximation. We have
\[ \nu(30) = \left\lfloor \frac{30}{2\pi} \right\rfloor = 2 \]
\[ p = \sqrt{\frac{30}{2\pi}} - \left\lfloor \sqrt{\frac{30}{2\pi}} \right\rfloor = 0.185 \ldots \]
\[ \psi(p) = \frac{\cos[2\pi(p^2 - p - 1/16)]}{\cos(2\pi p)} = 0.999 \ldots \]
\[ R(30) = (-1)^{\nu(30)-1}\left( \frac{30}{2\pi} \right)^{-1/4} c_0 \left( \sqrt{\frac{30}{2\pi}} - \nu(30) \right) \left( \frac{30}{2\pi} \right)^{-0/2} = -0.125 \ldots \]
\[ \vartheta(30) = \frac{30}{2} \log \frac{30}{2\pi} - \frac{30}{2} - \frac{\pi}{8} + \frac{1}{48 \times 30} = 8.057 \ldots \]

Therefore the first approximation of \( Z(30) \) is
\[ Z(30) \approx 2 \sum_{k=1}^{\nu(30)} \frac{1}{\sqrt{k}} \cos[\vartheta(30) - 30 \ln k] + R(30) \]
\[ = 2 \sum_{k=1}^{2} \frac{1}{\sqrt{k}} \cos[8.057 - 30 \ln k] - 0.125 \]
\[ = 0.86532 \ldots \]

Here the value of the main term is 0.5629 \ldots and the error term is -0.4260 \ldots. Similarly, we can figure out the value of \( Z(31) \) which equals -0.8093 \ldots with Error term -0.2845 \ldots. So \( Z(t) \) changes sign between 30 and 31. That means there is a root of \( Z(t) \) in this interval. Therefore \( \zeta(s) \) has a zero with \( 30 < \Re(s) < 31 \).
Chapter 2

The Pair Correlation of the Zeros of the Riemann Zeta Function

2.1 The Pair Correlation of the Zeros of the Riemann Zeta Function

After people have found so many zeros of the Riemann Zeta function, it is natural to use statistical tools to find relationships between these zeros. In 1973, Hugh Montgomery of the University of Michigan and quantum physicist Freeman Dyson of Princeton University discovered that there is an interesting relationship between the spaces of consecutive zeros of the Riemann Zeta function and the spaces of eigenvalues generated from a random matrix. Since then, a large amount of research has been done in this direction.

In the following sections we briefly go through Montgomery’s famous 1973 paper and see how he ended up with his Pair Correlation Conjecture. Then after providing the background knowledge of random matrix theory, which has been intensively studied by physicists, the exciting connection between zeros of the Riemann zeta function and eigenvalues of a random matrix is reviewed. People can also refer to [15], [16], [17] and [18] for more details. Random matrices are used to describe the energy levels of nuclei in theoretical physics and to some degree are related to the phenomena called "chaos". So studies in theoretical physics seem to provide research tools for number theory in pure mathematics. This is very interesting since normally in history it is mathematics that gives physics all the tools it needs.

As we know from Chapter One, the Riemann Zeta function has a simple pole with residue 1 at $s = 1$, trivial zeros at $s = -2n$, $n = 1, 2, 3, \ldots$, and non-trivial zeros

$$\rho = \beta + i\gamma, \ 0 < \beta < 1.$$
CHAPTER 2. THE PAIR CORRELATION OF THE ZEROS OF THE RIEMANN ZETA FUNCTION

The Riemann Hypothesis conjectures that $\beta = 1/2$ for all non-trivial zeros. We define

$$n(T) = \# \{ \gamma : 0 < \gamma \leq T \} , \quad N(T) = \frac{n(T + 0) + n(T - 0)}{2},$$

where $\#(S)$ means the number of elements inside set $S$ and $N(T)$ is the number of non-trivial zeros with $\beta = 1/2$ and $\gamma \leq T$. We have the asymptotic formula for $N(T)$:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \quad (2.1)$$

We first have the following definition.

**Definition 2.1.** Define for real $\alpha$ and $T \geq 2$,

$$F(\alpha) = F(\alpha, T) = \left( \frac{T}{2\pi} \log T \right)^{-1} \sum_{\gamma, \gamma' \in [0, T]} T^{\alpha(\gamma - \gamma')} \omega(\gamma - \gamma') \quad (2.2)$$

where $\gamma, \gamma'$ are heights (imaginary parts) of nontrivial zeros of $\zeta(s)$ and $\omega(u)$ is a certain weighting function.

This function plays an important role for Montgomery to reveal the properties of pair correlation of spacings of zeros of Zeta function. Here if $\omega(u)$ is real and symmetric in $u$, then $F(\alpha)$ is real and symmetric in $\alpha$. Indeed, we have

$$\overline{F(\alpha)} = \left( \frac{T}{2\pi} \log T \right)^{-1} \sum_{\gamma, \gamma' \in [0, T]} T^{-\alpha(\gamma - \gamma')} \omega(\gamma - \gamma') = F(\alpha) = F(-\alpha)$$

In Montgomery’s paper, $\omega(u) = 4/(4 + u^2)$.

For $F(\alpha)$, Montgomery has proved the following theorem in his famous paper [16].

**Theorem 2.2 (Montgomery).** Assume the Riemann Hypothesis is true. For real $\alpha$ and $T \geq 2$, $F(\alpha)$ is real and $F(\alpha) = F(-\alpha)$. For $\epsilon > 0$, there is $T_\epsilon > 0$ such that if $T > T_\epsilon$ then $F(\alpha) \geq -\epsilon$ for all $\alpha$. As $T \to \infty$,

$$F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$$

uniformly for fixed $0 < \alpha < 1 - \epsilon$.

The main objective in this section is to give a proof of Montgomery’s theorem.

We first establish some of the results about the variant sums over the zeros of $\zeta(s)$.

**Lemma 2.3.** Assume the Riemann Hypothesis is true. If $1 < \sigma < 2$ and $x \geq 1$ then

$$(2\sigma - 1) \sum_{\gamma} \frac{x^{\gamma(t - \gamma)}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} = -x^{-\frac{1}{2}} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1-\sigma+it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{\sigma+it} \right)$$

$$+ x^{\frac{1}{2} - \sigma + it} \log \tau + O(x^{-\frac{1}{2} - \sigma}) + O(x^{-\frac{1}{2} - \tau^{-1}})$$

where the summation over $\gamma$ is taken over all non-trivial zeros of $\zeta(s)$, $\tau = |t| + 2$ and the implicit constants depend only on $\sigma$. 

Proof. We start from the explicit formula for $\psi(x)$, e.g. (§17 of [8]), namely, for $x > 1$,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}. $$

Using the partial summation formula, we get

$$\sum_{n \leq x} \Lambda(n)n^{-s} = \frac{\zeta'(s)}{\zeta(s)} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n + s}$$

(2.3)

where $x > 1$ is not a power of primes and $s = \sigma + it$ with $s \neq 1, \rho, -2n$.

If we assume the Riemann Hypothesis is true, then we can write $\rho = 1/2 + i\gamma$ and $s = \sigma + it$ and we get from (2.3)

$$\sum_{\rho} \frac{x^{\rho-s}}{\rho - s} = \sum_{\gamma} \frac{x^{1/2+i\gamma-(\sigma+it)}}{1/2+i\gamma-(\sigma+it)} = -\left\{ \frac{\zeta'(s)}{\zeta(s)} - \frac{x^{1-s}}{1-s} + \sum_{n\leq x} \Lambda(n)n^{-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n + s} \right\}. $$

(2.4)

After multiplying both sides by $-x^{\sigma-1/2}$, we get

$$\sum_{\gamma} \frac{x^{i(\gamma-t)}}{1/2-i\gamma+\sigma+it} = x^{\sigma-1/2} \left\{ \frac{\zeta'(s)}{\zeta(s)} - \frac{x^{1-s}}{1-s} + \sum_{n\leq x} \Lambda(n)n^{-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n + s} \right\}. $$

(2.5)

By replacing $\sigma$ by $1 - \sigma$ in (2.4), we have

$$\sum_{\gamma} \frac{x^{i(\gamma-t)}}{1/2+i\gamma-(\sigma+it)} = x^{1/2-\sigma} \left\{ \frac{\zeta'(s)}{\zeta(s)} - \frac{x^{\sigma-it}}{\sigma-it} + \sum_{n\leq x} \Lambda(n)n^{\sigma-1-it} - \sum_{n=1}^{\infty} \frac{x^{-2n-1+\sigma-it}}{2n+1-\sigma+it} \right\}. $$

(2.6)

We next subtract (2.6) from (2.5). Since

$$-\frac{1}{\frac{1}{2}-i\gamma+\sigma+it} - \frac{1}{\frac{1}{2}+i\gamma-\sigma+it} = \frac{2\sigma-1}{(1/2-\sigma)^2 + (\gamma-t)^2},$$

we get

$$(2\sigma-1) \sum_{\gamma} \frac{x^{i\gamma}}{(\sigma-\frac{1}{2})^2 + (\gamma-t)^2} = -x^{1/2-\sigma+it} \frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)} - \frac{x^{1/2}(2\sigma-1)}{(\sigma-it)(\sigma-1+it)}

- \frac{x^{-1/2}}{2} \left( \sum_{n\leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1-\sigma+it} + \sum_{n> x} \Lambda(n) \left( \frac{x}{n} \right)^{\sigma+it} \right)

- x^{-1/2} \sum_{n=1}^{\infty} \frac{x^{-2n}(2\sigma-1)}{(2n+\sigma+it)(\sigma-1-2n-it)}.$$
and
\[ \frac{x^{1/2}(2\sigma - 1)}{(\sigma - it)(\sigma - 1 + it)} \ll x^{1/2} \tau^{-2}. \] (2.7)

Following from these estimates, we get
\[ (2\sigma - 1) \sum \frac{x^{it}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} = -x^{-\frac{1}{4}} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1 - \sigma + it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{\sigma + it} \right) + x^{\frac{1}{2} - \sigma + it} \log \tau + O(x^{\frac{1}{2} - \sigma}) + O(x^{1/2} \tau^{-1}) \]

and finish the proof of lemma.

