CONVERSATIONS ABOUT CONNECTIONS:
How secondary mathematics teachers conceptualize and contend with mathematical connections

by

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ABSTRACT

The importance of mathematical connections in learning and understanding mathematics is widely endorsed in both the research and the professional literature but teachers’ understanding of mathematical connections is underexplored. This study examined teachers’ conceptions of mathematical connections as knowledge at the interface of content knowledge and pedagogical content knowledge.

I had individual conversations with nine secondary mathematics teachers in a three-stage process of progressively more structured interviews. Interviews focussed on teachers’ explicit connections related to particular mathematical topics, including a common task about quadratic functions and equations. I coded transcribed interviews according to a model I developed that identified five types of connections – different representations, implications, part-whole relationships, procedures, and instruction-oriented connections.

Teachers’ thinking about connections was completely bound up with their thinking about teaching. They talked about real-world connections and connections to students’ prior knowledge, but only a few explicitly pointed out connections to their students.

Most teachers were enthusiastic in their approval of considering mathematics as an interconnected web of concepts. While some teachers saw mathematical connections as integral to the way they taught, others were conflicted, and expressed a tension between teaching concepts and teaching algorithms.
In the context of a structured task, teachers demonstrated knowledge of specific mathematical connections at a fine-grained level, but only with considerable effort. Teachers do have knowledge of specific mathematical connections but that knowledge is largely tacit.

Teachers described specific mathematical connections in all five categories of the model. The model proved robust in classifying connections across a range of mathematical topics and grain-size. The mathematical connections that teachers articulated dealt with a narrow range of content, and favoured connections that were explicitly described in their textbooks. Nevertheless, teachers were also able to identify certain connections as crucial to students’ understanding of a topic.

This systematic and detailed examination of the way that teachers view mathematical connections has laid a foundation for future research by demonstrating a methodology that facilitated the expression of teachers’ tacit knowledge, and by developing a model for classifying the explicit mathematical connections that teachers did express.
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CHAPTER 1: INTRODUCTION

My interest in exploring mathematical connections from a teacher’s perspective took root in the late 1980s when I worked as a faculty associate in Simon Fraser University’s (SFU) teacher education program. I first encountered the National Council of Teachers of Mathematics (NCTM) *Curriculum and evaluation standards for school mathematics* (1989) and *Professional standards for teaching mathematics* (1991) soon after their publication. In fact, during the late 1980s and early 1990s, I presented numerous workshops to pre-service teachers about them. After teaching mathematics and sciences for 20 years, the ideas in these documents struck a resonant chord. I became committed to them, particularly to the process standards – problem solving, reasoning and proof, communication, connections, and representation.

I carried my convictions about the importance of the Standards into my next career step - into educational administration. Part of my administrative assignment always involved responsibility for mathematics programs. As an administrator, I worked with in-service teachers, some of whom had lost the idealism of their pre-service days and were sometimes cynical about the teaching practices epitomized in the Standards. Their concerns were always about the process standards; teachers were skeptical about their practical relevance, agreeing that the Standards might be nice in an ideal world, but unworkable in the real one.

The British Columbia (BC) mathematics curriculum is heavily influenced by the National Council of Mathematics Teachers (NCTM) curriculum documents. Although *Curriculum and evaluation standards for school mathematics* (NCTM,
1989) and *Principles and Standards for School Mathematics* (NCTM, 2000) are American documents, they played an influential role in Canadian mathematics education in a variety of ways – through curriculum development, textbooks, and professional organizations.

The Western and Northern Canadian Protocol (WNCP) is an agreement among the western provinces and northern territories to produce and work within a common curriculum framework in several subject areas, including mathematics (McAskill & al., 2004). *Curriculum and evaluation standards for school mathematics* was an important resource for the development of the common curriculum framework for mathematics (Neel, 2001). The latest redesigns of the BC mathematics curriculum (BC Ministry of Education, 2000, 2001, 2006) were based on this common curriculum framework. Publishers of Canadian mathematics textbooks also relied on the NCTM Standards documents. For example, in their reference materials for teachers, Harcourt Canada and Thomson Nelson both repeatedly cited the NCTM documents (Doctorow, 2002; Harcourt Canada, n.d.). Furthermore, the BC Association of Mathematics Teachers (BCAMT) is an affiliate of the NCTM and publicizes the NCTM Standards to teachers and parents in the province. So, through their Integrated Resource Packages (IRPs) [the provincial curriculum guides], their textbooks, and their professional organization, BC teachers are exposed to the principles espoused by the NCTM.

Both the BC Mathematics IRPs and the NCTM Standards documents identify “connections” as a fundamental strand in the mathematics curriculum.

Students become aware of the usefulness of mathematics when mathematical ideas are connected to everyday experiences. Learning activities should help students relate mathematical concepts to realistic situations and allow them to see how one mathematical
idea can help them understand others (BC Ministry of Education, 2001).

Viewing mathematics as a whole... helps students learn that mathematics is not a set of isolated skills and arbitrary rules... An emphasis on mathematical connections helps students recognize how ideas in different areas are related. Students should come both to expect and to exploit connections, using insights gained in one context to verify conjectures in another... The opportunity to experience mathematics in context is important. Students should connect mathematical concepts to their daily lives, as well as to situations from science, the social sciences, medicine, and commerce. (NCTM, 2000, http://standards.nctm.org/document/chapter3/conn.htm)

In May, 2003, not long after I entered the PhD program, I attended the Canadian Mathematics Education Study Group (CMESG) conference at Acadia University in Nova Scotia. The conference is organized around working groups whose members participate in in-depth work on a particular topic every morning of the conference. I was part of a working group, led by Brent Davis (then of the University of Alberta) and Walter Whiteley (York University), that tackled the perceived problem of an overstuffed and over-engineered school curriculum in mathematics. The leaders of this group were carrying forward some work begun at an earlier Canadian Math Society (CMS) Mathematics Education Forum in 2003 in Montreal. In that forum, one of the working groups recommended that there should be a clear statement from the CMS about what students need from school and university mathematical experiences. The end-in-view was a curriculum that offered rich coherent mathematical experiences over many concepts, in contrast to what is often described as superficial and fragmented encounters with a great many disconnected topics (Whiteley & Davis, 2003a, p. 2).

Davis and Whiteley proposed that the CMESG working group prepare a draft "manifesto" for discussion with the CMS as a contribution to this process.
The manifesto identified the following as suitable aims for school mathematics instruction:

- students coming out of high school mathematics must be able to engage effectively with complex problems; they require the ability to ‘think mathematically’—that is, to investigate the mathematics in a situation, to refine, to expand, and to generalize;

- students’ mathematics concepts must be woven into a connected set of relationships;

- students must be able to independently encounter and make sense out of new mathematics.

These aims should have priority over any specific selection of content; and it is our judgment that it is impossible to achieve these objectives if teachers are required to cover each item on a curriculum list (Whiteley & Davis, 2003b, p. 1).

In the process of producing the manifesto, the working group met over three days, in small and large discussion groups. Some of the discussion necessarily dealt with the school mathematics curriculum, and mostly the mathematics curriculum in high schools. Implicit in these discussions and in the manifesto itself, was an acknowledgment of the role of teachers in meeting the aims of the curriculum. While our working group included quite a few people like me who were certified teachers and had taught high school mathematics before moving into other positions, we were not representative of the majority of current high school mathematics teachers. I began to wonder what practicing mathematics teachers would make of the issues raised in the manifesto, and especially, the aim that “students’ mathematics concepts must be woven into a connected set of relationships”.

Certainly, it appeared that the NCTM documents and the curriculum documents of my province were in agreement with the aims expressed by my working group at CMESG. But no one seemed to believe that the rhetoric was exemplified in
the reality of mathematics teaching in secondary schools. Moreover, most members of our CMESG working group had a hard time themselves articulating what mathematical ideas might compose the desired web of relationships.

Hence my interest in considering the point of view of secondary mathematics teachers. I decided to talk them about mathematical connections, to try to understand how they related mathematical ideas and topics themselves, what importance they saw pedagogically in the idea of making connections, and how they perceived their role in supporting their students to make mathematical connections.
CHAPTER 2: LITERATURE REVIEW AND PROPOSED THEORETICAL MODEL

The mathematics education literature is rife with articles about making connections. A simple search of the ERIC database for articles that mention both mathematics and connections in their abstracts generated a list of over 1100 pieces of writing. However, scanning the titles and abstracts of the articles found in the ERIC search, showed that the majority of them were about connections to the “real world”.

Curriculum and evaluation standards for school mathematics (NCTM, 1989) identified two types of connections:

modeling connections between problem situations that may arise in the real world or in disciplines other than mathematics and their mathematical representation(s); and

mathematical connections between two equivalent representations and between corresponding processes in each (NCTM, 1989, p. 146).

The first concerns the recognition and application of mathematics in contexts outside of mathematics, the second, the interconnections among ideas within mathematics. The latter, the area of mathematical connections, is underexplored in the research literature, and may be underrepresented in teachers’ consciousness (Businskas, 2005). It is this area of mathematical connections that is the focus of this study.

But what exactly is a “mathematical connection”? What is one actually doing when one is connecting mathematical ideas? What is the role of making connections
In learning/understanding/doing mathematics? What are the pedagogical implications?

In considering these questions, I first examine different ways that “mathematical connection” is conceptualized in the research literature and propose a model for describing mathematical connections more specifically. Then I review the role of mathematical connections in teaching and learning mathematics, particularly the teacher’s role vis-à-vis mathematical connections.

How is the term “mathematical connections” conceived in the mathematics education literature?

The Oxford English Dictionary presents nine definitions of “connection”; of these, the most appropriate in this context is “a causal or logical relationship or association; an interdependence” (Brown, 1993, p. 481). A definition like this is a helpful start, but far from adequate for any practical purpose. Starting with the idea of a mathematical connection as “a causal or logical relationship” between two mathematical entities, one faces some fundamental questions. For example, is a connection a feature of the subject matter or a feature of the learner’s understanding? If a connection is a feature of the learner’s understanding, is it an artefact or product of learning, or is making a connection an activity or process? Perhaps the answer is “all of the above”. A mathematical connection is variously referred to in the literature as a relationship between mathematical ideas (implying that it exists independently of the learner), as a relationship that is constructed by the learner, and as process that is part of the activity of doing mathematics. In what follows, I examine each one of these views.
**Mathematical connections: a feature of mathematics**

The NCTM statement that mathematics is “a web of closely connected ideas” is a position about the nature of mathematics. Mathematical ideas are linked by particular relationships and those connections can be identified *a priori* and independently of the learner.

**Mathematical ideas as connections**

Coxford (1995) conceptualized connections as very broad ideas/processes that can be used to link different topics in mathematics. He identified three categories of mathematical connections — unifying themes, mathematical processes, and mathematical connectors.

Unifying themes are themes that may be used to draw attention to the connected nature of mathematics. Coxford suggested change, data and shape as examples of these themes.

The notion of change may help connect algebra, geometry, discrete mathematics, and calculus... For example, how is a constant rate of change related to lines and linear equations? What changes occur in the graph of a function when a coefficient in the equation of the function is changed?... How does the perimeter or area of a plane shape change when it is transformed using isometries, size transformations, shears, or some unspecified linear transformation?... Each of these questions suggests opportunities to connect mathematical topics by relating them through the theme of change (Coxford, 1995, p. 4-5).

The second category included mathematical processes like representation, application, problem solving and reasoning.

For example, upper elementary school students should develop facility in moving back and forth among the concrete and pictorial models, the oral name, and the symbolic representation of any fraction or decimal. These connections are vital if students are to make
sense out of later operations on numbers (Coxford, 1995, p. 7).

Finally, the connectors are mathematical ideas like function, matrix, algorithm, graph, variable, ratio, transformation.

They are mathematical ideas that arise in relation to the study of a wide spectrum of topics. As such, they permit the student to see the use of one idea in many different and, perhaps, seemingly unrelated situations (Coxford, 1995, p. 10).

I would characterize Coxford’s view of mathematical connections as a view which is quite general. Although he provided specific examples of connections of the three types mentioned, the “grain-size” was relatively large. The particular relationships seemed to take the following forms:

- topic A is an instance of unifying idea B;
- process A can be applied to many topics to produce multiple ways of looking at a topic;
- topic A is an idea common to many topics.

Actually, the first and third categories – unifying themes and connectors, are similar in that they rely on the notion of a common theme to link topics. The distinction is in the “size” of the idea - in the first case, a small number of large overarching concepts that characterize mathematics in general, in the second case, a larger group of more specific mathematics concepts and procedures that are used in a variety of topics.

Following the “common theme” connection, Bosse (2003) considered conjunction and disjunction as a mathematical idea that could connect apparently disparate mathematical topics. He illustrated how the language of AND and OR is used in logic, intersection and union of sets, greatest common factor (GCF) and least common multiple (LCM), absolute value in algebra, and the probabilities of two
events. His contribution was one of laying out the mathematics and suggesting the conjunction/disjunction theme; however, he did not follow it to working with students or teachers. Similarly, Crites (1995) examined distance, speed and time through algebraic and geometrical interpretations, but did not extend his ideas to making instructional recommendations. Vonder Embse (1997) used parametric equations to demonstrate connections between trigonometric identities and graphs of ellipses and hyperbolas.

Crowley (1995) advocated for the inclusion of transformations as a unifying concept and gave examples in plane geometry, matrices, trigonometry. Again, this was essentially an explication of the mathematics. More examples were given by Hirschorn & Viktora (1995), including applications to statistics.

This “common theme” view was echoed by the NCTM’s Algebra Working Group –

... themes help students recognize important ideas and make connections (NCTM et al., 1998, p. 164).

Certainly, the notion of “common themes” or “big ideas” is an important and recurrent one in mathematics education. However, defining mathematical connections only in these terms leaves the idea of a mathematical connection at a quite general level.

Concept-to-concept links

In her studies of pre-service teachers’ understanding of number theory, Zazkis (2000) identified particular connections and looked for evidence that learners did or (more often) did not make them.

The mathematical connection among a factor, a divisor, and a multiple is expressed in the equivalence of the
following three statements, for any two natural numbers $A$ and $B$:

- $B$ is a factor of $A$;
- $B$ is a divisor of $A$;
- $A$ is a multiple of $B$ (Zazkis, 2000, p. 212).

In studying middle school children’s understanding of the connections between fractions and division, Weinberg (2001) also gave examples of particular connections –

- the numerator of a fraction is equivalent to the dividend;
- the denominator of a fraction is equivalent to the divisor.

These are both illustrations of a conceptualization of a connection as a particular type of relationship between two fine-grain sized mathematical concepts. Moreover, the particular relationships could be identified and theoretically, at least, the web of connected ideas could be built.

Considering mathematical connections as specific relationships between mathematical concepts is not a new idea, but it seems a productive definition for examining the role of mathematical connections in teaching and learning. Considering connections as concept-to-concept links is consistent with Richard Skemp’s model of learning mathematics as a hierarchy of concepts which are interconnected – “an integrated conceptual structure” (Skemp, 1987, p. 31).

**Equivalent representations in mathematics**

The notion of equivalent representations appeared in the elaborations of both of the previous perspectives; I now consider it in its own right. In the view of a mathematical connection as a recognition of the equivalence of two or more representations, we see an intersection of research into mathematical representations and research into mathematical connections. Researchers may invoke the linking of
mathematical ideas without using the term “connections”. Mathematical ideas can be presented in a variety of representations – concrete, pictorial, graphical, symbolic. For example, a straight line (concept) can be represented by an equation and by a graph (Hodgson, 1985). Binomial multiplication can be represented algebraically (symbolic) and with algebra tiles (concrete) (Chappell & Strutchens, 2001).

Consider “A is equivalent to B” as a type of relationship between two mathematical ideas. We can extend equivalence to a relationship between two different representations of the same mathematical concept. Elements of one representation map to elements in another. For example, in the equation, $y = mx + b$, characteristics of the graph of the line are equivalent to parts of the equation; slope is equivalent to $m$, y-intercept is equivalent to $b$.

The recognition and production of equivalent representations seems a particularly fruitful way of conceptualizing what a mathematical connection is, and the translation between equivalent representations, a useful process for both practicing and assessing that learners are making connections (Lloyd & Wilson, 1995; Reed & Jazo, 2002).

**Mathematical connections: a construction of the learner**

In all the examples considered so far, mathematical connections were considered as features of mathematics itself. Particular relationships among mathematical concepts “exist”, and the pedagogical task becomes that of making sure that those connections get into the mind of the learner. Another way of conceptualizing a mathematical connection is that it is an artefact of the learning process itself. In other words, making a mathematical connection is a process that
occurs in the mind of the learner(s) and the connection is something that exists in the mind of the learner; it is a mental construction of the learner.

Constructivism represents a variety of views, both about mathematics and about learning, with varying implications for teaching. The variants of constructivism to which most mathematics educators adhere have their roots in the works of Piaget and Vygotsky. In the following paragraphs, I consider some elements of constructivism from these two points of view in so far as they provide a theoretical framework for thinking about making mathematical connections.

Piaget states that people have two inherent tendencies – to organize behaviours and thoughts into coherent systems and to adapt, or adjust, to the environment (Woolfolk, 2000). Making mathematical connections is another way of saying that learners try to organize mathematical ideas into coherent systems or schemata. In the Piagetian framework, making connections is a natural activity. Furthermore, through the processes of assimilation and accommodation, learners either expand and elaborate existing schemata, or reconstruct them in the face of unfamiliar or contradictory information. Thus making connections can be viewed as the mechanism for assimilating new mathematical knowledge.

Another Piagetian process that seems related to conceptualizations of “mathematical connections” is abstraction. Noss and Hoyles (1996) define abstraction as the “transition from direct knowledge, knowledge in action, to reflective knowledge” (p. 16). As they review the ways that different mathematics education researchers view abstraction, abstraction is linked to connections over and over again. For example,

abstraction is a key component in the creation of a mathematical world populated by an interconnected set
of mathematical objects... abstraction comprises a shift in attention involving mental reconstruction after which the relationships between objects become central (p. 20).

Creating a new meaning means making a new connection.

Meaning can be maintained by involvement in the process of acting and abstracting, building new connections whilst consolidating old ones (Noss & Hoyles, 1996, p. 49).

The abstraction to which Noss and Hoyles refer seems to be a particular kind of connection between mathematical activities with concrete objects and mathematical ideas.

The Vygotskian view introduces a social element to constructivism.

Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals (Vygotsky, 1978, p. 57).

Cognitive development occurs through the learner’s conversations with more knowledgeable members of the community. A child can learn successfully when his/her existing knowledge is not too far removed from the knowledge of the community. Vygotsky (1978) postulates a zone of proximal development – an area of cognitive activity where a learner cannot make much progress on his/her own, but can succeed through interaction or collaboration with more capable others. The child then internalizes the strategies at play in the social dimension to become more capable individually.
Learning to make mathematical connections, considered from a Vygotskian perspective, requires the scaffolding provided by interaction with more capable peers or teachers.

Lopez (2001) used graphing calculators as a visualization tool “to make connections between mathematical concepts (lines, parabolas, and circles) and the construction of drawings” (p. 116). Visualization as a process that promotes the making of connections was also explored by Noss, Healy and Hoyles (1997), connecting visual and symbolic representations.

Also in the constructivist framework, Hazzan and Zazkis (1999) used “give an example” tasks to provide students with opportunities to make mathematical connections.

... while students are working on generating particular examples of a mathematical concept which satisfies certain properties, they construct a more general notion in their mind... when one constructs an example for a particular concept which satisfies certain properties, she also constructs a link among two (or more) concepts... the construction of these links contributes to the construction of a more complicated mathematical object – a schema (p. 4).

**Mathematical connections: a dynamic process**

In the previous sections, the key aspect of connections that was developed was that of connection as an idea, as a product of mental activity. The connection was the product, and could then be considered and judged with respect to truth value and usefulness in doing mathematics. Some researchers, however, focus on the activity of making connections.

... It seems to me that the act of observing relationships and drawing connections, whether between different functional representations or mathematical areas, is a key aspect of mathematical work, in itself, and should
not only be thought of as a route to other knowledge (Boaler, 2002, p. 11).

Much of the mathematics education literature that describes useful activities for promoting the making of connections seems based, at least implicitly, on a view of making connections as process. The salient characteristic is that the authors do not justify the importance of the particular relationships that are uncovered; the value seems to be in the doing. For example, Sullivan and Panasuk (1997) described their use of a “Fibonacci puzzle” with high school students, in which students started with squares whose side lengths were Fibonacci numbers, cut the squares into segments with lengths of the two preceding numbers, and formed new shapes from the sections. The authors used this activity, not to focus on any particular relationship, but as a medium for making general links between geometry and algebra; they did this by looking at squares and rectangles both geometrically and algebraically. At best, one could describe the mathematical connections that students are presumed to make as “geometric shapes can be described by using algebraic expressions and algebraic expressions can be represented geometrically”, in other words, equivalent representations expressed in very general terms.

Evitts (2004) studied the problem solving activity of a group of pre-service teachers as a process of making connections. He found that the pre-service teachers were engaged in making a variety of connections which he classified as follows:

Modeling: Was the subject attempting to find some aspect of his mathematical knowledge that could be used to portray some real-world component of the problem in a mathematical way?

Representational: Was the subject using two or more representations to talk about the same mathematical idea?

Structural: Was the subject discussing and using similarities she found between a real or mathematical
component of the problem and another real or mathematical situation?

Procedure-Concept: How did the subject describe or use procedures? Was his work rule- and formula-based? What indications were there of a conceptual basis for utilizing a procedure?

Between Strands of Mathematics: Was the subject “crossing over” from one strand of mathematics to another in her analysis of the problem? Were references made to other areas of mathematics? (p. 56).

Evitts framed his categories in terms of what the pre-service teachers were doing. But the categories can also be seen as the products of a process of drawing on prior knowledge.

All three ways of considering connections – as a feature of mathematics, as a construction of the learner, and as the process of making associations, are viable and I do not base my interpretations in this study on the philosophical differences among them. However, I mostly view a connection as a “product” – a mental object that can be remembered and talked about. In my own beliefs about the nature of mathematics, I am a constructivist; I believe that mathematics is a human creation, not a discovered reality. However, in dealing with school mathematics, the distinction between a connection considered as a viable construction made by a learner, or as a mathematical fact, has little consequence within the context of this study. As for the distinction between static and dynamic views, I see the distinction more as a duality; in our efforts to comprehend what a mathematical connection is, sometimes we think of a connection as an object, sometimes as a process.

Proposed model for thinking about mathematical connections

First of all, what counts as a mathematical concept? A concept is an idea. I take a mathematical concept to mean a class of mathematical objects, for example,
perfect square, number, variable, equation, parabola... In these examples, it is clear that some concepts are more complex than others, and could be seen as composites of simpler concepts. Skemp (1987) distinguishes between primary concepts, which can be derived from our experiences of the world, and secondary concepts, which are abstracted from other concepts. At this point, I acknowledge a range of complexity, but will use "concept" more loosely, and attend to distinctions when it is relevant to the discussion.

Next, what kind of relationship between two mathematical concepts is a connection? A trivial way of defining a relationship between two concepts is to accept any association a person might make between two ideas. Of course, people make idiosyncratic associations all the time, but I narrowed the focus to types of associations that might be more generally useful in improving mathematical understanding. I proposed the following set of seven categories as a preliminary framework for thinking about mathematical connections. I treat a mathematical connection as a true relationship between two mathematical ideas, A and B.

- **Alternate representation.** A is an alternate representation of B. The two representations are from different modes, like symbolic (algebraic), graphic (geometric), pictoral (diagram), manipulative (physical object), verbal description (spoken), written description. For example, the graph of a parabola is an alternate representation of \( f(x) = ax^2 + bx + c \) (geometric/algebraic). One of the McDonald's golden arches is an alternate representation of the graph of a parabola (physical/geometric).

- **Equivalent representation.** A is equivalent to B. I call concepts that are represented in different ways within the same form of
representation, equivalent, to distinguish them from representations in
different forms. For example, 3 + 2 is equivalent to 5;
f(x) = ax^2 + bx + c is equivalent to f(x) = a(x-p)^2 + q. In these
examples, both A and B are symbolic.

• Common features. A and B share some features in common. For
example, a square and a rectangle are linked by the common features
of 4 sides and right angles.

• Inclusion. A is included in (is a component of) B; B includes
(contains) A. This is a hierarchical relationship between two concepts.
For example, a vertex is a component of a parabola (and, a parabola
contains a vertex).

• Generalization. A is a generalization of B; B is a specific instance
(example) of A. This is another kind of hierarchical relationship. For
example, ax^2 + bx + c = 0 is a generalization of 2x^2 -7x + 3 = 0.

• Implication. A implies B (and other logical relationships). This
connection indicates a dependence of one concept on another in some
logical way. For example, the degree of an equation determines the
maximum number of possible roots.

• Procedure. A is a procedure used when working with object B. For
example, making a tree diagram is a procedure used to describe a
sample space (probability).

Making mathematical connections is a cognitive process that involves making
or recognizing links between mathematical ideas. Starting with such a
conceptualization of a mathematical connection, one could imagine mapping out a set
of mathematical ideas and all the pair-wise relationships among them. Thus, it should be possible to identify concepts at some chosen level of grain size and then explicitly describe all their relationships to diagram a web. Such a web could become the reference point for instruction and assessment.

This idea of a web relates to the well-known technique of concept mapping. In its most prosaic form, concept mapping simply draws link between nodes (concepts) in a way that is meant to represent the web of associations that a person makes in relation to a particular topic. In a more sophisticated version, the links are labelled, thus describing the particular relationships between concepts (Bartels, 1995).

On a large scale this kind of mapping of mathematical relationships is a daunting task, but on a small scale, it could provide a reference point for analyzing learners’ understandings of particular mathematical topics.

The *a priori* model that is presented here is based on mathematical properties, and does not include pedagogical ones. However, my interest in mathematical connections is from the point of view of a teacher. So, in the following section, I consider some features of pedagogy where research in the teaching of mathematics appears to have relevance for interpreting mathematical connections. In fact, as will be seen in Chapter 3, instructional considerations did influence how teachers thought of mathematical connections and led me to revise the initial model.

**Pedagogical considerations related to “mathematical connections”**

*Why is “making connections” important/valuable?*

The emphasis placed on making mathematical connections in North American curriculum documents indicates a prevailing belief that making connections is an important and valuable aspect of learning mathematics. Making connections is treated
as synonymous with (or perhaps, an indicator of) “deeper and more lasting understanding” (NCTM, 2000, p. 64). The NCTM documents claimed that making connections would allow students to better remember, appreciate and use mathematics.

Describing understanding in terms of making mathematical connections is evident in the work of several well-respected mathematics educators. For example, a facet of Liping Ma’s (1999) “profound understanding” of mathematics is the ability to connect ideas within a topic and to central concepts of the discipline. And, in describing models of understanding based on a Piagetian framework, Sierpinska (1996) describes the development of mathematical concepts in terms of cognitive structures and “cognitive connections” (p. 119).

**Instructional considerations**

Earlier, I described a variety of viewpoints from which mathematical connections could be defined. I chose the framework of a schema as described by Piaget (1970) generally, and Skemp (1987) particularly for the learning of mathematics, to support my model of what mathematical connections are. However mathematical connections are conceptualized, learners have to make them.

Are concepts in mathematics (such as graphs and equations) inherently connected or do the connections exist only in the minds of the learners? Although this is an interesting philosophical question, to me the answer is irrelevant. If students are unable to establish connections, then the connections cannot be used in problem situations regardless of whether they exist or not (Hodgson, 1995, p. 14-15).

However mathematical connections are defined, whether as aspects of mathematical structure that learners need to recognize or as mental objects that learners construct for themselves, their effectiveness for learning requires the activity
of the learner. Learners might make connections spontaneously, but “we cannot assume that the connection will be made without some intervention” (Weinberg, 2001, p. 26). The implied role for the teacher is to act in ways that will promote learners’ making of mathematical connections.

