Structure of Graph Homomorphisms

by

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Abstract

In this thesis we study finite graphs and graph homomorphisms from both, a theoretical and a practical view point. A homomorphism between two graphs $G$ and $H$ is a function from the vertex set of $G$ to the vertex set of $H$, which maps adjacent vertices of $G$ to adjacent vertices of $H$.

In the first part of this thesis we study homomorphisms, which are equitable. Suppose we have a fixed graph $H$. An equitable $H$-coloring of a graph $G$ is a homomorphism from $G$ to $H$ such that the preimages of vertices of $H$ have almost the same size (they differ by at most one). We consider the complexity of the following problem:

INSTANCE: A graph $G$.
QUESTION: Does $G$ admit an equitable $H$-coloring?

We give a complete characterization of the complexity of the equitable $H$-coloring problem. In particular, we show that the problem is polynomial if $H$ is a disjoint union of complete bipartite graphs, and it is NP-complete otherwise.

To get a better insight into a combinatorial problem, one often studies relaxations of the problem. The second part of this thesis deals with relaxations of graph homomorphisms. In particular, we define a fractional homomorphism and a pseudo-homomorphism as natural relaxations of graph homomorphism. We show that our pseudo-homomorphism is equivalent to a semidefinite relaxation, defined by Feige and Lovász. We also show that there is a simple forbidden subgraph characterization for our fractional homomorphism (the forbidden subgraphs are cliques). As a byproduct,
we obtain a simpler proof of the NP-hardness of the fractional chromatic number, a result which was first proved by Grötschel, Lovász and Schrijver using the ellipsoid method. We also briefly discuss how to apply these results to the directed case.

In the last part of this thesis we consider equivalence classes of graphs under the following equivalence: two graphs $G$ and $H$ are equivalent, if there exist homomorphisms from $G$ to $H$ and from $H$ to $G$. We study the multiplicative structure of these equivalence classes, and give a necessary and sufficient condition for the existence of a finite factorization of a class into irreducible elements. We also relate this problem to some graph theoretic conjectures concerning graph product.
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# Contents

Abstract .......................................................... iii
Acknowledgments .................................................. v
1 Introduction ...................................................... 1
   1.1 Basic Definitions and Notations ......................... 1
   1.2 Background ................................................ 8
2 Equitable Graph Homomorphism ............................... 15
   2.1 Introduction ................................................ 15
   2.2 Overview .................................................... 16
   2.3 Preliminary Results ....................................... 17
   2.4 Polynomiality ............................................... 19
   2.5 NP-hardness ............................................... 22
   2.6 Conclusion ................................................ 32
3 Relaxations of Graph Homomorphism .......................... 37
   3.1 Introduction ................................................ 37
   3.1.1 Notations ............................................... 38
   3.2 Overview .................................................... 40
   3.3 Relaxations of Homomorphism ............................ 40
   3.4 Fractional Homomorphisms ............................... 41
   3.4.1 Consequences .......................................... 44
   3.5 Pseudo-homomorphisms ................................... 46
   3.6 Examples ................................................... 51
   3.6.1 Graph Coloring ......................................... 52
   3.6.2 Digraph Coloring ....................................... 56
3.7 Conclusion ................................................. 58
4 Structure of Color Classes ............................... 60
  4.1 Introduction ............................................. 60
  4.2 Overview ............................................... 63
  4.3 Preliminary Results and Motivation ................... 64
  4.4 Ideals .................................................. 71
  4.5 Additive structure ..................................... 76
  4.6 Multiplicative structure ............................... 78
    4.6.1 Prime Spectrum .................................. 82
    4.6.2 Factorization ..................................... 90
  4.7 Conclusion ............................................. 94
5 Conclusions and Further Research ....................... 97
Bibliography ................................................. 99
Chapter 1

Introduction

1.1 Basic Definitions and Notations

Let \( Z, N, N_0, Q \) and \( R \) be the sets of all integers, positive integers, nonnegative integers, rational numbers and real numbers, respectively. For a set \( S \), we denote the cardinality of \( S \) by \( |S| \). For two sets \( S \) and \( T \), \( S \subseteq T \) denotes the strict, or proper inclusion, i.e., \( S \subseteq T \) and \( S \neq T \). The set of all subsets of a given set \( S \) is denoted by \( 2^S \).

A graph is an ordered pair \((V, E)\), where \( V \) is a (possibly empty) finite set (\(|V| < \infty\)) and \( E \) is a (possibly empty) collection of nonempty subsets of \( V \) of cardinality at most two. If \( G = (V, E) \) is a graph, then elements of \( V = V(G) \) are called vertices of \( G \) and the elements of \( E = E(G) \) are called edges of \( G \). The sets from \( E \) of cardinality one are also called loops. A graph is called simple if it does not have any loop. Two vertices \( g, g' \in V \) are called adjacent if \( \{g, g'\} \in E \) and we will write \( g \sim g' \).

A bijective mapping \( \alpha : V(G) \to V(H) \) such that for all \( g, g' \in V(G) \), \( g \sim g' \) if and only if \( \alpha(g) \sim \alpha(g') \) is called an isomorphism. Since we are interested only in classes of isomorphic graphs, we say that two graphs \( G \) and \( H \) are equal, denote \( G = H \), if there exists an isomorphism from \( G \) to \( H \).

Example 1 A graph \( G \) with \( k \in N \) vertices \( V(G) = \{1, \ldots, k\} \) and \( k - 1 \) edges
$E(G) = \{\{1,2\}, \{2,3\}, \ldots, \{k-1,k\}\}$ is called the path of length $k-1$ and it is denoted by $P_k$, see Figure 1.1 (B).

**Example 2** A graph $G$ with $k \in \mathbb{N}$ vertices $V(G) = \{1, \ldots, k\}$ and $k$ edges $E(G) = \{\{1,2\}, \{2,3\}, \ldots, \{k-1,k\}, \{k,1\}\}$ is called the cycle of length $k$ and it is denoted by $C_k$, see Figure 1.1 (A), (C).

**Example 3** A graph $G$ with $k \in \mathbb{N}$ vertices $V(G) = \{1, \ldots, k\}$ and $0$ edges $E(G) = \emptyset$ is called the independent set of size $k$ and it is denoted by $I_k$.

**Example 4** A graph $G$ with $k \in \mathbb{N}$ vertices $V(G) = \{1, \ldots, k\}$ and $\binom{k}{2}$ edges $E(G) = \{S \subseteq V(G) \mid |S| = 2\}$ is called the clique, or the complete graph of size $k$ and it is denoted by $K_k$, see Figure 1.1 (C).

**Example 5** A graph $G$ with $s + t \in \mathbb{N}$ vertices $V(G) = \{1, \ldots, s + t\}$ where $s, t \in \mathbb{N}$ and $st$ edges $E(G) = \{\{i,j\} \subseteq V(G) \mid i \leq s \text{ and } s < j \leq s + t\}$ is called the complete bipartite graph of size $s + t$ and it is denoted by $K_{s,t}$, see Figure 1.1 (D).

**Example 6** A graph $G$ with $1$ vertex $V(G) = \{1\}$ and $1$ edge $E(G) = \{\{1\}\}$ is called the loop, and it is denoted by $1$.

**Example 7** A graph $G$ with $0$ vertices $V(G) = \emptyset$ and $0$ edges $E(G) = \emptyset$ is called the empty graph, and it is denoted by $0$. For a technical reason we also define the complete graph $K_0 = 0$.

Given two graphs $G$ and $H$, we say that $G$ is a subgraph of $H$, if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. If at least one of these inclusions is proper, then $G$ is a proper subgraph of $H$ and if $E(G) = 2^{V(G)} \cap E(H)$, then $G$ is called an induced subgraph of $H$. For a subset $S \subseteq V(G)$ we denote by $G|S$ the induced subgraph of $G$ with the vertex set $V(G|S) = S$. If $h$ is a vertex of $H$, we also write $H - h$ for the subgraph $H|(V(H) - \{h\})$. Graph $G$ is called a spanning subgraph of $H$ if $G$ is a subgraph of $H$ and $V(G) = V(H)$.
A homomorphism is a mapping $\alpha : V(G) \rightarrow V(H)$ from the vertex set $V(G)$ of a graph $G$ to the vertex set $V(H)$ of a graph $H$, which maps adjacent vertices of $G$ to adjacent vertices of $H$, i.e., $g \sim g'$ in $G$ implies $\alpha(g) \sim \alpha(g')$ in $H$. We say that a graph $G$ is homomorphic to a graph $H$, if such a homomorphism exists, and in this case we will write $G \rightarrow H$, or $G \xrightarrow{\alpha} H$. If there is no homomorphism from $G$ to $H$, we write $G \not\rightarrow H$.

A homomorphism $G \xrightarrow{\alpha} G$ is called an endomorphism.

For every graph $G$, there is a unique homomorphism $\ell_G$ to the graph $1$ which maps all vertices of $G$ to the unique vertex of $1$. Also, for every graph $G$, there is a unique homomorphism $\emptyset_G$ from $0$ to $G$ which is just the empty mapping.

If $G \xrightarrow{\alpha} H$ is a homomorphism and $W$ is a subgraph of $G$, then a restriction of $\alpha$ to $W$, denote $\alpha|W$, is a homomorphism $W \xrightarrow{\alpha|W} H$ defined by $(\alpha|W)(w) = \alpha(w)$ for every $w \in V(W) \subseteq V(G)$.

A homomorphism $W \xrightarrow{\alpha} G$ is called injective, if $w \neq w' \in V(W)$ implies $\alpha(w) \neq \alpha(w') \in V(G)$, i.e., if $\alpha$ is one to one. An injective homomorphism is also called an monomorphism.
A homomorphism $G \rightarrow H$ is called surjective, if $h \in V(H)$ implies that there is a vertex $g \in V(G)$ such that $\alpha(g) = h$, i.e., if $\alpha$ is onto vertices. A surjective homomorphism is also called an epimorphism (sometimes epimorphism is referred to a mapping which is onto both vertices and edges, cf. [27], but we will require from epimorphism to be onto vertices only).

A homomorphism which is both injective and surjective, is called a bimorphism.

One can observe that there is a bimorphism from $G$ to $H$ if and only if $G$ is isomorphic to a spanning subgraph of $H$. Note that every isomorphism is an injective and surjective homomorphism and hence bimorphism. But not every bimorphism is an isomorphism, since one can define a bimorphism from $C_4$ to $K_4$, but these two graphs are not isomorphic.

An isomorphism from $G \rightarrow G$ is called an automorphism. The automorphism which maps every vertex of $G$ to itself is denoted by $1_G$, and it is also called the identity automorphism. Also, one can observe that for an endomorphism $G \rightarrow G$ the following three conditions are equivalent (since we only consider finite graphs):

- $\alpha$ is a monomorphism,
- $\alpha$ is an epimorphism,
- $\alpha$ is an automorphism.

If $G \rightarrow H$ and $H \rightarrow W$ are two homomorphisms, then their composition is the mapping $\beta\alpha$, where $(\beta\alpha)(g) = \beta(\alpha(g))$ for every $g \in V(G)$. One can easily observe that $G \rightarrow W$ is a homomorphism. Since the operation composition is a composition of mappings, it is associative. Moreover, $\beta 1_G = \beta = 1_H \beta$. Also, one can observe that if $\beta\alpha$ is surjective, then necessarily $\beta$ is surjective and if $\beta\alpha$ is injective, then necessarily $\alpha$ is injective. Moreover, $\alpha$ is injective if and only if $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$ and $\alpha$ is surjective if and only if $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$. 
Note that if $W$ is a subgraph of $G$, then the natural inclusion $\iota$ which maps vertices of $G$ to the corresponding vertices of $H$ is a monomorphism and $\alpha \iota = \alpha | W$.

Graphs $G$ and $H$ are called equivalent, denoted $G \leftrightarrow H$, if $G \to H$ and $H \to G$. One can observe that the relation $\leftrightarrow$ is an equivalence and if $G_1 \leftrightarrow G_2$, $H_1 \leftrightarrow H_2$ and $G_1 \to H_1$ then also $G_2 \to H_2$. Equivalent classes of graphs are also called color families by Weldz [54].

If $G \xrightarrow{\alpha} H$ and $H \xrightarrow{\beta} G$ are two homomorphisms such that $\beta \alpha = 1_G$, then we say that $G$ is a retract of $H$ and we call $\beta$ a retraction and $\alpha$ a coretraction, i.e., a retraction is a homomorphism which has a right inverse and a coretraction is a homomorphism which has a left inverse. Therefore every coretraction is injective and every retraction is surjective. Let $G \xrightarrow{\alpha} H$ be a homomorphism, a homomorphism $\alpha'$ is called a weak coretraction, or a weak coretraction belonging to $\alpha$, if $\alpha \alpha'$ is an automorphism on $H$. One can easily observe that $\alpha$ is a retraction if and only if there is a weak coretraction $\alpha'$ belonging to $\alpha$, since given a weak coretraction $\alpha'$ belonging to $\alpha$, $\alpha'(\alpha \alpha')^{-1}$ is a coretraction belonging to $\alpha$. A composition of two retractions is also a retraction, since if $\alpha \alpha' = 1$ and $\beta \beta' = 1$ then $(\beta' \alpha')(\alpha \beta) = 1$.

For a homomorphism $G \xrightarrow{\alpha} H$ and subsets $S \subseteq V(G)$, $T \subseteq V(H)$ we denote $\alpha(S) = \{\alpha(s) \in V(H) \mid s \in S\}$ and $\alpha^{-1}(T) = \{t \in V(G) \mid \alpha(t) \in T\}$. For a one element set $S = \{s\}$ we will write $\alpha(s)$ instead of $\alpha(\{s\})$ and $\alpha^{-1}(s)$ instead of $\alpha^{-1}(\{s\})$.

The product of two graphs $G$ and $H$, denoted $G \times H$, is a graph with $V(G \times H) = V(G) \times V(H)$, in which $\{(s,u),(t,w)\} \in E(G \times H)$ if and only if $\{s,t\} \in E(G)$ and $\{u,w\} \in E(H)$. The homomorphisms $G \times H \xrightarrow{p_1} G$ and $G \times H \xrightarrow{p_2} H$ defined by $p_1(s,u) = s$ and $p_2(s,u) = u$ are called natural projections. If $\alpha_1$ is a homomorphism from $G$ to $H_1$ and $\alpha_2$ is a homomorphism from $G$ to $H_2$, then the product of $\alpha_1$ and $\alpha_2$ is the homomorphism $G \xrightarrow{\alpha_1 \times \alpha_2} H_1 \times H_2$ mapping each vertex $g$ of $G$ to the vertex $(\alpha_1(g),\alpha_2(g))$ of $H_1 \times H_2$. One can easily check that the product of two homomorphisms is indeed a homomorphism. On the other hand, every homomorphism
CHAPTER 1. INTRODUCTION

Figure 1.2: $K_3 \vee 3, a \circ C_6$

$G \xrightarrow{\alpha} H_1 \times H_2$ is a natural product of homomorphisms $\alpha_1 = p_1 \alpha$ and $\alpha_2 = p_2 \alpha$. In particular, $G \rightarrow H_1$ and $G \rightarrow H_2$ is equivalent to $G \rightarrow H_1 \times H_2$.

Note that this product is not the familiar cartesian product of two graphs but it is the categorical product.

The coproduct or sum of two graphs $G$ and $H$, denoted $G + H$, is a disjoint union of $G$ and $H$, i.e., the vertex set $V(G + H) = (V(G) \times \{0\}) \cup (V(H) \times \{1\})$ and for $\{(s,i),(t,j)\} \in E(G + H)$ if and only if $i = j$ and $\{s,t\} \in E(G) \cup E(H)$. The homomorphisms $G \xrightarrow{i_1} G + H$ and $H \xrightarrow{i_2} G + H$ defined by $i_1(s) = (s,0)$ and $i_2(t) = (t,1)$ are called natural inclusions. If $\alpha_1$ is a homomorphism from $G_1$ to $H$ and $\alpha_2$ is a homomorphism from $G_2$ to $H$, then the sum of $\alpha_1$ and $\alpha_2$ is the homomorphism $G_1 + G_2 \xrightarrow{\alpha_1 + \alpha_2} H$ mapping vertices of $G_1$ by $\alpha_1$ and vertices of $G_2$ by $\alpha_2$. One can easily check that the sum of two homomorphisms is indeed a homomorphism. On the other hand, every homomorphism $G_1 + G_2 \xrightarrow{\alpha} H$ is a natural sum of homomorphisms $\alpha_1 = \alpha|_{G_1}$ and $\alpha_2 = \alpha|_{G_2}$ (the restrictions of $\alpha$ to $G_1$ and $G_2$). In particular, $G_1 \rightarrow H$ and $G_2 \rightarrow H$ is equivalent to $G_1 + G_2 \rightarrow H$.

For two graphs $G$ and $H$ with two distinguished vertices $g \in V(G)$ and $h \in V(H)$ we define the wedge (or the amalgam) $G \vee_{g,h} H$ to be the graph obtained from $G + H$ by identifying the vertices $g$ and $h$ (see Figure 1.2). The homomorphisms $G \xrightarrow{i_1} G \vee_{g,h} H$
and $H \cong G \uplus_{g,h} H$ defined by $i_1(s) = (s,0)$ and $i_2(t) = (t,1)$ are called natural inclusions. If $\alpha_1$ is a homomorphism from $G_1$ to $H$ and $\alpha_2$ is a homomorphism from $G_2$ to $H$ such that for $g_1 \in V(G_1)$ and $g_2 \in V(G_2)$, $\alpha_1(g_1) = \alpha_2(g_2)$, then the wedge of $\alpha_1$ and $\alpha_2$ is the homomorphism $G_1 \uplus_{g_1,g_2} G_2 \overset{\alpha_1 \uplus_{g_1,g_2} \alpha_2}{\longrightarrow} H$ mapping vertices of $G_1$ by $\alpha_1$ and vertices of $G_2$ by $\alpha_2$. One can easily check that the wedge of two homomorphisms is indeed a homomorphism. On the other hand, every homomorphism $G_1 \uplus_{g_1,g_2} G_2 \overset{\alpha}{\longrightarrow} H$ is a natural wedge of homomorphisms $\alpha_1 = \alpha|_{G_1}$ and $\alpha_2 = \alpha|_{G_2}$ (the restrictions of $\alpha$ to $G_1$ and $G_2$).

A core is a graph $G$, such that every endomorphism from $G$ to $G$ is an automorphism.

A graph $G$ is called bipartite if $G \rightarrow K_2$. Therefore if $G$ is bipartite, then its vertex set can be partitioned into two (possibly empty) independent sets $A$ and $B$, i.e., $V(G) = A \cup B$ (disjoint union) such that no two vertices of $A$ are adjacent as well as no two vertices of $B$ are adjacent. If $V(K_2) = \{1,2\}$ and $\alpha$ is a homomorphism from $G$ to $K_2$, then $A = \alpha^{-1}(1)$ and $B = \alpha^{-1}(2)$. We will say that $A$ and $B$ form a bipartition of $G$. One can observe that if $G$ is a connected bipartite graph, then it has exactly one such bipartition up to switching the roles of $A$ and $B$ (1 and 2). If this is the case, we will write $G = \{A,B\}$. Recall that if every vertex of the set $A$ is connected to every vertex of the set $B$, $G$ is called a complete bipartite graph and it is denoted by $K_{|A|,|B|}$, since it is determined by the sizes of the sets $A$ and $B$. We allow $A$ or $B$ to be empty and hence 0 and $I_n$ are also complete bipartite graphs.

A graph $G$ is called disconnected if $G = H + W$ where $H \neq 0$ and $W \neq 0$, and the subgraphs $H$ and $W$ are called components of $G$. A graph which is not disconnected is called connected. Since we only consider finite graphs, every graph is a sum of its connected components and this sum is unique up to permutation of summands.
1.2 Background

A homomorphism of a graph $G$ into a graph $H$ is also called an $H$-coloring of $G$ and it is a generalization of a $k$-coloring of $G$, which can be thought of as a homomorphism of $G$ into the complete graph $K_k$. We will refer to both $H$-coloring and $k$-coloring as graph coloring.

Graph coloring originated with the four color problem (1852) of Francis Guthrie:

Is every planar graph 4-colorable? (cf. [36]).

The positive answer to this question was given by Appel, Haken and Koch [1] after more than a hundred years of development of graph theory.

Graph coloring deals with the fundamental problem of partitioning a set of objects into classes according to some rules. Therefore time tabling, sequencing, constrained scheduling problems or resource allocation problems can often be formulated as graph coloring problems (cf. [44]). For these applications, one wants to determine whether a given input graph is $H$-colorable.

It is well known that the 2-coloring problem is polynomial time solvable, since one can greedily color vertices of a given graph by two colors (cf. [44]). For a connected graph, one can easily observe that this coloring is unique up to a permutation of colors. Hence if $G$ is connected and $G \xrightarrow{\phi} K_2$, $G \xrightarrow{\psi} K_2$ are two 2-colorings of $G$, then there is an automorphism $K_2 \xrightarrow{\alpha} K_2$ such that $\psi = \alpha \phi$. On the other hand $k$-coloring is known to be NP-complete for all $k > 2$ [17].

The computational complexity of the $H$-coloring problem was completely characterized by Hell and Nešetřil [30]. In particular, they proved that the $H$-coloring problem is polynomial time solvable when $H$ is a bipartite graph and it is NP-complete otherwise.
We would like to mention at this point that the $H$-coloring problem has proven to be much more difficult in the directed case. In [21] it is proved that there are oriented trees $H$ and also some oriented cycles for which the $H$-coloring problem is NP-complete. On the other hand, for all oriented paths and some oriented cycles $H$, the $H$-coloring problem is polynomial time solvable (see [34, 43]). According to the latest results of Hell, Zhu, and Nešetřil [32] and independently of Feder and Vardi [15], for digraphs $H$ with a property called bounded treewidth duality (for the definition see [32]), the $H$-coloring is polynomial time solvable. It is conjectured that the $H$-coloring problem is not polynomial otherwise [29]. This conjecture is a counterpart of the characterization in the undirected case, since bipartite graphs are the only graphs which have bounded treewidth duality in the undirected case [29].

Another class of problems, which are considered from the complexity point of view, is to determine whether a given input graph has some property. In particular, there is a polynomial time algorithm to determine whether a given graph is bipartite (this is the same as 2-coloring the graph). It is also well known that finding connected components of a given graph can be done in polynomial time [44] and therefore determining whether a given graph is connected is also polynomial time solvable. On the other hand, determining whether a given graph is a core was shown to be NP-hard by Hell and Nešetřil [31]. In Chapter 2 we will study the equitable $H$-coloring problem and give a complete complexity classification of this problem. The equitable $H$-coloring problem is a natural generalization of the equitable coloring problem studied in [40, 45, 10, 23]. We will prove that the equitable $H$-coloring problem is polynomial time solvable whenever $H$ is a disjoint union of complete bipartite graphs and is NP-hard otherwise.

With each loopless graph $G$, we associate two numbers $\chi$ and $\omega$ which are called the chromatic number and the clique number of the graph, respectively. The definitions are as follows:

$$\chi = \chi(G) = \min \{n \in \mathbb{N}_0 \mid G \rightarrow K_n\}$$
and
\[ \omega = \omega(G) = \max\{n \in \mathbb{N}_0 \mid K_n \rightarrow G\}. \]

Computing these numbers is known to be NP-hard [17]. If \( \beta \) is a homomorphism \( H \xrightarrow{\beta} K_{\chi(H)} \), then \( G \xrightarrow{\alpha} H \) implies \( G \xrightarrow{\beta \alpha} K_{\chi(H)} \) and hence \( \chi(G) \leq \chi(H) \). Therefore \( \chi \) is the same for two equivalent graphs. Similarly, if \( \gamma \) is a homomorphism \( K_{\omega(G)} \xrightarrow{\gamma} G \), then \( G \xrightarrow{\alpha} H \) implies \( K_{\omega(G)} \xrightarrow{\gamma \beta} H \) and hence \( \omega(G) \leq \omega(H) \). Therefore \( \omega \) is the same for two equivalent graphs.

The numbers \( \chi \) and \( \omega \) also have the following integer linear program formulations:

\[
\begin{align*}
\omega(G) & = \max \sum_v x_v \\
\sum_v x_v & \leq 1 \text{ for each } I \\
x_v & \in \{0,1\}
\end{align*}
\]

\[
\begin{align*}
\chi(G) & = \min \sum_I y_I \\
\sum_I y_I & \geq 1 \text{ for each } v \\
y_I & \in \{0,1\}
\end{align*}
\]

In these formulations we take \( v \) to range over the vertex set of \( G \) and \( I \) to range over all maximal independent sets in \( G \). If we replace the integrality conditions \( x_v \in \{0,1\} \), \( y_I \in \{0,1\} \) by \( x_v \geq 0 \), \( y_I \geq 0 \) respectively, we obtain linear programs with optima \( \omega_f(G) \), the fractional clique number of \( G \), and \( \chi_f(G) \), the fractional chromatic number of \( G \). One can observe that these linear programs are duals of each other and hence \( \omega_f(G) = \chi_f(G) \) by the duality theorem of linear programming [11]. Feasible solutions \( \{x_g\} \) and \( \{w_I\} \) of these linear programs are called fractional clique and fractional coloring respectively. Therefore we have the inequalities:

\[ \omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G). \]

If \( G \xrightarrow{\alpha} H \) and \( \{w_I\} \) is a fractional coloring of \( H \), then one can define a fractional coloring \( \{u_J\} \) of \( G \) by

\[
\begin{align*}
u_J = \begin{cases} w_I & \text{if } J = \alpha^{-1}(I) \\
0 & \text{otherwise.} \end{cases}
\end{align*}
\]

Therefore \( G \rightarrow H \) implies \( \omega_f(G) = \chi_f(G) \leq \omega_f(H) = \chi_f(H) \) and we conclude that \( \omega_f = \chi_f \) for equivalent graphs. To compute the fractional chromatic number is NP-hard. This result was first proved by Grötschel, Lovász and Schrijver in [20] proving the fact that if one can optimize over a polytope in polynomial time then (using the
ellipsoid method) one can optimize over its antiblocker (for the definition we refer interested readers to [20]). Since the antiblocker of the fractional clique polytope is the independent set polytope, this shows NP-hardness of fractional chromatic number. We will give an alternative algebraic/combinatorial proof of this result in Chapter 3.

Grötschel, Lovász and Schrijver in [20] also proved that one can compute in polynomial time a real number $\theta$, which satisfies

$$\omega(G) \leq \theta \leq \omega_f(G).$$

The definition for $\theta$ is as follows [42, 39]: Let $G$ be a graph with $|V(G)| = n$. An orthonormal corepresentation of $G$ is a system $\{v_g\}_{g \in V(G)}$ of unit vectors in a vector space $\mathbb{R}^r$, such that if $\{i, j\} \in E(G)$ then $v_i v_j^T = 0$. Every graph has an orthonormal corepresentation in $\mathbb{R}^n$, for instance an orthonormal basis of $\mathbb{R}^n$. Let $u = \{u_g\}_{g \in V(G)}$ range over all orthonormal corepresentations of $G$ and $e_1$ be the unit vector with first coordinate equal to 1, then

$$\theta(G) = \min_u \max_{g \in V(G)} \frac{1}{(e_1^T u_g)^2}.$$ 

One can observe that $G \cong H$ implies that $\theta(G) \leq \theta(H)$, since if $\{v_h\}_{h \in V(H)}$ is an orthonormal corepresentation of $H$, then $\{v_{\alpha(h)}\}_{g \in V(G)}$ is an orthonormal corepresentation of $G$. Therefore $\theta$ is also the same for equivalent graphs and one can consider $\theta(G) \leq \theta(H)$ to be a polynomially computable necessary condition for the existence of a homomorphism from $G$ to $H$.

Another polynomially computable necessary condition called a hoax was given by Feige and Lovász in [16]. They used a semidefinite relaxation of a quadratic program, which was obtained from a multi-prover interactive proof system for the graph homomorphism problem.

The model of interactive proofs was introduced by Goldwasser, Micali and Rackoff [19] for cryptographic applications, and by Babai [4] as a game-theoretic extension of
The model consists of a probabilistic polynomial-time verifier $V$ and a prover $P$ who tries to convince $V$ that the input $x$ is in a language $L$.

$L \in IP$ if there exists a verifier $V$ that is always convinced when $x \in L$, but if $x \notin L$ then any prover $P$ has only a small probability of convincing $V$ to the contrary.

The model of multi-prover interactive proofs was introduced by Ben-Or, Goldwasser, Kilian, and Wigderson [8].

The model consists of a probabilistic polynomial-time verifier $V$ communicating with two provers who cannot communicate with each other during the protocol. The provers try to convince the verifier that the input $x$ is in a language $L$.

$L \in MIP$ if there exists a verifier $V$ that is always convinced when $x \in L$, but if $x \notin L$ then the provers have only a small probability of convincing $V$ to the contrary.