Now if we put \( \sigma = 3/2 \), from Lemma 2.3 we have
\[ 2 \sum \frac{x^{it}}{1 + (t - \gamma)^2} = -x^{-1/2} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{-1/2 + it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2 + it} \right) + x^{-1 + it} \log \tau + O(x^{-1}) + O(x^{1/2} \tau^{-1}). \] (2.8)

Let \( L(x, t) = R(x, t) \) represents the left hand side and right hand side of the equation (2.8). Let \( \epsilon > 0 \) be given. Our next goal is to prove that if \( T > T_\epsilon \), then \( F(\alpha) \geq -\epsilon \) for uniformly \( \alpha \). Recall the definition of \( F(\alpha) \),
\[ F(\alpha) = \left( \frac{T}{2\pi \log T} \right)^{-1} \sum_{\gamma, \gamma' \in [0, T]} T^{2\alpha(\gamma - \gamma')} \omega(\gamma - \gamma'). \] (2.9)

So for \( L(x, t) = 2 \sum \frac{x^{it}}{1 + (t - \gamma)^2} \), we need to restrict the summation range to \([0, T]\). We first have
\[ \int_0^T |L(x, t)|^2 dt = \int_0^T L(x, t) \overline{L(x, t)} dt = 4 \int_0^T \sum_{\gamma, \gamma'} \frac{x^{it(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt. \] (2.10)

The following estimations and proposition are very useful for us to reach our goal.

**Lemma 2.4.** Under the Riemann Hypothesis, we have

(i) For \( t \in [0, T] \),
\[ \sum_{\gamma \in [0, T]} \frac{1}{1 + (t - \gamma)^2} \ll \left( \frac{1}{t + 1} + \frac{1}{T + 1} \right) \log T; \]

(ii) For \( t \in [0, T] \),
\[ \sum_{\gamma} \frac{1}{1 + (t - \gamma)^2} \ll \log T; \]

(iii) For \( t \in [T, \infty) \),
\[ \sum_{\gamma \in [0, T]} \frac{1}{1 + (t - \gamma)^2} \ll \frac{1}{t + 1} \log T; \]
(iv) For \( t \in (-\infty, 0] \),

\[
\sum_{\gamma \in [0,T]} \frac{1}{1 + (t - \gamma)^2} \ll \frac{1}{|t - T + 1|} \log T;
\]

Proof. We only give the proof of case (i). The other cases can be proved in a similar way. First we have

\[
\sum_{\gamma > T} \frac{1}{1 + (t - \gamma)^2} = \int_T^\infty \frac{1}{1 + (t - x)^2} dN(x).
\]

Since \( N(x) \sim \frac{\pi}{2x} \log x \), we get

\[
\sum_{\gamma > T} \frac{1}{1 + (t - \gamma)^2} = -\frac{N(T)}{1 + (t - T)^2} - 2 \int_T^\infty \frac{N(x)(t - x)}{(1 + (t - x)^2)} dx \ll \left( \frac{1}{t + 1} + \frac{1}{T - t + 1} \right) \log T.
\]

Now we study the integral \( \int_0^T |L(x,t)|^2 dt \). In view of (2.10)

\[
\int_0^T |L(x,t)|^2 dt = 4 \int_0^T \sum_{\gamma, \gamma'} (1 + (t - \gamma)^2)(1 + (t - \gamma')^2) \frac{x^{i(\gamma - \gamma')}}{1 + (t - \gamma)^2} dt
\]

\[
= 4 \int_0^T \sum_{\gamma, \gamma' \in [0,T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt
\]

\[
+ 4 \int_0^T \left( 2 \sum_{\gamma, \gamma' \in [0,T]} - \sum_{\gamma, \gamma' \in [0,T]} \right) \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt.
\]

Using Lemma 2.4, we have

\[
\left| \int_0^T \sum_{\gamma, \gamma' \in [0,T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt \right|
\]

\[
\leq \int_0^T \sum_{\gamma \in [0,T]} \frac{1}{1 + (t - \gamma)^2} \sum_{\gamma'} \frac{1}{1 + (t - \gamma')^2} dt
\]

\[
\ll \int_0^T \left( \frac{1}{t + 1} + \frac{1}{T - t + 1} \right) \log^2 T dt \ll \log^3 T.
\]

Similarly, we have

\[
\left| \int_0^T \sum_{\gamma, \gamma' \in [0,T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt \right| \ll \log^2 T.
\]

It then follows that

\[
\int_0^T |L(x,t)|^2 dt = 4 \int_0^T \sum_{\gamma, \gamma' \in [0,T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt + O(\log^3 T).
\]
Next we will show that
\[
\int_0^T |L(x, t)|^2 dt = 4 \int_{-\infty}^\infty \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt + O(\log^3 T).
\]
That means both
\[
\int_0^0 \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt
\]
and
\[
\int_T^\infty \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt
\]
are \(O(\log^3 T)\). In view of Lemma 2.4,
\[
\int_0^0 \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt \\ll \int_{-\infty}^0 \sum_{\gamma \in [0, T]} \sum_{\gamma' \in [0, T]} \frac{1}{(1 + (t - \gamma)^2)} dt \\ll \log^2 T \int_{-\infty}^0 \frac{dt}{(t - T + 1)^2} \\ll \log^2 T.
\]
Similarly
\[
\int_T^\infty \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt \ll \log^2 T.
\]
Thus
\[
\int_0^T |L(x, t)|^2 dt = 4 \int_{-\infty}^\infty \sum_{\gamma, \gamma' \in [0, T]} \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt + O(\log^3 T).
\]
We then interchange the summation and the integral to obtain
\[
\int_0^T |L(x, t)|^2 dt = 4 \sum_{\gamma, \gamma' \in [0, T]} \int_{-\infty}^\infty \frac{x^{i(\gamma - \gamma')}}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt + O(\log^3 T). \quad (2.11)
\]

**Proposition 2.5.** For real \(\gamma, \gamma' > 0\), we have
\[
\int_{-\infty}^\infty \frac{dt}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} = \frac{\pi}{2} \omega(\gamma - \gamma')
\]
where the function \(\omega\) is defined by \(\omega(u) = 4/(4 + u^2)\).

**Proof.** We prove this by using a complex integral along a semi-circular contour with the function of a complex variable \(z\) as
\[
\int_{\Gamma} f(z)dz = \int_{\Gamma} \frac{1}{(1 + (z - \gamma)^2)(1 + (z - \gamma')^2)} dz
\]
where $\Gamma$ is the line segment $[-R, R]$ and the semi-circle \{Re^{i\theta}: 0 \leq \theta \leq \pi\} in the counter-clockwise direction. Now we need to use the residue theorem to evaluate this integral. Since among the four poles $z_{1,2} = \pm i + \gamma$ and $z_{3,4} = \pm i + \gamma'$, only $i + \gamma$ and $i + \gamma'$ are inside contour $\Gamma$, and residues of these two poles are

$$\text{Res}_{i+\gamma} f(z) = -\frac{1}{2(\gamma - \gamma')(2 - i(\gamma - \gamma'))}$$

and

$$\text{Res}_{i+\gamma'} f(z) = -\frac{1}{2(\gamma' - \gamma)(2 - i(\gamma' - \gamma))}$$

respectively, by the residue theorem we have

$$\int_{\Gamma} f(z) dz = \pi \omega(\gamma - \gamma').$$

Since we know that

$$\int_{\Gamma} f(z) dz = \int_{-R}^{R} \frac{1}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt$$

$$+ \int_{\pi}^{0} \frac{iRe^{i\theta}}{(1 + (Re^{i\theta} - \gamma)^2)(1 + (Re^{i\theta} - \gamma')^2)} d\theta,$$

the integral over the semi-circle approaches 0 as $R \to \infty$. Therefore as $R \to \infty$ the original integral becomes

$$\int_{\Gamma} f(z) dz = \int_{-\infty}^{\infty} \frac{1}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt.$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt = \frac{\pi}{2} \omega(\gamma - \gamma').$$

Now after substituting it back into the equation (2.11), we have

$$\int_{0}^{T} |L(x,t)|^2 dt = \sum_{\gamma, \gamma' \in [0,T]} 4x^{i(\gamma - \gamma')} \int_{-\infty}^{\infty} \frac{1}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} dt + O(\log^{3} T)$$

$$= 2\pi \sum_{\gamma, \gamma' \in [0,T]} x^{i(\gamma - \gamma')\omega(\gamma - \gamma')} + O(\log^{3} T).$$

We let $x = T^{\alpha}$ for $T \geq 2$ and $\alpha$ be any real number. From the definition of $F(\alpha)$, we have

$$\int_{0}^{T} |L(x,t)|^2 dt = 2\pi \sum_{\gamma, \gamma' \in [0,T]} T^{i\alpha(\gamma - \gamma')\omega(\gamma - \gamma')} + O(\log^{3} T)$$

$$= F(\alpha) \cdot T \log T + O(\log^{3} T).$$

Since $\int_{0}^{T} |L(x,t)|^2 dt \geq 0$, we have

$$F(\alpha) \cdot T \log T + O(\log^{3} T) \geq 0$$
and hence
\[ F(\alpha) \geq -C\log^2 \frac{T}{T} \]
for some constant \( C > 0 \). Therefore we can find a \( T_\varepsilon > 0 \), such that
\[ F(\alpha) = F(\alpha, T) \geq -\varepsilon \]
for \( T > T_\varepsilon \) and any real \( \alpha \). This proves the first part of Theorem 2.2.

