QUANTUM LOCALIZATION IN MOMENTUM-SPACE
FOR TWO COUPLED KICKED ROTORS

by

Borzumehr Toloui Semnani
B.Sc., Sharif University of Technology, Tehran, Iran, 2000

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APPROVAL

Name: Borzumehr Tolouii Semnani
Degree: Master Of Science
Title of thesis: Quantum Localization in Momentum-Space for Two Coupled Kicked Rotors

Examining Committee: Dr. Karen Kavanagh
Professor, Department of Physics
Chair

Dr. Leslie Ballentine, Senior Supervisor
Professor, Department of Physics

Dr. Howard Trottier, Supervisor
Professor, Department of Physics

Dr. Andrew DeBenedictis, Supervisor
Senior Lecturer, Department of Physics

Dr. Malcolm Kennett, Internal Examiner
Assistant Professor, Department of Physics

Date Approved: 24 July 2007
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Abstract

A system of two periodically kicked coupled rotors is studied, to resolve an apparent contradiction regarding localization of states between previously published results, where different models and criteria were used. The system is mapped to an Anderson two-dimensional lattice. The quantum model is evolved numerically for a range of kicking parameters. To check for localization in momentum, the shape of the distribution and the state widths temporal growth are examined for each parameter set. Corresponding classical simulations are also performed as reference, to determine how their distribution widths grow with time, and whether they fall in regular or chaotic regions of classical phase space. The results demonstrate localized and apparently non-localized regimes. However the latter are shown to be consistent with the scaling theory of localization, which gives estimates for their localization lengths and times that far exceed the state size and time span of the simulations performed.
To my dear family
my parents and my sister.
To Azarjun, Mahgooneh, Mahin and Mandana
and to Milo!
“If we knew what it was we were doing, it would not be called research, would it?”

— Albert Einstein
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Chapter 1

Introduction

Studying the quantum evolution of systems whose classical counterparts are non-integrable has proven to be a very fruitful field. This field has come to be known by the somewhat paradoxical name of “quantum chaos”. The paradox in this name holds the key to the interest it has provoked. In classical chaotic systems, the quantum corrections, no matter how small, are amplified after a relatively short time span. This result is manifestly different from what takes place in the quantum regime. Quantum interference effects that arise, suppress classical chaos, and this is what creates the paradox in the name itself.

This property has made such systems ideal topics of investigation in the foundations of quantum mechanics. That’s because, the different classical and quantum behavior is directly relevant to the question of what a quantum state means. The answer to this question is what separates various so-called interpretations of quantum mechanics. In particular, it is relevant to the problem of the emergence of the classical world from a reality that is governed by the laws of quantum mechanics.

That is not, however, the sole source of interest in the study of such systems and their dynamics. The study of quantum chaos forms a solid basis for random-matrix theory, and through it, quantum chaos has come to play an increasingly important role in the new field of quantum computation. There are also a wide range of theoretical and experimental applications that have connections to this field of research. Plasma physics, solid state physics, electron transport theory, optics and the interaction of microwaves and matter are but a few examples [28] [42] [24].
An important class of systems within this theory are those in which the time evolution can be defined through discrete recurrence relations. These relate the coordinates and momenta of the system at each time step to those of the previous times. In particular the Chirikov map, or the so called standard map, has been the focus of many studies. One or more planar rotors that are kicked at regular time intervals, with kicking strengths that depend on their positions, constitute a typical system whose Hamiltonian is the generator of such a map.

In the quantum regime, and under certain conditions, this group of models can be mapped to a model of electrons moving in a (pseudo) random lattice. The momenta of the kicked rotor(s) then correspond to the position coordinates on the lattice. By a random lattice we mean a lattice with random potentials at each site. There is also a hopping potential from one site to others which is usually short range. It has been known for a long time that in truly random lattices electron wave-packets can manifest exponential localization around lattice sites. This phenomenon is known as Anderson localization [3]. In particular it is well known that in one dimension the eigenfunctions of a single electron in the lattice are always exponentially localized, and that in the absence of thermal effects, electron diffusion vanishes. In the lattice model counterparts of kicked rotors, the dimensionality of the lattice is determined by the number of degrees of freedom of the kicked rotor model, for instance, its number of independent momenta.

The question then arises whether the same phenomenon of localization can take place in the kicked rotor model, causing the state to be localized over its range of momenta. The answer is affirmative. This is rigorously shown for a certain class of single rotor models. The next interesting question would be whether the same is true for models of higher number of rotors. We know that in case of electrons in a three dimensional lattice, both localized and extended wave functions are allowed. It is noteworthy to observe that in two dimensions, the longitudinal direction has the same dimensionality as system's cross section. This means the conductance remains constant while system size is changed. A consequence of this is that in two dimensional lattices, the system is non-ohmic at all length scales. This is shown by arguments based on scaling theory. Hence, all states are ultimately localized. This leads to non-metallic resistance at temperatures near zero. For weak disorder and in shorter time spans, before the system is fully localized, or for non-zero temperatures, one could have a weakly localized system.

For these reasons, the study of a system of two kicked rotors is of particular interest. Do
two rotors also make a critical number? Are two rotor models the threshold that separates chaotic versus localized final states? Or would it still be true, that all systems are ultimately localized in angular momentum space?

At first glance one might expect the answer to last question to be affirmative. Indeed a natural extension of the one dimensional model to the case of two rotors has been studied, and its numerical results do confirm this expectation [16]. However the situation is not that straight forward, and there are other studies that seem, on the face of it, to contradict the above results [2]. These studies do not tackle the question of localization directly. Instead they were done in the context of quantum-classical correspondence. Evidence was presented to back the conjecture, that the existence of multiple rotors and their coupling can enable the system to retrieve diffusive growth of momentum state width when the kicking strengths are strong enough.

The situation is made more complicated by the fact that the two groups have chosen different forms for their kicking potentials. In [16], the kicking potential was chosen to be:

\[ k \cos \theta_1 \cos \theta_2 \]

While in [2] the potential had this form:

\[ k_1 \cos \theta_1 + k_2 \cos \theta_2 + \epsilon \cos(\theta_1 - \theta_2) \]

Moreover the kicking strengths used in the two reports were not chosen within the same range.

Finally, as was hinted earlier, the criteria used to determine the existence of localization, or lack thereof, are not the same. The first paper[16] looked for exponential fall in the momentum probability distributions of each rotor, while the second paper focused on how the root mean squared size of the wave function in momentum representation, grows with time, and marked the diffusive growth as the criteria for chaotic behavior and lack of localization.

There are different questions that could be asked, and different possible explanations for the apparent discrepancy can be proposed.

On the one hand it is reasonable to assume that the two systems should behave similarly. In the case of Anderson lattices the decisive factor is the dimensionality and symmetry class. In two dimensions all systems are localized, and the behavior does not depend as much on

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1Both are presented in suitable dimensionless units.
the particularities of the potentials, as long as there is disorder present. In other words, the particularities of the disorder is not expected to be a decisive factor. Although the two kicked rotor models have different kicking potentials that translate to different site energies in their lattice counterparts, they are of similar periodic trigonometric forms.

On the other hand the difference could be important in the kicked rotor model because the mapping leads only to correlations in the pseudo-random site potentials, not to truly independent ones. In the one dimensional case it was shown that this, nevertheless, suffices to produce localization. Could it be that the difference in kicking potential changes this situation from one model to the other? As was mentioned earlier, the strengths of the kicking parameters were not in the same range either. Could it be that higher kicking strengths can overcome the effects of localization in these models?

There is also the possibility of existence of hidden symmetries within one model that makes it a special case. In particular, the periodic boundary conditions for the lattice, determined by the kicking periods, could be playing a role.

And then there is the possibility, that the two different criteria used are not actually equivalent to each other. That is to say, it is possible, at least in principle, that a system can pass the exponential localization test according to one criterion, and yet contain the diffusive behavior in time of the second. The reason is that the parameter of importance in the second criterion is sensitive to local aspects of the distribution function. Such local aspects can be irrelevant in the overall exponential form. In this case there would be no common basis for direct comparison, and the conflicting results need not be tantamount to a contradiction.

Finally, it could be that the diffusive model has not been studied long enough, and that in longer times the system would diverge from the diffusive behavior and tend toward localization. This could be the equivalent of weak localization in the 2-dimensional lattice.

In this thesis, a more general model is studied which contains the two already studied systems as marginal cases. A general model is devised so that one can go from one model to the other by changing the relevant parameters. Here is the form of the general potential used:

\[ \lambda_1 \cos \theta_1 + \lambda_2 \cos \theta_2 + \lambda_3 \cos \theta_1 \cos \theta_2 + \lambda_4 \cos(\theta_1 - \theta_2) \]

This is of course not the most general model one can devise in two dimensions. It is, however, the simplest generalization of the two models discussed earlier. Furthermore, the form of the general periodic potential can always be written in terms of sine and cosine factors. In
that sense, our choice of model seems to entail all the features relevant to our discussion in this thesis. A wide range of parameters are applied to cover as much region as possible, while bearing in mind that too big a parameter set might result scales of change that are beyond the computational capacities and reasonable simulation running times available for such a study. The model is then numerically evolved for each set of parameters with two separate programs. Once using the equations of quantum evolution, and once using those of classical mechanics.

One reason for performing the classical evolution is to check whether the range of parameters are chosen such that the classical evolution is chaotic. That is to say, to determine whether possible localized behavior in the quantum regime are indeed due to the quantum nature of the evolution, and not, for instance, due to choice of parameters that would result in a regular and localized motion in the classical phase space as well.

The other reason is to compare the classical diffusion rates with those within the quantum regime, so long as and in so far as it does exhibit diffusive behavior. In particular, in case of actual diffusive evolution in the quantum case, to determine whether the diffusion rates are the same as their classical counterparts, and how the two compare with respect to each other. It is also instructive to be able to compare the quantum-classical differences for different sets of parameters, and to see how it relates to the size of the parameter.

The results are finally analyzed using both criteria mentioned above, in order to help further illuminate the situation and move towards a better understanding of the reasons for the apparently anomalous behaviors.
Chapter 2

Theoretical Background

2.1 Theory Of Chaos

2.1.1 Classical Chaos

It took a long time to realize that classical evolution of hamiltonian systems can exhibit two very different kinds of behavior.

The first, known and studied for a long time, is regular motion, in which the trajectories starting from nearby initial points in phase space stay more or less close to each other. Their separation grows as a power of time. Even though the experimental values are never exact, and are only determined within a range, this type of motion remains predictable in ordinary time spans.

There is, however, another type of motion possible in classical mechanics, in which trajectories are extremely sensitive to initial conditions, and the separation grows exponentially with time. Thus any small error is magnified by the dynamics of the system in relatively short time, resulting in completely different motion. Because of the inevitable inaccuracy in all initial data, this means that after a period of time the range of error grows so big that deterministic predictions of where the actual trajectory would be, become impossible. This motion has therefore been dubbed chaotic.

To make the above statements more quantitative, it is convenient to use the notion of Lyapunov exponents to characterize the rate of divergence of trajectories from one another along specific directions. To do so let’s take the case of a simple mapping $x(n+1) = F(x(n))$ where $x$ denotes a point in phase space, and $n$ counts the time steps. We take $d(n) = x_2 - x_1$.
to be the distance between two points, belonging to two nearby trajectories. When \( |d| \) is small the linearized equation of motion would be

\[
d_i(n) \approx M_{ij}d_j(n - 1)
\]

(2.1)

\[
M_{ij} = \left. \frac{\partial F_i}{\partial x_j} \right|_{x_1(t)}
\]

(2.2)

The largest Lyapunov exponent is

\[
\lambda_L = \lim_{t \to \infty} \lim_{|d(0)| \to 0} \frac{1}{t} \ln \left( \frac{|d(t)|}{|d(0)|} \right)
\]

(2.3)

This is a mathematical definition, and is exact in the limit \(|d(0)| \to 0\). In actual numerical calculations, performing the limiting process is not possible. Instead we must take \(|d(0)| > 0\) to be constant, for which case the equation is valid only as an approximation. The linearization then holds only for small \(|d(t)|\), and once it gets large the approximation (2.1) breaks down. As a result one has to renormalize \(|d(t)|\) periodically. In other words, the system is allowed to evolve only for a time \(T\). \(T\) must be small enough so that by the end of this time the two trajectories have remained relatively close. That is to say, the linearization of the dynamics of \(d(t)\) remains valid. Then the separation vector \(d(t)\) is brought back to its initial length, while its new direction is not altered. The procedure is then many times repeated for periods of \(T\). The Lyapunov exponent is then approximated, after \(N\) time steps, by the following sum:

\[
\lambda_L = \frac{1}{NT} \sum_{n=1}^{N} \ln \left( \frac{|d(nT)|}{|d(0)|} \right)
\]

(2.4)

Since Hamiltonian systems are area preserving, expansion in one direction must correspond to contraction in another, so the sum of Lyapunov exponents is always zero. Here the term “area” is being used in general to denote the hyper-volume of a subset of phase space. This feature of chaotic systems means that any small compact region of phase space, serving as an initial state, is expanded and contracted (while preserving its area). This leads to an alternative method of analyzing classical chaotic systems, by defining a probability density for an ensemble of trajectories and studying its evolution. This distribution evolves as

\[
\frac{\partial \rho(p,q)}{\partial t} = \{H, \rho\}
\]

(2.5)

\[\{H, \rho\} := \frac{\partial H}{\partial p} \frac{\partial \rho}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial \rho}{\partial p}\]
This is the Liouville equation. The overlap of two distributions is defined as

\[ A = \int \int \rho_1 \rho_2 dp dq \quad (2.6) \]

This quantity is preserved under the Liouville equations. Thus, unlike the case for trajectories, nearby distributions do not separate from one another. Since the area of the distribution remains fixed, it exhibits fine structure in phase space with time. Thus the Liouville representation proves to be a specially useful mode of analysis of classical chaos. Therefore this is the method I will be using in this work regarding classical evolutions.

### 2.1.2 Chaos In Quantum Systems

In order to have a point of reference for directly comparing the quantum and classical behavior, we need first to define a measure of separation or convergence in the quantum realm. A natural choice would be to use the Hilbert space inner product in the mathematical formulation of quantum theory. Pure states correspond to vectors in Hilbert space. Using the overlap of two states as a possible definition of their separation, however, leads nowhere. This is because the unitarity of quantum evolution ensures that the overlap remains constant over time:

\[ |\langle \psi'(t)|\psi(t)\rangle| = |\langle \psi'(0)|U_t^{-1}U|\psi(0)\rangle| = |\langle \psi'(0)|\psi(0)\rangle| \quad (2.7) \]

Due to the invariance of the trace function under cyclic permutation of its arguments, the same can be derived for the more general case of two state operators \( \rho_1 \) and \( \rho_2 \):

\[
\begin{align*}
Tr (\rho_1(t)\rho_2(t)) &= Tr (U_t \rho_1(0) U_t^{-1} U(t) \rho_2(0) U_t^{-1}) \\
&= Tr (U_t^{-1} U_t \rho_1(0) U_t^{-1} U_t \rho_2(0)) = Tr (\rho_1(0)\rho_2(0))
\end{align*}
\]

(2.8)

This feature of quantum states brings the classical distribution under the Liouville equation (2.5) to mind. Analogous to the quantum overlap of two states, the overlap of two classical distributions (2.6) also remains constant with time.

Besides sensitivity to initial conditions, classically chaotic systems also exhibit sensitivity to small perturbations to their Hamiltonians. This features seems to be equally valid in the quantum regime. In other words, if we prepare the two identical initial quantum states, but evolve them using slightly different parameters in the Hamiltonian, their overlap decreases very rapidly when (the classical counterpart of) the initial states are within a chaotic region of the classical phase space. On the contrary, when the initial state is in a regular section of
the classical phase space, the quantum overlap remains close to one [35] [36]. Again this is a feature quantum states share with classical phase space distributions evolving according to Liouville equations of motion.

Results like this suggest, rather strongly, that perhaps the natural basis of comparison between quantum and classical mechanics are probability distributions describing ensembles of equivalent systems, rather than single point trajectories. If that is the case, it is reasonable to ask how chaos manifests itself through distribution functions of such systems. Instead of the rates of separation of states, it is in the details of their fine structure that characteristics of chaos are to be found. Fine structure could also be manifest in aspects of quantum systems that do not have direct counterparts in classical mechanics. This opens new routes of investigation that are no longer bound by classical notion of trajectories [7] [9] [4] [5]. Hence it is possible to look for alternative criteria, solely in the quantum regime, to distinguish the two types of motion. That is, one can find criteria that are based on concepts and features that are completely quantum mechanical in nature. In this context, the measure of a good criterion would be the following: A system distinguished as regular by this criterion should approach a classical system with regular motion, in the limit $\hbar \to 0$. A system designated as entailing signatures of chaos, on the other hand, is expected to approach a classical chaotic system in the same limit. This can be investigated using powerful numerical simulations on modern computers.

The statistics of energy level spacings is one such property that has been investigated. Comprehensive research has shown that it contains important factors for determining signatures of chaos in quantum systems. Let $S$ be a generic energy level spacing in the quantum energy spectra of a class of Hamiltonian systems. Furthermore let $P(S)$ be the statistical distribution of level spacings, taken over an ensemble of systems in this class. For the case of a system made up of completely decoupled regular subsystems, there is a simple intuitive way to see that the distribution $P(S)$ for values of $S \to 0$ is not suppressed. In this case, there are many ways for each eigenvalue of the total system to be comprised of the sum of different eigenvalues of the subsystems. Thus, completely fortuitously, small level spacings may take up any value (including zero). For classically integrable systems, the typical level spacing distribution falls off exponentially $P(S) \sim e^{-S}$. The statistics is similar to that of an uncorrelated Poissonian random process.

The case of classically chaotic systems is more interesting. Almost all such Hamiltonians, whose matrix representation has large enough dimensions, have level spacing distributions
$P(S)$ that can be classified into universality classes defined by their group of canonical transformations\(^1\). Canonical transformations are transformations which change the Hamiltonian matrix, while leaving its eigenvalues the same, and which do not destroy its Hermiticity. They denote the symmetries of the Hamiltonian. These universality classes are studied in *random-matrix theory*. In this theory ensembles of Hermitian matrices are studied which have random elements with Gaussian (normal) distribution. Each ensemble has a specific group of canonical transformations. These matrices can be regarded as arbitrary Hamiltonians.

There are three main symmetry groups [32]. First group consists of Hamiltonians that are invariant under anti-linear transformations. As physical Hamiltonians, they include those systems that remain invariant under motion-reversal transformations $x \rightarrow -x$ and $p \rightarrow -p$. We take the dimensionality of the Hamiltonian’s Hilbert space to be $N$. Their group of canonical transformation is a subgroup of the $N \times N$ unitary matrices called $O(N)$, the group of orthogonal matrices. An orthogonal matrix is a square matrix $Q$ whose transpose $Q^T$ is also its inverse, so that

$$Q^T Q = Q Q^T = I \quad (2.9)$$

The next important group is that of symplectic transformations. Suppose $H$ is a skew-symmetric $2N \times 2N$ matrix. The group of symplectic transformation is comprised of $2N \times 2N$ matrices $M$ such that

$$M^T H M = H \quad (2.10)$$

Systems of spin $1/2$ particles with time-reversal symmetry are an example of this universality class. Finally, the third group is comprised of Hamiltonian random-matrices whose canonical transformations are neither of the groups mentioned above. Their corresponding group of canonical transformations are denoted as unitary.

One of the important features of classically chaotic systems is resistance against level crossings in their spectra. That is to say, adjacent energy levels tend to avoid becoming degenerate under perturbations of the Hamiltonian. This is the well-known phenomenon of level repulsion. This ability to resist level crossing depends on the number of free parameters that differentiate between two energy levels. The symmetry class of the Hamiltonian, in turn, determines how many free parameters are permitted. The resistance to level crossing affects the form of $P(S)$ as $S \rightarrow 0$ in each of the three universality classes introduced here.

---

\(^1\)There are few exceptions, but they are not typical cases.
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[32]. For the first class, orthogonal symmetry, we have

\[ P(S) = \left( \frac{\pi S}{2} \right) e^{-\pi S^2/4} \quad (2.11) \]

the group of unitary canonical transformations gives

\[ P(S) = \left( \frac{32 S^2}{\pi^2} \right) e^{-4 S^2/\pi} \quad (2.12) \]

and finally, for the symplectic group we have

\[ P(S) = \left( \frac{2^{18} S^4}{3^6 \pi^3} \right) e^{-64 S^2/9 \pi} \quad (2.13) \]

Here we have assumed that the average \( S \) has been normalized to 1. For all the chaotic cases above, the distribution \( P(S) \) vanishes in the limit \( S \to 0 \), then rises to a maximum for a value of \( S > 0 \). It then naturally falls off as \( S \) increases, since it has to be normalized.