Students should be made explicitly aware of mathematical connections.

As their school experiences with mathematics are broadened, their abilities to see the same mathematical structures in different settings should also improve (Thomas & Santiago, 2002, p. 485).

Just what pedagogical actions would be appropriate might vary with the way that teachers conceive of what mathematical connections are. However, conceptions of mathematical connections are rarely articulated and even more rarely, articulated in specific terms of relationships among concepts. Even in the Connected Mathematics Project (Senk & Thompson, 2003), a curriculum specifically designed to develop student understanding that is “rich in connections to other mathematical concepts and to real-world applications” (Cain, 2002, p. 224), instructional recommendations are very broadly stated. Essentially, the Connected Mathematics Project is based on a problem-solving approach (Zawojewski, Robinson, & Hoover, 1999).

In the sections dealing with making mathematical connections, the NCTM documents offer recommendations for teachers in the areas of planning, instruction and interactions with their students. Strategies for helping students make mathematical connections are reported as general strategies, that is, those that promote a mind set or create a climate in which the possibilities for connections are enhanced. Some examples of such strategies are:

- build on students’ previous experiences;
- listen to students in order to assess the connections students bring to their situation;
select problems that connect mathematical ideas;
capitalize on unexpected learning opportunities;
ask questions that direct students’ thinking;
use many concrete and pictoral models of concepts and procedures (NCTM, 1989, 2000).

Moreover, most of them are approaches that good teachers would use in any subject area. For example, relating new material to previously-taught material and to students’ own experiences and knowledge, asking students thought-provoking questions and listening to them to assess their thinking, taking advantage of the “teaching moment” (the unexpected opportunity), teaching students metacognitive strategies, setting rich, worthwhile tasks – these are all aspects of good teaching in general. Even the idea of using multiple representations, while very relevant in the mathematics context, is not unique to the teaching of mathematics. All in all, teachers are given only general guidance about instructional practices to promote mathematical connections.

**The role of teachers’ beliefs and knowledge**

There is an association between teachers’ conceptions of mathematics and their instructional practice (Thomson, 1992). Teachers’ strategies to help students make connections is presumably linked to their own knowledge and understanding of mathematics.

In their review of research on teachers’ knowledge, Fennema and Franke (1992) also considered the link between teachers’ knowledge and instruction and concluded that when teacher knowledge of content has been defined in a way that is congruent with the nature of mathematics and/or when a conceptual organization of knowledge was considered, a positive relationship was found between content knowledge of teachers and their instruction (pp. 152-153).
This finding pointed to the relevance of understanding teachers’ conceptions of mathematics and “the interrelationships of its major structural elements” (Fennema & Franke, 1992, p. 152), another way of saying mathematical connections.

Teachers’ understanding of mathematics as an interrelated web of ideas can be seen as a component of their mathematical content knowledge. But it can also be seen as the foundation for developing students’ understanding - “pedagogically useful understanding” (Ball & Bass, 2000, p. 89), or pedagogical content knowledge.

Lee Shulman characterized pedagogical content knowledge as

... the most useful forms of representation of those [content] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others; and

To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding of the structures of the subject (1986, p. 9).

Knowledge of how mathematical ideas are connected can be considered as an aspect of “understanding of the structures of the subject”. In fact, Liping Ma (1999) includes knowledge of how to link mathematical ideas as part the pedagogical content knowledge that teachers should have.

To help students make connections, teachers have to understand mathematics as an interrelated web of ideas themselves and know what strategies and examples will make it easier for students to do so.

Research Questions

Earlier in this chapter, I considered three perspectives for thinking about mathematical connections – as a feature of mathematics, as a construction of the
learner, and as a dynamic process. Further, I argued that, in the context of this study, views of connections as innate features of mathematics, or as constructions of the learner do not need to be differentiated to make progress in understanding how teachers think about mathematical connections. As a starting point, I treated a mathematical connection as a true relationship (whether constructed or discovered) between concepts. And, I proposed a model for thinking about mathematical connections based on this starting point.

I restricted my investigation to a specialist group, secondary mathematics teachers, for whom mathematical connections could be a facet of their own mathematical understanding and also an element of the way they teach mathematics to their students. As considered in this study, the matter of mathematical connections straddles the boundary (if there is one) between teachers’ own content knowledge and their pedagogical content knowledge. Even narrowing the perspective to consider mathematical connections from a teacher’s point of view, leaves a very wide scope for inquiry. Because the topic has not been extensively investigated, there are worthwhile questions still to be asked about many broad areas, for example, theoretical constructs, teachers’ own knowledge, their planning and instruction, teachers’ discourse in the classroom as they try to treat mathematics as a web of related concepts.

There were many places to begin. I chose to begin by examining teachers’ own understandings of various topics in school mathematics, and the ways that they connected mathematical ideas, thinking that I might, in future research, try to relate teachers’ own understandings to the ways that they taught.
The goal of this study was to identify emerging themes in teachers’ understanding of mathematical connections in the context of thinking about their practice. My inquiry was guided by the following two questions:

1. How do secondary mathematics teachers conceptualize “mathematical connections”?

2. What are the characteristics of the explicit mathematical connections that teachers are able to articulate?
CHAPTER 3: METHODS

In this study, I took a phenomenological stance - "grappling with a synthesis of phenomenological subjectivity and scientific objectivity" (Schwandt, 1994, p. 119).

Phenomenological researchers focus on what an experience means for persons who have had the experience and are able to provide a comprehensive description of it. The underlying assumption is that dialogue and reflection can reveal the essence – the essential, invariant structure or central underlying meaning – of some aspect of shared experience (Schram, 2003, p. 71).

The experience in this case was making connections with mathematical ideas. My aim was to uncover the essence of what mathematical connections and making connections signified for secondary mathematics teachers. I approached this task as a researcher but also as a secondary mathematics teacher myself, and as a teacher educator. The phenomenological attitude requires researchers to distance themselves from their own judgments and preconceptions (Schram, 2003). At the same time, I could not help but have my own ideas about the meaning of mathematical connections and about other teachers' perceptions of it. In working with the teachers, I tried to find a balance between my own a priori ideas, like possible categories and examples of connections, and allowing the phenomena to emerge from my conversations with teachers about their work. Hence my choice to pursue a qualitative inquiry.

The focus of a qualitative study unfolds naturally in that it has no predetermined course established or manipulated by the researcher such as would occur in a laboratory or other controlled setting... you engage study participants as much as possible in places and under conditions that are comfortable for and familiar to them... Being open and pragmatic to this degree requires that you possess a high comfort level with
ambiguity and uncertainty as well as trust in the ultimate value of what an emergent and largely inductive analytical process will provide (Schram, 2003, p. 7).

Research that attempts to uncover teachers' understandings of mathematical connections is a particular instance of research into teachers' mathematical thinking. Therefore, research designs that are commonly used to explore mathematical thinking were the ones I considered as suitable starting points for this research.

The variety of methods used in mathematics education research to study mathematical thinking are mostly based on having the subjects (usually students, but also teachers) work through some task (usually a problem solving task), and either through speaking, or writing (in its broadest sense) produce some explicit data from which the researchers can infer the subjects’ thinking. The most popular method is the clinical interview and less structured forms of interviewing. However, researchers also used journals (Liljedahl, 2004), open-ended writing (Aspinwall & Aspinwall, 2003), think-aloud protocols, and computer traces (Noss & Hoyles, 1996).

Ginsburg (1981) argued that the clinical interview is the most appropriate method for research into mathematical thinking. In the most scientific version of clinical interviewing (Goldin, 2000), as much as possible is pre-planned. A structured clinical interview is very useful to test models of students’ or teachers’ thinking, and to test definite hypotheses. However, it is a method that leaves little opportunity to explore the unanticipated response and is less useful for new work in a field. While connections have often been treated peripherally in other research, inquiry into mathematical connections as the central focus is in an early stage, complicated by the fact that the definition of mathematical connection varies with researchers, as described in Chapter 2. In fact, some researchers do not use the term at all while
studying processes like reflection, abstraction, visualization, concept formation, representation, which involve making links among mathematical ideas.

In a pilot study (Businskas, 2005), I interviewed three well-respected mathematics teachers about their conceptions of connections in mathematics. I asked questions that used the term, mathematical connection, in relation to teachers’ planning for instruction, to their teaching, and to the curriculum guides and textbooks that they used. Apart from referring to the recall of prior knowledge, the teachers appeared not to think explicitly about mathematical connections at all. Even with probing, they found it very difficult to name examples either of their own attempts to show a mathematical connection or of their students’ recognition of a mathematical connection. One possible explanation is that the teachers themselves do not see mathematics as a web of interrelated ideas but as a set of independent, unrelated topics and algorithmic procedures. An alternative hypothesis is that their understanding is tacit, and that they simply do it, that is, make mathematical connections, without articulating it, or that, when they do think about it, they think about it using different language, like some of the researchers to whom I referred earlier.

I saw my task as one of trying to make teachers’ implicit knowledge, explicit. In attempting this, opportunities to continually clarify meaning are essential. Hence, I chose to use semi-structured interviews. I prepared, ahead of time, questions that I thought were likely to elicit teachers’ articulation of their understanding of mathematical connections. I wanted to give teachers an opportunity to speak as naturally as possible about their work. I faced a constant internal tension, as interviewer, between giving teachers the freedom to say what they thought was important, which sometimes took the interview in unexpected directions, and
constantly probing for more elaboration when teachers directly or indirectly referred to connections.

Furthermore, because the results of the pilot study indicated that teachers, when they did speak about connections, did so in general terms, and had a lot of difficulty being specific, I decided on a three-stage interview process. Before the first interview, I asked teachers to complete a background questionnaire (Appendix A). Starting with an initial interview, in which my role was mostly to listen and ask for elaboration, the interviews became progressively more constrained. In the second interview, I required teachers to specifically discuss a particular mathematical topic. The third interview was more like a conventional task-based interview, in which all teachers used the same materials to complete a sorting task related to the same mathematical topic. I audiotaped all the interviews and later transcribed them.

I realize that by my very questions, I was ascribing significance to certain aspects of the experience. On the other hand, I was trying to ensure that I would evoke as full a description of teachers’ experience of this phenomenon as I could. For each interview, I had some prepared questions, and I asked additional questions during the course of the interview to follow up on things that the teachers said. And, I did my best to maintain a conversational style and ask my prepared questions at points where they seemed to flow from the conversation (Krathwohl, 1998; Smith & Osborn, 2003).

**Questionnaire**

I asked teachers who agreed to participate to complete a preliminary questionnaire (Appendix A). The questionnaire asked for some factual information like their mathematics preparation in university and their teaching experience, a self-
report of their confidence in their knowledge of the curriculum and the NCTM Standards and their effectiveness in teaching certain topics, and their opinions of topics that were easy or difficult for students to learn,

*Interview 1*

Teachers’ responses to this questionnaire were a starting point for the first interview. This was an acclimatizing interview and drew out information about teachers’ background and general views about teaching mathematics and the role of connections. I wanted to see how/if teachers would spontaneously talk about connections in relation to their own understanding of mathematics, and to their teaching.

I prepared a list of questions that I thought would get teachers talking about aspects of their work where references to making connections were likely to arise – their own backgrounds, their teaching aims, characteristics of topics that made them easy/hard for students to understand and teachers to teach. I introduced the term, connections, only in the latter part of the interview. The following is the list of prepared questions for Interview 1.

1. How did you get interested in becoming a mathematics teacher?

2. On the questionnaire, you said that your post secondary mathematics courses prepared you well/poorly for teaching high school math… Please tell me more about that.

3. When you are teaching Math, what are your goals for your students?

4. On the questionnaire, you said that X is a topic that you teach effectively… Please tell me how you approach this topic both in planning and in teaching.

5. What do you think makes certain topics easy or hard for students?

6. How might a student who understands a topic well think about it… what might be the student’s mental image of the topic?
7. I would like to hear your reaction to this statement which was made in a critique of current mathematics curricula in schools: students' mathematics concepts must be woven into a connected set of relationships. What do you think the statement means? Do you agree or disagree? What might be an example of a "connected set of relationships"?

8. When you are planning or teaching a lesson, how much do you specifically think about the way that mathematical ideas are connected?

Although the questions are presented above as a sequence, in practice, I tried to keep the interview as natural as possible, asking the first six questions at times that seemed appropriate in the conversation. I asked the last two questions only after the other issues had been discussed.

**Interview 2**

I focused the second interview on particular mathematics content to try to get the teachers to talk specifically about relationships of mathematical ideas. In preparation for the interview, I asked each teacher to choose a familiar topic that s/he thought was conceptual (i.e. not an algorithm) and had potential for linking to other mathematical topics. To reduce any pressure the teachers might feel about their knowledge of the topic, I told them that they could review beforehand, and even bring their books and materials to the interview, if they wished. Many of them made a pragmatic choice of a topic that was coming up soon in their teaching.

Again, I had prepared some core questions that I asked in reference to the teacher's chosen topic. Often the points of the questions came up spontaneously in my conversations with the teachers, so the questions became a guide to make sure I had given them opportunities to talk about mathematical connections.

1. Why did you choose this topic?

2. Please tell me about your understanding of this topic. What are the important concepts/ideas?
3. Please tell me how these ideas relate to each other or to other topics in mathematics.

4. From your point of view as a teacher, what are the most important concepts and procedures that you want your students to learn?

5. What are the mathematics ideas that your students must already know prior to learning this topic? How are these “prerequisite” concepts related to the ideas that you will teach?

I followed up teachers’ responses that included some reference to connections, relationships, or links, with questions asking them to elaborate. If teachers did not spontaneously make any references to connections, I asked further probing questions, and sometimes even leading questions in an attempt to get them to voice an opinion. After the interview, I asked teachers to show me their lesson plans/planning notes for their topics, which I examined for any references to connections.

Interview 3

Earlier studies (Businskas, 2005, 2007) indicated that teachers were very general in talking about mathematical connections and seemed to draw on very little specific mathematics content when providing examples of mathematical connections. In Interview 2, I gave teachers an opportunity to speak naturally about a mathematical topic of their choice. But I also wanted to create a situation that would constrain teachers to focus very precisely on fine-grained connections. I considered some other types of tasks used in mathematics education research. I thought that a problem-solving task as used, for example, by Evitts (2004) was unlikely to provoke the level of specificity that I was seeking. Other tasks, like having teachers critique a lesson plan or script, or create a lesson plan of their own, from a “connections” perspective, might provide the precision I sought. However, I was concerned that such tasks would bias teachers to consider the topic entirely from a teaching point of view, and thus would be more appropriate in future research that examined teachers’ practice.
So, I created a task to meet my needs. The third interview was a task-based interview in which all teachers dealt with the same topic – quadratic functions and equations. Having all the teachers focus on the same topic gave me an opportunity to examine commonalities and differences in the way they conceptualized connections. Teachers were given a set of 82 cards (Appendix B) containing mathematical terms, formulae and graphs related to this topic, gleaned from a selection of high school mathematics textbooks. I asked them to organize the cards in some way that showed the relationships among them and then to explain their organization.

When I tried the task, before administering it to the teachers, I found myself constantly thinking about multiple connections for many of the cards. Hence, I was hopeful that the task would allow teachers to produce unique organizations and focus on a wide range of connections, based on their own understanding of this and related topics.

I introduced the task by using the same script with all the teachers.

Today’s interview is the most structured of the ones we’ve had so far. This time the task is the same for everyone. The theme is Quadratic Functions and Equations. Just think about your understanding of the topic; don’t restrict yourself to how your students might approach it. I have a set of 82 cards [actually “sticky notes”] which I will ask you to organize. I will ask you talk about your organization and also ask you some questions about it.

These cards contain names of concepts and skills, diagrams, and symbolic notations. I have chosen them from high school mathematics text books. However, the card stack is not meant to be an exhaustive list. Your task is to organize these cards in some way that shows the relationships among them. Please group them in a way that will show how they are connected. The number of groups is up to you.

Please try to use as many of the cards as you can. Hopefully, you will use most of them, but you may not
necessarily use them all. Also, you may find that you want to include a card that is not in this set, in which case, you will be able to make a new one and use it. Please place the cards on the chart paper as the record of your organization; you can also attach cards to each other. You can write on the chart paper to identify relationships or anything else you think it’s important to record. At any time, you may refer to any reference materials that you brought with you.

Again, I had prepared some types of questions (Zazkis & Hassan, 1999), to ask the teachers while they talked about their organization. However, the use of these questions was much more fluid and *ad hoc* because the teachers needed different degrees of encouragement to identify the connections that they were making.

1. Please talk about what relates the cards in each group with the others [in the same group].
2. How are groups of cards related to each other? To what other mathematics topics might these groups be connected?
3. Please talk about any new cards that you included – how they are connected to ones already in the set, and why you think they are necessary.
4. For the cards you omitted, please explain why, one by one.
5. This task was designed to draw out your view of how mathematical ideas and skills in one topic area are connected. Please comment on the task, especially about the degree to which what you did truly reflects your understanding.

**Timelines**

I met with each teacher three times, at their convenience, and in their own schools. Each interview typically lasted 25-40 minutes, although a few ran longer because the teachers had so much to say. I was very cognizant of their generosity of time, and adapted my own schedule to theirs. The interviews for each teacher occurred at roughly one-month intervals, February to May, 2006.

**Participants**

The nine teachers who participated in this study were all volunteers from two high schools in the same school district, three from one school (population 900) and
six from the other (population 1800). The number of participating teachers was roughly 60% of the full-time mathematics teachers at each school. Seven participants were women; two were men. This ratio roughly reflected the gender split among mathematics teachers at the participating schools. They started teaching from 1994 to 2005, thus, at the time of the study, had from 1 to 12 years’ teaching experience. They formed two clusters with respect to experience – 6 teachers approaching mid-career with 8 to 12 years’ experience, and 3 beginning teachers with 1 to 3 years’ experience. Their teaching assignments ran the gamut from Grades 8 to 12 and included modified, regular, honours and Advanced Placement courses. One of them was a department head.

To seek participants for this study, I first approached the school district in which I had been both a teacher and, later, an administrator. The fact that I was known and respected in the district gave me a unique opportunity to approach teachers and be taken seriously. After obtaining approvals from the superintendent’s office and secondary school principals, I spoke at a mathematics department heads’ meeting to describe the proposed research and to ask for their support. I had immediate interest from two schools. I then visited those two schools and spoke with mathematics department members. As a result of those meetings, nine people volunteered to participate. I use pseudonyms for the names of both schools and teachers as I refer to them throughout this study.

Christine, Edward and Robert taught at Valley. Valley is one of the smallest schools in the district and takes pride in its community spirit. Lily, Sophie, Darcy, Nicole, Wendy and Josie taught at Seaside, which is one of the largest schools in the district. Seaside is known for its academic achievements and has become an unofficial
magnet school for the Advanced Placement program. Both schools are comprehensive public high schools, offering programs for students of all ability levels.

While the two schools represented the range of demographics across the district, the teachers who volunteered were not a representative sample. Despite my best efforts, I was not able to attract late-career teachers as participants. There did not seem to be any overwhelming reason, simply a lack of energy and a touch of cynicism about university research. They wished me well, but did not want to exert themselves.

The participating teachers shared some common characteristics. All but one were mathematics majors. They were all confident in their knowledge of the provincial curriculum for the courses they taught, but relied less on the provincial curriculum guide and more on their own department-developed course outlines and textbooks in day-to-day work. All of them began their careers after the spread of the NCTM Standards and might be expected to be “information-rich key informants” (McMillan & Schumacher, 1997, p. 397). The process standard of connections had already become part of the lexicon of these mathematics teachers when they started teaching, largely through the efforts of the British Columbia Association of Mathematics Teachers (BCAMT). However, while they all remembered being introduced to the NCTM standards in their teacher education programs, they claimed little to moderate knowledge of the Standards and did not refer to them in their day-to-day work. Their professional development activities were local; most attended the BCAMT conference that is held on a Province-wide Professional Day in the fall, but no others, and belonged to only one mathematics education professional association - the BCAMT.

I had known all of the participating teachers to some degree before this study, although I no longer work with any of them. Sophie and Darcy were two of the three
teachers I had interviewed in the pilot study mentioned earlier. As a result, we began
the interview process with a high level of trust on both sides. They felt that they could
speak freely and honestly, and I felt that I could rely on them to do so (Knight, 2002).
My audiotaping of the interviews did not concern them.

Data collection and analysis

For each teacher, I had the following data:

- questionnaire,
- Interview 1 transcript,
- Interview 2 transcript together with any notes or drawings the teacher
  made during the interview,
- a copy of the teacher's planning notes for the topic discussed in
  Interview 2,
- the teacher's organization chart for Quadratic Functions and Equations,
- Interview 3 transcript (discussion of the organization above),
- my own brief field notes.

With so much data to consider, interpretation “on the fly” was inescapable.

Experiences do not speak for themselves; nor do
features within a research setting directly or
spontaneously announce themselves as worthy of your
attention. As a qualitative fieldworker, you cannot view
your task simply as a matter of gathering or generating
‘facts’ about ‘what happened’. Rather, you engage in an
active process of interpretation: noting some things as
significant, noting but ignoring others as not significant,
and missing other potentially significant things
altogether (Schram, 2003, p. 9).

Teachers said a lot about issues in their teaching lives that appeared not to be
directly applicable to the goals of my study. As I listened to tapes and examined
transcripts, I did so through a self-imposed filter of looking for statements that might,
in the broadest sense, be teachers’ expressions of ideas about linking mathematical
ideas together. Certainly, I looked for terminology like “connection”, “relationship”,
“link”, “they need this to…”, “another way of looking”, “this is just the same as…”,

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“if they know this, then they can...”. However, I also tried to attend to a holistic interpretation of longer statements which might not have specifically mentioned connections but seemed to contain the essence of the idea, for example, when teachers talked about understanding, and “deep understanding”.

As the interviews progressively constrained teachers to be more specific and exacting, so too did the coding of the transcripts become more precise and rigorous. For Interview 1, I did a very simple coding, in which I identified statements made by teachers that were broadly about connections, regardless of the particular language used. As I read each transcript, I highlighted the relevant statements. I processed each transcript twice, with at least a week between readings. The selected statements became the raw material for developing summaries of each teacher’s views.

In analysing the transcripts for Interview 2, I began the same way as for the previous interview. I extracted statements that were broadly about connections, then coded the selected statements as general statements about connections, or as specific examples of connections. I tried to code the specific examples that each teacher gave using the categories of the model that I proposed in Chapter 2. Using an inductive process (Gall, Borg, & Gall, 1996; Huberman & Miles, 1994; Miles & Huberman, 1984), I also looked for salient characteristics of the teachers’ responses regardless of whether they were consistent with my connections model.

I had intended to similarly code teachers’ planning notes for the topics that we discussed in Interview 2. However, even though some teachers gave me detailed, multi-page teaching notes, they contained no explicit references to connections. So, I simply read their notes and inferred what I could about implicit indications of connections.
I analyzed the refined data from the first two interviews in two ways. I wrote a profile of each teacher that represented in more compact form, his/her positions about connections, using his/her own words to illustrate. And, I aggregated statements that teachers made about particular mathematics topics and examined them from a mathematical content perspective, that is, rather than focussing on what individual teachers said, I focussed on what was said about a particular topic, for example, conditional probability.

I designed the sorting task for Interview 3 to provide teachers with a specific mathematical vocabulary, which I hoped would encourage them to articulate many mathematical connections. While I did consider the broad features of the card arrangements they made, my focus was on what the teachers said about their reasons for grouping cards. In analyzing Interview 3, this task-based interview about Quadratic Functions and Equations, I started by extracting each teacher’s statements about connections. I then tried to code those statements according to my a priori model which had seven categories (described in detail in Chapter 2):

1. alternate representation
2. equivalent representation
3. common features
4. inclusion
5. generalization.
6. implication
7. procedure

Some categories were used by teachers rarely (#3, 4), and others were difficult to differentiate in practice (#4, 5). Sometimes teachers identified mathematics concepts as being related in terms of their role in teaching, rather than their
mathematical structure. Most commonly, this occurred when teachers identified certain ideas or skills as being prerequisites for learning the new topic.

Therefore, I revised the coding model to five categories of mathematical connections. The first four categories were based entirely on mathematical features. The fifth category was added to include relationships among mathematical objects based on teaching/learning. I emphasize the revision of the model that I undertook based on the initial analysis of the data. As I described in Chapter 2, my original focus for the study was at the interface of teachers’ content knowledge and pedagogical content knowledge. As I nudged teachers’ conversations toward my research questions by my queries, there was a reciprocal nudging of my focus by their responses. The teachers’ conversations clearly demonstrated how bound up their own knowledge of mathematics was with the mathematics that they taught – the two seemed inseparable. Moreover, mathematics and teaching mathematics seemed inseparable as well. To capture this very significant aspect of the way that teachers viewed mathematical connections, I added the category of instruction-oriented connections, where the linked objects are mathematical, but the reason for the association is pedagogical.

Here is the revised model:

1. **Different representation.** A is another representation of B. This category combined the alternate and equivalent distinctions (#1 and 2) that I made in the earlier scheme. The distinction did not seem to matter to teachers. However, I will still refer to alternate and equivalent representations at times where it is helpful to the analysis.

2. **Part-whole relationship.** A is a component or specific instance (example) of B; B contains A or is a generalization of A. This category combined #4
and #5 above. Both of the original statements have an impression of part/whole to them. Teachers predominantly made connections by identifying or using examples, occasionally by extending or generalizing, rarely by talking about components as described in the earlier #4.

3. **Implication.** A implies B.

4. **Procedure.** A is a procedure used to work with B.

5. **Instruction-oriented connection.** A and B are both concepts or skills that must be known in order to understand/learn C.

   Within each category, where the statement was specific enough, I identified the particular mathematical information that constituted the connection statement, for example, the $m$-value in the equation $y = mx + b$ is [an alternate representation of] the slope of the line. See Appendix D for the coding scheme.

   While I examined the actual layouts of the way that teachers organized their cards, I focussed less on what they put together, and more on their reasons – why they thought certain cards or groups of cards were related. Most of that information was contained in their comments about the organization.

   I also searched for patterns in what teachers did not connect by looking for common characteristics in the reasons teachers gave for leaving cards out of their layout, an indicator of what they saw as an absence of connection.
CHAPTER 4: TEACHERS’ INDIVIDUAL POINTS OF VIEW

Teacher profiles

Based on the conversations in Interview 1, I developed descriptions of the participating teachers. In these profiles, I tried to give a sense of what drew them into teaching mathematics, how they thought about their teaching, and the degree to which they spontaneously thought in terms of connections when discussing mathematics. Certainly, statements of fact, like years of teaching experience, are purely descriptive. However, in much of each profile, the portrayal of each teacher’s thoughts about teaching and especially, his/her thoughts about mathematical connections, is the distillation of the analysis of many pages of interview transcripts.

There was a considerable range in the way that teachers incorporated the idea of mathematical connections in their conversations. Some teachers seemed not think in terms of mathematical relationships until prompted (at least, not in any way that was evident in conversation). Others did not use the terminology of “connections”, but had their own metaphors for weaving a web of mathematical relationships. For some, thinking and speaking about connections was a pervasive and natural way of talking about mathematics, their teaching and their students’ learning.

The profiles that follow are arranged in a sequence to illustrate the range of teachers’ thinking about connections, starting with some teachers for whom the metaphor of connections was not the way that they naturally thought about mathematics. They are followed by others, whose conversations contained many spontaneous references to connections. However, the references were often very
general and seemed to be a general expression of “everything is connected to everything in mathematics”. Finally, there were a few teachers, for whom thinking in terms of relationships and patterns was a central aspect of their view of mathematics, and they were focussed on teaching their students to think in the same way. I used teachers’ own words to support the accounts. Sometimes a longer quotation was needed to provide an authentic and contextual sense of a teacher’s thoughts; in such cases I highlighted pertinent phrases by using boldface.