For the graph homomorphism problem, Feige and Lovász in [16] made the following two prover interactive proof system.

Let us say that the Verifier is trying to determine if a graph $G$ is homomorphic to a graph $H$. The Verifier chooses randomly and independently vertices $s$ and $t$ of $G$ and sends them, respectively to the provers $P_1$ and $P_2$. Then $P_1$ replies with a vertex $u$ from $H$ as the claimed image of $s$, and $P_2$ with a vertex $w$ from $H$ as the claimed image of $t$. The Verifier accepts just in the following situations:
1) If $s = t$ then $u = w$
2) If $s$ is adjacent to $t$ in $G$, then $u$ is adjacent to $w$ in $H$. 
CHAPTER 1. INTRODUCTION

This system can be viewed as an optimization problem over probabilistic strategies of the provers and therefore it has a quadratic programming formulation, whose convex (positive semidefinite) relaxation can be solved in polynomial time using the ellipsoid method. In Chapter 3 we will define a pseudo-homomorphism in terms of $\theta$ and a horn-product (to be defined) of graphs $G$ and $H$, and we will show that our pseudo-homomorphism is equivalent to the notion of hoax of Feige and Lovász.

The chromatic number is interesting also from the theoretical point of view. One of the most famous conjectures in graph theory is Hedetniemi’s conjecture [26]

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$ 

A graph $W$ is called multiplicative if for every two graphs $G$ and $H$,

$$G \times H \rightarrow W$$ 

implies that $G \rightarrow W$ or $H \rightarrow W$.

Hedetniemi’s conjecture is equivalent to the assertion that all complete graphs are multiplicative. One can easily prove that $K_1$ and $K_2$ are multiplicative and the proof of the multiplicativity of $K_3$ was accomplished by El-Zahar and Sauer [14]. This conjecture was extensively studied by Burr, Erdős and Lovász [9], Hell [28], Turzik [51], Duffus, Sands and Woodrow [12], Welzl [55] and others and they obtained some partial results. It is not even known whether the function

$$\mu(n) = \min\{m \mid G \nrightarrow K_n \text{ and } H \nrightarrow K_n \text{ but } G \times H \rightarrow K_m\}$$

tends to infinity or not. Hedetniemi’s conjecture is equivalent to the assertion that for all positive integers $n$, $\mu(n) = n + 1$. Poljak and Rödl [46] proved that $\mu(n)$ either tends to infinity with $n$, or is bounded above by 16. Duffus and Sauer [13] observed that one can naturally define the product and the sum of the equivalence classes of graphs (color families) and that they form a Heyting algebra. With every class they associated a particular Boolean algebra such that the class contains multiplicative graphs if and only if the corresponding Boolean algebra has exactly two elements. They proved that if all Boolean algebras associated with classes which contain complete graphs
are finite, then $\mu(n)$ tends to infinity with $n$. They also asked for conditions for the Boolean algebras to be finite. In Chapter 4 we will further follow this approach and show that the Boolean algebra associated with a particular color class is finite if and only if this class has a finite factorization into meet irreducible elements. We will also give a necessary and sufficient condition for an element of a Heyting algebra to have a finite factorization into meet irreducible elements.
Chapter 2

Equitable Graph Homomorphism

2.1 Introduction

In this chapter we consider the equitable coloring problem and a natural generalization of it, which we call the equitable $H$-coloring problem for simple graphs. Equitable colorings were studied in [40, 45, 10, 23], but not from the viewpoint of complexity. Here we give a complete complexity characterization of the equitable $H$-coloring problem.

A graph $G$ is said to be equitably colored with $k$ colors, if the vertices of $G$ are partitioned into $k$ classes $V_1, \ldots, V_k$ such that each $V_i$ is an independent set and the classes $V_i$, $i = 1, \ldots, k$ have almost the same size, i.e., $||V_i| - |V_j|| \leq 1$ for all $i$ and $j$. Recall that the graph $G$ is homomorphically to the graph $H$ ($G \rightarrow H$), if there is a mapping $\alpha : V(G) \rightarrow V(H)$ such that $(s, t) \in E(G)$ implies $(\alpha(s), \alpha(t)) \in E(H)$. Such mapping $\alpha$ is a homomorphism from $G$ to $H$. We say that a homomorphism $\alpha$ is equitable, if the sets $\alpha^{-1}(h) = \{g \in V(G) \mid \alpha(g) = h\}$ have almost the same size for all $h \in V(H)$, i.e., $||\alpha^{-1}(h)| - |\alpha^{-1}(h')|| \leq 1$ for all pairs $h, h' \in V(H)$. If there is an equitable homomorphism $\alpha$ from $G$ to $H$, we write $G \sim H$ or $G \sim^\alpha H$. An equitable homomorphism $G \sim H$ is also called an equitable $H$-coloring of $G$. Given a graph $H$, by the equitable $H$-coloring problem we mean the following decision problem:

INSTANCE: A graph $G$

QUESTION: Is $G \sim H$?
In this chapter we will prove that the equitable $H$-coloring problem is polynomial time solvable when $H$ is a disjoint union of complete bipartite graphs, and it is NP-hard otherwise. A similar result holds if we restrict the instances of this problem to be connected graphs. In particular, the restricted version of the equitable $H$-coloring problem is polynomial time solvable whenever $H$ is either not connected (for essentially trivial reasons), or is a complete bipartite graph, and it is NP-hard otherwise. The restriction of equitable $H$-coloring problem to connected instances will be called the \textit{connected equitable $H$-coloring problem}.

\section*{2.2 Overview}

In this chapter, we give a complete complexity characterization of the equitable $H$-coloring problem. In Section 2.3, we introduce some definitions and make some easy observations about equitable homomorphisms which will be needed further. In Section 2.4, we observe that the connected $H$-coloring problem is trivial in the case when $H$ is disconnected and it is polynomial time solvable if $H$ is a complete bipartite graph. Consequently we prove that the equitable $H$-coloring problem is polynomial time solvable if $H$ is a disjoint union of complete bipartite graphs. In Section 2.5, we will show that both problems are NP-hard in all other cases. In particular we give a Turing reduction from the connected $H$-coloring problem to the connected equitable $H$-coloring problem which settles the case when $H$ is not a bipartite graph. Then we give a polynomial reduction from the balanced complete bipartite subgraph problem to the connected equitable $P_4$-coloring problem which settles the case of the smallest noncomplete bipartite graph. Then we will settle the case when $H$ is a connected bipartite graph which is not a complete bipartite graph. This will be done by showing a Turing reduction from the connected equitable $P_4$-coloring problem. To settle the last case, i.e., the equitable $H$-coloring problem when $H$ is not a disjoint union of complete bipartite graphs, we will show a polynomial reduction to $H$ from the equitable $H'$-coloring problem where $H'$ is a connected component of $H$ which is
not a complete bipartite graph. In Section 2.6, we try to give some insight into ideas which are behind our reductions.

2.3 Preliminary Results

In this section, we will make some simple observations about equitable homomorphisms. First of all, let us notice that if the number of vertices of $G$ is $n$ and the number of vertices of $H$ is $m$, then $n$ can be uniquely written as $n = km + r$, where $0 \leq r < m$ and $k = \lceil \frac{n}{m} \rceil$.

**Lemma 1** $G \xrightarrow{\alpha} H$ is an equitable homomorphism if and only if for all $h \in V(H)$:

$$\frac{n}{m} \leq |\alpha^{-1}(h)| \leq \lfloor \frac{n}{m} \rfloor.$$

**Proof:** If for some $h \in V(H)$ we had $|\alpha^{-1}(h)| \leq \lfloor \frac{n}{m} \rfloor - 1$ then

$$n \leq (\lfloor \frac{n}{m} \rfloor - 1) + (m - 1)\lfloor \frac{n}{m} \rfloor = m\lfloor \frac{n}{m} \rfloor - 1 \leq m\frac{n}{m} - 1 = n - 1,$$

which is a contradiction. Similarly if for some $h \in V(H)$ we had $|\alpha^{-1}(h)| \geq \lceil \frac{n}{m} \rceil + 1$ then

$$n \geq (\lceil \frac{n}{m} \rceil + 1) + (m - 1)\lceil \frac{n}{m} \rceil = m\lceil \frac{n}{m} \rceil + 1 \geq m\frac{n}{m} + 1 = n + 1,$$

which is a contradiction. \(\square\)

**Corollary:** $G \xrightarrow{\alpha} H$ is an equitable homomorphism if and only if for $r$ vertices $h$ of $H$ we have $|\alpha^{-1}(h)| = k + 1$, and for $n - r$ vertices $h$ of $H$ we have $|\alpha^{-1}(h)| = k$.

For a given graph $H$ we say that $w : V(H) \to N$ is a weight function on $H$, i.e., if $w$ maps vertices of $H$ to positive integers. Let us fix the graph $H$ and a weight function $w$ on $H$; we write $w(H) = \sum_{h \in V(H)} w(h)$. We say that $\alpha$ is a $w$-weighted equitable homomorphism to $H$, written as $G \xrightarrow{\alpha_w} H$, if for all $h \in V(H)$

$$w(h)\left\lfloor \frac{|V(G)|}{w(H)} \right\rfloor \leq |\alpha^{-1}(h)| \leq w(h)\left\lceil \frac{|V(G)|}{w(H)} \right\rceil.$$
We also write \( G \sim_w H \) if there is a \( w \)-weighted equitable homomorphism from \( G \) to \( H \).

Let \( H \) be a graph. We write \( N_H(h) \) for the neighborhood of the vertex \( h \) in \( H \), i.e., \( N_H(h) = \{ h' \in V(H) \mid h \sim h' \} \). We say that two vertices \( h, h' \in V(H) \) are similar, denoted \( h \equiv h' \), whenever \( N_H(h) = N_H(h') \). This relation is an equivalence relation on the vertices of the graph \( H \) and partitions them into disjoint equivalence classes. Let \( H^r \) be a graph, whose vertex set is equal to the set of equivalence classes of the vertices from \( H \), and such that two equivalence classes \( c \) and \( c' \) are adjacent whenever there are two vertices \( h \in c \) and \( h' \in c' \) such that \( h \sim h' \) in \( H \). One can easily check that \( H^r \) is a well defined loopless graph and that there is a natural homomorphism \( \pi \) from \( H \) to \( H^r \) mapping a vertex \( h \in V(H) \) to the equivalence class \( c \) with \( h \in c \).

**Lemma 2** Let \( G \) and \( H \) be two graphs with \( |V(G)| = n \) and \( |V(H)| = m \). Let \( w \) be a weight function on \( H^r \) defined by \( w(c) = |c| \) for every equivalence class \( c \). Then \( G \sim H \) if and only if \( G \sim_w H^r \).

**Proof:** Let \( \alpha \) be an equitable homomorphism from \( G \) to \( H \) and let \( \pi \) be the natural homomorphism from \( H \) to \( H^r \). Then for all \( h \in V(H) \)

\[
\left[ \frac{n}{m} \right] \leq |\alpha^{-1}(h)| \leq \left[ \frac{n}{m} \right].
\]

Therefore summing for all \( h \in c \), we obtain

\[
|c| \left[ \frac{n}{m} \right] \leq |(\pi \alpha)^{-1}(c)| \leq |c| \left[ \frac{n}{m} \right].
\]

On the other hand, let us assume that there is a homomorphism \( G \sim_{\alpha}^w H^r \). This implies, that we can (arbitrarily) partition vertices from \( \beta^{-1}(c) \) into \( |c| \) disjoint sets \( A^c_1, \ldots, A^c_{|c|} \) such that \( \left[ \frac{n}{m} \right] \leq |A^c_i| \leq \left[ \frac{n}{m} \right] \) for all \( i = 1, \ldots, |c| \). If we denote the vertices of the class \( c \) by \( h^c_i, \ldots, h^c_{|c|} \), then mapping all vertices of \( A^c_i \) to the vertex \( h^c_i \) defines an equitable homomorphism from \( G \) to \( H \). \( \square \)

On the other hand, for a weight function \( w \) on \( H \) we can define a graph \( H_w \) with the vertex set \( V(H_w) = \bigcup_{h \in V(H)} \{ h \} \times \{ 1, \ldots, w(h) \} \) for which two vertices \((h, i) \) and
(h', j) are adjacent if and only if h and h' are adjacent in H. If w is a constant function, i.e., there is a constant k such that w(h) = k for all h ∈ V(H), then we will simply write w = k. (see Figure 2.1). Note that we have replaced each vertex h with w(h) copies of itself. One can observe that H = H_w for w = 1 and there is a natural homomorphism π from H_w to H mapping the vertex (h, i) to the vertex h. The following lemma is now a straightforward consequence of these definitions.

**Lemma 3** Let G and H be two graphs. Then

(a) G →_w H if and only if G → H_w, moreover if G →_H H_w, then G →_w H, where π is the natural homomorphism.

(b) G → H if and only if there is a bimorphism from G to some H_w such that \[ \lfloor \frac{n}{m} \rfloor \leq w(h) \leq \lceil \frac{n}{m} \rceil \] for all h ∈ V(H).

### 2.4 Polynomiality

In this section, we will observe that the connected equitable H-coloring problem is trivially polynomial time solvable if H is not connected. Then we will show that if H is a complete bipartite graph, then the connected equitable H-coloring problem is polynomial time solvable. Finally we will prove that the equitable H-coloring problem
is in \( \mathbb{P} \) whenever \( H \) is a disjoint union of complete bipartite graphs.

First, let us consider the connected equitable \( H \)-coloring problem and assume that \( H \) has at least two connected components, i.e., \( H = H_1 + \ldots + H_q \), \( q \geq 2 \). Then any connected graph \( G \) must be mapped to one of \( H_i \), say \( H_1 \). Therefore, if \( G \preceq H \), then \( |\alpha^{-1}(h)| = 0 \) for all \( h \in V(H_i) \) for \( 2 \leq i \leq q \). Since \( \alpha \) is an equitable homomorphism,

\[
\left\lfloor \frac{n}{m} \right\rfloor \leq |\alpha^{-1}(h)| = 0 \leq \left\lceil \frac{n}{m} \right\rceil,
\]

where \( n = |V(G)| \) and \( m = |V(H)| \). We conclude that \( \left\lfloor \frac{n}{m} \right\rfloor = 0 \) and \( \left\lceil \frac{n}{m} \right\rceil \leq 1 \). Therefore no two vertices can be mapped to the same vertex, and hence the homomorphism is injective. This implies that \( G \) is a subgraph of \( H_1 \) and therefore also of \( H \). Since \( H \) is fixed, it can be decided by brute force whether \( G \) is a subgraph of \( H \) in constant time (since \( G \) can contain at most \( |V(H)| \) vertices to be equitably homomorphic to \( H \)). We conclude that in the case that \( H \) is not connected, the connected equitable \( H \)-coloring problem is trivially polynomial time solvable.

Next, let us consider the connected equitable \( H \)-coloring problem and assume that \( H \) is a complete bipartite graph \( K_{s,t} \). Since \( K_{s,t} = K_2 \), our problem is to find \( w \)-equitable homomorphism from \( G \) to \( K_2 \) with weight function \( w(1) = s \) and \( w(2) = t \). Let \( G \) be a connected graph. One can decide in linear time whether \( G \) is bipartite and find a 2-coloring if there is one. Therefore, if \( G \) is bipartite, then we can get the unique bipartition \( \{A, B\} \) of \( G \) in linear time. Hence if \( G \sim w K_2 \), then all vertices of \( A \) have to be mapped to the vertex 1 and all vertices of \( B \) have to be mapped to the vertex 2 or vice versa. Therefore \( G \sim K_{s,t} \) if and only if either

\[
s\left[ \frac{|A| + |B|}{s + t} \right] \leq |A| \leq s\left[ \frac{|A| + |B|}{s + t} \right] \quad \text{and} \quad t\left[ \frac{|A| + |B|}{s + t} \right] \leq |B| \leq t\left[ \frac{|A| + |B|}{s + t} \right],
\]

or

\[
t\left[ \frac{|A| + |B|}{s + t} \right] \leq |A| \leq t\left[ \frac{|A| + |B|}{s + t} \right] \quad \text{and} \quad s\left[ \frac{|A| + |B|}{s + t} \right] \leq |B| \leq s\left[ \frac{|A| + |B|}{s + t} \right],
\]

which can be decided in polynomial time. Therefore we have just proved the following lemma.
Lemma 4 If $H$ is a complete bipartite graph, then the connected equitable $H$-coloring problem is in $P$.

Now we can prove the main result of this section.

Theorem 5 If $H$ is a disjoint union of complete bipartite graphs then the equitable $H$-coloring problem is in $P$.

Proof: We will modify the dynamic programming approach to the partition problem from Garey and Johnson [17] p. 90. Assume the connected components of $H$ are $K_{u_1,v_1}, \ldots, K_{u_r,v_r}$ and the connected components of $G$ are $G_1, \ldots, G_k$. One can observe that the graph $H'$ consists of $r$ disjoint copies of $K_2$ and we will denote the two vertices of the $j$-th copy by $a^j$ and $b^j$. By Lemma 2, $G \rightsquigarrow H$ if and only if $G \rightsquigarrow_w H'$, where $w$ is defined by $w(a^j) = u_j$ and $w(b^j) = v_j$. Let $\alpha$ be a homomorphism from $G$ to $H'$ and let $s^j_\alpha = |\alpha^{-1}(a^j)|$ denote the number of vertices mapped to $a^j$ by $\alpha$ and $t^j_\alpha = |\alpha^{-1}(b^j)|$ denote the number of vertices mapped to $b^j$ by $\alpha$. According to our definition, $\alpha$ is a $w$-equitable homomorphism if and only if for all $q = 1, \ldots, r$ the following inequalities are satisfied:

$$u_q \left\lceil \frac{|V(G)|}{|V(H)|} \right\rceil \leq s^q_\alpha \leq u_q \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor$$

and

$$v_q \left\lceil \frac{|V(G)|}{|V(H)|} \right\rceil \leq t^q_\alpha \leq v_q \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor.$$

By Lemma 2 we conclude that $G \rightsquigarrow H$ if and only if there is a homomorphism $\alpha$ from $G$ to $H'$, such that these conditions are satisfied. For integers $1 \leq m \leq k$, $0 \leq j_1 \leq u_1 \left\lceil \frac{|V(G)|}{|V(H)|} \right\rceil, \ldots, 0 \leq j_r \leq u_r \left\lceil \frac{|V(G)|}{|V(H)|} \right\rceil, 0 \leq l_1 \leq v_1 \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor, \ldots, 0 \leq l_r \leq v_r \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor$, let $t(m, j_1, \ldots, j_r, l_1, \ldots, l_r)$ denote the truth value of the statement: "there is a homomorphism $\alpha$ from $G_1 + \ldots + G_m$ to $H'$, such that for all $q = 1, \ldots, r$, $s^q_\alpha = j_q$ and $t^q_\alpha = l_q"$. One can observe that $G \rightsquigarrow H$ if and only if there is an entry $t(k, j_1, \ldots, j_r, l_1, \ldots, l_r)$ which has value $T$ and for all $q = 1, \ldots, r$, $j_q \geq u_q \left\lceil \frac{|V(G)|}{|V(H)|} \right\rceil$ and $l_q \geq v_q \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor$. We will show a recursive procedure that can be used to compute $t(k, j_1, \ldots, j_r, l_1, \ldots, l_r) = T$. First of all, one can observe that $t(1, j_1, \ldots, j_r, l_1, \ldots, l_r) = T$.
if and only if there is a \( q \in \{1, \ldots, r\} \), such that \( j_q = |A_1|, \ l_q = |B_1| \) and \( j_p = l_p = 0 \) for \( p \neq q \), or \( j_q = |B_1|, \ l_q = |A_1| \) and \( j_p = l_p = 0 \) for \( p \neq q \), which encodes that \( G_1 \) is mapped to the \( q \)-th copy of \( K_2 \) and either the vertices of \( A_1 \) are mapped to the vertex \( a^q \) and the vertices of \( B_1 \) are mapped to the vertex \( b^q \) or the vertices of \( A_1 \) are mapped to the vertex \( b^q \) and the vertices of \( B_1 \) are mapped to the vertex \( a^q \). Having computed all \( t(m, j_1, \ldots, j_r, l_1, \ldots, l_r) \) with given \( m \), \( t(m+1, j_1, \ldots, j_r, l_1, \ldots, l_r) = T \) if and only if there is a \( q \in \{1, \ldots, r\} \), such that either

\[
t(m, j_1, \ldots, j_q - |A_{m+1}|, \ldots, j_r, l_1, \ldots, l_q - |B_{m+1}|, \ldots, j_r) = T,
\]

or

\[
t(m, j_1, \ldots, j_q - |B_{m+1}|, \ldots, j_r, l_1, \ldots, l_q - |A_{m+1}|, \ldots, j_r) = T,
\]

which encodes that \( G_{m+1} \) is mapped to the \( q \)-th copy of \( K_2 \). The number of tuples \((k, j_1, \ldots, j_r, l_1, \ldots, l_r)\) is bounded by \(|V(G)|^{2r+1}\) and therefore one can decide whether \( G \sim H \) in polynomial time. □

### 2.5 NP-hardness

Both the equitable \( H \)-coloring problem and the connected equitable \( H \)-coloring problem are clearly in NP. In this section, we will focus our attention on finding reductions from known NP-hard problems. Since the connected equitable \( H \)-coloring problem is a restriction of the equitable \( H \)-coloring problem, if we prove that for a given graph \( H \) the connected equitable \( H \)-coloring problem is NP-hard, then an immediate corollary is that the equitable \( H \)-coloring problem is also NP-hard. Similarly as in the equitable case, the version of the \( H \)-coloring problem restricted to the connected instances will be called the connected \( H \)-coloring problem.

**Theorem 6** There is a Turing reduction from the connected \( H \)-coloring problem to the connected equitable \( H \)-coloring problem.
Proof: Let us assume that the vertex set of $H$ is $V(H) = \{1, \ldots, m\}$. Let $G$ be an instance of the connected $H$-coloring problem, that is, $G$ is connected. Let us fix a vertex $g' \in V(G)$ and let $|V(G)| = n$. We will show that the following two conditions are equivalent:

(a) $G \rightarrow H$

(b) $G \forall_{g', h} H_w \sim H$ for at least one $h \in V(H)$ and at least one weight function $w$ on $H$ such that $w(1) + \ldots + w(m) = mn + 1$

The implication (b) $\Rightarrow$ (a) follows from the fact that $G \rightarrow G \forall_{g', h} H_w$ for all $h \in V(H)$ and all weight functions $w$ on $H$. It remains to prove the implication (a) $\Rightarrow$ (b). Let $\alpha$ be a homomorphism from $G$ to $H$. If we denote $x_i = |\alpha^{-1}(i)|$ for $i = 1, \ldots, m$, then $x_1 + \ldots + x_m = n$. Therefore $mn = (m + 1)n - n = ((m + 1) - x_1) + \ldots + ((m + 1) - x_n)$. Without loss of generality we may assume that $\alpha(g') = 1$. It follows, that for $w$ such that $w(1) = (m + 1) - x_1 + 1$ and $w(h) = (m + 1) - x_h$ for $h \neq 1$ we have $G \forall_{g', 1} H_w \xrightarrow{\alpha_{g', 1}} H$ where $\pi$ is the natural homomorphism from $H_w$ to $H$. (We added 1 to the first element, since $g'$ and 1 are identified by the wedge.) Since $G$ and $H$ are connected, $G \forall_{g', 1} H_w$ is also connected. To see that our reduction is polynomial, one can observe that the graph $G \forall_{g', h} H_w$ has $(m + 1)n$ nodes which is $O(n)$ for the fixed graph $H$. The number of graphs $G \forall_{g', h} H_w$ is $m \binom{mn}{n-1}$, since $m$ is the number of ways we can choose $h \in V(H)$ and $\binom{mn}{m-1}$ is the number of solutions to the equation $w_1 + \ldots + w_m = mn + 1$ in positive integers. For a fixed graph $H$, this is $O(n^{m-1})$ and we conclude that our reduction is indeed polynomial.

Hell and Nešetřil [30] have shown that for nonbipartite $H$, $H$-coloring is NP-complete. For connected $H$, $G \rightarrow H$ if and only if for all connected components $G'$ of $G$, $G' \rightarrow H$. Therefore there is a Turing reduction from the $H$-coloring problem to the connected $H$-coloring problem. This implies the following corollaries.

**Corollary 7** If $H$ is a connected nonbipartite graph, then the connected equitable $H$-coloring problem is NP-hard.

**Corollary 8** If $H$ is a connected nonbipartite graph, then the equitable $H$-coloring problem is NP-hard.
The following problem, known as the \textit{balanced complete bipartite subgraph} problem is NP-complete, see [17]:

\textbf{INSTANCE}: Bipartite graph \( G \) and a positive integer \( m \leq \frac{1}{2}|V(G)| \).

\textbf{QUESTION}: Does \( G \) have a subgraph isomorphic to \( K_{m,m} \)?

We will actually need the following restriction of the balanced complete bipartite subgraph problem.

The \textit{restricted balanced complete bipartite subgraph} problem:

\textbf{INSTANCE}: Bipartite graph \( G \), its bipartition \( \{A, B\} \) and a positive integer \( m \) such that \( \frac{1}{2} \max\{|A|,|B|\} < m \leq \frac{1}{2}|V(G)| \).

\textbf{QUESTION}: Does \( G \) have a subgraph isomorphic to \( K_{m,m} \)?

\textbf{Lemma 9} The restricted balanced complete bipartite subgraph problem is NP-complete.

\textbf{Proof}: Membership in NP is trivial. We will give a polynomial-time reduction from the balanced complete bipartite subgraph problem. Let \( G \) and \( m \) be an instance of the balanced complete bipartite subgraph problem. It is well known that we can find a bipartition \( \{A, B\} \) of \( G \) in polynomial time. Let the number of vertices of \( G \) be \( n = |V(G)| \). Let us construct a new graph \( G' \) with bipartition \( \{A', B'\} \) as follows. First, we create \( 2n \) new vertices. Let \( A' \) contain all of the vertices of \( A \) and \( n \) of the new vertices and let \( B' \) contains all vertices of \( B \) and the remaining \( n \) new vertices. We connect \( a \in A' \) with \( b \in B' \) if and only if \( a \) is a new vertex, or \( b \) is a new vertex, or \( a \sim b \) in \( G \). One can observe that \( G \) has a subgraph isomorphic to \( K_{m,m} \) if and only if \( G' \) has a subgraph isomorphic to \( K_{m+n,m+n} \). Moreover \( \frac{1}{2}|V(G')| = \frac{1}{2}|V(G)| + n \geq m + n > n = \frac{n}{2} + \frac{|V(G)|}{2} \geq \frac{n}{2} + \frac{1}{2} \max\{|A|,|B|\} = \frac{1}{2} \max\{n + |A|, n + |B|\} = \frac{1}{2} \max\{|A'|,|B'|\} \).

Let us recall that \( P_4 \) denotes the path of length three on four vertices, i.e., the graph obtained from \( K_{2,2} = C_4 \) by removing one edge (see Figure 1.1 (B)). Note that \( P_4 \) is the smallest connected graph which is not a complete bipartite graph. The following proposition will serve as a basis for our inductive proof.

\textbf{Proposition 10} The connected equitable \( P_4 \)-coloring problem is NP-complete.
Proof: Recall that the vertex set of $P_4$ is $V(P_4) = \{1, 2, 3, 4\}$ and the edge set is $E(P_4) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. We will give a polynomial-time reduction from the restricted balanced complete bipartite subgraph problem. Let bipartite graph $G$, its bipartition $\{A, B\}$, and a positive integer $m$ such that $\frac{1}{2} \max\{|A|, |B|\} < m \leq \frac{1}{2}|V(G)|$, be an instance of the restricted balanced complete bipartite subgraph problem. We construct a connected bipartite graph $G'$ with a bipartition $\{A', B'\}$ as follows. First, we create $4m - |V(G)|$ new vertices. Let $A'$ contains all vertices of $A$ and $2m - |A|$ of the new vertices and let $B'$ contains all vertices of $B$ and the remaining $2m - |B|$ new vertices. We connect $a \in A'$ with $b \in B'$ if and only if $a$ is a new vertex, or $b$ is a new vertex, or $a \not\sim b$ in $G$. The restrictive condition on $m$ implies that we have added a positive number of new vertices to both $A$ and $B$ and therefore our graph $G'$ is connected. We now claim that $G$ has a subgraph isomorphic to $K_{m,m}$ if and only if $G' \sim P_4$. Let us assume that $G$ has a subgraph isomorphic to $K_{m,m}$. This implies that there are vertices $a_1, \ldots, a_m \in A$ and vertices $b_1, \ldots, b_m \in B$ such that for all $i = 1, \ldots, m$ and $j = 1, \ldots, m$, $a_i \sim b_j$ in $G$. Define a mapping $A' \cup B' \to \{1, 2, 3, 4\}$ so that the $m$ vertices $a_i$ are mapped to 1 and the remaining $m$ vertices from $A'$ are mapped to 3. Similarly, the vertices $b_j \in B'$ are mapped to 4 and the rest of the vertices from $B'$ are mapped to 2. One can easily observe that this mapping is an equitable homomorphism from $G'$ to $P_4$.