On the other hand, energy levels of integrable systems show no particular resistance to crossings. They show neither clustering nor repulsion. That is because, integrability sets a strong constraint on the Hamiltonian, and renders the results of random-matrix theory inapplicable. This also marks a big difference between regular and classically chaotic systems regarding the general shape of the level space distributions. In regular systems, unlike the chaotic case, there is no suppression of small values of \( S \) in the distribution.

It should be noted here that the results discussed above are for the case of systems with more than one dimensions. One dimensional systems are special cases, where small level spacings are necessarily suppressed to lack of any other degree of freedom.

All that was discussed so far, have been in terms of autonomous Hamiltonian systems. It is possible to treat periodically driven systems in a similar manner. That is to say, the evolution of the system from one discrete step to the next can be described by a unitary operator. These operators are called “Floquet” operators \( F \). Since these operators are unitary, their eigenvalues are unimodular. If \( \Phi \) is an eigenvector of \( F \), then \( F \Phi = \lambda \Phi \), where \( \lambda = e^{-i\phi} \). The phase of the eigenvalue, \( \phi \), is also known as “quasi-energy”, and in many ways, it has the same role, and contains the same information, as energy eigenvalues do for isolated Hamiltonian systems. The evolution operator for longer times would result from consecutive application of the same operator. The kicked rotor is of this type.

Similar to the discussion above about random-matrix theory, ensembles of unitary matrices with random elements are also studied with their symmetry groups. They also fall
into universality classes based on their symmetries, where each member of these ensembles can represent a Floquet-operator. Again for classically chaotic systems, the level spacing distribution of the spectra can be determined by the universality class that the operator belongs to. There exists, however, an important difference between isolated Hamiltonian systems and systems with periodically driven Hamiltonians. Unlike energies, the quasi-energy eigenvalues are defined on a circle. In other words, being simply phases by nature, they are defined within the interval $[0, 2\pi]$. The fact that their space is bounded could have bearing on the results of level-spacing for such generic systems. It is not evident that they would be able to regulate their separations freely, as is the case of energy eigenstates in Hamiltonian systems.

One important exception to the above result is the category of systems exhibiting localization in position, such as the Anderson lattice models. They show no level repulsion even when the classical motion is chaotic. This is because states are localized, and eigenstates fall off exponentially. Thus, two states centered around values far from each other have negligible overlap, and negligible corresponding matrix elements. This means the energy eigenstates could be close to each other. Consequently, the level statistics has actually a Poissonian distribution, similar to the case of regular systems! As will be seen in the next section, the system under study in this work, the kicked rotor model, can be mapped on to an Anderson lattice, with angular momenta becoming the counterpart of lattice sites. So there is a possibility of quantum localization in angular momentum space. Despite the similarity, it is not clear whether the same arguments would work in the case of the kicked rotor model because, unlike lattice sites, the region of momentum space that the system can cover would be limited, depending on the initial energy of the system. So the overlap of localized eigenstates might not become negligible in this case.

Of course, these criteria can’t be meaningfully applied to classical systems directly, just as the classical concept of Lyapunov exponents, as separation rates of phase space trajectories, are not applicable to the quantum case.

It is important to note here that none of the above have been shown to be conclusive markers of chaotic behaviors in all systems of interest. Rather, each have been deduced from a particular set of studied systems, and proven to be effective in those contexts. There are no known sets of conditions that are both necessary and sufficient for categorizing quantum systems in regular and chaotic classes; that is to say, in two distinct classes, such that members of one class exhibit regular evolution in the classical limit, while those of the other
evolve chaotically in that same limit.

2.1.3 Classical Limit

The essential question regarding the classical limit is this: Once the classical limit is reached, what elements of quantum mechanics transform to what elements of classical mechanics. This is where quantum chaos theory connects to studies of foundations of quantum mechanics and quantum-classical correspondences.

Ehrenfest Correspondence principle

The first approach to the problem of quantum-classical correspondence is to consider quantum expectation values as the counterparts to the classical dynamical variables. This approach is based on the famous theorem by Ehrenfest [17]. Consider the Hamiltonian system with $H = \frac{p^2}{2m} + V(q)$. The expectation values follow

$$
\frac{d}{dt} \langle q(t) \rangle = \frac{\langle p(t) \rangle}{m}
$$

$$
\frac{d}{dt} \langle p(t) \rangle = \langle F(q) \rangle
$$

In the above, $F$ is the operator corresponding to the force, and is defined as $F = -\partial V/\partial q$. The above constitutes a closed set of equations of motion for $\langle q(t) \rangle$ and $\langle p(t) \rangle$, only if the approximation $\langle F(q) \rangle \approx F(\langle q \rangle)$ is valid. In that case we can replace $\langle F(q) \rangle$ with $F(\langle q \rangle)$ to get the centroids of the quantum state to follow equations similar to those of the Newtonian mechanics. This approximation becomes an equality only for special Hamiltonians with linear or quadratic potentials\footnote{The harmonic oscillator is one example.}. Otherwise the equations will be good approximations, only for narrow states in position.

In order to go further, we need to quantify the above statement, and to define the relevant scale for narrowness. Let us expand $\langle F(q) \rangle$ around $\langle q \rangle$.

$$
\langle F(q) \rangle = F(\langle q \rangle) + \frac{\Delta q^2}{2} \left. \frac{d^2 F}{dq^2} \right|_{q=\langle q \rangle} + \cdots
$$

$\Delta q := \langle (q - \langle q \rangle)^2 \rangle^{1/2}$ is the width of the state. States whose $\Delta q \ll L$ should be considered narrow. $L$ is defined as a suitable length scale, such that the quadratic term in the Taylor
expansion becomes of the same order as $F(\langle q \rangle)$. That is,

$$L^2 \sim \left. \frac{d^2 F}{dq^2} \right|_{q = \langle q \rangle}$$

(2.16)

In other words, $L$ is basically the length scale of the variation of the potential. Note that the Taylor expansion has no $\hbar$ dependence. What matters is clearly the width of the probability distribution, which is in general completely independent of $\hbar$. Indeed, it is worth noting at this stage, that nowhere in our arguments did we use any specifically quantum mechanical calculation. All that was said is equally valid for a classical ensemble with phase space distribution $P_c$, where the symbol $\langle \rangle$ is now re-interpreted as denoting classical averages [8]

$$\langle q \rangle = \int \int q \rho_c(q,p,t) \, dq \, dp$$

$$\langle p \rangle = \int \int p \rho_c(q,p,t) \, dq \, dp$$

$$\langle F(q) \rangle = \int \int F(q) \rho_c(q,p,t) \, dq \, dp$$

In other words, if the initial distribution is not narrow, then the centroids of a classical ensemble won't follow Hamilton's equations of motion either!

At this stage, one can ask whether all macroscopic systems should necessarily have narrow states? This is a relevant question, since one might at first make the naïve assumption, that the classical world and the macroscopic world should be considered one and the same thing. Indeed, in most popular accounts of quantum mechanics, one often comes by the claim that quantum mechanics is needed where one works in the atomic and subatomic scales.

So the question is, what factors determine the state size of a given system? To shed some light on this problem, it is helpful to separate the initial conditions from the dynamical evolution in later times.

The width of the initial state of a system has no logical dependence on the scale of the system, or its dynamical parameters. The initial state of a system depends only on the way it is prepared. The constraint of narrow initial state is unnecessarily restrictive. It is completely feasible to consider macroscopic systems prepared with state size larger than the scale of variation of the potential, $\Delta q(0) > L$, under which it will later evolve with time. By the same logic, it is also possible to prepare a microscopic system in a narrow initial
state $\Delta q(0) \ll L$. Thus, macroscopic scale$^3$ is neither a necessary nor a sufficient condition for classical behavior. Even if the quantum state is initially narrow, so that Ehrenfest’s correspondence holds, the state size will generally increase as time goes by. As a result, the correspondence won’t continue to hold. Thus, instead of comparing spatial scales, it makes more sense to derive a time-scale, during which the above correspondence remains valid. The question then becomes whether this time-scale is long enough so that, in practice, states remain narrow during the entire period of interest.

We define Ehrenfest breaking time, $t_{Ehr}$, to be the time when the state width reaches the length scale of the potential energy, so that the Ehrenfest correspondence doesn’t hold afterwards. In other words $\Delta q(t_{Ehr}) = L$. Furthermore consider the initial state to be narrow $\Delta q(0) \ll L$. Let’s consider the case where the initial state lies in a region of phase space that exhibits classical chaos. It has been shown that a classical distribution with the same $\Delta q(0)$ as the quantum wave-packet will actually grow$^4$ as $e^{\lambda t}$ within the Ehrenfest regime $[7][5][6][8][9][21][20][19]$. Thus we have

$$t_{Ehr} \sim \frac{1}{\lambda} \log \left( \frac{L}{\Delta q(0)} \right)$$

(2.17)

Suppose we assume a minimum-uncertainty initial state such that

$$|\Delta q(0)| \sim |\Delta p(0)| \sim |h|^{1/2}$$

(2.18)

This means

$$t_{Ehr} \sim \frac{1}{\lambda} \log \left( \frac{L}{h^{1/2}} \right)$$

(2.19)

or$^5$

$$t_{Ehr} \sim -\frac{1}{2\lambda} \log |h|$$

(2.20)

In case of regular motion, however, $\Delta q(t)$ will grow only as some power of $t$.

$$\Delta q(t) \approx \Delta q(0) + \int_0^t \frac{\Delta p(t')}{m} \, dt'$$

(2.21)

$^3$That is, the macroscopic scale of initial conditions.

$^4\lambda$ is the largest Lyapunov exponent.

$^5$By this, it is meant only their numerical values. Since otherwise the equation is not dimensionally correct.
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We can approximate that by

$$\Delta q(t) \approx a + \frac{b \Delta p(0)}{m} |t|^{\alpha'}$$

(2.22)

The system mass is $m$. Recall $\Delta q(t_{Ehr}) = L$. This gives the following estimate for the break time

$$|t_{Ehr}| \sim \left| \frac{b \Delta p(0)}{Lm} \right|^{-\frac{1}{\alpha'}}$$

(2.23)

Using the minimum uncertainty state as in (2.18), with $\alpha := 1/2\alpha'$, this gives

$$|t_{Ehr}| \sim |\hbar|^{-\alpha}$$

(2.24)

Numerical results show $\alpha \approx 1$. Of course, the minimum uncertainty initial state need not necessarily be symmetrical in $\Delta q$ and $\Delta p$. As long as they are proportional to powers of $h$, and their product $\Delta p \Delta q \sim h$, the $h$ dependence of $t_{Ehr}$, is a power law for regular motion, and logarithmic for chaotic systems. To sum up, we have

$$|t_{Ehr}| \sim |h|^{-\alpha} \quad (\text{Regular motion})$$

(2.25)

$$|t_{Ehr}| \sim \frac{1}{\lambda} \log \left( \frac{L}{h^{1/2}} \right) \quad (\text{Chaotic motion})$$

(2.26)

So in case of such systems with classical regular motion, the time span during which Ehrenfest correspondence holds can be very long compared to the chaotic case.

As $h$ decreases the difference between the two grows. For all practical purposes, the centroids of such probability distributions will follow classical equations of motion[17]. Unlike the regular case, the logarithmic dependence of $t_{Ehr}$ on $h$ for classically chaotic systems means that it is possible for the state to grow beyond the Ehrenfest scale within physically feasible time spans\(^7\). Zurek et al. have shown this for the chaotic motion in a system modeling one of Jupiter’s moons [45]. However this $h$ dependence of the above relations is arbitrary. It comes about only because of the initial state size, and not from the system’s dynamics. Indeed it has now become evident that Ehrenfest break time has no intrinsic $h$ dependence at all. By the same token, as was said earlier, a classical distribution with the same $\Delta q(0)$ would diverge from Hamilton’s equations in a similar way as quantum wave-packet [7] [5] [6] [8] [9] [20][19].

\(^6\)Again, only the numerical values are meant.

\(^7\)By Ehrenfest scale, it is meant the scale where Ehrenfest regime is valid.
Liouville Correspondence Principle

The Ehrenfest principle proved not to be an appropriate measure to determine the threshold of classical behavior. In the last section it was observed that the centroids of both quantum states and classical distributions behave the same way. Be it in classically regular or chaotic regions. In other words, the Ehrenfest correspondence principle does not differentiate between quantum and classical evolutions. This suggests that it would be better to take the phase-space distribution of an ensemble of classical trajectories to correspond to quantum states.

This comparison can be made most easily in the Wigner representation [41]. This representation makes it possible to express the quantum state and the expectation values in a manner that is very similar to that of a phase-space distribution. For a quantum operator \( \hat{\rho} \) it is defined through:

\[
\rho_w(p, q) = \frac{1}{2\pi\hbar} \int e^{ipq'/\hbar} \left\langle q - \frac{q'}{2} \right| \hat{\rho} \left| q + \frac{q'}{2} \right\rangle dq'
\]  

(2.27)

It could equally be defined by

\[
\rho_w(p, q) = \frac{1}{2\pi\hbar} \int e^{-ipq'/\hbar} \left\langle p - \frac{p'}{2} \right| \hat{\rho} \left| p + \frac{p'}{2} \right\rangle dp'
\]

Thus, in a sense, it could be seen as an intermediate representation between the position and momentum representations. We define a corresponding number for every quantum operator \( \hat{A} \) as

\[
A_w(p, q) = \frac{1}{2\pi\hbar} \int e^{ipq'/h} \left\langle q - \frac{q'}{2} \right| \hat{A} \left| q + \frac{q'}{2} \right\rangle dq'
\]  

(2.28)

The quantum expectation value \( \langle \hat{A} \rangle = Tr (\hat{\rho} \hat{A}) \) is

\[
\langle \hat{A} \rangle = \int \int \rho_w(p, q) A_w dp dq
\]  

(2.29)

This makes the connection to classical averages clear. It is important to note that, in general, the Wigner function can take on negative values. Thus, it can’t be regarded as a probability distribution. There is a special case where the Wigner function turns out to be non-negative. Consider the following Gaussian wave packet, centered around the points \( q_0 \) and \( p_0 \):

\[
\psi(q) = \left( \frac{1}{2\pi\alpha^2} \right)^{1/4} e^{-(q-q_0)^2/4\alpha^2} e^{ip_0q/\hbar}
\]  

(2.30)
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Its Wigner function is

$$\rho_w(p,q) = \frac{1}{\pi \hbar} \exp\left(-\frac{(q-q_0)^2}{2(\Delta q)^2}\right) \exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right)$$

Where \(\Delta q := a\), and \(\Delta p := \hbar/2a\) are the half-widths of position and momentum respectively. The Wigner function is itself a product of Gaussian position and momentum probability distributions. Gaussians are the only pure states with non-negative Wigner function [29].

The time evolution of the Wigner function under the Hamiltonian \(\hat{H} = \hat{p}^2/2m + V(\hat{q})\) is [41]

$$\dot{\rho}_w = \{H, \rho_w\} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \hbar^{2n} \partial^{2n+1} V(q)}{2^{2n}(2n+1)!} \frac{\partial^{2n+1} \rho_w}{\partial q^{2n+1}}$$

The first term on the r.h.s is the same as that given by Liouville equations. The quantum corrections are due to the extra terms on r.h.s, collectively known as Moyal terms. From equation (2.28), it is evident that the \(p\)-dependence of the Wigner function is made of superposition of oscillation terms, the fastest oscillating component going approximately as \(\exp(-q f(p)/\hbar)\), where \(f(p)\) is the state width. This gives an upper bound for the derivative with respect to momentum for each term in the Moyal sum in (2.32), as follows

$$\frac{\partial^{2n} \rho_w}{\partial p^{2n}} \leq \left(\frac{\Delta q}{\hbar}\right)^{2n} \rho_w$$

Thus each term in the Moyal sum is at most of the order of

$$\frac{(\Delta q)^{2n} \partial^{2n} V(q)}{\hbar \partial q^{2n}} \rho_w$$

As a result, in the Ehrenfest regime (\(\Delta q \ll L\), or \(V\) a linear or constant function of \(q\)) we can ignore the Moyal terms and derive the classical Liouville equation.

We now wish to estimate the magnitude of the Moyal terms. When these are non-negligible, the Liouville equation will no longer hold. The quantum nature of the Moyal corrections in (2.32) are evident in their \(\hbar\) dependence. The direct proportionality to powers of \(\hbar\) might give the impression that as \(\hbar \to 0\), the quantum corrections vanish and the classical motion is retrieved. However that is not necessarily the case, and the situation is more subtle. There may exist an implicit dependence on \(\hbar\) because the upper bounds (2.33), show that each derivative \(\partial_p\) in the Moyal terms contributes a factor of the order \(\Delta q/\hbar\) which can neutralize the explicit \(\hbar\) factors in (2.32). The Liouville equation is area preserving, so, in case of positive Lyapunov exponent \(\lambda\) for classical motion, the distribution
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will spread exponentially as $e^{\lambda t}$. As the distribution stretches, it has to become narrower. This means that the distribution has to vary over finer and finer scales as time increases. Suppose $\sigma$ is the scale of variation of the distribution parallel to the fastest shrinking. Then $\sigma(t) \sim \exp(-\lambda t)$. Let $\sigma_p(t) \approx \exp(-\lambda t) \sigma_p(0)$ as well, where $\sigma_p$ is the projection of $\sigma$ on the momentum axis. The scale $L$, the length scale of potential non-linearities, and $\sigma_p$ are related to the Wigner function evolution by noticing that $dV/dq \approx V/L$ and $\partial \rho_\omega/\partial p \approx [1/\sigma_p(t)] \rho_\omega$, or equivalently $\partial \rho_\omega/\partial p \approx [1/\sigma_p(0)] \exp(\lambda t) \rho_\omega$. Each term of the Moyal corrections are thus of the scale

$$h^{2n} \frac{d^{2n+1}V(q)}{dq^{2n+1}} \frac{\partial^{2n+1} \rho_\omega}{\partial p^{2n+1}} \approx \frac{1}{L \sigma_p(t)} \left( \frac{h}{L \sigma_p(t)} \right)^{2n} \rho_\omega$$

(2.35)

On the other hand, the Poisson brackets, that correspond to the classical Liouville equation, are of the scale

$$\{H, \rho_\omega(p, q, t)\} \approx \frac{1}{L \sigma_p(t)} \rho_\omega$$

(2.36)

This means that the time it takes for the lowest Moyal term to become non-negligible is of the order

$$t_b \sim \log \left( \frac{L \sigma_p(0)}{h} \right)$$

(2.37)

Note that at $t = 0$ we can take $\sigma_p(0)$ to be the variance of the initial quantum state in the momentum representation $\Delta p(0)$. This time can be regarded as the break time for Liouville correspondence[46].

As we have seen, the Liouville break-time is similar to that of Ehrenfest, although it is derived within a very different context. If we regard the width of the initial state and $\hbar$ as separate parameters, then the break-time for Ehrenfest correspondence is not sensitive to $\hbar$, whereas the Liouville break-time has $\hbar$ dependence. On the other hand, the Ehrenfest break-time depends on the initial width of the distribution, whereas the difference between the quantum and classical probability distributions does not show sensitivity to the initial conditions. This has been shown for various models by Ballentine et al. [7] [5] [6].

The main limitation of the Liouville correspondence principle is that the arguments used to derive the form of the break-time only work for Wigner distributions that are initially close to the shape of a classical distribution, and it remains valid as long as the two remain close. As we know the break-time is the time when the Moyal terms have grown too large to
be negligible. What guarantees are there that by this time, the differences have not already become so large, as to break down the approximate similarity in shape between Wigner distribution and a classical probability distribution? If that is the case, the argument based on area preservation given above need not be valid anymore. That’s because, the Wigner function can, in general, have negative values of any magnitude, and the very notion of an area will not always be meaningful.

It is important to note that as \( h \) is decreased, the quantum interference patterns, which give rise to negative values for Wigner function, and which have a periodic character in phase space do not go away. Instead, although the amplitude of the interference pattern does not go to zero, its wavelength does. In other words the oscillations between positive and negative values of the Wigner function come closer together. Suppose the classical limit could, in principle, be defined for our system, by defining a set of similar models in which the relative size of \( h \) can be made small indefinitely. Since any measuring device has a limited power of detection, there inevitably comes a scale on which interference patterns become so fine that they won’t be observed. In other words the detecting instrument performs, in effect, a smoothing on the Wigner function. For instance, the entanglement to the environment, a phenomenon that is known as decoherence, can be seen as limiting the detection power of the instruments, and thus, contributing to the smoothing out that would get rid of negative values and quantum interference effects. This will then lead to essentially classical Liouville evolution of the detected distribution.