**Sophie**

Sophie has taught mathematics for eleven years, all of them at Seaside, after a short-term contract at another school. She was drawn to teaching mathematics as a career by her love of mathematics. It is the logic of mathematics that appealed to her and she excelled at it from an early age. What she remembered most about her high school experiences in mathematics was “a lot of teacher at the board” and having to work things out for herself or with her friends.

She felt that her university mathematics courses had prepared her “fairly well” for her present work, particularly the courses that involved proofs. She described how these courses had left her with a certain disposition to the way she approached mathematics, an attitude that she tried to pass on to her own students:

... the best way to solve something, and showing your steps, and being exact and not assuming things ... I’m always saying things like that to my students – don’t assume things.

She modelled her own teaching on that of professors she had had who were very organized and sequential, because that was the way that she learned best – “I make sure that I’m very organized, my notes are very structured”. In fact, her
planning notes were very detailed, like scripts for her role in the lesson. She wanted her students to become confident in their study of mathematics.

In trying to clarify her goals, I asked her this question, "if I asked students what they learned in your class, what would it please you to hear?". This question perplexed her.

I don’t know... Hard work pays off... what they learned, that’s tough... I know what they’ll say, they’ll say being neat... But what did they learn? Maybe I don’t think about that enough.

In many of her descriptions of her work with her students, she was very concerned with how she presented information and with teaching students good habits, like showing the steps, and, of course, being neat. She did a lot of direct instruction in her teaching. She felt a great sense of responsibility to prepare her students well.

In teaching trigonometric identities, a topic that Sophie thought she taught effectively, she was very thorough, but also wanted her students to share some of her enjoyment.

I make sure that the examples have all the main strategies that are used. So during the examples, I point out these strategies, and then at the end, we write a summary of the strategies. And... even like a little pep talk – there’s not one correct way, you just try things, just don’t make any mathematical mistakes, don’t do anything you couldn’t do with regular algebra, and it works out. Like, don’t worry about it, just try it, it’s kind of fun, and you know, when you get them to work out, it’s fun.

Sophie could describe what she did, but had a hard time explaining why it worked. In fact, she concluded that she had taught this topic effectively “because the students do OK in it.”
Sophie started to talk about connections when describing topics that students found easy and hard. She ascribed students’ finding the topic of permutations and combinations easy to the fact that “it does relate to real life things and you can visualize”. And, logarithms was a difficult topic because “there isn’t really anything in real life that they [students] can relate it to easily”. This belief arose from her own learning style – “I like to visualize things, and you can visualize real-life things... I’m constantly relating it.”

Beyond that, she seemed puzzled by students’ difficulties with the topic of logarithms – recognizing the problem but not knowing how to deal with it.

A logarithm is an exponent, and changing back and forth between log form and exponential form, those are the main things. Actually I don’t understand what they don’t understand... I feel like I’m doing my best, but it doesn’t quite work... I have to do something different.

Sophie valued understanding in her students and drew a distinction between conceptual understanding and the memorization of procedures and examples.

Solving equations... The big picture is isolating the variable, understanding that you’re undoing, that multiplying and dividing are the opposites, inverse of each other, that’s conceptual, but you could make a mistake every time... if your integer skills aren’t good. Then there’s the memorizers, those are the procedural people... developing conceptual understanding... that’s the most important thing.

In offering examples to illustrate conceptual understanding, she started talking more about relationships. She gave an example of understanding a parabola.

It’s a graph of an equation with an x squared in it, the direction of opening, the shifting... they [students] have to be able to visualize and connect any equation with any graph, concept of maximum and minimum... and you could relate it to a maximum and minimum problem... that’s what they have trouble with, relating.
She invoked the algebraic form and the graph as alternate representations, but in general terms. Her one specific example described the relationship between the coefficient $a$ in the equation and the amount of expansion/compression compared to a standard parabola – “he knew that a 2 in the front was an expansion”.

There are hints in the ways that Sophie talked about this topic of a conflict between her stated valuing of conceptual understanding and her recognition that she regularly slipped into the procedural mode herself by simply teaching algorithms for doing questions – “at some point, you have to help them be able to pass the test”. She claimed that time constraints limited her.

On the whole, Sophie had trouble saying much about relationships. She described connections in terms of visualization, which sounded like she might be thinking of alternate representations where one forms a mental image of an idea. However, on further probing, it turned out that what she meant by visualization was “just have the steps outlined in your mind before you start writing them down”, akin to a mental rehearsal of a solution.

The term, connections, was not naturally part of Sophie’s language. She said very little about connections spontaneously. When I pressed Sophie to consider connections, for her, connections meant applications to real world situations.

**Wendy**

Wendy has been teaching for eight years and is delighted to be at Seaside. She had volunteered at the school before entering her teacher education program. In fact, she was inspired by one of the other mathematics teachers in this department [not one of the participants in this study], whom she saw as a model.
She was so... clear in her instructions. Her notes were fabulous and she was just nice too... she was approachable... She made stuff seem easier and more interesting than other teachers.

For Wendy, mathematics was easy and fun. She enjoyed the excitement of problem solving and the thrill of "getting the answer". When asked about her university mathematics courses, she could not remember many of the courses. Some courses, like history of mathematics, she found useful because she could draw on what she had learned when helping students with projects. Others, like calculus and linear algebra, she saw as helpful because they gave her some familiarity, a "general idea", that made it easier to prepare for her high school courses.

Wendy found her teacher education program, where she was in a pilot program "just for math teachers", more helpful than her university mathematics courses. She recalled a methods course which emphasized that students "should be problem solving, not... just getting question/answer", and spoke enthusiastically about this approach.

Here's a problem and in life there's not always a solution to every single problem... you should approach it by what are the possibilities. So just to get the kids to think broader than what's the way to get the answer... to get them to think more in a wider range.

She re-iterated that broadening students' mathematical thinking, that is, encouraging them to consider and explore multiple possibilities in a mathematical situation, was one of her teaching goals. Nevertheless, she categorized open-ended problem solving and projects as "fun stuff" that she might have time for in the junior grades, but not with the seniors. Furthermore, social goals of teaching were at least as important to Wendy. She wanted her students to be good citizens.

I really don't believe that I'm just teaching math. I'm teaching sociability as well... responsibility, all that
I find a lot of kids seem to lack a lot of common sense... so I try to get them to think, you know what, you come here, you have to make sure you’re responsible, you know, what you say is important too, so I’ll make sure you’re not offending anyone, that kind of stuff.

As far as learning mathematics was concerned, Wendy wanted her students to enjoy mathematics, to have them believe “you know, this is not so bad... I don’t have to hate it... I can actually even get through”. This desire seemed to be the driving force for her teaching – “I always try to make my lessons as simple and clear as possible for them”.

In discussing mathematical topics that she taught effectively, she named trigonometric functions but could offer little insight into what she did.

**I think I do a lot of examples with them**... like I dissect it a little bit. I don’t just say OK, let’s graph it. **We talk about it**... I felt that I taught that effectively just because the marks are really great... I just felt that I must have done a good job for them to actually get it... **I don’t have anything specific** though.

Talking about her use of examples was the closest Wendy came to articulating connections. She described using a progression of examples, starting with “very simple ones” and gradually working up to more complicated and difficult ones. Each example was a small extension from the previous one.

The first one might be... just graphing a sine graph... and then of course, there’s the shape of it, compared to the cosine one and then we tag on like, the shifting from left to right, up and down... and then we talk about the one with the amplitude...Basically... the vertical first, and then we do the horizontal, cause that’s the toughest one cause it deals with the period... I just tell them the order they have to go through. If you go in this particular order, your answer will come out nicely.
When asked why she linked her examples the way she did, she thought it was “just easier”. Although Wendy often mentioned students’ understanding, she also often mentioned how she shows them or tells them what to do.

All in all, Wendy made little specific reference to connections. When she did, she saw concepts and procedures as being “tied together”, and saw this connection as a characteristic of “the really bright kids”. She equated “procedural style” to “showing the work”.

I think students still need to be shown how to do procedural. I’m sure they have a concept in their heads. Of course, or how else would they get the answer. But a lot of them couldn’t express it… I actually had to teach them concepts to actually “show the work”… Now they can actually literally explain to me why the problem is the way it is and how the solution is.

Wendy grasped the necessity of her role in making students’ implicit knowledge explicit, by teaching them to articulate and link the steps in their thinking.

In discussing topics that students found difficult, Wendy talked about logarithms as being “so out there… so different for them [students]… the concept of logs, what is it?”. Asked to answer the “what is it?” question, she said:

It’s an exponential graph actually, right? It’s tied in with exponents, right?… When I explain that to them I think it’s hard… cause log is a word… With a graph, you see numbers, you know, not just a word. So for them, I think it’s hard for them to get a head around that it’s an actual graph and log is a number. Cause, a lot kids, you say log 5 and they just think it’s log 5 and they can’t graph it as a decimal number.

And later, in trying to explain how a good student might understand logarithms, she said:

The concept of the graph, to understand what the logs look like and the relationship between logs and
exponents... to actually see the connection and the relationship. I think that might make them understand logs a little more, that it’s something that’s a number.

Wendy recognized the importance of connections and the role of connections in understanding.

I don’t feel like I’ve done a very good job in connecting it together, and that’s why I think they don’t do as well.

Her solution would be to “spend more time with it”, doing more examples and practice. In fact, she saw it as a flaw of the current textbook, that it had “minimal drill”. Even though Wendy had a possible way of helping students in mind, she felt “we don’t have time... to tie in the connections or relationships”.

In fact, a recurring theme in Wendy’s conversation, was that external expectations, like provincial examinations and department-wide testing, and lack of time prevented her from taking a problem solving approach and from emphasizing connections.

Wendy’s explicit notions of connections seemed rudimentary. When she mentioned relationships, she expressed them in general terms. While she believed that connections were a valuable indicator of understanding, she seemed unaware of how to help students to see them – “It’ll kind of just sink in eventually, hopefully.”

Nicole

Nicole was in her third year of teaching at the time of the study. Even within the short time she had been at Seaside, she was seen by her colleagues as having leadership potential. She was drawn to teaching by the enjoyment that she herself found in mathematics as well as by a desire to help others find that sense of fun.

I love doing math, so just to be able to do math every day is fun for me, but I also like helping them [students]
see that and helping them kinda get to that point as well
is something else that I like.

Being available to help her students was very important to Nicole. Her undergraduate experiences strongly influenced the type of teacher that she has become, particularly the contrast that she saw between professors who were always “open to answering any questions and helping you wherever necessary” and those who gave little time or interest to their students outside of class. In her own teaching, she went out of her way to urge students to ask questions and come for help.

Another important value in Nicole’s teaching was her emphasis on effort and “work ethic”. Regardless of their capabilities, Nicole wanted all her students to do their best, to set a goal and to achieve it.

I say, do all your homework. If you can do the homework and if you finish it and if you get all the answers right, you’re going to do well in the class. It means you’re understanding it. But some of them don’t do their homework and don’t even give themselves a chance in that. But I would want them to do the best that they can.

Getting her students to develop organizational skills and a strong work ethic were just as important to her as “learning the material”.

What she gained from her university experiences in mathematics influenced her motivation and teaching style, but “the math you take in university has not much relation to what you’re doing mathwise in the school”. She reiterated this idea, that the mathematics content she learned in university had no relevance to her high school teaching, in that she made no direct use of it, several times. She enjoyed her undergraduate mathematics courses for their own sake, and thought they gave her “a broader sense of math”, but claimed to have forgotten a lot of it – “I don’t remember
90% of it anymore”. She has learned or re-learned the mathematics of the high school curriculum through teaching it.

She identified trigonometric functions as a topic that she taught effectively, but could not articulate what made her teaching effective. She thought the cause might have been that this was a topic that she remembered from her own time in high school. This discussion contained the first hint of Nicole’s thinking in terms of connections. It is an instance of her considering her own mathematical understanding.

I remember doing this, so something I can build that I knew before and I could kind of work from that and figure out things that I remembered doing and how I did them... I guess because I had that previous knowledge it kind of helps make that a stronger area for me.

She implied that her ability to link the mathematics in the curriculum to her own prior knowledge of trigonometric functions made her teaching more effective. As well, implicit in the statement above, is Nicole’s operational view of what understanding means. If students can do the questions, then they obviously understand the work. Nicole described a conversation that she had with Lily about a question on a Mathematics 12 test.

We’re looking at the same question and we did it completely differently, and she said I don’t know why I do it this way and I said, I don’t know why I do it this way, but we just did it completely different methods. And she teaches it to them that way and I teach it the other way.

Again there is a procedural tone of teaching methods to “do” questions, a concentration on how to arrive at an answer to the problem, rather than to explore and understand the wider mathematical context of the problem. Moreover, when I asked Nicole if she and Lily had discussed the two methods in relation to each other, she simply concluded that the difference “must be from ways that we learned it”, and did
not pursue the topic. This discussion did not lead either of them to incorporate teaching both methods.

Nicole identified permutations and combinations as a topic that students find easy, and invoked connections as the explanation – “because it’s something that they can relate to a bit more” [than trig functions or logarithms]. The connections she referred to were “real world” connections – “something that they deal with every day and they can relate to it”. But she also introduced the notion of connections among mathematical ideas, where new ideas are thought of as extensions of other ones.

The topics kind of relate to each other as you go through, so it’s not a whole bunch of information that they have to learn. It’s kind of one or two topics you kind of just change up a little bit.

Nicole identified logarithms as a topic difficult for students. She was very confident teaching it and was puzzled why students had so much difficulty with it. She speculated that students might find logarithms hard because the topic is new, something students haven’t seen before, or that, because the topic comes early in the year, students haven’t “built up that work ethic” which would enable them to persist, or because “there’s a lot of information that they need to remember”. She thought that, even after studying the topic, students did not really understand what a logarithm was.

The kids say, ‘what is a logarithm?’ Well, I think, ‘what is a logarithm?’ I don’t even know if there is a definition for logarithms.

When I asked her for her definition, she defined a logarithm as “an anti-exponent”, and wanted her students to be able to move between logarithmic and exponential forms, that is, between two forms of representation, but did not believe that her students really understood what a logarithm was.
She distinguished between rote application of “the method of how to do the questions” and understanding how to do a question, but admitted that she mostly taught rules. Although she did try to “throw out some questions in the end where they actually have to think about it”.

In her teaching, connections meant connections to prior knowledge.

To build on those things that they’ve learned before – it’s something they’ve learned before and we’re just adding something new on to it.

Moreover, her approach was ad hoc. She did not include connections in her planning. Rather, she spoke in terms of tying in things that they’ve learned before “if I can see something that’s connected to something we’ve done before”.

When I asked Nicole how she tied in prior knowledge, she gave an example from her teaching of equation solving in Grade 8 as an illustration.

Let’s do an easy one, \( x - 3 = 5 \), and let’s do an easy one where we just collect our variables, or like terms, or let’s do an easy one where we just do the distributive property. Let’s... put a couple of them, and maybe let’s just add another thing in here. And so, you kind of just take one step at a time and instead of just throwing a hard question at them, you’re just kind of building on to something from before and you make it just a little bit harder as you go along... So it’s just that one extra step you’re adding in and all of a sudden it’s the same as before.

She described starting her lessons with a warm up in which she reviewed relevant prior learning with the students “to get them to see where those things come from”. This notion of linearly connecting one idea to another was also reflected in her planning. She wrote out her lesson plans to guarantee that I haven’t missed, forgot to say anything, and then I can have these good examples that go along from each other. And I always try to lay them out nicely so that they do work in that kind of sequence.
as they go along instead of trying to come up with them on the top of my head. At least I know that I’ve got good examples down that they can see the connection of as we go through.

For Nicole, examples were the primary vehicle for demonstrating connections. In all the illustrations that she offered, the underlying structure was -- start with something the student already knows, and gradually add one simple idea or step at a time to achieve learning a new idea or procedure. She seemed to have an implicit belief that the careful choice and ordering of her examples would allow students to see connections spontaneously.

Josie

Josie was in her first year of teaching. Teaching was her first career and she was in her first job. The year before, she had completed her teaching practicum at another school in the same district. Josie has a Bachelor’s degree with a major in Mathematics. She saw herself as being very well prepared for teaching high school mathematics both in terms of her knowledge of the content and her knowledge of the BC curriculum and the NCTM standards. Josie is an immigrant, having come to Canada while she was in elementary school.

Josie was drawn to mathematics for the fun of it, and enjoyed it as a hobby.

I like those non-routine problems, problem solving. I don’t like calculus, because I think it’s just memorizing formulas, so I like number theory, I like proving and I like trying to solve problems, and logically, cause I think math is just, math is a language where it requires logic.

She was motivated to enter teaching by the example of one of her own teachers.
I wanted to become like him, to try and touch people's lives, not only in the subject area, but also to try to help young people, try to help them grow.

While she saw little direct relationship between her university mathematics courses and what she taught in high school, she valued her university experience because of the depth of understanding that it gave her. Although Josie rarely used words like connection, relationship, link, she talked a lot about the importance of understanding and described a metaphor that exemplifies mathematics as a web of closely connected ideas.

I know the map of the city. I've not just memorized the map, but... I've been there and I lived in this neighbourhood, so... I know every building and I know everything around in the city, right? So that if I want to go from A to B, I can take any routes I want, but if there's someone who doesn't know this city very well, they will memorize oh, you turn left here, you turn right here... If they get lost in one place, they wouldn't be able to turn around and go. But I know the whole thing, it's like I'm standing on a mountain, and I can see the whole thing. I know how each road will go. I think that's my way of saying deeper understanding... Math is about understanding the whole thing, and then you can go from one place to another.

Her metaphor made me recall Greeno's description - understanding mathematics is like “knowing your way around in the environment and knowing how to use its resources” (1991, p. 175).

Josie repeatedly distinguished between “understanding” and “just the steps”.

I try to teach the core more...not the steps, but the reason behind it and the concepts.

As an example, Josie offered:

Solving equations with variables on both sides... One way to do it, is to collect x first... or you can bring the numbers to the other side first, also two ways to write it... So this I think is different ways to go to the answer.
When challenged to differentiate between concepts and procedures, she said

**They’re all concepts** right? First of all, \( y = mx + b \) is a concept. It’s a concept because it’s a relationship between \( x \) and \( y \)... every line has an equation, and it can be changed, it will be in the form of \( y = mx + b \). So it’s not just a memorized thing... I just use the formula. But I have to teach why it works first... **Even though they [students who memorize the formula] get the same answer, their understanding is very different.**

Josie described how, in her teaching, she tried to link what students were learning now to what they had learned earlier, and to forecast how the current concept might fit in to later work. She gave domain and range as examples of concepts that recurred throughout the curriculum in relation to different types of functions and relations – lines in Grade 10, parabolas in Grade 11, and circles and ellipses in Grade 12.

When she gave an example of teaching effectively, she ascribed her success to showing her work clearly and using colours to illustrate. She went on at length about how she used colour coding to teach graphing linear functions. At first, she seemed rather obsessive about using colours, but eventually it became clear that the ultimate purpose of her colour coding was to reinforce connections. For example, she put \( b \) of the equation and the \( y \)-intercept of the graph in the same colour to show that they were the same feature of \( y = mx + b \), just represented in algebraic and graphic forms.

Josie’s model of understanding was based on reasoning from some basic principles. While she could not articulate how thinking like this differed from applying rote-learned procedures, she was convinced of the difference. She was deeply committed to teaching for understanding, but saw herself as fighting an uphill battle. In fact, one of the recurring themes in Josie’s conversations was her focus on perceived obstacles.

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I kind of struggle, I don’t want to teach just the steps... the reality is that some students... don’t really want to connect them... I still have... my will to teach for understanding... we have common tests, so it’s really restricting your teaching... the order of the textbook is really bad... I’m already in trouble because I have to teach that order and also don’t have freedom, don’t have as much freedom to go over something that I find more interesting than what the text has to say.

*Lily*

Lily has been teaching at Seaside for three years. She wanted her students to be engaged in her mathematics classes and to experience some of the fun that she saw in mathematics. Moreover, it was important to Lily that her students think for themselves.

Don’t just accept because I said so. Think why, where did this come from,... always be questioning.

She spoke movingly about how she developed this perspective as a result of methods courses that she took as a student teacher.

One of the courses I took, I think with Rina actually, I don’t even remember what it was called. It was an extra course on problem solving. It was really good. It really made you open up and think about, asking why, why and I don’t even, I couldn’t even tell you a specific problem but that whole mentality, she really instilled that in us – keep asking questions, let’s look at this a little bit more, cause it kind of just opened your eyes. It just made you see things in a wider perspective. And I guess the other methods course was I guess, Designs for Learning whatever it is, Peter taught, and that was fantastic. It was so inspiring. And that’s what it was. You would sit there for four hours. You wouldn’t even mind sitting there for four hours because it was just constant, OK, how would you do this, how would you do that, and it made you realize how different it could be, I guess. And really how your perspective could be changed, well, mine did, anyway. So those really changed me.
She did not look back so favourably on her undergraduate mathematics courses. She struggled in her classes until she learned to “teach myself”. She credited her own study habits and what she learned in her teacher education program with preparing her for teaching high school mathematics.

Lily was enthusiastic about the importance of connections –

I get really excited... even if it’s the smallest little thing, the fact that they can make those connections without any prompting.

Despite her zeal for thinking and process as opposed to memorization, Lily had a hard time being specific. She talked about relating to prior knowledge, building on students’ knowledge in general terms. At times, she seemed unsure of her own role. When asked about explicitly stating connections to prior knowledge, she said:

I would like to say that I would try every lesson, but I probably don’t. I try as much as I can... actually I do it a lot, now that I think about it, ... I probably do it and don’t even notice.

Lily’s last statement raised the possibility of some teaching behaviours becoming automatic and tacit, and therefore, ones which would not be captured in this study with its focus on features of which teachers are explicitly aware.

With continued probing to talk about examples of topics that are easy/hard, she offered some examples. She thought that students in Grade 12 found learning trigonometric functions of special angles easy because it was an “easy transition”.

It’s really familiar to them, the idea of ratios and that sort of thing. So it’s really easy to start with a unit circle and take the ratios in a unit circle because they’ve also done with circles, they know the radius is one... and I’m just going, well, what if we just extend it, and look at quadrants... So I just feel like that it’s really easy to transition them cause I’m not really asking that much more of them than they already know.
She introduced an interesting notion here – something like the length of a connection – the new information was just a little bit more than they already knew, so they could see the connection. Lily did not mention Vygotsky; but her description is certainly reminiscent of learning within a student's zone of proximal development. While Nicole did not describe her strategies in terms of connections, she essentially did the same thing as Lily – built up her students' knowledge one little step at a time. For both Nicole and Lily, a small step, or “short” connection, seemed to be an indicator that students would spontaneously grasp the connection.

One of Lily's examples illustrated the importance of the teacher's intervention for students to make connections.

They can't add... π + π/4 and I have to go back... sometimes I actually have to show them. OK, let's pretend π is not there, what's 1 + ¼? and they can do that. You kind of have to, sometimes you have to take them through it, sometimes they just need that one little prompt and then they're fine.

In her “one little prompt”, Lily explicitly drew students' attention to thinking of π + π/4 as an extension of 1 + ¼ (which they did know how to interpret). But later, while talking about why students find the topic of logarithms so hard, she was not so sure how to intervene.

Logarithms, I don’t know, it’s just one of those things. And I don’t know what I can do better to help them.

She thought the key prior knowledge that students needed was the laws of exponents, and that many of them didn’t really know those laws. As for the mathematical ideas involved in this topic, she identified the definition of a logarithm and “a firm understanding of switching back and forth from exponential form and logarithmic form, back and forth”. She recognized the connection between
exponential and logarithmic representations as important for students’ understanding of the topic.

*Christine*

Christine was the most experienced teacher in this group, having taught for twelve years and at three different high schools. She was attracted to mathematics “purely for the fun of it”. She recalled doing binary mathematics at home with her father when she was in Grade 5; her favourite university mathematics course was Number Theory, in part because she saw it as free from applications and meaning in the “real world”. While she really enjoyed her university mathematics experience, she found little use for what she had learned in teaching high school mathematics. She said that, with the exception of Calculus,

> the math you take in post secondary is so far beyond what you’re teaching in high school, sometimes it’s hard to see an actual direct connection.

She did see value in her mathematics background in helping her “to have a deep and complete understanding of what it is you’re teaching”, but did not see a mathematics background as being essential. She claimed to know several excellent high school mathematics teachers who had little or no post secondary mathematics training.

> They know the high school curriculum backwards and forwards, can do all of it, have a deep understanding of it and can convey that and get the kids to understand it without having gone beyond that themselves.

Entry requirements for a teaching specialization in mathematics are evidence of the accepted wisdom that high school mathematics teachers need a postsecondary course of study in mathematics. Thus, Christine’s view was unusual. It seemed that
the mathematical knowledge that she thought was essential, was “pedagogically useful mathematical understanding” (Ball & Bass, 2000, p. 89).

I asked her what she meant by a “deep understanding”, to which she replied:

I think a deep understanding ... indicates seeing how things connect to one another, how the different topics interweave into each other and how they’re connected and how one can lead into the next. Also being able to explain things in different ways. If you, if the only way that you can explain it is, for example, the way the textbook explains it, and then a student says ‘I don’t get that at all”, you need to have a deep understanding to be able to say, let’s go at it from a different direction and be able to approach it differently. So you need to really understand a topic to be able to circle around it, to see where you can get the student to understand it.

Christine defined understanding a mathematical topic in terms of connections. It is clear from her statement that she was talking about her own understanding of mathematics. But she saw her knowledge entirely in pedagogical terms, in that she valued a connected understanding because it gave her the facility to choose effective strategies and explanations. She talked little about how she connected mathematical ideas independently of teaching, rather she concentrated on how she would make links to help her students learn.

Although Christine mentioned the idea of topics interweaving, implying a metaphor of a net, her model for connections was essentially linear, linking the current topic to prior knowledge and to a lesser degree, forecasting how it would reappear in future work.

I... like to remind them of prior skills that they’ve learned... when you can make connections with what they’ve done before, it helps them remember the process... So I generally lead them through, starting with prior skills and then working up to the new material... I also connect it to the future... to show them why they need to be able to do it.
Christine used the term, connections, extensively, but she spoke quite generally.

Q Can you think of an example of this interconnectedness?

A ... Well, in Grade 8 we do fractions, ... if you can’t add one-half and three-quarters together, how are you going to be able to add two algebraic fractions together? So, it’s connected. Math 8 and Math 10 are forever connected and those, the basic skills lead to higher level skills. And then in Grade 10, doing the, you know, rational expressions leads you on to higher level mathematics in which you have to deal with rational functions, and everything connects, even if it seems like it’s a very separate topic.

Her example sounded similar to Lily’s example of adding $\pi + \pi/4$, but Christine’s point of view was more general. She linked doing operations with rational numbers to doing operations with rational expressions, and then to rational functions.

Again she described making these links in a linear fashion - reviewing prior knowledge, extending to new material, followed by a (possible) general “look ahead” to future material. This passage illustrated another characteristic of Christine’s thinking about connecting mathematical ideas. When she spoke generally, she began with the language of “ideas” and “concepts”. However, as soon as she got specific, she talked about “skills” and “doing questions”.