For the converse, let us assume that $\alpha$ is an equitable homomorphism from $G'$ to $P_4$. It follows that $|\alpha^{-1}(i)| = m$ for each $i = 1, 2, 3, 4$. Without loss of generality we may assume that $\alpha(A') = \{1, 3\}$ and $\alpha(B') = \{2, 4\}$. Let $\alpha^{-1}(1) = \{a_1, \ldots, a_m\} \subseteq A'$ and $\alpha^{-1}(4) = \{b_1, \ldots, b_m\} \subseteq B'$. Since $1 \not\sim 4$ in $P_4$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, m$, we must have $a_i \not\sim b_j$ in $G'$. Therefore none of $a_1, \ldots, a_m, b_1, \ldots, b_m$ can be a new vertex, as new vertices from $A'$ or $B'$ are connected to all of the vertices from the other set. This implies that $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$, and that $a_i \sim b_j$ in $G$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, m$. We conclude that $G$ has a subgraph isomorphic to $K_{m,m}$. □

Corollary 11 The equitable $P_4$-coloring problem is NP-complete.
CHAPTER 2. EQUITABLE GRAPH HOMOMORPHISM

The following lemma will make our proofs a little bit easier.

**Lemma 12** There is a Turing reduction from the connected equitable $H$-coloring problem to the restriction of the connected equitable $H$-coloring problem to connected instances $G$ such that $|V(H)|$ divides $|V(G)|$.

**Proof:** Let connected graph $G$ be an instance of the connected equitable $H$-coloring problem, $|V(G)| = n$ and $|V(H)| = m$. Let $n = km + r$, where $0 \leq r < m$. If $\alpha$ is an equitable homomorphism from $G$ to $H$, then for $r$ vertices $h$ of $H$: $|\alpha^{-1}(h)| = k + 1$ and for the remaining $m - r$ vertices $h$ of $H$: $|\alpha^{-1}(h)| = k$. For every subset $S \subseteq V(G)$ of cardinality $m - r$, define $w_S$ to be a weight function on $G$ given by

$$w_S(g) = \begin{cases} 2 & \text{if } g \in S \\ 1 & \text{otherwise.} \end{cases}$$

All graphs $G_{w_S}$ are connected and they have $(k + 1)m$ vertices and obviously $m$ divides $(k + 1)m$. We now claim that $G \leadsto H$ if and only if for at least one $S$, $G_{w_S} \leadsto H$. If $G \leadsto H$, then for $m - r$ vertices $h$ of $H$ we have $|\alpha^{-1}(h)| = k$. Let us choose one $g \in \alpha^{-1}(h)$ for each such $h$ to form the set $S$ of cardinality $m - r$. One can observe that if we define $\beta((g, i)) = \alpha(g)$ from $G_{w_S}$ to $H$, then for each $h \in V(H)$, $|\beta^{-1}(h)| = k + 1$ and hence $G_{w_S} \leadsto H$.

On the other hand, let us assume that for some $S$, we have $G_{w_S} \leadsto H$. Let $G \leadsto G_{w_S}$ be the coretraction defined by $\pi^1(g) = (g, 1)$ for all $g \in V(G)$. Then $G \leadsto H$.

Since there are $\binom{n}{m-r} = O(n^m)$ subsets of $V(G)$ of cardinality $r$ and $m$ is fixed, our reduction is polynomial. □

For a technical reason we also need the following four lemmas. They will enable us to give a Turing reduction from the connected equitable $P_4$-coloring problem to the connected equitable $H$-coloring problem in the case when $H$ is connected bipartite graph which is not complete bipartite.

**Lemma 13** The natural homomorphism $\pi$ from $H_w$ to $H$ is a retraction.
Proof: To see this, one can define a coretraction \( \pi' \) from \( H \) to \( H_w \) by mapping each vertex \( h \in V(H) \) to any one of the vertices \((h,1),(h,2),\ldots,(h,w(h)) \in V(H_w)\).

**Lemma 14** Let \( n \) be a positive integer, \( H \) be a graph on \( m \) vertices and \( w \) be a weight function on \( H \) such that \( w(h) \geq mn \) for all \( h \in V(h) \). Then any homomorphism \( H_w \rightarrow H \) such that \( |\phi^{-1}(h)| \leq (m + 1)n \) for all \( h \in V(H) \) is a retraction. Moreover, there exists a weak coretraction \( \pi' \) belonging to \( \phi \) such that \( \pi' \) is a coretraction belonging to the natural homomorphism \( \pi \) from \( H_w \) to \( H \).

Proof: Let us consider sets \( B_h = \{(h,1),\ldots,(h,w(h))\} \subseteq V(H_w) \) and \( A_h = \{\phi((h,1)),\ldots,\phi((h,w(h)))\} \subseteq V(H) \) for all \( h \in V(H) \). We claim that the sets \( A_h \) for \( h \in V(H) \) have a system of distinct representatives, i.e., vertices \( a_1,\ldots,a_m \in V(H) \) such that each \( a_i \in A_i \). According to Hall’s theorem [24], given a collection of \( m \) sets, there is a system of distinct representatives if and only if, for all \( 1 \leq s \leq m \), the union of any \( s \) of the sets has cardinality at least \( s \). For a contradiction, suppose that this is not the case, i.e., that there are \( s \leq m \) sets \( A_{h_1},\ldots,A_{h_s} \) such that

\[
|\bigcup_{i=1}^{s} A_{h_i}| \leq s - 1.
\]

We may assume that \( s > 1 \), since all sets \( A_h \) are nonempty. This means that all vertices from \( \bigcup_{i=1}^{s} B_{h_i} \) are mapped to at most \( s - 1 \) vertices from \( H \). The sets \( B_{h_i} \) have cardinality at least \( mn \) each and they are disjoint. Therefore at least \( \frac{smn}{s-1} \) vertices of \( H_w \) are mapped to the same vertex, i.e., there is a vertex \( h \in V(H) \) with

\[
|\phi^{-1}(h)| \geq \frac{smn}{s-1} = mn + \frac{mn}{s-1} > mn + \frac{mn}{m} = mn + n,
\]

which is a contradiction. We conclude that the sets \( A_h \) for \( h \in V(H) \), have a system of distinct representatives \( a_h, h \in V(H) \). We may assume that \( a_h = \phi((h,i_h)) \) for some \( i_h \in \{1,\ldots,w(h)\} \); we now define \( \pi'(h) = (h,i_h) \). One can observe that \( \pi' \) is a homomorphism, since \( h \sim h' \) implies \((h,i_h) \sim (h',i_{h'})\). Moreover \( (\pi\pi')(h) = \pi(h,i_h) = h \) and hence \( \pi' \) is a coretraction belonging to \( \pi \). Finally, since \((\phi\pi')(h) = a_h \) and since the \( a_h, h \in V(H) \) are distinct, we conclude that \( \phi\pi' \) is an injective endomorphism
and hence an automorphism. Hence \( \pi' \) is a weak coretraction belonging to \( \phi \) and we conclude that \( \phi \) is a retraction. \( \Box \)

**Lemma 15** Let \( w \) be a weight function on \( H, H_w \xrightarrow{\phi} H \) a retraction, \( H_w \xrightarrow{\pi} H \) the natural homomorphism and for \( h \in V(h), B_h = \{(h,1), \ldots, (h,w(h))\} \subseteq V(H_w) \). Then for every coretraction \( H \xrightarrow{\phi'} H_w \) belonging to \( \phi \) and for every coretraction \( H \xrightarrow{\pi'} H_w \) belonging to \( \pi \):

(a) \( \pi'(h) = (h, i) \) for every \( h \in V(H) \) and some \( i \in \{1, \ldots, w(h)\} \),

(b) \( N_{H_w}(h^*) = N_{H_w}((\pi'\pi)(h^*)) \) for every \( h^* \in V(H_w) \),

(c) \( N_H(h) \subseteq N_{H_w}(\phi(\pi'(\phi'))(h)) \) for every \( h \in V(H) \), and

(d) \( N_H(h) \subseteq N_{H_w}(\phi(h^*)) \) for every \( h \in V(H) \) and every \( h^* \in B_{(\pi\phi)(h)} \).

**Proof:**

(a): Let us assume that \( \pi'(h) = (h', i) \in V(H_w) \) for some \( i \in \{1, \ldots, w(h')\} \). Since \( \pi\pi' = id_H \), we have \( h = (\pi\pi')(h) = \pi(h', i) = h' \). We conclude that \( \pi'(h) = (h, i) \).

(b): If \( h^* = (h, j) \) for some \( j \in \{1, \ldots, w(h)\} \), then by (a) \( \pi'(h) = (h, i) \). Therefore \( (\pi'\pi)(h^*) = \pi'(h) = (h, i) \). From the definition of the graph \( H_w \) it follows that \( N_{H_w}((h, i)) = N_{H_w}((h, j)) \) and we conclude that \( N_{H_w}(h^*) = N_{H_w}((\pi'\pi)(h^*)) \).

(c): Since \( \phi' \) is a homomorphism, \( u \in N_H(h) \) implies \( \phi'(u) \in N_H(\phi'(h)) = N_H((\pi'\pi)(\phi'(h))) \) by (b). Therefore \( u = \phi(\phi'(u)) \in N_H(\phi((\pi'\pi\phi')(h))) \), since \( \phi \) is a homomorphism and \( \phi\phi' = id_H \).

(d): Let \( \phi'(h) = (h', i) \) for some \( i \in \{1, \ldots, w(h')\} \). Then \( (\pi\phi')(h) = \pi((h', i)) = h' \) and if \( h^* \in B_{(\pi\phi')(h)} = B_{h'} \), then \( h^* = (h', j) \) for some \( j \in \{1, \ldots, w(h')\} \). Let \( \pi' \) be a coretraction belonging to \( \pi \) defined by

\[
\pi'(h) = \begin{cases} 
(h', j) = h^* & \text{if } h = h' = \pi(h^*) \\
(h, 1) & \text{otherwise.}
\end{cases}
\]

By part (c) we have \( N_H(h) \subseteq N_{H_w}((\phi\pi'\pi\phi')(h)) = N_H((\phi\pi'\pi)((h', i))) = N_{H_w}(\phi(h')) = N_{H_w}(\phi(h^*))) = N_H(\phi(h^*)). \) \( \Box \)
**Lemma 16** Let $G$ and $H$ be two graphs, $w$ a weight function on $H$ such that $w(h) \leq \left\lceil \frac{|V(G + H_w)|}{|V(H)|} \right\rceil$ for all $h \in V(H)$, $H_w \xrightarrow{\gamma} H$ the natural homomorphism, and $G + H_w \xrightarrow{\beta} H$ an equitable homomorphism. Let $\phi = \beta|H_w$, and for some coretraction $\pi'$ belonging to $\pi$, $\phi\pi'$ is an automorphism on $H$. Then there exists an equitable homomorphism $G + H_w \xrightarrow{\gamma} H$ such that $\gamma|H_w = \pi$.

**Proof:** Let us denote $B_h = \{(h, 1), \ldots, (h, w(h))\} \subseteq V(H_w)$ for $h \in V(H)$ and $\alpha = (\phi\pi')^{-1}$ be the inverse of the automorphism $\phi\pi'$, i.e., $\phi\pi'\alpha = id_H$. Thus $\phi$ is a retraction and $\phi' = \pi'\alpha$ is a coretraction belonging to $\phi$. We will prove the lemma by induction on the number $q = q_\beta = \sum_{h \in V(H)} |\{h^* \in B_{\alpha(h)} \mid \phi(h^*) \neq h\}|$.

For $q = 0$, all vertices from each $B_{\alpha(h)}$ are mapped to the vertex $h$ by $\phi$ and hence $\alpha\phi = \pi$. Since $\alpha$ is an automorphism, it follows that for $\gamma = \alpha\beta$ we have $G + H_w \xrightarrow{\gamma} H$ and $\gamma|H_w = \pi$.

Assume that $q > 0$. Let $h^* \in B_{\alpha(h)}$ be such that $\phi(h^*) \neq h$. Since $\beta$ is an equitable homomorphism,

$$|\beta^{-1}(h)| \geq \left\lceil \frac{|V(G + H_w)|}{|V(H)|} \right\rceil \geq w(\alpha(h)) = |B_{\alpha(h)}|.$$  

Since $h^* \in B_{\alpha(h)}$ and $h^* \notin \beta^{-1}(h)$, there is a vertex $w \in \beta^{-1}(h)$ such that $w \notin B_{\alpha(h)}$. Note that $w$ can be a vertex of $G$. Define a mapping $\beta_1$ from $G + H_w$ to $H$ by switching the images of $w$ and $h^*$ in $\beta$, i.e., for $u \in V(G + H_w)$ define

$$\beta_1(u) = \begin{cases} h & \text{if } u = h^* \\ \phi(h^*) & \text{if } u = w \\ u & \text{otherwise.} \end{cases}$$  

Obviously, since $\beta$ is equitable, $\beta_1$ is also equitable and $q_{\beta_1} < q_{\beta}$. We will show that $\beta_1$ is a homomorphism. Since $B_{(\pi\phi')(h)} = B_{(\pi\phi')(\alpha(h))} = B_{\alpha(h)}$ and $h^* \in B_{\alpha(h)}$, it follows from part (d) of Lemma 15 that $N_H(h) \subseteq N_H(\phi(h^*))$. If $u, v \in V(G + H_w)$ and $u \sim v$, then we distinguish the following four cases:

Case 1: $\{u, v\} \cap \{w, h^*\} = \emptyset$. In this case $\beta_1(u) = \beta(u) \sim \beta(v) = \beta_1(v)$, since $\beta$ is a homomorphism.

Case 2: $\{u, v\} \cap \{w, h^*\} = \{w\}$. We may assume that $w = v$ and let us recall that
\[ \beta(w) = h, \text{ since } w \in \beta^{-1}(h). \] Hence \[ \beta_1(u) = \beta(u) \sim \beta(v) = \beta(w) = h \] and therefore \[ \beta_1(u) \in N_H(h) \subseteq N_H(\phi(h^*)) = N_H(\beta_1(w)) = N_H(\beta_1(v)). \] We conclude that \[ \beta_1(u) \sim \beta_1(v). \]

Case 3: \( \{u,v\} \cap \{w,h^*\} = \{h^*\}. \) We may assume that \( v = h^* \in V(H_w) \) and since \( u \sim v, \) also \( u \in V(H_w). \) Therefore we have \( u \sim h^* \) in \( H_w \) and hence \( u \in N_{H_w}(h^*) = N_{H_w}(\pi(h^*)) \) by part (b) of Lemma 15. Since \( \alpha \phi \) is a homomorphism, this implies that \( (\alpha \phi)(u) \in N_H((\alpha \phi \pi')(h^*)) = N_H(\pi(h^*)), \) since \( \alpha = (\phi \pi')^{-1}. \) From our assumption, \( h^* \in B_{\alpha(h)} \) and hence \( \pi(h^*) = \alpha(h). \) Hence \( (\alpha \phi)(u) \in N_H(\alpha(h)), \) and applying homomorphism \( \alpha^{-1} = \phi \pi' \) we conclude that \( \phi(u) \in N_H(h). \) Therefore \( \beta_1(u) = \beta(u) = \phi(u) \sim h = \beta_1(h^*) = \beta_1(v). \]

Case 4: \( \{u,v\} \cap \{w,h^*\} = \{w,h^*\}. \) We will show that this case cannot occur, otherwise \( u \sim v \) implies that \( w \sim h^*. \) Therefore \( h = \beta(w) \sim \beta(h^*) = \phi(h^*) \) and hence \( \phi(h^*) \in N_H(h) \subseteq N_H(\phi(h^*)), \) which is a contradiction, since \( H \) is bipartite and hence loopless. \( \square \)

Now we are ready for the proof of the main theorem of this section.

**Theorem 17** Let \( H \) be a connected bipartite graph which is not complete bipartite. Then the connected equitable \( H \)-coloring problem (and hence also the equitable \( H \)-coloring problem) is NP-hard.

**Proof:** One can observe that a connected bipartite graph is a complete bipartite graph if and only if it does not contain an induced subgraph isomorphic to \( P_4 \). Let the vertices of \( H \) be \( V(H) = \{1, \ldots, m\} \) and we may assume that the first four vertices form an induced \( P_4 \). We may also assume \( m > 4, \) since if \( m = 4 \) then \( H = P_4. \) We will give a Turing reduction from the connected equitable \( P_4 \)-coloring problem. Let a connected graph \( G \) be an instance of the connected equitable \( P_4 \)-coloring problem. By Lemma 12 we may assume that \( |V(G)| = n = 4k \geq 4. \) Let us define the weight function \( w \) on \( H \) by

\[
    w(j) = \begin{cases} 
        mn & \text{if } j \leq 4 \\
        mn + k & \text{otherwise.}
    \end{cases}
\]

\[ \square \]
We claim that $G \sim P_4$ if and only if $G + H_w \sim H$.

Let $H_w \xrightarrow{\pi} H$ be the natural homomorphism. One implication of our claim is obvious, since if we assume that $G \xrightarrow{\gamma} H'$, then $G + H_w \xrightarrow{\gamma+\pi} H$.

Note that $G + H_w$ has $n + 4nm + (m - 4)(nm + k) = (nm + k)m$ vertices, and hence $\frac{|V(G+H_w)|}{|V(H)|} = nm + k$.

If we assume that $G \xrightarrow{\gamma} P_4$ and $H_w \xrightarrow{\pi} H$ is the natural homomorphism, then $G + H_w \xrightarrow{\gamma+\pi} H$.

On the other hand, let us assume that $G + H_w \xrightarrow{\beta} H$. We will show that the conditions of Lemma 16 are satisfied. Since $G + H_w$ has $n + 4mn + (m - 4)(mn + k) = m(mn + k)$ vertices, $\frac{|V(G+H_w)|}{|V(H)|} = mn + k$. Since $\beta$ is an equitable homomorphism, exactly $mn + k$ vertices of $G + H_w$ must be mapped to the same vertex of $H$. Let $\psi = \beta|G$ and $\phi = \beta|H_w$. Thus $|\phi^{-1}(h)| = mn + k = mn + \frac{n}{4} \leq mn + n$ for all $h \in V(H)$.

From Lemma 14 it follows that there is a coretraction $\pi'$ belonging to $\pi$ such that $\phi'\pi'$ is an automorphism. For the weight function $w$ we have $w(h) \leq mn + k = \frac{|V(G+H_w)|}{|V(H)|}$. Therefore Lemma 16 implies that there is an equitable homomorphism $G + H_w \xrightarrow{\gamma} H$ such that $\gamma|H_w = \pi$. Since $\pi^{-1}(h) = mn + k$ for $h \notin \{1, 2, 3, 4\}$ and $\pi^{-1}(h) = mn$ for $h \in \{1, 2, 3, 4\}$, no vertex from the graph $G$ is mapped to any of the vertices from $V(H) - \{1, 2, 3, 4\}$. Moreover, for $h \in \{1, 2, 3, 4\}$ we have $|(\gamma|G)^{-1}(h)| = k$ and we conclude that $\gamma|G$ is an equitable homomorphism from $G$ to $P_4$.

To finish our reduction, let us fix a vertex $g \in V(G)$. For every vertex $h^* \in V(H_w)$, let $W_{h^*}$ be the graph obtained from $G + H_w$ by connecting the vertex $g$ with the vertex $h^*$ with an edge. One can easily observe that all $W_{h^*}$ are connected. We claim that $G \sim H'$ if and only if for at least one of $h^*$, $W_{h^*} \sim H$.

If $G \xrightarrow{\gamma} H'$, then for any $h \in N_H(\gamma(g))$, $W_{(h,1)} \sim H$, if we map vertices of $G$ by $\gamma$ and the vertices of $H_w$ by the natural homomorphism. On the other hand, if $W_{h^*} \xrightarrow{\delta} H$ and $i$ denotes the obvious inclusion of $G + H_w$ in $W_{h^*}$, then $G + H_w \xrightarrow{\gamma+\delta} H$ and hence $G \sim H'$. \(\square\)
Finally, we will prove the following theorem which concludes the complete characterization of the complexity of the equitable $H$-coloring problem.

**Theorem 18** If $H$ is not a disjoint union of complete bipartite graphs, then the equitable $H$-coloring problem is NP-hard.

**Proof:** Let $H$ have connected components $H_1, \ldots, H_k$. Without loss of generality we may assume that $H_1$ is not a complete bipartite graph. We will give a polynomial reduction from the equitable $H_1$-coloring problem to the equitable $H$-coloring problem. Let $G$ be an instance of the equitable $H_1$-coloring problem. Let us denote $n$ to be the number of vertices of $G$ and $m$ to be the number of vertices of $H_1$. We may also assume that $m$ divides $n$ by Lemma 12. Let $w = \frac{n}{m}$ be the constant weight function.

We will prove that $G \sim H_1$ if and only if $G + (H_2)_w + \ldots + (H_k)_w \sim H$. The implication \(\Rightarrow\) is obvious. Let us assume that $G + (H_2)_w + \ldots + (H_k)_w \sim H_1 + H_2 + \ldots + H_k$. Clearly no two graphs are mapped to the same graph. Therefore there is a permutation $\sigma$ on $k$ elements such that the $i$th graph from the left hand side is mapped to the $\sigma(i)$th graph from the right hand side. Let us decompose $\sigma$ into a product of disjoint cycles. Let $1$ be in the cycle $(1, i_1, \ldots, i_r)$, i.e., $G \sim H_{i_1}, (H_{i_1})_w \sim H_{i_2}, \ldots, (H_{i_{r-1}})_w \sim H_{i_r}, (H_{i_r})_w \sim H_1$. By Lemma 3, from $(H_{i_1})_w \sim H_{i_{r+1}}$ we get a bimorphism $(H_{i_1})_w \rightarrow (H_{i_{r+1}})_w$ and their composition gives us a bimorphism $G \rightarrow (H_{i_1})_w$. By Lemma 3 we conclude that for some $\alpha G \sim^\sim_{\sigma} H_1$ and since $w$ is a constant, $G \sim^\sim H_1$ for the natural homomorphism $\pi$. $\square$

2.6 Conclusion

In this section we would like to clarify the ideas behind our NP-hardness proofs. Recall that a graph $G$ is called a **core** if it is not homomorphic to any of its proper subgraphs, or equivalently any homomorphism $G \rightarrow G$ is an isomorphism. Welzl in [54] proved that every graph has a retraction to uniquely defined graph which is a core.
Hell and Nešetřil in [31] proved that determining whether a graph is a core is NP-hard.

We define a graph \( H \) to be an apparent core, if for every weight function \( w \) on \( H \) and every two retractions \( H_w \xrightarrow{\phi} H \) and \( H_w \xrightarrow{\psi} H \), there is an automorphism \( H \xrightarrow{\alpha} H \) on \( H \) such that \( \alpha \phi = \psi \). Therefore there is an automorphism \( \alpha \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
H_w & \xrightarrow{\phi} & H \\
\downarrow & & \downarrow \alpha \\
H & & H
\end{array}
\]

We will show later that every core is an apparent core and we will also provide examples of apparent cores which are not cores. The motivation of the definition of apparent cores follows from the proof of the following proposition.

**Proposition 19** If \( H \) is a connected apparent core which contains \( P_4 \) as an induced subgraph, then the equitable \( H \)-coloring problem is NP-complete.

**Proof:** Let the vertices of \( H \) be \( V(H) = \{1, \ldots, m\} \) and we may assume that the first four vertices form an induced \( P_4 \). We also may assume \( m > 4 \), since if \( m = 4 \) then \( H = P_4 \). We will give a polynomial reduction from the connected equitable \( P_4 \)-coloring problem. Let a connected graph \( G \) be an instance of the connected equitable \( P_4 \)-coloring problem. By Lemma 12 we may assume that \( |V(G)| = n = 4k \geq 4 \). Let us define the weight function \( w \) on \( H \) by

\[
w(j) = \begin{cases} 
mn & \text{if } j \leq 4 \\
mn + k & \text{otherwise.} \end{cases}
\]

We will prove that \( G + H_w \xrightarrow{\gamma} H \) if and only if \( G \xrightarrow{\gamma} P_4 \). Note that \( G + H_w \) has \( n + 4mn + (m - 4)(mn + k) = (nm + k)m \) vertices, and hence \( \frac{|V(G + H_w)|}{|V(H)|} = nm + k \).

If we assume that \( G \xrightarrow{\gamma} P_4 \) and \( H_w \xrightarrow{\tau} H \) is the natural homomorphism, then \( G + H_w \xrightarrow{\gamma \circ \tau} H \).

On the other hand, let us assume that \( G + H_w \xrightarrow{\beta} H \). Let \( \phi = \beta|H_w \) be the restriction of \( \beta \) to \( H_w \). Since \( \beta \) is an equitable homomorphism, exactly \( nm + k \) vertices
must be mapped to the same vertex. Therefore \( |\phi^{-1}(h)| = mn + k = mn + \frac{n}{4} \leq mn + n \).

From Lemma 14 we conclude that \( \phi \) is a retraction. Since \( H \) is an apparent core, there is an automorphism \( \alpha \) such that the following diagram is commutative,

\[
\begin{array}{ccc}
H_w & \xrightarrow{\phi} & H \\
\downarrow \alpha & & \downarrow \pi \\
H & & \\
\end{array}
\]

i.e., \( \alpha \phi = \pi \). Let us consider the homomorphism \( G + H_w \xrightarrow{\alpha \beta} H \). Since \( \alpha \) is an automorphism, \( \alpha \beta \) is also an equitable homomorphism. Moreover, \( \alpha(\beta | H_w) = \alpha \phi = \pi \) and hence vertices from \( (h,1), \ldots, (h,w(h)) \) are mapped to the vertex \( h \). Therefore \( \pi^{-1}(h) = mn + k \) for \( h \not\in \{1,2,3,4\} \) and \( \pi^{-1}(h) = mn \) for \( h \in \{1,2,3,4\} \). This implies that no vertex from the graph \( G \) is mapped to any of the vertices from \( V(H) - \{1,2,3,4\} \). Moreover, for \( h \in \{1,2,3,4\} \) we have \( k = |(\alpha(\beta | G))^{-1}(h)| \) and we conclude that \( \alpha(\beta | G) \) is an equitable homomorphism from \( G \) to \( P_4 \).

Therefore the NP-hardness proof for apparent cores is quite straightforward and it motivated their definition. We will give an alternative characterization of apparent cores but first we will prove the following simple lemma.

**Lemma 20** Let \( w \) be a weight function on a graph \( H \) and let \( H_w \xrightarrow{\phi} H \) be a retraction. Then if \( \phi \neq \pi \), there is a coretraction \( H \xrightarrow{\pi'} H_w \) which belongs to \( \pi \) but does not belong to \( \phi \).

**Proof:** Assume that \( \phi \neq \pi \), i.e., there is a vertex \( (h',i) \in V(H_w) \) such that \( \phi((h',i)) \neq h' \). Let us consider a mapping \( \pi' \) such that

\[
\pi'(h) = \begin{cases} 
(h',i) & \text{if } h = h' \\
(h,1) & \text{otherwise.}
\end{cases}
\]

One can observe that \( \pi' \) is a coretraction belonging to \( \pi \) but it does not belong to \( \phi \), since \( (\phi \pi)(h') = \phi((h',i)) \neq h' \). \( \square \)
Theorem 21 A graph $H$ is an apparent core if and only if for all distinct vertices $h, h' \in V(H)$, $N_H(h) \not\subseteq N_H(h')$.

Proof: Let the vertex set of $H$ be $V(H) = \{1, \ldots, m\}$.

First, let us assume that there are two distinct vertices $h, h' \in V(H)$ such that $N_H(h) \subseteq N_H(h')$. Let $w$ be a weight function on $H$ defined by

$$w(j) = \begin{cases} 2 & \text{if } j = h \\ 1 & \text{otherwise.} \end{cases}$$

Define $\phi((j,1)) = j$ for $j = 1, \ldots, m$ and $\phi((h,2)) = h'$. For every endomorphism $H \xrightarrow{\alpha} H$ such that $\alpha \phi = \pi$ we have $\alpha(h) = \alpha(\phi((h,1))) = \pi((h,1)) = h = \pi((h,2)) = \alpha(\phi((h,2))) = \alpha(h')$ and hence $\alpha$ cannot be an automorphism.

On the other hand, let us assume that for all distinct vertices $h, h' \in V(H)$, $N_H(h) \not\subseteq N_H(h')$.