In view of (2.8), we have
\[ R(x, t) = -x^{-1/2} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{-1/2+it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2+it} \right) + x^{-1+it} \log T + O(x^{-1}) + O(x^{1/2}e^{-1}). \]

We are going to show
\[ \int_0^T |R(T^\alpha, t)|^2 dt = ((1+o(1))T^{-2\alpha} \log T + \alpha + o(1))T \log T \quad (2.12) \]
where \( T \geq 2 \) and \( 0 \leq \alpha \leq 1-\varepsilon \). In order to estimate \( \int_0^T |R(T^\alpha, t)|^2 dt \), we first prove that

**Lemma 2.6.** Assume the Riemann Hypothesis. For \( T \geq 2, x \geq 1 \), we have
\[ \frac{1}{x} \int_0^T \left| \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{-1/2+it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2+it} \right|^2 dt = T \log x + O(T + x \log x). \]

**Proof.** By using Parseval’s identity for a Dirichlet series (e.g. [16])
\[ \int_0^T \left| \sum_{n=1}^\infty a_n n^{-it} \right|^2 dt = \sum_{n=1}^\infty |a_n|^2 (T + O(n)). \]

Let \( T_1 \) be the integral in the lemma and by Parseval’s identity
\[ T_1 = \frac{1}{x} \sum_{n \leq x} \Lambda^2(n) \left( \frac{x}{n} \right)^{-1} (T + O(n)) + \frac{1}{x} \sum_{n > x} \Lambda^2(n) \left( \frac{x}{n} \right)^{3} (T + O(n)) \]
\[ = Tx^{-2} \sum_{n \leq x} \Lambda^2(n)n + O \left( x^{-2} \sum_{n \leq x} \Lambda^2(n)n^2 \right) \]
\[ +Tx^2 \sum_{n > x} \frac{\Lambda^2(n)}{n^2} + O \left( x^2 \sum_{n > x} \frac{\Lambda^2(n)}{n^2} \right). \]

Under the Riemann Hypothesis, we have (e.g. P.113 of [8])
\[ \psi(x) = x + O(x^{1/2} \log^2 x). \]
By the partial summation formula, we get
\[ \sum_{n \leq x} \Lambda(n)^2 = \psi(x) \log x - \int_1^x \frac{\psi(t)}{t} \, dt = x \log x - O(x^{3/2} \log^3 x). \] (2.13)

Thus, we have
\[ \sum_{n \leq x} \Lambda(n)^2 = \frac{x^2}{2} \log x + O(x^2) \] (2.14)
and
\[ \sum_{n \leq x} \Lambda(n)^2 n^{-2} \ll x^3 \log x. \] (2.15)

Similarly,
\[ \sum_{n > x} \Lambda(n)^2 n^{-3} = \int_x^\infty \frac{1}{t^3} d \sum_{n \leq t} \Lambda(n)^2 = \frac{1}{2} \log x + O(x^{-2}) \] (2.16)
and
\[ \sum_{n > x} \Lambda(n)^2 n^{-2} \ll \frac{\log x}{x}. \] (2.17)

Therefore, it follows from (2.13)-(2.17) that
\[ T_1 = T \log x + O(T + x \log x). \]

Since \( \tau = |t| + 2 \), we have
\[ \int_0^T |x^{-1+it} \log \tau|^2 \, dt = \int_0^T |x^{-1+it} \log(|t| + 2)|^2 \, dt = \int_0^T x^{-2} \log^2(t + 2) \, dt = \frac{T}{x^2} (\log^2 T + O(\log T)). \]

Finally, we have proved that
\[ \int_0^T |R(x, t)|^2 \, dt = T \log x + O(T + x \log x) + \frac{T}{x^2} (\log^2 T + O(\log T)). \]

Now we can set \( x = T^\alpha \) for any \( 0 \leq \alpha \leq 1 - \epsilon \) with \( \epsilon > 0 \) arbitrarily small. Then we obtain
\[ \int_0^T |R(T^\alpha, t)|^2 \, dt = \alpha T \log T + T^{1-2\alpha} \log^2 T + O(T) = (\alpha + (1 + o(1)) T^{-2\alpha} \log T + o(1)) T \log T. \]
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Now going back to the equation (2.8)

\[
\int_0^T |L(T^\alpha, t)|^2 dt = \int_0^T |R(T^\alpha, t)|^2 dt
\]

we obtain

\[
(\alpha + (1 + o(1))T^{-2\alpha} \log T + o(1))T \log T = F(\alpha)T \log T + O(\log^3 T).
\]

After simplification, we arrive at the main result of Montgomery’s theorem: as \( T \to \infty \),

\[
F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)
\]

uniformly for fixed \( 0 < \alpha < 1 - \epsilon \). We complete the proof of Theorem 2.2.

Based on his main result, Montgomery made the following conjecture.

**Conjecture 2.7.** Assume the Riemann Hypothesis. We have

\[ F(\alpha) = 1 + o(1) \]  \hspace{1cm} (2.18)

for \( \alpha \geq 1 \), uniformly in bounded intervals as \( T \to \infty \).

This conjecture is actually equivalent to the pair correlation conjecture.

We demonstrate some corollaries of Montgomery’s main theorem which will help us to understand more the ideas behind Montgomery’s conjecture.

**Corollary 2.8.** Assume the Riemann Hypothesis. For fixed \( 0 < a < 1 \), we have

\[
\sum_{\gamma, \gamma' \in [0, T]} \left( \frac{\sin(a(\gamma - \gamma') \log T)}{a(\gamma - \gamma') \log T} \right) \omega(\gamma - \gamma') \sim \left( \frac{1}{2a} + \frac{a}{2} \right) \frac{T}{2\pi} \log T
\]

and

\[
\sum_{\gamma, \gamma' \in [0, T]} \left( \frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma') \sim \left( \frac{1}{a} + \frac{a}{3} \right) \frac{T}{2\pi} \log T
\]

as \( T \to \infty \).

**Corollary 2.9.** Assume the Riemann Hypothesis. As \( T \to \infty \),

\[
\sum_{\gamma \in [0, T]} \geq \left( \frac{2}{3} + o(1) \right) \frac{T}{2\pi} \log T
\]

where the sum is over all simple zeros of \( \zeta(s) \) with heights \( \gamma \) within the interval \([0, T]\) on the critical line.
The following Fourier transform will be employed to prove the above three corollaries:

\[ \hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u) e^{-2\pi i \alpha u} du \]

and the inverse formula that

\[ r(\alpha) = \int_{-\infty}^{\infty} \hat{r}(u) e^{2\pi i \alpha u} du. \]

By this, we first obtain the following convolution formula.

**Theorem 2.10.** We have

\[
\sum_{\gamma, \gamma' \in [0, T]} r\left( \frac{\gamma - \gamma'}{2\pi} \right) \omega(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}(u) du.
\]

**Proof.** By the definition of \( F(u) \), we have

\[
\frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}(u) du = \int_{-\infty}^{\infty} \sum_{\gamma, \gamma' \in [0, T]} T^{i u (\gamma - \gamma')} \omega(\gamma - \gamma') \hat{r}(u) du
\]

\[
= \sum_{\gamma, \gamma' \in [0, T]} \omega(\gamma - \gamma') \int_{-\infty}^{\infty} T^{i u (\gamma - \gamma')} \hat{r}(u) du.
\]

Since

\[ T^{i u (\gamma - \gamma')} = e^{2\pi i u (\gamma - \gamma') \log T/(2\pi)}, \]

by the inverse formula for the Fourier transform we have

\[
\frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}(u) du = \sum_{\gamma, \gamma' \in [0, T]} \omega(\gamma - \gamma') \int_{-\infty}^{\infty} e^{2\pi i u (\gamma - \gamma') \log T/(2\pi)} \hat{r}(u) du
\]

\[
= \sum_{\gamma, \gamma' \in [0, T]} r\left( \frac{\gamma - \gamma'}{2\pi} \right) \omega(\gamma - \gamma').
\]

By choosing suitable \( r(u) \) and employing this convolution formula, we can prove the previous corollaries. First we recall the Fourier transforms for functions

\[ r_1(u) = \frac{\sin(2\pi a u)}{2\pi a u} \]

and

\[ r_2(u) = \left( \frac{\sin(\pi a u)}{\pi a u} \right)^2. \]

For any \( 0 < a \leq 1 \), it is easy to show that

\[ r_1(\xi) = \frac{1}{2a} \chi_0(\xi) \]
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and

\[ r_2^2(\xi) = \frac{1}{a^2}(a - |\xi|)\chi_a(\xi) \]

where \( \chi_a \) denotes the characteristic function of the interval \([-a, a]\).

Now we are ready to prove Corollary 2.8. Applying this to the convolution formula we have

\[
\sum_{\gamma, \gamma' \in [0,T]} r_1 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') = \sum_{\gamma, \gamma' \in [0,T]} \left( \frac{\sin(a(\gamma - \gamma') \log T)}{a(\gamma - \gamma') \log T} \right) \omega(\gamma - \gamma') \\
= \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}_1(u) du \\
= \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \frac{1}{2a} \chi_a(\xi) du \\
= \frac{T}{4a\pi} \log T \int_{-a}^{a} F(u) du.
\]

Since \( F(u) \) is symmetric in \( u \) and when \( 0 < u < 1 \), from Montgomery's main theorem, we have

\[ F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1) \]

as \( T \to \infty \). Thus we have

\[ \int_{-a}^{a} F(u) du = 2 \int_{0}^{a} ((1 + o(1))T^{-2u} \log T + u + o(1)) du \\
\sim 2 \int_{0}^{a} (e^{-2u} \log T + u) du \\
\sim 1 + a^2. \]

Therefore

\[
\sum_{\gamma, \gamma' \in [0,T]} \left( \frac{\sin(a(\gamma - \gamma') \log T)}{a(\gamma - \gamma') \log T} \right) \omega(\gamma - \gamma') = \frac{T}{4a\pi} \log T \int_{-a}^{a} F(u) du \\
\sim \left( \frac{1}{2a} + \frac{a}{2} \right) \frac{T}{2\pi} \log T
\]

as \( T \to \infty. \)

For the second part of this corollary, we use

\[ r_2(u) = \left( \frac{\sin(\pi u)}{\pi u} \right)^2. \]

and its Fourier transform

\[ r_2(\xi) = \frac{1}{a^2}(a - |\xi|)\chi_a(\xi) \]

into the convolution formula. Following a similar argument as above, we get

\[
\sum_{\gamma, \gamma' \in [0,T]} r_2 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') = \sum_{\gamma, \gamma' \in [0,T]} \left( \frac{\sin(a/2(\gamma - \gamma') \log T)}{a/2(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma') \\
= \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}_2(u) du \\
= \frac{T}{2\pi} \log T \int_{-a}^{a} F(u) \frac{1}{a^2}(a - |u|) du.
\]
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Again, by the symmetry of $F(u)$ and the main theorem of Montgomery, we have

\[
\int_{-a}^{a} F(u) \frac{1}{a^2} (a - |u|) du = \frac{2}{a} \int_{0}^{a} F(u) du - \frac{2}{a^2} \int_{0}^{a} u F(u) du
\]

\[
= \frac{2}{a} \int_{0}^{a} ((1 + o(1)) T^{-2u} \log T + u + o(1)) du
\]

\[
- \frac{2}{a^2} \int_{0}^{a} u((1 + o(1)) T^{-2u} \log T + u + o(1)) du
\]

\[
\sim \frac{1}{a} + a - \frac{2a}{3} = \frac{1}{a} + \frac{a}{3}
\]

as $T \to \infty$. Therefore we obtain the desired result

\[
\sum_{\gamma, \gamma' \in [0,T]} \left( \frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma') \sim \left( \frac{1}{a} + \frac{a}{3} \right) \frac{T}{2\pi} \log T.
\]

Now we can use Corollary 2.8 to prove Corollary 2.9. Corollary 2.9 reveals that if Riemann Hypothesis is true, more than two thirds of the zeros of $\zeta(s)$ are simple. It is conjectured that all zeros of $\zeta(s)$ are simple.