One way to mathematically model this procedure is to form a non-negative distribution out of Wigner functions by smoothing them with a Gaussian function in both position and momentum. This will transform the Wigner function to another distribution known as Husimi distribution. The reason why a phase space probability distribution that matches the quantum state at all times does not exist, is because position and momentum operators do not commute, and so they can’t have a joint eigenvector. The next best thing to an eigenvector of position and momentum, that is allowed in quantum mechanics is a minimal uncertainty state localized in phase space, with its centroids being the pair \((q, p)\) and, with Gaussian distributions in both position and momentum. We denote these type of states with the general notation \( |q, p\rangle \). Such states are not unique, and as long as their rms half-widths satisfy the minimum uncertainty relation \( \Delta_q \Delta_p = \hbar/2 \), each of \( \Delta_q \) and \( \Delta_p \) can take

\[ \Delta_q \Delta_p \leq \frac{\hbar}{2} \]

\[ \Delta_q \text{ and } \Delta_p \text{ are the standard deviations in position and momentum, respectively.} \]

\[ \hbar \text{ is the reduced Planck constant.} \]

---

\[ \text{Provided, of course, that it is close to the classical evolution.} \]
different values. If we choose $\Delta_s q = s$, their position representation would be of the form

$$\langle x | q, p \rangle = (2\pi s^2)^{-1/4} e^{-(x-q)^2/4s^2} e^{ipx/h}$$  \hbox{(2.38)}$$

These states are not orthonormal, but they form an overcomplete set, and satisfy

$$\int \int dq dp \ |q, p\rangle \langle q, p\rangle = 2\pi \hbar$$  \hbox{(2.39)}$$

The Husimi distribution for the state operator $\hat{\rho}$ is defined as

$$\rho_H(p, q) = \frac{1}{2\pi \hbar} \langle q, p | \hat{\rho} | q, p \rangle$$  \hbox{(2.40)}$$

for the case of a pure state, $\hat{\rho} = |\psi\rangle \langle \psi|$, this becomes

$$\rho_H(p, q) = \frac{1}{2\pi \hbar} |\langle q, p | \psi\rangle|^2$$  \hbox{(2.41)}$$

The Husimi distribution can be interpreted as a probability density. In this interpretation, it would be the probability density of finding the system within a region in phase space centered over the pair $(q, p)$, with half-widths $\Delta q_s = s$ and $\Delta p_s = \hbar/2s$. As a probability distribution it is clear that it is nowhere negative. Expanding the equation (2.40), using the definition of $|q, p\rangle$ and recalling (2.31), gives

$$\rho_H(p, q) = \frac{1}{\pi \hbar} \int \int \rho_w(q', p') \exp \left(\frac{-(q' - q)^2}{2s^2}\right) \exp \left(-\frac{(p - p')^2}{\hbar^2}\right) dq' dp'$$  \hbox{(2.42)}$$

The Husimi distribution can be interpreted as a Gaussian smoothing of the Wigner function for the state operator $\hat{\rho}$, using the Gaussian $|q, p\rangle$ as the smoothing function. This way, one can get a smoothed non-negative probability distribution from the Wigner function.

The time evolution equation for the Husimi function is much more complicated than that of the Wigner function. For a Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{Q})$, the Husimi distribution evolves as [34]:

$$\frac{\partial}{\partial t} \rho_H(q, p, t) = \frac{\partial^2 \rho_H}{\partial q^2} + \frac{\partial^2 \rho_H}{\partial p^2}$$  \hbox{(2.43)}$$

where

$$\frac{\partial \rho_H}{\partial t} = -\frac{p}{m} \frac{\partial \rho_H}{\partial q} - \frac{(\Delta_s p)^2}{m} \frac{\partial^2 \rho_H}{\partial p^2}$$  \hbox{(2.44)}$$

is the potential independent part of the equation that comes from the kinetic energy term of the Hamiltonian, and

$$\frac{\partial \rho_H}{\partial t} = \sum_{n, m, k \geq 0} \left(\frac{ih}{2}\right)^{2n} \frac{(\Delta_s q)^{2m-2k}}{2^n k! (2n)! (m - 2k)!} \frac{\partial^{2n+1+m} V(q)}{\partial q^{2n+1+m}} \frac{\partial^{2n+1}}{\partial p^{2n+1}} \frac{\partial^{m-2k} \rho_H}{\partial q^{m-2k}}$$  \hbox{(2.45)}$$
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is the potential dependent part of the time evolution equation, and comes from the potential energy term of the Hamiltonian. Here $m - 2k \geq 0$. Upper limits can be placed on the size of the partial derivatives of $\rho_H$, that is

$$
\frac{\partial^{2n+1} \rho_H}{\partial q^{2n+1}} \leq \frac{\rho_H}{(\Delta_s q)^{2n+1}} \tag{2.46}
$$

$$
\frac{\partial^{2n+1} \rho_H}{\partial p^{2n+1}} \leq \frac{\rho_H}{(\Delta_s p)^{2n+1}} \tag{2.47}
$$

Therefore the second term in (2.44) is at most of the order of $\Delta_s p/p_{typ}$ relative to the first term. $p_{typ}$ here is a typical value of momentum in that state. Also all the terms with indices $n$ and $m$ in (2.45) will be at most around

$$
\frac{(\Delta_s q)^{2n+1+m}}{\hbar} \frac{\partial^{2n+1+m} V(q)}{\partial q^{2n+1+m}} \rho_H \tag{2.48}
$$

Hence, for $\Delta_s p \ll p_{typ}$ and $\Delta_s q \ll L$, the smallest term in the sum (2.45) will be for $n = m = k = 0$. This gives

$$
\frac{\partial \rho_H}{\partial t} = -\frac{p}{m} \frac{\partial \rho_H}{\partial q} + \frac{\partial V(q)}{\partial q} \frac{\partial \rho_H}{\partial p} \tag{2.49}
$$

Which is the Liouville equation for classical distributions. If we define $S := p_{typ} L$ as the typical action of the system, then the classical regime can be expressed as $\hbar \ll S$.

It should be noted that $\Delta_s q$ and $\Delta_s p$ are the half-width of the smoothing function and are different from the width of the actual state $\Delta q$ and $\Delta p$. It should also be noted that the Wigner function still could exhibit interference patterns, and contain negative values. It is only after the “smoothing”, interpreted as resulting from the limits of detection of the fine structure, that interference patterns go away and the classical Liouville equation practically describes the time evolution of the system.

There are two major approaches to the origin and the physical justification of the smoothing process. One way such smoothing could come about is through coupling with the environment, and the phenomenon of decoherence [45].

But even for an isolated quantum system, it has been noted that smoothing could still be justified, since all practical devices and measurements processes perform an effective coarse-graining similar to the smoothing effect discussed above, and that it is perfectly reasonable to take that into account in the quantum-classical correspondence schemes [44].
2.2 Kicked Rotors

Classical Kicked Rotor

A very important class of models, studied extensively in Chaos theory, are the so-called "kicked planar rotor" models [31]. In classical mechanics they can be described by a simple iteration map, that has become the standard mapping in the study of relevant topics. The model has only one parameter, exhibits both regular and chaotic motion, and the transition from one to the other for the parameter is well understood. The system is defined on a cylindrical phase space. The chaotic behavior is comprised of fast random-like changes in the angle variable running around the cylinder, accompanied by slow diffusive growth of the conjugate angular momentum (action) variable along the axis of the cylinder.

In the quantum regime, the corresponding group of systems has naturally been one of the first topics of research in the quantum dynamics of chaotic systems. Interestingly, the kicked rotor systems have revealed a fascinating feature in their quantum evolution. The purely quantum interference effects have been shown to suppress the diffusion of angular momentum in one dimensional systems, and thus manifest completely different behavior than their classical counterparts, namely quantum localization effects [26] [27] [16] [18] [23]. This feature and its characteristics in the two-coupled rotors are the main topic of this thesis. Therefore it is useful to cover the main points of this class of systems before discussing the actual model and results of this project.

The time-dependent Hamiltonian of a system of classical kicked rotors can be written as

\[ H = T_0(\{p_i\}) + V(\{\theta_i\}) \sum_{n=-\infty}^{+\infty} \delta(t - n\tau) \]  

(2.50)

Here \( \tau \) is the period of kicking. \( \{\theta_i\} \) and \( \{p_i\} \) are the set of angles and their conjugate angular momenta of the system's rotors. \( T_0 \) is the general function for kinetic energy.

\[ T_0(\{p_i\}) = \sum_i \frac{p_i^2}{2I_i} \]  

(2.51)

\( I_i \) is the moment of inertia of the \( i \)th rotor. \( V \) is determines the form of the potential, and is assumed to be a periodic function in angle variables with period \( 2\pi \). The motion can be described by the map

\[ \theta_i(n + 1) = \theta_i(n) + \frac{\tau}{I_i} p_i(n) \pmod{2\pi} \]
\[ p_i(n + 1) = p_i(n) - \frac{\partial}{\partial \theta_i} V \left( \{ \theta_k(n + 1) \} \right) \] (2.52)

Here \( p_i(n) \) and \( \theta_i(n) \) are the angle and angular momentum of the \( i \)th rotor after the \( n \)th kick.

We now focus on systems with a single rotor. This way we can study the most important aspects of the kicked rotor model in its simplest form and without unnecessary complications. The generalization to the case of multiple rotors afterwards is trivial.

To simplify the equation, we rewrite the Hamiltonian, in units of the system action scale, \( I/\tau \), and the time period \( \tau \), as a dimensionless quantity. We further assume that the potential is of the form

\[ V(\theta) = K \cos \theta \] (2.53)

Here \( K \) is a constant, acting as the measure of the kicking strength.

\[ H(t) = \frac{p^2}{2} + K \cos \theta \sum_{n=-\infty}^{+\infty} \delta(t - n) \] (2.54)

All the symbols are now dimensionless quantities. This gives the Chirikov-Taylor standard map [31]:

\[ \theta_{n+1} = \theta_n + p_n \quad (\text{mod} \ 2\pi) \] (2.55)

\[ p_{n+1} = p_n + K \sin \theta_{n+1} \] (2.56)

This map is characterized by one parameter only, namely \( K \), the dimensionless kicking strength. Depending on initial conditions, the motion can be either regular or chaotic no matter what \( K \) is. However for \( K \) smaller than a critical value \( K_c \approx 0.97164 \), the motion is completely bounded in momentum. The chaotic regions are isolated and Kolmogorov-Arnol’d-Moser (KAM) trajectories separate them from each other. At \( K_c \) the remaining KAM trajectories disappear. For \( K \gg K_c \) the motion becomes ergodic, exhibiting exponential instability\[13\] \[22\]. The maximum Lyapunov exponent per step is

\[ \Lambda \approx \log \left( \frac{K}{2} \right) \] (2.57)

In this range, there is diffusion in angular momentum-space. That is to say, \( p \)-values perform a random walk for large enough time-steps, so that

\[ (\Delta p)^2 \propto n \tau \]
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Where $(\Delta p)^2$ is the average of the squared change in angular momentum after $n$ kicking steps. The proportionality constant is known as the diffusion coefficient. In this case, it is $D \approx K^2/2$ to the lowest order in $K$, so that

$$(\Delta p)^2 \approx \frac{K^2}{2} n\tau$$

One special case of this model is that of a periodically driven pendulum of mass $m$ and length $\ell$, with natural frequency of its small displacements given by $\omega_0^2 = g/\ell$ ($g$ free fall acceleration). The potential then would be $V(\theta) = m\ell^2\omega_0^2 \cos \theta$ and moment of Inertia would equal $M = m\ell^2$. This is the original model from which this important and well studied group of generalized systems actually takes its name from.

For this special form of $V$, it is possible to have a qualitative understanding of the diffusion of angular momentum in the chaotic regime for strong kicking strengths. It is worth going through that briefly. In equation (2.56), once $p_n$ becomes of the order of $2\pi$ or larger, due to a couple of kicks for large values of $K$, the successive values of $\theta_n$ will no longer be correlated. Hence the sign of $\sin \theta$ becomes random and the $p_n$ sequences will perform a random walk henceforth.

**Quantum Kicked Rotor**

We now study the quantum version of the same time dependent system of periodically kicked rotors. Recall that the Hamiltonian is of the form

$$H = T_0(\{\hat{\theta}_i\}) + V(\{\hat{\theta}_i\}) \sum_{n=-\infty}^{+\infty} \delta(t - n\tau)$$

Now we define $\{\hat{\theta}_i\}$ and $\{\hat{p}_i\}$ as the operators corresponding to angle and angular momenta of the rotors. Thus, we have $\hat{\theta}_i \equiv \imath \hbar \partial/\partial \theta_i$.

Again $T_0(\{\hat{p}_i\}) = \sum_i \hat{p}_i^2 / 2I_i$ is the general form of the kinetic energy, and the functional form of $V$ is periodic with period $2\pi$. $\tau$ is the kicking period. The Schrödinger equation in angle representation is

$$\imath \hbar \frac{\partial}{\partial \tau} \psi(\{\theta_i\}, t) = \left( T(\{\hat{p}_i\}) + V(\{\theta_i\}) \sum_{n=-\infty}^{+\infty} \delta(t - n) \right) \psi(\{\theta_i\}, t)$$

Here time is measured in units of $\tau$ and $T(\{\hat{p}_i\}) := \tau T_0(\{\hat{p}_i\})$ is the rescaled kinetic energy.
We now focus on the single rotor case in order to extract the most important results of this class of models. The case where the potential form can be expressed as $V(\theta) = k \cos \theta$, with $k$ a constant kicking strength, is particularly important. Unlike the classical case, the quantum system needs two dimensionless parameters to specify its motion.

We take them to be

\[
\alpha := \frac{\hbar \tau}{T}
\]
\[
\beta := \frac{k}{\hbar}
\]

Their product $K = \alpha \beta$ has no $\hbar$ dependence, and is equivalent to the classical dimensionless kicking parameter. The classical limit corresponds therefore to the case $\alpha \to 0$ and $\beta \to \infty$, while keeping the product $\alpha \beta$ constant and equal to $K$ of the classical counterpart system.

Taking the above parameters as the natural units of the system, we can write the time-dependent Schrödinger equation in these units

\[
i \frac{\partial}{\partial t} \psi(\theta, t) = \left(-\frac{\alpha}{2} \frac{\partial^2}{\partial \theta^2} + \beta V(\theta) \sum_{n=-\infty}^{+\infty} \delta(t - n)\right) \psi(\theta, t)
\]

However in the following section I will use the more general form, so that the transition to models containing two or more rotors is made easier.

The system is a free rotator between kicks, and thus the evolution is a constant rate rotation. Within these ranges, it is useful to express the wave-function in angular momentum representation,

\[
\psi_n := \langle n | \psi \rangle
\]

where

\[
\hat{\mathbf{P}} |n\rangle = n |n\rangle
\]

Thus we have

\[
\psi_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \; e^{-i n \theta} \psi(\theta)
\]
\[
\psi(\theta, t) = \sum_n e^{i n \theta} \psi_n(t)
\]

At the kicking times the kinetic energy becomes unimportant, due to the delta functions, and we need only consider infinitesimal times just before and after the kick. We denote the wave function just before the $k$th kick as $\psi^-(k)$ and just after the kick as $\psi^+(k)$. 

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Between the kicks, the motion is
\[ \psi_n^-(k+1) = e^{-i\omega t} \psi_n^+(k) \] (2.65)

The effect of the kick is
\[ \psi^+(\theta, k) = e^{-iV(\theta)} \psi^-(\theta, k) \] (2.66)

Combing the two and expressing everything in the angular momentum representation gives
\[ \psi_n^+(k+1) = \sum_n J_{n-m} e^{-i\lambda T(n)} \psi_n^+(k) \] (2.67)

where the function \( J_{n-m} \) is defined as
\[ J_{n-m} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i(m-n)\theta} e^{-iV(\theta)} \] (2.68)

In the important case \( V(\theta) = \beta \cos \theta \), \( J_{n-m} \) equals \( J_{n-m} \), Bessel function of the first kind. This can be seen, using the identity
\[ e^{i\beta \cos \theta} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{i\theta} \] (2.69)

Where \( J_n \) denotes the Bessel function of the first kind of order \( n \).

2.2.1 Transformation To Anderson Model

In this section we derive the mapping that takes the kicked rotor model to a tight binding model for electron transportation between the sites of a lattice. This is known as the Anderson Model.

The Hamiltonian is periodic in time \( H(t+1) = H(t) \) (in dimensionless units), hence the solutions to the Schrödinger equation can be classified by the way their wave-function transforms under translations in time. This way, we can derive a new quantum number that acts in many ways similar to energy for isolated Hamiltonian systems. That is to say, like energy eigenvalues, this quantum number determines the time dependence of the solutions to the Schrödinger equation. For this reason it has been termed as quasi-energy [18] [23] [43]. This number is the only useful quantum number that can be found for the system in question.

The quasi-energy states, are those whose quasi-energy is \( \omega \), having the form
\[ \psi_\omega = e^{-i\omega t} u_\omega(\theta, t) \] (2.70)
where $u_\omega(\theta, t + 1) = u_\omega(\theta, t)$. States with different quasi-energies are orthogonal [43]. It is possible to write the propagator of the systems evolution between each kick, as a unitary operator. The discussion here is analogous to the Bloch-Floquet theorem, only here, instead of spatial periodicity, we have periodicity in time. Therefore propagators such as ours have come to be known as Floquet-operators. In our case it is

$$F = e^{-iV(\theta)} e^{-i\theta^2/2} \quad (2.71)$$

$$|\psi_\omega(t + 1)\rangle = F |\psi_\omega(t)\rangle \quad (2.72)$$

States $\psi_\omega$ are also known as Floquet-states. Since the quasi-energy states are periodic, it suffices to consider their values right before and after each kick. As before, we define $u^\pm$ to denote the state just before and after a kick, respectively. Note that there is no time dependence anymore. Similarly, we define $u^\pm_n$ as the angular momentum representation of the state just before and after a kick.

$$u^\pm_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \; e^{-im\theta} u^\pm(\theta) \quad (2.73)$$

Between the kicks, the evolution is simple

$$u^+_m = e^{i\omega} e^{-iT(n)} u^+_m \quad (2.74)$$

For determining the effect of the kick, it is better to return to the angle representation

$$u^+(\theta) = e^{-iV(\theta)} u^-(\theta) \quad (2.75)$$

Combining the two, we get

$$u^+_m = e^{i\omega} \sum_n J_{m-n} e^{-im^2/2} u^+_n \quad (2.76)$$

Where the function $J_{m-n}$ is defined in (2.68).

Next we define

$$|\bar{u}\rangle := \left( |u^+\rangle + |u^-\rangle \right)/2 \quad (2.77)$$

as the average between the state immediately before, and after the kick. We also define

$$W := -\tan \left( \frac{V(\theta)}{2} \right) \quad (2.78)$$
or equivalently we have

\[ e^{-iV(\theta)} = \frac{1 + iW}{1 - iW} \]  \hspace{1cm} (2.79)

Using the above in (2.75) and writing both sides in terms of \( \bar{u} \), we get

\[ u^{-}(\theta) = \bar{u}(\theta) \left( 1 - iW(\theta) \right) \]
\[ u^{+}(\theta) = \bar{u}(\theta) \left( 1 + iW(\theta) \right) \]  \hspace{1cm} (2.80)

Their Fourier Transform is

\[ u_{m}^{-} = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} d\theta \, e^{-im\theta} \bar{u}(\theta) \left( 1 - iW(\theta) \right) \]
\[ u_{m}^{+} = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} d\theta \, e^{-im\theta} \bar{u}(\theta) \left( 1 + iW(\theta) \right) \]  \hspace{1cm} (2.81)

Using the convolution theorem, we can re-write the above as

\[ u_{m}^{-} = \bar{u}_{m} - i \sum_{r} \bar{u}_{r} W_{m-r} \]
\[ u_{m}^{+} = \bar{u}_{m} + i \sum_{r} \bar{u}_{r} W_{m-r} \]  \hspace{1cm} (2.82)

Where \( \bar{u}_{n} \) is the \( n \)th Fourier component of \( \bar{u}(\theta) \), and \( W_{n} \) is the \( n \)th Fourier component of \( W(\theta) \). From now on, for simplicity, we use \( u \) to stand for \( \bar{u} \). We substitute them on both sides of (2.74) to get

\[ u_{m} - i \sum_{r} W_{r} u_{m+r} = e^{iE_{m}} \left( u_{m} + i \sum_{r} W_{r} u_{m+r} \right) \]  \hspace{1cm} (2.83)

\[ E_{m} := \omega - T(m) \]  \hspace{1cm} (2.84)

Next we isolate the term \( r = 0 \) in each sum to one side, and rewrite the equation, we obtain

\[ T_{m} u_{m} + \sum_{r \neq 0} W_{r} u_{m+r} = E u_{m} \]  \hspace{1cm} (2.85)

where we define

\[ E := -W_{0} \]  \hspace{1cm} (2.86)
\[ T_{m} := i \frac{1 - e^{i(\omega \cdot m^2/2)}}{1 + e^{i(\omega \cdot m^2/2)}} = \tan \left( \frac{E_{m}}{2} \right) \]  \hspace{1cm} (2.87)
Table 2.1: The correspondence between kicked rotor and the Anderson lattice. The quantities in the same row are equal to each other.