First we will prove the claim for $\psi = \pi$. Let $\phi$ be a retraction from $H_w$ to $H$ and $\phi' : H \rightarrow H_w$ be a coretraction belonging to $\phi$. Let us define $\alpha = \pi \phi'$. Obviously $H \xrightarrow{\alpha} H$ is an endomorphism. To show that it is an automorphism, it is enough to prove that it is injective. Let $\pi'$ be any coretraction belonging to $\pi$. By Lemma 15, $N_H(h) \subseteq N_H((\phi' \pi' \phi')(h))$ for every $h \in V(H)$ and we conclude that $h = (\phi' \pi' \phi')(h)$ for every $h \in V(H)$ which means that $\phi' \pi' \phi' = id_H$. Therefore $\phi' \pi'$ is a left inverse to $\alpha = \pi \phi'$ and we conclude that $\alpha$ is an injective endomorphism and hence an automorphism. It remains to prove that $\alpha \phi = \pi$. Since $\alpha$ is an automorphism, every left inverse has to also be a right inverse and hence $\alpha \phi \pi' = id_H$. This implies that $\alpha \phi$ is a retraction and every coretraction $\pi'$ belonging to $\pi$ belongs to $\alpha \phi$. We conclude that $\alpha \phi = \pi$ by Lemma 20.

To finish the proof, let us assume that $\phi$ and $\psi$ are two retractions from $H_w$ to $H$. Let $\alpha$ and $\beta$ be two automorphisms on $H$ such that $\alpha \phi = \pi$ and $\beta \psi = \pi$. Then $\alpha \phi = \beta \psi$ and hence $\psi = (\beta^{-1} \alpha) \phi$ for the automorphism $\beta^{-1} \alpha$ on $H$. \qed
CHAPTER 2. EQUITABLE GRAPH HOMOMORPHISM

We say that an endomorphism is simple, if it differs from the identity automorphism on at most one vertex. If a graph $H$ contains two vertices $h, h'$ such that $N_H(h) \subseteq N_H(h')$, then one can define a simple endomorphism $H \xrightarrow{\phi} H$ by

$$
\phi(j) = \begin{cases} 
    h' & \text{if } j = h \\
    j & \text{otherwise}.
\end{cases}
$$

This simple endomorphism is clearly not an automorphism. Therefore apparent cores are exactly those graphs which do not have any simple endomorphism which is not an automorphism. This also implies that every core must be an apparent core. Note that every simple endomorphism $H \xrightarrow{\phi} H$ induces a retraction from $H$ onto induced subgraph $\phi(H)$ which has $|V(H)| - 1$ vertices. Therefore one can find an induced subgraph which is an apparent core in a polynomial time by applying simple retractions at most $|V(H)|$ times, while there is a pair $h, h'$ of distinct vertices such that $N(h) \subseteq N(h')$. Examples of apparent cores are cycles and complete graphs. Even cycles are examples of graphs which are apparent cores but not cores.

One can observe that our proof of Theorem 17 was inspired by the proofs of Proposition 19 and Theorem 21. Moreover, from our discussion above it follows that apparent cores, unlike cores (c.f. [31]) can be recognized in polynomial time. Moreover it also follows that for a given graph one can find a retract which is an apparent core in polynomial time and therefore we think that they may be of independent interest.

We would also like to mention that keeping the target graph $H$ fixed, we have obtained polynomial time algorithms for the equitable $H$-coloring problem in the case $H$ is a disjoint union of complete bipartite graphs and for the connected equitable $H$-coloring problem in the case when either $H$ is disconnected or $H$ is a disjoint union of complete bipartite graphs. In both of these cases the provided algorithms are exponential in $|V(H)|$ and this may be a target for further improvement.
Chapter 3

Relaxations of Graph Homomorphism

3.1 Introduction

Determining whether $G \to H$ for input graphs $G$ and $H$ is known to be an NP-complete problem (cf. [30]). This problem is a generalization of a large number of well-studied problems. For example, determining if a graph $G$ is $k$-colorable is equivalent to setting $H$ to be a complete graph on $k$ vertices and determining if $G \to H$. Determining if a graph $H$ has a $k$-clique is equivalent to setting $G$ to a complete graph on $k$ vertices and determining if $G \to H$. Therefore it is desirable to identify families of instances for which a polynomial algorithm exists. Recent research in semidefinite optimization shows that it is a powerful tool towards obtaining approximation algorithms for NP hard problems (see [18, 38]). We use semidefinite programming to study the graph homomorphism problem. We define the notions pseudo-homomorphism and fractional homomorphism which are, respectively, semidefinite and linear relaxations of a certain integer program corresponding to the graph homomorphism problem. In fact, the pseudo-homomorphism is defined in terms of the classical $\theta$ number of Lovász [42]. As semidefinite programs can be solved in polynomial time, pseudo-homomorphism is a polynomial time notion. It turns out, as we will show in a later section, that our notion of pseudo-homomorphism coincides with another semidefinite
relaxation defined by Feige and Lovász in [16]. Feige and Lovász used this relaxation to show polynomial time solvability of various special cases of graph homomorphism. Moreover $G$ is homomorphic to $H$ implies $G$ is pseudo-homomorphic to $H$ and this further implies that $G$ is fractional homomorphic to $H$. It turns out that fractional homomorphism is a co-NP complete problem. Therefore, if for some subset of graphs one can reverse any of these implications, then for this subset the corresponding problems are polynomial time solvable. One such example is the 2-coloring problem and another is the digraph homomorphism problem restricted to instances which can be solved using consistency checks.

Although the general graph homomorphism does not admit a simple forbidden subgraph characterization, we show a surprisingly simple such characterization for the fractional homomorphism, which also yields several other interesting consequences. First, we obtain a much simpler proof of the NP hardness of fractional clique number, a result which was first proven in [20] using the ellipsoid algorithm. A longstanding conjecture in graph theory is the graph product conjecture [22]. Using our characterization of the fractional homomorphism, we show that the above conjecture is true, if we replace homomorphism by fractional homomorphism. The main results of this chapter appeared in [6] and they are a joint work with Sanjeev Mahajan.

### 3.1.1 Notations

All graphs in this chapter are finite, undirected and without loops, unless specified otherwise. We say that two vertices are incident, if they are either adjacent or equal.

Let $G$ and $H$ be two graphs. The hom-product $G \circ H$ is defined as the graph with $V(G \circ H) = V(G) \times V(H)$, in which $\{(s,u),(t,w)\} \in E(G \circ H)$ if and only if $(s \neq t)$ and $(\{s,t\} \in E(G)$ implies $\{u,w\} \in E(H))$.

Recall, that the clique number, $\omega(G)$, and the chromatic number, $\chi(G)$, of a graph $G$, have the following integer linear program formulations. In these formulations we take $v$ to range over the vertex set of $G$ and $I$ to range over all maximal independent
sets in $G$.

$$\omega(G) = \max \sum_v x_v \quad \chi(G) = \min \sum_I y_I$$

$$\sum_v x_v \leq 1 \text{ for each } I \quad \sum_{I \ni v} y_I \geq 1 \text{ for each } v$$

$$x_v \in \{0, 1\} \quad y_I \in \{0, 1\}$$

If we replace the integrality conditions $x_v \in \{0, 1\}, y_I \in \{0, 1\}$ by $x_v \geq 0, y_I \geq 0$ respectively, we obtain linear programs with optima $\omega_f(G)$, the fractional clique number of $G$, and $\chi_f(G)$, the fractional chromatic number of $G$. One can observe that these linear programs are duals of each other and hence $\omega_f(G) = \chi_f(G)$ by the duality theorem of linear programming (see [11]). Therefore we have the inequalities:

$$\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G).$$

To compute any of these numbers is NP-hard. The NP-hardness of the fractional chromatic number was proved by Grötschel, Lovász and Schrijver in [20]. They also proved that one can compute in polynomial time a real number $\theta$ (see also [39]) which satisfies

$$\omega(G) \leq \theta \leq \omega_f(G).$$

One possible definition for $\theta$ is as follows [50):

Let $S$ be the set of all $|V(G) \times V(G)|$ positive semidefinite matrices, let $I$ denote the unit matrix and $J$ denote the matrix of all 1's. The $\bullet$ product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the number $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$. Then

$$\theta = \theta(G) = \max \{B \bullet J | B \in S; B \bullet I = 1; \forall (s, t) \notin E(G) : b_{st} = 0\}.$$ 

Another semidefinite programming upper bound for $\omega$ is the real number $\theta_{1/2}$ of Schrijver [47], which can be expressed as:

$$\theta_{1/2} = \theta_{1/2}(G) = \max \{B \bullet J | B \in S; B \bullet I = 1; \forall (s, t) \notin E(G) : b_{st} = 0; \forall s, t \in V(G) : b_{st} \geq 0\}.$$ 

Szegedy in [50] observed that $\theta_{1/2}$ is the same as the vector chromatic number whose definition can be find in [38].
3.2 Overview

In Section 3.3, we define three relaxations of graph homomorphism and their relations. In Section 3.4, we give a forbidden subgraph characterization of fractional homomorphism and its consequences which include an easier NP-hardness result for the fractional clique number problem and the fractional version of Hedetniemi’s conjecture. In Section 3.5, we show that our notion of pseudo-homomorphism is equivalent to the notion of hoax of Feige and Lovász [16] and give a necessary condition for the existence of a pseudo-homomorphism. We will also show that homomorphism, fractional homomorphism and pseudo-homomorphism are not equivalent in general. In Section 3.6, we will give some examples to demonstrate how our theory can be applied to give a polynomial time algorithm for some particular instances of graph homomorphism problem. We conclude this chapter with Section 3.7 where we discuss other possibility how to define the pseudo-homomorphism.

3.3 Relaxations of Homomorphism

In this section we define three relaxations of graph homomorphism. Our motivation follows from the following theorem.

Theorem 22 For any two graphs $G$ and $H$ we have the following inequalities:

$$
\omega(G \circ H) \leq \theta_{1/2}(G \circ H) \leq \theta(G \circ H) \leq \omega_f(G \circ H) = \chi_f(G \circ H) \leq \chi(G \circ H) \leq |V(G)|,
$$

and $\omega(G \circ H) = |V(G)|$ if and only if there is a homomorphism from $G$ to $H$.

Proof: The first and the second inequalities follow from [50]. The third inequality follows from Theorem 10 in [42]. From the definition of a fractional clique and from the duality of linear programming, it follows that $\omega_f = \chi_f \leq \chi$. The last inequality follows from the fact that we can color the vertices of $G \circ H$ by the vertices of the graph $G$, by assigning to a vertex $(s, u) \in V(G \circ H)$ the color $s \in V(G)$. From the definition of the hom-product it follows that no two adjacent vertices obtain the same color. It remains to prove that $G \rightarrow H$ if and only if $\omega(G \circ H) = |V(G)|$. Let us
assume that $\alpha$ is a homomorphism from $G$ to $H$. One can easily observe that vertices $(s, f(s)), s \in V(G)$ form a clique in $G \circ H$ of size $|V(G)|$. On the other hand if we have a clique $C$ in $G \circ H$ of size $|V(G)|$, then the first coordinates have to span all the vertices of $G$ because $(s, u) \not\sim (s, w)$, and hence $C$ can be regarded as a function $V(G) \to V(H)$. We define a homomorphism $\alpha$ from $G$ to $H$ as follows: $\alpha(s) = u$ if and only if $(s, u) \in C$. To prove that $\alpha$ is a homomorphism, let $s, t$ be vertices in $G$. Then $(s, \alpha(s))$ and $(t, \alpha(t))$ are from the clique $C$ and hence they are adjacent. From the definition of the hom-product, it follows that $s \sim t$ implies $\alpha(s) \sim \alpha(t)$. We conclude that $\alpha$ is a homomorphism. □

The previous theorem suggests the following relaxations of homomorphism:

Let $G$ and $H$ be two graphs.

**Definition 23** We say that $G$ is fractionally homomorphic to $H$ (denoted by $G \rightarrow_f H$) if $\omega_f(G \circ H) = |V(G)|$.

**Definition 24** We say that $G$ is pseudo-homomorphic to $H$ (denoted by $G \rightarrow_p H$) if $\theta(G \circ H) = |V(G)|$.

**Definition 25** We say that $G$ is pseudon $1/2$-homomorphic to $H$ (denoted by $G \rightarrow_{p/2} H$) if $\theta_{1/2}(G \circ H) = |V(G)|$.

**Corollary 26** $(G \rightarrow H) \Rightarrow (G \rightarrow_{p/2} H) \Rightarrow (G \rightarrow_p H) \Rightarrow (G \rightarrow_f H)$.

Later we will show that we cannot reverse the first and the last implications in general. This is not surprising because $G \rightarrow_p H$, and $G \rightarrow_{p/2} H$ are polynomial, while both $G \rightarrow H$ and $G \rightarrow_f H$ are NP hard. However, for certain families of graphs we can prove the reverse implications, and so obtain a polynomial algorithm for these families.

### 3.4 Fractional Homomorphisms

In this section, we will give a forbidden subgraph characterization of fractional homomorphism. In particular we will prove that $G$ is fractionally homomorphic to $H$ if and
only if $G$ does not contain a subgraph isomorphic to $K_{\omega_f(H)+1}$. This will enable us to prove a fractional version of the graph product conjecture [22]. As a consequence we will obtain an alternative proof of the NP-hardness of computing the fractional clique number of a graph.

**Theorem 27** $G \to_f H$ if and only if $\omega(G) \leq \omega_f(H)$.

**Proof:** Assume first that $G \to_f H$. For a contradiction let us also assume that there is a clique $C$ with $|\omega_f(H) + 1|$ vertices in $G$. We will show that $\omega_f(G \circ H) < |V(G)|$ by constructing a dual fractional coloring smaller than $|V(G)|$. Let $w$ be a minimum fractional coloring of $H$. Its value is $\chi_f(H) = \omega_f(H)$. All independent sets in $G \circ H$ are of the form $I = \bigcup_{i=1}^k s_i \times A_i$ where $s_1, \ldots, s_k$ form a clique in $G$ and for $i \neq j, \forall a_i \in A_i, a_j \in A_j : a_i \not\sim a_j$. This is so, because two vertices $(s, u), (t, w)$ from $G \circ H$ are not connected by an edge whenever $(s = t) \land (u \neq w)$, or $(s \sim t) \land (u \not\sim w)$.

Now we construct a fractional coloring $w^*$ of $G \circ H$ as follows: $w^*_{C \times T} = w_T$ for any maximal independent set $T$ from $H$, $w^*_{u \times V(H)} = 1$ for $u \not\in C$, and $w^* = 0$ otherwise. One can observe that this is a feasible fractional coloring and its objective value is at most $\omega_f(H) + (|V(G)| - |\omega_f(H) + 1|) < |V(G)|$.

On the other hand if we assume that $\omega(G) \leq \omega_f(H)$, we will show that $\omega_f(G \circ H) = |V(G)|$. Let $x$ be a maximum fractional clique in $H$, and $w$ a maximum fractional coloring of $H$ dual to $x$. From complementary slackness [11] it follows that

$$x_u = x_v \sum_{J \ni v} w_J,$$

where $J$ runs through the maximal independent sets in $H$. We construct a fractional clique $x^*$ in $G \circ H$ as follows: Let $x^*_{(s, u)} = \frac{x_u}{\omega_f(H)}$, where $s \in V(G), u \in V(H)$. Let us denote $x_S = \sum_{s \in S} x_s$ for any subset $S$ of vertices from $V(G)$ and similarly $x^*_T = \sum_{(s, u) \in T} x^*_{(s, u)}$ for any subset $T$ of vertices from $V(G \circ H)$. To prove that $x^*$ is a fractional clique, we have to show that for every independent set $I \subseteq G \circ H$, $x^*_I \leq 1$. For an independent set $I = \bigcup_{i=1}^k s_i \times A_i$, let us write $B = \bigcup_{i \neq j} A_i \cap A_j$, where $i, j = 1, \ldots, k$ and let $B_i = A_i - B$. Then $I \subseteq \bigcup_{i=1}^k s_i \times (B_i \cup B)$, and therefore

$$x^*_I \leq \sum_{i=1}^k x^*_{s_i \times B_i} + x^*_{\{s_1, \ldots, s_k\} \times B} = \frac{1}{\omega_f(H)} \sum_{i=1}^k x_{B_i} + \frac{k}{\omega_f(H)} x_B.$$
By our assumption \( k \leq \omega_f(H) \), therefore the following inequality holds:

\[
x^*_f \leq \frac{1}{\omega_f(H)} \sum_{i=1}^{k} x_{B_i} + x_B = \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_{v \in B_i} x_v + \frac{1}{\omega_f(H)} \sum_{J} w_J x_B = \\
= \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_{J \ni v} w_J \sum_{J \ni v} x_v + \frac{1}{\omega_f(H)} \sum_{J} w_J x_B = \\
= \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_{J} w_J x_{J \cap B_i} + \frac{1}{\omega_f(H)} \sum_{J} w_J x_B = \\
= \frac{1}{\omega_f(H)} \sum_{J} w_J (\sum_{i=1}^{k} x_{J \cap B_i} + x_B),
\]

where \( J \) runs through the maximal independent sets of \( H \). One can observe that the set \( B \cup \bigcup_{i=1}^{k} (J \cap B_i) \) is an independent set in \( H \) and therefore \( \sum_{i=1}^{k} x_{J \cap B_i} + x_B \leq 1 \). Hence

\[
x^*_f \leq \frac{1}{\omega_f(H)} \sum_{J} w_J = 1.
\]

It remains to show that the size of this fractional clique is \( |V(G)| \).

\[
\sum_{(s,u) \in V(G \circ H)} x^*_f = \sum_{(s,u) \in V(G) \times V(H)} \frac{x_u}{\omega_f(H)} = |V(G)| \sum_{u \in V(H)} \frac{x_u}{\omega_f(H)} = |V(G)|.
\]

\( \square \)

This theorem shows that although the graph homomorphism problem is \( \text{NP-complete} \), for any fixed non-bipartite graph \( H \) [30], the fractional graph homomorphism problem is polynomial, for every fixed graph \( H \), since one can use brute force to check for a clique of size \( [\omega_f(H)] + 1 \) in \( G \). The theorem also shows that if \( H \) is part of the input, then fractional homomorphism is \( \text{co-NP complete} \). Another interesting consequence of this theorem is that for a given \( k \) and a graph \( G \), it is \( \text{NP-hard} \) to determine if \( G \) has fractional chromatic number at least \( k \). This result was originally proved in [20] by the ellipsoid method.
3.4.1 Consequences

Theorem 27 has many interesting consequences and we devote this subsection to them.

**Proposition 28** The following problem is NP-hard:

- **Instance:** A graph $G$ and a number $n$
- **Question:** Is $\omega_f(G)$ at least $n$?

**Proof:** We give a reduction from the clique problem. Let $G$ and $n$ be an instance of the clique problem. Since $n - 1 = \omega_f(K_{n-1})$, by our theorem $\omega(G) \geq n$ if and only if $G \not\rightarrow_f K_{n-1}$, which is equivalent to $\omega_f(G \circ K_n) < |V(G)|$. \(\square\)

A longstanding conjecture in graph theory states that whenever $G \not\rightarrow K_n$ and $H \not\rightarrow K_n$, then also $G \times H \not\rightarrow K_n$ cf. [22].

**Proposition 29** $G \not\rightarrow_f W$ and $H \not\rightarrow_f W$ imply $G \times H \not\rightarrow_f W$

**Proof:** By Theorem 27 $G \not\rightarrow_f W$ and $H \not\rightarrow_f W$ imply $\omega(G) > \omega_f(W)$ and $\omega(H) > \omega_f(W)$. One can easily observe that $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$ and therefore also $\omega(G \times H) > \omega_f(W)$ which means $G \times H \not\rightarrow_f W$. \(\square\)

The fractional version of the product conjecture follows by taking $W = K_n$.

We would like to mention some properties of homomorphism which are not shared by fractional homomorphism and vice versa.

The binary relation $\rightarrow_f$, like homomorphism, is not antisymmetric. Its quite bad behavior can be demonstrated by an example of two graphs with different chromatic number which are fractionally homomorphic to each other, for instance $K_2$ and $C_5$.

Unlike homomorphism, fractional homomorphism is not even transitive. As an example let us choose an arbitrary graph $G$ such that $\omega(G) = 2$ and $\omega_f(G) \geq 3$. Then $K_3 \rightarrow_f G$, $G \rightarrow_f K_2$ but $K_3 \not\rightarrow_f K_2$. An example of such a graph $G$ is the Mycielski
CHAPTER 3. RELAXATIONS OF GRAPH HOMOMORPHISM

graph $M_{11}$ from Figure 3.1.

On the other hand, fractional homomorphism is dichotomic, i.e., for every two graphs $G$ and $H$, either $G \rightarrow_f H$ or $H \rightarrow_f G$. This follows from the fact that $H \not\rightarrow_f G$ implies that $\omega(H) > \omega_f(G)$ and therefore we have $\omega(G) \leq \omega_f(G) < \omega(H) \leq \omega_f(H)$, which means that $G \rightarrow_f H$.

Welzl in [54] proved that for every two graphs $G$ and $H$ such that $G \rightarrow H$ and $H \not\rightarrow G$ there exists a graph $W$ such that $G \rightarrow W \rightarrow H$, $W \not\rightarrow G$ and $H \not\rightarrow W$. This property is called density property of homomorphism. Similarly one can ask whether graphs are dense with respect to the fractional homomorphism, i.e., whether for every two graphs $G$ and $H$ such that $G \rightarrow_f H$ and $H \not\rightarrow_f G$ there exists a graph $W$ such that $G \rightarrow_f W \rightarrow_f H$, $W \not\rightarrow G$ and $H \not\rightarrow W$. First of all, we have already observed that $H \not\rightarrow_f G$ implies $G \rightarrow_f H$ and therefore we can omit $G \rightarrow_f H$ and $G \rightarrow W \rightarrow_f H$ parts of the question. The following proposition answers this question.

**Proposition 30** If $H \not\rightarrow_f G$ then there exists a graph $W$ such that $W \not\rightarrow_f G$ and $H \not\rightarrow_f W$ (and hence $G \rightarrow_f W \rightarrow_f H$) if and only if $\omega(H) - \omega_f(G) > 1$. Moreover, if this condition is satisfied, we can choose $W$ to be a clique.

**Proof:** $H \not\rightarrow_f G$ is equivalent to $\omega(H) > \omega_f(G)$. Therefore we have the following inequalities:

$$\omega(G) \leq \omega_f(G) < \omega(H) \leq \omega_f(H).$$

$W \not\rightarrow_f G$ and $H \not\rightarrow_f W$ if and only if $\omega(W) > \omega_f(G)$ and $\omega(H) > \omega_f(W)$. Therefore there exists $W$ which satisfies these conditions if and only if the following inequalities are satisfied:

$$\omega(G) \leq \omega_f(G) < \omega(W) \leq \omega_f(W) < \omega(H) \leq \omega_f(H).$$

The necessity of our condition is now clear, since $\omega(W)$ and $\omega(H)$ are integers. The sufficiency follows, since if the condition is satisfied, we can choose $W = K_{\omega(H)-1}$. \qed
3.5 Pseudo-homomorphisms

In this section we will relate pseudo-homomorphism to the concept of a hoax introduced by Feige and Lovász [16].

Let $V$ be an incidence matrix of the graph $G \circ H$ and $C = \frac{1}{|V(G)|} V$. Feige and Lovász [16] considered the following optimization problem with convex constraints, which we denote by $(\ast)_{G,H}$:

\[
\begin{align*}
\text{maximize} & \quad \sum_{s,u,t,w} C_{su,tw} Q_{su,tw} \\
\text{s.t.} & \quad Q \text{ is positive semidefinite} \\
& \quad Q \text{ is symmetric} \\
& \quad \forall s, t : \sum_{u,w} Q_{su,tw} = 1 \\
& \quad \forall s, t, u, w : Q_{su,tw} \geq 0.
\end{align*}
\]

The ellipsoid algorithm can be used to solve this optimization problem which is a semidefinite program. This program is based on a two-prover interactive proof system. For more details we refer the interested reader to paper [16]. A feasible solution to $(\ast)_{G,H}$ for which the objective function has value 1 is called a hoax (c.f. [16]) and in their paper [16], Feige and Lovász proved that if there is a homomorphism from $G$ to $H$ then $(\ast)_{G,H}$ has a hoax. They also gave the following necessary and sufficient conditions for $(\ast)_{G,H}$ to admit a hoax.

**Lemma 31** [16] The system $(\ast)_{G,H}$ has a hoax if and only if there exists a system of vectors $v_{su}, s \in V(G), u \in V(H)$ satisfying the following conditions:

\[
\hat{v} = \sum_u v_{su} \tag{3.1}
\]

is independent of $s$,

\[
\hat{v} v_{su}^T = \|v_{su}\|^2 \tag{3.2}
\]

\[
\|\hat{v}\| = 1 \tag{3.3}
\]
CHAPTER 3. RELAXATIONS OF GRAPH HOMOMORPHISM

\[ \forall s, u, t, w : v_{su}v_{tw}^T \geq 0 \]  \hspace{1cm} (3.4)

and

\[ v_{su}v_{tw}^T = 0 \text{ whenever } V_{su, tw} = 0. \]  \hspace{1cm} (3.5)

The optimization problem obtained from \((*)_{G,H}\) by removing the last, nonnegativity condition, \(\forall s, t, u, w : Q_{su, tw} \geq 0\) will be denoted by \((**)_{G,H}\) and any feasible solution to \((**)_{G,H}\) for which the objective function has value 1 will be called a semi-hoax.

One can similarly obtain the following necessary and sufficient conditions for \((**)_{G,H}\) to admit a semi-hoax.

**Lemma 32** The system \((**)_{G,H}\) has a semi-hoax if and only if there exists a system of vectors \(v_{su}, s \in V(G), u \in V(H)\) satisfying the conditions (3.1), (3.2), (3.3), and (3.5).

We now prove that our notions of pseudohomomorphism and pseudo-homomorphism coincide with the notions of hoax and semi-hoax respectively.

**Theorem 33** The system \((*)_{G,H}\) has a hoax if and only if \(G \rightarrow_{p/2} H\).

**Proof:** Let \(Q\) be a hoax of the system \((*)_{G,H}\). By Lemma 31, there exists a system of vectors \(v_{su}\) satisfying (1)-(5). One can observe that \(Q_{su, tw} = v_{su}v_{tw}^T = 0\) whenever \((s, u)\) is not adjacent to \((t, w)\) in \(G \circ H\). Also,

\[ \sum_{s \in V(G), u \in V(H)} Q_{su, su} = \sum_{s, u} v_{su}v_{su}^T = \sum_{s, u} \hat{v}_u^T = \hat{v}(\sum_{u} v_{su})^T = \hat{v}(\sum_{u} \hat{v})^T = |V(G)|. \]

Moreover,

\[ \sum_{s, t \in V(G), u, w \in V(H)} Q_{su, tw} = \sum_{s, t, u, w} v_{su}v_{tw}^T = \sum_{s, t} \hat{u}v^T = |V(G)|^2. \]

Taking \(B = \frac{1}{|V(G)|} Q\), we have proved that \(\theta_{1/2}(G \circ H) \geq |V(G)|\), and from Theorem 22 it follows that \(G \rightarrow_{p/2} H\).
On the other hand let us assume that $G \rightarrow_{p/2} H$ and let $\theta_{1/2}(G \circ H) = |V(G)|$. Let $B = AA^T$ be the positive semidefinite matrix with nonnegative entries such that $B \bullet I = 1$, for which $(s, u) \not\preceq (t, w)$ implies $b_{su, tw} = 0$ and

$$\theta_{1/2}(G \circ H) = |V(G)| = \sum_s a_s a_t^T.$$ 

We notice that $B_{su, sw} = 0$ for $u \neq w$ and therefore the $a_{su}$'s (the rows of matrix $A$) are orthogonal when $s$ is fixed. If we denote $a_s = \sum u a_{su}$, then from the condition $B \bullet I = 1$ we have

$$1 = \sum_{s \in V(G), u \in V(H)} B_{su, su} = \sum_{s, u} a_{su} a_{su}^T = \sum_{s, u} \|a_{su}\|^2 = \sum_s \sum_u \|a_{su}\|^2 = \sum_s \|a_s\|^2,$$

and therefore from $B \bullet J = \theta_{1/2}(G \circ H)$, it follows that

$$\theta_{1/2}(G \circ H) = \sum_{s, t \in V(G), u, w \in V(H)} B_{su, tw} = \sum_{s, t, u, w} a_{su} a_{tw}^T = \sum_s a_s a_t^T \leq \sum_{s, t} \|a_s\| \|a_t\| = \left(\sum_s \|a_s\|^2\right)^{1/2} \leq \sum_s 1^{1/2} \sum_s \|a_s\|^2 = |V(G)|.$$

The last two inferences are a consequence of Cauchy's inequality. Because $\theta_{1/2}(G \circ H) = |V(G)|$, it follows that both inequalities have to be equalities. That is,

$$\left(\sum_s 1^{1/2} \|a_s\|^2\right)^2 = \sum_s \|a_s\|^2,$$

and this happens if and only if all $a_s$'s have the same lengths. From $B \bullet I = 1$ it follows that for all $s \in V(G)$

$$\|a_s\| = |V(G)|^{-1/2}.$$

Also the inequality

$$\sum_{s, t} a_s a_t^T \leq \sum_{s, t} \|a_s\| \|a_t\|$$

must be an equality, i.e., the angle between $a_s$ and $a_t$ must be zero for each $s, t \in V(G)$. Therefore all $a_s$'s are equal. Taking $Q = |V(G)|B$, we have proved the conditions (1) and (3). The conditions (4) and (5) follow from the definition of $B$. From the orthogonality of the $a_{su}$'s for a fixed $s$, we have

$$a_s a_s^T = \sum_w a_{su} a_{su}^T = a_{su} a_{su}^T = \|a_{su}\|^2,$$
implying (2). Therefore we can deduce from Lemma 31 that \( Q \) is a hoax of the system 
\((*)_{G,H}\). □

One can similarly prove the following theorem.