To prove Corollary 2.9, we first observe that

\[
\sum_{\gamma, \gamma' \in [0,T]} \frac{1}{\gamma - \gamma'} = \sum_{\gamma \in [0,T]} m_{\rho}
\]

where $m_{\rho}$ denotes the multiplicity of $\rho$. Let's first look at the left hand side of (2.19). Suppose $\rho$ is of multiplicity $n$, i.e., there are $n$ zeros at the same height. These $n$ zeros can be formed into $n^2$ pairs satisfying the condition $\gamma = \gamma'$. So these zeros are counted $n^2$ times in the left hand side of (2.19). On the right hand side, for the same height with multiplicity $n$, each zero contributes $n$ times to the sum; there are $n$ such zeros, so altogether $n^2$ zeros are counted into the sum at this height. Therefore the equation (2.19) holds.

Next we claim that

\[
\sum_{\gamma, \gamma' \in [0,T]} \frac{1}{\gamma - \gamma'} = \sum_{\gamma \in [0,T]} m_{\rho} \left( \frac{\sin((a/2)(\gamma - \gamma') \log T)}{a/2(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma').
\]

This is because when $\gamma = \gamma'$, by definition of $\omega(u) = 4/(4 + u^2)$, we have $\omega(\gamma - \gamma') = \omega(0) = 1$. Also we use the limit

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

Therefore when $\gamma = \gamma'$ we have

\[
\left( \frac{\sin((a/2)(\gamma - \gamma') \log T)}{a/2(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma') \sim 1.
\]
Now we can establish the following inequality
\[
\sum_{\gamma, \gamma' \in [0, T], \gamma = \gamma'} 1 = \sum_{\gamma \in [0, T]} m_\rho \leq \sum_{\gamma, \gamma' \in [0, T]} \left( \frac{\sin(a/2(\gamma - \gamma')) \log T}{a/2(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma'),
\]
since on the right hand side more non-negative terms are added. Now by applying the second statement of Corollary 2.8,
\[
\sum_{\gamma, \gamma' \in [0, T]} \left( \frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 \omega(\gamma - \gamma') \sim \left( \frac{1}{a} + \frac{a}{3} \right) \frac{T}{2\pi} \log T.
\]
Let \( a = 1 - \epsilon \) for small enough \( \epsilon \). Then we have
\[
\sum_{\gamma \in [0, T]} m_\rho \leq \left( \frac{1}{1-\epsilon} + \frac{1-\epsilon}{3} \right) \frac{T}{2\pi} \log T = \left( \frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T.
\]
Now we introduce the summation
\[
\sum_{\gamma \in [0, T]} (2 - m_\rho).
\]
In this summation simple zeros are counted as 1, zeros with multiplicity 2 are counted as 0 and zeros with multiplicity more than 2 will give a negative number to the sum. So we have
\[
\sum_{\gamma \in [0, T]} 1 \geq \sum_{\rho \text{ simple}} (2 - m_\rho) = 2 \sum_{\gamma \in [0, T]} 1 - \sum_{\gamma \in [0, T]} m_\rho.
\]
Notice that \( \sum_{\gamma \in [0, T]} 1 \) is the counting function of the zeros of \( \zeta(s) \) and it is
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]
Hence we have
\[
\sum_{\gamma \in [0, T]} m_\rho \leq \left( \frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T,
\]
and obtain the desired result
\[
\sum_{\gamma \in [0, T], \rho \text{ simple}} 1 \geq 2 \sum_{\gamma \in [0, T]} 1 - \sum_{\gamma \in [0, T]} m_\rho
\]
\[
= 2 \left( \frac{T}{2\pi} \log T \right) - \left( \frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T
\]
\[
= \left( \frac{2}{3} + o(1) \right) \frac{T}{2\pi} \log T.
\]
as \( T \to \infty \). This completes the proof of Corollary 2.9.

Corollary 2.9 tells us that out of all the zeros that lie on the critical line, more than two thirds of them are simple. Later this result has been improved by Conrey, Ghosh and Gonek [6] to
\[
\sum_{\gamma \in [0, T], \rho \text{ simple}} 1 \geq \left( \frac{19}{27} - \epsilon \right) N(T)
\]
using the Generalized Lindelöf Hypothesis and Riemann Hypothesis.
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2.1.1 Montgomery’s Pair Correlation Conjecture

Now we will go through the arguments Montgomery used to make his celebrated pair correlation conjecture on zeros of the Riemann Zeta function. Let us go back to the main theorem first. It states that

\[ F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1) \]

uniformly for fixed \(0 < \alpha < 1 - \epsilon\). What happens if \(\alpha \geq 1\)? Montgomery gave the following conjecture.

Conjecture 2.11. Assume the Riemann Hypothesis. We have

\[ F(\alpha) = 1 + o(1) \]

for \(\alpha \geq 1\), uniformly in bounded intervals, as \(T \to \infty\).

Recall when we are proving Montgomery’s main theorem, we have

\[ R(x, t) = -x^{-1/2} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1/2+it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2+it} \right) + x^{-1+it} \log x + O(x^{-1}) + O(x^{1/2}x^{-1}). \] (2.20)

However, in order to estimate the integral

\[ \int_0^T |R(T^\alpha, t)|^2 dt \]

for \(\alpha > 1\), we need a more precise estimate than (2.20).

In view of (2.7), we know that term \(O(x^{1/2}x^{-1})\) in (2.20) in fact is the term

\[ -\frac{2x^{1/2}}{(1/2 + it)(3/2 - it)}. \]

Then we now consider a new term \(T_0\) given by

\[ T_0 = \int_0^T \left| x^{-1/2} \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1/2+it} + x^{-1/2} \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2+it} \right|^2 \frac{2x^{1/2}}{(1/2 + it)(3/2 - it)} \frac{2x^{1/2}}{(1/2 + it)(3/2 - it)} dt \]

\[ = \int_0^T \left| \frac{1}{x} \sum_{n \leq x} \Lambda(n) \left( \frac{1}{n} \right)^{-1/2+it} + x \sum_{n > x} \Lambda(n) \left( \frac{1}{n} \right)^{3/2+it} \right|^2 \frac{2x^{1/2}}{(1/2 + it)(3/2 - it)} \frac{2x^{1/2}}{(1/2 + it)(3/2 - it)} dt. \]

Montgomery used the 2-tuple conjecture from Hardy and Littlewood [11] to bound this integral by

\[ T_0 \ll T \log T \]

where \(x = T^\alpha\) for all \(\alpha \geq 1\). Similar as before, we get

\[ \int_0^T |R(x, t)|^2 dt = (1 + o(1))T \log T. \]
Then we use the equation
\[ \int_0^T |R(T^\alpha, t)|^2 dt = \int_0^T |L(T^\alpha, t)|^2 dt \]
to obtain the conjecture

\[ F(\alpha) T \log T + O(\log^3 T) = T \log T (1 + o(1)) \]
\[ F(\alpha) = 1 + o(1) \]
when \( T \to \infty \) for \( \alpha \geq 1 \).

Now by assuming Hardy and Littlewood 2-tuple conjecture and the Riemann Hypothesis, we then have \( F(\alpha) \) for any \( \alpha \) as following

\[ F(\alpha) = \begin{cases} 
(1 + o(1))T^{-2|\alpha|} \log T + |\alpha| + o(1) & \text{for } |\alpha| < 1 \\
1 + o(1) & \text{for } |\alpha| \geq 1 
\end{cases} \]
as \( T \to \infty \). Next we derive another form for \( F(\alpha) \) when \( \alpha \leq 1 \).

Let us first recall the definition of Dirac delta function.

**Definition 2.12.** Function \( \delta(t) \) is Dirac's delta function satisfying the following property

\[ \delta(t) = \begin{cases} 
0 & \text{for } t \neq 0; \\
\infty & \text{for } t = 0; 
\end{cases} \]

with

\[ \int_{t_1}^{t_2} \delta(t) dt = \begin{cases} 
1 & \text{for } 0 \in [t_1, t_2]; \\
0 & \text{otherwise}. 
\end{cases} \]

From the definition we see that \( \delta(t) \) has an infinite peak at \( t = 0 \) with the total area of unity. This function could be also viewed as a limit of a Gaussian

\[ \delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-t^2/2\sigma^2}. \]

This function has an important property as follows

\[ \int_{t_1}^{t_2} f(t) \delta(t) dt = f(0) \quad (2.21) \]

for any function \( f(t) \). This is easy to see since \( \delta(t) \) is zero anywhere except \( t = 0 \) and \( f(0) \) is a constant, so one can pull it out from the integral, then the equation (2.21) follows from the definition.