<table>
<thead>
<tr>
<th>Kicked Rotor</th>
<th>Anderson Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$: quasi-energy</td>
<td></td>
</tr>
<tr>
<td>$T(m)$: kinetic energy</td>
<td>$T_m$: on-site potential</td>
</tr>
<tr>
<td>$\tan\left(\frac{\omega - T(m)}{2}\right)$</td>
<td></td>
</tr>
<tr>
<td>$V(\theta)$: potential energy</td>
<td>$W_r$: hopping potential</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta e^{-i\theta} \tan\left[\frac{V(\theta)}{2}\right]$</td>
<td>$E$: lattice energy eigenvalue</td>
</tr>
<tr>
<td>$u_+^\pm (\theta)$: quasi-energy eigenvectors</td>
<td>$= -W_0$</td>
</tr>
<tr>
<td>$\bar{u}(\theta) = \frac{u^+ (\theta) + u^- (\theta)}{2}$</td>
<td></td>
</tr>
<tr>
<td>$u_+^\pm = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta e^{-i\theta} u_\omega^\pm (\theta)$</td>
<td></td>
</tr>
<tr>
<td>$\left[u_m^+ + u_m^-\right]/2$</td>
<td>$u_m$: amplitude on the lattice</td>
</tr>
</tbody>
</table>

Equation (2.85) is of the form of the Schrödinger equation for a particle on a one-dimensional lattice, with $T_m$ the potential of each site, and $W_r$ the amplitude of hopping to the $r$-th neighbor. This is the Anderson hopping on a tight-binding model [3]. Thus we have established a connection between our dynamical model and a solid-state model. As is seen here, the angular momentum in the kicked rotor model corresponds to the lattice sites in the tight-binding model. The nature of correspondence is interesting, and is worth investigating further. Table 2.1 shows the connection between the parameters of the two systems.

The energy of the lattice model is determined by the form of the kicking potential. It has actually turned out to be simply the zeroth Fourier component of a straightforward function of this potential $W = -\tan(V/2)$ (2.78).

The quasi-energy $\omega$ of the original kicked rotor model, on the other hand, has not been mapped to the energy eigenvalue of the Anderson model. Instead it has become a parameter for the lattice site potentials $^9 T_m = \tan\left(\frac{\omega - T(m)}{2}\right)$.

The relationship between the sequence $T_m$ and the localization of the solutions on the lattice (and hence in the angular momentum space of the dynamical model) is of importance.

---

^9To avoid confusion, it is worth reminding that $T(m)$ is the kinetic energy in the kicked rotor, while $T_m$ is the on-site potential in the Anderson lattice.
If \( \{T_m\} \) is a periodic sequence, then the solutions on the lattice behave as the Bloch-wave functions. This corresponds to quantum resonance in the kicked rotor model [13]. On the other extreme, if each element of \( \{T_m\} \) is totally independent of the others, and is produced by a source with a specific probability distribution\(^{10}\), then we have Anderson localization in one-dimension [30]. The interesting case arises when \( \{T_m\} \) are not totally independent from each other, although they can still be viewed as instances of a probability distribution\(^{11}\).

It has been shown numerically, that in a generic case of single kicked rotor systems, the correlations between the elements of \( \{T_m\} \) is small enough for Anderson localization to take place [23]. Mott and Twose [33] had already shown analytically, back in 1961, that any break in translational symmetry in the one-dimensional lattice, for instance due to introduction of impurities, is enough to produce localization. Whether analogous results are true for all cases of the two-rotor systems, with various forms of the kicking potentials in the Hamiltonian, is the main question behind this work.

The surprising result is that the quasi-energy of the kicked rotor system, \( \omega \), merely chooses among more or less equivalent sets of already existing random-like numbers. In a given kicked rotor model, the form of the potential is of course already fixed. Fixed form of potential is equivalent to a fixed energy \( E \) for the mapped Anderson lattice, as we saw above.

However, it is beneficial to look at the tight-binding equation (2.85) to see what it says about the mathematical relationship between the two parameters \( \omega \) (residing in the values of \( T_m \), as noted above) and \( V(\theta) \) (represented by \( E = -W_0 \)). That is to say, (2.85) can be regarded in two distinct and complementary contexts. We can look at it within the context of the Anderson lattice. Then the system can have different values of energy \( E \). In this case, \( \omega \) becomes merely a parameter in the tight-binding Hamiltonian. The relationship between the two has a rather simple form. The Feynman-Hellman theorem relates the energy eigenvalues of a time-independent Hamiltonian to the set of parameters that specify that Hamiltonian. The general form of the theorem states that

\[
\frac{\partial E_n}{\partial \gamma} = \int ds \psi^*_n(s) \frac{\partial E_n}{\partial \gamma} \psi_n(s)
\]

Here \( \psi_n \) and \( E_n \) are the \( n \)th eigenvalue and eigenvector of the Hamiltonian, \( \gamma \) is the relevant

\(^{10}\)The existence of these two characteristics together is usually referred to as randomness. This term is used, however, in different contexts with different meanings in mind.

\(^{11}\)Such a sequence is referred to as pseudo-random is some texts.
Table 2.2: Comparison between the interpretation of variables in the two contexts: kicked rotor and Anderson lattice. The two columns are complementary to each other in each context.

<table>
<thead>
<tr>
<th>Kicked Rotor</th>
<th>Anderson Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ : index for quasi-energies of the rotor</td>
<td>$\nu$ : index for lattice energy eigenvalues</td>
</tr>
<tr>
<td>$\omega_\mu(E)$ : $\mu$th quasi-energy as a function of $E$ (a given parameter of the system)</td>
<td>$E_\nu(\omega)$ : $\nu$th lattice energy eigenvalue as a function of $\omega$ (a given parameter of the system)</td>
</tr>
<tr>
<td>$\omega_\mu(V)$ : $\mu$th quasi-energy as a function of $V$ (rotor potential energy-fixed)</td>
<td>$E_\nu({T_m})$ : $\nu$th lattice energy eigenvalue as a function of ${T_m}$ (sequence of lattice-site potentials-fixed)</td>
</tr>
</tbody>
</table>

parameter in question, and $ds$ means integration over the domain of the eigenvectors. We can use the equivalent of the above theorem for the lattice model, which is discrete. In that case, we should treat $\omega$ as the parameter for determining the lattice energy value $E$. The Hamiltonian equation (2.85) then gives

$$
\frac{dE}{d\omega} = \sum_m \frac{1}{(2\cos^2(\omega - T(m)))} > 0
$$

(2.89)

So $E$ and $\omega$ vary monotonically together. The low symmetry of the system usually prevents degenerate eigenvalues $E_\nu$ to exist for different values of $\omega$. $E_\nu$ is the energy of the eigenstate with index $\nu$ of the Anderson lattice. Because of these features, $E_\nu$ can be inverted to give the functional dependence of various values of $\omega$ for a given $E$.

This means that we are now looking at the relation (2.85) in a different context now. That is to say, the equations are now describing the kicked rotor model, in other words, a time-dependent system with allowed quasi-energies $\omega$. The quantity that differentiates one system from another is now $V(\theta)$, that is, the functional form of the kick in the potential energy term in the Hamiltonian of the kicked rotor. Since $E$ is a function only of $V$, we could consider $E$ as the parameter that specifies the system, and as a result of, that its spectra of quasi-energies $\omega_\mu$, and their corresponding eigen-states $u_\mu(\theta)$. The index $\mu$ counts the quasi-energies of the kicked rotor.

One can look at the relationship between $E$ and $\omega$ in either of these two contexts: Anderson lattice or the kicked rotor. Table 2.2 summarizes the main points of these two complementary contexts. In other words, the features of the spectrum of the quasi-energies
in the kicked rotor model are similar to those of the spectrum of energy eigenvalues of the Anderson lattice [23].

Another interesting question is the shape of the quasi-energy eigenstates, and the role of each parameter in determining it. Due to Anderson localization, the quasi-energy eigenstates fall off exponentially around some value of the angular momentum. The localization length of each eigenstate is determined only by \( E \), its corresponding energy eigenvalue. Due to the equivalence between the two models, this means that the quasi-energy eigenstates also fall off exponentially around some value of the angular momentum, and the localization length \( \zeta \) defined by \( u \sim e^{-|p-p_0|/\zeta} \) is determined solely by the energy eigenvalue of the lattice \( E \), and therefore by the shape of the potential. It is not determined by the kicking period \( \tau \) or the value of the quasi-energy at all. The center of localization is determined by the quasi-energy eigenvalue of that state. States of near quasi-energies are usually centered around values of angular-momentum that are far apart from each other. Oppositely, eigenstates that are close to each other have eigen-energies that are not close to each other. \( \zeta(E) \) gives a scale for this separation. If the center values are close compared to \( \zeta \), their eigen-energies spacing is of the order of \( \langle |W_r| \rangle / \zeta \).

It is worthwhile to mention briefly here how this is played out in the actual behavior of the kicked rotor, and to see how these parameters affect the time evolution of that system, we expand the kicked rotor state function in terms of the quasi-energy eigenstates. Since we know that such a state\(^{12} |u^+\rangle \) is localized over values of momentum, we can index it with the momentum around which it is centered \( m \): \( |u^+\rangle_{(m)} \) with quasi-energy \( \omega_m \). We can write \( |\psi\rangle \) in terms of these states

\[
|\psi(t)\rangle = \sum_m a_m |u^+\rangle_{(m)} e^{-i\omega_m t}
\]

The coefficient \( a_m \) is determined by the initial state. Next we project \( |\psi\rangle \) onto the momentum eigenstates

\[
\psi_n^+(t) = \sum_m a_m u^+_n e^{-i\omega_m t}
\]

\[
u^+_n := \langle n | u^+ \rangle_{(m)}
\]

The index \( m \) is the index for quasi-energy and \( n \) is the index for angular momentum. Let’s assume the state started as an eigenstate of angular momentum around value \( s \), that is

\(^{12}\)We take the state immediately after each kick for simplicity.
CHAPTER 2. THEORETICAL BACKGROUND

\( \psi_n^+(0) = \delta_{ns} \). Then after time \( t \),

\[
\psi_n^+(t) = \sum_m (u_{sm}^+)^* u_{nm}^+ e^{-i\omega_m t}
\]  \hspace{1cm} (2.93)

Taking the Fourier transform gives the spectrum of quasi-energies

\[
A_s^\mu(\omega) = \sum_m (u_{sm}^+)^* u_{nm}^+ \delta(\omega - \omega_m)
\]  \hspace{1cm} (2.94)

To see what the spectrum is around the initial angular-momentum we have to look at

\[
A_s^\mu(\omega) = \sum_m |u_{sm}^+|^2 \delta(\omega - \omega_m)
\]  \hspace{1cm} (2.95)

This quantity has been named "local density of quasi-energy states" by Fishman et al [23]. Local since it is a weighted average. In the corresponding lattice, the contribution of each site is weighted by the square of the state function. The local density of states is proportional to \( \zeta \), the localization length, since due to the exponential localization, only a few states, within the range of \( \zeta \) contribute significantly to the local density of states. These are the states centered along angular momenta that lie within this range around the initial state. Each quasi-energy eigenstate can be identified with the value of angular momentum around which it is centered. Because of the delta functions in the sum, the local density of states is practically finite and consists of about \( \zeta \) peaks. This shows that the system evolution must be almost-periodic. This means, one way the presence of localization in momentum in the kicked rotor system shows itself is in the way the expectation values of the observables change with time. In case of localization, their change is expected to be almost periodic. This could be a useful point for the experimental realizations of the kicked rotor model. Since this feature is not directly related to our present work, we do not investigate it further.

In this section we showed the close relationship between a system of periodically kicked and coupled rotors, and the Anderson lattice. We also investigated the effects of the parameters of each system on its localization properties, as well as the way they relate to each other in the two systems.
Chapter 3

The Model

3.1 General Hamiltonian

We will study the system of two coupled kicked rotors, defined by the Hamiltonian

\[
H = \frac{1}{2} \alpha_1 m_1^2 + \frac{1}{2} \alpha_2 m_2^2 + [\lambda_1 \cos \theta_1 + \lambda_2 \cos \theta_2 + \lambda_3 \cos \theta_1 \cos \theta_2 + \lambda_4 \cos (\theta_1 - \theta_2)] \sum_{n=-\infty}^{+\infty} \delta(t - n)
\]

(3.1)

The parameters are all dimensionless. They are defined as follows

\[
\alpha_{1,2} := \frac{h \tau}{I_{1,2}}
\]

(3.2)

Here \( \tau \) is the time span between kicks, and \( I_{1,2} \) are the moments of inertia of rotor one and two. \( \lambda_j \) are dimensionless kicking strengths. The angular momenta \( p_i \) are measured in units of \( I_i/\tau \) \( (i = 1, 2) \).

3.2 Quantum Dynamics

The quantum operators corresponding to \( p_1 \) and \( p_2 \) are \( \hat{p}_1 = -i \partial / \partial \theta_1 \) and \( \hat{p}_2 = -i \partial / \partial \theta_2 \) respectively. Like the one-dimensional case studied in the previous chapter, we have the Floquet-operator

\[
\hat{U} = e^{-i\hat{V}(\theta_1, \theta_2)} e^{-iT(\hat{p}_1, \hat{p}_2)}
\]

(3.3)
with the eigenstates
\[ \psi_{\omega} = e^{-i\omega t} u_{\omega}(\theta_1, \theta_2, t) \] (3.4)
Here \( T(\hat{p}_1, \hat{p}_2) = \frac{1}{2}\alpha_1 \hat{p}_1^2 + \frac{1}{2}\alpha_2 \hat{p}_2^2 \) are the kinetic energy terms in the Hamiltonian. \( \hat{V}(\theta_1, \theta_2) \) denotes the potential energy terms.

Again, let \( u^\pm \) be the values of the state just before and after each kick, and as before, define \( \bar{u} = (u^+ + u^-)/2 \). We take its Fourier transform and obtain
\[ u_m := \langle m_1 m_2 | \bar{u} \rangle \]
\[ u_m = \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta_1 d\theta_2}{2\pi} e^{-i(m_1\theta_1 + m_2\theta_2)} \bar{u}(\theta_1, \theta_2) \] (3.5)
Here \( m := (m_1, m_2) \) is a two dimensional vector. Following the same arguments as in the previous section, we arrive at the tight-binding model on the two-dimensional Anderson lattice
\[ T_m u_m + \sum_{r \neq 0} W_r u_{m+r} = E u_m \] (3.6)
where we have
\[ T_m := \tan\left(\frac{|\omega - E_m|}{2}\right) \]
\[ E_m := \frac{1}{2} \left( \alpha_1 m_1^2 + \alpha_2 m_2^2 \right) \]
\[ W_r := \langle r_1 r_2 | W(\theta_1, \theta_2) \rangle \]
\[ W_r = \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta_1 d\theta_2}{2\pi} e^{-i(r_1\theta_1 + r_2\theta_2)} W(\theta_1, \theta_2) \]
\[ W(\theta_1, \theta_2) = \tan\left[ V(\theta_1, \theta_2)/2 \right] \]
\[ E := -W_0 \] (3.7)
These are the equivalent quantities to those discussed in chapter 2 for the one dimensional lattice.

3.2.1 The Method for Numerical Calculations

The numerical calculations in the quantum regime are performed as follows: The propagator (3.3) is applied in two parts. First we start in the momentum space, and apply the free
motion propagator $e^{-iT(\hat{\theta}_1, \hat{\theta}_2)}$ to the state. This propagator is diagonal in this representation, and performing it is easy. Then we use the Fast Fourier Transform (FFT) algorithm to go to the angle representation. Now we apply the kicking propagator $e^{-i\tilde{V}(\theta_1, \theta_2)}$. This step is done, because in this representation the propagator is once more diagonal. Next step is to perform the inverse Fast Fourier Transform, to get back to the momentum representation, and repeat this process for the next kicking cycle.

The momentum values are of course discrete and set a grid for the numerical realizations. The Fast Fourier Transforms are instances of Discrete Fourier Transforms. The angular values are thus also set on the same grid points by this type of algorithms. The numerical calculations is limited by two types of errors. The first are usual errors due to truncations. There is however the possibility of a second type of systematic error that is accumulative. This type of error occurs once the width of the state reaches the limit of the grid used. The results of discrete Fourier transforms beyond the grid size will then reflect back to the points inside grid and give higher contributions than the true case. This error has to be avoided. Hence the grid size sets a limit on the total simulation time. The simulation can run only until the state size reaches the end points of the grid.

### 3.3 Classical Dynamics

The classical counterpart of the model consists of an initial probability distribution in the 4-dimensional classical phase space, and its dynamical evolution. The distribution is determined by the corresponding initial quantum state. The classical marginal distribution in angles and momenta should be equal to the square of the state function. The distribution is a function of 4 independent degrees of freedom. Instead of numerically evolving it using the corresponding Liouville equations, it is easier, more feasible, and equally valid, to use an ensemble of initial points with the same distribution\(^1\). Each initial point $(\theta_1, \theta_2, p_1, p_2)$ evolves separately on its classical trajectory under point-wise classical mapping. The distribution is approximated at later times from the new values of each member of the ensemble.

Similar to the quantum case, it is convenient to express everything in terms of dimensionless variables. Here also, angular momenta $m_i$ are measured in units of $I_i/\tau$, where $I_i$ are the moments of inertia, and $\tau$ is the period between kicks. This way, the kicking period

\(^1\)The number of points in the ensemble must be large enough for the approximation to be statistically valid.
in the dimensionless version of the model is normalized to one. Recalling the definition of 
the quantum parameters $\alpha_{1,2}$ from (3.2), and assuming $\alpha_1 < \alpha_2$, we define

$$
\begin{align*}
C_{1j} &:= \alpha_1 \lambda_j \\
C_{2j} &:= \frac{\alpha_2}{\alpha_1} \lambda_j \quad \text{for } j = 1, 2, 3, 4
\end{align*}
$$

The factors of $\hbar$ cancel each other, and the parameters $c_{ij}$ remain as the only relevant 
dimensionless dynamical parameters for the classical system. The classical mapping for 
each point $(\theta_1, \theta_2, p_1, p_2)$ of the phase space, becomes

$$
\begin{align*}
{p'_n}^{(1)} & = p_n^{(1)} + c_{11} \sin \theta_n^{(1)} + c_{13} \sin \theta_n^{(1)} \cos \theta_n^{(2)} + c_{14} \sin(\theta_n^{(1)} - \theta_n^{(2)}) \\
{p'_n}^{(2)} & = p_n^{(2)} + c_{22} \sin \theta_n^{(2)} + c_{23} \cos \theta_n^{(1)} \sin \theta_n^{(2)} - c_{24} \sin(\theta_n^{(1)} - \theta_n^{(2)}) \\
{\theta'}_{n+1}^{(1)} & = \theta_n^{(1)} + p_{n+1}^{(1)} \\
{\theta'}_{n+1}^{(2)} & = \theta_n^{(2)} + p_{n+1}^{(2)}
\end{align*}
$$

3.4 Two Special Cases Of the Hamiltonian

In this section, we introduce and analyze the results of two limiting cases of our general 
Hamiltonian. These two cases were already studied by two Doron and Fishman [16], and by 
Adachi, Toda and Ikeda [2] respectively. They are the motivation for the general form of the 
chosen Hamiltonian in this work. On the face of it, the two papers seem to be presenting 
opposite results, with respect to quantum localization, for their respective models and range 
of parameters. They however have studied two different quantities as measures of the growth 
of the state width. In the present study, we have performed numerical calculations using 
the general form of the Hamiltonian, and calculated both of those measures, that we have 
utilized as localization criteria. We have used a variety of parameters between these two 
special cases.

In this section, we reproduce and analyze the relevant results of these two special Hamil­
tonians. This will also serve to verify the calculation methods we have used, by comparing 
the results with the relevant results already reported in the two papers [16] [2].