**Theorem 34** The system \((**)_{G,H}\) has a semi-hoax if and only if \( G \rightarrow_p H \).

Although we do not have a forbidden subgraph characterization for pseudo-homomorphism,
as we have for fractional homomorphism, we can prove at least a necessary condition
for existence of a pseudo-homomorphism which will be sufficient for our purposes.

First, let us recall some definitions and results from [39].

If \( G \) and \( H \) are two graphs, then their strong product \( G \star H \) is defined as the graph
with \( V(G \star H) = V(G) \times V(H) \), in which \((s,u)\) is incident to \((t,w)\) if and only if \( s \)
is incident to \( t \) in \( G \) and \( u \) is incident to \( w \) in \( H \). Let \( \overline{G} \) denote the complement of \( G \)
and \( \vartheta(G) = \theta(\overline{G}) \). The following lemma is the monotonicity property from [39] p. 7.

**Lemma 35** [39] Given two graphs \( G \) and \( G' \), \( G \subseteq G' \) implies \( \vartheta(G) \geq \vartheta(G') \).

The following lemma is from [39] p. 22.

**Lemma 36** [39] Given two graphs \( G \) and \( G' \), \( \vartheta(G \star G') = \vartheta(G)\vartheta(G') \).

**Proposition 37**

\[ \vartheta(\overline{G \circ H}) \leq \vartheta(G)\vartheta(H) \].

**Proof:** Since \( G \star H \subseteq \overline{G \circ H} \), the inequality follows from the previous two lemmas. □

**Corollary 38** If \( G \rightarrow_p H \) then \( \vartheta(G)\vartheta(H) \geq |V(G)| \).

In the last part of this section we prove that in general one cannot reverse implications in the Corollary 26. We need the following results (see [42, 39]):

for odd \( n \)

\[ \vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} \],
\[ \psi(C_n) = \frac{1 + \cos(\pi/n)}{\cos(\pi/n)}, \]

and for every \( n \)

\[ \begin{align*}
\psi(K_n) &= 1, \\
\psi(K_n^c) &= n
\end{align*} \]

where \( C_n \) denotes the cycle on \( n \) vertices and \( K_n \) denotes the complete graph on \( n \) vertices.

**Proposition 39** For positive integers \( n, m \) such that \( n < m \), \( C_{2m+1} \not\rightarrow_p K_2 \) and \( C_{2n+1} \not\rightarrow_p C_{2n+1} \).

**Proof:** The proof follows from the fact that \( \psi(C_{2n+1}) \) is an increasing function of \( n \geq 1 \), while \( \psi(C_{2n+1}) \) is a decreasing and their product is \( 2n+1 \). Also, we have \( \psi(C_5) = \sqrt{5} \). Therefore \( \psi(C_{2n+1})\psi(C_{2m+1}) < \psi(C_{2n+1})\psi(C_{2n+1}) = 2n+1 \) and also \( \psi(C_{2m+1})\psi(K_2) = 2\psi(C_{2m+1}) < \sqrt{5}\psi(C_{2m+1}) = \psi(C_5)\psi(C_{2n+1}) \leq \psi(C_{2m+1})\psi(C_{2m+1}) = 2m+1 \). Corollary 38 implies the assertion of our proposition. \( \square \)

Since \( C_5 \rightarrow f K_2 \), and the above theorem shows that \( C_5 \not\rightarrow_p K_2 \), the pseudo-homomorphism is strictly stronger than the fractional homomorphism, i.e., the existence of the pseudo-homomorphism implies the existence of the fractional homomorphism but not vice versa.

The next proposition shows that homomorphism is strictly stronger than pseudo\( _{1/2} \)-homomorphism. Let \( M_{11} \) be Mycielski's graph from Figure 3.1.

**Proposition 40** \( M_{11} \not\rightarrow K_3 \) but \( M_{11} \rightarrow_{p/2} K_3 \).

**Proof:** The first assertion follows from the fact that \( M_{11} \) is not three colorable. The odd cycle \( 1, 2, 3, 4, 5 \) requires at least three colors and the vertices \( 6, 7, 8, 9, 10 \) are forced to use all three colors. Hence the vertex 11 needs a fourth color. The second
assertion follows from the fact that the following matrix $Q$ is the hoax. We will write its $s,t$ blocks:

$$Q_{s,s} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix},$$

$$Q_{s,t} = \begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix},$$

for $s \sim t$, and

$$Q_{s,t} = \begin{pmatrix} 1/6 & 1/12 & 1/12 \\ 1/12 & 1/6 & 1/12 \\ 1/12 & 1/12 & 1/6 \end{pmatrix},$$

for $s \not\sim t$.

One can check that all conditions for $Q$ to be a hoax hold. $\Box$

### 3.6 Examples

In this section we will demonstrate how one can possibly reverse implications in the Corollary 26 for some families of graphs and hence have a polynomial algorithm for
these families. Unfortunately, for all our examples there is already a known polynomial algorithm by some other means. For a better readability, we will use the following notations.

For a graph $H$ let us denote

$$(\rightarrow H) = \{G \mid G \rightarrow H\},$$

and for a family of graphs $C$, let us denote

$$(C \rightarrow) = \{H \mid G \rightarrow H \text{ for some } G \in C\}$$

and if $G \not\rightarrow H$ for all $G \in C$, then we will write $C \not\rightarrow H$.

We will use similar notations in the case of pseudo-homomorphism and fractional homomorphism.

### 3.6.1 Graph Coloring

We will need a slight modification of Theorem 8.1 in [16].

**Proposition 41** Let $C$ be a family of graphs such that $C \not\rightarrow_p H$. Then the class $(\rightarrow_p H)$ is in $P$, and separates $(C \rightarrow_p)$ and $(\rightarrow H)$.

*Proof:* Obviously, $(\rightarrow H) \subseteq (\rightarrow_p H)$. Assume, there is a graph $G \in (C \rightarrow_p) \cap (\rightarrow_p H)$. Then there is a graph $G' \in C$ such that $G' \rightarrow G$, and hence also $G' \rightarrow_p G$. We have also $G \rightarrow_p H$ and from the transitivity of $\rightarrow_p$ (see [16], for the transitivity of "has a hoax" relation), $G' \rightarrow_p H$ and hence $C \rightarrow_p G$ which is a contradiction. $\square$

The class of all odd cycles will be denoted by $C_{odd}$.

**Proposition 42** For $H$ bipartite, $G \rightarrow H$ if and only if $G \rightarrow_p H$. 

CHAPTER 3. RELAXATIONS OF GRAPH HOMOMORPHISM

Proof: The graph $H$ is bipartite if and only if it does not contain any odd cycle and this is if and only if $H$ is 2-colorable. Therefore we have

$$(\text{C}_{\text{odd} \rightarrow}) = (\rightarrow K_2).$$

By Proposition 39, $\text{C}_{\text{odd} \not \rightarrow} K_2$ and hence from Proposition 41 it follows that the class $(\rightarrow_p K_2)$ is in $P$, and separates $(\rightarrow K_2)$ (all bipartite graphs) from $(\text{C}_{\text{odd} \rightarrow})$ (graphs which contain an odd cycle, i.e. nonbipartite graphs). Therefore $(\rightarrow_p K_2) = (\rightarrow K_2)$. But for bipartite graph $H$, one can easily observe that $G \rightarrow H$ if and only if $G \rightarrow K_2$ and hence the theorem follows. □

Note that this example also confirms the well known results that 2-colorability is polynomial. Similarly, one can use Proposition 39 to show that finding the odd girth of a graph (size of the smallest odd cycle) is polynomial.

One can notice that 2-colorability has a duality property given by the equation (3.6).

If

$$(\text{C}_{\rightarrow}) = (\rightarrow H)$$

then we say that $H$ has $\text{C}$-duality property (c.f. [32]).

**Proposition 43** If $H$ satisfies the duality property $(\text{C}_{\rightarrow}) = (\rightarrow H)$, and $\text{C}_{\not \rightarrow_p} H$ for some family of graphs $\text{C}$, then $(\rightarrow_p H) = (\rightarrow H)$ and hence $H$-colorability is polynomial.

*Proof:* This is because $(\text{C}_{\rightarrow}) \cap (\rightarrow_p H) = \emptyset$ implies $(\rightarrow_p H) = (\text{C}_{\rightarrow}) \cap (\rightarrow_p H) = (\rightarrow H) \cap (\rightarrow_p H)$, and hence $(\rightarrow_p H) \subseteq (\rightarrow H)$. The other direction is trivial. Therefore in this case, the semidefinite program $(*)_{\rightarrow H}$ gives us a polynomial algorithm for the $H$-coloring problem. □

We can slightly generalize this as follows. Let an *almost complete graph* be a graph obtained from a complete graph by removing one edge.

Let us define a *k-string* as a graph $G_k$ with the following properties:

a/ $G_k$ is a union of almost complete graphs on $k$ vertices $H_i$, $i = 1, \ldots, m$ and some edges between them
b/ If \((x_i, y_i)\) is the missing edge from \(H_i\) then \(y_i = x_{i+1}\) for \(i = 1, \ldots, m - 1\), and \((x_1, y_m)\) is an edge in \(G_k\).

One can observe that if there is a \(k + 1\)-string as a subgraph of a graph \(G\), then \(G\) is not \(k\)-colorable, otherwise the vertices \(x_i\), and \(y_i\) would have to have the same color and b/ would imply that \(x_1\), and \(y_m\) have the same color but they are adjacent. Let \(S_k\) denote the set of all \(k\)-strings, and

\[
S = \bigcup_{k \geq 1} S_k,
\]

then the class of graphs with the following duality property is denoted by \(C_{st}\):

\[
\forall n : (S_{n+1} \rightarrow) = (\rightarrow K_n).
\]  

(3.7)

One can easily observe that \((S_3 \rightarrow) = (C_{odd} \rightarrow)\). Therefore \(C_{st}\) contains the family of all bipartite graphs (see Example 1). \(C_{st}\) contains also all graphs with \(\omega = \chi\) because \(K_n \in S_n\).

**Example 8** \(C_{2n+1} \in S_3\).

**Example 9** \(K_n \in S_n\).

**Lemma 44** Let \(v_{su}\) be vectors satisfying the conditions of Lemma 31 and \(O\) be a subset of these vectors which are mutually orthogonal. Then for \(\tilde{v}_O = \sum_{u \in O} u\) we have

\[
\| \tilde{v}_O \| \leq 1,
\]

and the equality holds if and only if \(v_O = \tilde{v}\).

**Proof:**

\[
\| v_O \|^2 = \sum_{u \in O} \| u \|^2 = \sum_{u \in O} \tilde{v}_O u^T = \tilde{v}_O \tilde{v}_O^T \leq \| \tilde{v} \| \cdot \| v_O \| = \| v_O \|,
\]

which implies \(\| v_O \| \leq 1\). We can also see that if the equality holds then

\[
1 = \| v_O \|^2 = \tilde{v}_O \tilde{v}_O^T,
\]

and hence the angle between the vectors \(\tilde{v}\), and \(v_O\) is 0. But both of them have their length equal to 1 and so they are equal. On the other hand if they are equal it implies \(\| v_O \| = 1\). This finishes our proof. □
Lemma 45 Let \((*)_{G,H}\) has a hoax, \(v_{su}\) be vectors satisfying the conditions of Lemma 31 and \(K_n\) be a complete subgraph of \(G\) with \(n = |V(H)|\) vertices. Then \(\forall u \in V(H)\): \(\sum_{s \in K_n} v_{su} = \tilde{v}\).

Proof: For a fixed vertex \(u \in V(H)\), vectors \(v_{su}, s \in K_n\) are orthogonal. From the previous lemma it follows that \(\|\sum_{s \in K_n} v_{su}\| \leq 1\). Therefore

\[
n = |V(H)| \geq \sum_{u \in V(H)} \|\sum_{s \in K_n} v_{su}\| \geq \|\sum_{u \in V(H)} \sum_{s \in K_n} v_{su}\| = \|n\tilde{v}\| = n.
\]

Combining this with Lemma 2 yields our assertion. \(\Box\)

Now we can generalize Proposition 42 as follows.

Proposition 46 For \(G \in C_{st}\), \(G \rightarrow K_n\) if and only if \(G \rightarrow_p K_n\).

Proof: Since \(G \rightarrow K_n\) implies \(G \rightarrow_p K_n\), it follows from Proposition 43 that it is enough to prove \(S_{n+1} \not\rightarrow_p K_n\). To get a contradiction, assume \(G_{n+1} \rightarrow_p K_n\) for some \(G_{n+1} \in S_{n+1}\) and hence \((*)_{G_{n+1},K_n}\) has a hoax. Let \(v_{su}\) be vectors given by Lemma 31. The vectors \(v_{su}, u \in V(K_n)\) are mutually orthogonal, since \(s\) cannot be mapped on distinct nodes, and hence

\[
\sum_{u \in V(K_n)} \|v_{su}\|^2 = \|\tilde{v}\|^2 = 1.
\]

Since \(G_{n+1} \in S_{n+1}\), it satisfies conditions a/ and b/ of the definition of \(n + 1\)-string and we will use notation from these conditions. The vectors \(v_{su}, s \in V(H_i) - \{x_i\}\) are also mutually orthogonal, since any two distinct nodes from \(V(H_i) - \{x_i\}\) are adjacent and hence they must be mapped on distinct nodes. Similarly the vectors \(v_{su}, s \in V(H_i) - \{y_i\}\) are mutually orthogonal. On the other hand, \(|V(H_i)| = n + 1\) and hence from previous lemma follows

\[
1 = \|\sum_{s \in V(H_i) - \{x_i\}} v_{su}\|^2,
\]

and similarly

\[
1 = \|\sum_{s \in V(H_i) - \{y_i\}} v_{su}\|^2.
\]
From Lemma 44 it follows that \( \sum_{s \in V(H) \setminus \{x_i\}} v_{su} = \hat{\nu} = \sum_{s \in V(H) \setminus \{y_i\}} v_{su} \), and therefore \( \hat{\nu} - v_{x_iu} = \hat{\nu} - v_{y_iu} \) which implies \( v_{x_iu} = v_{y_iu} \). From condition b/ we have obtained \( v_{x_1u} = v_{y_mu} \). But vertices \( x_1 \), and \( y_m \) are adjacent and hence cannot be mapped to the same vertex \( u \). Therefore \( v_{x_1u} \) and \( v_{y_mu} \) have to be at the same time orthogonal, which is a contradiction, because their sum over all \( u \) is the unit vector \( \hat{\nu} \). 

### 3.6.2 Digraph Coloring

A directed graph \( G \) is said to be homomorphic to a directed graph \( H \), if there exists a function \( f : V(G) \to V(H) \) called a homomorphism, such that whenever \( (s, t) \) is an edge in \( G \), \( (f(s), f(t)) \) is an edge in \( H \). We can now similarly define the hom-product of two directed graphs \( G, H \) as the undirected graph \( G \circ H \) with 

\[
V(G \circ H) = V(G) \times V(H)
\]

and \( (s, u) \sim (t, w) \) if and only if \( (s, t) \in E(G) \Rightarrow (u, w) \in E(H) \) and \( (t, s) \in E(G) \Rightarrow (w, u) \in E(H) \).

Hell, Nešetřil, and Zhu found polynomial algorithms for classes of graphs which have trees-duality in [32, 34, 33], i.e., \( C \)-duality where \( C \) is the family of all trees. They showed that the following labeling algorithm works for \( H \)-coloring (the problem whether \( G \to H \) for a fixed graph \( H \)) for graphs \( H \) which have trees-duality. We will later refer to this algorithm as the consistency test (cf. [32]).

**Instance:** A digraph \( G \);

**Question:** Is \( G \to H \)?

Define labels \( L_k : V(G) \to 2^{V(H)}, k \geq 0 \) by induction as follows:

\[
L_0(s) = V(H) \text{ for all } s \in V(G)
\]

\[
L_{k+1}(s) = \{u \in L_k(s) \mid \forall t \in V(G) \exists w \in L_k(t) : ((s, t) \in E(G) \Rightarrow (u, w) \in E(H)) \land ((t, s) \in E(G) \Rightarrow (w, u) \in E(H)) \}.
\]

For any vertex \( s \), the set \( L_{k+1}(s) \subseteq L_k(s), k \geq 0 \), and \( L_0(s) = V(H) \). Therefore
there is some \( i \) such that \( L_i(s) = L_{i+1}(s) \).

Output: \( G \) is \( H \)-colorable if for all \( s \in V(G) \): \( L_i(s) \neq \emptyset \).

We now prove that the class of graphs with trees-duality also admits a polynomial time solution by the theory described in this thesis.

**Proposition 47** Let \( C \) be the class of all graphs which have trees-duality. Then for every graph \( H \in C \) and every graph \( G \), \( G \rightarrow_H H \) if and only if \( G \rightarrow_f H \).

**Proof:** For any \( H \in C \) the \( H \)-coloring problem can be solved using the consistency test [32]. Therefore it is sufficient to show that if there is an \( s \in V(G) \) : \( L_i(s) = \emptyset \), then \( G \not\rightarrow_f H \).

Let us assume that \( G \rightarrow_f H \), and that a fractional clique \( x \) satisfies

\[
\omega_f(G \circ H) = \sum_{(s,u) \in V(G \circ H)} x_{(s,u)} = |V(G)|. \tag{3.8}
\]

First we will show by induction on \( k \geq 1 \) that if \( u \notin L_k(s) \), then the corresponding weight \( x_{(s,u)} = 0 \).

If \( k = 1 \), and \( u \notin L_1(s) \), then there exists \( t \in V(G) \) such that for all \( w \in L_0(t) = V(H) : (s \sim t) \land (u \not\sim w) \). Therefore for all \( w \in V(H) : (s, u) \not\sim (t, w) \). Hence \( (s, u) \) together with vertices \( (t, w) \), \( w \in V(H) \), form an independent set, and therefore

\[
x_{(s,u)} + \sum_{w \in V(H)} x_{(t,w)} \leq 1.
\]

The equality (3.8) implies that \( \sum_{w \in V(H)} x_{(t,w)} = 1 \) and hence \( x_{(s,u)} = 0 \).

Assume that \( u \notin L_k(s) \) implies \( x_{(s,u)} = 0 \). For \( u \notin L_{k+1}(s) \) we have \( t \) such that for all \( w \in L_k(t) : (s \sim t) \land (u \not\sim w) \), which means also that vertices \( (s, u) \), and \( (t, w), w \in L_k(t) \) form an independent set. For \( w \notin L_k(t) \) we can use our induction hypothesis that \( x_{(t,w)} = 0 \), and therefore we can apply the equations from the basic case.

If \( L_i(s) = \emptyset \), then for all \( u \in V(H) : x_{(s,u)} = 0 \), which contradicts (3.8). \( \square \)
CHAPTER 3. RELAXATIONS OF GRAPH HOMOMORPHISM

3.7 Conclusion

In the first chapter, we have proved that \( G \rightarrow H \) implies \( \omega(G) \leq \omega(H) \), \( \theta(G) \leq \theta(H) \), \( \chi_f(G) \leq \chi_f(H) \) and \( \chi(G) \leq \chi(H) \). Therefore, if \( \theta(G) \leq \theta(H) \), then we have the following inequalities:

\[
\omega(G) \leq \theta(G) \leq \theta(H) \leq \omega_f(H),
\]

and therefore \( G \rightarrow H \) implies \( \theta(G) \leq \theta(H) \) and this further implies \( \omega(G) \leq \omega_f(H) \), which is equivalent to \( G \rightarrow_f H \). Hence an alternative definition of (polynomially decidable) pseudo-homomorphism could be that \( G \) is pseudo’-homomorphic to \( H \) if and only if \( \theta(G) \leq \theta(H) \). We do not know if this is equivalent to our definition of pseudo-homomorphism and hence to the hoax of Feige and Lovász. Although, for \( G \) with the transitive automorphism group, \( \theta(G) \theta(G) = |V(G)| \) (c.f. [42]). Hence it follows from our necessary condition (Corollary 38) that for graphs \( G \) with transitive automorphism group, \( G \rightarrow_p H \) implies \( \theta(G) \leq \theta(H) \). Our investigation was inspired by work of Feige and Lovász [16] and this determined our choice. The alternative definition is in terms of forbidden subgraphs.

Finally, from the proof of Theorem 22 it follows that there is a bijective correspondence between cliques of size \( |V(G)| \) in \( G \circ H \) and homomorphisms from \( G \) to \( H \). Therefore one can identify these cliques and homomorphisms from \( G \) to \( H \). Let us define \( \theta(0) = 0 \). From our definition of \( G \circ H \) it follows that \( G \circ H \) is always loopless, even if we extend the definition to all graphs. If one consider a pseudo-homomorphism to be any symmetric, positive semidefinite matrix \( \Lambda \) of dimension \( |V(G \circ H)| \times |V(G \circ H)| \) such that \( \Lambda \cdot I = 1 \), for all \( \{(s, u), (t, w)\} \notin E(G \circ H) \): \( \Lambda_{(s,u),(t,u)} = 0 \) and \( \theta(G \circ H) = \Lambda \cdot \Lambda \), then one may consider a category \( \text{PFG} \) of graphs and pseudo-homomorphisms, if one define a composition of pseudohomomorphisms \( G \xrightarrow{\Lambda} H \) and \( H \xrightarrow{\Lambda'} W \) to be \( G \xrightarrow{\Lambda''} W \) such that \( \Lambda''_{(s,u),(t,w)} = \frac{1}{|V(H)|^2} \sum_{a,b \in V(H)} \Lambda_{(s,a),(t,b)} \Lambda'_{(a,u),(b,w)} \) (compare with [16], Lemma 5.22). The category \( \text{FG} \) of all finite graphs and homomorphism is actually embedded in \( \text{PFG} \). To see this, we need a faithful functor \( T \) which maps homomorphisms to pseudo-homomorphisms. Let \( G \cong H \) be a homomorphism. Then
define \( T(\alpha) = A \) such that
\[
A_{(s,u),(t,w)} = \begin{cases} 
\frac{1}{|V(G)|} & \text{if } u = \alpha(s) \text{ and } w = \alpha(t) \\
0 & \text{otherwise.}
\end{cases}
\]

We conclude that \( T \) is an embedding of \( FG \) to \( PFG \) and one can observe that this embedding is isomorphic to the embedding of \( FG \) to the category of graphs and hoaxes of Feige and Lovász, which is implicit in their paper [16].
Chapter 4

Structure of Color Classes

4.1 Introduction

In this chapter we will consider all finite graphs and we will denote the set of all finite graphs by $G$.

A graph $W$ is called *multiplicative* if for every two graphs $G$ and $H$,

$$ G \times H \rightarrow W \text{ implies that } G \rightarrow W \text{ or } H \rightarrow W $$

(4.1)

Hedetniemi [26] proposed the following conjecture.

**Conjecture 48** *(Hedetniemi [26])* All complete graphs are multiplicative.

One can easily prove that $K_1$ and $K_2$ are multiplicative and the proof of the multiplicativity of $K_3$ was accomplished by El-Zahar and Sauer [14]. An alternative form of (4.1) is the following:

$$ G \not\rightarrow W \text{ and } H \not\rightarrow W \text{ imply that } G \times H \not\rightarrow W. $$

(4.2)

If one further restricts graphs $G \not\rightarrow W$ and $H \not\rightarrow W$ of (4.2), then it is sometimes possible to conclude that $G \times H \not\rightarrow W$. This approach was considered by Burr, Erdős and Lovász [9], Hell [28], Turzík [51], Duffus, Sands and Woodrow [12], Welzl [55] and others. Particularly, they proved the following results:
CHAPTER 4. STRUCTURE OF COLOR CLASSES

Proposition 49 [9] If $G \not\sim K_n$, $H \not\sim K_n$ and each vertex of $G$ is contained in a complete subgraph of order $n$, then $G \times H \not\sim K_n$.

Proposition 50 [12, 55] If $G$ and $H$ are connected, $K_n \to G \not\sim K_n$ and $K_n \to H \not\sim K_n$, then $G \times H \not\sim K_n$.

Note that one can assume connectivity of $G$ and $H$, since if there are $G$ and $H$ such that $G \not\sim K_n$, $H \not\sim K_n$ but $G \times H \to K_n$, then for some connected component $G'$ of $G$, $G' \not\sim K_n$ and similarly for some connected component $H'$ of $H$, $H' \not\sim K_n$ but $G' \times H' \to K_n$.

Proposition 51 [51] If $G \not\sim K_n$, $H \not\sim K_n$ and for every pair of edges $e$, $e'$ of $G$ there exists an edge $e''$ of $G$ which intersects both $e$ and $e'$, then $G \times H \not\sim K_n$.

For a graph $G$, let $\Phi(G)$ be the simplicial complex whose vertices are the vertices of $G$ and whose simplexes are those subsets of $V(G)$ which have a common neighbor. A homomorphism $\alpha$ from $G$ to $H$ induces a simplicial mapping, since a graph homomorphism takes a set of vertices with a common neighbor to a set of vertices with a common neighbor. It was shown by Lovász [41] that if the geometric realization $|\Phi(G)|$ of $\Phi(G)$ is $(n-2)$-connected then $G \not\sim K_n$. A homological version of this result was given by Walker in [52]. A topological space $X$ is called $n$-acyclic mod 2, if the $j$-th reduced homology group of $X$ with coefficients in $\mathbb{Z}/2$ vanishes for each $j \leq n$. Walker [52] proved that if $G$ is a graph such that $\Phi(G)$ is $(n-2)$-acyclic mod 2, then $G \not\sim K_n$. It follows that if $|\Phi(G \times H)|$ is $(n-2)$-connected then $G \times H \not\sim K_n$ (cf. [28]). However, it is not the case that $|\Phi(G \times H)| = |\Phi(G) \times \Phi(H)|$ (cf. [28]). In fact, Walker [52] showed that $|\Phi(G) \times \Phi(H)|$ has the same homotopy type as $|\Phi(G \ast H)|$, where $G \ast H$ is the join of $G$ and $H$, i.e., the graph obtained from $G + H$ by connecting all vertices of $G$ with all vertices of $H$ (cf. Proposition 4.1. in [52]).

Other multiplicative graphs were studied by Häggkvist, Hell, Miller and Neumann-Lara [22] and they proved:

Proposition 52 [22] Each cycle is multiplicative.
It is not even known whether the following function

\[ \mu(n) = \min\{m \mid G \not\in K_n \text{ and } H \not\in K_n \text{ but } G \times H \rightarrow K_m \text{ for some } G, H \} \]

tends to infinity or not. Note that the Hedetniemi’s conjecture is equivalent to the assertion that for all positive integers \( n \), \( \mu(n) = n + 1 \). Poljak and Rödl [46] proved that \( \mu(n) \) either tends to infinity with \( n \), or is bounded above by 16.

Duffus and Sauer [13] proposed another approach. Let us recall that a graph \( G \) is equivalent to a graph \( H \), denoted by \( G \leftrightarrow H \), if \( G \rightarrow H \) and \( H \rightarrow G \). The relation \( \leftrightarrow \) is an equivalence, since \( \rightarrow \) is reflexive and transitive relation. In their paper [13], Duffus and Sauer studied the equivalence classes of graphs determined by the relation \( \leftrightarrow \). They observed that if \( \langle G \rangle \) denotes the class of graphs equivalent to the graph \( G \), then for any two equivalence classes of graphs \( \langle G \rangle \) and \( \langle H \rangle \) one can naturally define their meet \( \langle G \rangle \times \langle H \rangle \) to be the class \( \langle G \times H \rangle \), their join \( \langle G \rangle + \langle H \rangle \) to be the class \( \langle G + H \rangle \) and their relative pseudocomplement \( \langle H \rangle((\langle G \rangle)) \) to be the class \( \langle H(G) \rangle \), the class containing the map graph \( H(G) \) defined in [22] (we will provide the definition later). They have observed, that the set \( \mathcal{G} = \mathcal{G}/\leftrightarrow \) of these equivalence classes with just mentioned operations form a Heyting algebra with the zero element \( (0) \) and the unit element \( (1) \). Respectively, we will use the symbols \( \times \) and \( + \) for operations meet and join in distributive lattices, unless specified otherwise. Recall that Heyting algebra is a distributive lattice with the minimum element such that for every two elements \( a \) and \( b \) there exists an element \( x \) for which \( a \times c \leq b \) is equivalent to \( c \leq x \) for every element \( c \). It follows that if such an element \( x \) exists then it is unique and it is called the relative pseudocomplement of \( a \) with respect to \( b \). We will denote the relative pseudocomplement of \( a \) with respect to \( b \) by \( b(a) \).