Specific examples were rare. In fact, Christine came up with only one sustained example –

... the shape of the sine and cosine graph... when you multiply a function by -1, it reflects it through the x-axis and just flips it upside down... **what I do is connect those graphs, again, to prior knowledge** because in Math 11, they’ve done a thorough unit on parabolas, and they’re familiar with quadratics and, except for changing the period, everything else connects to sinusoidal functions. As far as where the number occurs in the equation, it does exactly the same thing, we’re
vertically compressing it, we’re shifting it, and all the same things happen and so I compare each change to what happens with the parabola, and then we practice and go through, and I have them just scale the axes to fit the curve. So they start with a ready-made curve and just put the information on to it. But because it’s connecting to the prior knowledge of the parabola, and they can see that they numbers are going in the same familiar places, by the time you get down to changing the period, it’s the only thing left to do, they can handle that.

She articulated one mathematical connection very specifically – multiplying a function by -1 (an algebraic transformation) is also represented by reflecting the graph of the function through the x-axis (a geometric transformation). Without listing them individually, Christine also implied that the specific connections between coefficients in the equations for parabolas and aspects of their graphs could be extended to sinusoidal functions. Once again, as she talked about a specific example, her language gained a procedural tone.

Christine identified conics as a topic that was easy for students.

I think kids like something that always works the same way. And conics are pretty simple once you’ve got the equations of each of the four shapes…. It’s very straightforward and very formulaic… There’s no systems, there’s no word problems.

Most of the teachers believed that students found topics easy when they were obviously related to their lives. In contrast, Christine’s reason was that this topic allowed students to manage simply by using procedural skills. This was another instance of a characteristic of Christine’s conversation – the emphasis on doing procedures.

Logarithms was a hard topic for Christine’s students but she did not know why.
I wish I knew. It’s detail oriented and I think they get tangled up in the details... Though conceptually, too, a lot of them have trouble with the basic concept that the logs are just exponents... I think the graphing, the log and exponential graphs, they’re fine with, and that’s fairly easy procedurally, and they’re used to functions, so even conceptually, and they know already what the inverse is, so you can work off of inverses to see the relations between the two graphs. So the graphs are not hard. It’s the actual manipulation of the log laws, and it is procedurally that they have trouble.

Once again, Christine’s reference point seemed to be procedures.

_Darcy_

Darcy got her first teaching job at Seaside and has been there for ten years. She was drawn to teaching by the success and enjoyment she got helping her friends in mathematics. She majored in mathematics and described the mathematics courses she took in university as preparing her “pretty well” for teaching high school mathematics, but indirectly. Rather than using “the stuff I learned in university” in her classes, Darcy saw its value as giving her “a better understanding” generally. The exception was a course in the history of mathematics. It gave her an understanding of “where the math was developed” and enabled her to regularly link topics that she was teaching to the history of their discovery. She saw the value of making these connections as motivational; it helped her to keep the students interested in math, for example, by telling them stories about Pythagoras and secret math societies, and about Pascal’s triangle actually being used by Chinese mathematicians in the thirteenth century.

Her goals for her students were:

For them to succeed... and also for them to enjoy math. I’d like that math makes sense to them - it’s not just a bunch of random concepts, but they understand that things relate in some way... just see how all the
algebraic stuff ties together and how the concepts really
do remain the same, rather than it just being these are
steps you need to do for this particular type of question.

Darcy repeatedly expressed frustration with what she saw as students’ lack of
interest in understanding concepts in favour of learning procedures. She described
how students seemed to leap into applying an algorithm in response to some surface
feature of the question.

... like the difference between an expression and an
equation. And even in Grade 12, when they’re given
something where they need to simplify the expression,
they go and solve it at the end. Even though it’s not
equal to zero, they just assume it is.... I just find that
they’re like, OK, if I see something that looks quadratic,
automatically they jump to factoring and solving, so
they don’t really have an understanding of what it is.

In our discussion of mathematical topics that were hard or easy to teach or
learn, Darcy provided a number of examples of connections. She thought that she
taught trigonometry effectively, and described using “diagrams” [graphs of
trigonometric functions], rather than just doing it “algebraically”, because she
believed that it would help students to “see what it is that we’re finding”.

I think it’s just that some students learn more visually...connecting it to something like an abstract equation,
then they can see it. That works for some of them. I do
both cause I know that [for] some students, the equation
[alone] makes sense. Some students just want the rule,
like 180 - \( \theta \) is the other one. [Darcy was referring
to the sine property that, where \( \theta \) is one root of \( \sin \theta = k \),
\( \theta_2 = \pi - \theta \) is another root.] Some students really need to
see it on the diagram why that’s true and all that.

While it appeared that Darcy was focussing on making links between two
ways of representing, she was just as concerned with attending to students’ different
learning styles. Nevertheless, she presented two different ways of looking at the topic
to students – the algebraic formula and the graph of the sine function, and talked
about how they showed the same thing.
In teaching combinatorics, a topic that Darcy’s students found easy, she developed the formulae for \( nP_r \) and \( nC_r \) with her students through examples. She used a “real world” connection to Lotto 6/49 to show “how the combination relates to the permutations”. Her technique was to induce the formulae from sets of examples, rather than to do a formal derivation, an instance of relating specific instances to the general case of the formula.

Darcy quickly identified logarithms as a topic that students find difficult. She named two key ideas that she believed were essential to understanding logarithms – that the logarithm represents an exponent, and that a logarithmic function is the inverse of an exponential function. In exploring students’ difficulties, Darcy gave examples that illustrated how false or superficial connections might impede students’ learning.

They all like to just use log as if it’s a variable or something. One of the big problems when they’re solving equations and it’s \( \log (x + 3) \), they all think that’s \( \log x \) plus \( \log 3 \) cause they don’t really have the concept of it being a function and not a variable, not something that you can just distribute … Like just the fact that the log represents an exponent, I think, they have trouble seeing because they’re used to, OK, an exponent is a number that’s a superscript… log (something) isn’t an exponent in their minds.

Darcy recognized that making a connection that “log” is an instance of a variable, rather than a symbol for a function, was an example of a false connection which led students to make errors. She believed that linking the concept of exponent to a surface feature – the fact that it is written as a superscript, impeded students’ ability to recognize that the superscript form and “log” form were equivalent representations of the same idea. Unlike most of the other teachers, Darcy seemed to have some hypotheses about why students struggled with the topic of logarithms.
In this kind of discussion of mathematical topics, Darcy equated the ability to recognize relationships as an indicator that students understood the topic, in contrast to a procedural approach where students just want to know how to get the answer. And Darcy admitted that she often gave in to students’ “demand” for “just the steps”. She cited the pressures imposed by preparing students for provincial examinations, the lack of time, and the departmental schedules, where all teachers of a course are expected to be at roughly the same spot in the curriculum, as hindrances to teaching for understanding.

Darcy expressed a great faith in the connectedness of mathematics, but saw identifying connections as difficult to do. She spontaneously introduced the notion of some ideas being more closely related than others, but could not come up with any examples at all. She insisted throughout that everything is connected in mathematics, but really struggled when trying to be more specific.

Darcy spoke strongly about the importance of students’ seeing the relationships, but did not pay attention to it in her planning. Because of her years of teaching experience, she did not find it necessary to make detailed notes. She did however, write out the examples that she planned to use while teaching – not to give herself a guide for pointing out connections, but “to cover all the types of questions that they’ll need to know how to do”.

Robert

Robert has been teaching mathematics for nine years. Before teaching at Valley, he taught for several years in a district alternate program for behaviourally-challenged students. Robert has a Bachelor’s degree with a major in Mathematics and a Master’s degree in education. He was confident in his knowledge of the provincial
curriculum, and familiar with the NCTM Standards. Robert was attracted to teaching because of the satisfaction he had found in coaching children’s sports, and chose mathematics because it was fun and “more conceptual” than a “memory-based program”.

He saw the value of his university mathematics courses as giving him “depth”.

I think it’s important for a math teacher to have a really deep understanding of math and to see a bunch of different branches of mathematics.

Asked to elaborate on “deep understanding”, Robert said:

I think what it means to really understand is... not just an application and just being able to see a question and know that it’s this type of question so I have to do this kind of algorithm. I think it’s more like having a broad understanding of why, what the question is basically asking,... I don’t know, to understand theoretically why you’re being asked to do this, not just you’re blindly putting down the answer and then having no connection between the question and the answer. If you can kind of connect the two and you can realize, of course, it has to be like that ... then they’re starting to get a really good understanding of what’s going on.

Robert saw understanding in opposition to rote learning and application of algorithms. He spontaneously used the terminology, connections. While Robert started to talk about his own understanding, he unconsciously shifted to talking about his students’ understanding and continued to talk of his students.

He gave an example from teaching quadratic functions and equations to illustrate what he meant by understanding. He described teaching how to graph quadratic functions and find the zeros. Later, when teaching how to solve quadratic equations algebraically, he pointed out that solving the equation is the same as finding the zeros of the function.
There are two different methods of the same solution, and they’re connected – what they’re finding are those x-intercepts, those zeros. But a lot of kids don’t connect to that, cause they don’t have an understanding, cause they’re just blindly doing the question, or they’re blindly just graphing it, and not understanding that the algebraic expression is exactly the same as the graphical one.

Robert’s goal in teaching was to promote his students’ understanding of the topic but he worried that “not all the kids are going to pick up on all the conceptual knowledge, a lot of kids just learn procedural skills”.

I’m striving to teach them a conceptual understanding of the mathematics. Some kids will get the procedural and some will get the whole thing. **I think that the best I can do is to give them a conceptual understanding. I don’t want to teach them just the procedures or algorithms.** I want them to understand what they’re doing and why they’re doing it… I think **any time that you can make a connection, I think that’s a broader value for learning.** They’re able to see those connections and see those patterns. That’s what we do as humans in our day to day life is look for connections and patterns. So I think that, yeah, that’s one specific skill and one specific connection, but it’s also kind of a way of thinking, to be able to look for those connections. I think that’s really important for kids coming out of high school.

For Robert, connections were far more than an aspect of his own understanding of mathematics and he actively attended to pointing out connections to his students. He strove to connect new ideas to prior knowledge by asking students to “generate ideas of what they already know about this topic” so that they could “easily connect” to the related new topic. He also stressed real-world connections – identifying applications of the new mathematics concept. He called what he was doing “facilitating this connection, bridge between two topics”. He believed that students who were making an effort were likely to see connections and that this ability would improve with practice.
In describing the value of making connections, Robert believed in a general improvement – “they’ll understand better”, and also that students might improve in their problem-solving skills.

I think it’s because of the connections, because they’re working at it and they’re actively looking to try to find connections.

Here, and in other things that Robert said, he joined the notions of making connections and making an effort.

Robert identified transformations as a topic that he taught effectively. While he wasn’t sure what made his teaching effective, he emphasized that he tried to “stress the conceptual idea of transformation”.

Instead of looking at the equation and having them figure out what it looks like from the equation, I really stress that they look at the base equation, like, if it’s a square root function, then they’d look at \( \sqrt{x} \), and then they go from there.

He was working with transformations as variants of “the base equation”. He wanted his students to have an image of the graph of a simple equation, like \( y = \sqrt{x} \), and to link the numeric coefficients of a more complicated equation like, \( y = 2\sqrt{x} - 3 \), to transformations of the graph of \( y = \sqrt{x} \), an instance of connecting algebraic and graphical representations.

Robert thought that students were held back in their attempts to understand by their weak algebra skills.

I think that they’re weak at most everything to do with algebra. I don’t think they have a very good knowledge of combining like terms, or even multiplying, or factoring out.

Continuing to expound on the example of factoring polynomials, Robert linked the processes of factoring and division:
I think they don’t have a conceptual understanding of factoring at all. I don’t think they understand that when you factor something out and you get these two binomials, that those are actually the divisors of the original quadratic.

Later, in talking about trigonometric identities and equations as a topic that students found difficult, Robert again referred to “algebra skills” as an obstacle to students’ understanding.

Robert did not write formal lesson plans, but he did think through the content for the lesson. He focussed on two areas – “ways to engage the kids to work through the material” and “how I can connect it to what we’ve already learned”.

Edward

Edward has been teaching for eleven years, and has always taught at Valley. What he remembers about his own high school experiences in mathematics is being a less than exemplary student.

I remember having a very limited understanding of the concepts back then... just trying to grind my way through the test and stuff like that, knowing as little as possible, trying to get away with doing as little as possible.

He had always found mathematics easy, but discovered a love for it about half-way through his undergraduate work. He got interested in teaching mathematics as a career through the example of a respected family member who was a mathematics professor. In talking about how his university mathematics courses influenced him as a high school mathematics teacher, Edward said:

It’s not the particular courses... because of course you do some things in university third and fourth year that you would never actually use in a Math 12 or in a calculus class, things like complex analysis and stuff like that. But I think it’s just the whole idea of making connections between all of the topics, and I guess the
overall approach to doing mathematics, that’s what I think I apply, not the particular concepts but just the whole approach to it, seeing the world in a mathematical way.

Edward was very articulate about his own mathematical understandings and made many specific references to mathematical ideas, launching into lengthy and detailed discussions about systems of inequalities and linear optimization, polynomials, and Riemann sums. He also spontaneously expressed a variety of mathematical connections. Some of his statements were at a very general level, for example,

I think the algebra is sort of the fundamental language that underlies all the bigger stuff... it’s the foundation upon which you can build other stuff.

In talking about various examples, he also made the following specific statements of mathematical connections:

- zeroes of a function [are an alternate representation of] the places on a curve where you cross the x-axis;
- factoring [is a procedure that] could be used to find zeroes of a function;
- a positive leading coefficient [implies] that the graph of the function goes uphill; “negative coefficients end going downhill”;
- an nth degree polynomial can have at most n roots [implication].

But the connections that mattered the most to him were the connections to the real world. He came back to this theme repeatedly.

Where this is used in the real world... if you look at the process of graphing an inequality on its own, it really doesn’t reveal anything to them [students] that is useful for their everyday lives... get them to understand that a polynomial arises out of a real world situation... all
these things we learn about quadratics, polynomials, trigonometric functions, are all just mathematical things that we apply to real life.

He saw these connections as important largely for motivational purposes.

It just gives them a little bit of a sense of purpose… If I give them an idea of where in the real world it might be used, then it might give them a little bit of a sense of purpose.

Mostly he talked about mathematical connections in a pedagogical context, as something that he wanted students to see or grasp, particularly about linking to prior knowledge, but in fairly general terms.

They need to be able to see that that connects back to something that they’ve learned before – that they can bring that piece of knowledge into a new setting.

For Edward, connections were a facet of understanding:

The people [who] have a strong understanding really understand things on a much simpler level. I really believe that they actually have a lot less that they’re working with in there. Whereas the people who have a very low understanding are coming into the test with a whole bunch of stuff in their head that they’ve got plugged into their short-term memory, but really, they don’t have a lot of, they’re not making connections between a lot of these little tidbits of things that they’ve memorized for the test… it’s almost like a less is more type of thing [i.e. less memorizing means more understanding].

Edward was emphatic that he tried to teach “through discovery”, by having students explore and look for patterns. He believed that students should be able to deduce much of what they needed to know from a few key facts, akin to reasoning from first principles. He named “min/max problems” as a difficult topic for students for two reasons – “the anxiety that goes along with doing word problems”, and students’ reliance on a procedural approach:
They’re memorizing, they’re trying to do it algorithmically instead of doing it by understanding what’s actually happening... It’s the setting up of the equation that’s the hard part, and usually that comes from not really understanding how to set up the variables.

He suggested representing the situation in the problem by diagrams as a helpful technique – setting up an alternate representation to the “word story”. He also spoke about the importance of engagement - “you have to care to understand it”, for students to be willing to take the time to think through the problem situation before starting to calculate.

“Students’ mathematics concepts must be woven into a connected set of relationships.”

At some point in the latter part of the first interview, I asked the teachers to respond to this statement, asking them to elaborate and agree/disagree. There were two salient points in the quotation in the context of this study. First, “mathematics concepts” pointed to an emphasis on mathematical ideas and topics rather than applications. Second, “connected set of relationships” hinted at interconnectedness and a web-like metaphor.

For Nicole, Sophie and Josie, my question was the first time that some version of the term, connections, really entered the conversation (Sophie made a passing comment about relating to real life before I posed the quotation). All of the other teachers spontaneously introduced the term, or a synonym like “relationship”, earlier in the discussion. The teachers’ responses varied from very short and unelaborated (Josie) to extended responses with specific examples (Edward). Nevertheless, there was considerable similarity in the content of the responses.
I start with excerpts from the teachers’ responses to give the reader a sense of how they approached interpreting this explicit exhortation about connections. The excerpts are evocative of the teachers’ personalities and often indicate other concerns that were important to them. I follow each group of quotations with an analysis of the common features of teachers’ responses.

First I consider the responses of the teachers who had not mentioned connections before this point in the interview – Sophie, Josie, and Nicole.

Sophie:

Woven into a connected set of relationships... so that could be anything, could be real life, could be their other courses, which I guess is real life. Well, I take that to mean that it has to make sense in terms of visualizing real life situations. I don’t know... I don’t understand the statement... connected to prior learning, connected to a real life situation... something that they can picture.

Josie:

What it means to me? I agree. That’s what I would strive in doing when I teach. But the reality is that some students, even if they’re smart, they don’t really want to connect them and they just want to get good marks. So I will still try my best to do it... When I was teaching functions, what I really think was important in this chapter was to model a real situation using linear functions, cause... I’m trying to relate. Later on in Grade 11, they learn how to model a situation using parabolas, and Grade 12, they learn even more, they learn different functions, like trig functions, they get different situations that follow this pattern.

Nicole:

One concept should be connected to all the other things that they’re learning... so they can kind of build on that as they go through... connecting within topics... I would want to connect things to their everyday life so that they can kind of see... how that makes sense rather than just teaching them something
that they’re never going to know why they’re learning it… Sometimes that’s impossible to do, some of the curriculum is impossible to relate… maybe you’re not going to use it now, but we’re going to build up to a point where you could use it… A lot of the stuff that they do in the younger grades isn’t all that applicable just yet, but it will be when they get into the higher grades… I think it will help later on.

Although Nicole mentioned topic to topic connections, they were not her priority. For all three of them, the important connections were the connections to real life, expressed as some sort of application that was either personally relevant to the individual or meaningful to people in general. All of them mentioned connections to prior knowledge as well. Josie’s example indicated that she was extending modelling using linear functions to more complex functions. Her description is reminiscent of spiral curriculum notions of revisiting earlier themes and developing them a step further. Nicole was so committed to connections as applications that she construed any linking of ideas or skills that did not have an immediate practical use as an intermediate linear process of building up skills, which ultimately had to have a practical result.

The importance of connecting to the real world and connecting to prior knowledge, the predominant way that teachers talked about mathematical connections, were recurring motifs in what the rest of the teachers had to say as well. But additional themes also appeared. Here’s what the other six teachers had to say about the statement “students’ mathematics concepts must be woven into a connected set of relationships”.

Wendy:

The stuff you’re learning should tie all together… not every topic is… necessary totally intertwined, but there’s some topics, where [it’s] closely related… Combinations and permutations is related to
probability... it’s sort of tied into statistics too. So, there is a range of certain sections that are tied in. But you get to logs. I don’t find it as interconnected... logs is kind of like totally separate. So if you don’t tie that in, explain that all that stuff you learned you still need, you need this concept in order to do this concept... well then it’s ‘why did I bother learning it?’ Or they throw it out of their mind... So at least it gives them a purpose... I’m learning it cause I’m going to need it for later on.

Lily:

We need to build on what we have and show that, oh, when you were in Grade 9, you did this, now look how this ties in here, and it’s going to tie in again... In the future we’re going to add this to it, so you’re building... I agree with it, but that’s not what we do. Particularly with Grade 10... I’m really struggling with them because I can’t make all those connections... The topics have very little to do with each other, or they don’t flow nicely. I can’t just tie them together... I think kids could learn a lot more that way if it was more continuous... Building on what you know, like a little piece of thread going all the way through, same colour thread and it kind of just trailed along... just a little link, just something familiar.

Christine:

I guess that that would mean that they’re wanting students to have a big picture of how the different topics that we’re teaching them connect together and are, even though they seem to be separate, really do have connections. ... And I would agree with that, because I think that if students don’t see a connection to something else that they’re doing, they value it less... There’s connections, and especially, not even connections to things in math, but connections to the sciences, when we’re studying sinusoidal curves, or we’re studying exponential functions and there’s connections into physics, and there’s connections into chemistry and biology... Some kids don’t place value on something unless you can show them how it connects to something else.

Darcy:
We should be showing them that everything connects together in math... Do I agree? ... I think it’s hard to show the relationships between some topics with the understandings that they... It would be nice, but I think it would be very hard to do... Some concepts are more directly connected than others, but... it is all kind of connected... some things more strongly than others... to find those meaningful connections is hard to do. I guess if they can see that everything’s connected, they’ll see more the value of every individual topic – that it’s not just a topic on its own, that it’s something that relates to... math as a whole... If they have a context to put it in, it might make more sense to them... In Grade 12, we’re trying to teach them how to do everything they need to know for the provincial and... go off on how they’re all related... might involve bringing other things in and there probably isn’t time to do that.

Robert:

Woven into a connected set of relationships... Concepts should be imbedded not just topic by topic in math but an understanding that shows how all those topics can be connected or how they can be interconnected and not just probably with the math areas but with all the other areas that we teach day to day... And I would agree with that statement, they [students] do need to understand how things are connected. Math is not just an isolated topic, it’s something that’s got a rich history in our world and it’s got a rich application and they should be able to start to see how mathematics can be used in the world and within how all the different branches are connected and how it’s connected to their learning in high school... I try to build that connection between the graphical and the algebraic model... Or solving systems of equations – you solve them in a bunch of different ways. And I teach them this one first, this one, this one, this one. But some kids don’t pick it up ‘til the very end, and then all of a sudden they understand all of it. Or... they understand one really well and then hopefully they’ll get the connection to the other ones.

Edward:

It means that when they’re studying... and something pops up that requires them to have some piece of prior knowledge... They need to be able to see that that
connects back to something that they’ve learned before – that they can bring that piece of knowledge into a new setting... I’m thinking of an example from Math 8... I would say that a web of connections would be understanding that you can represent a relation with an equation, you can represent it with a table of values, with an ordered pair, with a graph, so I kind of see the relation is the central idea and then, the ways of representing them are the branches sort of, off of the web, and just sort of be able to see that it’s all those things... Calculus, Riemann sums... What do Riemann sums allow us to do? They allow us to chop up an area into rectangles or trapezoids and find the area under a curve. They also allow us to chop up a 3-dimensional figure into discs or washers or cylindrical cells and add them all up to get the volume. So that the central idea of being able to take things and add them up can be applied to volumes, can be applied to areas, can be applied to ... real situations like total change theorem... So the central idea is... the Riemann sum, and it kind of branches out to all these other topics.

These six teachers also saw real-life applications as very important. But they drew a distinction between these applications and mathematical connections, the “topic to topic” relationships. Connecting mathematical ideas was almost always expressed as connecting new material to prior learning, often as building up more complex ideas or skills from previously-learned simpler ones. This implicit model is analogous to Skemp’s (1987) hierarchical model of building up complex concepts from “primary” ones. Christine and Lily extended the connection to prior knowledge into the future by indicating to students how what they were currently learning would be used later.

The two most common metaphors that the teachers used spontaneously in their description of connections were those of tying and building. However, teachers used them interchangeably, so I doubt that the metaphors were distinct in their minds. They used them more as synonyms. The underlying image in both cases was one of linking the current topic to a previously learned one, usually one at a time, occasionally
combining several previously learned ideas at once, akin to Piaget’s process of assimilation. Assimilation is the process of incorporating new information to expand an existing schema. Although no teacher referred to Piaget or the process of assimilation, their image of how connections are involved in learning, are consistent with this view of assimilation.

Lily and Edward both described unique metaphors. Lily interpreted mathematical connections as a common thread – an idea that was common to several topics. Her metaphor reminded me of Coxford’s notion of connections as common themes that cut across mathematical topics. But Lily was unable to elaborate or give examples of what that thread might mean to her.

I don’t even know. I never even thought about it. It’s something that’s frustrated me a lot.

Edward, on the other hand, had no trouble at all giving examples. His metaphor was that of a starburst – a central idea or representation as a node that other ideas branch from. Common to both metaphors is the notion of some central or core idea to which other ideas are linked.

Both Robert and Edward identified mathematical connections as alternate representations. Where Edward referred to multiple representations, Robert emphasized two - algebraic and graphical representations. They were the only two teachers to spontaneously consider connections as alternate representations.

While she spoke about connections quite generally, Wendy introduced the idea of degrees of connectedness, identifying some topics as more closely related than others. Her notion seemed very similar to Lily’s and Nicole’s ideas about the length of a connection, a way of describing metaphorical distances between concepts in a person’s schema of a mathematical topic.
Josie and Nicole thought that identifying connections and showing them to students were hard to do. Darcy and Lily shared the same sentiments. Teachers named two different reasons for this difficulty. First, what the connection actually was might be obscure, or even “impossible to relate” [Nicole]. For Josie, Darcy, and Lily, perceived obstacles, like students’ lack of interest, poor curriculum design, or lack of time, were a second factor. They saw “teaching connections” as something that required them to work against the prevailing ethos of mathematics teaching in their school.

For the most part, teachers dealt with connections in an *ad hoc* manner. Most wrote out the examples they planned to use in their teaching. To some degree, they did this to prepare a teaching sequence that they believed would allow students to see connections for themselves. But an equally important factor was the desire to make sure that they would cover all the variants of the problem types that students would encounter in homework and tests. They saw discussions about the relationships of ideas as time-consuming and difficult, and admitted that they didn’t spend much time on them.

Teachers generally relied on teacher-generated and presented examples to demonstrate how new information was linked to what students already knew. Only rarely did teachers explicitly point out the connections. They expected students to make the connections for themselves, and assisted them by making the connections “short”.

Finally, several teachers spoke about the value of making connections. Darcy and Christine believed that students would value more highly knowledge that they saw as connected to something else. Making connections would make mathematics
more interesting and therefore motivate students [Nicole, Wendy], and help students to understand or remember [Nicole, Sophie, Christine, Darcy, Lily].

**Summary of emerging themes in Interview 1**

The discussions in the first interview were far-ranging. I had a set of questions that I posed to teachers but the sequence and actual wording varied to some degree from teacher to teacher. Individuals emphasized different aspects of their work; some teachers went on apparent tangents to make points that were important to them. All in all, I tried to keep the first interview as close to a natural conversation as possible.

Two types of themes emerged from these conversations. First, there was a set of themes about the teachers’ beliefs and values related to teaching in general.

- All professed a sincere commitment to teaching for understanding, but in practice, talked about “doing questions” and “showing how”. This apparent disconnect might just be an artefact of the shorthand that teachers use in talking about their work. Or it might be an indicator of a conflict between stated beliefs and actual practices.
- “I know I taught it well because students did well on the test”. [Nicole, Sophie, Wendy]. The teachers seemed to have little grasp of what made their teaching effective. They treated the mechanism of the effects of their teaching as a “black box”, where the sum of their teaching behaviours was the input and their students’ results on tests were the output.
- Teachers generally lacked awareness of what blocked students’ learning [Sophie]. For the most part, teachers were stymied when
asked for specifics of blocks to students’ understanding, a category of pedagogical content knowledge (Shulman, 1987).

- Other goals of teaching were at least as important as advancing students’ knowledge of mathematics [Wendy, Nicole]. Imbuing students with virtues like social responsibility, persistence, work ethic, and the disposition to like and enjoy mathematics had a prominent place in teachers’ thinking about their work. Most of them believed that achieving these goals would have a more lasting effect on their students’ lives than mastering the mathematics content.

Second, there were some common themes in what the teachers said about connections.

- All could speak generally about connections, but had a hard time being specific. Even the teachers who did not spontaneously introduce the idea of connections into the conversation, talked about relationships among mathematical ideas at some point, often while they were talking about deep understanding. In response to probing, most were able to offer examples of mathematical topics that were related. However, except for Edward, they found it difficult to make their examples specific or to identify what the relationships were.