Now we claim that in the main theorem

\[ F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1), \]
the function \( T^{-2\alpha} \log T \) behaves like the Dirac delta function \( \delta(a) \) as \( T \to \infty \). Since

\[
\lim_{T \to \infty} T^{-2|\alpha|} \log T = 0
\]

when \( \alpha \to 0 \), we only need to check that

\[
\lim_{T \to \infty} \int_{t_1}^{t_2} T^{-2|\alpha|} \log T d\alpha = 1 \tag{2.22}
\]

for \( 0 \in [t_1, t_2] \). Suppose \( f(\alpha) \) is a smooth function in \([-1, 1]\) that all its derivatives exist. We consider the integral

\[
\int_{-1}^{1} T^{-2|\alpha|} (\log T) f(\alpha) d\alpha.
\]

Since \( f(\alpha) \) has Taylor's expansion at \( \alpha = 0 \) as

\[
f(\alpha) = f(0) + \alpha f'(0) + \frac{1}{2} \alpha^2 f''(0) + \cdots,
\]

we have

\[
\int_{-1}^{1} T^{-2|\alpha|} (\log T) f(\alpha) d\alpha = \int_{-1}^{1} T^{-2|\alpha|} (\log T) f(0) d\alpha = f(0)
\]

by (2.22) and the fact that

\[
T^{-2|\alpha|} \log T \text{ does behave like the Dirac delta function } \delta_0 \text{ as } T \to \infty.
\]

Therefore, in the limit, \( F \) behaves as

\[
F(\alpha) = |\alpha| + \delta_0(\alpha), \quad \text{for } |\alpha| < 1.
\]

Recall that \( r_2(\alpha) = (1 - |\alpha|) \chi_1(\alpha) \) with \( a = 1 \). So

\[
F(\alpha) = (1 - r_2(\alpha)) + \delta_0(\alpha), \quad \text{for } |\alpha| < 1.
\]

Since \( \delta_0 = 1 \),

\[
\hat{F} = \frac{1 - r_2}{1 - \frac{\sin \pi u}{\pi u}} + 1.
\]
By the convolution formula, we have
\[ \sum_{\gamma, \gamma' \in [0, T]} r_2 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(\alpha) r_2(\alpha) d\alpha \]
\[ = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} \hat{F}(\alpha) r_2(\alpha) d\alpha \]
\[ = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 + \delta_0 \right) r_2(\alpha) d\alpha \tag{2.23} \]

as \( T \to \infty \). The summation in the left hand side of (2.23) can be rewritten as
\[ = r_2(0) \sum_{\gamma, \gamma' \in [0, T]} 1 + \sum_{\gamma, \gamma' \in [0, T]} r_2 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') \]
\[ \sim r_2(0) \frac{T}{2\pi} \log T + \sum_{\gamma, \gamma' \in [0, T]} r_2 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') \]
because of
\[ \sum_{\gamma, \gamma' \in [0, T]} 1 = N(T) \sim \frac{T}{2\pi} \log T. \]

We then conclude that
\[ \sum_{\gamma, \gamma' \in [0, T]} r_2 \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 + \delta_0 \right) r_2(\alpha) d\alpha \tag{2.24} \]

since \( \int_{-\infty}^{\infty} \delta_0 r_2 d\alpha = 1 \).

Finally, if we replace \( r_2(\alpha) \) in (2.23) by
\[ r(\alpha) = \chi_{[a, b]}(\alpha), \]
using a similar argument, we get
\[ \sum_{\gamma, \gamma' \in [0, T]} r \left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 + \delta_0 \right) r(\alpha) d\alpha \]
which is
\[ \sum_{\gamma, \gamma' \in [0, T]} \omega(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{a}^{b} \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 + \delta_0 \right) d\alpha. \]

In order to make \( \gamma \) and \( \gamma' \) satisfy the condition
\[ a \leq (\gamma - \gamma') \frac{\log T}{2\pi} \leq b \]
as \( T \to \infty \) for fixed \( a < b \), the difference \( \gamma - \gamma' \) must tend to 0. Hence as \( T \to \infty \), the term \( \omega(\gamma - \gamma') \) tends to \( \omega(0) = 1 \). Therefore we obtain

\[
\sum_{\gamma, \gamma' \in [0, T], a \leq (\gamma - \gamma') \frac{\log T}{2\pi} \leq b} 1 \sim \frac{T}{2\pi} \log T \int_a^b \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 + \delta_0 \right) d\alpha
\]

as \( T \to \infty \). If we assume \( a > 0 \), then the interval does not contain 0 so that \( J_a^b \delta_0 = 0 \). Hence we have

\[
\sum_{\gamma, \gamma' \in [0, T], a \leq (\gamma - \gamma') \frac{\log T}{2\pi} \leq b} 1 \sim \frac{T}{2\pi} \log T \int_a^b \left( 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right) d\alpha
\]

as \( T \to \infty \). Thus we come up to the final form for Montgomery's celebrated conjecture.

**Conjecture 2.13 (Montgomery Pair Correlation Conjecture).** Assume the Riemann Hypothesis. For fixed \( 0 < a < b < \infty \), as \( T \to \infty \),

\[
\sum_{\gamma, \gamma' \in [0, T], a \leq (\gamma - \gamma') \frac{\log T}{2\pi} \leq b} 1 \sim \frac{T}{2\pi} \log T \int_a^b 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 du.
\]

### 2.2 The Pair Correlation of Eigenvalues of The Gaussian Unitary Ensemble

#### 2.2.1 Random Matrix and The Gaussian Unitary Ensemble

After Heisenberg's discovery of the uncertainty principle which originates in quantum mechanics, classic mathematical analysis methods invented by Newton that deal with larger objects in the laboratory are not suitable any more for a world consisting of tiny atoms, particles and waves. According to Heisenberg's theory, uncertainties and imprecision always turned up if one tried to measure the position and the momentum of a particle simultaneously. It also happened when one tried to find the energy and the time variables of a particle at the same time. This is not the fault of the measure methods nor it is caused by the person who conducts the measurement. It is the nature of quantum mechanics. Physicists thus need new mathematical tools to help them describe and study these new phenomena that arise from quantum mechanics. Theories of probability and statistics thus play a more important role in the research of quantum mechanics. Random matrix theory is an example.

Physicists use random matrices, especially high dimensional random matrices, to describe the energy levels of a physical system in quantum mechanics. The entries of random matrices are taken randomly from probabilistic distributions. This is why they are called "random matrices". If the probabilistic distribution from which the entries are chosen is set to be a Gaussian distribution, and
its entries are symmetric to the diagonal line which reflects the symmetry properties of the physical
system, this type of random matrix is called a Gaussian invariant ensemble. Here ensemble means a

Here ensemble means a

ensemble means a collection of matrices with a probability distribution attached to provide a measurement for it. There

collection of matrices with a probability distribution attached to provide a measurement for it. There

are several types of Gaussian invariant ensembles; the one we will examine in detail is known as the

gaussian

Gaussian Unitary Ensemble (GUE), denoted as $E_{2G}$. The other two main types are the Gaussian

Gaussian Symplectic Ensemble (GSE) and the Gaussian Orthogonal Ensemble (GOE). Matrices of the GUE
type can be used to describe physical systems which are not invariant under time-reversal. These
two objects are connected by direct correspondence between eigenvalues of the matrix and energy

level of energy levels of the system. Because of this correspondence, energy levels of a system can be referred to

as eigenvalues of the random matrix. There are some good textbooks about random matrix theory,
such as [15]. People can refer to them for more details.

Before we introduce the definition of GUE, we first recall that the Hermitian matrix is an $N \times N$

matrix with complex entries so that the matrix is equal to its own conjugate transpose. That is, if

$A = (a_{ij})$ is Hermitian, then $a_{ij} = \overline{a_{ji}}$ for $1 \leq i \leq j \leq N$, or

$$A^* = A$$

where $X^*$ is the conjugate transpose of $A$. For example,

$$
\begin{pmatrix}
1 & 1 + i \\
1 - i & 3
\end{pmatrix}
$$
is a Hermitian matrix. Clearly the diagonal entries of Hermitian matrices are real. One of the most

important properties of Hermitian matrices is that all their eigenvalues are all real and Hermitian

matrices are diagonalizable by unitary matrices.

Now we consider a $N \times N$ Hermitian matrix $H = (a_{ij})$ so that $a_{jj}$ are real and $a_{ij} = \overline{a_{ji}}$

and regard these entries $a_{jj}, 1 \leq j \leq N$ and $\Re(a_{ij}), \Im(a_{ij}), 1 \leq i < j \leq N$ as independent random

variables. We give the definition for the Gaussian Unitary Ensemble (GUE) as follows.

**Definition 2.14.** The Gaussian Unitary Ensemble is defined to be the set of $N \times N$ Hermitian

matrices $H = (a_{ij})$ such that

- the diagonal elements $a_{jj}$ and the real and the imaginary parts of the off-diagonal elements

  $a_{ij}$ for $1 \leq i < j \leq N$ are independent random variables. Therefore the distribution function

  $P(H)$ of $H$ is the product of the distribution functions of each random variable as

  $$P(H) = \prod_{1 \leq i \leq N} f_i(\Re(a_{ii})) \prod_{1 \leq i < j \leq N} g_{ij}(\Im(a_{ij}))$$

  where $f_i$, $g_{ij}$ are the distribution functions for $\Re(a_{ii})$ and $\Im(a_{ij})$.

- $P(H)dH = P(H')dH'$ if $H' = U^{-1}HU$ for any unitary matrix $U$. 
We know from the definition that all the elements of a matrix of the GUE type are random and independently distributed and the distributions of these elements are invariant under unitary transformations.

The joint probability density function of the (real) eigenvalues of an $N \times N$ matrix in the GUE is

$$P_N(x_1, \ldots, x_N) = C_N \cdot \exp\left(-\sum_{j=1}^{N} x_j^2\right) \prod_{1 \leq i < j \leq N} |x_j - x_i|^2.$$  

Here the constant $C_N$ can be shown (e.g. in [21]) to be

$$C_N^{-1} = 2^{-\frac{N^2}{4}(N-1)}\pi^\frac{N^2}{2} \prod_{j=1}^{N} j!$$  

so that $P_N$ satisfies

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \ldots, x_N) dx_1 \cdots dx_N = 1.$$  

The $n$-point correlation functions ($n = 1, \ldots, N$) of the GUE is defined by

$$R_n(x_1, \ldots, x_n) := \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N.$$  

The probabilistic meaning of this function is as follows. Let $[x_1, x_1 + dx_1], \ldots, [x_n, x_n + dx_n]$ be $n$-infinitesimally short disjoint intervals. Then the probability that there are eigenvalues inside each of these intervals is

$$R_n(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$  

We are interested in the 2-point (pair) correlation function $R_2(x_1, x_2)$ for an $N \times N$ matrix. We will prove the following theorem.

**Theorem 2.15.** The pair correlation function for the Gaussian Unitary Ensemble is

$$\frac{1}{\alpha_1 \alpha_2} R_2(x_1, x_2) \sim 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$$

as $N \to \infty$ where $u = |x_1/\alpha_1 - x_2/\alpha_2|$ and $\alpha_j = \frac{\pi}{\sqrt{2N-x_j^2}}$ is the mean local spacing of the eigenvalues at $x_j, j = 1, 2$.