Duffus and Sauer [13] also observed that if we fix a class \( w \in \mathcal{G} \), then the set \( B_w = \{w(g) \mid g \in \mathcal{G} \} \) is a Boolean algebra with meet \( x \land y = x \times y \), join \( x \lor y = w(w(x + y)) \), complement \( x^c = w(x) \), the zero element \( w \) and the unit element \( (1) \) - the class of all graphs with a loop. They also proved that if \( B(K_n) \) is finite for all \( n \), then the function \( \mu(n) \) tends to infinity with \( n \). Consequently they asked for
which graphs $W, B_{(W)}$ is finite.

We say that $W$ has a finite factorization, if there are multiplicative graphs $W_1, \ldots, W_n$, such that $W \leftrightarrow W_1 \times \ldots \times W_n$. From the definition it follows that if $W$ is multiplicative and $W \leftrightarrow W'$ then $W'$ is also multiplicative. Therefore if $W$ is multiplicative, then all graphs in $(W)$ are multiplicative. We say that a class $w \in \mathcal{G}$ is multiplicative, if graphs from $w$ are multiplicative. We will prove that the boolean algebra $B_{(W)}$ is finite if and only if $W$ has a finite factorization.

One can observe that multiplicative classes of graphs are meet irreducible elements in the lattice $\mathcal{G}$. Therefore $B_w$ is finite if and only if $w$ is a meet of finitely many meet irreducible elements of $\mathcal{G}$. This suggests that the finite factorizations of elements of $\mathcal{G}$ are closely related to Hedetniemi's conjecture, and that understanding of the multiplicative structure of $\mathcal{G}$ would give us some insight about the distribution of multiplicative elements. In the following sections we focus our attention on the study of the multiplicative structure of $\mathcal{G}$, as a result we give necessary and sufficient conditions for a graph $W$ to have a finite factorization. These conditions are presented in terms of ideals, to be defined later. Using ideals to study factorization is a standard approach to this problem in other areas of mathematics, such as commutative algebra.

We will also show that even the existence of one graph $W$ which has a finite factorization and satisfies $K_4 \rightarrow W$ would imply that $\mu(n) \rightarrow \infty$. This is a generalization of the result of Duffus and Sauer in [13], who proved that if all $B_{(K_n)}$ are finite, then $\mu(n) \rightarrow \infty$. By our results, their assumption is equivalent to the assumption that all $K_n$ have finite factorizations.

4.2 Overview

In Section 4.3, we will make our statements from Section 4.1 more precise and the results of this section were our motivation for investigating finite factorizations. In Section 4.4, we formally define ideals and make simple observations about operations
on them. We will also prove the counterpart of the prime avoidance theorem from
commutative algebra. Section 4.5 contains several quite trivial observations about
additive structure of $\mathcal{G}$. In Section 4.6, we study the multiplicative structure of ideals.
We define prime ideals and develop some tools which are useful in later sections. Many
of our proofs were motivated by the corresponding results from commutative algebra.
In Section 4.6.1, we endow the set of all prime ideals with Zariski topology and we
investigate topological properties of some of its subset, which are closely related with
the factorization problem. In Section 4.6.2, we give necessary and sufficient conditions
for the existence of irredundant and finite factorizations. Finally, in Section 4.7, we
summarize our results and interpret some of them in graph theoretical terms.

4.3 Preliminary Results and Motivation

In this section we will prove most of our claims from Section 4.1. We will use results
and constructions from [22], [46] and [13].

Probably the most important construction is the following one from [22].

Definition 53 [22] The map graph of two graphs $G$ and $W$, denoted $W(G)$, is defined
as follows: the vertices of $W(G)$ are all possible mappings $\phi : V(G) \to V(W)$ and
$\{\phi, \psi\}$ is an edge of $W(G)$ whenever $\{s, t\} \in E(G)$ implies $\{\phi(s), \psi(t)\} \in E(W)$.

The following properties of the map graph were also observed in [22].

Proposition 54 [22]
(a) $W(G)$ has a loop if and only if $G \to W$.
(b) $G \times W(G) \to W$.
(c) $G \to W(W(G))$.
(d) $W \to W(G)$.
(e) $G \times H \to W$ if and only if $H \to W(G)$.
(f) $W$ is multiplicative if and only if $G \not\to W$ implies $W(G) \to W$.
(g) $G \to G'$ implies $W(G') \to W(G)$.
(h) $W \to W'$ implies $W(G) \to W'(G)$.
Therefore $W$ is multiplicative if and only if $W(G) \leftrightarrow W$ or $W(G) \leftrightarrow 1$ for all graphs $G$. We will need also the following properties of the map graph.

**Proposition 55.**

(i) $W(G_1 + G_2) \leftrightarrow W(G_1) \times W(G_2)$,

(j) $(W_1 \times W_2)(G) \leftrightarrow W_1(G) \times W_2(G)$,

(k) if $G$ is a connected nonbipartite graph then $(W_1 + W_2)(G) \leftrightarrow W_1(G) + W_2(G)$,

(l) $W$ is multiplicative if and only if $G \not\leftrightarrow W$ implies $W(G) \to W$ for all connected nonbipartite graphs $G$.

**Proof:**

(i): This was proved by Duffus and Sauer [13].

(j): Since $W_1 \times W_2 \to W_1$, Lemma 54 (h) implies that $(W_1 \times W_2)(G) \to W_1(G)$. Similarly $(W_1 \times W_2)(G) \to W_2(G)$ and hence $(W_1 \times W_2)(G) \to W_1(G) \times W_2(G)$. On the other hand, the mapping $\alpha$ from $W_1(G) \times W_2(G)$ to $(W_1 \times W_2)(G)$ defined by $\alpha((\phi, \psi))(g) = (\phi(g), \psi(g))$ is easily seen to be a homomorphism.

(k): Since $W_1 \to W_1 + W_2$, Lemma 54 (h) implies that $W_1(G) \to (W_1 + W_2)(G)$. Similarly $W_2(G) \to (W_1 + W_2)(G)$ and hence $W_1(G) + W_2(G) \to (W_1 + W_2)(G)$. On the other hand let us fix an arbitrary vertex $\psi$ from the graph $W_1(G) + W_2(G)$. Let us define a mapping $\alpha$ from the vertices of $(W_1 + W_2)(G)$ to the vertices of $W_1(G) + W_2(G)$ as follows:

$$\alpha(\phi) = \begin{cases} 
\phi & \text{if } \phi(V(G)) \subseteq V(W_1) \text{ or } \phi(V(G)) \subseteq V(W_2) \\
\psi & \text{otherwise.}
\end{cases}$$

We will show that $\alpha$ is a homomorphism. It is enough to prove that vertices $\phi$ of $(W_1 + W_2)(G)$ such that $\phi(V(G)) \not\subseteq V(W_1)$ and $\phi(V(G)) \not\subseteq V(W_2)$ are isolated vertices, i.e., their neighborhoods are empty. For a contradiction let us assume that for some $g, g' \in V(G)$ we have $\phi(g) \in V(W_1)$ but $\phi(g') \in V(W_2)$, and that $\phi \sim \phi'$ for some vertex $\phi'$ of $(W_1 + W_2)(G)$. Since $G$ is a connected nonbipartite graph, there are (not necessarily distinct) vertices $g = g_1, g_2, \ldots, g_{2k+1} = g' \in V(G)$ such that $g_i \sim g_{i+1}$ for $i = 1, \ldots, 2k$. Since $\phi \sim \phi'$, it follows that $\phi(g) = \phi(g_1) \sim \phi'(g_2) \sim \phi(g_3) \sim \ldots \sim \phi'(g_{2k}) \sim \phi(g_{2k+1}) = \phi(g')$. This implies that $\phi(g)$ and $\phi(g')$ belong to
the same connected component of \( W_1 + W_2 \), which is a contradiction.

(l): One can observe that all bipartite graphs are multiplicative, since every bipartite graph is either empty, or it is equivalent to either \( K_1 \) or \( K_2 \) and all these graphs are multiplicative. If \( W \) is multiplicative than by Lemma 54 (f) it follows that \( G \not\rightarrow W \) implies \( W(G) \rightarrow W \) and hence this implication is true for all connected nonbipartite graphs \( G \). On the other hand, if \( W \) is not a multiplicative graph than \( W \) is not bipartite and there are graphs \( G, H \) such that \( G \not\rightarrow W, H \not\rightarrow W \) but \( G \times H \rightarrow W \).

Let \( G_1, \ldots, G_n \) be the connected components of \( G \) and \( H_1, \ldots, H_m \) be the connected components of \( H \) i.e., \( G = G_1 + \ldots + G_m \) and \( H = H_1 + \ldots + H_m \). Then \( G \not\rightarrow W \) and \( H \not\rightarrow W \) imply that at least one connected component of \( G \) and \( H \) is not homomorphic to \( W \). We may assume that \( G_1 \not\rightarrow W \) and \( H_1 \not\rightarrow W \). But \( G \times H \rightarrow W \) implies \( G_1 \times H_1 \rightarrow W \) and hence by Proposition 54 (e), \( H_1 \rightarrow W(G_1) \). Since \( H_1 \not\rightarrow W \), also \( W(G_1) \not\rightarrow W \) and therefore \( G_1 \) is a connected nonbipartite graph such that \( G_1 \not\rightarrow W \) and \( W(G_1) \not\rightarrow W \).

We will show later that the connectivity of a graph is a dual notion of the multiplicativity of a graph. In this sense, corollary of the conditions (k) and (l) is the following dual of the fact that the product of two connected graphs is a connected graph if and only if at least one of them is not bipartite (see [53]).

**Corollary 56** The sum of two multiplicative graphs is multiplicative.

The next two constructions are from [46].

**Definition 57** [46] The arc graph of a graph \( G \), denote \( \delta G \), is defined as follows: the vertices of \( \delta G \) are the ordered pairs \((x, y)\) such that \( x \sim y \) in \( G \), and \((x, y)\) is adjacent to \((u, v)\) whenever \( y = u \).

**Lemma 58** [46] \( \delta(G \times H) = \delta G \times \delta H \).

**Definition 59** [46] For each positive integer \( n \), let \( S_n \) be the following symmetric graph. The vertices of \( S_n \) are the subsets of \( \{1, \ldots, n\} \) and two vertices \( A, B \) are adjacent whenever \( A \) is not a proper subset of \( B \) and \( B \) is not a proper subset of \( A \).
Lemma 60 [46] $S_n \leftrightarrow K_n^{\lceil \frac{n}{2} \rceil}$.

Proposition 61 [46] $\delta G \rightarrow K_n$ if and only if $G \rightarrow S_n$.

We will prove a more general version of this proposition as Proposition 64. But even the results mentioned above enable us to prove the following.

Proposition 62 If $K_n$ is multiplicative, then $K_n^{\lceil \frac{n}{2} \rceil}$ is also multiplicative.

Proof: Assume that $K_n$ is multiplicative and let $G \times H \rightarrow K_n^{\lceil \frac{n}{2} \rceil}$. This is equivalent to $\delta(G \times H) \rightarrow K_n$ and hence $\delta G \times \delta H \rightarrow K_n$. Since $K_n$ is multiplicative by our assumption, either $\delta G \rightarrow K_n$ or $\delta H \rightarrow K_n$. But this is equivalent to that either $G \rightarrow K_n^{\lceil \frac{n}{2} \rceil}$ or $H \rightarrow K_n^{\lceil \frac{n}{2} \rceil}$ and we conclude that $K_n^{\lceil \frac{n}{2} \rceil}$ is multiplicative. □

Note that if $n \geq 4$, then $n < \binom{n}{\lceil \frac{n}{2} \rceil}$ and hence if there is an $n \geq 4$ such that $K_n$ is multiplicative, then there are infinitely many $n$ such that $K_n$ is multiplicative.

The following construction generalizes the already mentioned graph $S_n$ from [46].

For each graph $H$, let $S_H$ be the graph such that the vertices of $S_H$ are all subsets of $V(H)$ and two vertices $A, B$ are adjacent in $S_H$ whenever the following two conditions are satisfied:

(a) there exists $a \in A - B$ such that for all $b \in B$, $a \sim b$,
(b) there exists $b \in B - A$ such that for all $a \in A$, $b \sim a$.

One can observe that $S_n$ is equal to $S_{K_n}$.

Lemma 63 $G \rightarrow H$ implies $S_G \rightarrow S_H$.

Proof: If $G \xrightarrow{\alpha} H$ then let us map $A$ to $\alpha(A)$. This mapping is a homomorphism, since if $A \sim B$ then there exists $a \in A - B$ such that for all $b \in B$, $a \sim b$ and hence $\alpha(a) \in \alpha(A) - \alpha(B)$ and for all $\alpha(b) \in \alpha(B)$, $\alpha(a) \sim \alpha(b)$. □

Now we can prove the following proposition, which is a generalization of Proposition 61.
Proposition 64 \( \delta G \rightarrow H \) if and only if \( G \rightarrow S_H \).

Proof: If \( \delta G \overset{\alpha}{\rightarrow} H \) then define \( \beta(g) = \alpha(\{g\} \times N_G(g)) \). If \( g \sim g' \), then \( (g, g') \in \beta(g) - \beta(g') \) and \( (g, g') \sim (g', g'') \in \beta(g') \) which implies that \( \beta(g) \sim \beta(g') \).

On the other hand if \( G \overset{\beta}{\rightarrow} S_H \) then define \( \alpha(x, y) = h \in \beta(x) - \beta(y) \) such that for all \( h' \in \beta(y), h \sim h' \). If \( (x, y) \sim (y, z) \) then \( \alpha(x, y) = h \sim h' = \alpha(y, z) \in \beta(y) \). \( \square \)

The following proposition shows how to construct new multiplicative graphs from old ones.

Proposition 65 If \( W \) is multiplicative, then also \( S_W \) is multiplicative.

Proof: Assume that \( W \) is multiplicative and let \( G \times H \rightarrow S_W \). This is equivalent to \( \delta(G \times H) \rightarrow W \) and hence \( \delta G \times \delta H \rightarrow W \). Since \( W \) is multiplicative by our assumption, either \( \delta G \rightarrow W \) or \( \delta H \rightarrow W \). But this is equivalent to that either \( G \rightarrow S_W \) or \( H \rightarrow S_W \) and we conclude that \( S_W \) is multiplicative. \( \square \)

Proving these results, let us recall that a graph \( G \) is equivalent to a graph \( H \), denoted \( G \leftrightarrow H \), if \( G \rightarrow H \) and \( H \rightarrow G \). The relation \( \leftrightarrow \) is an equivalence, since \( \rightarrow \) is reflexive and transitive relation. Following [13] we are going to study the equivalence classes of graphs determined by the relation \( \leftrightarrow \). If \( \langle G \rangle \) denotes the class of graphs equivalent to the graph \( G \), then one can define the meet \( \langle G \rangle \times \langle H \rangle \) to be the class \( \langle G \times H \rangle \), the join \( \langle G \rangle + \langle H \rangle \) to be \( \langle G + H \rangle \) and the relative pseudocomplement of \( \langle W \rangle \) and \( \langle G \rangle \) is \( \langle W(G) \rangle \) to be the class containing the map graph \( W(G) \). We have already mentioned in Section 4.1 that the set \( \mathcal{G} = \mathcal{G} \leftrightarrow \) of these equivalence classes with just mentioned operations form a Heyting algebra. We would like to point out that they are well defined, since \( G \leftrightarrow G' \) and \( H \leftrightarrow H' \) imply that \( G \times H \leftrightarrow G' \times H' \) and \( G + H \leftrightarrow G' + H' \). From Lemma 54 \((g), (h)\) it also follows that \( W \leftrightarrow W' \) and \( G \leftrightarrow G' \) imply that \( W(G) \leftrightarrow W'(G') \). Moreover, if considered as a poset, \( \langle G \rangle \leq \langle W \rangle \) if and only if \( G \rightarrow W \).

We have also mentioned that if we fix a class \( w \in \mathcal{G} \), then the set \( B_w = \{w(g) \mid g \in \mathcal{G} \} \) is a Boolean algebra with meet \( x \wedge y = x \times y \), join \( x \vee y = w(w(x + y)) \), complement
$x^c = w(x)$, zero element $w$ and unit $(1)$ - the class of all graphs with a loop [13]. Duffus and Sauer [13] proved that if $B(K_n)$ is finite for all $n$, then the function $\mu(n)$ tends to infinity with $n$. Consequently they asked for which graphs $W$, $B(W)$ is finite. Now we can provide an answer to this question.

**Proposition 66** The boolean algebra $B(W)$ is finite if and only if $W$ has a finite factorization.

**Proof:** Let us assume that $B(W)$ is finite. By Stone's representation theorem (see [49]), every finite Boolean algebra is isomorphic to a Boolean algebra of all subsets of some finite set. Therefore the zero element is equal to the meet of all coatoms. Duffus and Sauer [13] have shown that all coatoms in $B(W)$ are multiplicative classes (the same conclusion is implicit in Proposition 99) and hence $W$ has a finite factorization. On the other hand, let us assume that $W \leftrightarrow W_1 \times \ldots \times W_n$. Then $W(G) \leftrightarrow (W_1 \times \ldots \times W_n)(G) \leftrightarrow W_1(G) \times \ldots \times W_n(G)$. But since each $W_i$ is multiplicative, we have $(W_i(G)) \in \{(W_i), (1)\}$, which implies that there are at most $2^n$ different classes of graphs in the set $\{W(G) \mid G \in \mathcal{G}\}$ and therefore $B(W)$ is finite. $\square$

We have also the following proposition.

**Proposition 67** If $G$ and $H$ have a finite factorizations, then both $G \times H$ and $G + H$ also have a finite factorizations. Moreover, if $W$ has a finite factorization then the map graph $W(G)$ has a finite factorization, for each graph $G$.

**Proof:** Let $G \leftrightarrow P_1 \times \ldots \times P_n$ and $H \leftrightarrow Q_1 \times \ldots \times Q_m$ be finite factorizations of $G$ and $H$, i.e., assume that all graphs $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ are multiplicative. Then $G \times H \leftrightarrow P_1 \times \ldots \times P_n \times Q_1 \times \ldots \times Q_m$ and hence $G \times H$ has a finite factorization. Also $G + H \leftrightarrow (P_1 \times \ldots \times P_n) + (Q_1 \times \ldots \times Q_m) \leftrightarrow \Pi_{i,j}(P_i + Q_j)$ (see Lemma 74), and by Corollary 56 all sums $P_i + Q_j, i = 1, \ldots, n, j = 1, \ldots, m$ are multiplicative; we conclude that also $G + H$ has a finite factorization. For the second part let us assume that $W \leftrightarrow W_1 \times \ldots \times W_r$ be a finite factorization of $W$. Then by Proposition 55, $W(G) \leftrightarrow W_1(G) \times \ldots \times W_r(G) \leftrightarrow \Pi_{G \neq W_i} W_i$, since $W_i$'s are multiplicative; we
conclude that $W(G)$ has a finite factorization. □

By the previous proposition, classes of graphs which have a finite factorization form a Heyting subalgebra of the Heyting algebra $\mathcal{G}$.

Now we are going to prove that it is enough to assume a finite factorization of one graph which contains $K_4$ for the proof that $\mu$ tends to infinity. We need the following proposition.

**Proposition 68** If for every graph $G$ there exists a multiplicative graph $W$ such that $G \rightarrow W$, then the function $\mu(n)$ tends to infinity with $n$.

**Proof:** Assume that for every graph $G$ there exists a multiplicative graph $W$ such that $G \rightarrow W$. Let us denote $X_n = \{W \mid K_n \rightarrow W, W \text{ multiplicative}\}$. The assumption implies $X_n \neq \emptyset$ and therefore $\gamma(n) = \min\{m \mid W \rightarrow K_m \text{ for some } W \in X_n\}$ is a well-defined, positive integer valued function. In other words, $\gamma(n)$ is the minimum chromatic number of all multiplicative graphs containing $K_n$. We will prove that $\gamma(\mu(n)) > n$. Let $W_n \in X_n$ be one of the multiplicative graph such that $W_n \rightarrow K_{\gamma(n)}$. Let $G \not\rightarrow K_n$, $H \not\rightarrow K_n$ and $G \times H \rightarrow K_{\mu(n)}$. Since $W_{\mu(n)} \in X_{\mu(n)}$, $G \times H \rightarrow K_{\mu(n)} \rightarrow W_{\mu(n)}$. By assumption $W_{\mu(n)}$ is multiplicative and hence $G \rightarrow W_{\mu(n)}$ or $H \rightarrow W_{\mu(n)}$. On the other hand $W_{\mu(n)} \rightarrow K_{\gamma(\mu(n))}$, which implies that $\gamma(\mu(n)) > n$. One can observe that $\mu$ is a nondecreasing function and if it was bounded, say by $c$, then for all $n$ we would have $n < \gamma(\mu(n)) \leq \max\{\gamma(1), \ldots, \gamma(c)\}$, which is a contradiction. □

Note that if there is a sequence of multiplicative graphs $\{P_i\}_{i=0}^{\infty}$ such that for all $n$ there is $P_i$ which contains $K_n$, then the conditions of Proposition 68 are satisfied. Moreover, if $K_4 \rightarrow W$, and $W$ has a finite factorization, then there exists a multiplicative graph $P$, such that $W \rightarrow P$ ($P$ can be chosen to be any graph from the factorization of $W$). If we denote $P_0 = P$ and $n_0 = 4$, then obviously $K_{n_0} \rightarrow P_0$. By induction let us define $n_{i+1} = \left(\frac{n_i}{2}\right)$ and $P_{i+1} = S_{P_i}$. Assuming $K_{n_i} \rightarrow P_i$ it follows
that $K_{n+1} \rightarrow S_{K_n} \rightarrow S_{P_i} = P_{i+1}$. Since $n_0 = 4$, it follows that $n_i < n_{i+1}$ and all $P_i$'s are multiplicative. Therefore we have the following corollary.

**Corollary 69** If there is a graph $W$ which has a finite factorization and $K_4 \rightarrow W$, then $\mu(n) \rightarrow \infty$.

This corollary is a motivation for our investigation of finite factorizations in the consequent sections.

### 4.4 Ideals

In this section, we will define ideals and operations on them. We will also make some simple observations which are useful in later sections.

**Lemma 70** For any graphs $G$ and $H$, $G \times H \rightarrow G \rightarrow G + H$.

*Proof:* A homomorphism $\alpha$ from $G \times H$ to $G$ can be defined as $\alpha(g, h) = g$. A homomorphism $\beta$ from $G$ to $G + H$ can be defined as $\beta(g) = (g, 0)$. □

**Lemma 71** The following conditions are equivalent:

1. $G \leftrightarrow G \times H$
2. $G \rightarrow H$
3. $G + H \leftrightarrow H$.

*Proof:* 

(i) $\Rightarrow$ (ii): From $G \rightarrow G \times H$ and $G \times H \rightarrow G$ (according to the Lemma 70) we conclude $G \rightarrow H$.

(ii) $\Rightarrow$ (iii): Let $\alpha_1$ be a homomorphism of $G$ to $H$ and let $\alpha_2$ be any homomorphism of $H$ to $H$ (for instance the identity). A homomorphism $\beta$ from $G + H$ to $H$ can be defined as $\beta((s, i)) = \alpha_1(s)$ for $(s, i) \in V(G + H)$.

(iii) $\Rightarrow$ (ii): From $G + H \rightarrow H$ and $G \rightarrow G + H$ (according to the Lemma 70) we conclude $G \rightarrow H$. 

(ii) ⇒ (i): Let \( \alpha \) be a homomorphism of \( G \) to \( H \). A homomorphism \( \beta \) of \( G \) to \( G \times H \) can be defined as \( \beta(s) = (s, \alpha(s)) \) for \( s \in V(G) \).

Therefore we have the following corollary.

**Corollary 72** For all graphs \( G, G \leftrightarrow G \times G \leftrightarrow G + G \).

Let us recall that a graph \( G \) is called a core if it is not homomorphic to any of its proper subgraphs, or equivalently if any endomorphism \( G \rightarrow G \) is an automorphism.

**Lemma 73** [54](cf. also [31]) Every equivalence class of graphs has a uniquely defined graph which is a core.

Recall that we consider two graphs equal if and only if they are isomorphic. One can easily observe that \( + \) and \( \times \) are commutative and associative operations and we have the following distributive laws.

**Proposition 74** For \( G, H, F \in \mathcal{G} \)

(i) \( G \times (H + F) = (G \times H) + (G \times F) \)

(ii) \( G + (H \times F) \leftrightarrow (G + H) \times (G + F) \)

**Proof:**

(i): One can check that the mapping which maps the vertex \( (g, (s, i)) \) of \( G \times (H + F) \) to the vertex \( ((g, s), i) \) of \( (G \times H) + (G \times F) \) is an isomorphism.

(ii): The mapping, which maps the vertex \( (g, 0) \) to the vertex \( ((g, 0), (g, 0)) \) and the vertex \( ((h, f), 1) \) to the vertex \( ((h, 1), (f, 1)) \) is a homomorphism from \( G + (H \times F) \) to \( (G + H) \times (G + F) \). On the other hand the mapping which maps the vertex \( ((g, 0), (s, j)) \) to the vertex \( (g, 0) \), the vertex \( ((h, 1), (g, 0)) \) to the vertex \( (g, 0) \) and the vertex \( ((h, 1), (f, 1)) \) to the vertex \( ((h, f), 1) \) is a homomorphism from \( (G+H)\times(G+F) \) to \( G + (H \times F) \). □

As in [13], we identify equivalent graphs, and study the multiplicative structure of the classes of equivalent graphs \( \overline{\mathcal{G}} \). As we have mentioned above, \( \overline{\mathcal{G}} \) is a Heyting
algebra, and hence a distributive lattice with meet $\times$ and join $+$. Also we will use symbols $0$ and $1$ for the minimum and the maximum elements of any lattice and all lattices considered are assumed to have $0$ and $1$. We will not use any specific property of the Heyting algebra $\mathcal{G}$ and hence we will state all results for a general distributive lattice $\mathcal{L}$ with $0$ and $1$. To interpret these results for graphs, the reader only has to think of $\mathcal{L}$ as the lattice $\mathcal{G}$. The only exception to this rule is in the next section. Recall that we use symbols $\times$, $+$ and $\leq$ for meet, join and the partial order relation respectively.

We now formally define ideals.

A subset $a$ of a lattice $\mathcal{L}$ is an ideal, if $a$ satisfies the following properties:

1. $A \in a$ and $G \in \mathcal{L}$ imply $A \times G \in a$ and
2. $A, B \in a$ implies $A + B \in a$.

One can observe that the first condition is equivalent to

1'. if $A \in a$ and $B \leq A$ then $B \in a$.

We will use this condition and condition (1) interchangeably.

A subset $f$ of a lattice $\mathcal{L}$ is a filter, if $f$ satisfies the following properties:

1. $A \in f$ and $G \in \mathcal{L}$ imply $A + G \in f$ and
2. $A, B \in f$ implies $A \times B \in f$.

Again the first condition is equivalent to

1'$. if $A \in a$ and $A \leq B$ then $B \in a$.

Let $\mathcal{L}$ be a distributive lattice. Note that the set $\{0\}$ is an ideal and it is contained in every ideal of $\mathcal{L}$. It is easy to see that the intersection of ideals in $\mathcal{L}$ is always an ideal and that $\mathcal{L}$ itself is an ideal. We say that an ideal is proper, if it is not equal to $\mathcal{L}$. Let $S$ be a nonvoid subset of $\mathcal{L}$, and let $(S)$ denotes the smallest ideal containing $S$ (the intersection of all ideals containing $S$). The ideal $(S)$ is called the ideal generated by $S$. For $S = \emptyset$ we define $(S) = \{0\} = (\{0\})$. An ideal is called principal if it is
generated by a one element set and \textit{finitely generated} if it is generated by finite set. In this case we will write \((A_1, \ldots, A_n)\) instead of \((\{A_1, \ldots, A_n\})\). In particular, we have \((1) = \mathcal{L}\).