- Topics that are easy for students are familiar or relate to real life [Sophie, Nicole, Lily]. The two most common reasons that teachers gave for students finding a topic easy - familiarity and relevance, are both instances of making connections. When students find a topic familiar, they are making a small extension from prior knowledge to
new material. When they see the relevance of a topic, they are making a connection to their own lives.

- Topics that are hard for students are hard because they do not relate to other topics or do not have immediate applications [Sophie, Wendy, Nicole, Lily]. Teachers overwhelmingly identified logarithms as a difficult topic. I discuss logarithms further in Chapter 5.

- Teachers used examples extensively to demonstrate connections [Wendy]. Teaching examples played a central role in the way that teachers brought in prior knowledge when starting to work on new material.

- Perceived external obstacles prevented teachers from teaching in ways that they said they valued [Sophie, Wendy, Josie, Lily]. Most of the teachers made assumptions that trying to incorporate a specific emphasis on making connections would be time-consuming and would detract from preparing students to achieve high scores on tests.

Finally, I draw attention to an emergent tension that became apparent in the way that teachers talked about their work. The tension was typically expressed as some instance of conflict between what a teacher wanted to do, or believed was the right thing to do, and what the teacher felt compelled or obliged to do. For example, while all teachers spoke of the importance of conceptual understanding and of their desire to teach for conceptual understanding, they described situations where they abandoned a conceptual approach and taught only algorithms. Sometimes they felt compelled to save time; sometimes they felt that students expected or demanded a procedural approach. A hint of the same kind of tension surfaced when teachers talked
about connections — they thought connections were valuable, but felt obliged to focus on covering the curriculum.
CHAPTER 5: EXPLORING CONNECTIONS THROUGH DISCUSSIONS OF MATHEMATICAL CONTENT

To focus our conversations more specifically on mathematical connections, I asked teachers to choose a topic that they thought was rich in connections. In this chapter, I consider how they described mathematical connections in curriculum topics that they had chosen as particularly apt for this discussion. The topic sections vary considerably in length. Sometimes more than one teacher chose the same topic, so their comments are considered together. Some teachers were very terse in their comments; others were more talkative. Some spoke through examples that needed to be presented to the reader to provide the context. Nevertheless, each topic section fully summarizes the key aspects of our conversation about that topic.

First, I address the different topics discussed by the teachers. Then, I summarize the types of mathematical connections that emerged according to the revised framework described in Chapter 3. Finally, I identify the common themes that surfaced in these conversations.

Since this chapter is organized by mathematical topic, I include here a summary table describing the participating teachers, to aid the reader in remembering the “cast of characters”.
<table>
<thead>
<tr>
<th>Teacher</th>
<th># yrs teaching</th>
<th>Currently teaching</th>
<th>school</th>
<th>Undergraduate/Graduate work</th>
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<td>&quot;visualizing real life situations&quot;</td>
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<td>8</td>
<td>Math 9, 10, 11, 12</td>
<td>Seaside</td>
<td>Math major</td>
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<tr>
<td></td>
<td>&quot;you need this concept in order to do this concept&quot;</td>
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<tr>
<td></td>
<td>&quot;connect things to their everyday life&quot;</td>
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<td></td>
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<tr>
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<td>12</td>
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<td>Valley</td>
<td>Math minor</td>
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<td>Math 9, 10, 11, 12</td>
<td>Valley</td>
<td>Math major M Ed</td>
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<tr>
<td></td>
<td>&quot;all the different branches are connected&quot;</td>
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<td>Valley</td>
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<tr>
<td></td>
<td>&quot;relation is the central idea...ways of representing are the branches&quot;</td>
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Logarithms

This topic actually arose in teachers’ conversations in the first interview about topics that students found difficult. Seven of the teachers were teaching Principles of Mathematics 12 at the time of the study; only Josie and Edward were not. Six of these seven teachers identified logarithms as a difficult topic. Although no teacher chose to discuss logarithms in detail in the second interview, I consider it here briefly because it was almost unanimously identified as a very difficult topic by the teachers who taught it. Teachers believed that students found a topic difficult when they could see few connections.

All of the teachers agreed that the fundamental idea that students needed to understand was that a logarithm is an exponent, and the fundamental skill was making conversions between logarithmic and exponential forms of expressions and equations. The mathematical connection entailed here is that of equivalent representations both in algebraic form. However, the transformation required to produce this equivalent representation goes beyond the usual arithmetic transformations using addition, subtraction, multiplication and division, that students are typically engaged in when doing algebra. It requires taking the inverse of a function. Only Wendy and Darcy referred to logarithms as functions, and only Darcy specifically identified the understanding that a logarithmic function is the inverse of an exponential function, as crucial to students’ learning.

Teachers believed that lack of procedural skill in applying laws of exponents or “log laws” became obstacles for students. Even more strongly, they attributed students’ difficulties to their perception that the topic of logarithms was unrelated to other mathematical topics that students had learned. It is this position that makes
discussions about the difficulty of certain topics relevant when considering connections. Essentially, teachers were hypothesizing that the lack of obvious connections, mathematical or real-world, impeded students’ learning.

**Conditional probability [Sophie, Wendy, Lily]**

Although I had asked teachers to choose a mathematical topic for our discussion that they believed had rich potential for making connections, Sophie, Lily and Wendy made their choice for much more pragmatic reasons. As Sophie said, “because it’s the next thing coming up that students have trouble with, so I thought I could get some ideas”. At first glance, it seemed an improbable coincidence that three of the teachers would choose the identical topic. However, I believe two factors contributed to this coincidence. First, this topic was one that all three teachers found problematic to teach. Second, Sophie, Lily and Wendy were all at the same place in the book (Alexander & Kelly, 1999) at the time of the interview.

All three teachers agreed that conditional probability was a topic that their students found hard, but differed in their own confidence in the topic. Sophie found all aspects of probability easy – “for me, it’s so obvious”. Wendy felt she had come to understand the topic through teaching it, rather than through her coursework which was very formula-oriented - “it was a lot of formulas… and there was a lot of memorization”. Lily was insecure in her own knowledge of conditional probability. Her difficulties surfaced at times as apparent confusion, but she was tenacious in working through the ideas she struggled with.

I asked the teachers to start by talking about the topic – the key concepts involved and how they were related. Sophie defined conditional probability as
limiting your sample space... your total number of possible outcomes is being decreased, limiting the sample space ... you have to be given a condition.

Mostly she talked about conditional probability through examples, and those examples were questions from the textbook. Wendy and Lily had even more difficulty talking about conditional probability in the abstract and immediately introduced examples so that they could discuss the topic in concrete terms.

Wendy talked about conditional probability only in terms of her teaching of the topic and did not talk about conditional probability theoretically at all. There was a kind of circularity to the way Wendy expressed her thinking. Several times, she defined conditional probability as “probability depending on a condition that they [the textbook authors] first set”, and kept using the terms “condition” and “probability” almost exclusively in trying to articulate the concept. She jumped almost immediately to giving examples, and trying to use aspects of the example to illustrate what she meant.

Lily also tried to explain what conditional probability was through an example, and seemed to falter in her explanation, but this may have been an artefact of her hesitant manner. While she could not initially describe conditional probability in any abstract/mathematical terms, eventually, she named these key prerequisite concepts to the learning of conditional probability – the definition of probability as “number of successes over total number of outcomes”, complementarity, and related and unrelated events - “A AND B, A OR B kind of things”. She also introduced Venn diagrams as way of representing the union and intersection of sets (events).

The central reference point for all three teachers was the “question”, the textbook problem that they did as a worked example or assigned to their students. They agreed that the most crucial aspect for students’ success was the mathematical
modelling aspect, that is, correctly representing the problem in terms of probabilities.

And they agreed that this was the root of students’ difficulties; students found it hard to connect a description of a situation to mathematical formulae.

Sophie:

It’s the specific questions, the language of the questions that leads to trouble, like, what is the given, or what is the sample space. The students have a lot of difficulty deciding what’s the probability I’m trying to find, what’s the given, and I have trouble explaining it very well.

Wendy:

If they can look at a problem and realize it’s conditional probability, then, I think it’s been successful, they’ve learned the topic... understanding how to use tree diagrams, maybe the formula to understand what conditional probability is first, and then you know, applying those ideas, and then, realizing what method to use. And I think that’s pretty much it.

Lily:

I want them to be able to read that question and say, hey, OK, this is what this means, like I understand what is being asked and even when, yeah, there’s a formula, Bayes’ formula, right?, yeah, Bayes’ Law that, even if they don’t have it memorized, it’s not such a big deal.

Of the three, Lily placed the most value on a problem-solving approach in her teaching.

I see it in a really broad way. I see it in terms of - you can’t just have tunnel vision when approaching any problem... Open it up just a little bit and kind of look at what are the other possibilities. I think that’s just problem solving in general.

Lilly’s valuing of problem solving was a constant in the way that she described her approach to teaching conditional probability, but she saw the connections that she
could/did make quite generally. For example, she talked about using tree diagrams to “get a visual” and relating the tree diagram to solving the problem using Bayes’ Law. It seemed that she used a tree diagram as an intermediary representation to link the situation to the formula.

Doing worked examples was Sophie’s only teaching strategy for this topic. She used Venn diagrams and tree diagrams as alternate representations of the situations in the problems and showed the students examples using both types of visual representations. However, not knowing what to do beyond showing and talking through examples was a recurring theme in Sophie’s description of her teaching. At times she made reference to linking to prior knowledge, for example, formulae and definitions, permutations and combinations, and “trying to relate it to real life”, but she didn’t “talk about it in the big sense” with her students. Several times during the interview, she recognized that her teaching of this topic was very much a teaching of algorithms, yet she didn’t have an algorithm for translating the questions into probabilities.

Of course, yeah, I want them to understand it, but I would really expect they should just be able to do that type of question without understanding, and yet, they’re still not able to, a lot of the students who have trouble, they still can’t even do it.

Sophie hoped that her students would understand, but the most important thing was that they could do the questions. Throughout the interview, Sophie was remarkably honest and often questioned herself and the way that she did things. It seemed that the interview was a catalyst for her to reconsider her practice. While she did mention some relationships, they were rare, and most as the result of probing and sometimes quite leading questions.
It became very obvious in this interview, but also showed up in the first interview, that Wendy thought of doing mathematics as doing questions. She repeatedly referred to "doing examples" as her main teaching technique. The examples were sample questions, generally from the textbook, that she worked through in a lecture or discussion format with her students. For Wendy, "understanding the concept" was tantamount to knowing how to do the question. She seemed to regard understanding in an instrumental way – defining it by its product, correctly solving a question.

Wendy constantly drew on examples expressed in question format to illustrate her thinking and was able to think of examples "on the fly" as she needed them. It was through her discussion of particular examples that Wendy articulated some notions of relationships of other mathematical concepts and conditional probability. For example, she linked conditional probability to choosing without replacement, and to permutations and combinations as part of her explanation of "putting everything together... so many different aspects of probability".

In response to probing, she tried to elaborate on the relationship between permutations and combinations and probability.

Using permutations and combinations to find probability... that's right after conditional probability, and that's sort of a different concept, but not really, they're all somewhat linked in a way. You know, they're sort of separate ideas to get to different problems, different routes to get to different questions, I suppose.

Q what is it about combinations and permutations that links it to probability?

I think permutations and combinations is linked to conditional probability, part of it. What part of it? I think when we're dealing with permutations and combinations, it deals with larger values whereas the
other probability we're doing is quite small, like say
tossing a coin or rolling a die, those are fairly small
numbers you got to deal with. But... what's the chance
you're going to win the lottery? It’s going to be one in
14 million... you’re going to need permutations and
combinations to help you get to those bigger
numbers... So, I think with permutations and
combinations, it helps out with generating the
numbers.

Wendy always referred to “permutations and combinations” almost as a single
word. The relationship that she saw was essentially procedural – using formulae from
combinatorics to calculate the number of possible outcomes in a situation. She gave,
“what’s the probability of picking 3 aces and 2 kings?”, as an example of a question
in which applying “permutations and combinations” would be useful. Unfortunately, I
didn’t probe further to see how she would relate this more specifically to probability,
and it would have been interesting.

“All tree diagrams are related to all of probability”. The strongest connection
that Wendy made in discussing conditional probability was in using tree diagrams to
represent the complete set of outcomes, and identifying particular branches in the tree
diagram as positive outcomes. For her, tree diagrams were a visual way of
representing what would be difficult for students to “do in their heads”. She also
invoked using a table as a procedure to find and display the set of outcomes, but did
not draw any link between the table and the tree diagram.

An interesting feature of Wendy’s thinking was the degree to which she
focussed on decoding the language of word problems in her teaching.

I find that year after year... I mention the word,
difference of squares, they have no idea what it
means... I ask them, what do you see, when you see the
word, difference, what do you see when you see
squares. And, I find that if they recognize the word,
they match it up and it sinks in a little more too. So, I
find that the meanings of the words actually do help
them to understand what they’re actually using...
Most of my teaching I try to do that so they understand
what they’re learning, then it makes more sense.

She was teaching students to connect words in the question to particular
operations or methods for solving the problem.

At one point, Wendy tried to illustrate the connection between a tree diagram
and the formula for conditional probability using a problem from the textbook as an
example.

A company has two factories that make computer chips. Suppose 70% of the chips come from factory 1 and 30%
coming from factory 2. In factory 1, 25% of the chips are
defective; in factory 2, 10% of the chips are defective.

Suppose it is not known from which factory a chip
came. What is the probability that the chip is defective?

Suppose a defective chip is discovered. What is the
probability that the chip came from factory 1?

If the question says, suppose a defective chip is
discovered, what’s the probability that it came from
Factory 1? So, the conditional is that it is defective.
What’s the probability that it came from the first
factory? So the formula would be – probability of
factory 1 AND it’s defective over the probability that
it’s defective. Right? So the tree diagram I would draw
first is well, OK, it comes from Factory #1 or it can
come from Factory #2. The probability of that, they just
told us is 70%, so 0.7 and 0.3 for Factory #2. And then,
you either have a defective chip or you don’t have a
defective chip, so you have a defective chip, or you
don’t have a defective chip for Factory #2. And then it
also tells us that 25% in Factory 1 are defective, so then
you put 0.25, so the other one’s got to be 0.75. And in
Factory 2, it’s 10% defective, so it’s 0.1 and 0.9. So
once I have all the numbers listed out, it’s very easy...
now it’s all number crunching. So, if you have the
picture here, it’s easy once you have the formula to help
you.

I asked Wendy if it was possible to represent the problem correctly by starting
the tree diagram with the branch defective/not defective. [In fact, the situation in the
problem can be represented by a tree diagram that starts either way. However, the information given in the problem constrains which representation is useful because the calculations can only be done starting with Factory 1/Factory 2.] While she first concluded that it was just “easier” to start with F1/F2, she eventually articulated a connection that allowed her to decide how to start the tree diagram –

Understanding that they [the probabilities] have to add to 1. That would be, I think, a big indicator of where you should start. Yeah. The complement idea. So again that’s totally related to probability, and I find that if you understand that, you’ll know where to kind of start off with.

Of the three teachers, Wendy was the only one to articulate the specific connection of complementarity to selecting the starting point of the tree diagram – i.e. the starting branch for the tree diagram should be the one for which both probabilities are given, or can be calculated using complementarity. Nevertheless, Wendy’s notions of mathematical connections did not seem to be well-developed. She expressed the relationships in general terms, usually identifying that ideas were related but not how. In her teaching, she relied heavily on teaching students to decode the language of a word problem, by showing them how English words connected to arithmetic relationships.

Sophie and Lily chose the same problem from the text as the example through which they outlined their understanding of conditional probability and its connections. In their textbook, problems are classified as A, B or C, depending on their level of difficulty. The problem below is a C problem, the most difficult type:

A new medical test for glaucoma is 95% accurate. Suppose 0.8% of the population have glaucoma. What is the probability of each event?

A randomly selected person will test negative.
A person who tests negative has glaucoma.

A person who tests positive does not have glaucoma (Alexander & Kelly, 1999, p. 440).

Sophie and Lily both talked through the same problem but differed greatly in the degree to which they talked about connections.

In representing the glaucoma problem, Sophie, who had said earlier that she found conditional probability problems easy, started with a tree diagram whose initial branch was glaucoma/not glaucoma, each with sub-branches – positive test/ negative test. I asked her if the problem could also be correctly represented by starting with the test rather than the disease. This led her into a prolonged and frustrating discussion with herself.

In these medical test questions, you have the disease, you have your test outcome, and then you have if the test is accurate or not, so there are three things with these. So I have to deal with the fact now, does this person test positive or negative? And so, if the test is accurate, and they have the disease, they’re going to test positive, and so on… Well, now I’m wondering. So a person who tests positive does not have glaucoma. So the given is, that they test positive so it’s going to be not accurate… wait a minute, now I’m confused … OK, probability of NOT glaucoma, or NOT the disease, given that they test positive. Oh, oh see, I forgot something there myself. See, I just did what I said the students have trouble with, mixing up the probability of not having the disease and testing positive with simply just the probability of not having the disease. See, I need to have the formula in front of me.

She quickly identified the three sets of branches, but then tied herself into knots as she tried to conceptualize the problem without using Bayes’ Law (“the formula”). In exploring the problem, she made three different attempts of representations, making the initial branch glaucoma/not glaucoma, or, positive test/negative test, or accurate/inaccurate. I asked her if all three versions were correct ways of approaching the problem.
I believe that they’re all correct, um, they might be a little more difficult conceptually to figure out, because, again, there are three, three things you have to have the third related and so, to me, the first one, where you start with the disease first and positive, to me, that’s easy to relate whether the test is accurate or not. But it could be because that’s what I’m used to. I mean, for each of these I’d have to figure out the third thing, positive, you have the disease, so therefore the test is accurate. Oh, OK, so then but what’s going to go along here? Oh, ah, no, testing positive and negative up here yeah, you can’t just fill something in. When you have to, what would go there? OK, like, you test positive and you have the disease, the disease is accurate and so testing positive … oh, I’m really confused. Ok, this is accurate/not accurate. You test negative but you have the disease, so it’s inaccurate. Wow … probability of having the disease and accurate or not accurate. Now I’m thinking, … I don’t know about that.

Her statement, “you can’t just fill something in”, was a reference to the lack of given information about the probabilities of certain events. Even for Sophie, who had taught this topic many times, reasoning through the problem proved difficult. In the end, she recognized that she had been dismissing alternate approaches to representing the problem and was upset at how difficult she found it to make different representations. But she did not extend her thinking to identify how she would judge which representations were viable for solving the problem.

In contrast, Lily started with a more “big picture” approach.

I think that the key to remember, to beginning the question is to know that tests aren’t right all the time. You can’t just rely on, you know, you could test positive and not have it, or just kind of a real-life situation, not even as far as what do I do with the numbers, but just you know, or you could have something and test negative, and so you have to, going in, understand that things aren’t always, I guess, accurate. And I think that’s a big part of it, a big part of the topic that I think people struggle with.
She seemed to stumble in explaining what she meant, and sounded like she was paying attention only to surface features, for example, that a medical test could be inaccurate. As she continued to respond to probes, it became clear that she was, in fact, trying to illustrate what was most important to her as a teaching goal – that students would really understand the situation and be able to explore it.

I want them to be able to read that question and say, hey, OK, this is what this means, like I understand what is being asked.

Talking about the glaucoma problem, Lily said:

This is kind of the total outcome… I know it’s not number of outcomes, but the same idea… we’re talking about just testing negative in general… You can test negative given you have glaucoma but you can also test negative if you don’t have glaucoma. So, I see this… it’s total number of outcomes, I relate those two… probability of negative in general… it includes everything, every possible way you could be negative which is also the denominator of this [Bayes’ Law]… I don’t think they made that jump.

Lily struggled to express herself, but she actually made several mathematical connections in this part of her explanation. She extended the idea of the sample space to help her conceptualize \(P(\text{testing negative})\) as an instance of considering “all possible outcomes” – all the people who tested negative. Then she related \(P(\text{testing negative})\) to the denominator in Bayes’ Law.

Later, she talked about representing \(P(\text{testing negative})\), the denominator, as the union of the sets, \{people who test negative and who have glaucoma\} OR \{people who test negative and who don’t have glaucoma\} and also invoked complementarity to find the value of \(P(\text{testing negative})\). And she related the conditional probability to the intersection of the sets, \{people who have glaucoma\} AND \{negative test\}. 
Related events - when I talk about related events being important for conditional probability, it’s because you have to look at, I’ll just refer to the same question, so glaucoma AND testing negative… Where A OR B, this would be the denominator part, denominator part of here because you’re looking at… probability of having glaucoma and testing negative OR not having glaucoma and testing negative. So that’s the A OR B part… And the complement just comes down to if I tell you the test is 95% accurate I expect you to pick up on your own that that means the test is 5% inaccurate.

Lily brought up a variety of mathematical connections in talking about the glaucoma problem. I then asked her whether she would point out some of these connections to her students. Her answers indicated that she did not place much importance on explicitly talking about connections like these with her students.

I wouldn’t spend a huge amount of time on it [connections to Bayes’ Law]… I don’t even think they would notice it was so much a link [between related events and conditional probability].

Lily saw Venn diagrams, whether actually drawn or imagined, as a representation, or common theme, that linked the topics of permutations and combinations, and probability together and believed that this was an easy connection that students made on their own. As she continued discussing probability more generally, she referred to “a nice flow and that’s because there’s a lot of stuff they’re familiar with already”. The “nice flow” referred to the organization of the Chapter in the Grade 12 textbook. Further references to probability “in Grade 8 and Grade 9” indicated that Lily thought about probability as a topic that was defined as curriculum, rather than in strictly mathematical terms.

In summary, although Sophie, Wendy and Lily had, by their own report, very different levels of confidence in their knowledge of probability, that did not seem to be a factor when it came to articulating mathematical connections. All three of them
named the topic as it was represented in their textbook; they sometimes referred to conditional probability as a “section” and talked about how it related to other sections in the book. Not surprisingly, then, they mentioned the same strategies, particularly tree diagrams, Bayes’ formula and Venn diagrams. They talked about aspects of conditional probability only in relation to teaching the topic, and not in terms of their own understanding independent of teaching. Interestingly, it was Lily, who was least confident in her knowledge, who spoke the most about mathematical connections.

**Geometry and trigonometry topics [Nicole, Josie]**

Nicole chose the broad topic of Geometry for her topic interview. Actually, she wanted to focus on surface area and volume of 3-dimensional geometric figures. She chose this topic because she had just finished “doing it” with her Grade 12 Applications of Mathematics class and found that her students had struggled with it. In this course, students work on application problems.

> They don’t really know what the question is asking, so it’s that conceptual understanding of it rather than just knowing how to plug things into formulas… but once they understand the question and what it’s asking, then they know how to do it.

This theme, that students have their greatest difficulty in modelling, that is, in representing real-life situations in mathematical ways, also appeared in conversations with Sophie, Lily and Wendy about conditional probability. Nicole repeatedly stressed the importance of “conceptual understanding” rather than “memorizing the formula”. Using surface area as an example, she described teaching her students to mentally unfold a 3-D shape and draw or visualize the resulting “net diagram”. Finding the surface area then became a matter of adding up the “pieces” – typically some combination of triangles, rectangles and circles.
Formulas you forget, you don’t remember the formulas. But if you actually have an understanding of what it means to find the surface area, find those different pieces, then you’ll always remember how to do it… I’ll show them the formulas so that they know the formulas, they can use that. But I still want them to get the idea of how, where that formula comes from and how, what it actually means to find the surface area, instead of saying, here’s a cone, here’s the formula, do it. Here’s a cone, well if we break it apart, here’s what it looks like, this is where that formula comes from. You can use that formula to find it, but at least you know where it’s coming from.

However, she believed that only some of her students had the ability to understand.

There’s going to be a bunch of kids in the class that aren’t going to be able to follow how to get there… They’re not the type that are going to conceptually understand what’s going on.

Even so, she was determined to teach the topic “conceptually”. Her teaching notes listed her examples. All of them showed the 3-D figure involved redrawn as a net, and the calculations as a sum of the areas of the component shapes. This was the only representation that she used.

In describing her choice of topic, Nicole had actually mentioned perimeter, area, surface area, and volume as related quantities. When I asked her how these quantities were related, she was able to clearly articulate some connections.

Finding the area of a 2-dimensional shape, you’re just looking at one specific shape. When you’re getting into 3 dimensions, you have all these little area pieces that have to be put together to create this 3-dimensional shape. So what I was talking about before,… unfolding this 3-dimensional shape to have now it being 2-dimensional, and you’re finding the area of all those components and putting them together to create that surface area. So … to find surface area, you basically have to have an understanding of area. You can’t do surface area without understanding area.
In her own understanding, she saw area as a component of surface area; surface area could be considered an extension of area. But entwined with her own understanding, was the instruction-oriented connection – area as a prerequisite concept for learning surface area.

In relating area and volume, Nicole used a metaphor to show volume as an extension of area.

Basically you’re taking a 2-dimensional shape and you’re kind of lifting it up and creating this 3-dimensional shape.

She had a harder time with perimeter. She linked perimeter and area by the fact that they were both properties of the same shape. She used a parallel description to link surface area and volume.

Perimeter involves a 2-dimensional shape as does the area but perimeter is finding the distance around the shape rather than how much space it’s occupying… I guess that’s similar cause surface area’s just kind of the outside of your shape and how much area it encloses for the whole outside shape but volume is how much can be contained inside that shape.

Nicole described her teaching sequence as perimeter → area → surface area → volume. This was the sequence in the textbook, and Nicole agreed with it, seeing it as a sequence that enabled students to “make that connection”, especially from area to surface area.

Josie chose solving right triangles as an example of a topic that is conceptually rich. While she strained to express herself clearly, it was evident that she was using a framework reminiscent of Skemp for describing her knowledge –

There’s lots of different small concepts, and also bigger concepts, like, I would say, the main concept is similar triangles and they have the ratios of the sides are all the same … we’re starting to learn trig ratios, tangents,
sine and cosine. And these three ratios... they stem from, like the concept that similar triangles have the same ratio of, for example, opposite to adjacent... So if you draw two different right triangles that are similar, if you divide, if you find the ratio between the opposite and adjacent, they will be the same... That's a big concept there... I think that's what I want to do is teach them the concept first instead of just teaching what is SOHCAHTOA.

In talking about this topic, Josie articulated the following explicit connections;

the first three are versions of $A$ implies $B$, the latter three are versions of $A$ is a procedure used in working with $B$.

- The biggest angle is opposite the biggest side.
- The hypotenuse is opposite to the right angle.
- The smallest angle is opposite the shortest side.
- Pythagorean theorem is... just a method for us to find... one of the sides.
- “Angles’ sum adds up to 180” is just a tool to get the answer.
- This notation [SOHCAHTOA] is... just a shortcut.

What Josie re-iterated over and over again was her wish to teach “bigger” concepts, like similarity and ratio, which students could then apply in a variety of ways. Subsumed within these broader concepts, she saw smaller concepts, like the Pythagorean theorem, the sum of angles in a triangle is 180°, which she thought of as procedures – “shortcuts... some ways for them to get the answer”.

Both Nicole and Josie chose topics within geometry that could be taught procedurally – by having students memorize and apply formulae. Both of them rejected the idea of teaching just formulae and repeatedly emphasized the importance of conceptual understanding. But they also both believed that students resisted the conceptual approach, some because they weren’t willing to exert the effort to
understand, others because they were not capable of understanding. Nevertheless, both Nicole and Josie stressed finding solutions by applying broader principles, not just memorized formulae.

*Algebra topics: functions [Christine, Robert]*

Christine chose the topic of functions as one “that we could really delve into”. The following quotation illustrates three noteworthy characteristics of the way that Christine saw the connections in this topic.