Notice that this function is the same as the distribution function that appeared in Montgomery’s pair correlation conjecture of zeros of the Riemann Zeta function. We first need the very important oscillator wave functions. We give their definition and important properties in the following section.
2.2.2 Oscillator Wave Function

We recall here the Hermite polynomials. For $n \geq 0$, the Hermite polynomial, $H_n(x)$, is defined by

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$ 

It is well-known that ([21] or [10])

$$H_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}, \quad n \geq 0 \tag{2.25}$$

and

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \omega^n = e^{2x\omega - \omega^2}$$

and for $n \geq 1$,

$$H_n'(x) = 2nH_{n-1}(x),$$
$$H_{n+1}(x) = 2xH_n(x) - H_n'(x).$$

For $n \geq 0$, we also define the Hermite function, $h_n(x)$, by

$$h_n(x) := H_n(x) e^{-1/2x^2}.$$ 

The set $\{h_n(x)\}_{n=0}^{\infty}$ forms an orthogonal family in $L^2(\mathbb{R}, dx)$ and it can be shown that for every $n \geq 1$,

$$\left(\frac{d}{dx} + x\right)h_n(x) = 2nh_{n-1}(x),$$
$$\left(\frac{d}{dx} - x\right)h_n(x) = h_{n+1}(x),$$

and

$$2(n - m)h_n(x)h_m(x) = h_n(x)h''_m(x) - h_m(x)h''_n(x). \tag{2.26}$$

**Definition 2.16.** For each $n \geq 0$, the Oscillator Wave Function, $\phi_n(x)$, are Hermite functions with a normalization coefficient $(\sqrt{\pi}/2n!)^{-1/2}$, i.e.,

$$\phi_n(x) := (-1)^n(\sqrt{\pi}/2n!)^{-1/2} e^{1/2x^2} \frac{d^n}{dx^n} e^{-x^2} = (\sqrt{\pi}/2n!)^{-1/2} h_n(x).$$

**Lemma 2.17** (Mehler's Formula). For $|\omega| < 1$,

$$\sum_{n=0}^{\infty} \frac{h_n(x)h_n(y)}{2^n n!} \omega^n = (1 - \omega^2)^{-1/2} e^{-\frac{1}{2}(\xi^2 + \eta^2)} e^{\frac{1}{4}(\frac{1}{\xi^2} + \frac{1}{\eta^2})(x^2 + y^2) + (\frac{1}{\xi^2} + \frac{1}{\eta^2})xy}.$$
CHAPTER 2. THE PAIR CORRELATION OF THE ZEROS OF THE Riemann Zeta Function

Proof. Since

$$h_n(x) = H_n(x) e^{-x/2} = (-1)^n e^{1/2x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and we know from Fourier analysis that

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} e^{2iux} du,$$

we have

$$h_n(x) = (-1)^n e^{1/2x^2} \frac{d^n}{dx^n} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} e^{2iux} du$$

$$= (-1)^n e^{1/2x^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} e^{-u^2} e^{2iux} du$$

$$= \frac{(-1)^n e^{1/2z^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{u^2} (2iu)^n e^{2iux} du$$

$$= \frac{(-2i)^n e^{1/2z^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^n e^{-u^2 + 2iux} du.$$

Therefore we have

$$\sum_{n=0}^{\infty} \frac{h_n(x)h_n(y)}{2^nn!} \omega^n$$

$$= \frac{1}{\pi} e^{1/2(z^2+y^2)} \sum_{n=0}^{\infty} \left( \int_{-\infty}^{\infty} u^n e^{-u^2 + 2iux} du \right) \left( \int_{-\infty}^{\infty} v^n e^{-v^2 + 2iyv} dv \right) \frac{(-2)^n}{n!} \omega^n$$

$$= \frac{1}{\pi} e^{1/2(z^2+y^2)} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(-2uv\omega)^n}{n!} e^{-u^2 + 2iux - v^2 + 2iyv} dudv$$

$$= \frac{1}{\pi} e^{1/2(z^2+y^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(-2uv\omega)^n}{n!} e^{-u^2 + 2iux - v^2 + 2iyv} dudv.$$

Since

$$\sum_{n=0}^{\infty} \frac{(-2uv\omega)^n}{n!} = e^{-2uv\omega}$$

and the Fourier inverse is

$$\int_{-\infty}^{\infty} e^{-v^2-2uv\omega} e^{2iyv} dv = \sqrt{\pi} e^{-(y+i\omega)^2},$$

we have

$$\sum_{n=0}^{\infty} \frac{h_n(x)h_n(y)}{2^nn!} \omega^n$$

$$= \frac{1}{\pi} e^{1/2(z^2+y^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 + 2iux - v^2 + 2iyv} dudv$$

$$= \frac{1}{\pi} e^{1/2(z^2+y^2)} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} e^{-u^2 + 2iyv - 2uv\omega} dv \right] e^{-u^2 + 2iux} du$$

$$= \frac{1}{\sqrt{\pi}} e^{1/2(z^2-y^2)} \int_{-\infty}^{\infty} e^{-(1-\omega^2)u^2 - 2iyu\omega + 2iux} du$$

$$= (1-\omega^2)^{-1/2} e^{-\frac{1}{2} \left( \frac{1+2\omega^2}{1-\omega^2} \right)(z^2+y^2)} + (1-\omega^2)xy.$$
This proves Mehler's Formula.

**Proposition 2.18.** The family \( \{ \phi_n \}_{n=0}^{\infty} \) is an orthonormal set in \( L^2(\mathbb{R}, dx) \)

**Proof.** Since \( \{ \phi_n \}_{n=0}^{\infty} \) is in family of \( \{ h_n \}_{n=0}^{\infty} \), it is orthogonal, we only need to show

\[
\int_{-\infty}^{\infty} (\phi_n(x))^2 dx = 1.
\]

From Mehler's formula we let \( x = y \) to have

\[
\sum_{n=0}^{\infty} \frac{(h_n(x))^2}{2^n n!} \omega^n = (1 - \omega^2)^{-1/2} e^{-\frac{1}{1+\omega^2} x^2}.
\]

Integrating both sides with respect to \( x \) and \( |\omega| < 1 \) we have

\[
\sum_{n=0}^{\infty} \left( \int_{-\infty}^{\infty} (h_n(x))^2 dx \right) \frac{\omega^n}{2^n n!} = (1 - \omega^2)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{1+\omega^2} x^2} dx.
\]

Since by definition

\[
\phi_n(x) = (\sqrt{\pi} 2^n n!)^{-1/2} h_n(x),
\]

we have

\[
h_n(x) = (\sqrt{\pi} 2^n n!)^{1/2} \phi_n(x).
\]

Thus we get

\[
\sum_{n=0}^{\infty} \left( \int_{-\infty}^{\infty} (\phi_n(x))^2 dx \right) \omega^n = \frac{1}{\sqrt{\pi}} (1 - \omega^2)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{1+\omega^2} x^2} dx
\]

\[
= \frac{1}{1 - \omega}
\]

\[
= \sum_{n=0}^{\infty} \omega^n.
\]

So we obtain the desired result

\[
\int_{-\infty}^{\infty} (\phi_n(x))^2 dx = 1
\]

by comparing the coefficient of the function.

**Corollary 2.19.** For \( |\omega| < 1 \),

\[
\int_{-\infty}^{\infty} (h_n(x))^2 dx = \int_{-\infty}^{\infty} (H_n(x))^2 e^{-x^2} dx = \sqrt{\pi} 2^n n!.
\]
Proposition 2.20 (Christoffel-Darboux). Let \( \{p_n(x)\}_{n=1}^{\infty} \) be an orthonormal family with respect to a weighting function \( \omega(x) \) on \((a, b)\) so that

\[
(p_n, p_m) := \int_a^b p_n(x)p_m(x)\omega(x)dx = \begin{cases} 
0 & \text{if } n \neq m, \\
1 & \text{if } m = n.
\end{cases}
\]

Let \( a_n \) be the coefficient of \( x^n \) in \( p_n(x) \) and \( c_n = (p_n, p_n) \). Then we have

\[
\sum_{n=0}^{N} c_n^{-1} p_n(x)p_n(y) = \frac{a_N}{a_{N+1} c_N} \frac{p_{N+1}(x)p_N(y) - p_N(x)p_{N+1}(y)}{x - y}
\]

and

\[
\sum_{n=0}^{N} c_n^{-1} (p_n(x))^2 = \frac{a_N}{a_{N+1} c_N} \left( p_{N+1}'(x)p_N(x) - p_N'(x)p_{N+1}(x) \right).
\]

Proof. This is a classical result in orthogonal polynomials. See the proof in §3.2 in [21].

Now we will use the Christoffel-Darboux formulae to derive some useful formulae for Oscillator Wave function. Our goal is to find the invariants for the Hermite polynomials. Then we can find explicit formulae for Oscillator Wave functions.

Recall that by definition Oscillator Wave functions are just Hermite functions \( h_n \) with a normalization constant \( (\sqrt{\pi}2^n n!)^{-1/2} \) and Hermite functions are just Hermite polynomials multiplied by \( e^{-1/2x^2} \) and orthogonal with respect to the inner product

\[
(H_n, H_m) = \int_a^b H_n(x)H_m(x)e^{-x^2}dx.
\]

Then we can use the previous properties we proved for Hermite polynomials to find the coefficients that exists in the Christoffel-Darboux formulae in order to obtain formulae for Oscillator Wave functions.

For \( c_n = (H_n, H_m) \), from Corollary 2.19, we know

\[
c_n = (H_n, H_m) = \int_{-\infty}^{\infty} (H_n(x))^2 e^{-x^2}dx = \sqrt{\pi}2^n n!
\]

and the leading coefficient \( a_n \) of the \( n \)-th Hermite polynomial is \( 2^n \) from (2.25). Applying these values into the Christoffel-Darboux formulae for Hermite polynomials we have

\[
\sum_{n=0}^{N} (\sqrt{\pi}2^n n!)^{-1} H_n(x)H_n(y) = \frac{2^N}{2^{N+1}(\sqrt{\pi}2^n n!)} \frac{H_{N+1}(x)H_N(y) - H_N(x)H_{N+1}(y)}{x - y}.
\]

Multiplying both sides by \( e^{-1/2x^2}e^{-1/2y^2} \) and re-organizing the coefficients we have

\[
\sum_{n=0}^{N} (\sqrt{\pi}2^n n!)^{-1} H_n(x)e^{-1/2x^2}H_n(y)e^{-1/2y^2} = \frac{2^N e^{-1/2x^2}e^{-1/2y^2} H_{N+1}(x)H_N(y) - H_N(x)H_{N+1}(y)}{2^{N+1}(\sqrt{\pi}2^n n!)}.\]
Since
\[ \phi_n(x) = (-1)^n (\sqrt{\pi} 2^n n!)^{-1/2} e^{1/2 x^2} \frac{d^n}{dx^n} e^{-x^2} = (\sqrt{\pi} 2^n n!)^{-1/2} e^{-1/2 x^2} H_n(x), \]
we have the following result.