Similarly, the set \(\{1\}\) is a filter and it is contained in every filter of \(\mathcal{L}\). It is easy to see that the intersection of filters in \(\mathcal{L}\) is always a filter and \(\mathcal{L}\) is a filter. We say that a filter is \textit{proper}, if it is not equal to \(\mathcal{L}\). Let \(S\) be a nonvoid subset of \(\mathcal{L}\). The smallest filter containing \(S\) (the intersection of all filters containing \(S\)) is called the \textit{filter generated by} \(S\).

The following observations are quite trivial but we will use them later.

**Proposition 75** If \(H \in \mathcal{L}\) then \((H) = \{G \in \mathcal{L} \mid G \leq H\}\).

\textit{Proof:} Let \(a = \{G \mid G \leq H\}\). First we will show that \(a\) is an ideal. The condition (1) follows from the transitivity of \(\leq\), since \(G \in a\) implies that \(G \leq H\), and therefore if \(G' \leq G\) then necessarily also \(G' \leq H\). The condition (2) follows, since \(G \in a\) and \(G' \in a\) imply that \(G \leq H\) and \(G' \leq H\) so we conclude \(G + G' \leq H\) and hence \(G + G' \in a\). Since \(H \leq H\), it follows that \(H \in a\) and therefore we have proved that \((H) \subseteq a\). The condition (1) implies the other inclusion. \(\Box\)

**Proposition 76** Every finitely generated ideal of \(\mathcal{L}\) is principal.

\textit{Proof:} Let \(a = (G_1, \ldots, G_n)\) be a finitely generated ideal. From (2) it follows that the principal ideal \((G_1 + \ldots + G_n) \subseteq a\). On the other hand \((G_1 + \ldots + G_n)\) contains all \(G_i, i = 1, \ldots, n\), since \(G_i \leq G_1 + \ldots + G_n\) for all \(i = 1, \ldots, n\) and we conclude that \(a = (G_1, \ldots, G_n) \subseteq (G_1 + \ldots + G_n)\). \(\Box\)

The following proposition shows that the existence of a homomorphism can be translated to inclusion between corresponding ideals in the lattice \(\mathcal{G}\).

**Proposition 77** Let \(G, H \in \mathcal{L}\). Then \(G \leq H\) if and only if \((G) \subseteq (H)\).
CHAPTER 4. STRUCTURE OF COLOR CLASSES

Proof: \( G \leq H \) implies that \( \{ F \mid F \leq G \} \subseteq \{ F \mid F \leq H \} \) and by Proposition 75 it follows that \( (G) \subseteq (H) \). On the other hand, let us assume that \( (G) \subseteq (H) \). Since \( G \in (G) \), this implies that \( G \in (H) \) and consequently by Proposition 75, \( G \rightarrow H \). \( \square \)

Note that the union of two ideals is not an ideal in general. For two ideals \( a \) and \( b \) of \( \mathcal{L} \) we define

\[
a + b = \{ G \mid G = A + B, \ A \in a, \ B \in b \}.
\]

One can easily check that \( a + b \) is an ideal.

**Proposition 78** If \( a \) and \( b \) are two ideals then \( a + b = (a \cup b) \).

**Proof:**
If \( G \in a + b \), then \( G = A + B \) for some \( A \in a \) and \( B \in b \). Therefore \( G \leq A + B \in (a \cup b) \) and hence \( G \in (a \cup b) \). On the other hand \( (a \cup b) \) is the smallest ideal containing both \( a \) and \( b \) and therefore \( (a \cup b) \subseteq a + b \). \( \square \)

**Proposition 79** \( (G) \cap (H) = (G \times H) \) and \( (G) + (H) = (G + H) \).

**Proof:** From the previous proposition we have the second equality. Since \( G \times H \leq G \), we have \( (G \times H) \subseteq (G) \), and similarly \( (G \times H) \subseteq (H) \), which implies \( (G \times H) \subseteq (G) \cap (H) \). On the other hand if \( W \in (G) \cap (H) \), then \( W \leq G \) and \( W \leq H \), which implies that \( W \leq G \times H \), and hence \( W \in (G \times H) \). \( \square \)

One can observe that \( \cap \) and \( + \) are commutative and associative binary operations on the set of all ideals \( I_\mathcal{L} \) of \( \mathcal{L} \), which satisfy both distributive laws and hence form a distributive lattice with the minimum \( \{0\} = (0) \) and the maximum \( \mathcal{L} = (1) \).

A system of sets \( \mathcal{C} \) is called a chain if for all \( a, b \in \mathcal{C} \), either \( a \subseteq b \) or \( b \subseteq a \). The intersection of the chain \( \mathcal{C} \) is the set \( \bigcap_{a \in \mathcal{C}} a \) and the union of the chain \( \mathcal{C} \) is the set \( \bigcup_{a \in \mathcal{C}} a \).

Since the intersection of ideals is always an ideal, the intersection of a chain of ideals is also an ideal. One can easily observe that the union of a chain of ideals is
also an ideal. Similarly one can observe that the intersection and the union of a chain of filters are filters.

We will also need the following theorem, whose proof was inspired by the proof of Theorem 81 from [37].

**Theorem 80** If \( a \) is an ideal and \( \mathcal{B} \) is a finite system of ideals, then

\[
\text{a} \subseteq \bigcup_{b \in \mathcal{B}} b \text{ implies } \text{a} \subseteq \text{b} \text{ for some } \text{b} \in \mathcal{B}.
\]

**Proof:** By induction on \( n = |\mathcal{B}| \). The proposition is obviously true for \( n = 1 \). Assume that \( n \geq 2 \). For a contradiction, let us assume that for all \( b \in \mathcal{B} \), \( a \not\subseteq b \). For every \( c \in \mathcal{B} \) we may assume that otherwise we use the induction hypothesis to conclude that \( a \subseteq \text{b} \) for some \( \text{b} \in \mathcal{B} \).

Therefore for every \( c \in \mathcal{B} \) there exists \( G_c \in a \) such that \( G_c \not\subseteq \bigcup_{b \neq c} b \). Since \( a \) is an ideal and \( n \geq 2 \), for any two distinct \( c \neq c' \in \mathcal{P} \), \( G = G_c + G_c' \in a \). We claim that \( G \not\subseteq \bigcup_{b \in \mathcal{B}} b \). For a contradiction assume that \( G \in b \) for some \( b \in \mathcal{B} \). Since \( c \) and \( c' \) are distinct, either \( b \neq c \) or \( b \neq c' \). We may assume that \( b \neq c \). Since \( G_c \leq G_c' \), it follows that also \( G_c \in a \), which is a contradiction proving our claim. Therefore \( G \in a \) and \( G \not\subseteq \bigcup_{b \in \mathcal{B}} b \), which is a contradiction with our assumption \( a \subseteq \bigcup_{b \in \mathcal{B}} b \). \( \Box \)

The previous theorem has a counterpart in commutative algebra where it is called prime avoidance theorem, since in the case of commutative rings at most two ideals of \( \mathcal{B} \) may not be primes.

### 4.5 Additive structure

To warm up, let us first have a brief look at the additive structure of ideals in \( \overline{G} \). Loosely speaking, it turns out that (join) irreducible elements correspond to connected graphs and hence every graph is the sum of finitely many (join) irreducible
CHAPTER 4. STRUCTURE OF COLOR CLASSES

elements. To be more precise, let us recall that the graph with empty vertex and edge sets is denoted by 0. For every graph $G$, $0 + G = G + 0 = G$, $0 \to G$ and $0 \times G = 0$.

Recall that a graph $W$ is connected, if there are no two nonzero graphs $G$ and $H$ such that $W = G + H$.

Definition 81 We say that an ideal $a$ is additive, if $a \subseteq b + c$ implies $a \subseteq b$ or $a \subseteq c$.

Lemma 82 An ideal $a$ is not additive if and only if there are ideals $b$ and $c$ such that $a = b + c$ and $b \nsubseteq c$ and $c \nsubseteq b$.

Proof: If there are ideals $b$ and $c$ such that $a = b + c$ and $b \nsubseteq c$ and $c \nsubseteq b$, then it is sufficient to show that $a \nsubseteq b$ and $a \nsubseteq c$ in order to prove that $a$ is no additive. Let us assume that this is not true and say $a \subseteq b$. Then

$$a \subseteq a + c \subseteq b + c = a$$

which implies that $a = a + c$, and we conclude that $c \subseteq a \subseteq b$.

On the other hand, let us assume that $a$ is not additive and hence there are two ideals $b$ and $c$ such that $a \subseteq b + c$ but $a \nsubseteq b$ and $a \nsubseteq c$. Therefore $a = a \cap b + a \cap c$. If $a \cap b \subseteq a \cap c$, then $a = a \cap c$ which implies that $a \subseteq c$, a contradiction. We conclude that $a \cap b \nsubseteq a \cap c$ and similarly $a \cap c \nsubseteq a \cap b$. □

Proposition 83 A principal ideal $(G) \subseteq \overline{G}$ is additive if and only if the class $G$ contains a connected core.

Proof: By Lemma 73 we may assume that $G$ is a core. If $G$ is connected, then $(G) \subseteq b + c$ implies that $G \in b + c$ and therefore $G \to H + H'$ for some $H \in b$ and $H' \in c$. Since $G$ is connected, this implies that $G \to H$ or $G \to H'$ and hence $G \in b$ or $G \in c$. Therefore $(G) \subseteq b$ or $(G) \subseteq c$.

Let us assume that $G$ is not connected and $G_1, \ldots, G_n$ are its connected components. Therefore $(G) = (G_1) + (G_2 + \ldots + G_n)$. If $(G_1) \subseteq (G_2 + \ldots + G_n)$ or equivalently $G_1 \to G_2 + \ldots + G_n$, then $G \leftrightarrow G_2 + \ldots + G_n$, which is a contradiction with our
assumption that $G$ is a core. Similarly $(G_2 + \ldots + G_n) \subseteq (G_1)$ would imply that $G \leftrightarrow G_1$, which is again a contradiction. □

This proposition shows that connectivity is a dual of multiplicativity in a sense that join irreducible elements of $\overline{G}$ have connected cores and meet irreducible elements of $\overline{G}$ have multiplicative cores.

**Proposition 84** Every principal ideal in $\overline{G}$ can be uniquely written as a finite sum of additive principal ideals and hence every element is a finite sum of join irreducible elements.

**Proof:** This follows from the fact that every graph has a unique core, whose connected components have to be again cores and therefore generate additive principal ideals. □

### 4.6 Multiplicative structure

In this section, we will define prime ideals and give necessary tools to deal with them. We assume all ideals to be ideals of a distributive lattice $\mathcal{L}$ with 0 and 1.

For any two ideals $a, b$ we define

$$(a : b) = \{ G \in \mathcal{L} \mid (G) \cap b \subseteq a \}.$$ 

**Lemma 85** Let $a, b, c, \varnothing, \alpha$ be ideals. Then

(a) $(a : b)$ is an ideal,

(b) $(a : b) = (1)$ if and only if $b \subseteq a$,

(c) $(a : (1)) = a$,

(d) $a \subseteq (a : b)$,

(e) $b \subseteq (a : (a : b))$,

(f) $b \cap (a : b) = a \cap b$, 


(g) \((a : b) : c) = (a : (b \cap c))\),

(h) \((a : b) \cap c \subseteq (a : (b : c))\),

(j) \((a : (\cup_i a_i)) = \cap_i (a : a_i)\),

(k) \((\cap_i a_i : a) = \cap_i (a_i : a)\),

(l) \(a \subseteq b \text{ and } c \subseteq d \implies (a : d) \subseteq (b : c)\),

(m) \(a = (a : b) \cap (a : (a : b))\).

**Proof:**

(a) Let us assume that \(G \in (a : b)\) and \(H \leq G\). If \(B \in b\) then \(H \times B = (H \times G) \times B = H \times (G \times B) \leq G \times B \in a\) and hence \(H \in (a : b)\). To prove the second condition, let us assume that \(G, H \in (a : b)\). Let \(B \in b\). Then \(G \times B\) and \(H \times B\) are in \(a\) and therefore also their sum \(G \times B + G \times H\) is in \(a\). But \((G \times B) + (H \times B) = (G + H) \times B\) and therefore \(G + H \in (a : b)\).

(l) Let \(G \in (a : d)\) and \(C \subseteq C \subseteq d\). Then \(G \times C \in a \subseteq b\) and therefore \(G \in (b : c)\).

(b) If \((a : b) = (1)\), then \(1 \in (a : b)\) and therefore for \(B \in b\), \(B = 1 \times B \in a\). On the other hand if \(b \subseteq a\), \(B \in b\) and \(G \in (1)\), then \(G \times B \leq B \in b \subseteq a\).

(d) Let \(A \in a\) and \(B \in b\). Then \(A \times B \leq A \in a\) implies that \(A \in (a : b)\).

(c) From (d) it follows that \(a \subseteq (a : (1))\). Let \(G \in (1)\) and \(A \in a\). Then \(A \times G \leq A \in a\) which implies \(A \in (a : (1))\).

(e) Let \(B \in b\) and \(G \in (a : b)\). Then \(B \times G \in a\) and therefore \(B \in (a : (a : b))\).

(f) Let \(G \in b \cap (a : b)\), then \(G \in b\) and \(G \in (a : b)\). Therefore \(G = G \times G \in a\).

(g) Let \(G \in ((a : b) : c)\) and \(H \in b \cap c\). Then \(G \times H \in (a : b)\) since \(H \in c\). Also \(H \in b\) and hence \(G \times H = (G \times H) \times H \in a\). We conclude that \(G \in (a : (b \cap c))\). On the other hand let us assume that \(G \in (a : (b \cap c))\) and \(H \in c\). Then if \(F \in b\), then \(F \times H \in b \cap c\) and hence \(G \times (F \times H) \in a\) which implies that \(G \times H \in (a : b)\) and we conclude that \(G \in ((a : b) : c)\).

(h) Let \(G \in (a : b) \cap c\). Then \(G = G \times G\) such that \(G \in (a : b)\) and \(G \in c\). Let \(H \in (b : c)\). Then \(G \times H = (G \times G) \times H = G \times (G \times H)\). Since \(G \times H \in b\), \(G \times (G \times H) \in a\) and hence \(G \in (a : (b : c))\).

(j) Let \(G \in (a : (\cup_i a_i))\) and \(H \in a_i\). Then \(G \times H \in a\) which implies that \(G \in a_i\) for all \(i\). On the other hand let us assume that \(G \in \cap_i (a : a_i)\) and \(H \in \cup_i a_i\). Then there is an \(i\) such that \(H \in a_i\) which implies that \(G \times H \in a\) and we conclude that
CHAPTER 4. STRUCTURE OF COLOR CLASSES

\[ G \in (a : (\bigcup_i a_i)) \].

(k) Let \( G \in (\bigcap_i a_i : a) \). Then for \( H \in a_i \), \( G \times H \in \bigcap_i a_i \subseteq a_i \). Therefore \( G \in (a_i : a) \) for all \( i \), which implies that \( G \in \bigcap_i (a_i : a) \). On the other hand assume that \( G \in \bigcap_i (a_i : a) \).

Then \( G \in (a_i : a) \) for all \( i \) and therefore for \( H \in a_i \), \( G \times H \in a_i \) for all \( i \), which implies that \( G \times H \in \bigcap_i a_i \). Therefore \( G \in (\bigcap_i a_i : a) \).

(m) From (d) it follows that \( a \subseteq (a : b) \cap (a : (a : b)) \). The other inclusion follows from (f). \( \square \)

This Lemma particularly implies that the set of all ideals \( \mathcal{I}_L \) is a Heyting algebra with relative pseudocomplement \( a(b) = (a : b) \).

An ideal \( p \) is called prime if it is proper and for any two ideals \( a \) and \( b \), \( a \cap b \subseteq p \) implies \( a \subseteq p \) or \( b \subseteq p \). An equivalent condition is that for every \( G, H \in \mathcal{L}, G \times H \in p \) implies \( G \in p \) or \( H \in p \), which means that the complement \( \mathcal{L} - p \) is a filter.

Looking at the complements, one can easily observe that both the intersection and the union of a chain of prime ideals is a prime ideal.

By induction it follows that if \( p \) is a prime ideal and \( A_f \) is a finite system of ideals, then

\[ \bigcap_{a \in A_f} a \subseteq p \text{ implies } a \subseteq p \text{ for some } a \in A_f. \]

The proof of the following lemma is essentially the proof of Theorem 1 from [37].

Lemma 86 Let \( a \) be an ideal, \( \mathfrak{f} \) be a filter, \( a \cap \mathfrak{f} = \emptyset \) and \( b \) be an ideal containing \( a \), which is maximal with respect to the exclusion of \( \mathfrak{f} \). Then \( b \) is prime, i.e., the complement of \( b \), (i.e. \( \mathcal{L} - b \)) is a filter.

Proof: Given \( G \times H \in b \) we must show that either \( G \) or \( H \) belongs to \( b \). Suppose the contrary. Then the ideal \( (G) + b \) is strictly larger than \( b \) and therefore intersects \( \mathfrak{f} \). Thus there exists an element \( S_1 \in \mathfrak{f} \) such that \( S_1 = A_1 + G', G' \in (G) \) and \( A_1 \in b \).

Similarly we have \( S_2 \in \mathfrak{f} \) such that \( S_2 = A_2 + H', H' \in (H) \) and \( A_2 \in b \). Hence
\[ S_1 \times S_2 = (A_1 + G') \times (A_2 + H') = (A_1 \times A_2) + (A_1 \times H') + (G' \times A_2) + (G' \times H') \in \mathfrak{b}, \]

since \( G' \times H' \leq G \times H \in \mathfrak{b} \) and the other three terms obviously belong to \( \mathfrak{b} \). Since \( \mathfrak{f} \) is a filter, \( S_1 \times S_2 \in \mathfrak{f} \cap \mathfrak{b} \), which is a contradiction. \( \square \)

We will also need the following lemma. It is a dualization of the previous lemma, and hence its proof can be obtained from it by interchanging the roles of + with \( \times \), \( \leq \) with \( \geq \), and ideal with filter.

**Lemma 87** Let \( a \) be an ideal, \( \mathfrak{f} \) be a filter, \( a \cap \mathfrak{f} = \emptyset \) and \( \mathfrak{e} \) be a filter containing \( \mathfrak{f} \), which is maximal with respect to the exclusion of \( a \). Then the complement \( \mathcal{L} - \mathfrak{e} \) is an ideal.

An ideal \( \mathfrak{m} \) is called **maximal** if it is proper, and if there is no proper ideal \( a \) such that \( \mathfrak{m} \subset a \subset (1) \).

Given any ideal \( a \) disjoint from a filter \( \mathfrak{f} \), by Zorn’s lemma we can expand \( a \) to an ideal \( \mathfrak{p} \) maximal with respect to disjointness from \( \mathfrak{f} \), and by the previous lemma it follows that \( \mathfrak{p} \) is prime. In particular if we take \( a = \{0\} \) and \( \mathfrak{f} = \{1\} \), then the resulting ideal is a maximal ideal. Hence we have the following proposition.

**Proposition 88** Every lattice with \( 0 \neq 1 \) has at least one maximal ideal.

**Proposition 89** If \( \mathfrak{m} \) is maximal ideal in \( \mathcal{L} \), then \( \mathfrak{m} \) is prime.

**Proof:** Follows from Lemma 86 taking \( \mathfrak{f} = \{1\} \) and \( a = \mathfrak{m} \). \( \square \)

In our particular case there is exactly one maximal ideal in \( \overline{\mathcal{G}} \), the ideal of classes of equivalent graphs which contain loopless graphs, i.e. \( \mathfrak{m} = \overline{\mathcal{G}} - \{(1)\} \) (and we do not need Zorn's lemma in this case).

**Proposition 90** The following conditions are equivalent:

(i) \( \mathfrak{p} \) is prime,

(ii) \( (\mathfrak{p} : a) \in \{\mathfrak{p}, (1)\} \) for all ideals \( a \),

(iii) \( (\mathfrak{p} : a) \in \{\mathfrak{p}, (1)\} \) for all principal ideals \( a \).
Proof: (i) $\Rightarrow$ (ii) follows from Lemma 85 (m), (a) and (ii) $\Rightarrow$ (iii) is obvious. To prove the implication (iii) $\Rightarrow$ (i) let us assume that $p$ is not prime and hence there are $G, H \in \mathcal{L}$ which do not belong to $p$ but $G \times H$ belongs to $p$. Then $H \in (p : (G))$ and hence $(p : (G)) \neq p$, since $H \not\in p$ and also $(p : (G)) \neq (1)$, since $G \not\in p$. □

In particular, this proposition and Lemma 85 (b) implies that for prime ideal $p$,

$$
(p : a) = \begin{cases} 
(1) & \text{if } a \subseteq p \\
 p & \text{otherwise.}
\end{cases}
$$

The proof of the following lemma is essentially the proof of Theorem 10 from [37].

**Lemma 91** Let $a$ be an ideal which does not contain $G$. Then $a$ is contained in a prime ideal which also does not contain $G$. Moreover, there exists a prime ideal $p$ satisfying these conditions which is minimal with respect to the inclusion.

**Proof:** Let $\mathcal{f}$ be the smallest filter which contains $G$, i.e., $\mathcal{f}$ contains exactly those elements, which are greater or equal to $G$. One can observe that $a$ and $\mathcal{f}$ are disjoint and by Zorn’s lemma we can expand $a$ to an ideal $p$ which is maximal with respect to exclusion of $\mathcal{f}$, and hence prime by Lemma 86. Let us embed $p$ in a maximal chain $\mathcal{P}$ of prime ideals containing $a$ (using Zorn’s lemma). The intersection of the chain $\mathcal{P}$ is a prime ideal which is clearly minimal. □

### 4.6.1 Prime Spectrum

In this section we study topological properties of prime ideals. It turns out, as we will see in the next section, that they are related to the factorization problem.

Let $\mathcal{L}$ be a distributive lattice with 0 and 1 and $Spec(\mathcal{L})$ be the set of all prime ideals of $\mathcal{L}$. For each subset $E$ of $\mathcal{L}$, let $V(E)$ denote the set of all prime ideals which contain $E$. For $G \in \mathcal{L}$ we write $V(G)$ instead of $V(\{G\})$. One can observe that (i) if $a = (E)$ is the ideal generated by $E$, then $V(E) = V(a)$,
(ii) $V(0) = \text{Spec}(\mathcal{L})$, $V(1) = \emptyset$,

(iii) $\bigcap_{i \in I} V(E_i) = V(\bigcup_{i \in I} E_i)$,

(iv) $V(a) \cup V(b) = V(a \cap b)$,

(v) $V(b) \subseteq V(a)$ if and only if $a \subseteq b$.

We will prove only the condition (v). Assume that $a \subseteq b$. It follows that if $p \subseteq b$ then $p \subseteq a$ and hence $V(b) \subseteq V(a)$. On the other hand let us assume that $a \not\subseteq b$. Then there is $G \in a$ such that $G \not\in b$. By Lemma 91 it follows that there is $p \in V(b)$ such that $G \not\in p$ and hence $p$ cannot contain $a$ as a subset. Therefore $p \not\in V(a)$ and we conclude that $V(b) \not\subseteq V(a)$.

Conditions (i) – (iv) imply that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space $\text{Spec}(\mathcal{L})$ is called the prime spectrum of $\mathcal{L}$.

The open sets of the form $\text{Spec}(\mathcal{L}) - V(G)$, $G \in \mathcal{L}$ are called basic open sets. They are denoted by $D(G)$ and they form a basis for the open sets of the Zariski topology, since every closed set is of the form $V(E) = \bigcap_{G \in E} V(G)$. Moreover, one can observe that

(i) $D(G) \cap D(H) = D(G \times H)$,

(ii) $D(G) = \emptyset$ if and only if $G = 0$,

(iii) $D(G) = \text{Spec}(\mathcal{L})$ if and only if $G = 1$,

(iv) $D(G) = D(H)$ if and only if $G = H$,

(v) $D(G)$ is compact, i.e., every open covering of $D(G)$ has a finite subcovering.

In particular, (v) and (iii) imply that $\text{Spec}(\mathcal{L})$ is compact.

For an arbitrary subset $A \subseteq \text{Spec}(\mathcal{L})$, the ideal

$$J(A) = \bigcap_{p \in A} p$$

is called the ideal of $A$. One can easily observe that $J(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} J(A_i)$.

**Lemma 92** For every $E \subseteq \mathcal{L}$ and $A \subseteq \text{Spec}(\mathcal{L})$ we have $E \subseteq J(A)$ if and only if $A \subseteq V(E)$.
Proof: Let us assume that \( E \subseteq J(A) \). It follows by the condition (v) that \( V(J(A)) \subseteq V(E) \). But \( A \subseteq V(J(A)) \), since if \( p \in A \) then \( J(A) = \bigcap_{q \in A} q \subseteq p \) and hence \( p \in V(J(A)) \). Therefore \( A \subseteq V(J(A)) \subseteq V(E) \). On the other hand let us assume that \( A \subseteq V(E) \). This means that for all \( p \in A \), \( p \supseteq E \) and hence also \( J(A) = \bigcap_{p \in A} p \subseteq E \). \( \Box \)

Proposition 93 \( V(J(A)) = \overline{A} \) is the closure of \( A \) in \( \text{Spec}(L) \) for every \( A \subseteq \text{Spec}(L) \), and \( J(V(E)) = (E) \) is the ideal generated by \( E \) in \( L \) for every \( E \subseteq L \).

Proof: The first equality follows, since
\[
\overline{A} = \bigcap_{V(E) \supseteq A} V(E) = \bigcap_{E \subseteq J(A)} V(E) = V(J(A)).
\]
For the second equality let us denote \( a = (E) \), the ideal generated by \( E \). The first equality and \((i)\) imply that \( V(a) = V(E) = \overline{V(E)} = V(J(V(E))) \) and hence by \((v)\) we have \( (E) = a = J(V(E)) \). \( \Box \)

Let us recall that a topological space \( X \) is Hausdorff if for every two distinct points \( x, y \in X \) there exist two disjoint open sets \( U_x(y), V_y(x) \subseteq X \) such that \( x \in U_x(y) \) and \( y \in V_y(x) \). In particular if \( X \) is Hausdorff, then all points \( x \in X \) are closed, since \( X - \{x\} = \bigcup_{y \neq x} V_y(x) \) is open. The spectrum \( \text{Spec}(L) \) with the Zariski topology is not Hausdorff, since any two nonempty open sets have nonempty intersection by \((i)\). Moreover, it contains nonclosed points, which is due to the fact that \( \text{Spec}(L) \) includes not just maximal ideals, but all prime ideals. Let us determine the closure of a point of \( \text{Spec}(L) \). If the point is the prime ideal \( p \) then its closure is by Proposition 93 \( V(J(p)) = V(p) \) and hence it consists of all prime ideals \( q \) with \( p \subseteq q \). In particular, a point of \( \text{Spec}(L) \) is closed if and only if it is a maximal ideal.

Let \( a \) be a proper ideal in \( L \) and let us denote the set of all minimal prime ideals containing \( a \) (with respect to the inclusion) by \( M_a \), i.e., \( M_a = \{ p \in \text{Spec}(L) \mid p \supseteq a \text{ and } p \supseteq q \supseteq a \text{ implies } p = q \text{ for all } q \in \text{Spec}(L) \} \). Lemma 91 implies that \( M_a \neq \emptyset \) (take \( G = 1 \)). \( M_a \) is a subset of \( \text{Spec}(L) \) and we endow \( M_a \) with the induced topology.
of $\text{Spec}(\mathcal{L})$, i.e., $Y \subseteq M_\mathfrak{a}$ is closed in $M_\mathfrak{a}$ if and only if $Y = V \cap M_\mathfrak{a}$ for some closed subset $V$ of $\text{Spec}(\mathcal{L})$. One can observe that all closed sets are of the form $V_\mathfrak{a}(E) = V(E) \cap M_\mathfrak{a}$ for some $E \subseteq \mathcal{L}$ and the sets $D_\mathfrak{a}(G) = D(G) \cap M_\mathfrak{a}$, $G \in \mathcal{L}$ form a basis for the open sets of the induced topology on $M_\mathfrak{a}$.

It follows that all points of $M_\mathfrak{a}$ are closed in $M_\mathfrak{a}$. We will show that $M_\mathfrak{a}$ is actually Hausdorff space, but first we will prove a criterion for a prime to belong to $M_\mathfrak{a}$. The proof of the following lemma was inspired by the proof of Theorem 2.1 from [35].

**Lemma 94** Let $\mathfrak{a}$ be a proper ideal and $\mathfrak{p}$ a subset of $\mathcal{L}$. Then the following conditions are equivalent:

(i) $\mathfrak{p} \in M_\mathfrak{a}$,

(ii) $\mathcal{L} - \mathfrak{p}$ is a filter that is maximal with respect to missing $\mathfrak{a}$,

(iii) $\mathfrak{a} \subseteq \mathfrak{p}$, $\mathfrak{p}$ is a prime ideal and for every $G \in \mathfrak{p}$, there is an $H \notin \mathfrak{p}$ such that $G \times H \in \mathfrak{a}$.