A function is any, is a relationship that’s expressed in any form in which you have a number to begin with, an input, do something to it, and get a number out as a result...all the ways that it can be represented are all connected in that you can represent something one way, and from that represent it then in other different ways, because a function can be expressed in words, as a statement of a relationship, it can be written as an equation, which then, of course, can produce a graph, it can be a list of ordered pairs, it can be summed up as a table of values.

First, Christine had a variety of alternate representations for functions. The idea of connection as alternate representation was her predominant way of looking at functions; she considered verbal, numeric, algebraic and graphical representations. Interestingly, her algebraic representations were “equations” [her word]; she did not refer to function notation at all. Perhaps the reason was that she constrained the topic of functions to the curriculum of Mathematics 10. But she also mentioned later in the interview, that she was never taught function notation in high school, and that caused her problems in university mathematics courses.

A second characteristic is that Christine described the alternate representations in very general terms, for example, “if I’m given a written rule to begin with, I automatically form an equation ...from that equation, ... what the graph would look
like”. Throughout the interview she repeated the idea of alternate representations, particularly algebraic/graphical. Only when pushed did she describe a “fine-grained” connection – “in slope-intercept form... you can see that the slope of line is the coefficient of the x and the y-intercept is the... constant”.

Finally, Christine talked about connections from a pedagogical stance –

... one of the most important things is for them to actually understand by the end of it, that it is all connected... I think it’s really important by the end of that functions unit that they actually do see that they[concepts like domain and range] are all connected and dependent on one another, and that given one, it can lead them to other places.

While she repeatedly stressed the importance of students making connections and expressed concern that students don’t see them, she specifically focussed on the making of connections only occasionally, through a review activity involving concept-mapping.

Christine identified her topic as functions in general, though most of her examples were of linear functions. Robert chose as his topic a particular kind of function - quadratic functions and equations, coincidentally, the topic I had chosen for the common task. He chose this topic because

I think [it] has a lot of interesting connections, like visual and algebraic connections and lots of connections to real physical problems.

In fact, he described what a quadratic function was in terms of throwing a baseball; the trajectory was a parabola. As he talked about quadratic functions further, he listed some component concepts, like maximum, minimum, symmetry, and “skills” like factoring, but mostly spoke in terms of relationships.

The important relationship is to understand that a graph, the picture view of the equation or the formula, is the
same as the algebraic and how they’re just two
different versions of the same thing, and that they’re
not two distinct types of mathematics, they’re one, and
you use both to help you. So, when I solve the problem
with the kids, I draw the picture, I do the algebra, I
show, I tell them that these things go together… The
graph is just another way of showing all the different
pairs of x and y that work in the equation.

Robert repeatedly referred to the algebraic expression and the graph as
alternate representations of the same thing, and usually at this level of generality.

When pressed to be more precise, he articulated the following connections:

- line of symmetry of this parabola would have to be right
  in between the two [zeros];
- minimum point, the x-value of that minimum point, has
to be in between the two zeros.

He returned to the theme of connecting to real-world situations through
working problems but acknowledged that the students found these textbook problems
difficult. His explanation included both procedural and conceptual obstacles; he
identified weak arithmetic and algebra skills, particularly in calculations with
fractions and decimals, and factoring.

Where I’m not giving them an equation but I’m giving
them some information that they synthesize and put
together and make an equation or make a picture of it,
they have some difficulty with that as well… They do
not have a strong enough conceptual understanding of
what they’re doing.

Asked to describe a student who did have a good conceptual understanding of
the topic, Robert spoke in terms of how the student would approach doing a problem.

I think that they would be able to look at the type of
problem, and realize what the question’s asking. I
think, basically with these types of questions there’s
two things that they’re looking for – either a maximum
or a minimum, or for a zero. And so then, the kid who
has a good conceptual understanding would understand,
OK, it’s a quadratic function, I’m looking for the
maximum point, so I need to find a vertex. And then, they could take the information that’s given to them and understand that those two pieces of information given to them are two points on the parabola and they need to, using those two points, how do I describe what the parabola is. I think also, a kid with a good conceptual understanding would understand that the two points given would also give them two other points, because of the symmetry and that from there, that they can easily build ... once they have that they can build out the maximum’s going to occur, or the x-value or whatever the variable, the maximum will occur at and then, and then, be able to piece it together to find all the necessary pieces to draw the function and to solve the problem.

Imbedded in this description is the metaphor of building – of putting together components to make a greater whole, like drawing the parabola. Also, he mentioned several instances of connections as implications, for example, that two given points on the parabola, actually become four known points because of symmetry. Moreover, he described students who were good at math as being “better at seeing patterns and connections”. “The kids that are good at math understand that what we’ve been doing will help us to what we’re doing today” [another instance of the connection to prior knowledge].

My goal when I’m teaching this topic is to get the kids, all the kids to understand what this topic is about. And then when we move to the next one, I say, OK, here’s something that’s connected to what we’ve just learned, and how they’re connected and then we work through that.

With respect to connections, there were two common themes in Robert’s planning – connecting to prior knowledge, and emphasizing alternate representations.

Both Christine and Robert strongly emphasized mathematical connections as alternate representations, and almost always as graphical and algebraic representations.
Darcy chose integers as the mathematical topic that she wanted to explore in Interview 2 because “students always seem to have trouble grasping the concept of positives and negatives”. What she wanted her students to learn, however, she expressed in procedural terms – “to be able to do the four main operations with integers”. In talking about teaching the topic of integers, Darcy referred to two types of connections. First, she made connections to real world topics, specifically, money and temperature. Second, she related integers to the number line and taught students how to represent addition of integers as moving back and forth along the number line.

Darcy offered the following example to illustrate how she used the analogy of transactions with money (she called it a “physical representation”) to introduce addition and subtraction of integers.

If they look at -9 + 6, I want them to be able to figure out that that should be -3 because the negative number is bigger and ... just be able to think of it... if you owed 9 dollars and you got 6 dollars somehow and you were able to pay that back, you'd still owe 3, I guess... Some of these students would look at -9 + 3 and they'll just go 12, negative 12, negative, positive, or whatever, just be randomly guessing all the answers, cause they’re not really, they’re not making a connection to what the negative means... Well, subtraction ... if you owe somebody 5 dollars, but somebody takes away that debt, that actually... increased the money you have really, cause now you don't owe 5 dollars, you’ve got zero. That's how I would introduce subtraction to them.

She used the conventional meanings of addition and subtraction, and aspects of money ownership to illustrate positive and negative – money in hand was “positive”, money owed was “negative”. She used the money analogy only for addition and subtraction. Darcy could not offer real world analogies to multiplication and division with integers, nor additional analogies for addition and subtraction.
The mathematical connection that she made was to represent integers and addition and subtraction of them on the number line.

We start with the number line and just looking at where zero is, and how negative 2 is less than negative 1 and that and then looking at, if you're adding, you're increasing the number, so you're moving to the right on the number line. [For subtraction] you also look at the number line, so you're going to do the opposite of when you're adding. So I teach it [subtraction], I guess I just teach it as the opposite of adding.… More difficulties with subtracting, especially the concept of subtracting a negative, how that is actually like adding a positive number - they just see negative, whether it’s a plus negative 6 or minus negative 6, they want to do the same thing.

But she was quick to have her students move away from the number line.

So we start with that and then I guess try to develop a rule how you could figure it out without actually counting on the number line.

She saw using the number line as a way of helping students visualize what addition and subtraction of integers meant, but not as an efficient procedure for actually doing the operations. Moreover, even though she saw that students were having trouble with subtraction, she felt that she could not slow down and take “even a day” because of the pressures to complete the curriculum, even though this was Grade 8.

When it came to multiplication and division, which she said the students found much easier than addition and subtraction, she used no alternate representations.

Multiplication, ah, yes, I relate that to adding. So if you’re doing 5 times negative 2, you’ve got 5 negative 2’s, so you can look at it as negative 2, plus negative 2, 5 times. So I try to relate it to what they already know, but then when it gets to negative 5 times negative 2, that I find harder to really explain why a negative times a negative is positive.
It's like a pattern ... so if 4 times negative 3 is negative 12, and 3 times negative 3 is negative 9, so we've actually added 3, 2 times negative 3 is negative 6, so again, added 3, so every time that we are decreasing the number of times we multiply by negative 3, we're actually adding 3 ... So, if we kept decreasing that, so we had negative 1 times negative 3 ... just by the pattern, we'd still be adding 3, so that would be positive 3. So usually I would just introduce it that way, but then I would just go to the rule.

She started by relating multiplication to addition by showing multiplication as repeated addition. She then showed a pattern that equated multiplication by a negative integer to repeated addition of a positive integer and extended that pattern to allow for multiplications of negative x negative. Here too, she wanted to move as quickly as she could to a rule that students could apply. In fact, she described teaching division of integers by just telling students that it followed “the same rule as multiplying”.

There was an interesting and unspoken conflict in Darcy’s thinking. She constantly stressed the importance of understanding, and bemoaned students’ jumping to blindly applying rules. At the same time, she herself, moved as quickly as she could to teaching students rules so that students could move on to more complex calculations.

*Calculus [Edward]*

For the second interview, Edward chose to discuss the fundamental theorem of calculus because “it’s not a procedural thing, it’s more a conceptual thing”. He focussed on the curriculum of the Advanced Placement (AP) Calculus course; he saw the fundamental theorem of calculus as a central idea that was used “to tie together a lot of the ideas that they [the students] have learned all through the course”. In fact, the metaphor of tying ideas together was a recurring one. Consider the example below.
So, where you have a function, \( g \), defined as an integral of say \( f(t)dt \). So, you get this kind of function, a function defined as an integral, right? And then they [the textbook authors] ask a whole bunch of questions about the function, so they ask you to evaluate, say, \( g(3) \)… so \( g(3) \) you’re looking at an area and then when you start talking about \( g'(3) \) then you’re starting to talk about the function itself, but you’re also talking about the increasing and decreasing behaviour of \( g \), right? Then when you start talking about the concavity of \( g \), then you’re analyzing the derivative of \( f \). So, you see… it really ties together, \( g \) itself talks about areas, which is integration… But then when you start getting into \( g' \) and \( g'' \), then you’re tying in all the ideas from earlier in the year about max and min and inflection points and concavity and all that stuff. So, I just like the fact that you know, with one question, you can almost tie together everything they’ve learned in the course.

He was describing alternate representations and implications, but at a fairly general level. Although he discussed mathematics content in detail in this interview, his point of view throughout was that of a teacher; he could not separate the mathematics and his teaching of the mathematics. He talked a lot about his teaching and how his students respond. He called what he did “learning by discovery”.

Well, what I would normally do with this kind of stuff, I would definitely get them to work in groups of 2 or 3. So by the time I would talk about, sort of get into detail about the fundamental theorem, they would know that the integral sign means the area under the curve from \( a \) to \( b \) kind of thing… **I probably try to get them to learn through [doing] the question as opposed to just saying this is how you do it.** I’d probably get them into groups of 2 or 3 and give them a question to do with a bunch of sub-parts and just kind of circulate around the room and have as many conversations as possible with them and see how they’re doing… **I think the question itself would totally lead them to the understanding.**

He then followed it up with a summary discussion and writing notes on the board.
I say something like, \( g' > 0 \), what does that imply? that implies that \( f \) is greater than zero, that also implies that \( f' \) is greater than 0. Then I would point to this and say, if \( f \) is greater than zero, then what does that mean? Then they could say, well, gee, it's increasing. If \( f' \) is bigger than zero, what does that mean? that means \( f \) is increasing, it also means \( g \) is concave up, OK? Those are some key things. Then we could talk about \( g' \). What happens if \( g' \) is negative, or, how do I want to analyze \( g' \) negative? Well, then I can look at \( f \) again... so those are the kind of things I would summarize on the board so they can see how they relate to each other.

He was careful to explicitly draw his students into the process of making implications from some basic facts to others. In fact, another favourite theme of Edward's was "less is more", (on which he expounded in the first interview). "The students that are good, they understand that the less they know, the better". In other words, the fewer facts they memorize, and the more they rely on reasoning with those facts, the better. He was adamant in his opposition to his students' memorizing many mathematical facts. Instead, he wanted them to be able to recognize relationships and to reason from some basic knowledge to find new information.

Edward offered two other detailed examples that to him illustrated the making of connections. One was his discussion with a student who posed the question – "is position the same as displacement?". The other was an example from a different topic – how he taught logic, specifically the positive and contrapositive. In each, he demonstrated following a line of reasoning, and specifically focussing his students on finding a relationship. When he talked about reasoning/implication, he gave quite specific examples, but was more general when he talked about alternate representations.

All in all, Edward's conversation was steeped in specific mathematical examples that he was able to consider in depth. He agreed with the importance of
connections, but told me, outside the formal interviews, that the really important connections were the personal connections he made with his students.

**Summary of mathematical connections articulated by teachers**

The following table summarizes the types of mathematical connections that teachers articulated in discussions of their chosen topics, with illustrative examples. As mentioned earlier, some teachers grouped certain mathematical ideas as related, not because of inherent mathematical structures, but because of their role in teaching the topic. So, while these instruction-oriented connections are not mathematical connections in the strict sense, they are included in the table for the sake of completeness – as a variant of mathematical connections that is likely unique to teachers.
<table>
<thead>
<tr>
<th>Type of mathematical connection</th>
<th>Examples</th>
</tr>
</thead>
</table>
| Different representation | Logarithmic and exponential forms  
Venn diagrams an alternate representation of union and intersection of sets [probability]  
Tree diagrams an alternate representation of all possible outcomes [probability]  
A function can be expressed in different forms – e.g. equation, graph table of values  
Movement on a number line as a representation of addition  
An integral in calculus is a representation of an area under a curve |
| Implication | Complementarity in probability: \( P(A) \rightarrow P(\text{not } A) \)  
The hypotenuse is opposite the right angle  
Because a parabola is symmetrical, coordinates of a point on one arm \( \rightarrow \) coordinates of a point on the other arm  
\( F' > 0 \rightarrow \) the function is increasing |
| Part-whole relationship | Using examples to illustrate concepts  
Conditional probability outcomes as a subset of “all possible outcomes”  
Surface area as sum of areas of component shapes  
Perimeter and area are both properties of a shape  
SOHCAHTOA as a particular instance of ratios of sides in similar triangles |
| Procedure | Use combinatorics formulae to calculate probabilities  
Tree diagrams  
Pythagorean theorem as a method to find sides of a right triangle |
| Instruction-oriented | Prerequisite concepts to the learning of conditional probability [Lily, Sophie]  
Area as prior knowledge to understand surface area |
Two noteworthy characteristics of teachers' mathematical connections can be seen in the table above. First, when teachers articulated mathematical connections, they often did so in general terms. Second, the variety of connections did not appear to be constrained by the topic.

In the examples above, tree diagrams are treated in two ways – as a procedure to find and count all possible outcomes and as a way of representing all possible outcomes. This example raises the possibility that two objects, in this case “tree diagram” and “all possible outcomes” can be related by connections of different types.

Summary of emerging themes in Interview 2

In Chapter 4, I drew attention to a tension between what teachers thought they ought to do and what they felt obliged to do because of external pressures. That tension was also evident in this set of conversations. As before, teachers claimed that they wanted to teach conceptually, but actually taught algorithms. This tension was also expressed as an opposition of attending to student learning and attending to covering the curriculum, seen most poignantly in Darcy’s rush through “Integers” even though she was teaching Grade 8.

Another noteworthy feature of these conversations was that the teachers talked almost exclusively about teaching the topic. Only Christine and Josie made a few statements that indicated they were exploring the mathematical content for its own sake. Given that teachers mostly chose curriculum topics that they were in the process of teaching at the time of this interview, this emphasis is not surprising. But the fact that they spoke solely about teaching is worth mentioning.
As with the themes identified from the conversations in Interview 1, some of the common ideas that teachers expressed were about teaching mathematics in general, and some were specifically about making connections. The most pervasive general themes were:

- Teachers want students to understand the mathematics that they are learning. All the teachers expressed this as an important goal. Although their individual conceptions of understanding varied, they all favoured understanding over memorization of algorithms.
- Teachers claimed a commitment to teaching concepts rather than algorithms. A few of them actively avoided teaching formulae. A few ruefully stated that they succumbed to teaching algorithms for doing questions even though they valued the conceptual approach.
- Most of the teachers assumed that students would resist a conceptual approach.
- The teachers believed that many students lacked “basic skills”; this lack became an obstacle to their learning.

The most important themes related to connections were:

- Teachers believed that the hardest thing for students, by far, was modelling a situation (usually as described in a textbook word problem) in mathematical terms. In other words, making specific, fine-grained “real-world” connections was very hard.
- In their own descriptions of mathematical content, teachers used examples extensively and talked less commonly about mathematical ideas in abstract terms.
• Teachers varied considerably in the degree to which they spontaneously articulated mathematical connections.

• In addition to modelling (real-world) connections and the strictly mathematical connections that were the starting point of this study, teachers also made another kind of connection, which I called an "instruction-oriented connection", where they identified mathematical objects as related based on their role in teaching, not on their mathematical associations.

• When teachers did describe mathematical connections, they spontaneously articulated several types, but mostly, alternate representations.
CHAPTER 6: EXPLORING CONNECTIONS THROUGH A COMMON TOPIC – QUADRATIC FUNCTIONS AND EQUATIONS

I asked teachers to work with a set of 82 cards that contained names, formulae and graphs representing ideas associated with quadratic functions and equations, as described more fully in Chapter 3. Their task was to arrange the cards in some way that showed the relationships among them; we used the arrangement as a basis for discussion of the mathematical connections that they saw. Teachers were able to take the time they needed to complete the sorting task. Class periods in both schools were 77 minutes long; no teacher came close to needing that much time. The teachers completed the task in 25 to 40 minutes, except for Wendy who finished in just 13 minutes. Although she left out 27 of the cards (the highest number), she was confident that she had truly represented her thinking about the topic.

All of the teachers proceeded confidently. Occasionally, they changed their minds about the placement of a card and sometimes ruminated because they didn’t immediately recognize a term or an algebraic expression. Although teachers sometimes (rarely) admitted to not knowing some concept represented on a card, no teacher made any mathematical mistakes that could be identified through their card arrangements or subsequent discussions. At the end, only Josie thought that the activity itself was not natural to her way of thinking about the subject. Nevertheless, she did believe that the final product reflected the way that the objects represented on the cards were related.
It is reasonable to suppose that each teacher will have a unique schema. Their representations ranged from a linear flow chart [Sophie] to a complex 10-cluster grouping [Josie]. I was careful not to use the language of “concept map” or “mind map” in an effort to keep from imposing a particular way of dealing with the task. But several of the teachers did use that language while they were working, Christine in particular. Because the focus of my study was to examine how teachers could explicitly formulate mathematical connections, I was more interested in the relationships that they could articulate, than the simple grouping of objects as “connected”. So, I considered mostly what the teachers said about why they placed certain cards together and how they described the relationships. Appendices C and E show details, including photographs, of the teachers’ card arrangements.

**Individual teachers’ approach to the task**

First, I describe how each teacher dealt with the task – his/her general approach and overall pattern, how s/he chose what cards to add or leave out, and the main kinds of connections that s/he articulated. Then I consider the specific mathematical connections, regardless of who mentioned them, to describe the landscape of relationships associated with quadratic functions and equations for this group of teachers.

**Sophie**

Sophie produced a linear arrangement that represented her teaching sequence for this topic. She started with “cone” because she introduced the topic of quadratic functions and equations through conics, moved to consider the algebraic properties of quadratic equations, then graphing, then solving quadratic equations. She forced herself to use all the cards, and ended her sequence with a group of terms that she
thought were related to quadratics, but that she did not use in her teaching. She added a few cards, not to introduce new ideas, but to duplicate existing ones so that she could maintain her linear sequencing of the cards.

A striking characteristic of Sophie’s thinking about this topic was that she was completely absorbed in thinking about the sequence in which she would teach the concepts and procedures illustrated on the cards.

The instructions were to keep in mind everything I knew about quadratics but I had trouble doing that. I was really thinking with each card, when do I teach this, how do I teach it, how does it relate to other things I teach. I really had trouble going beyond that.

Even when her own mathematical understanding indicated a relationship, she resisted the connection because it wasn’t part of the way she taught the topic. For example, talking about linking the remainder theorem and the factor theorem, Sophie said:

See that’s the thing. I had trouble using this even in quadratics because, as a teacher, I don’t use these theorems with the quadratics, I use them with cubics and so I’m having trouble relating them, even though I know, oh yes, we easily could use these theorems.

Sophie found talking about her organization of the cards difficult – not in identifying what ideas were related, but how they were related. It was common for her to start with statements like “they’re just related”, or to refer to surface features, for example, “these letters $a$, $b$, and $c$, … they’re all related, they’re all from the equations”. Nevertheless, with probing, Sophie articulated mathematical connections of all five types.

Another salient feature of Sophie’s conversations was the revelation of her self-examination of her thinking and her practice.
I had trouble connecting this... with anything else. Which really makes me start thinking, we teach all these little topics and then what happens with them... This helps me, seeing these things down here [her organization of the cards]. I started to think about - are they connected? Why do I teach them to my students? Do I have, even if... the lesson doesn’t make the connection, do I even have the connection in my own brain?

What Sophie had to say in relation to this task is consistent with the struggles that she had in the first two interviews. The fact that she was able to articulate a variety of mathematical connections when pressed, indicated that she had the mathematical knowledge. But thinking in terms of connections was so different from Sophie’s natural linear approach, that early conversations made it seem that she didn’t see the connections at all.

**Wendy**

Wendy did this task very quickly – in roughly half the time of most of the others. She was confident and rarely moved a card once she had placed it. She organized the cards into two clusters. There was a small one containing nine cards that were related to factoring, because “we do teach the factoring stuff first”. The rest of the cards were placed in a large cluster “related to graphing the parabola”. In discussion, Wendy carved off a subgroup, including relation, function, co-ordinates, point and table of values, which she identified as prerequisite knowledge for understanding the rest, which specifically dealt with quadratics.

Like Sophie, Wendy’s organizing principle was the sequence in which she taught the topic. As she talked her way through her pattern, she constantly referred to teaching aspects – “when we talk about...”, “it gets them to think about...”, “it leads us to...”, and so on. She spoke freely about relationships, yet almost all of the mathematical connections that she articulated were different representations, almost
equally algebraic/graphic representations, and equivalent algebraic representations, for example, considering different forms of the equation for parabolas.

Wendy added no new cards and left out 27. Even with probing, she was reluctant to reconsider the large number of cards that she left out. She left out cards for two predominant reasons. She identified a list of terms that she considered very broad.

The reason I didn’t put in things like derive and simplify, cause it’s so broad that you can use it for anything. When you say simplify, it can mean like, oh, add/subtract fractions. So I didn’t really want to put it in to specifically this because I don’t want to imply that this is the only time we use the word simplify.

It seemed that Wendy was thinking in “anti-connection” terms, by wanting to restrict what she taught to ideas that were unique to the particular topic. My asking her if she could see any relationships involving the left-out terms, even though she might not use them in her teaching, prompted a long sigh, but she did try to extend her thinking. Continuing to talk about simplifying, Wendy said:

Simplify can mean the same thing, even, like simplify the solution or simplify the answer... so there could be some connection. I’m not looking at it like specific, like they’re completely different, there could be some sort of cross-over... There could be some overlap in the meaning, just a little bit. And I find that’s what most students have trouble with - is when they see, ‘simplify’ and they’re like, what do you mean? They don’t know exactly because it’s such a broad term.

In her description of simplifying, Wendy seemed to be treating it as a procedure common to many mathematical topics. Yet rather than seeing this connection as helpful, she saw it as an obstacle to students’ learning. So, rather than a unifying principle, simplification seemed to be a collection of procedures, whose similarity interfered with students’ ability to distinguish them. For example, it seemed
that Wendy did not view procedures like collecting like terms and reducing a fraction to lowest terms as instances of an overarching idea.

Wendy was opposed to teaching formulae.

Like I said before, I don’t like to use a lot of formulas or a lot of big words cause I know the kids won’t remember them in the first place. So I don’t really teach in a sense of formulas. I don’t stress on the formulas… I’ve seen them before, but I don’t teach it that way because I don’t find memorizing certain formulas are useful. Understanding is more important to me.

This statement was Wendy’s rationale for leaving out half of the formula cards. Again, I pushed to see if she could/would identify some relationships even though she did not teach this way. She acknowledged that she could not remember some formulae, and related others on the basis of surface features, for example, “I’m guessing that the $p$ and the $x = p$ are related, ‘cause they’re both $ps$."

**Nicole**

Nicole’s organization of the cards looked like a flowchart. She started with “algebra” and added cards as she thought of relationships. In describing her display however, she started with “quadratic” with branches to an “equation side” and a “graph side” and focussed on transforming from one kind of representation to another.

When I think quadratics, there’s kind of two branches I took, the equations and the graphs, so that’s probably where I started from. And the left side here is mostly how to deal with equations and how to solve them and stuff. The right side is mostly centered on the graphing… When you have your equation, if you can put it in completing the square, in that form, then you can graph from it. So I kind of got that down the middle. It’s how this moves into this, I guess.

Her description sounded like she was creating a script for “the story of quadratics” – “of understanding how quadratics work and what they are and how they
follow a pattern”. Nicole made many statements articulating connections with little probing, and mentioned mathematical connections of many types. She found many of the relationships “obvious”, was unsure of some, but after some thought, “did manage to find ways to put them in”.

Nicole left out only six cards, five of them formulae, for the simple reason that she did not recognize them. Nicole made a point of saying that she had enjoyed the sorting task.

Josie

We had just started working on the sorting task during Josie’s spare period when she was called to do an emergency supervision. As we walked to the nearby classroom, we decided that it would be preferable for Josie to continue the task, rather than interrupt and start over another day. So, Josie worked at the back of the classroom, while I supervised a French class doing seatwork. Josie was quite unperturbed, and thought there was no impact on her work from the short break. However, the division of labour meant that I was not able to monitor Josie at work as closely as I did the other teachers.

Josie produced the most unique and complicated organization of all nine teachers. She arranged the cards in ten distinct clusters, with some overlaps, and one group as a subgroup of another. She had an arching group that she called “a rainbow kind of thing” that represented background knowledge. She further separated groups on the basis of “leans toward equations” and “leans toward functions”, a distinction that no other teacher made. However, other teachers did seem to distinguish between equations and functions in that they spoke of solving equations, but graphing functions. Josie emphasized the “important connection between function and
equation”, although she had a hard time specifying what the connection was.

Generally, though, Josie had little difficulty talking about mathematical connections either within groups, or between groups. The mathematical connections that she articulated were predominantly different representations, or procedures.

Josie left out 20 cards – 2 formulae which she did not recognize, and 18 terms which she deemed too simple to include.

But some of them, even though... they should be put in here, but I didn’t put it because they’re so simple. Like I thought... they should already know it very well. So I didn’t even bother putting them in. You should really have known that too long ago that I wouldn’t really be considering... teaching it if I were to teach it.

Josie referred to some things as “too simple” to include; Wendy excluded many terms because they were “too broad”. But they had thirteen excluded terms in common. This exclusion was an unusual kind of “connection”, like a negative instruction-oriented connection.

_Lily_

In organizing the cards, Lily started with ‘algebra’ and put together cards that she saw as “background information”. Like the other teachers, Lily viewed organizing the cards in terms of how she would teach the topic of quadratic functions and equations. She identified roughly a third of the cards she used as prerequisite knowledge.