**Proposition 2.21.** We have
\[ \sum_{n=0}^{N-1} \phi_n(x) \phi_n(y) = \left( \frac{N}{2} \right)^{1/2} \phi_{N-1}(x) \phi_N(y) - \phi_N(y) \phi_{N-1}(x) \]
and
\[ \sum_{n=0}^{N-1} (\phi_n(x))^2 = N(\phi_N(x))^2 - (N(N+1))^{1/2} \phi_{N-1}(x) \phi_{N+1}(x). \]

**Proposition 2.22.** We have
\begin{enumerate}
  \item[(i)] \((-1)^m m^{1/4} \phi_{2m}(x) \sim \frac{1}{\sqrt{\pi}} \cos(2m^{1/2} x)\)
  \item[(ii)] \((-1)^m m^{1/4} \phi_{2m+1}(x) \sim \frac{1}{\sqrt{\pi}} \sin(2m^{1/2} x)\)
\end{enumerate}
as \(m \to \infty\).

**Proof.** This follows from the corresponding formula of the Hermite polynomials which are given in [10]
\[ \lim_{m \to \infty} \frac{(-1)^m m^{1/2}}{2^{2m} m!} H_{2m} \left( \frac{x}{2m^{1/2}} \right) = \frac{1}{\sqrt{\pi}} \cos(x) \]
and
\[ \lim_{m \to \infty} \frac{(-1)^m}{2^{2m} m!} H_{2m+1} \left( \frac{x}{2m^{1/2}} \right) = \frac{2}{\sqrt{\pi}} \sin(x). \]

\[\square\]

### 2.2.3 Pair Correlation Function for the GUE

We now have all the tools in hand and we are ready to derive the pair correlation function for the GUE.

**Proposition 2.23.** Let \(x_1, x_2, \cdots, x_N\) be \(N\) real variables and \(f: \mathbb{R}^2 \to \mathbb{R}\) be a real-valued function on \(\mathbb{R}^2\). Suppose \(J_N(x_1, \cdots, x_N) = J_N = (f(x_i, x_j))_{1 \leq i, j \leq N}\) is a \(N \times N\) matrix satisfying
\begin{enumerate}
  \item[(i)] \(\int_{-\infty}^{\infty} f(x, x) dx = A,\)
  \item[(ii)] \(\int_{-\infty}^{\infty} f(x, y) f(y, z) dy = f(x, z).\)
\end{enumerate}
Then
\[ \int_{-\infty}^{\infty} (\det J_N)\, dx_N = (A - N + 1) \det(J_{N-1}) \]

where \( J_{N-1} \) is the \((N - 1) \times (N - 1)\) matrix obtained from \( J_N \) by removing the \( N\)th row and \( N\)th column.

**Proof.** Let \((J_N)_{i,j}\) be the \((N - 1) \times (N - 1)\) matrix obtained from \( J_N \) by removing the \( i\)th row and \( j\)th column. Then we have

\[ \det(J_N) = (-1)^{N-1} \{ f(x_N, x_1) \det(J_N)_{(N,1)} - f(x_N, x_2) \det(J_N)_{(N,2)} + \cdots + (-1)^{N-1} f(x_N, x_N) \det(J_N)_{(N,N)} \}. \]

Note that only the last column of \((J_N)_{(N,1)}, (J_N)_{(N,2)} \cdots (J_N)_{(N,N-1)}\) involves the variable \( x_N \) and they are all equal to

\[ \begin{pmatrix} f(x_1, x_N) \\ f(x_2, x_N) \\ \vdots \\ f(x_{N-1}, x_N) \end{pmatrix}. \]

So using (ii), we have

\[ \int_{-\infty}^{\infty} f(x_N, x_1) \det(J_N)_{(N,1)} dx_N = \det \begin{pmatrix} f(x_1, x_2) & \cdots & f(x_1, x_N) & f(x_1, x_1) \\ f(x_2, x_2) & \cdots & f(x_2, x_N) & f(x_2, x_1) \\ \vdots & \cdots & \vdots & \vdots \\ f(x_{N-1}, x_2) & \cdots & f(x_{N-1}, x_N) & f(x_{N-1}, x_1) \end{pmatrix} = (-1)^{N-2} \det(J_N)_{(N,N)} = (-1)^{N-2} \det(J_{N-1}). \]

Similarly, we have

\[ \int_{-\infty}^{\infty} f(x_N, x_j) \det(J_N)_{(N,j)} dx_N = (-1)^{N-j-1} \det(J_{N-1}). \]

Therefore, we have

\[ \int_{-\infty}^{\infty} \det(J_N) dx_N = - \det(J_{N-1}) - \cdots - \det(J_{N-1}) + \int_{-\infty}^{\infty} f(x_N, x_N) \det(J_{N-1}) dx_N = (A - N + 1) \det(J_{N-1}). \]

Now we apply this proposition to \( P_N(x_1, x_2, \ldots, x_N) \). Let \( M = M(x_1, x_2, \ldots, x_N) = (M_{ij}) \) be the \( N \times N \) matrix such that \( M_{ij} = \phi_{i-1}(x_j) \) for \( 1 \leq i, j \leq N \) where \( \phi_n(x) \) is the oscillator wave function. We have the following
Proposition 2.24.

\[ P_N(x_1, x_2, \ldots, x_N) = \frac{1}{N!} \det(M)^2. \]

Proof. We first have the Vandermonde determinant

\[ \Delta(\vec{x}) = \Delta(x_1, x_2, \ldots, x_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ x_1^2 & x_2^2 & \cdots & x_N^2 \\ \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{vmatrix} = \prod_{1 \leq i < j \leq N} (x_j - x_i). \quad (2.27) \]

Recall that for each \( n \geq 0 \)

\[ H_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}. \]

As \( H_n(x) \) is just a polynomial of \( x \), multiplying the appropriate coefficients to each the \( i \)th rows with \( i < j \) and adding them to the \( j \)th row, we can replace the \( j \)th row of the determinant of (2.27) by the following row in terms of Hermite polynomials but without changing the right hand side of (2.27)

\[ \left( H_{j-1}(x_1), H_{j-1}(x_2), \ldots, H_{j-1}(x_N) \right). \]

We next transform the resulting determinant matrix in terms of the oscillator wave function \( \phi_{j-1}(x) \) by multiplying each \( k \)th column by \( e^{-x_k^2/2} \) and multiplying each \( j \)th row by the factor \((\sqrt{2^{j-1}(j-1)!})^{1/2}\). We finally get

\[ \exp \left( -\frac{1}{2} \sum_{k=1}^{N} x_k^2 \right) \Delta(x) = c \cdot \det M \]

for some constant \( c \). So by the definition of \( P_N(x_1, x_2, \ldots, x_N) \), we have

\[ P_N(x_1, \ldots, x_n) = C_N \cdot \exp \left( -\sum_{j=1}^{N} x_j^2 \right) \prod_{1 \leq i < j \leq N} |x_j - x_i|^2 = c' (\det M)^2. \quad (2.28) \]

It remains to calculate the coefficient \( c' \) explicitly. We define \( K_N(x, y) \) for each fixed \( N \geq 1 \) by

\[ K_N(x, y) := \sum_{n=0}^{N-1} \phi_n(x) \phi_n(y) \]

and an \( N \times N \) matrix \( K_N \) by

\[ K_N := \left( K_N(x_i, x_j) \right)_{1 \leq i, j \leq N}. \]
So

\[ K_N = M^T M. \]

Now let us verify that \( K_N \) satisfies the conditions of Proposition 2.23. In view of the orthonormality of the family \( \{ \phi_n \} \), we have

\[
\int_{-\infty}^{\infty} K_N(x, x) \, dx = \int_{-\infty}^{\infty} \sum_{j=0}^{N-1} (\phi_j(x))^2 \, dx = \sum_{j=0}^{N-1} \int_{-\infty}^{\infty} (\phi_j(x))^2 \, dx = N
\]

and

\[
\int_{-\infty}^{\infty} K_N(x, y) K_N(y, z) \, dy = \int_{-\infty}^{\infty} \left( \sum_{i=0}^{N-1} \phi_i(x) \phi_i(y) \left( \sum_{j=0}^{N-1} \phi_j(y) \phi_j(z) \right) \right) \, dy
\]

\[
= \sum_{i=0}^{N-1} \int_{-\infty}^{\infty} \phi_i(x) \phi_i(y) \left( \phi_j(y) \phi_j(z) \right) \, dy
\]

\[
= \sum_{i=0}^{N-1} \phi_i(x) \phi_i(z) \int_{-\infty}^{\infty} \phi_i(y) \, dy
\]

\[
= \sum_{j=0}^{N-1} \phi_j(x) \phi_j(z)
\]

\[ = K_N(x, z). \]

It is now legitimate to apply Proposition 2.23 to the matrix \( K_N \). Since

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \ldots, x_N) \, dx_1 \cdots dx_N = 1,
\]

it follows that

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\det M)^2 \, dx_1 \cdots dx_N
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det(K_N) \, dx_N \cdots dx_1
\]

\[
= (N - N + 1) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det(K_{N-1}) \, dx_{N-1} \cdots dx_1
\]

\[ \vdots \]

\[ = (N - 1)! \int_{-\infty}^{\infty} \det(K_1) \, dx_1
\]

\[ = N!. \]

Therefore, we have

\[ c' = \frac{1}{N!} \]

and this proves our result.
Recall that the \( n \)-point correlation function \( R_n \) is defined as
\[
R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N.
\]

So
\[
R_n(x_1, \ldots, x_n) = \frac{1}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det(K_N) dx_N \cdots dx_{n+1} \]
\[
= \frac{N-N+1}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det(K_{N-1}) dx_{N-1} \cdots dx_{n+1} \]
\[
= \cdots \]
\[
= \frac{1 \cdot 2 \cdots (N-(n-1))}{(N-n)!} \int_{-\infty}^{\infty} \det(K_{n+1}) dx_{n+1} \]
\[
= \det K_n.
\] (2.29)

**Theorem 2.25** (Wigner's Semi-Circle Law). The density \( R_1(x) \) of eigenvalues for the Gaussian Unitary Ensemble satisfies
\[
R_1(x) \sim \frac{1}{\pi} \sqrt{2N - x^2}
\]
as \( N \to \infty \).