3.4.1 Special Case 1

The results of the first special case are based on the work of Doron and Fishman [16]. In 
their paper, the form of the kicking potential was chosen as $\cos \theta_1 \cos \theta_2$. In our model this
corresponds to

\[ \lambda_1 = \lambda_2 = \lambda_4 = 0 \]  \hspace{2cm} (3.10)

Other parameters are: \( \alpha_1 = 1, \alpha_2 = \sqrt{2}, \hbar = 1 \). The size of the computational grid (corresponding to the size of the accessible momentum space) is 256 x 256. The initial condition was chosen sharply centered around the origin in momentum space. The results for various values of \( \lambda_3 \) were calculated. Their momentum probability distribution was plotted in logarithmic scale, and it was verified that they all fall exponentially away from the center. This is an indication of Anderson localization. The relevant measure used is the localization length \( \zeta \) defined as

\[ u_m(n) \sim \exp\left(-\frac{|m-n|}{\zeta}\right) \]  \hspace{2cm} (3.11)

The characteristic criterion of localization was considered to be as follows: the state function in momentum representation\(^2\), retains its over-all shape of exponential fall, away from the center of localization. The localization length remains more or less constant and does not diverge. Figure 3.1 shows one such state for \( \lambda_3 = 3.5 \). It refers to FIG.1 p.869 in [16]. Figure 3.2 plots the numerical results of the localization length of the momentum distribution for values of \( \lambda_3 = \{0.5,1.0,1.5,...,4\} \) in semi-log scale. The best fit confirms exponential growth of localization length with \( \lambda_3 \), the only non-zero coupling strength here. (compare with FIG.2 p.869 in [16]) This was seen as verification of the result of scaling theory of localization for this particular model.

\(^2\)or its square- the momentum probability distribution.
Figure 3.1: Momentum probability distribution along the diagonal (in momentum space ie. $m_1 = m_2$) along the axis of rotor 1 ($m_2 = 0$), along the axis of rotor 2 (($m_1 = 0$). $\alpha_1 = 1$, $\alpha_2 = \sqrt{2}$, $\lambda_1 = 0.0$, $\lambda_2 = 0.0$, $\lambda_3 = 3.5$, $\lambda_4 = 0.0$. At $t=1024$ steps.
Figure 3.2: Localization length $\zeta$ versus coupling strength $\lambda_3$ in Logarithmic scale. Numerical results are in circles and the continuous line is the best fit for exponential growth. $\alpha_1 = 1, \alpha_2 = \sqrt{2}, \lambda_1 = \lambda_2 = \lambda_4 = 0.0$
3.4.2 Special Case 2

The second special case is based on the work of Adachi, Toda and Ikeda[2]. The form of potential in this case, in term of our parameters, corresponds to

$$\lambda_3 = 0.0$$  \hspace{1cm} \text{(3.12)}

Other parameters are: \(J_1 = J_2 = 1\), \(\tau = 1\), \(\alpha_1 = \alpha_2 = h\) and \(h = \frac{41}{512} \times 2\pi\). The size of the computational grid (corresponding to the size of the accessible momentum space) is chosen smaller than \(41 \times 2\pi\). The context of the paper is not localization. The main focus is on investigating the possibility of recovering chaotic classical behavior from quantum systems with degrees of freedom higher than one, and the coupled kicked rotor is chosen as a model in that respect. The result that is relevant to the present work concerns calculating the variance of the momentum probability distribution\(^3\)

$$\Delta P := \left\langle (P - \langle P \rangle)^2 \right\rangle$$  \hspace{1cm} \text{(3.13)}

This changing pattern of this quantity, as a measure of the width of the momentum state, is observed with time. For various values of \(\lambda_4\), the calculation was repeated, and for higher values of \(\lambda_4\), linear growth of this quantity was observed, that is an indication of diffusion of the momentum probability distribution, and could be regarded as an indication of classical-like chaotic behavior. What matters here is that, if this is a essential feature of this system, and not a temporary behavior, it would rule out the possibility of quantum localization. The results of our simulation, for the same range of parameters, can be seen in Figure 3.3. In Figure 3.3 we have plotted the momentum standard deviation (instead of the variance) versus the square root of time \(\sqrt{t}\). We will use standard deviation throughout the present thesis. Our results are consistent with those reported in their original paper. [Compare with FIG1. (a) and (b), p 660 [2]]

\(^3\)In the paper [2], the quantity is defined as \(\langle \psi_0 | (\hat{p}_t - \hat{p}_0)^2 | \psi_0 \rangle\). \(\psi_0\) is the initial wave function and \(\hat{p}_t\) is the momentum operator at time \(t\) (in Heisenberg picture). This is strictly speaking not always equal to the variance, since the operator at two different times need not commute. However based on the rest of the article and their description of their numerical method, it is very probable that this is what was meant. In any case we use the variance of the momentum distribution in this thesis.
Figure 3.3: Standard variation of momentum for rotor 1. \( \alpha_1 = \alpha_2 = \frac{41}{512} \times 2\pi \), \( \lambda_1 = \lambda_2 = 0.97 \), \( \lambda_3 = 0.00 \). 

- \( \lambda_4 = 0.01 \), \( \lambda_4 = 0.05 \), \( \lambda_4 = 0.10 \), \( \lambda_4 = 0.50 \), \( \lambda_4 = 1.00 \), \( \lambda_4 = 2.00 \).
The choice of $\alpha = 41/512 \times 2\pi$ in the paper needs further discussion. As mentioned in chapter 2, in a truly random Anderson lattice, the only situation were localization has been analytically shown to exist, the choice of $\alpha$ commensurate with $2\pi$ constitutes an important special case\(^4\). In this case, the lattice potential is no longer random, but in fact it becomes periodic. The corresponding wave functions therefore would be like Bloch-functions, and therefore extended. In such a case no quantum localization takes place. If the analogy with the kicked rotor can be extended in such situations, the rotor counterpart of this situation is the phenomenon of quantum resonance. However, as was seen in chapter 2, the mapping that takes the kicked rotor model to that of the Anderson lattice, does not generate truly independent and random on site potentials. The existence of such correlations can create complications, and it is no longer obvious that in such cases localization can still not take place, depending among other things, on the form of the kicking potential and the range of kicking and coupling strengths. Since the focus of the authors of this paper was to study possible chaotic behavior in two or more quantum coupled kicked rotors, it makes sense that they should choose $\alpha$ this way to best avoid possible localizations (though as was said, due to correlations this can't be known with certainty.) Since we are interested in possibility of localization, we avoid this complication. In the rest of this thesis, $\alpha = 1$ has been chosen. However this is an interesting topic of research for future extension of this type of research.

\(^4\) $\alpha$ is a dimensionless quantity carrying information regarding kicking period, moment of inertia and the value of $\hbar$. 
Chapter 4

Numerical Results And Calculations

In this chapter we present the main calculations and results for the general model. These include many numerical simulations in a wide range of parameters, both for the classical and quantum evolution. We present further calculations to derive meaningful conclusions regarding the main question of this thesis regarding quantum localization for the general Hamiltonian.

4.1 Parameters

Recall the potential terms in the Hamiltonian

\[ V(\theta_1, \theta_2) = \lambda_1 \cos \theta_1 + \lambda_2 \cos \theta_2 + \lambda_3 \cos \theta_1 \cos \theta_2 + \lambda_4 \cos (\theta_1 - \theta_2) \]  

(4.1)

At this stage, it is useful to distinguish between the first two parameters \( \lambda_1 \) and \( \lambda_2 \), which are measures of the strength of single-rotor periodic kicks. We call them the kicking parameters hence forth. The last two parameters, \( \lambda_3 \) and \( \lambda_4 \) denote the strength of the periodic couplings, and we refer to them as the coupling parameters.

As discussed in chapter 3.2, Doron and Fishman [16] only include the coupling parameter \( \lambda_3 \), ranging\(^1\) from 0.0 to 4.5 in their numerical calculations. We choose \( \lambda_3 \) to vary from 0.0

\(^1\)Refer to FIG.2 in [16] and its discussion. Given the size of momentum space in their simulations, wavefunctions resulting from higher values of the coupling parameter \( k \), corresponding to \( \lambda_3 \), contained systematic errors.
to 3.0, in increments of 0.5, in the present work. The range of $\lambda_4$ is also chosen to be the same. This range lies within the range chosen by Adachi et. al [2].

The choice of kicking parameters, in all the results of Doron and Fishman, was $\lambda_1, \lambda_2 = 0$. Ikeda et. al chose the following values $\lambda_1 = \lambda_2 = 0.97$, $\lambda_1 = \lambda_2 = 3.0$ and $\lambda_1 = 3.0$, $\lambda_2 = 0.8$. These values proved to be too large for the scope of practical simulations, given the limitation on the size of the grid$^2$ and the time span of the computations. Indeed, one possible resolution why no localization was detected by Ikeda et. al still is that their size of the grid and computation time was too small for the values they had chosen. Therefore, it makes sense to use smaller values for kicking parameters $\lambda_3$ and $\lambda_4$, and to go to higher values of these parameters in case no diffusion or regimes of non-localization were detected for the smaller values. This is even more relevant in our case, due to two different couplings in our model. We have chosen the following values in this work: $\lambda_1 = \lambda_2 = 0.0$, $\lambda_1 = 3.0$, $\lambda_2 = 0.25$ and $\lambda_1 = 3.0$, $\lambda_2 = 0.5$. As will be discussed in section 4.2, these choices have turned out to be large enough to give non-localized results for certain values of the coupling strengths, as well as regimes of localization.

The kinetic energy terms contain two dimensionless parameters as well, namely $\alpha_1$ and $\alpha_2$.

$$T(m_1, m_2) = \frac{1}{2} \alpha_1 m_1^2 + \frac{1}{2} \alpha_2 m_2^2$$

(4.2)

It was discussed in sections 3.2 and 3.3 that the choice of $\alpha_1, \alpha_2$ in the report of Ikeda et al.[2], as a ratio of $2\pi$, can lead to periodic boundary conditions in the corresponding Anderson lattice and to special types of wave-functions over the lattice. As was noted, theoretically speaking, for the truly random and uncorrelated lattices of the Anderson model, exponential localization is expected for values of $\alpha$ that are incommensurate with $2\pi$. Doron and Fishman [16] used such an $\alpha$, ie. $\alpha = 1$, to investigate localization. To avoid the complications of a commensurate $\alpha$, and in line with the premises of Doron and Fishman, we have chosen the value of $\alpha_1 = \alpha_2 = 1$ in the main computations of this chapter.

In order to make things easier and to avoid unnecessary complications, we'd like the initial ensemble corresponding to the wave-function, to fall completely in either the regular or chaotic regions of the classical phase space. To determine what region in phase space the initial states fall in, Lyapunov exponents were calculated for each set of kicking parameters. Based on the above results, the corresponding initial quantum state was chosen to

$^2$That is, the grid comprised of the values of momentum.
be centered at angle $\theta_0 = 0$ and $\hbar m_0 = 0$ for each rotor, and have a Gaussian distribution in momentum space. We choose it to be a minimum uncertainty state. Furthermore we choose the width in momentum to be $\hbar \Delta m$, with $\Delta m$ a constant dimensionless number. So we have $\hbar \Delta m \Delta \theta_0 = \hbar/2$. The width of the distribution in angle is thus independent of $\hbar$.

Table 4.1 gives the initial parameters:

Therefore, the initial state has the following form in angle $\theta$ representation, as the product

$$
\psi(\theta_1, \theta_2) = \psi(\theta_1) \psi(\theta_2)
$$

$$
\psi(\theta) = \sum_{m} a_m e^{i m (\theta - \theta_0)}
$$

$$
a_m = e^{-\left(\frac{m-m_0}{2\Delta m}\right)^2}
$$

This choice for the initial state of the system was found to have positive Lyapunov exponents for all values of non-zero coupling strengths. Only for the case of $\lambda_3 = \lambda_4 = 0$, turning the system to that of two independent single kicked rotors was the system in a regular region, not surprisingly.

Finally, size of the grid was chosen to be $2^{11}$ grid points, ranging from -1023 to 1024. This grid size was 4 times larger than those used in Doron and Fishman [16] and Ikeda et al. [2]. The simulation was continued for longer times, without the support of the state function reaching the end points of the grid, which would have generated unavoidable reflections of the values falling outside the grid back onto those inside. The time span for each calculation was 30000 kicking steps, compared to 1024 time steps used by Doron and Fishman [16], and 2000 time steps used by Ikeda et al. [2]. It was verified that for this time span, the state functions for all the runs remained within the grid size, and the systematic error mentioned above was avoided.
CHAPTER 4. NUMERICAL RESULTS AND CALCULATIONS

Figure 4.1: Initial Guassian momentum probability distribution for each of the rotors. $m_0 = 0.0$, $\theta_0 = 0.0$, $\Delta m_0 = 1.25786$, $\Delta \theta_0 = 0.0$. 
**4.2 Classical Results**

We are looking for the equivalent of Anderson localization which is a completely quantum phenomenon. It is important to distinguish this from situations where the classical system remains in a region of phase space depending on its initial energy and does not cover the corners of the available phase space. Therefore it is important to run the classical system with the same kicking and coupling parameters as the quantum system, for reference. If the classical counterpart covers the whole available phase space and its momentum distribution width, that is, the standard deviation of the momentum distribution, grows to the grid size of the simulation, while the quantum system essentially remains localized over a proper sub-region of the momentum axis, we can be sure that an instance of quantum localization has been realized\(^3\).

Another reason why the results of the classical evolution are needed is for the cases that both the classical and the quantum momentum state width show diffusive growth with time. Having the classical diffusion results makes comparison between the diffusion coefficient of the classical and quantum growth possible. This is important, because if the two are the same over a relatively long span of time, then this suggests nearly classical behavior of the quantum system, and is strong evidence against quantum localization of the system. If they differ, this result can’t, by itself, be used against either case, but it at least shows decidedly quantum behavior of the system and leaves room for a possible localization in later times.

For the classical evolution, an ensemble of 1,000,000 initial states of points in phase space is created, such that each member has the same distribution in angles and momenta as the distribution obtained from the quantum initial state as in 4.1. The initial quantum state is defined as a Gaussian in the momentum representation. To determine the shape of the probability distribution for each angle, we have to perform a Fourier transform to obtain the distribution for each angle, and then calculate the cumulative probability distribution there and use inverse interpolation on it to get the right distribution for each member of the ensemble.

The results of the calculation show similar behavior over the entire range of parameters

---

\(^3\)Note also that \(\alpha\) can also be regarded as a dimensionless measure of quantum behavior of the system as a dimensionless \(\hbar\), that is, as the ratio of the system action size \(I\) is the moment of inertia, and \(\tau\) the kicking period. \(I/\tau \sim h\). Since Anderson-like localization is essentially a quantum feature, \(\alpha = 1\) is relatively large. Thus, the system is far from its classical limit, and the quantum aspects of the system’s evolution are expected to be discernible without the need to wait for long times.
for the sets $\lambda_1 = \lambda_2 = 0.25$ and $\lambda_1 = \lambda_2 = 0.50$. Except for $\lambda_3 = \lambda_4 = 0.0$, the uncoupled cases, the growth of the momentum state width was diffusive, and the state vector covered the entire grid before the end of the simulation. Figures 4.2 and 4.3 show the change of the momentum standard deviation versus $\sqrt{t}$, the square root of time, for typical values of the coupling parameters.

Figure 4.2: standard deviation of momentum for classical rotor. a $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. b $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 1.0$, $\lambda_4 = 2.0$. c $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 1.0$, $\lambda_4 = 0.0$.

\(^4\)that is, momentum standard deviation.
Figure 4.3: standard deviation of momentum for classical rotor. a $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 3.0, \lambda_4 = 3.0$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 1.0, \lambda_4 = 2.0$. c $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 1.0, \lambda_4 = 0.0$.

The linear shape of the graph denotes diffusion. In case of diffusion we have

$$\langle p \rangle^2 = D_v t$$  \hspace{1cm} (4.4)

Therefore the slope of the graphs of standard deviation versus $\sqrt{t}$, we name it $D_s$ also entails the same information.

$$D_s = \sqrt{D_v}$$ \hspace{1cm} (4.5)

$D_s$ shows clear dependence on the coupling parameters $\lambda_3$ and $\lambda_4$, and increases as these parameters are increased. $D_s$ has the same behavior with the kicking parameters $\lambda_3$ and $\lambda_4$ too, but here the dependence seems to be weaker.
CHAPTER 4. NUMERICAL RESULTS AND CALCULATIONS

We investigate this dependence of $D_v$ on the kicking and coupling parameters further in the next section\(^5\).

4.2.1 Classical Diffusion Coefficient

There are analytical results for the single kicked rotor with one kicking parameter $K$, i.e. the standard map.

\[
\begin{align*}
    p_{n+1} &= p_n + K \sin \theta_n \\
    \theta_{n+1} &= \theta_n + p_{n+1}
\end{align*}
\]  

(4.6)

that relate the ensuing diffusion coefficient to this parameter[31]. In particular, one can get a first approximation to the functional form of $D_v$ by assuming a uniform probability distribution for the angle, and integrating $(p - \langle p \rangle)$ over this uniform distribution. We obtain

\[ D_v \approx \frac{K^2}{4} \]  

(4.7)

or equivalently

\[ D_v \approx \frac{K}{2} \]  

(4.8)

This approximation is referred to as the first order\(^6\) approximation. Using the same assumption we obtain the following form for the first order approximation for our model for rotors 1 and 2 respectively

\[
\begin{align*}
    D_{v(1)} &= \frac{(\lambda_3 + \lambda_4)^2}{4} + \frac{\lambda_3^2}{4} + \frac{\lambda_4^2}{2} \\
    D_{v(2)} &= \frac{(\lambda_3 + \lambda_4)^2}{4} + \frac{\lambda_3^2}{4} + \frac{\lambda_4^2}{2}
\end{align*}
\]  

(4.9)

It is not obvious that the assumption of a uniform probability distribution for $\theta_{1,2}$ is valid and that it does indeed give the first order approximation. This result can be recovered using a rigorous analysis based on the Fourier transforms. The Fourier method can also give higher order corrections for the standard mapping using Bessel functions. However in our model, since there are more than one kicking and coupling parameter, this method can’t yield a closed form for higher corrections. The reason for this, as well as more detailed

\(^5\)or equivalently the dependence of $D_v$ on the parameters.

\(^6\)or sometimes quasi-linear.
explanations of the simplest approximation, and the more sophisticated Fourier method is given in Appendix B. The result (4.9) is derived there explicitly.

At this stage, we give numerical verification that the approximation (4.9) is indeed a good approximation to the actual diffusion coefficients. \( D_s^{(1,2)} \) is

\[
D_s^{(1)} = \sqrt{\frac{(\lambda_3 + \lambda_4)^2}{4} + \frac{\lambda_4^2}{4} + \frac{\lambda_1^2}{2}}
\]

\[
D_s^{(2)} = \sqrt{\frac{(\lambda_3 + \lambda_4)^2}{4} + \frac{\lambda_3^2}{4} + \frac{\lambda_2^2}{2}}
\]

(4.10)

we can rewrite these as follows

\[
D_s^{(1)} = \frac{\lambda_3 + \lambda_4}{2} \left[ 1 + \frac{\lambda_3^2 + 2\lambda_3 \lambda_4}{(\lambda_3 + \lambda_4)^2} \right]^{1/2}
\]

\[
D_s^{(2)} = \frac{\lambda_3 + \lambda_4}{2} \left[ 1 + \frac{\lambda_3^2 + 2\lambda_3 \lambda_4}{(\lambda_3 + \lambda_4)^2} \right]^{1/2}
\]

(4.11)

If we plot the \( D_s^{(1,2)} \) versus axes \( \lambda_3 \) and \( \lambda_4 \), for fixed values of kicking strengths \( \lambda_1 \) and \( \lambda_2 \), the graph would be approximately planar. The correction terms in the brackets on the r.h.s of (4.11) are not negligible, but they remain small. Figures 4.6-4.7 show this result for the case \( \lambda_1 = \lambda_2 = 0.5 \). We have done the numerical simulation of the classical evolution for time span of 1000 time steps. We have extended the range of parameters for \( \lambda_3 \) and \( \lambda_4 \) to be in the interval [0, 30] to observe the validity of the approximation over a long range. This way the oscillation of the numerical results around the first order approximations can be clearly observed. The first order approximation is calculated using (4.10).
Figure 4.4: Numerical results for the diffusion coefficient $D_s^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. 
Figure 4.5: First order approximation (4.10) to the diffusion coefficient $D_{10}^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. 

$\Lambda_1$ $\Lambda_2$

$\Lambda_3$ $\Lambda_4$
Figure 4.6: Diffusion coefficient $D_1^{(1)}$ together with the first order approximation $D_{30}^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. 
Figure 4.7: Diffusion coefficient $D^{(1)}_6$ together with the first order approximation $D^{(1)}_{x0}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. Different perspectives of the same set of parameters.
Figure 4.8: Diffusion coefficient $D_5^{(1)}$ together with the first order approximation $D_{s0}^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. Different perspectives of the same set of parameters.
Figure 4.9: Contour map of the diffusion coefficient $D_s^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$.