Before you can do anything with quadratics, like factor them, or solve or anything, you have to have a strong idea of the background, language, and what you can and cannot do, such as like terms, just things you need to know before you even start to think about doing something with quadratic equations and functions.
She named a second cluster as “equations, relations, functions, all that sort of thing”, including concepts that applied to functions in general, not just to quadratics. Her third group dealt very specifically with quadratics. She did not separate algebraic and graphic representations in laying down the cards, but made numerous references to the ways that algebraic and graphic representations were connected when she talked about them.

Lily added no cards; of the eighteen cards that she left out, seven were formulae (half of the formulae cards). Her primary reason for excluding them was that she was not sure of them – “I honestly don’t remember what we used them for”, or “I didn’t know where to put it”.

The mathematical connections that Lily was able to articulate were almost exclusively alternate representations and procedures. When I asked her whether this sorting task led to a reasonably accurate reflection of her thinking, she answered,

I think this how I think about them, but I think, in teaching, you kind of just go with the text, like you kind of just go with the textbook… I know some of the ways I organize things is not the way we teach it.

This spontaneous reference to the way that she felt bound to teach the topic in the textbook sequence in spite of finding a different sequence more natural, echoed the concerns expressed by Josie about the constraints under which teachers felt they operated.

Christine

Christine arranged the cards in three clusters, a large cluster representing the algebraic view of functions, and two smaller clusters that included graphic representations.
This was hard, cause there’s just so many ways to connect things… It’s all interconnected but it’s quite muddled.

Such was Christine’s view of the sorting task. While other teachers alluded to considering other possibilities in arranging their cards, Christine saw the most complexity in the activity. Her final arrangement was not her first choice in showing the relationships.

I was trying to organize it more like a tree branching out, and then I couldn’t get my thoughts to organize themselves into any sort of hierarchical thing that was making sense.

Her starting point seemed to be to impose a structure on the information with which she was presented. When she was talking about her arrangement, she had a clear organizing principle in mind – algebraic and geometric views of functions from an instructional perspective. Like others, she talked about items being related because they were instances of knowledge prerequisite to learning about quadratic functions and equations – “basic concepts”, “basic vocabulary”, “basic operations”. She was able to articulate a variety of mathematical connections, especially different representations and part/whole examples.

Christine was one of the few teachers to add cards to her arrangement. She added ‘standard form’ to label the equation, and the term, ‘reflection’ to make the connection with a diagram of a parabola reflected through the x-axis, and with the coefficient, $a = -1$. She left out three cards, all of them formulae, stating that she couldn’t remember them – “I’m not really a formula person”.

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**Darcy**

Darcy started her card arrangement with methods of solution because algebraic ways of solving just “jumped out” at her. She made two large clusters and organized each into subgroups.

I guess I mainly sorted it into graphing stuff and algebraic stuff. I know they’re related... just sorting out the kinds of operations that we ask students to do with quadratics, between the graphing and solving equations.

Like most of the other teachers, Darcy’s basis for thinking about how the ideas on the cards were related, was her sequence for teaching this topic. She was able to talk about connections with little prompting, and described relationships confidently. She articulated many mathematical connections of various types. She left out just a few cards because they “just didn’t really seem to fit anywhere”.

Darcy thought that this sorting task allowed her to truly reflect her thinking about quadratic functions and equations.

I guess I think of quadratics of being broken up into, OK, graphing stuff and algebraic stuff, and they do them together, but I do think of it as two different things in my mind, the graphing from the actual solving, algebraic stuff, although there is overlap and stuff.

**Robert**

Robert arranged his cards in two clusters – “the algebra side and the geometry side”. Unlike the others, Robert spent some time shuffling and examining the cards before he started to place them. I wondered about his thinking when doing so, but he said that he had no plan in mind; he was “just going with the flow”.

When talking about his arrangement, Robert made many statements like “they all go together”, “they all relate together”, “those all kind of go together”. I kept asking him to elaborate, but he had difficulty being more specific. He was able to
articulate a variety of mathematical connections, but relatively few in comparison to the other teachers.

Robert left out twenty-three of the cards, five formulae because he didn’t know what they were, and many terms that he named “basic skills” or thought were irrelevant.

I didn’t use them because I didn’t think that they added anything new… To me it just didn’t feel like it added anything to the understanding of the concept… Perfect square, exponent, value, square root, square, radical, … none of these things I thought were key. Well, I think they’re all kind of ideas that kids need to know before they start looking at parabolas, but I don’t think it adds anything to the understanding. Like if they don’t understand it, they can’t understand parabolas, but this is kind of like basic skills, and then there’s the understanding of quadratic functions beyond that.

Robert’s reasoning was similar to Josie’s and Wendy’s rationale for leaving out a large number of terms. In fact, these three teachers omitted the highest numbers of terms and had eight omitted terms in common. Their rationales for dismissing a significant number of the objects they had to work with raises two alternative hypotheses – that teachers might be ignoring some connections to students’ detriment, or that some connections might not be pedagogically useful.

In spite of his struggles with articulating mathematical connections, Robert was very committed to the importance of connections. When asked to comment on how realistically the sorting task depicted his understanding, Robert said:

I think it reflects my understanding in that I’ve tried to show that there are two different sides to it… either algebraically or using the graph… I think without an explanation people might assume that I believe that they’re two separate ideas, but I don’t think that they are. I think that they’re interconnected… in that from one, you could start with algebra, up to the zeros and
work your way back to get to the graph, or you could go up graph, down here, back up to help you with the algebra. **Connections kind of arise going forwards and backwards.**

There is a noteworthy distinction between the ways that Robert and Darcy, who also described her view of quadratics as having two basic components – the algebraic and the graphic, valued connections. For Darcy, the connectedness seemed peripheral, an “overlap” that she did not particularly attend to; for Robert, the connectedness was central to the way he thought about the alternate representations.

**Edward**

Edward was the only one of the teachers to choose “thinking aloud” as he did the task rather than talking about it after completion. He began by saying:

> What I'll try to do is organize it according to topic, the way, I guess they would be presented in the text book. That’s where I’ll start. I’ll see where it goes from there. So, that would be quadratic formula.

While all the teachers organized their cards with regard to teaching the topic, Robert’s comment was the most overt reference to the influence of textbooks on his organization. Edward produced three clusters, and drew lines connecting items across groups. He spontaneously articulated mathematical connections of many types, but also linked some cards together on the basis of them being prerequisite knowledge – “nuts and bolts stuff”, and identified cards as being closely related because they were “the first thing that you learn about when you learn about algebra”.

Edward’s work on the task was unique in two ways compared to the other teachers. In thinking about how to place cards, he looked forward, identifying a connected idea and looking for the corresponding card, for example,

> When I saw the word discriminant on a piece of paper, the next thing I looked for was the number of roots and
then the next thing I would have looked for would have been single root, double roots, two distinct real roots. But then I guess you could take that one step further and talk about imaginary roots.

No one else talked about thinking this way. This difference might just be an artefact of Edward’s thinking aloud; he was the only teacher to do so. But it also hints at the possibility of a forward thinking strategy, where Edward thought of a connection, then looked for the appropriate card. A more reliable characteristic was Edward’s grappling with formulae that he did not know. Rather than simply discarding items with which he was unfamiliar, Edward started working on them algebraically, trying to transform them into something he might recognize. Nicole also made some effort to deal with unknown items before discarding them, but her method was to try to look them up in a textbook.

Like Christine, Edward found this a “tough task” and for the same reason:

...because everything’s kind of related to everything. I mean I’ve got that stuff farthest away from this stuff, graphs farthest away from roots, but really they could have been put right beside each other.

Other teachers also referred to possibilities of multiple connections, sometimes by adding a duplicate card, sometimes by drawing lines to show that a group placed in one spot was related to a group in another area. In these ways, they worked around the limitations imposed by a two-dimensional format.

**Types of mathematical connections**

Every teacher approached this task as though they were planning for teaching, not as though they were considering their own knowledge of quadratic functions and equations. Some, like Sophie, recognized that they were diverging from what I had
asked them to do; others seemed quite unaware. My attempts to gently redirect teachers were ineffective. So, I went with them on the course that they set.

The first of my research questions was “How do secondary mathematics teachers conceptualize ‘mathematical connections’?” The short answer is an emphatic “almost exclusively in terms of their teaching”. This view was apparent in our conversations in Interview 2 and emerged even more strongly here. I had taken pains to include in the set of cards, some terms, graphs and formulae that were outside of the current high school curriculum, for example, focus, directrix, 

\[ x^2 - (r_1 + r_2)x + r_1r_2 = 0, \]

and a large number of terms that were more broadly used in mathematics than with respect to quadratic functions and equations. Nevertheless, teachers focussed on how they would teach the topic.

Some teachers talked a lot about their thinking in this task, and others, less so. Some teachers spoke quite freely and spontaneously about connections; others made many of their statements about connections in response to probing. Moreover, most teachers repeated themselves at some point. So, I have given no weight to how many statements about connections the teachers made. Rather, I identified the categories of connections to which they referred, and noted the specific relationships that they mentioned. In general, teachers articulated most of the types of connections described in my model, but sometimes favoured certain types of connections over others. For example, almost all of the mathematical connections that Wendy articulated were different representations of the same concept.

The conversations about this sorting task demonstrated that, when pressed, teachers were able to articulate mathematical connections of different types, but most of them did not do so spontaneously. Certain types of connections were articulated more often than others. Most often, teachers linked different representations of a
concept, or invoked related procedures. Neither of these findings is particularly surprising. Different representations are the type of connection that is emphasized in textbooks. For example, transformations of parabolas are clearly illustrated graphically and algebraically in the Grade 11 textbook (Alexander & Kelly, 1998), and the two representations are explicitly linked.

Given teachers' imbedded view of mathematics as "doing questions", the prevalence of procedures is to be expected. The pervasiveness of procedures and different representations in teachers' discussion of this task is consistent with the types of mathematical connections that teachers talked about when they discussed other mathematical topics as well (Chapter 5).

I now move to a finer-grain-size to discuss the particular mathematical connections that teachers identified. "What are the characteristics of the explicit mathematical connections that teachers are able to articulate?" was my second research question. Teachers conversations about quadratic functions and equations yielded many statements about the way that mathematical ideas were linked, that I was able to interpret according to my model of mathematical connections. I first consider the four types of strictly mathematical connections – different representations, implications, part-whole connections, and procedures. I then discuss the category of instruction-oriented connections.

**Connections as different representations**

The predominant type of connection that teachers articulated was between algebraic and geometric/graphic representations of aspects of quadratic functions and equations. Sometimes the connections were general, linking some form of a quadratic equation to one or more of the parabola graphs, for example,
y = ax^2 + bx + c is a parabola [Lily],

graph of a quadratic is a parabola [Darcy],

this trinomial is literally the parabola, the quadratic equation [Wendy],

the function is the expression and parabola is the shape of the quadratic function [Josie].

When it came to specific statements of mathematical connections, teachers referred to five sets of alternate representations, and mentioned them repeatedly; they are listed in the following table.
Table 3: Teachers’ alternate representation connections - quadratics

<table>
<thead>
<tr>
<th>Alternate representation connection</th>
<th>Teachers said… (examples)</th>
</tr>
</thead>
</table>
| (p,q) is the algebraic representation of the vertex of a parabola | … vertex (p,q) [Darcy]  
… vertex, which is of the form, (p,q) and it’s a point on the graph [Nicole] |
| The minimum or maximum of a quadratic function is represented by the vertex of the graph. | … the vertex is a max or min [Lily]  
… the functions having a maximum and minimum, which is the vertex… [Christine] |
| The roots of a quadratic equation can be represented as the x-intercepts of the graph, the zeros of the function, or the zeros of the graph. | … the intercept is another way of saying the roots [Sophie]  
… what’s considered a root – the graph crossing the x-axis [Wendy]  
… roots go with the equation, zeros go with function [Edward]  
… zero of the function is the roots of the equation [Josie]  
… solving the equation means finding the zeros of the function, which are also the roots of the function, which are also the x-intercepts [Nicole] |
| x = p is the equation of the axis of symmetry. | … x = p is the equation of the axis of symmetry [Sophie]  
… x equals p and tells you your axis of symmetry [Nicole] |
| The table of values represents the coordinates of the points on the graph. | … if you get a table of values, you list it out and graph it, it will produce the parabola looking shape. [Wendy]  
… your most basic parabola ever and a table of values showing you the coordinates of those points [Christine] |

Equating the minimum/maximum with the vertex is not strictly correct – the minimum or maximum of a quadratic function is equivalent to the y-coordinate (or q-value). However, none of the teachers made this distinction aloud. The teachers’ language when describing alternate representations of roots was rather loose. They switched between the terms “roots” and “zeros” for no apparent reason, and, at times, talked about the zeros of an equation and the roots of the function. All but one of the
relationships listed were mentioned by seven of the nine teachers. Only Nicole and Sophie mentioned the symmetry connection. It appeared that a small group of particular mathematical connections was deeply ingrained in these teachers’ thinking.

In addition, teachers made connections between equivalent representations, that is, different representations expressed in the same mode. They linked different algebraic representations of quadratic functions, particularly \( y = ax^2 + bx + c \) and \( y = a(x-p)^2 + q \), but only Darcy identified finer-grained relationships, namely,

\[
(p,q), \frac{-b}{2a}, \frac{c - b^2}{4a}
\]

because from the original formula, \( y = ax^2 + bx + c \), that’s what the values of \( p \) and \( q \) would be if we complete the square,

\[
x=p \text{ and } -b/2a \text{ which is just the x-coordinate of the vertex from both of the forms of the equation.}
\]

Finally, teachers expressed many equivalent relationships as definitions or identified terms as synonyms, for example,

- a parabola can be made by all the points that are equidistant from a point and a line [Darcy],
- inverse … there’s a reflection over the \( y=x \) line [Wendy],
- isolate the variable and solve, I use them interchangeably [Lily],
- symmetry … where the left hand side will mirror the right hand side [Wendy].

**Implications**

Teachers articulated only a small number of particular implications, though they repeated each one several times.
**Table 4: Teachers' implication connections - quadratics**

<table>
<thead>
<tr>
<th>Implication connection</th>
<th>Teachers said… (examples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The value of $a$ in $y = ax^2 + bx + c$ (or another algebraic version, like $y = a(x-p)^2 + q$) implies compression/expansion and reflection. If the value of $a$ is negative, the standard parabola is reflected over the x-axis.</td>
<td>… coefficient $a$ in that equation tells us whether we have a compression or expansion in the graph [Nicole] … value of $a$ is your expansion or compression factor, and if it’s negative, being reflected because the coefficient is negative [Christine]</td>
</tr>
<tr>
<td>The $p$ and $q$ values in $y = a(x-p)^2 + q$, indicate the amount of translation.</td>
<td>… as far as what the values of $p$ and $q$ do, those are translations; there’s a horizontal translation which is determined by the $p$ value [Darcy] … it’s $x^2 + 2$, so it’s been shifted up two units, so it’s also translation [Nicole] … we use the $ps$ and $qs$, those are the letters we use to translate, so we’re talking about shifting left, right [Lily]</td>
</tr>
<tr>
<td>The value of the discriminant, $(b^2 - 4ac)$, determines the properties of the roots. If $D=0$, then the equation has one real root. If $D&gt;0$, then the equation has two real roots. If $D&lt;0$, then the equation has no real roots.</td>
<td>… properties of roots which we figure out by the discriminant, $b^2 - 4ac$ [Darcy] … a piece of the quadratic formula and it allows you to tell the number of roots [Edward] … you can use the discriminant to find out if they have zeros [Robert] … if the discriminant is greater than zero, there are two real roots, if it’s is equal to zero, there are no real roots [Christine] … if it’s a negative number, or a negative value, you’re not going to get any roots, it’s impossible [Wendy]</td>
</tr>
</tbody>
</table>

Most of the implication connections that teachers mentioned were the latter two in the table, linking $p$ and $q$ values to translations of the standard parabola, and linking the value of the discriminant to properties of roots. Teachers referred to these two relationships repeatedly in their conversations. Perhaps this emphasis is a reflection of the curricular emphases in their textbooks. Their Grade 11 textbook devotes several sections to transformations of quadratic graphs, specifically linking the coefficients $a$, $p$ and $q$ to types of transformations of the graphs (Alexander &

I was surprised to see that the stress on properties of roots did not carry over to making a connection to the graphical version, that is, to continue the chain of implication that “one real root” means that the parabola touches the x-axis at one point, “two real roots” means the parabola intersects the x-axis at two points, and so on.

**Procedure connections**

Teachers made many comments about procedures. Often the topic of procedures came up when teachers were talking about prerequisite skills and student weaknesses. Summarized below are comments about procedures that have a distinct flavour of connections, that is, they are statements that describe a procedure linked to some mathematical idea.
Table 5: Teachers' procedure connections - quadratics

<table>
<thead>
<tr>
<th>Procedure connection</th>
<th>Teachers said... (examples)</th>
</tr>
</thead>
</table>
| Completing the square | … completing the square because that’s how we would get that equation into the standard form, \( y = a(x-p)^2 + q \) [Darcy]  
… completing the square kind of allows you to derive a formula which will allow you to find the vertex [Nicole]  
method of completing the square is used to derive the quadratic formula [Sophie] |
| Factoring | … \( x^2 - (r_1 + r_2)x + r_1 r_2 = 0 \) because that’s basically showing the product-sum idea which is what they use to find the factors [Darcy]  
… you know how to solve the quadratic equation by factoring it to get this kind of equation [Josie]  
… factor to find the zeros of a function [Wendy] |
| FOIL (first, outside, inside, last)  
This mnemonic is used to recall the four products in the multiplication of binomials. | … FOIL is a way of expanding binomials [Sophie]  
… when you’re FOILing out to get something to general form, then that would connect to just your basic operations, distributive property [Christine] |
| Using the quadratic formula | … other ways to find intercepts,… I think in the quadratic formula in terms of … \( x = (-b \pm \sqrt{b^2 - 4ac})/2a \) [Lily]  
… those variables, \( a, b \) and \( c \)… you can plug them into this quadratic formula and solve [Nicole]  
… you can also get the quadratic formula to find the roots for you [Wendy] |
| Arithmetic operations | … nuts and bolts of doing algebra, so associative property, commutative, distributive property, zero property of multiplication [Edward]  
… operations… adding, subtracting, multiplying, dividing,… those are the kind of things we do with variables [Lily]  
… trinomial… maybe simplify, or collect like terms, or tell me the coefficients [Lily]  
… substitution is closely related to coming up with a table of values and trying to find the points that work in a relation [Edward] |

At first glance, I was surprised that all the procedural connections I had identified in the teachers’ conversations were algebraic or numeric methods, and none
were graphing methods. Arguably, a connection like the one between a table of values and graph might be interpreted as a procedure, that is, considering constructing a table of values as the first step in producing a graph. However, in talking about graphs, teachers essentially talked about the meaning of various aspects; in contrast, much of what they had to say about the algebra was focussed on simplifying and solving.

**Part-whole connections**

Teachers made a variety of connections that focussed on the idea of a hierarchy of complexity (Skemp, 1987), including references to one object being part of a more complex or “larger” one, and extensions of an object, usually a procedure, to a wider field. I have described this type of connection in more general terms than the previous ones because it was the category of connection that teachers made the most rarely.
### Table 6: Teachers' part-whole connections - quadratics

<table>
<thead>
<tr>
<th>Part-whole connection</th>
<th>Teachers said... (examples)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inclusions</strong>: an object is a component of a larger idea or set ([a \in \mathbf{B}]).</td>
<td>... connected to the quadratic formula is the discriminant which is the part inside the radical [Christine] ... properties of the graph, like the domain, the range, intercept, co-ordinates [Darcy] ... discriminant is part of the quadratic formula, it's a piece of the quadratic formula [Edward] ... when I say coordinates, what I'm thinking in my mind is that's a relation, that's a small point of a bigger picture [Lily] ... parabolas is just a small part of conics [Wendy]</td>
</tr>
<tr>
<td><strong>Examples</strong>: an object is a specific instance of a more general concept.</td>
<td>... starting to connect some of these things, there were examples of graphs and parabolas that showed first of all the relationships [Christine] ... cone being a geometric shape [Darcy] ... ((x+4)^2) would be a perfect square [Nicole] ... here are some more examples of graphs with different vertices and different compressions [Sophie]</td>
</tr>
<tr>
<td><strong>Extensions/generalizations</strong>: an object is a component of a larger idea or set ([\mathbf{B} \text{ contains } \mathbf{a}]).</td>
<td>... guess and check, really can be applied to any type of equation [Darcy] ... roots, the zeros, again, doesn't necessarily have to be quadratics. Any kind, any kind of equation that you have [Nicole] ... applying the same translation, compression, inverse, this kind of graphical thing, to basically any function, so a cubic, or a square root function [Lily] ... domain, range, coordinates, that could go with just any graph [Darcy] ... the transformations are not obviously just moving parabolas around but you could move any type of graph or any type of geometric movement [Robert] ... the remainder theorem here, so that takes [us] into degrees higher than two [Edward]</td>
</tr>
</tbody>
</table>

I have classified inclusion relationships into two types – a component and an example. A component is at a simpler level of complexity than its parent concept, but still expressed in the abstract. An example is a specific instance, one of many
members of a family without implying that it is less complex. Teachers often referred to examples in their conversations with me.

The reverse perspective, of generalizing rather than specifying, seems closely related to Coxford's notion of connectors. Teachers often made extensions to other mathematical topics, illustrating that a particular concept or, more commonly, procedure, cut across a range of other areas.

*Instruction-oriented connections*

Teachers approached the card task from the perspective of “content for teaching” (Ball & Bass, 2000). Statements like the following were common:

I was really thinking with each card, when do I teach this, how do I teach it, how does it relate to other things I teach [Sophie];

[What] I’ll try to do is organize it according to topic, the way, I guess they would be presented in the text book. [Edward].

I note that teachers took the teaching perspective in spite of requests to consider their own knowledge. They seemed not to differentiate between their personal knowledge of mathematics and their knowledge for teaching; they treated the mathematical knowledge that they taught as all of their knowledge of the topic.

I targeted my questions in this interview very specifically to mathematics rather than teaching, so it is noteworthy that teachers invoked curriculum and instruction-based reasons for identifying cards as connected. In addition to general statements, like those quoted above, teachers made three kinds of instruction-oriented connections – curriculum, vocabulary and prerequisite knowledge, illustrated in the table below.
Instruction-oriented connections, even though they link mathematical ideas, seem to be distinct from the other four categories of mathematical connections.

Connecting terms because they are examples of vocabulary, and connecting topics because they come together in a curriculum sequence, are clearly different from the other four categories in my model; they can only arise in the context of teaching and learning. But I speculate that prerequisites could be interpreted in more mathematical terms. Saying that students who don’t understand concepts like exponent, value, radical can’t understand parabolas, as Robert did, implies the existence of a connection. The mathematical connection is implicit and perhaps too “long” or
indirect to be easily specified. Nevertheless, there is a connection, or chain of connections, that links a prerequisite to new knowledge.

Earlier in this chapter, I discussed, teacher by teacher, cards that they added or left out in their arrangements. Here I re-iterate a significant pattern in the cards that teachers left out. Except for Sophie, who used all the cards, all teachers left out a few cards because they didn’t know or couldn’t remember what they signified. Four teachers, Lily, Josie, Wendy and Robert, left out a large number of cards. Lily left out her cards for one reason – she was unsure of what they were. For Lily, the omitted cards were an indicator of her lack of confidence in her own knowledge.

However, Josie, Wendy and Robert left out cards because they saw the terms on the cards as too far removed from the topic of quadratic functions and equations to consider in terms of connections. They variously referred to them as too broad, too simple or too basic, or as concepts that students should have learned long ago. It seemed that the length of the connections between these ideas and quadratic functions and equations were too great to be useful. Josie’s, Wendy’s and Robert’s lists overlapped. The following terms were mentioned by at least two of them; terms in boldface were mentioned by all three - **associative**, **commutative**, **derive**, directrix, **exponent**, **expression**, focus, formula, geometry, inequality, methods of solution, **operations**, properties of roots, radical, **square**, square root, **substitute**, value, variable. Josie, Wendy and Robert all acknowledged that the omitted ideas all had some, albeit distant, connection to quadratic functions and equations. But it seemed as if they wanted to draw a boundary around “quadratic functions and equations” and keep it as a self-contained topic.

There is a hint in the statements of these three teachers of another tension - between a view of mathematics as a connected web of ideas, strongly expressed by
Josie and Robert, and a need to compartmentalize the topic, perhaps to keep teaching it manageable. Yet, treating fundamental ideas as unconnected to a curriculum topic may, in fact, be a disservice to students who may be struggling to see that many processes are common to many topics.

**Summary of emerging themes in Interview 3**

The following common theme arose in teachers’ descriptions of the relationships on which they based their card organization.

- While teachers articulated many types of mathematical connections in conversation, the specific relationships mentioned were relatively few and the content encompassed quite narrow.

The rest of the themes illustrate how the teaching perspective dominated both how teachers approached the task and the mathematical connections they talked about.

- Even though I had asked the teachers to consider all their knowledge of quadratics, regardless of whether it was included in the high school curriculum or not, they talked almost entirely about aspects of quadratics that are currently in the BC curriculum for Grades 11 and 12. It seemed that they were no longer actively aware of other things they had learned about quadratics earlier in their own studies. This response is consistent with what teachers had reported in the questionnaires at the beginning of the study – that they had forgotten much of the mathematics that they had learned in university and rarely relied on what they learned in university mathematics courses in their own teaching.
• Most of the teachers responded to the sorting task as though they were organizing information for instruction, rather than considering mathematical relationships in their own right. The organizing principle for grouping and sequencing terms was related to how they would be presented to students in lessons.

• For some of them, the organization of terms was constrained not only by the teaching perspective, but also by the textbook's organization of the material [Lily, Edward].

• In addition to mathematical connections, teachers did considerable linking of terms on the basis that the concepts and procedures named were prerequisite knowledge.

• Three teachers, Wendy, Josie and Robert, dismissed a large number of terms as too broad, too simple or too basic to discuss as related to quadratic functions and equations. At the same time, they and other teachers lamented students' apparent lack of understanding of basic concepts and skills. I wonder if teachers are too quick to assume that something is obvious to students.
CHAPTER 7: CONCLUSIONS AND IMPLICATIONS

Conclusions

My work in this study has been akin to exploring on foot, terrain that has been overflown by air and known in its broad features. In this ground-level journey, familiar large elements were recognized, and close inspection exposed details that were previously unknown.

My conversations with the nine participating teachers revealed a group of teachers who were well-prepared in their subject, caring about the welfare of their students and exerting their best efforts in fulfilling their responsibilities. They were staunch supporters of teaching mathematics for understanding. They felt that they did not always teach up to the standard of their values and felt constrained by their curriculum, their textbooks, and the organization of their school and departments. Yet they spoke movingly about their efforts to keep striving to teach for understanding. As a group, they presented a more hopeful picture than the dreary portrait painted by Stigler and Hiebert (1999) of American Grade 8 mathematics teachers, who seemed content to focus on rote learning.

The teachers’ were more pedagogically than mathematically oriented. For these mathematics-specialist teachers, the only mathematics that concerned them was the mathematics that they taught. Their own studies in mathematics went well beyond the high school curriculum. Yet they rarely drew on this knowledge.

Within this landscape, I return to my research questions.
How do secondary mathematics teachers conceptualize mathematical connections?

The findings of this study illuminate a number of facets that were common to all participants in this study. First of all, teachers’ thinking about connections, and apparently their thinking about mathematics in general, was completely bound up with their thinking about teaching. When speaking spontaneously, teachers talked about real-world connections and connections to students’ prior knowledge. In this way, they exemplified to some degree both categories of connections as defined by the NCTM – modelling and mathematical connections. Teachers’ predominant strategy of presenting a sequence of examples to develop a new concept or procedure was intended to make it easy for their students to connect new information to previous knowledge, but only a few teachers explicitly pointed out the connections.