**Proof:** (i) $\Rightarrow$ (ii): If $\mathfrak{p} \in M_\mathfrak{a}$ then $\mathfrak{p}$ is prime and hence $\mathcal{L} - \mathfrak{p}$ is a filter. By Zorn's lemma one can expand $\mathcal{L} - \mathfrak{p}$ to a filter $\mathfrak{f}$ that is maximal with respect to missing $\mathfrak{a}$. If $\mathfrak{q}$ is an ideal containing $\mathfrak{a}$ that is maximal with respect to being disjoint from $\mathfrak{f}$ then $\mathfrak{q}$ is prime by Lemma 86. Note that $\mathfrak{q}$ is disjoint from $\mathcal{L} - \mathfrak{p}$ which implies that $\mathfrak{q} \subseteq \mathfrak{p}$ and hence the minimality of $\mathfrak{p}$ implies that $\mathfrak{q} = \mathfrak{p}$. Thus $\mathcal{L} - \mathfrak{p} = \mathfrak{f}$.

(ii) $\Rightarrow$ (iii): Since $\mathfrak{f} = \mathcal{L} - \mathfrak{p}$ is a filter that is maximal with respect to missing $\mathfrak{a}$, $\mathfrak{a} \subseteq \mathfrak{p}$. Moreover, by Lemma 87, $\mathfrak{p}$ is an ideal and since its complement is a filter it is a prime ideal. Let $G \in \mathfrak{p}$ and let $\mathfrak{f} = \{G \times H \mid H \in \mathcal{L} - \mathfrak{p}\}$. Then $\mathfrak{f}$ is a filter that properly contains $\mathcal{L} - \mathfrak{p}$. So there is some $H \in \mathcal{L} - \mathfrak{p}$ such that $G \times H \in \mathfrak{a}$.

(iii) $\Rightarrow$ (i): Assume that $\mathfrak{a} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, where $\mathfrak{q}$ is a prime ideal. If there exist some $G \in \mathfrak{p} - \mathfrak{q}$, then there is an $H \notin \mathfrak{p}$ such that $G \times H \in \mathfrak{a} \subseteq \mathfrak{q}$, which is a contradiction because $\mathfrak{q}$ is a prime ideal. Therefore $\mathfrak{p} = \mathfrak{q}$. $\Box$

Now we can prove the following theorem.

**Theorem 95** The topological space $M_\mathfrak{a}$ is Hausdorff.
CHAPTER 4. STRUCTURE OF COLOR CLASSES

Proof: If $a$ is a prime ideal, then $M_a$ has only one point and our theorem is true. Therefore we may assume that $a$ is not a prime ideal. Let $p$ and $q$ be two distinct points of $M_a$. Since both of them are minimal and distinct, we have $p \not\subseteq q$ and $q \not\subseteq p$. Therefore there exist $P, Q$ such that $P \in p$, $P \not\subseteq q$, $Q \in q$ and $Q \not\subseteq p$. Moreover by Lemma 94 there exists $G \not\subseteq p$ such that $P \times G \in a$. Since both $Q$ and $G$ do not belong to $p$, also $Q' = Q \times G$ does not belong to $p$ but $Q'$ belongs to $q$, since $Q$ does. Hence $p \in D(Q')$, $q \in D(P)$ and $D(P) \cap D(Q') = D(P \times Q')$. But $D(P \times Q') \cap M_a = \emptyset$, since $P \times Q' = P \times G \times Q \subseteq P \times G \in a$ and hence every minimal prime ideal which contains $a$ has to also contain $P \times Q'$. We conclude that the open sets $D_a(P) = D(P) \cap M_a$ and $D_a(Q') = D(Q') \cap M_a$ are disjoint, $p \in D_a(Q')$, $q \in D_a(P)$ and hence $M_a$ is a Hausdorff space. □

Let us recall that a topological space is discrete if all of its subsets are clopen, i.e., they are both closed and open. If $X$ is a finite Hausdorff space, then all subsets of $X$ are closed (and hence also open) since all points are closed, and therefore $X$ is discrete. Note that a topological space is discrete if and only if all of its points are clopen. It turns out, as we will see in the next section, that discreetness of $M_a$ is closely related to the factorization of $a$. Therefore our aim is to identify all points of $M_a$ which are open (and hence clopen) in $M_a$.

**Proposition 96** $V_a(J(A))$ is the closure of $A$ in $M_a$ for every $A \subseteq M_a$, $J(M_a) = a$, and $J(D_a(G)) = (a : (G))$ for every $G \in \mathcal{L}$.

Proof: From Proposition 93 it follows that $V_a(J(A))$ is the closure of $A$ in $M_a$. To prove the equality $J(M_a) = a$, let us denote $b = J(M_a) = \cap_{p \in M_a} p$. Obviously $a \subseteq b$ since $a$ is a subset of every $p$ from $M_a \neq \emptyset$. Assume that $G \not\subseteq a$ belongs to $b$. By Lemma 86 there is a minimal prime ideal containing $a$ which does not contain $G$, which is a contradiction, since $b$ is intersection of all minimal prime ideals containing $a$. We conclude that $a = b$. 
CHAPTER 4. STRUCTURE OF COLOR CLASSES

The last equality follows, since

\[ J(D_a(G)) = \bigcap_{p \in D_a(G)} p = \bigcap_{p \in M_a \land G \not\subseteq p} p = \bigcap_{p \in M_a} (p : (G)) = (\bigcap_{p \in M_a} p : (G)) = (J(M_a) : (G)) = (a : (G)). \]

We say that an ideal \( b \) is \( a \)-divisorial, if \( b = (a : (a : b)) \). We denote the set of all prime ideals which are \( a \)-divisorial by \( O_a \). We will prove later that \( O_a \) is a subset of \( M_a \) and that \( O_a \) is the set of all clopen points of \( M_a \).

**Lemma 97** The ideal \( b \) is \( a \)-divisorial if and only if there exists an ideal \( c \) such that \( b = (a : c) \).

**Proof:** If \( b \) is \( a \)-divisorial, then \( b = (a : (a : b)) \) and therefore we can take \( c = (a : b) \). On the other hand let us assume that \( b = (a : c) \) for some ideal \( c \). Then Lemma 85 (e) implies that \( c \subseteq (a : (a : c)) = (a : b) \) and hence by Lemma 85 (1) we have \( (a : (a : b)) \subseteq (a : c) = b \). The reverse inclusion follows from Lemma 85 (e). \( \square \)

It follows from Lemma 85 (b) that \( L = (1) = (a : a) \) and from Lemma 85 (c) we have also \( a = (a : (1)) \). Therefore \( a \) and \( (1) \) are always \( a \)-divisorial. Lemma 85 (d) also implies that for every \( a \)-divisorial ideal \( b \), \( a \subseteq b \subseteq (1) \).

**Lemma 98** If \( p \) is a prime ideal such that \( a \subseteq p \), then \( a \subseteq (a : p) \) if and only if \( p \) is \( a \)-divisorial.

**Proof:** Let us assume that \( a \subseteq (a : p) \). Then we have \( (a : p) \cap (a : (a : p)) = a \subseteq p \) and since \( p \) is prime \( (a : p) \subseteq p \) or \( (a : (a : p)) \subseteq p \). But \( (a : p) \subseteq p \) would imply that \( (a : p) \cap (a : p) = a \cap p = a \) which is a contradiction with our assumption \( a \subseteq (a : p) \). We conclude that \( (a : (a : p)) \subseteq p \), which together with the (e) part of Lemma 85 imply that \( p \) is \( a \)-divisorial.

On the other hand let us assume that \( p \) is \( a \)-divisorial. For a contradiction let \( a = (a : p) \). Then \( p = (a : (a : p)) = (a : a) = (1) \), which is a contradiction. \( \square \)
An $a$-divisorial ideal $b$ is called a *maximal $a$-divisorial ideal*, if it is proper and if there is no proper $a$-divisorial ideal $c$ such that $b \subset c \subset (1)$.

**Proposition 99** $p \in O_a$ if and only if $p$ is a maximal $a$-divisorial ideal.

**Proof:** Assume that $p$ is prime $a$-divisorial ideal but it is not maximal $a$-divisorial ideal, i.e., $p \subset b$ for some $a$-divisorial ideal $b \neq (1)$. Lemma 85 (1) implies that $(a : b) \subseteq (a : p)$. Since $p$ is $a$-divisorial, $a \subseteq p$ and Lemma 85 (f) implies that $b \cap (a : b) = a \cap b \subseteq a \subseteq p$. But $p$ is prime and hence $(a : b) \subseteq p$, since $b \subseteq p$ is impossible due to our assumption $b \supset p$. The inclusions $(a : b) \subseteq (a : p)$ and $(a : b) \subseteq p$ imply $(a : b) \subseteq p \cap (a : p) = p \cap a = a$ and hence by Lemma 85 (b) and (1), $(1) = (a : a) \subseteq (a : (a : b)) = b$, which is a contradiction.

On the other hand assume that $p$ is maximal $a$-divisorial ideal and suppose $b$ and $c$ are ideals such that $b \cap c \subseteq p$ but $b \nsubseteq p$ and also $c \nsubseteq p$. It can be assumed that $p \subseteq b$ and $p \subseteq c$, otherwise one can replace $b$ with $p + b$ and $c$ with $p + c$. Then $p \subseteq b \subseteq (p : c) \subseteq (1)$. If we prove that $(p : c)$ is $a$-divisorial, then these inequations would imply that $(p : c)$ is $a$-divisorial ideal properly containing $p$ and therefore it is equal to $(1)$. This would imply that $c \subseteq p$, which is a contradiction with our assumptions. Therefore, to finish the proof that $p$ is prime, it is enough to show that $(p : c)$ is $a$-divisorial. But this follows from the equalities

$$(p : c) = ((a : (a : p)) : c) = (a : ((a : p) \cap c)),$$

and Lemma 97. □

**Lemma 100** $O_a \subseteq M_a$.

**Proof:** Assume that $p \in O_a$. If $p$ is not minimal, then there exists a prime ideal $q$ such that $a \subseteq q \subseteq p \subset (1)$. Lemma 85 (1) and Lemma 98 imply that $a \subseteq (a : p) \subseteq (a : q)$ and hence by Lemma 98 it follows that $q$ is $a$-divisorial. But this is a contradiction, since by Lemma 99 $q$ is also maximal $a$-divisorial ideal. □

For open subsets of $M_a$ we have the following lemma.
Lemma 101  If a subset \( A \subseteq M_a \) is open in \( M_a \) then \( J(A) \) is \( a \)-divisorial ideal. On the other hand, if \( b \) is \( a \)-divisorial ideal, then there exists a subset \( A \subseteq M_a \) which is open in \( M_a \) and \( b = J(A) \).

Proof:  Let us assume that the subset \( A \subseteq M_a \) is open in \( M_a \). Therefore \( A = \bigcup_{i \in I} D_a(G_i) \) and hence by the Proposition 96

\[
J(A) = J(\bigcup_{i \in I} D_a(G_i)) = \bigcap_{i \in I} J(D_a(G_i)) = \bigcap_{i \in I} (a : (G_i)) = (a : (\bigcup_{i \in I} (G_i))).
\]

Therefore by Lemma 97, \( J(A) \) is \( a \)-divisorial ideal. On the other hand if \( b \) is \( a \)-divisorial ideal, then by Lemma 97 there exists an ideal \( c \) such that \( b = (a : c) \). Therefore

\[
b = (a : c) = (a : (\bigcup_{G \in \mathcal{C}} (G))) = \bigcap_{G \in \mathcal{C}} (a : (G)) = \bigcap_{G \in \mathcal{C}} J(D_a(G)) = J(\bigcup_{G \in \mathcal{C}} D_a(G)).
\]

□

Now we are ready to identify all open (and hence clopen) points of \( M_a \).

Theorem 102  Let \( p \) be a point of \( M_a \). Then the following conditions are equivalent:

(i)  \( \{p\} \) is clopen in \( M_a \),

(ii)  \( p \in O_a \),

(iii)  \( \{p\} = D_a(G) \) for some \( G \in \mathcal{L} \), i.e., \( \{p\} \) is a basic open set in \( M_a \).

Proof:  (i) \( \Rightarrow \) (ii): Let us assume that \( \{p\} \) is clopen and hence open in \( M_a \). From Lemma 101 it follows that \( J(\{p\}) = p \) is \( a \)-divisorial ideal and hence \( p \in O_a \).

(ii) \( \Rightarrow \) (iii): Let us assume that \( p \in O_a \). Since \( p \) is \( a \)-divisorial, \( p = (a : (a : p)) \).

Moreover, by Lemma 99 \( p \) is maximal \( a \)-divisorial ideal and by Lemma 98 \( a \subseteq (a : p) \).

Therefore there exists \( G \in (a : p) \) such that \( G \not\subseteq a \). Then \( a \subseteq a + (G) \subseteq (a : p) \) and hence \( p = (a : (a : p)) \subseteq (a : (a + (G))) \subseteq (1) \). Since \( p \) is maximal, \( p = (a : (a + (G))) = (a : a) \cap (a : (G)) = (1) \cap (a : (G)) = (a : (G)) = J(D_a(G)) \) by Proposition 96. We claim that \( D_a(G) = \{p\} \), since if \( q \in D_a(G) \), then \( p = J(D_a(G)) \subseteq q \) and hence \( q = p \).

(iii) \( \Rightarrow \) (i): Assume that \( \{p\} \) is a basic open set in \( M_a \). Since \( M_a \) is Hausdorff, \( \{p\} \) is also closed in \( M_a \) and we conclude that \( \{p\} \) is clopen in \( M_a \). □

It follows from this theorem that \( M_a \) is discrete if and only if \( M_a = O_a \), i.e., all points of \( M_a \) are clopen.
In this section we define and give necessary and sufficient conditions for an existence of factorizations in distributive lattice with 0 and 1 and consequently in Heyting algebras.

Let $a$ be a proper ideal. The representation

$$a = \bigcap_{p \in A} p$$

of $a$ as an intersection of prime ideals is called a factorization. If $A$ is finite, than it is called a finite factorization. The factorization is called irredundant if

$$a \neq \bigcap_{p \neq q} p$$

for each $q \in A$.

The existence of a factorization follows from Proposition 93, since we have $a = J(V(a)) = \bigcap_{p \in V(a)} p$. Unless $a$ is an intersection of maximal ideals, there are ideals $p, q$ in $V(a)$ such that $p \subseteq q$ and hence this factorization is obviously redundant. Another factorization follows from the Proposition 96, since we have $a = J(M_a) = \bigcap_{p \in M_a} p$ and we will call this factorization the natural factorization.

**Lemma 103** If a proper ideal $a$ has a finite factorization, then it has an irredundant (finite) factorization.

**Proof:** If $a$ has a finite factorization, then one can omit the redundant factors from finite factorization to obtain an irredundant one. □

We will show that an irredundant factorization is unique (if it exists), but we need first the following lemma.

**Lemma 104** If $a = \bigcap_{p \in A} p$ is an irredundant factorization, then $A \subseteq M_a$.

**Proof:** Let us assume that $q \in A$, but $q \not\in M_a$. Then by Zorn’s Lemma there is $q' \in M_a$ such that $q' \subseteq p$. Since $q \cap \bigcap_{p \neq q} p = a \subseteq q'$ and $q \not\subseteq q'$, it follows that
$\cap_{p \neq q} p \subseteq q'$ and therefore $\cap_{p \neq q} p \subseteq q' \subseteq q$, which is a contradiction with our assumption that the factorization is irredundant. □

Now we will prove the uniqueness of irredundant factorization.

**Theorem 105** The factorization $a = \cap_{p \in A} p$ is irredundant if and only if $A = O_a$. In particular, the irredundant factorization is unique.

**Proof:** Let us assume that the factorization $a = \cap_{p \in A} p$ is irredundant. To prove that $O_a \subseteq A$, let us assume that $q \in O_a$. Then $q = (a : (G))$ for some $G \in \mathcal{L}$ by Proposition 102 and hence

$$q = (a : (G)) = \left( \bigcap_{p \in A} (p : (G)) \right) = \bigcap_{p \in A} (p : (G)) = \bigcap_{G \neq p} p \subseteq p$$

for all $p$ such that $G \not\in p$. If $G \in p$ for all $p \in A$, then $q = (1)$, which is a contradiction. Since by Lemma 104, $A \subseteq M_a$ we conclude that $q = p \in A$. To prove the reverse inclusion $A \subseteq O_a$, let us assume that $q \in A$. Since the factorization is irredundant, $\cap_{p \neq q} p \not\subseteq q$ and hence there is $G \in \cap_{p \neq q} p$ such that $G \not\in q$. Therefore

$$(a : (G)) = \left( \bigcap_{p \in A} (p : (G)) \right) = \bigcap_{p \in A} (p : (G)) = \bigcap_{G \neq p} p = q.$$  

This implies that $q$ is $a$-divisorial and hence $q \in O_a$.

To prove the other implication we need to show that if $a = \cap_{p \in O_a} p$, then this factorization is irredundant. Let $q \in O_a$. Then by Lemma 98

$$a \subset (a : q) = \left( \bigcap_{p \in O_a} p : q \right) = \bigcap_{p \in O_a} (p : q) = \bigcap_{p \neq q} p$$

and therefore this factorization is irredundant. □

An obvious corollary is the following.

**Corollary 106** The natural factorization of $a$ is irredundant if and only if $M_a = O_a$.

Therefore an irredundant factorization of $a$ exists (and is unique) if and only if $J(O_a) = a$ and we have the following theorem.
Theorem 107 A proper ideal \(a\) has an irredundant factorization if and only if the set \(O_a\) is dense in \(M_a\).

Proof: If \(a\) has the irredundant factorization then \(J(O_a) = a\) and hence by Proposition 96 the closure of \(O_a\) in \(M_a\) is equal to \(V_a(J(O_a)) = V_a(a) = M_a\). On the other hand if the set \(O_a\) is dense in \(M_a\), then its closure in \(M_a\) is \(M_a\) and hence by Proposition 96 \(M_a = V_a(J(O_a))\). This implies that \(a = J(M_a) = J(V_a(J(O_a))) = J(O_a)\) by Proposition 93. \(\Box\)

We have the following equivalent conditions for the natural factorization to be irredundant.

Theorem 108 The following conditions are equivalent:

(i) the natural factorization of \(a\) is irredundant,

(ii) for each \(q \in M_a\), \(\bigcap_{p \neq q} p \not\subseteq q\),

(iii) \(M_a = O_a\), i.e., \(M_a\) is a discrete space.

Proof: \(i \iff (ii)\), since \(a = \bigcap_{p \in A} p = \bigcap_{p \neq q} p\) if and only if \(\bigcap_{p \neq q} p \subseteq q\). \(i \iff (iii)\) is Corollary 106. \(\Box\)

For finite factorizations we need the following lemma.

Lemma 109 If \(a\) has a finite factorization then \(M_a = O_a\).

Proof: If \(a\) has a finite factorization then it has an irredundant factorization and hence \(O_a\) is finite by Theorem 105. We have already proved that \(O_a \subseteq M_a\). To prove the other inclusion, assume that \(q \in M_a\). Then \(\bigcap_{p \in O_a} p = a \subseteq q\) and since \(O_a\) is finite, there is \(p \in O_a\) such that \(p \subseteq q\). From the minimality of \(q\) it follows that \(q = p \in O_a\). \(\Box\)

Now we can prove the following equivalent conditions for the existence of a finite factorization.
Theorem 110 Let \( a \) be a proper ideal. Then the following conditions are equivalent:

(i) \( a \) has a finite factorization,

(ii) for each \( q \in M_a \), \( q \not\subseteq \bigcup_{p \neq q} p \),

(iii) the natural factorization of \( a \) is finite, i.e., \( M_a \) is finite.

Proof: We will prove that (i) \( \iff \) (iii) and (ii) \( \iff \) (iii).

(i) \( \Rightarrow \) (iii): If \( a \) has a finite factorization then it has an irredundant factorization and hence \( O_a \) is finite by Theorem 105. Moreover, \( M_a = O_a \) by Lemma 109 and hence \( M_a \) is finite.

(iii) \( \Rightarrow \) (i): This implication is obvious.

(iii) \( \Rightarrow \) (ii): This implication follows from Theorem 80.

(ii) \( \Rightarrow \) (iii): Assume that for each \( q \in M_a \), \( q \not\subseteq \bigcup_{p \neq q} p \). For a contradiction, let us assume that \( M_a \) is infinite. Since \( V_a(1) = \emptyset \) and \( V_a(G) = M_a \) for every \( G \in a \), the set

\[ f = \{ G \in \mathcal{L} \mid |V_a(G)| < \infty \} \]

is a subset of \( \mathcal{L} \) which is disjoint from \( a \) and \( 1 \in f \). We claim that \( f \) is a filter. This claim follows, since if \( G \in f \) and \( G \leq H \) then \( V_a(H) \leq V_a(G) \), which implies that \( H \in f \). If \( G, H \in f \) then \( V_a(G \times H) = V_a(G) \cup V_a(H) \), which implies that \( G \times H \in f \). By Zorn’s lemma we can enlarge \( f \) to a maximal filter \( \mathcal{e} \) which is disjoint from \( a \) and by Lemma 94, \( \mathcal{e} = \mathcal{L} - q \) for some minimal prime ideal \( q \in M_a \). Since \( q \not\subseteq \bigcup_{p \neq q} p \), there is \( G \in q \) such that \( G \not\subseteq p \) for each \( p \neq q \) from \( M_a \). Since \( G \in q \), \( G \not\subseteq \mathcal{e} = \mathcal{L} - q \). On the other hand \( V_a(G) = \{ q \} \) which implies \( G \in f \subseteq \mathcal{e} \), which is the desired contradiction. \( \Box \)

The following proposition follows from definitions.

Proposition 111 If \( \mathcal{H} \) is a Heyting algebra, then for \( W,G \in \mathcal{H} \), \( ((W) : (G)) = (W(G)) \).

This means that if \( a \) and \( b \) are principal ideals in a Heyting algebra, then also \( (a : b) \) is a principal ideal. Together with Proposition 102 (iii) this implies that if \( a \) is principal, then all \( a \)-divisorial prime ideals are principal, since \( \{ p \} = D_a(G) \) implies that \( p = J(\{ p \}) = J(D_a(G)) = (a : (G)) \) by Proposition 96. This means that if \( a \)
is principal, then all ideals from $O_a$ are principal. The following proposition is also immediate from the definition.

**Proposition 112** A principal ideal $(P) \subseteq \mathcal{G}$ is prime if and only if the class of equivalent graphs $P$ is multiplicative.

Now we can give the promised necessary and sufficient conditions for finite factorization of principal ideals in a Heyting algebra (and hence graphs up to the relation $\leftrightarrow$).

**Theorem 113** Let $\mathcal{H}$ be a Heyting algebra, $1 \neq W \in \mathcal{H}$ and let $a = (W)$. Then the following conditions are equivalent:

(i) $a$ has a finite factorization,

(ii) $Ma = O_a$,

(iii) all minimal prime ideals containing $W$ are principal.

**Proof:**

(i) $\Rightarrow$ (ii): This implication follows from Lemma 109.

(ii) $\Rightarrow$ (iii): Let us assume that $Ma = O_a$. Since all ideals from $O_a$ are principal also all minimal ideals containing $W$, i.e., ideals from $Ma$ are principal.

(iii) $\Rightarrow$ (i): Let us assume that all minimal ideals containing $W$ (i.e. ideals from $Ma$) are principal. We will show that the condition (ii) of Theorem 110 is satisfied. We may assume that $|Ma| = \infty$, otherwise the natural factorization is a finite factorization. For a contradiction assume that for some $q \in Ma$, $q \subseteq \bigcup_{p \neq q} p$. Since all minimal prime ideals containing $W$ are principal, $q = (G)$ for some $G \in \mathcal{H}$. Therefore $G \in q \subseteq \bigcup_{p \neq q} p$ and hence $G \in p$ for some $p \neq q$. It follows that $q = (G) \subseteq p$, which is a contradiction with the minimality of $p$. $\square$

4.7 Conclusion

We will summarize what we have proved and interpret the results for graphs. Let us notice that there are Heyting algebras for which not every element is a meet of finitely
many meet irreducible elements. An example is any infinite Boolean algebra. Let us consider the following properties of a distributive lattice $\mathcal{L}$:

$P1$. Every element is a join of finitely many join irreducible elements.

$P2$. Every element is a meet of finitely many meet irreducible elements.

$P3$. The meet of any two join irreducible elements is join irreducible.

$P4$. The join of any two meet irreducible elements is meet irreducible.

It follows from Proposition 84, [53] and Corollary 56 that the Heyting algebra $\mathcal{G}$ is countable distributive lattice and satisfies $P1$, $P3$ and $P4$. In [7] it is shown that $P1$ and $P3$ imply $P4$ and that $P1$, $P2$, and $P3$ are both necessary and sufficient conditions for a countable $\mathcal{L}$ to be projective in the variety of all distributive lattices (for definitions we refer interested readers to [7]). Therefore every graph has a finite factorization if and only if $\mathcal{G}$ is projective in the variety of distributive lattices. We do not know whether this is indeed the case. We do not even know whether $K_4$ has a finite factorization. Another condition satisfied by $\mathcal{G}$ is:

$P5$. If $G \in \mathcal{G}$ is a join irreducible element, then $(W_1 + W_2)(G) = W_1(G) + W_2(G)$ for all $W_1, W_2 \in \mathcal{G}$.

This condition follows from Proposition 55. The conditions $P1$ and $P2$ are not true in any infinite Boolean algebra. The conditions $P3$ and $P4$ are not true in any Boolean algebra and the condition $P5$ is true in every Boolean algebra. We do not know whether every Heyting algebra which satisfies four conditions $P1$, $P3$, $P4$, and $P5$ has to also satisfy the condition $P2$.

To interpret the last theorem in the graph theoretical terms, let us fix a graph $W$. For a set $S$ of graphs which contains the graph $W$ we say that $S$ is $W$-minimal, if it satisfies the following four conditions

1. (1st ideal condition) $G \in S$ and $H \rightarrow G$ imply $H \in S$,
2. (2nd ideal condition) $G \in S$ and $H \in S$ imply $G + H \in S$,
3. (primality condition) $G \times H \in S$ implies $G \in S$ or $H \in S$,
4. (W-minimality condition) $G \in S$ if and only if $W(G) \notin S$.

For a set $S$ of graphs we say that $S$ is principal, if it satisfies the condition
5. (principality condition) there exists $G \in S$ such that $H \to G$ for all $H \in S$.

It follows that $W$ has a finite factorization if and only if every $W$-minimal set of graphs is principal.
Chapter 5

Conclusions and Further Research

In this thesis we have studied graph homomorphisms, equitable graph homomorphisms, and the corresponding $H$-coloring problems from both theoretical and practical view points.

In Chapter 2, we have completely characterized the complexity of the equitable $N$-coloring problem and its restricted version, the connected equitable $H$-coloring problem. In particular, we have proved that both problems are polynomial if $H$ is a disjoint union of complete bipartite graphs. Since the complexity of our algorithm is exponential in the number of connected components of $H$, one possible area for further research is to improve our algorithm or to find a better one.

In Chapter 3, we have proposed relaxations of graph homomorphism. In particular we defined pseudo-homomorphism and fractional homomorphism as, respectively, semidefinite and linear relaxations of a certain integer program corresponding to the graph homomorphism problem. We have shown a simple forbidden subgraph characterization for the existence of fractional homomorphism but we could prove only a necessary condition for the existence of pseudo-homomorphism. It would be nice to have a similar characterization for the pseudo-homomorphism. Another possible area for further research is to give a nontrivial example of a class of graphs for which either fractional homomorphism or pseudo-homomorphism would give a polynomial
algorithm, since all of our examples are already known to be polynomial time solvable by other means. It seems that using semidefinite programming is overkill for these particular examples.

In Chapter 4, we have studied the multiplicative structure of equivalence classes of graphs. In particular, we have considered the problem of finite factorization. We gave necessary and sufficient conditions for the existence of a finite factorization. However, we are not able to use these conditions even for simple graphs like $K_4$. Currently it is known that $K_1$, $K_2$, $K_3$ and all cycles are multiplicative and that if $G \not\sim H$ and $H \not\sim G$ then $G \times H$ is not multiplicative (cf. [22]). Therefore it is easy to construct a graph which is not multiplicative, e.g., $K_3 \times M_{11}$, where $M_{11}$ is the Mycielski graph from Figure 3.1. We have shown that if $K_4$ is multiplicative then there are infinitely many complete graphs which are multiplicative. We have also shown that if $K_4$ has a finite factorization then $\mu(n)$ goes to infinity with $n$. (Whether $\mu(n)$ goes to infinity with $n$ is an open problem.) It is possible that $K_4$ does not even have a finite factorization. Another possible problem for further research is to answer the factorization question for $K_4$. 
Bibliography


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