Figure 4.9 and Figure 4.10 show the contour map of the numerical and quasi-linear results respectively. The numerical results oscillate around the first order approximation, and they approach it as the coupling parameters are increased. Figure 4.11 shows the ratio of the numerical results to the quasi-linear approximation. This graph is the equivalent for our model of that of the standard mapping. (see graph 5.13, p335 [31])
Figure 4.10: Contour map of the quasi-linear $D_{s0}^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. 
Figure 4.11: Ratio of $D_s^{(1)}/D_{s0}^{(1)}$ versus coupling parameters $\lambda_3$ and $\lambda_4$. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. 
4.3 Quantum Results

At this stage we present the numerical results of the quantum system. Our method in this thesis has been to fix the values of \( \lambda_1 \) and \( \lambda_2 \) for each set of runs and to vary \( \lambda_3 \) and \( \lambda_4 \) in each set. Table 4.2 shows the arrangement of results in one such group with \( \lambda_1 \) and \( \lambda_2 \) already fixed. Thus we have three sets of numerical results each containing 49 cases. For each case, the following are computed:

1. The variance of the momentum distribution of each rotor for every kicking step.
2. The state vector and its square in momentum representation over the grid range\(^7\) a) along the axis \( m_1 \) (i.e. \( m_2 = 0 \)) and b) along the diagonal \( m_1 = m_2 \). The results are stored for every 1000th kicking step.

The two are not completely equivalent. It is possible, in principle, to have non-usual forms of distribution with a long tail and a central peak, where the standard deviation is dominated by the central peak only, and not by the long tail that extends at either side.

Regarding the dependence of the momentum standard deviation\(^8\), the results can be categorized into three distinct regions. i) A localized regime where the momentum width is saturated, ii) the regime of almost diffusive behavior, and iii) the intermediate regime between the two. Table 4.2 shows these regimes for the two sets of runs \( \lambda_1 = \lambda_2 = 0.25 \) and \( \lambda_1 = \lambda_2 = 0.5 \). The results of the set \( \lambda_1 = \lambda_2 = 0.0 \) were all localized.

\(^7\)Since \( \sigma_1 = \sigma_2 \) and the initial state is symmetrical over the axes, the two rotors behave similarly, so the results along the axis \( m_2 \) are redundant.

\(^8\)Which is a measure of the state width.
CHAPTER 4. NUMERICAL RESULTS AND CALCULATIONS

Table 4.2: The time behavior of each group with fixed kicking parameters $\lambda_1$ and $\lambda_2$. D: Almost-diffusive regime. I: Intermediate Regime. L: Localized Regime. Above $\lambda_1 = \lambda_2 = 0.50$. Below $\lambda_1 = \lambda_2 = 0.25$.

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In each group with fixed values of $\lambda_1$ and $\lambda_2$, the behavior for $\lambda_3$ and $\lambda_4$ small is clearly indicative of localization. Figure 4.12 shows a typical case. It shows the momentum standard deviation versus square root of time for a small value of $\lambda_3$ and $\lambda_4$, for each set $\lambda_1 = \lambda_2 = 0.25$ and $\lambda_1 = \lambda_2 = 0.5$ for comparison. These values correspond to the upper right corner of the 4.2.
Figure 4.12: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 2.5, \lambda_4 = 0.5$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 2.5, \lambda_4 = 0.5$. 
The second category shows erratic behavior that ranges from small oscillations, large oscillations, and a combination of oscillations and over all slowly saturating increase in the over all envelope of the erratic changes in state width. They correspond to the middle section of the table 4.2. The left part of this mid section clearly show saturation in the overall momentum state width, while as we move to the right of the table, the width of the system behaves less and less as one with a saturated envelope and tends to increase until the end of the simulation time span. Figures 4.13-4.16 are typical graph of this category, for each set $\lambda_1 = \lambda_2 = 0.25$ and $\lambda_1 = \lambda_2 = 0.5$ for comparison.

Figure 4.13: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 2.5, \lambda_4 = 1.5$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 2.5, \lambda_4 = 1.5$. 
Figure 4.14: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 1.5, \lambda_4 = 2.5$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 1.5, \lambda_4 = 2.5$. 
Figure 4.15: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 1.0, \lambda_4 = 2.5$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 1.0, \lambda_4 = 2.5$. 
Figure 4.16: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 2.5, \lambda_4 = 1.0$. b $\lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 1.0, \lambda_4 = 2.0$. 
The last category, corresponding to the lower right corner of the table 4.2 of large $\lambda_3$ and $\lambda_4$, tend to increase almost in a monotone fashion until the end of the simulation time, with little or no oscillations. In other words they grow almost diffusively, with the diffusion coefficient slowly decreasing with time. Figures 4.17-4.18 are typical graphs from this category, for each set $\lambda_1 = \lambda_2 = 0.25$, $\lambda_1 = \lambda_2 = 0.25$ and $\lambda_1 = \lambda_2 = 0.0$ for comparison.

Note that the results for $\lambda_1 = \lambda_2 = 0.0$, the momentum width is clearly localized even for the largest values of $\lambda_3$ and $\lambda_4$. That is the reason why the results of this group were not included for other ranges of $\lambda_3$ and $\lambda_4$.

Figure 4.17: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 2.0$, $\lambda_4 = 3.0$. b $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 2.0$, $\lambda_4 = 3.0$. c $\lambda_1 = 0.0$, $\lambda_2 = 0.0$, $\lambda_3 = 2.0$, $\lambda_4 = 3.0$. 
Figure 4.18: standard deviation of momentum for Quantum rotor. a $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 2.0$. b $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 3.0$, $\lambda_4 = 2.0$. c $\lambda_1 = 0.0$, $\lambda_2 = 0.0$, $\lambda_3 = 3.0$, $\lambda_4 = 2.0$. 
The behavior of the momentum probability distribution at different instances of time shows a similar pattern. The results for small $\lambda_3$ and $\lambda_4$ have a clear exponential fall, whose slope and overall envelope remain the same at different snapshots $t$ different times. While those falling in the second category have exponential drop offs whose envelope widens with time, and whose slope decreases at future time steps. Eventually as the coupling parameters $\lambda_3$ and $\lambda_4$ are increased, the shape rests less and less within an exponentially decaying envelope. Finally for large $\lambda_3$ and $\lambda_4$, the shape of the momentum distribution no longer exhibits an exponential drop off, at least within the time span of the simulation. Figures (4.19-4.22) show a typical example of each category. Each figure contains a super-position of three snapshots taken at $t = 1000$, $t = 5000$ and $t = 30000$ of the momentum probability distribution along the axis $m_1$ of the first rotor ($m_2 = 0$).

Finally 4.23 shows a comparison between the classical and quantum growth of the momentum width for $\lambda_1 = \lambda_2 = 0.5$. Figure 4.24 shows this for $\lambda_1 = \lambda_2 = 0.25$ coupling parameters residing in the quasi-diffusive parameter range, as a typical example of this behavior. The two grow in an almost identical fashion for a time span that is short compared to the simulation time. (At the very early times, the quantum actually grows faster!) Although the quantum system continues in an almost diffusive fashion for much longer times, and until the end of the simulation, it falls short of the classical diffusion at this early stage. Therefore, even in this strong regime of quasi-diffusive growth of momentum state width, it is not behaving classically. In the next section we analyze this almost diffusive behavior with a gradual fall of the diffusion coefficient in more detail.
Figure 4.19: Momentum probability distribution along the axis for rotor 1. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.0$, $\lambda_4 = 3.0$. a at $t=1000$ steps, b at $t=5000$ steps, c at $t=30000$ steps.
Figure 4.20: Momentum probability distribution along the axis for rotor 1. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 2.5$, $\lambda_4 = 1.0$. a at $t=1000$ steps, b at $t=5000$ steps, c at $t=30000$ steps.
Figure 4.21: Momentum probability distribution along the axis for rotor 1. $\lambda_1 = 0.05$, $\lambda_2 = 0.5$, $\lambda_3 = 1.5$, $\lambda_4 = 3.0$. a at $t=1000$ steps, b at $t=5000$ steps, c at $t=30000$ steps.
Figure 4.22: Momentum probability distribution along the axis for rotor 1. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. a at $t=1000$ steps, b at $t=5000$ steps, c at $t=30000$ steps.
Figure 4.23: **Above.** a standard deviation of momentum for classical rotor during early time steps. **b** standard deviation in momentum for quantum rotor during early time steps. \( \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 3.0, \lambda_4 = 3.0. **Below.** Difference between the standard deviation of momentum for classical rotor and the quantum rotor during early time steps. \( \lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 3.0, \lambda_4 = 3.0. \)
Figure 4.24: Above. a standard deviation of momentum for classical rotor during early time steps. b standard deviation in momentum for quantum rotor during early time steps. $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. Below. Difference between the standard deviation of momentum for classical rotor and the quantum rotor during early time steps. $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. 
4.3.1 Scaling Results

It was observed in the last section, that for large values of coupling parameters $\lambda_3$ and $\lambda_4$, in each set of fixed kicking parameter $\lambda_1$ and $\lambda_2$, the momentum state width grows in an almost linear fashion with $\sqrt{t}$, with the slope of the line decreasing very slowly in the time span of the numerical calculations. Therefore, at least for this category, the question of quantum localization remains still unanswered.

In order to test for localization, we used the results of the scaling theory for Anderson localization. The main ideas behind scaling theory, as it applies to Anderson localization, and the way it relates to our kicked rotor model is presented in Appendix A.

Our approach in this section is as follows: We assume localization and follow the scaling theory of localization, which is expected to hold in case of localization, to see whether we reach results that would contradict the assumption. If we do not encounter such contradictions, in other words if the state width in momentum in the almost diffusive regime of large coupling parameters can be scaled according to this theory, then that can be seen as a strong evidence in favor of localization in this regime. Since this category contains the largest values of $\lambda_3$ and $\lambda_4$, for each run with fixed kicking parameter $\lambda_1$ and $\lambda_2$, the rest of the instances investigated in this thesis which have smaller values of $\lambda_3$ and $\lambda_4$ are also expected to be ultimately localized. Furthermore, this scaling can be used to estimate the localization length, and the time it would take for the momentum state width to exhibit saturation. These can be compared with the grid size and the time span of the simulations to see whether the assumed localization could apply within the estimated time span as well as the size of the momentum range.

Based on the discussion in Appendix A, we use the average displacement in momentum after the first kick for a sharp initial state at the origin, $l$, as the scaling parameter $^9(A.14)$.

\[
l^2 = \sum_{r_1,r_2=-\infty}^{\infty} (r_1^2 + r_2^2) |U_{0,r}|^2 \tag{4.12}
\]

We know that the diffusion coefficient $D_n$, changes with time$^{10}$. In line with scaling theory, we assume $D_n$ at any time to be primarily a function of only $L$, the momentum state width at that time. In other words, we assume that it is only through the time dependence of the

$^9$Here $U_{0,r}$ is the propagator matrix element between pairs $(r_1, r_2)$ and the origin in momentum space.

$^{10}$corresponding to the linear slope of the change of momentum variance with time.
momentum state width \( L \), that the diffusion coefficient changes with time. Since we are in the almost diffusive regime, we can use the perturbation theory to obtain the following equation for the diffusion coefficient, at time step \( t_L \) in our simulations (see A.8)

\[
\frac{d \log D_v}{d \log L} = -\frac{a}{D_v}
\]  

(4.13)

for a specific value of \( a \), not yet known. If we further assume, for simplicity, that \( D_v \) changes at discrete times steps \( t_n \) and corresponding momentum state width \( L_n \) only, and that in the time interval between these points,

\[
\Delta T_n := t_{n+1} - t_n
\]  

(4.14)

the state size grows linearly with the fixed value of \( D_n \) at the beginning of the interval, then we have

\[
D_n = D_0 - c \log \left\lfloor \frac{L_n}{l} \right\rfloor
\]  

(4.15)

Here \( D_0 \) is the value of diffusion coefficient during the first kicking step. \( c \) is an unknown constant. Since \( D_n \Delta T_n = (L_{n+1} - L_n)^2 \), we obtain the following recursion relationship

\[
L_{n+1} = L_n + \sqrt{\Delta T_n} \times \sqrt{\left( f l - \frac{f}{b} \log \left\lfloor \frac{L_n}{l} \right\rfloor \right)}
\]  

(4.16)

Although the (4.16) is valid in for the cases where \( \Delta T_n \) change with \( n \), in the actual calculations we set them all equal to

\[
\Delta T_n =: \Delta T
\]  

(4.17)

for simplicity. We use \( \Lambda \), the so-called saturation length, to denote values of the momentum standard deviation where the diffusion coefficient \(^{12}\) approaches zero. \( \Lambda \) scales exponentially with \( l \), the scaling parameter (4.12)

\[
\Lambda = l e^{bl}
\]  

(4.18)

\(^{11}\)Which as far as our state function at that time is concerned can be taken as the length of the system at that time. The rest of the grid, not yet accessed by the state function, is expected to be irrelevant as far as the state function at that time is concerned.

\(^{12}\)Or the average diffusion coefficient over time, when the envelope of the momentum width stops growing with time.
This is easily seen by setting the right hand side of (4.16) to zero. The constants of (4.15) have been written in the form (4.16), as \( f \) and \( b \), to get directly to the form of the scaling dependence of \( \Lambda \) on \( l \) (4.18).

We calculated \( l \) numerically for each set of kicking and coupling parameters, by setting the initial condition to be sharp over the origin \( (m_1 = m_2 = 0) \), and running the numerical simulation for one kicking step. The results are represented in table 4.3. To get an approximation on \( b \) and \( f \) we divided the entire simulation into equal intervals \( \Delta T \) of small size, namely 5, 10 or 15 kicking steps. We used the equation (4.16) to fit a curve to the curve of the momentum standard deviation versus \( \sqrt{t} \), for a range of parameters \( b \) and \( f \). We found the best fit, and consequently the approximate values of \( b \) and \( f \), by minimizing the area under the square of the difference between the fitting curve and the standard deviation. The results of the best fit for \( b \) and \( f \) are shown in table 4.3. For the almost diffusive regime, the lower right corner of the tables, the fitted value for \( b \) is about the same for the range of coupling parameters \( \lambda_3 \) and \( \lambda_4 \). This is what we would have expected if the scaling theory, and hence localization, was true. The results were therefore consistent with the principles of scaling theory. It suggests that the scaling theory of localization does indeed give a consistent and reasonable result. Figures 4.25-4.27, show the scaling fit for some of the results in the range of almost-linear growth of momentum width.
Table 4.3: Numerical values of \( I \), \( b \) and \( f \) [See (4.16)] for each run with \( \lambda_1 = \lambda_2 = 0.50 \)

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</table>

\( f \)
Figure 4.25: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. 
Figure 4.26: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 2.5$, $\lambda_4 = 3.0$. 
Figure 4.27: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 2.0$. 
For the regime of non-linear behavior with oscillations, i.e., the intermediate section of table 4.2, as we move towards the coupling parameters of the localized regime, the value of $b$ decreases, but still remains almost constant along the diagonal of table 4.2.

Figures 4.28-4.31 show the scaling fit for some of the results in this range. Notice that the fit to the graph is no longer good. So the best approximation for $b$ differed from those of the almost-diffusive regime as well. This is not unexpected. The equation (4.13) is based on perturbation, and is presumed to be valid only in regimes near the complete diffusion limiting point. From the graph of the curves in this section of coupling parameters, it is already clear that the behavior of momentum state width is not nearly linear at all.

Figure 4.28: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 2.0$, $\lambda_4 = 2.0$. 
Figure 4.29: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 1.0$, $\lambda_4 = 2.0$. 
Figure 4.30: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T_n} = 10$ [see (4.14)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 1.0$. 
Figure 4.31: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 1.5$. 
Table 4.4: Numerical values of $l$, $b$ and $f$ [See (4.16)] for each run with $\lambda_1 = \lambda_2 = 0.25$

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<td>$\lambda_3$</td>
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<tr>
<td>$\lambda_3$</td>
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Note also that the for cases where the system width is saturated, the fit does follow the overall envelope of the momentum width in later times. This is not surprising either, since the system is already saturated. So it is expected that any viable approximating function should follow the saturation behavior.

Table 4.4 Shows the results for the near diffusive and intermediate runs, for the case of kicking parameters $\lambda_1 = \lambda_2 = 0.25$. Since this set of results are further away from the diffusive regime, than the previous set, the best fit results are not as good. Figures 4.32 - 4.35 show the best fit to numerical data for some of the results in this set.
Figure 4.32: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. 

\[\text{standard deviation in momentum of first rotor}
\text{Theoretical approximation}\]
Figure 4.33: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ (see (4.17)). $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 2.5$, $\lambda_4 = 2.5$.  

![Standard deviation in momentum of first rotor and theoretical approximation](image)
Figure 4.34: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 2.5$, $\lambda_4 = 2.0$. 
Figure 4.35: Standard deviation of momentum for the quantum rotor and the best fit based on scaling theory for $\sqrt{\Delta T} = 10$ [see (4.17)]. $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 1.5$, $\lambda_4 = 3.0$. 
Table 4.5: Numerical values of $\Lambda$ [See (4.18)] for each run. Above. $\lambda_1 = \lambda_2 = 0.50$. Below. $\lambda_1 = \lambda_2 = 0.25$. Compare with Table 4.2.

$\begin{array}{cccccc}
\lambda_4 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 \\
\lambda_3 \\
1.0 & & & & & 62 \\
1.5 & & & 50 & 182 & \\
2.0 & & 41 & 144 & 400 & \\
2.5 & 35 & 118 & 318 & 601 & \\
3.0 & 31 & 100 & 262 & 487 & 905 \\
\end{array}$

$\begin{array}{cc}
\lambda_4 & 2.5 & 3.0 \\
\lambda_3 \\
1.5 & - & 56 \\
2.0 & 35 & 112 \\
2.5 & 94 & 240 \\
3.0 & 130 & 212 \\
\end{array}$

Next we calculate the expected localization length $\Lambda$ for each set of parameters using (4.18). The results are presented in table 4.5. The approximated values for localization length, in case of almost-diffusive regime of large $\lambda_3$ and $\lambda_4$, is much larger than the system width at the end of the simulation. It is therefore not surprising that the numerical results of our computations did not show distinctive saturating behavior within the time span of the simulation. Using the recursion relation (4.16) we can construct the best fit and extend it for enough time to approach the localization length scale for the momentum state width. This way we can get an estimate for the scale of time needed to detect localization. Figure 4.36 shows this calculation performed for the run with the largest kicking and coupling parameters. Again the time spans thus estimated way exceed the simulation times of our numerical results. The initial assumption of this section, considering the results as lying within localization range of the kicking parameters, has been strengthened with regard to the scaling estimates presented here.
Figure 4.36: Standard deviation of momentum for the quantum rotor and the best fit based on Scaling theory for time steps $t = 1,210,000$ ($\sqrt{t} = 1100$). $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 3.0$, $\lambda_4 = 3.0$. 
Chapter 5

Concluding Discussion

In this final chapter, we will present a general discussion of our work, how the results relate to the questions posed at the beginning of this thesis, and some of the issue surrounding them. We will also propose and discuss possible directions that can be taken for further work in the future, and related problems that need to be addressed before a full understanding of this field of study can emerge.

This work was based, in a large part, on the extensive work of S. Fishman, E. Doron, D.R. Grempel and R.E. Prange concerned with the presence of quantum localization in the periodic kicked rotor sets of models [16][18][23]. The issue arises since it was discovered that a mapping exists between these models and Anderson lattices, where random and uncorrelated lattice-site potentials induce localization of the quantum state. Fishman et al. studied a typical single kicked rotor model [23], analogous to the classical standard mapping, and analyzed, in some detail, what part various parameters and physical quantities of this typical model play in determining localization features of momentum, and the time-dependent behavior of momentum amplitudes. They proposed that the condition of independent random site potentials in the related Anderson lattice is too restrictive and can be relaxed. In particular they argued that existence of short range correlations, which is inevitable for lattices that are derived from the kicked rotor models, is sufficient to induce localization in a typical kicked rotor system. They presented numerical results for a range of kicking strengths that corroborated this.

The next natural extension was to a system of two coupled rotors, corresponding to a two-dimensional Anderson lattice. In a later paper [16], Fishman and Doron studied a simple generalization of their previous model to two dimensions, and provided results
of numerical simulations that once again supported this claim. However, in the 2-rotor systems, the choice of kicking form, as function of the angle, is not as restricted as in a single rotor. They have argued that, in fact, the properties of the kicking potential should play no particular role in the outcome of localized states, as long as they are well-behaved and typical functions. Nevertheless this claim can be shown rigorously, only in the case of truly random and uncorrelated lattices. Here, we generalized the two rotor model by including other terms, both of independent kicks to each rotor, as well as a different form of coupling in order to investigate this claim using concrete numerical data for the same range of parameters.

The motivation for the extra terms for the kicking form came from a separate paper from an independent team, Ikeda et al. [2], working on a different problem, namely whether the coupling can induce classical chaotic behavior and diffusive growth of the momentum variance with time. Their results displayed diffusion for a (rather large) set of parameters, at least within the time span of their results. This seemed to be in clear contrast to the results of Fishman and Doron. It was this apparent contradiction that caught our attention, and made us interested in this topic. This work began to find a possible resolution to this situation. Beside differences in the kicking potential in the two models, the two papers were focused on different aspects of their numerical results. In other words, their criteria for localization were different. Doron and Fishman used the exponential decay of the momentum probability distribution, and an almost constant corresponding localization length as their criterion for localization, where as Ikeda et. al had focused on the root-mean-square momentum state width, and its change with time to look for diffusive behavior.