Wigner's Semi-Circle Law tells us the density function of the positions of the eigenvalues approaches a semi-circle with center at the origin and the radius \( 2\sqrt{2N} \) as \( N \to \infty \).

**Proof.** Since
\[
R_1(x) = \det K_1 = \sum_{n=0}^{N-1} \phi_n(x)^2
\]
so the theorem follows from this and Proposition 2.21 and 2.22. \( \square \)

In view of Wigner's Semi-Circle Law, the mean spacing between consecutive eigenvalues is \( \sim \frac{1}{\pi} \sqrt{2N - x^2} \). We define normalized eigenvalues by \( y_j = x_j/\alpha_j \) with
\[
\alpha_j := \frac{1}{\pi} \sqrt{2N - x_j^2}.
\]

Let \( u = |y_1 - y_2| \). We would like to know the distribution of \( u \). We come to the proof of Theorem 2.15. From (2.29), we have
\[
R_2(x_1, x_2) = \det(K_2)
\]
\[
= \left( \sum_{n=0}^{N-1} \phi_n^2(x_1) \right) \cdot \left( \sum_{n=0}^{N-1} \phi_n^2(x_2) \right) - \left( \sum_{n=0}^{N-1} \phi_n(x_1) \phi_n(x_2) \right)^2.
\]
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In view of Propositions 2.21 and 2.22, we have
\[
\frac{1}{\alpha_1\alpha_2} \sum_{n=0}^{N-1} \phi_n(x_1)\phi_n(x_2)
= \frac{1}{\alpha_1\alpha_2} \left( \frac{N}{2} \right)^{1/2} \frac{\phi_{N-1}(x_1)\phi_N(x_2) - \phi_N(x_2)\phi_{N-1}(x_1)}{x_1 - x_2}
\sim \frac{\sin(\pi(y_2 - y_1))}{\pi(y_1 - y_2)}
\]
and
\[
\sum_{n=0}^{N-1} \phi_n^2(x_j) \sim \frac{\pi}{\sqrt{2N - x_j^2}} = \alpha_j.
\]
Therefore we obtain
\[
\frac{1}{\alpha_1\alpha_2} R_2(x_1, x_2) \sim 1 - \left( \frac{\sin u}{u} \right)^2
\]
as \(N \to \infty\). This completes the proof.

2.3 Numerical Support for Montgomery’s Pair Correlation Conjecture and further work

After Montgomery’s pair correlation conjecture was published, people have used data of the zeros of the Riemann zeta function to verify this conjecture. Riemann was the first one who carried on the numerical computation to find out the zeros of \(\zeta(s)\) with the formula now known as the Riemann-Siegel Formula. But as he expressed in his paper, the major goal of his research was not in the Zeta function, so he just computed a few of zeros of \(\zeta(s)\) and then gave up. In the following seventy-two years mathematicians made some progress on computing the zeros of \(\zeta(s)\), but the method they used was the Euler-Maclaurin theorem which is less efficient than the Riemann-Siegel formula especially when \(t\) is relatively large. Siegel’s rediscovery of Riemann’s method expedited the computations on this topic and more and more zeros were computed based on it. When people entered into the computer age, the powerful aids from super computation abilities of modern large scale computers along with the more efficient algorithms find out billions of zeros of \(\zeta(s)\). More or less though, they are all based on the Riemann-Siegel formula.

However, in order to find statistical properties of the zeros of the Riemann Zeta function, we need a large amount of zeros on the critical line. Also since Montgomery’s pair correlation conjecture is a conjecture about the limiting case, this also needs a large amount of zeros to test the conjecture. Starting from 1987, A. Odlyzko [17] has obtained many zeros with very high heights to empirically test the pair correlation conjecture and GUE predictions. According to Odlyzko, the GUE prediction is that when \(N \to \infty\) and the eigenvalues of the matrices from GUE are suitably normalized, then their pair correlation becomes \(1 - ((\sin(\pi u))/u)^2\).
Recall Montgomery’s pair correlation conjecture that assuming the Riemann Hypothesis, for fixed $0 < a < b < \infty$, as $T \to \infty$,
\[
\sum_{\gamma, \gamma' \in [0, T], \frac{\alpha}{\log T} \leq (\gamma - \gamma') \leq \beta} 1 \sim \frac{T}{2\pi} \log T \int_a^b 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du.
\]

Odlyzko has rewritten Montgomery’s pair correlation conjecture as that for any fixed $\alpha, \beta$ with $0 < \alpha < \beta < \infty$,
\[
N^{-1} \# \{(n, k) : 1 \leq n \leq N, k \geq 0, \delta_n + \delta_{n+1} + \cdots + \delta_{n+k} \in [\alpha, \beta]\}
\sim \int_{\alpha}^{\beta} 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du.
\]
Here $\delta_n$ is the normalized spacing between consecutive zeros $\frac{1}{2} + i\gamma_n$ and $\frac{1}{2} + i\gamma_{n+1}$ to be
\[
\delta_n = (\gamma_{n+1} - \gamma_n) \frac{(\log \gamma_n)/(2\pi)}{2\pi}
\]
with mean value 1.

In his tests, he actually tested the following even stronger pair correlation conjecture
\[
M^{-1} \# \{(n, k) : N \leq n \leq N + M, k \geq 0, \delta_n + \delta_{n+1} + \cdots + \delta_{n+k} \in [\alpha, \beta]\}
\sim \int_{\alpha}^{\beta} 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du
\]
as $N, M \to \infty$ with $M$ not too small compared to $N$. This is because if we assume Montgomery’s pair correlation conjecture holds, this one also holds.

He first use two sets of zeros
\[
\{\gamma_n : \gamma_n > 0, \zeta(1/2 + \gamma_n) = 0, 1 \leq n \leq 10^5\}
\]
and
\[
\{\gamma_n : \gamma_n > 0, \zeta(1/2 + \gamma_n) = 0, 10^{12} + 1 \leq n \leq 10^{12} + 10^5\}.
\]
The graphs of the tested pair correlations of these two data sets and the graph of the function $1 - ((\sin(\pi u))/(\pi u))^2$ demonstrates a good fit when $\alpha, \beta$ are small but is less persuasive as $\alpha, \beta$ becomes larger. However, after obtaining $8 \times 10^6$ zeros at $n = 10^{20}$ and using this data set to test the pair correlation again, his recent results are amazing: there is an almost perfect fit between data and the conjecture. These tests strongly suggest the validity of Montgomery’s pair correlation conjecture.

Montgomery and Odlyzko’s work have inspired a large amount of research in related fields. Because of the similarity between the pair correlation of zeros of the Riemann Zeta function from number theory and the pair correlation of eigenvalues of matrices in GUE, people now try to find
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more relations between these two fields. Since random matrix theory has been studied by theoretical physicists for many years and there are already many results, people can use them to predict the corresponding behaviors of the Riemann zeta function in number theory. For example, Keating and Snaith [14] conjectured that the moments and value distribution of $\zeta(s)$ on the critical line can be expressed from results of random matrix theory. They argued as follows. Let $Z(U, \theta)$

$$Z(U, \theta) := \det(I - U e^{-i\theta})$$

be the characteristic polynomial of a unitary $N \times N$ matrix $U$. Also $U$ is considered to be a random variable in the circular unitary ensemble (CUE) which is the unitary group $U(N)$ equipped with the unique translation invariant probability measure. Then by using Selberg’s integral, the expected values of the moments of $|Z|$, denoted as $\langle |Z(U, \theta)| \rangle_{U(N)}$, are given by

$$\langle |Z(U, \theta)| \rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + s)}{\Gamma(j + s/2)^2}.$$  

Then one can obtain the function

$$f_{CUE}(\lambda) := \lim_{N \to \infty} \frac{1}{N^{\lambda^2}} \langle |Z(U, \theta)| \rangle_{U(N)}^{\lambda} = \frac{G^2(1 + \lambda)}{G(1 + \lambda)}$$

where $G$ denotes the Barnes G-function.

From the theory of the Riemann Zeta function it has been expected that

$$\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\lambda} dt \sim f(\lambda) a(\lambda) (\log T)^{\lambda^2}$$

where $a(\lambda)$ is

$$a(\lambda) = \prod_{p} \left\{ (1 - 1/p)^{\lambda^2} \sum_{m=0}^{\infty} \frac{\Gamma(\lambda + m)}{m!\Gamma(\lambda)} p^{-m} \right\}$$

and $f(\lambda)$ unknown. Then they use the result from the CUE of random matrix theory to conjecture that

$$f(\lambda) = f_{CUE}(\lambda).$$

This conjecture agrees with the known values for $\lambda = 1, 2$ and conjectured value $f(3) = 42/9!$ by J.B.Conrey and A.Ghosh [5] and $f(4) = 24024/16!$ by J.B.Conrey and S.M.Gonek [7]. There are some other similar results as [1] and [2]. Also people can refer to the reviews in [1] and [13] for most of the results and more details.

Mathematicians have also studied higher order correlations of the zeros of the Riemann Zeta function. First Hejhal [12] studied the three-point case. He showed that the triple correlation of zeros of the zeta function is the same as the one from the GUE computed by Dyson [9]. Rudnick and Sarnak [20] and [19] improved the results to the case of $n$-point correlations of the zeros of the Riemann zeta function and the corresponding results from a random unitary matrix.
People also extended the pair correlation conjecture on the Riemann Zeta function to the more general $L$-functions. We know that the Riemann Zeta function is just a special case of $L$-functions and they all satisfy generalizations of the Riemann Hypothesis. For any individual $L$-function, Rudnick and Sarnak [20][19] proved that there exists a wide generalization of Montgomery's law for all n-level correlations for zeros of $L$-functions of an arbitrary cuspidal automorphic representation of $GL_n(G)$.

Although there is lots of empirical data supporting the pair correlation conjecture and many other results derived from it, we still do not know why zeros of the Zeta function behave like eigenvalues from random matrices. But the relationship between the zeros of the Riemann Zeta function and eigenvalues of random matrices do provide strong support for the old idea that goes back to Hilbert and Polya. That is, one should look for a quantum mechanical system whose Hamiltonian has eigenvalues given by the Riemann zeta-function zeros. Self-adjointness of the Hamiltonian then corresponds to the Riemann Hypothesis. In conclusion, this area will continue to be prosperous and probably the proof of Riemann Hypothesis will come out of it.
Bibliography