Second, most teachers were enthusiastic in their approval of considering mathematics as an interconnected web of concepts. Yet this stance seemed to be more a general position statement - a statement of a positive disposition toward connections, rather than a reflection of expert knowledge at a specific level. While some teachers saw mathematical connections as integral to the way they taught, others were conflicted, assuming that an emphasis on connections would be time consuming and detract from their responsibilities to “cover the curriculum” and to prepare their students for external assessments.

Finally, in the context of a structured task, teachers were able to demonstrate a knowledge of specific mathematical connections at a fine-grained level, but only with considerable effort. This finding indicated that teachers do have knowledge of mathematical connections as defined in this study, but that knowledge is largely tacit.
What are the characteristics of the explicit mathematical connections that teachers are able to articulate?

As mentioned earlier, teachers typically saw mathematical connections as knowledge that became a strategic element of their teaching. This view was evident in two ways – as knowledge that formed the basis for making instructional decisions, and as a way of thinking that they wanted their students to acquire. Initially, teachers talked about mathematical connections simply in terms of connections to students’ prior knowledge. In further discussions of mathematical topics, they articulated mathematical connections as different representations of the same concept, as implications, as relationships between a part and its whole, and as procedures. In addition, they made instruction-oriented connections that linked mathematical objects on the basis of their role in teaching and learning.

While teachers identified specific mathematical connections in a variety of categories, those connections dealt with a narrow range of content, and favoured connections that were explicitly described in their textbooks. Nevertheless, teachers were also able to identify certain connections as crucial to students’ understanding of a topic.

Contributions

While “connections” has become an ubiquitous term in mathematics education, there has been relatively little research specifically focussed on mathematical connections and even less on teachers’ conceptualizations of mathematical connections. This study broke new ground in two ways – in exploring teachers’ ideas about mathematical connections in the context of thinking about teaching, and in examining explicit connections that teachers articulated.
In answering the research questions about teachers’ mathematical connections, this study made contributions in the following areas – theory, methodology, mathematical content, and pedagogy. I comment on each of these areas in turn.

**Theory**

The teachers participating in the study thought of connections in both mathematical and instructional terms. In Chapter 2, I considered that my research might be situated at the boundary between content knowledge and pedagogical content knowledge. I expected to gain insight into teachers’ own mathematical understandings separate from teaching, and some hints of how those understandings impacted on the way they saw their teaching. In fact, I found that the mathematics that teachers thought about was the school mathematics that they taught. It appears that school mathematics, as a subset of teachers’ content knowledge, is so powerful a factor that it pushes their other mathematical knowledge out of their awareness.

Furthermore, teachers’ mathematical knowledge seems completely intertwined with the teaching of it. Thus, it seems that teachers’ conceptualization of mathematical connections is firmly embedded in pedagogical content knowledge.

I have developed a model for thinking about the kinds of mathematical connections that teachers articulate. The model is based on schema theory (Piaget, Skemp), that is, the notion that a person’s knowledge is organized as a mental structure of related concepts. In Skemp’s (1987) formulation, the schema is hierarchical, but it does not have to be; it could also be a web. My model postulates a set of potential mathematical connections that might be made across many mathematical topics. In a small way, the model elaborates the notion of a schema for a mathematical topic by proposing the characteristics of the links. The model considers
five types of connections between mathematical concepts/ideas; four of them are mathematically-based and the fifth is instruction-oriented, as listed below:

*Different representations.* The same concept is represented in two or more ways. Alternate representations are those in different modes of representation. Equivalent representations are those in the same mode.

*Implications.* One concept leads to another in a logical form, IF..., THEN...

*Part-whole relationships.* One concept is linked to another in some sense of part and whole. Part-whole relationships include examples, inclusions and generalizations.

*Procedures.* An algorithmic procedure is associated with a particular concept.

*Instruction-oriented connections.* Mathematical objects are linked not because of any mathematical association but because they share some pedagogical purpose. Instruction-oriented connections manifested in two main forms. First, teachers made general reference to the importance of linking the new topic to students' prior knowledge. Often, the specific connections to prior knowledge could be described as extensions of what students already knew. Second, groups of mathematics concepts and procedures were linked together as prerequisites – concepts, skills or vocabulary that students should have mastered before embarking on the new topic.

This model has three important characteristics. First, it has been field-tested (at least in a preliminary way). In applying this model to analyse my conversations with teachers, I revised it, adding or collapsing categories to arrive at a model that both captured the range of teachers’ thinking and also was simple to apply. Second, the model appears to be robust, in that it captured the full range of mathematical
connections that teachers were able to articulate, across a span of mathematical topics from number theory, geometry, trigonometry, algebra and calculus. Third, mathematical connections as described in this model have a dimension of grain-size. Teachers sometimes connected two very basic and specific concepts (fine-grain), and sometimes, ideas that were much broader and general in scope (large-grain). The model is applicable to associations at different grain sizes. The same terminology can be used to describe large grain-size connections, like “conics is part of geometry” and very fine-grained connections like “(p,q) is an alternate representation of the vertex of a parabola”. Identifying fine-grained connections was persistently harder for teachers than making large-grained ones.

Methodology

The three-stage interview process that I used in this study afforded opportunities to examine teachers’ thinking about mathematical connections in different ways, from their native naturalistic and spontaneous views, to views that were focussed on mathematical connections through a concentration on particular mathematics content, to an in-depth consideration of a common topic.

There are two significant aspects to the method. First, the three-stage interview process itself allowed me to hone in on my second research question, “What are the characteristics of the explicit mathematical connections that teachers are able to articulate?”, without imposing constraints on teachers’ free expression until after they had exhausted their innate personal descriptions. Moreover, the three interviews, with different foci, and spaced a month apart, provided a check of the consistency of teachers’ statements. It is likely that weakly-held or inauthentic positions would
reveal themselves in contradictory statements over time. The fact that there were no contradictions is an indicator that teachers' responses were genuine.

My second contribution was the development of a structured task that forced teachers to examine a particular mathematical topic at a fine-level of detail. The task is described in detail in Chapter 3, and materials used are shown in Appendix B.

Teachers had difficulty in talking about mathematical connections in specific terms. An essential purpose of the sorting task was to break through the generalities, and give teachers a vocabulary at a fine-grain size with which to talk about mathematical connections. I gave teachers a group of 82 cards that contained a wide range of terms, algebraic formulae, and graphs, that included both terms used in the current textbook, and other sources. I then asked them to arrange the cards in ways that showed how they were related. Sorting tasks themselves are not new; the application to mathematical connections is. This one was distinct in that the analysis was based not on what teachers put together (the "mapping") but on what they said about their reasons.

A card task like this can be adapted for use in further studies about mathematical connections in a variety of ways. One example is to have teachers work on a task similar to the one in this study in groups that are required to reach consensus. The discussion among teachers would provide additional insights into their thinking, and the final product may indicate specific connections that teachers regard as particularly important.

Content

This study has made a beginning in identifying particular mathematical connections that teachers recognize and deem important for several topics. For one of
those topics, quadratic functions and equations, an aggregate list of connections was compiled with contributions from all the teachers. It clearly is not a picture of all the mathematical connections that could be made in relation to this topic. But it is a starting point for mathematics educators to consider, from a mathematical perspective, what are the important and productive relationships of which learners should be made explicitly aware.

While teachers were able to articulate a variety of types of mathematical connections, the actual connections were relatively sparse. Chapter 6 gives the details of mathematical connections that teachers identified when working with quadratic functions and equations. There were two aspects of the topic that were emphasized in teachers’ mathematical connections:

- Characteristics of the quadratic equation are linked to characteristics of the graph, particularly roots/zeros, vertex, maximum/minimum and transformations;
- Using the quadratic formula and factoring are procedures to solve quadratic equations. The discriminant in the quadratic formula provides information about the properties of the roots.

Conversations about other topics yielded fragments that seemed central to teachers’ consideration of mathematical connections:

- Probability: Bayes’ Law is interpreted in terms of union and intersection of sets, and outcomes. Combinatorics formulae, tree diagrams and Venn diagrams are used to identify possible outcomes and to calculate probabilities;
• Geometry/trigonometry: Surface area of 3-D shapes is modelled as a decomposition into 2-D shapes. Trigonometric ratios (SOHCAHTOA) are treated as an instance of properties of similar triangles;
• Functions: Functions can be represented algebraically and graphically;
• Integers: Physical and graphical (number line) models are used to show addition and subtraction, but multiplication and division are represented symbolically;
• Calculus: Reasoning is used to build up properties of functions from the properties of their integrals and derivatives.

The most striking feature of these content summaries is how skimpy they are. I think this paucity is an illustration of how difficult teachers find it to articulate mathematical connections. By extension, it demonstrates the need for a concerted expert effort to describe the important mathematical connections related to a topic, rather than leaving it to individual teachers. That such an approach would be effective is indicated by the finding that the particular mathematical connections that teachers spoke of the most, are precisely those that are explicitly demonstrated in their textbooks.

**Pedagogy**

This study does not make any immediate recommendations for practice; such recommendations would be premature. What it does is lay a foundation – identifying some factors that should be considered in any program to help teachers promote mathematical connections.

I discussed situating “mathematical connections” as a component of pedagogical content knowledge in Chapter 2. This study demonstrated that teachers
do think of mathematical connections in both mathematical and instructional terms. In future research, which moves to a consideration of teaching practice, rather than teachers’ own understanding, situating “mathematical connections” firmly as a component of pedagogical content knowledge is likely to prove fruitful.

I did not set out to study teachers’ pedagogical content knowledge per se. The findings to which I refer in this respect emerged from discussions about mathematical content, so I am hesitant to draw conclusions about the broader issues. Nevertheless, this study pointed to some apparent weaknesses in teachers’ pedagogical content knowledge. The teachers in this study seemed to have little knowledge of what made topics easy or difficult for students, or what constituted obstacles to their students’ learning. They judged the effectiveness of their teaching largely by test results and could not name the features of their teaching that made it successful. Findings like these raise the possibility that these more global issues may need to be addressed before work on mathematical connections can become more productive.

Teachers attend to the real, not the rhetorical, expectations of the educational system. Thus, they target their efforts to teaching students the mathematics that will help the students achieve high scores on external examinations. Even though curriculum documents pay lip-service to the importance of mathematical processes like making connections, ILOs and examination questions rarely do. Rather than exhortations to teachers, a more likely route to success is to emphasize in assessment what is valued in the rhetoric.

Furthermore, teachers depend on their textbooks. Textbooks that attended to overtly identifying specific mathematical connections would go a long way to giving teachers the tools with which to work.
This study's finding that explicitly identifying mathematical connections is hard for teachers points to the necessity of substantial input from teachers and mathematics educators to identify the most important and productive mathematical connections that should be part of a teacher’s pedagogical content knowledge.

Finally, the finding that teachers sometimes dismiss certain connections as too simple and therefore, do not emphasize them with students, should be considered as a possible obstacle for students who do not see such connections on their own.

**Limitations**

There was a high level of trust between the participating teachers and myself. I am confident that they spoke freely and honestly about their experiences, sometimes choosing mathematical topics to discuss that they knew they were struggling with and sometimes being quite critical of their own practice. However, I was a single researcher in these conversations, and examined them through a single lens. I listened and understood from the perspective of an experienced and reform-minded mathematics teacher myself. But a single lens has its drawbacks. There was no triangulation—I used no formal process of checking back with teachers after I had done the analysis, nor did I involve others in coding the interview data. So, I cannot point to evidence of reliability and validity. The question remains whether others would see what I saw. Did biases of which I was unaware creep into my interpretations because I knew these teachers so well? Would others who were not their professional colleagues make different judgments?

The participating teachers were a self-selected group who were interested enough in exploring their own thinking to devote 3-4 hours each to our conversations. They represented new to mid-career teachers who were exposed to the NCTM
Standards. I was not able to get any participants who were in the late stage of their career – those who started teaching before the publication of the Standards. It would be interesting to see if their views and abilities to articulate connections are any different. The lack of very experienced teachers in the study has consequences for generalizability. And perhaps, in a study like mine, where the goal was to identify emergent themes, there are themes that would have arisen with senior teachers that did not arise in this group. For example, I wonder if the most experienced teachers might have articulated greater insight into students' learning difficulties and teachers' effectiveness, or if they might have shown less knowledge and interest in the NCTM Standards.

The description of teachers' practice is based on their self-reports. I did not do any classroom observations to see if what they did matched what they said they did. Hence, the findings of this study must be viewed as findings about teachers' personal perceptions.

This study must be seen as a first step in a multi-step process of understanding how teachers view mathematical connections, what they do in their teaching to promote making connections, and ultimately, how mathematics educators can help them to do so.

**Future Research**

This study breaks new ground in its focus on teachers' explicit thinking about mathematical connections in the teaching context. Further studies are needed to confirm its findings. Thus, one future research direction is to maintain the focus on teachers' own mathematical connections, but work with larger and more varied groups of teachers, and extend it to more mathematical topics.
Another, and crucial, research direction is to explore the issues raised in this study in the context of teacher's actual practice. I see two logical next steps – to look into teachers' planning, and to investigate their work in classrooms. Classroom observation is needed to determine what references are made to mathematical connections and how they are made, by teachers and students. The classification model developed here could be used as a tool to analyze classroom discourse with respect to connections and the model could perhaps be further revised and strengthened.

A different line of research, combining mathematics and curriculum, is to examine the high school curriculum from the point of view of identifying connections that are worth emphasizing. Many reform-based curricula exist, some specifically written to afford opportunities for making connections, and could perhaps become the starting point for this work.

Finally, this study's finding that teachers do not put much emphasis on mathematical connections in their practice, points to the need for investigating what is the added value to teaching and learning of emphasizing connections, and ultimately, to find professional development models that would enhance this aspect of teachers' practice.

Closing thoughts

I embarked on this degree program after a long and successful career in public education. I started my research with some questions about connections that germinated slowly while I worked in public education, and came to the forefront about five years ago. It seemed that everyone in the academic community of mathematics educators was talking about connections – talking about connections as

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though the meaning was obvious. I had flashbacks to a first-year mathematics course, where the professor always responded the same way when we, his students, got lost and could not follow his reasoning. “It’s intuitively obvious”, he would say. But what people meant when they talked about connections was not obvious to me. And so, a research topic was born.

My success as a researcher strongly depended on the cooperation of the teachers participating in my study. The relationship between the teachers (“the subjects”) and myself (“the researcher”) was something that occupied my thoughts as much as my research plan. I felt an enormous ethical responsibility to the teachers, to respect their time limitations and their privacy, to get below the surface, to really understand their positions. As I coded my data and drew inferences, bouts of soul-searching slowed me down – was I being fair? was I being superficial? was I portraying them even-handedly? Of course, questions of honesty, fairness, authenticity, are questions that every teacher faces every day. But somehow, knowing that my portrait of these teachers and their thinking would become public, that they could become the subjects of discussion and possibly judgment by people who didn’t know them as well as I, made those questions weigh more heavily. My “data” were people – complex, sometimes unpredictable, always worth listening to. And I cared about them well beyond any principles of scientific honesty. That in turn, reminded of Nel Noddings and her work on caring as it applied to educational research (Noddings, 1986). I was reminded of the centrality of the duty of care to any activity in education.

My study produced some interesting findings, ones that can be followed up in further research. And now that I have some sense of the terrain, I want to have conversations like these again, not simply dialogues this time, but discussions with
groups of teachers as we try to describe what mathematical connections are important for learners of mathematics and why.
REFERENCE LIST


Appendix A: Background Questionnaire

The purpose of this questionnaire is to collect some information about your mathematics and teaching background and to identify some potential topics of school mathematics to pursue in further work.

In writing up the research, I will refer to you (when necessary) using a first-name pseudonym. If you would like to choose your own, please do so below.

Name: ___________________________ Preferred pseudonym: ___________________________ (please print) (optional)

Date completed: ___________________________

<table>
<thead>
<tr>
<th></th>
<th>Question</th>
<th>Answer</th>
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<tbody>
<tr>
<td>1</td>
<td>In what year did you start teaching?</td>
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<td>2</td>
<td>If teaching is not your first career, what did you do before teaching?</td>
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<td>3</td>
<td>In how many schools have you taught? Please count only contract assignments, not TOC.</td>
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<td>4</td>
<td>Please list the math courses that you are teaching this year and the number of classes of each.</td>
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<td>5</td>
<td>Please list any other math courses that you have taught in the past but are not teaching now.</td>
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<td>6</td>
<td>Please list any particular math courses that you haven't taught yet but want to teach in the future.</td>
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<td>7</td>
<td>What is your highest academic credential in mathematics (e.g. math minor, masters in math, etc.)?</td>
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<td>8</td>
<td>How well do you think your post-secondary math courses prepared you for teaching high school math?</td>
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<td>9</td>
<td>To what math-related professional organizations do you belong (e.g. BCAMT)?</td>
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<td>10</td>
<td>What conferences or other math-related professional development events do you attend?</td>
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</table>
Please complete the table below about post-secondary math courses that you have taken.

**Topic**: The math topics listed below come from the names of undergraduate mathematics courses and represent the wide range of courses offered.

**# courses**: How many courses did you take with a similar title?
- If you have never taken a course in this area, leave the space blank.
- For topics that you have studied, please indicate how many courses you’ve taken in it. For example, if you took a course called “Ordinary Differential Equations” and another called “Partial Differential Equations”, you’ve taken 2 courses in Differential Equations.
- If you think you might have studied a topic, but can’t remember for sure, put a ? in the space.

**Relevance**: On the whole, is the content that you learned in this course relevant or helpful to you in teaching high school mathematics? Answer Y or N for the courses that you have taken.

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<tr>
<th>Topic</th>
<th># courses</th>
<th>Relevance</th>
<th>Topic</th>
<th># courses</th>
<th>Relevance</th>
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<td>Boundary Value Problems</td>
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<td>Linear Programming</td>
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<td>Calculus</td>
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<td>Mathematical Logic</td>
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<td>Coding Theory/ Cryptography</td>
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<td>Mathematical Modelling</td>
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<td>Combinatorial theory/ Combinatorics</td>
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<td>Mathematical Proof</td>
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<td>Complex Analysis</td>
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<td>Mathematics for Computing</td>
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<td>Complex Variables</td>
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<td>Mathematics for Teaching</td>
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<td>Computer Algebra</td>
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<td>Matrix Algebra</td>
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<td>Fluid Dynamics</td>
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<td>Graph theory</td>
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<td>Groups and Fields</td>
<td></td>
<td></td>
<td>Stochastic Processes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Groups and Symmetry</td>
<td></td>
<td></td>
<td>Topology</td>
<td></td>
<td></td>
</tr>
<tr>
<td>History of Mathematics</td>
<td></td>
<td></td>
<td>Vectors</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the blanks above, please list any topics you have studied that are not covered in the printed list.
Please rate your knowledge of each of the following sets of documents by marking X on each line.

- **BC IRPs for the courses you teach.**
  - no _X_ ___________ expert
  - knowledge ___________ knowledge

- **BC IRPs for other math courses**
  - no _X_ ___________ expert
  - knowledge ___________ knowledge

- **BC Graduation Program**
  - no _X_ ___________ expert
  - knowledge ___________ knowledge

- **Western and Northern Canadian Protocol (WNCP) for Mathematics**
  - no _X_ ___________ expert
  - knowledge ___________ knowledge

- **NCTM Principles and Standards**
  - no _X_ ___________ expert
  - knowledge ___________ knowledge

Please add any other information that you would like me to know about your mathematics background or teaching experience. (optional)

Please complete the table below for each math course that you teach this year.

**Course Name:**

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A course topic that students generally find easy to master</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A course topic that students generally find very difficult</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A course topic that you feel confident that you teach very effectively</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A course topic that you feel you don't teach effectively enough</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Teachers were given as many copies of this table as they needed.
Appendix B: List of Cards used in the Task-based Interview

**TERMS**

algebra
associative property
coefficient
commutative property
complete the square
compression
cone
co-ordinates
curve
derive
directrix
discriminant
distributive property
domain
equation
expansion
exponent
expression
factor
factor theorem
focus
FOIL
formula
function
geometry
graph
guess and check
inequality
intercept
inverse
isolate the variable
maximum
methods of solution
minimum
operations
parabola
perfect square point

**FORMULAE**

- properties of roots: \((p, q)\)
- quadratic: \(b^2 - 4ac\)
- quadratic formula: \(p \pm \sqrt{-q/a}\)
- radical: \((-b/2a, (c - b^2/4a))\)
- range: \(x = p\)
- remainder: \(x = -b/2a\)
- theorem: \(x = (-b \pm \sqrt{b^2 - 4ac})/2a\)
- root: \(x = 2c/(-b \pm \sqrt{b^2 - 4ac})\)
- simplify: \(ax^2 + bx + c = 0\)
- solve: \(x^2 - (r_1 + r_2)x + r_1\)
- square root: \(r_2 = 0\)
- substitute: \(a(x-p)^2 + q = 0\)
- symmetry: \(y = ax^2 + bx + c\)
- vertex: \(y = a(x-p)^2 + q\)

**GRAPHS**

- GR: standard parabola
- table of values: example
- GR: 3 example parabolas
- GR: horizontal translation
- GR: vertical translation
- GR: compression examples
- GR: inverse

**FORMULAE**

- a, b, c
### Appendix C: Summary of Quadratic Functions and Equations Task

14 formulae, 8 graphs, 60 terms

<table>
<thead>
<tr>
<th>Sophie purple</th>
<th>Started with cards...</th>
<th>Left out cards...</th>
<th>Added cards...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = ax^2 + bx + c$</td>
<td>0</td>
<td>$y = x^2, a=1, b,c=0$</td>
<td></td>
</tr>
<tr>
<td>$y = x^2, a=1, b,c=0$</td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wendy yellow</th>
<th>Co-ordinates, graph, parabola</th>
<th>27</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = p$, $x = -b/2a$, $x = 2c/(-b \pm \sqrt{b^2 - 4ac})$</td>
<td>graph showing focus, directrix, algebra, associative, coefficient, commutative, derive, directrix, exponent, expression, focus, geometry, guess and check, methods of solution, operations, simplify, square, square root, substitute, translation, value, variable</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nicole mauve</th>
<th>Algebra, a,b,c, variable</th>
<th>6</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \pm \sqrt{(-q/a)}$, $(-b/2a, c - b^2/4a)$, $x = p$, $x = -b/2a$, $x = 2c/(-b \pm \sqrt{b^2 - 4ac})$</td>
<td>associative property</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$
<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Right cards...</th>
<th>Added cards...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Josie gray</td>
<td>Symmetry, discriminant</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>associative, commutative, co-ordinates, derive, exponent, expression, focus, formula, geometry, inequality, methods of solution, operations, properties of roots, radical, square, substitute, value, variable</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p \pm \sqrt{-q/a}$, $x = 2c / (-b \pm \sqrt{b^2 - 4ac})$</td>
<td></td>
</tr>
<tr>
<td>Lily Pink</td>
<td>Equation, table of values, TOV chart</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>graph showing focus, directrix, graph showing 3 sample parabolas</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>associative, commutative, directrix, discriminant, factor theorem, focus, remainder theorem, zero property</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b^2 - 4ac$, $p \pm \sqrt{-q/a}$, $(-b/2a, c - b^2 / 4a)$, $x = p$, $x = -b/2a$, $x = 2c / (-b \pm \sqrt{b^2 - 4ac})$, $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$</td>
<td></td>
</tr>
<tr>
<td>Christine Hot pink</td>
<td>algebra, geometry</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>standard form, vertex, reflection</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = p$, $(-b/2a, c - b^2 / 4a)$, $x = 2c / (-b \pm \sqrt{b^2 - 4ac})$</td>
<td></td>
</tr>
<tr>
<td>Darcy chartreuse</td>
<td>Methods of solution, graph</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>associative, commutative, exponent, operations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>complete the square $q = c - b^2 / 4a$</td>
<td></td>
</tr>
<tr>
<td>Started with cards...</td>
<td>Left out cards...</td>
<td>Added cards...</td>
<td></td>
</tr>
<tr>
<td>-----------------------</td>
<td>-------------------</td>
<td>---------------</td>
<td></td>
</tr>
<tr>
<td>Robert Pale green</td>
<td>algebra, graph</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td></td>
<td>associative, commutative, derive, directrix, distributive property, exponent, expression, formula, inequality, operations, perfect square, properties of roots, radical, square, square root, substitute, value, variable</td>
<td>Edward Roots, zeros of a function</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p \pm \sqrt{-q/a}$, $(-b/2a, c - b^2/4a)$, $x = -b/2a$, $x = 2c/(-b \pm \sqrt{b^2 - 4ac})$, $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Edward Bright blue

Roots, zeros of a function

5 graph showing 2 vertical parabolas cone, directrix $\pm \sqrt{-q/a}$, $x = 2c/(-b \pm \sqrt{b^2 - 4ac})$
Appendix D: Coding Scheme for Interview 3

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIFF REP</td>
<td>different representations of the same idea</td>
</tr>
<tr>
<td>ALT</td>
<td>alternate representation</td>
</tr>
<tr>
<td>ALG/GR</td>
<td>algebraic/graphic generally</td>
</tr>
<tr>
<td>PQ/VER</td>
<td>(p,q)/vertex</td>
</tr>
<tr>
<td>ROOT/INT</td>
<td>root/intercept</td>
</tr>
<tr>
<td>TOV/GR</td>
<td>table of values/graph</td>
</tr>
<tr>
<td>VER/MM</td>
<td>vertex/minimum or maximum</td>
</tr>
<tr>
<td>ZERO/RT</td>
<td>zero/root</td>
</tr>
<tr>
<td>EQUIV</td>
<td>equivalent representation</td>
</tr>
<tr>
<td>ALG</td>
<td>algebraic</td>
</tr>
<tr>
<td>DEF</td>
<td>definition</td>
</tr>
<tr>
<td>SYM</td>
<td>synonym</td>
</tr>
<tr>
<td>IMP</td>
<td>implication</td>
</tr>
<tr>
<td>A COMP</td>
<td>‘a’ → compression</td>
</tr>
<tr>
<td>D RT PROP</td>
<td>discriminant → root properties</td>
</tr>
<tr>
<td>DIR/MM</td>
<td>direction parabola opens → minimum or maximum</td>
</tr>
<tr>
<td>EXP2/QUAD</td>
<td>exponent=2 → quadratic</td>
</tr>
<tr>
<td>PQ/TRANS</td>
<td>values of p, q → translation</td>
</tr>
<tr>
<td>PT/WH</td>
<td>part/whole</td>
</tr>
<tr>
<td>XAMP</td>
<td>example</td>
</tr>
<tr>
<td>XTEN</td>
<td>extension/generalization</td>
</tr>
<tr>
<td>PART</td>
<td>part/component</td>
</tr>
<tr>
<td>PROC</td>
<td>procedure</td>
</tr>
<tr>
<td>COMP SQ</td>
<td>complete the square</td>
</tr>
<tr>
<td>FACTOR</td>
<td>factor</td>
</tr>
<tr>
<td>FOIL</td>
<td>FOIL (first, outside, inside, last)</td>
</tr>
<tr>
<td>FORM</td>
<td>use formula</td>
</tr>
<tr>
<td>OPS</td>
<td>use operations</td>
</tr>
<tr>
<td>SOLV</td>
<td>solve</td>
</tr>
<tr>
<td>SUB</td>
<td>substitute</td>
</tr>
<tr>
<td>INSTR</td>
<td>instruction-oriented</td>
</tr>
<tr>
<td>PREREQ</td>
<td>prerequisite</td>
</tr>
</tbody>
</table>

The coding scheme for Interview 2 was a simplified version of the above, using the first and sometimes second level of codes only.
The interpretation of Interview 1 was more holistic, identifying statements as about connections or not.
Appendix E: Photos of Teachers’ Organizations of Quadratic Functions and Equations Cards

Sophie