In order to clarify the situation, we performed a large number of numerical calculations. Our numerical results consisted of a rather wide range of parameters. Furthermore, we used a grid size that was almost 4 times larger than those used by either of the two groups. This enabled us to continue the calculations for much longer times, since this way the errors of finite grid size would become significant only in much later times. This was important in order to go beyond any temporary uncharacteristic behaviors at early times. We checked both the envelope of the momentum distribution for exponential fall, which is indicative of strict localization, as well as momentum standard deviation and how its temporal variation, in order to detect evidence of possible diffusion. These two quantities incorporate the two criteria in each of the two papers [16] [2].

We also performed numerical simulations for the classical counterparts of these systems,
to be able to compare them with the quantum results. The classical system exhibited diffusive growth of their momentum width with time. We went beyond the question of resolving the contradiction between the two quantum cases, and derived first order approximations to the diffusion coefficients, as a simple function of the kicking and coupling strengths. The numerical results were shown to oscillate around this approximation, with the amplitude of oscillations decreasing for higher kicking and coupling strengths. This result conforms with the well known results of the standard mapping of the single kicked rotor.

The quantum outcomes were interesting. First, the results showed that the two localization criteria were in agreement for all cases. We did recover localization for lower values of the kicking and coupling parameters based on both criteria. However we also had a regime, consisting of runs with higher values of these parameters, that demonstrated almost linear growth of the momentum standard deviation with square root of time. This suggested an almost-diffusive regime. However, the slope of the standard deviation, which denotes the square root of the effective diffusion coefficient exhibited a very slow decrease with time. Moreover, comparison with the classical results showed that the rates of diffusion are different, and thus it was established that the quantum evolution did not approach the classical limit. There was an intermediate regime of medium ranged parameters between these two cases where the change of standard deviation with time was erratic, and ranged from small to very noticeable oscillations. For the lower range of these parameters, the state width displayed saturation in the envelope of its oscillations, while for the larger parameters, it did not seem to saturate in any discernible fashion.

The scaling theory of localization for charged particles in Anderson lattices raised yet another possibility to clarify the situation. The system was two dimensional and for two dimensional Anderson lattices, there is no critical value of disorder to cross over from localized to diffusive regimes, so that, even in the limit of ohmic behavior, such systems do not generate diffusion. Using the scaling theory of Anderson lattices seemed specially appropriate at this stage, since Fishman and Doron [16] had done similar analysis on their results. Fitting the numerical data to the expressions derived from scaling theory of localization near this limit produced reasonable results. In particular, the calculated fitting parameter $b$ in $\Lambda \sim e^{bl}$ (4.18), was approximately constant for all the results in the almost-diffusive regime, as it should have been according to scaling theory. $\Lambda$ is the saturation size of the momentum width, and $l$ is the relevant scaling parameter as defined in chapter 4.3.1. Moreover the estimates of scaling theory for $\Lambda$, based on our results, implied that $\Lambda$ grows exponentially
with the kicking and coupling parameters. This explains why we didn't see saturation of the state size in this regime of large kicking and coupling parameters. The saturation scale, and the time span needed to reach it went far beyond the scope of the grid size and calculation time of our numerical simulations. For the numerical results in the localized regime, the fitting did reproduce the right trend at the saturation limit, but this is not surprising and has no special significance. Any form of fitting would have accomplished that. The fitting results for the intermediate region of parameters was not good. The fitting parameter for these results were also found to vary from their constant value in the almost-diffusive regime. This was to be expected, since the scaling form (4.18) was based on perturbation corrections to the ohmic regime. The intermediate region is sufficiently away from that limit, so that this perturbation is no longer reliable. We have no theory for the scaling properties of this intermediate regime. Nevertheless, residing between the already localized regime, and the almost-diffusive one that does match localization conditions of scaling theory, makes it reasonable to conclude that this intermediate regime must also be localized, even though its detailed form has not been determined.

There are some issues that need to be investigated further. In the scaling theory of the Anderson lattices, the size of the lattice is fixed and well defined. In our case, where momentum degree of freedom is studied, the notion of lattice size is not as well defined. In this work, we have made the assumption that the momentum standard deviation at any given time, can be substituted as the size of the system at that point of time. The intuitive rational behind this assumption is as follows. As long as the wave function, starting from a sharp initial gaussian, has not accessed the larger values along the momentum axis, these extra unaccessible regions of momentum space should not matter for the state function and the its diffusion rate. However we must emphasize here that this is merely an intuitive assumption and we have not confirmed it in a rigorous fashion. Finding the truth of this assumption is a possible direction for further research.

Another issue is the special choice of parameters by Ikeda et al. The relevant dimensionless parameters is in the kinetic energy term of the Hamiltonian, and contains the information regarding the size of $\hbar$, the kicking period and the moments of inertia of each rotor. Their choice for this parameter was a rational fraction of $2\pi$. It was discussed in chapter 2, that for Anderson lattices a value of this parameter commensurate with $2\pi$ amounts to periodic lattice boundary conditions. Hence the eigenstates of the system would have the characteristics of Bloch states, and are extended, rather than localized. This has been
demonstrated for the one-dimensional lattices. The extra degrees of freedom for higher di­
dimensions might produce complications. In any case, we have ignored this extra complication in this work, but it presents an interesting topic of study. In particular, a periodic boundary condition defines a well defined system size at all times. This could, in turn, be related to the problem of systems on tori, which have their own scaling properties.

Yet another interesting direction for further study is to compare the relative effect of each of the kicking and coupling parameters on the growth rate of the momentum width. Our results showed that the state size depended significantly on the kicking and coupling parameters. The classical functional dependence of the diffusion coefficient, in its first approximation, to the kicking and coupling strengths, can provide a good analogy. It would be advantageous to arrive at an explicit form for the quantum results that relates the state size to these parameters. This hasn’t been achieved in the present work. Based on the scaling results, establishing a link between the scaling parameter and the kicking and coupling strengths could be an important step in this direction.

Finally, we mention a possible connection of this model to experiment. Casati et. al [10] have shown how a single kicked rotator with extra time-dependent kicking functional terms with incommensurate frequencies can be mapped to a model of two (or more) coupled kicked rotors without explicit time dependence in their kicking potential, as in our model. Experimental realizations of single kicked rotors are possible, for instance atoms in standing microwave radiation with periodic pulses [37]. Adding time dependent modulations to the “kicking” waves is feasible. This way, physical experiments can be brought to bear evidence on the localization for such systems. They can be done for longer times, inaccessible to numerical simulations.
Appendix A

We saw in chapter 2.2.1 that the problem of localization in the kicked rotor model is related to that of a hopping particle on a lattice. The primary context of study of such systems has been that of conducting behaviors of charged particles in various situations and dimensions. Approaching the topic of scaling and localization in connection with charged particles and conduction has many merits. Most significantly, conduction of electrical charges provides a directly measurable physical quantity, namely conductance $G$, or more specifically dimensionless conductance $[39]$

$$g := \frac{G}{e^2/h} \quad (A.1)$$

This can be linked to our dynamical problem. For instance, as long as conductivity $\sigma$ is well-defined, it can be immediately linked to the diffusion coefficient $D$ of the hopping particle $\sigma = e^2 N_0 D$ where $e$ is the charge of electron and $N_0$ electron density. More generally the link to conductance can be made through a generalization of the notion of “mean free path”. We will come back to this and make the link to the kicked rotor model in the last section of this Appendix.

A.1 Scaling Theory of Localization of Charged particles in a Lattice

The main issue in scaling theory is to learn about measures of localization through manner in which the dimensionless conductance $g$ (A.1) scales with the system size $L$.

---

1Not conductivity $\sigma$ which is an intensive quantity. Conductance on the other hand is an extensive quantity, and thus a good measure for determining diffusion or localization.

2or other possible scale variables of the system.
Consider a charged particle moving in a disordered medium. The average distance over which the phase of its wave function fluctuates over a full period is the “mean free path” $l$ of the particle. For larger distances than $l$, the motion of the electron would be diffusive, and can’t be considered free anymore. Thus $l$ is the cut off length from below of all the scaling theory calculations. The conductance at this length, $g_0$, acts as the “microscopic” measure of disorder of the system. Large disorder makes for small $g_0$, and vice-versa.

We can now increase the size of the system $L > l$. Our central problem is to determine how $g$ changes with $L$ in terms of a small subset of the system parameters. In our present discussion, these will be $l$ and $g_0$. In the most general case, $g(L)$ can approach two asymptotic forms depending on $g_0^3$, as well as the number of dimensions of the system. If the disorder is small, then $l$ is large with respect to the distance between scattering potentials. There is a well defined conductivity $\sigma$, whose magnitude can be calculated based on conventional transport theory, and for large values of $L \gg l$, the conductance of the $d$-dimensional system follows Ohm’s law,

$$g(L) = \sigma L^{d-2}$$  \hspace{1cm} (A.2)

In the other extreme, for random potentials, the system is localized. Hopping happens only from an occupied state to an unoccupied one with near energy, which is far apart in distance, and thus the hopping elements are exponentially small. We can define a localization length, $\zeta$, as the relevant length scale in this situation, and we have

$$g(L) \sim e^{-L/\zeta}$$  \hspace{1cm} (A.3)

Abrahams et. al [1] argued, based on perturbation theory, that the logarithmic derivative of $G$ with respect to $L$ is a function of $g$ only,

$$\beta(g) = \frac{d \log g}{d \log L}$$  \hspace{1cm} (A.4)

The asymptotic form of $\beta(g)$ for the Ohmic region is

$$\beta(g) = (d - 2)$$  \hspace{1cm} (A.5)

and for the localized region it is

$$\beta(g) = \log \left( \frac{g}{g_0} \right)$$  \hspace{1cm} (A.6)

$^3$ie. the disorder in the environment
There exists a critical value for $g_0$ where the large scale behavior changes from Ohmic to localized.

For weak disorder, it is possible to use perturbation theory to calculate the corrections to the Ohmic region, yielding

$$\beta(g) = (d - 2) - \frac{a}{g} \quad (A.7)$$

It is clear that the case of $d = 2$ is a special case. In the so called Ohmic region, $\beta(g) = 0$, and conductivity and conductance remain numerically equal. This means that in two (and one) dimensions there is no conductive regime, and 2 is the critical dimension.

We have

$$\frac{d \log g}{d \log L} = -\frac{a}{g} \quad (A.8)$$

Starting from length $l$, we have

$$g(L) = g_0 - c \log \left[ \frac{L}{l} \right] \quad (A.9)$$

The localization length, $\zeta$, is defined as the length scale for which the scaling theoretical correction becomes comparable to $g_0$, that is

$$\zeta \approx l \exp \left[ \frac{g_0}{c} \right] = l e^{bl} \quad (A.10)$$

where $b$ is constant for a category of systems with similar range of parameters.

### A.2 Dynamical Localization for the Kicked Rotor

Recall that the propagator of the kicked rotor model between each two consecutive kicks is

$$\hat{U} = e^{-i\hat{T}_0} e^{-i\hat{V}} \quad (A.11)$$

$T_0(m_1, m_2)$ is the kinetic energy and $V(\theta_1, \theta_2)$ is the potential energy, naturally expressed in momentum and angle representation respectively. Defining

$$U_{n,k} = \langle n_1, n_2 | \hat{U} | k_1, k_2 \rangle \quad (A.12)$$

We will have

$$U_{n,k} = e^{-iE_n} R(r, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad (A.13)$$
Where $E_n = \frac{1}{2}(\alpha_1 m_1^2 + \alpha_2 m_2^2)$ and $T_m = \tan \left[ \frac{1}{2}(\omega - E_n) \right]$ is the on-site potential. Since the values $T_m$ behave in many respects as a random sequence, this means that the phase of the wave-function is randomized between each kick. As a result, the average displacement in momentum after the first kick

$$l^2 = \sum_{r_1, r_2 = -\infty}^{\infty} (r_1^2 + r_2^2) |U_{0,r}|^2$$  \hspace{1cm} (A.14)

can be regarded as the counterpart of the mean free path in the previous chapter. Hence it is expected that, in case of localization, the scaling takes the same form as was derived before for large values of the parameters and nearly diffusive regime, with the saturation length $\Lambda$ corresponding to $\zeta$ in the previous discussion. In other words

$$\Lambda = l e^{b l}$$  \hspace{1cm} (A.15)

We use this choice in chapter 5 with our numerical results.
Appendix B

We want to study the dependence of the classical diffusion coefficient on the kicking and coupling parameters of the kicked rotor model. In this appendix, our approach closely follows the method described in [31]. The method was originally worked out for a known and simple model of a single kicked rotor, the so called “standard map”. We first introduce the method by applying it to the standard map, which has a simpler form. We then use it for our present model and derive the approximations and discuss the results.

B.1 The Diffusion Coefficient In The Standard Mapping

This is the standard mapping

\[
\begin{align*}
\theta_{n+1} &= \theta_n + p_n + K \sin \theta_n \\
p_{n+1} &= \theta_n + p_n + K \sin \theta_n
\end{align*}
\] (B.1)

In the simplest approximation, we assume that the probability distribution for \( \theta \) is uniform, and as a result we can obtain the quasi-linear diffusion coefficient for momentum as follows

\[
D_Q = \frac{1}{2\pi} \int_0^{2\pi} (p_{n+1} - p_n)^2 d\theta_n
\] (B.2)

For the standard map, it gives

\[
D_Q = \frac{1}{2\pi} \int_0^{2\pi} (K \sin \theta_n)^2 d\theta_n = \frac{K^2}{2}
\] (B.3)

The assumption of a uniform distribution for \( \theta \) has no rigorous backing. We now develop a rigorous method to obtain higher order corrections for the diffusion coefficient, and to verify the simple result obtained here.

---

1See in particular section 5.5 in the book.

2The standard map is explained in more detail in Chapter 2.2 of the present thesis.
B.1.1 The Fourier Method [31]

We first define the diffusion coefficient after \( n \) steps to be

\[
D_n = \frac{1}{n} \langle (p_n - p_0)^2 \rangle_{p_n, \theta_n}
\]

Where \( \langle \rangle_{p_n, \theta_n} \) denotes the average over all the points in phase space \((p_n, \theta_n)\) that the system can reach after \( n \) steps. In order to obtain higher order approximations to the diffusion coefficient, write it in terms of the conditional probability \( W \), that the system starting at \((p_0, \theta_0)\) will reach the point \((p_n, \theta_n)\) after \( n \) kicks. 

\[
D_n = \frac{1}{n} \int dp_n d\theta_n (p_n - p_0)^2 W (p_n, \theta_n | p_0, \theta_0) \quad (B.5)
\]

We want to obtain a recursion relation. It is evident that \( W (|) \) can be written as

\[
W (p_n, \theta_n | p_0, \theta_0) = \int dp_{n-1} d\theta_{n-1} W (p_n, \theta_n | p_{n-1}, \theta_{n-1}) W (p_{n-1}, \theta_{n-1} | p_0, \theta_0) \quad (B.6)
\]

The form of the mapping shows itself in this recursion relation.

\[
W (p_n, \theta_n | p_{n-1}, \theta_{n-1}) = \delta (p_n - p_{n-1} - K \sin \theta_{n-1}) \\
\times \delta (\theta_n - \theta_{n-1} - p_{n-1} - K \sin \theta_{n-1}) \quad (B.7)
\]

The initial condition can be entered in terms of delta functions.

\[
W = \delta (p - p_0) \delta (\theta - \theta_0) \quad (n = 0) \quad (B.8)
\]

Since we are interested in \( D_n \) for large \( n \), and since \( p_n \sim \sqrt{n} \), we can approximate (B.5) as follows

\[
D_n = \frac{1}{n} \int dp_n d\theta_n p_n^2 W (p_n, \theta_n | p_0, \theta_0) \quad (B.9)
\]

If we try to calculate \( D_n \) directly, using the above recursion relation, we will soon find that the equations become very complicated after a few iterations. Instead, we need a method in which the calculations of the recursion relation for later iterations become simpler. The trick is to make a Fourier transform of \( W \) and work in the Fourier space. We define \( a(m, q) \) as the Fourier transform of \( W(\theta, p) \),

\[
a_n (m_n, q_n) = \frac{1}{4\pi^2} \int dp_n d\theta_n \exp (-im_n \theta_n - iq_n p_n) W (p_n, \theta_n | p_0, \theta_0) \quad (B.10)
\]

\(^3\) We can consider \((p_0, \theta_0)\) the centroid of a very sharp initial Gaussian distribution, whose width is small compared to our desired accuracy.
APPENDIX B.

Here \(a_n(m, q)\) is also a function of \(\{p_0, \theta_0\}\), but for brevity, expressing this dependence is suppressed in the following discussion. Using (B.10) in (B.9), integrating the delta functions and using integration by parts we obtain the following relation between \(D_n\) and \(a_n\):

\[
D_n = -\frac{(2\pi)^4}{n} \left. \frac{\partial^2 a_n(0, q_n)}{\partial q_n^2} \right|_{q_n=0}
\]  

(B.11)

Next we take the Fourier transform of both side of (B.6), so that we could reach a recursion relation for the consecutive values of \(a_n\). Thus we have

\[
a_n(m_n, q_n) = \frac{1}{4\pi^2} \int dp_n d\theta_n \exp(-im_n\theta_n - iq_n p_n) \times \int dp_{n-1} d\theta_{n-1} \delta(p_n - p_{n-1} - K \sin \theta_{n-1}) \times \delta(\theta_n - \theta_{n-1} - p_{n-1} - K \sin \theta_{n-1}) \times \int dq_{n-1} \sum_{m_{n-1}} \exp(im_{n-1}\theta_{n-1} + iq_{n-1} p_{n-1}) a_{n-1}(m_{n-1}, q_{n-1})
\]  

(B.12)

Integrating \(\theta_n\) and \(p_n\) out, then integrating over \(q_{n-1}\), simplifying the results with the arising delta functions, we finally reach

\[
a_n(m_n, q_n) = \frac{1}{2\pi} \sum_{m_{n-1}} \int d\theta_{n-1} a_{n-1}(m_{n-1}, q_{n-1}) e^{[i(m_n - m_{n-1})\theta_{n-1} - q_{n-1} K \sin \theta_{n-1}]} \]  

(B.13)

with the condition

\[
q_{n-1} = q_n + m_n
\]  

(B.14)

At this stage we need to use the identity

\[
\exp(-i X_i Y_j \sin \theta) = \sum_{l_{ij} = -\infty}^{\infty} J_{l_{ij}}(|X_i| Y_j) \exp[-i l_{ij} \theta \sgn(X_i)]
\]  

(B.15)

Here \(J_n\) is the Bessel function of the first kind of order \(n\). \(\sgn\) is the signum function. Integrating over \(\theta_{n-1}\) in (B.13) then gives

\[
a_n(m_n, q_n) = \frac{1}{2\pi} \sum_{l_n, m_{n-1}} a_{n-1}(m_{n-1}, q_{n-1}) J_{l_n}(|q_{n-1}| K) \delta_k(m_{n-1} - m_n - l_n \sgn(q_{n-1}))
\]  

(B.16)

Finally by taking the sum over \(m_{n-1}\), we arrive at the following recursion relation

\[
a_n(m_n, q_n) = \sum_{l_n = -\infty}^{\infty} a_{n-1}(m_{n-1}, q_{n-1}) J_{l_n}(|q_{n-1}| K)
\]  

(B.17)
with these conditions

\[
\begin{aligned}
    m_n &= m_{n-1} - l_n \text{ sgn}(q_{n-1}) \\
    q_n &= q_{n-1} - m_n
\end{aligned}
\]  

(B.18)

In other words, after \( n \) iterations we will get \( a_n \) in terms of \( a_0 \)

\[
a_n(m_n, q_n) = \sum_{l_n, \ldots, l_1} a_0(m_0, q_0) J_{l_n} J_{l_{n-1}} \cdots J_{l_1}
\]  

(B.19)

The initial value \( a_0 \) can be calculated by Fourier transforming the initial condition for \( W \) (B.8), which gives

\[
a_0 = \frac{1}{4\pi^2} \exp(-i q_0 p_0 - i m_0 \theta_0)
\]  

(B.20)

These results can be interpreted in the following manner. The set of integers \( \{l_1, \ldots, l_n\} \) act as free parameters that define a “path” in the Fourier space \((m, q)\). Each value of \( l_r \) gets us from \((m_{r-1}, q_{r-1})\) to a particular pair \((m_r, q_r)\) using (B.18). Each choice of \( \{l_1, \ldots, l_n\} \), i.e. each “path”, corresponds to a term in the sum (B.19). Ultimately, we need to calculate \( D_n \), from \( a_n \). The form of (B.11) shows that the paths must end at the origin in the last step, i.e. we need to have \( m_n = q_n = 0 \). The argument of the Bessel function is \( K |q| \). For large \( K \), the Bessel functions fall as \( K^{-1/2} \) except for the region \( q \to 0 \). In this limit, the Bessel functions of orders higher than zero are small. The idea is to sort the terms in the sum (B.19), starting with lowest values of \( l_r \), with the restriction that each set should form a closed path of \((m_r, q_r)\) pairs, starting and ending at the origin \((0,0)\).

So the first order approximation, the case where only the zeroth order contributes, corresponds to a path that remains at the origin for all the steps, a path of length zero. For this path

\[
a_n(0, q_n) = \frac{1}{4\pi^2} [J_n (|q| K)]^n \exp(-i q_n p_0)
\]  

(B.21)

We know that

\[
J_0(x) \approx 1 - \frac{x^2}{4}
\]  

\( x \to 0 \)  

(B.22)

Using this expansion in \( a_n(0, q) \) and taking the derivative with respect to \( q \) we get

\[
D_n = -\frac{(2\pi)^4}{n} \left( 1 - \frac{nK^2q_n^2}{4} + \ldots \right) \left( 1 - i q_n p_0 + \ldots \right) \bigg|_{q_n=0}
\]  

(B.23)
APPENDIX B.

Table B.1: Values of \( m, q \) (B.10) and \( l \) (B.18) for shortest nonzero path as described in the text.

<table>
<thead>
<tr>
<th>Step</th>
<th>( l )</th>
<th>( m )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( r+1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( r+2 )</td>
<td>-2</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>( r+3 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( r+4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Where we have shown terms up to quadratic that survive the differentiation. In the limit of large \( n \) we finally obtain

\[
D_{n0} = \frac{K^2}{2} \tag{B.24}
\]

Which is the same as the simple, intuitive approximation of assuming a uniform distribution of \( \theta \).

B.1.2 Higher orders

The next step is to consider paths that do leave the origin \((q, m) = (0, 0)\) and return to it. The shortest such path comprises of three steps. Suppose they are done in iterations \( r, r + 1, \) and \( r + 2 \). There are two such paths, one the inversion of the other with respect to the origin of the \((m, q)\) space. The values of the parameters for one of the two paths are shown in Table B.1. For the other, the values of \((m, q)\) are the negatives of these values. The resulting term for these paths are

\[
a_n = \frac{1}{4\pi^2} \left[ J_0(Kq_n) \right]^{n-3} \left[ J_1(Kq_n) \right]^2 J_2(K) \exp(-q_n p_0) \times 2n \tag{B.25}
\]

The \( 2n \) factor is because one can pick \( r \) to be any of the \( n \) steps, and for each \( r \) we have two paths, as mentioned above. We have used the fact that for Bessel functions of the first kind, negative orders can be made positive

\[
J_{-l} = J_l \times (-1)^l \quad \text{For} \quad l > 0 \tag{B.26}
\]
Again, expanding the Bessel functions and keeping only the quadratic terms in $q$, this simplifies to

$$a_n(0, q_n) = \frac{2n}{4\pi^2} \frac{K^2 q_n^2}{4} J_2(K)$$  \hfill (B.27)

Hence we have

$$D_{n1} = \frac{K^2}{2} [1 - 2 J_2(K)]$$  \hfill (B.28)

Note that since the standard map has only one kicking parameter $K$, which enters the argument of the Bessel function, these two shortest non-zero paths do indeed produce the next order correction in a simple fashion. In other words, there is only one Bessel term without $q$ dependence, and there is no remaining freedom regarding its order. We shall see in the following sections that in case of multiple kicking parameters, this is no longer the case.

## B.2 The 2 Coupled Kicked Rotor Model

We use the following mapping

$$\begin{cases}
  p_{n+1}^{(1)} = p_n^{(1)} + \lambda_1 \sin \theta_n^{(1)} + \lambda_3 \sin \theta_n^{(1)} \cos \theta_n^{(2)} + \lambda_4 \sin \left( \theta_n^{(1)} - \theta_n^{(2)} \right) \\
  p_{n+1}^{(2)} = p_n^{(2)} + \lambda_2 \sin \theta_n^{(2)} + \lambda_3 \cos \theta_n^{(1)} \sin \theta_n^{(2)} - \lambda_4 \sin \left( \theta_n^{(1)} - \theta_n^{(2)} \right) \\
  \theta_{n+1}^{(1)} = \theta_n^{(1)} + p_{n+1}^{(1)} \\
  \theta_{n+1}^{(2)} = \theta_n^{(2)} + p_{n+1}^{(2)}
\end{cases}$$  \hfill (B.29)

The values $p_n^{(1,2)}$ and $\theta_n^{(1,2)}$ are the momenta and angles (B.29) after the $n$th kick, belonging to rotors 1 and 2 respectively. We assume we are in a chaotic region of phase space. This means that the variance of momenta of each rotor will grow diffusively with time. The simplest approximation for the diffusion coefficient of the either rotor, is, once again, obtained by assuming uniform distribution over both the angles. We calculate all the quantities of for the first rotor. The calculations for the second rotor are similar. We have

$$D_Q^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} \left( p_{n+1}^{(1)} - p_n^{(1)} \right)^2 \, d\theta_n^{(1)} d\theta_n^{(2)}$$  \hfill (B.30)
This yields

\[ D_{q}^{(1)} = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta_{n}^{(1)} d\theta_{n}^{(2)} \left[ \lambda_{1} \sin \theta_{n}^{(1)} + \lambda_{3} \sin \theta_{n}^{(1)} \cos \theta_{n}^{(2)} + \lambda_{4} \sin \left( \theta_{n}^{(1)} \theta_{n}^{(2)} \right) \right]^{2} \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta_{n}^{(1)} d\theta_{n}^{(2)} \times \]

\[ \left[ \lambda_{1}^{2} \sin^{2} \theta_{n}^{(1)} + \lambda_{3}^{2} \sin^{2} \theta_{n}^{(1)} \cos^{2} \theta_{n}^{(2)} + \lambda_{4}^{2} \sin^{2} \left( \theta_{n}^{(1)} - \theta_{n}^{(2)} \right) \right. \]

\[ + 2\lambda_{1}\lambda_{3} \sin^{2} \theta_{n}^{(1)} \cos \theta_{n}^{(2)} + 2\lambda_{1}\lambda_{4} \sin \theta_{n}^{(1)} \sin \left( \theta_{n}^{(1)} - \theta_{n}^{(2)} \right) \]

\[ + 2\lambda_{3}\lambda_{4} \sin^{2} \theta_{n}^{(1)} \cos^{2} \theta_{n}^{(2)} - 2\lambda_{3}\lambda_{4} \sin \theta_{n}^{(1)} \sin \theta_{n}^{(2)} \cos \theta_{n}^{(2)} \]

\[ + \lambda_{1}\lambda_{3}\lambda_{4} \sin^{3} \theta_{n}^{(1)} \cos \theta_{n}^{(2)} - \lambda_{1}\lambda_{3}\lambda_{4} \sin^{2} \theta_{n}^{(1)} \sin \theta_{n}^{(2)} \cos \theta_{n}^{(2)} \right] \]

\[ = \frac{\lambda_{1}^{2}}{2} + \frac{\lambda_{3}^{2}}{4} + \frac{\lambda_{4}^{2}}{2} + 0 + 0 + \frac{\lambda_{3}\lambda_{4}}{2} - 0 - 0 \]

\[ = \frac{(\lambda_{3} + \lambda_{4})^{2}}{4} + \left[ \frac{\lambda_{3}^{2}}{2} + \frac{\lambda_{4}^{2}}{4} \right] \]  

(B.31)

To put this result on firm ground, and to possibly obtain higher order corrections, we apply the Fourier method to our model, following the route taken in the case of the standard mapping.

### B.2.1 The Fourier Method For Two Coupled Rotors

We define the diffusion coefficient corresponding to each rotor after \( n \) kicks as

\[ D_{n}^{(i)} = \frac{1}{n} \left\langle \left( p_{n}^{(i)} - p_{0}^{(i)} \right)^{2} \right\rangle_{p_{n}^{(1,2)}, \vartheta_{n}^{(1,2)}} \quad (i = 1, 2) \]  

(B.32)

Here \( \langle \cdot \rangle_{p_{n}^{(1,2)}, \vartheta_{n}^{(1,2)}} \) denotes the average over all the points in phase space \( \left( p_{n}^{(1,2)}, \vartheta_{n}^{(1,2)} \right) \) that the system can reach after \( n \) kicks. In order to obtain higher order approximations to this diffusion coefficient, we write them in terms of the conditional probability density \( W(\cdot) \) that an initial state \( \left( p_{0}^{(1,2)}, \vartheta_{0}^{(1,2)} \right) \) can reach the point \( \left( p_{n}^{(1,2)}, \vartheta_{n}^{(1,2)} \right) \) after \( n \) kicks 4.

\[ D_{n}^{(i)} = \frac{1}{n} \int dp_{n}^{(1,2)} d\theta_{n}^{(1,2)} \left( p_{n}^{(i)} - p_{0}^{(i)} \right)^{2} W \left( p_{n}^{(1,2)}, \vartheta_{n}^{(1,2)} \left| p_{0}^{(1,2)}, \vartheta_{0}^{(1,2)} \right. \right) \]

\[ (i = 1, 2) \]  

(B.33)

\[ ^{4}\text{We can consider} \left( p_{0}^{(1,2)}, \vartheta_{0}^{(1,2)} \right) \text{the centroid of a very sharp initial Gaussian distribution, whose width is small compared to our desired accuracy.} \]
Just like the calculations for single kicked rotor in previous section, we can define similar recursion relations, initial conditions, and their Fourier transform $a_n(m_{1,2}^{(n)}, q_{1,2}^{(n)})$. Again, we obtain

$$D_n^{(1)} = -\frac{(2\pi)^4}{n} \frac{\partial^2 a_n(0,0,q_1^{(n)},0)}{\partial q_1^{(n)} 2} \bigg|_{q_1^{(n)}=0}$$  \hspace{1cm} (B.34)

Following the same line of reasoning, and similar calculations we finally reach at the recursion relation, corresponding to the first rotor, for $a_n$,

$$a_n(m_{1,2}^{(n)}, q_{1,2}^{(n)}) = \sum_{l_{11}^{(n)}, l_{12}^{(n)}, l_{13}^{(n)}, l_{14}^{(n)}} a_{n-1}(m_{1,2}^{(n-1)}, q_{1,2}^{(n-1)}) \mathcal{J}_{l_{11}^{(n)}} \left( |q_1^{(n-1)}| \frac{\lambda_3}{2} + \lambda_4 \right) \mathcal{J}_{l_{13}^{(n)}} \left( |q_1^{(n-1)}| \lambda_1 \right) \mathcal{J}_{l_{14}^{(n)}} \left( |q_2^{(n-1)}| \lambda_3 \right) \mathcal{J}_{l_{13}^{(n)}} \left( |q_2^{(n-1)}| \lambda_4 \right)$$

$$\times \mathcal{J}_{l_{22}^{(n)}} \left( |q_2^{(n-1)}| \frac{\lambda_3}{2} + \lambda_4 \right) \mathcal{J}_{l_{24}^{(n)}} \left( |q_2^{(n-1)}| \lambda_3 \right) \mathcal{J}_{l_{24}^{(n)}} \left( |q_2^{(n-1)}| \lambda_4 \right)$$

(B.35)

The sums are taken over all integer values. The following relation between the consecutive points $\{m_{1,2}^{(r)}, q_{1,2}^{(r)}\}$ and $\{m_{1,2}^{(r+1)}, q_{1,2}^{(r+1)}\}$, for $r = 1...n$, which defines the allowed "paths" in Fourier space to go through, similar to the case of the standard mapping. Each "path" corresponds to one term in the recursion relation

$$
\begin{align*}
m_1^{(n)} &= m_1^{(n-1)} - \left( l_{11}^{(n)} + l_{12}^{(n)} + l_{13}^{(n)} + l_{14}^{(n)} \right) \text{sgn}(q_1^{(n-1)}) - \left( l_{23}^{(n)} - l_{24}^{(n)} \right) \text{sgn}(q_2^{(n-1)}) \\
m_2^{(n)} &= m_2^{(n-1)} - \left( l_{22}^{(n)} + l_{23}^{(n)} + l_{24}^{(n)} \right) \text{sgn}(q_2^{(n-1)}) - \left( l_{13}^{(n)} - l_{14}^{(n)} \right) \text{sgn}(q_1^{(n-1)}) \\
q_1^{(n)} &= q_1^{(n-1)} + m_1^{(n)} \\
q_2^{(n)} &= q_2^{(n-1)} + m_2^{(n)}
\end{align*}
$$

(B.36)

The terms $l_{ij}^{(n)}$ act as parameters in the above calculations, and can take all possible values. Again, because we need to calculate $D_n^{(1)}$, the equation (B.34) implies that we need consider only paths that end in $m_{1,2}^{(n)} = 0$ and $q_{1,2}^{(n)} = 0$. Similar to the standard mapping, the arguments of the Bessel functions in (B.35) are products of the parameters $\lambda_{1,2,3,4}$, and the values of $|q^{(r)}|$. So, for non zero $q^{(r)}$, they fall as powers of $\lambda^{-1/2}$ for large values of $\lambda$. Hence for large values of $\lambda_{1,2,3,4}$ the dominant terms must have all their $q^{(r)} \rightarrow 0$. So following the same route, we begin with the first order approximation, where only the term with all
$l_{ij} = 0$ is kept in the sum. This corresponds to a path that remains in the origin of the Fourier space for all the $n$ steps, as defined in (B.36). We have

$$a_n^{(0)}(0, 0, q_1^{(n)}, 0) = \left(\frac{1}{2\pi}\right)^4 \exp\left(-2\frac{q_1^{(n)} p_0}{p_0}\right) \times J_0^n \left(|q_1^{(n)}| \lambda_1\right) J_0^n \left(|q_1^{(n)}| \frac{\lambda_3}{2}\right) J_0^n \left(|q_1^{(n)}| \left[\frac{\lambda_3}{2} + \lambda_4\right]\right) \times J_0^n (0) \times J_0^n (0) \times J_0^n (0) \tag{B.37}$$

Recalling (B.22), and expanding to quadratic terms, we get

$$a_n^{(0)}(0, 0, q_1^{(n)}, 0) \approx \left(1 - n \frac{\lambda_3 q_1^{(n)} 2}{4} - n \frac{\lambda_4 q_1^{(n)} 2}{416} - n \frac{\lambda_3^2 + \lambda_4^2 q_1^{(n)} 2}{4}\right) - q_1^{(n)} p_0^{(1)} + h.o.t. \tag{B.38}$$

Replacing this into (B.34), we obtain

$$D_{n0}^{(1)} = \frac{\lambda_3^2}{2} + \frac{\lambda_4^2}{8} + \frac{[\lambda_3/2 + \lambda_4]^2}{2} \tag{B.39}$$

After simplification it gives

$$D_{n0}^{(1)} = \frac{(\lambda_3 + \lambda_4)^2}{4} + \left[\frac{\lambda_2^2}{2} + \frac{\lambda_3^2}{4}\right] \tag{B.40}$$

Thus we finally recover the quasi-linear result. Similar calculations gives the diffusion coefficient of the second rotor as.

$$D_{n0}^{(2)} = \frac{(\lambda_3 + \lambda_4)^2}{4} + \left[\frac{\lambda_2^2}{2} + \frac{\lambda_3^2}{4}\right] \tag{B.41}$$

or equivalently, the first order approximation for the standard deviation for each rotor is

$$D_{n0}^{(s(1))} = \sqrt{\frac{(\lambda_3 + \lambda_4)^2}{4} + \left[\frac{\lambda_2^2}{2} + \frac{\lambda_3^2}{4}\right]}$$

$$D_{n0}^{(s(2))} = \sqrt{\frac{(\lambda_3 + \lambda_4)^2}{4} + \left[\frac{\lambda_2^2}{2} + \frac{\lambda_3^2}{4}\right]} \tag{B.42}$$

We used this equation in chapter 4 to approximate the numerical results.
Table B.2: Values of $m_1$ and $q_1$ (B.35) for shortest nonzero path as described in the text.

<table>
<thead>
<tr>
<th>Step</th>
<th>$m_1$</th>
<th>$q_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>r+1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>r+2</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>r+3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**B.2.2 Higher Order Terms**

At this section, we once again turn our attention to paths of non-zero length that leave the origin and come back to it at some point during the iterations. As in the example of the standard mapping, we consider the shortest such path, again comprising of three steps. Table B.2 shows the values of the pair of quantities comprising such a path. The parameters of the path $l_{ij}$ have the following restrictions.

\[
\begin{align*}
(l_{11}^{(r+1)} + l_{13}^{(r+1)} + l_{14}^{(r+1)}) &= 1 \\
(l_{14}^{(r+2)} + l_{14}^{(r+2)} + l_{14}^{(r+2)}) &= -2 \\
(l_{14}^{(r+3)} + l_{14}^{(r+3)} + l_{14}^{(r+3)}) &= 1
\end{align*}
\]  

(B.43)

Because many values of the $l_{ij}$'s satisfy these equations, there are many terms here, whereas for a single rotor, there was only one term, $l_r$ for each value on the right hand side.

\[
a_n^{(1)} = 2n \sum_{l_1,l_2,l_3} \sum_{l_1',l_2',l_3'} \sum_{i,j,k} J_0 \left( \left[ \lambda_1 q_1^{(n)} \right] \right)^{n-3} J_0 \left( \left[ \frac{\lambda_3}{2} q_1^{(n)} \right] \right)^{n-3} J_0 \left( \left[ \frac{\lambda_3}{2} + \lambda_4 \right] q_1^{(n)} \right)^{n-3} \\
\times A (l_{1,2,3}, l_{1',2,3}) F_{i,j,k} (l_{1,2,3}) + \frac{1}{4\pi^2} \exp(-u_1^{(n)} p_0^{(1)})
\]  

(B.44)

where

\[
A (l_{1,2,3}, l_{1',2,3}) := J_{l_1} \left( \left[ \lambda_1 q_1^{(n)} \right] \right) J_{l_1'} \left( \left[ \lambda_1 q_1^{(n)} \right] \right) J_{l_2} \left( \left[ \frac{\lambda_3}{2} q_1^{(n)} \right] \right) J_{l_2'} \left( \left[ \frac{\lambda_3}{2} q_1^{(n)} \right] \right) \\
\times J_{l_3} \left( \left[ \left( \frac{\lambda_3}{2} + \lambda_4 \right) q_1^{(n)} \right] \right) J_{l_3'} \left( \left[ \left( \frac{\lambda_3}{2} + \lambda_4 \right) q_1^{(n)} \right] \right)
\]  

(B.45)
and

\[ F_{i,j,k} := \mathcal{J}_i(\lambda_1) \mathcal{J}_j \left( \frac{\lambda_3}{2} \right) \mathcal{J}_k \left( \frac{\lambda_3}{2} + \lambda_4 \right) \]  \hspace{1cm} (B.46)

The sums over \( l_{1,2,3} \) and \( l'_{1,2,3} \) are only over the following values.

\[
\begin{aligned}
&l_{ij}^{(r+1)} = 0, 1 - 1 \\
&l_{ij}^{(r+3)} = 0, 1, -1 \quad \text{For} \quad i = 1, j = 1, 3, 4
\end{aligned}
\]  \hspace{1cm} (B.47)

That’s because for Bessel terms whose arguments are coupled to \( q \), we only need to keep up to the quadratic terms. We know that

\[ \mathcal{J}_l(X) \sim X^l \quad \text{For} \quad l > 0 \]  \hspace{1cm} (B.48)

However, note that there are no such restrictions on the three Bessel terms that have no \( q \) dependence in their arguments. We expand the terms and take the derivative (B.34).

\[ D_{n_1}^{(1)} = - (\lambda_1 + \lambda_2 + \lambda_3)^2 \sum_{i,j,k} \mathcal{J}_i(\lambda_1) \mathcal{J}_j \left( \frac{\lambda_3}{2} \right) \mathcal{J}_k \left( \frac{\lambda_3}{2} + \lambda_4 \right) \]

\[ \forall i, j, k \quad i + j + k = 2 \]  \hspace{1cm} (B.49)

Since there are no restrictions on the indices \( i, j, k \), this term alone does not include all the second order corrections. That’s because, each of these indices can be zero in infinitely many terms of the sum, each contributing a Bessel function of order zero with nonzero arguments that do not vanish as we go to the regime of large \( \lambda_{1,2,3,4} \). Therefore this approach can’t give us a closed form for the higher correction terms.
Bibliography


BIBLIOGRAPHY


