GRADE 11 STUDENTS' UNDERSTANDING
OF CIRCLE GEOMETRY IN A
COMPUTER ENVIRONMENT

by

Karen Reed
B.A., University of Western Ontario, 1989

THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Faculty
of
Education

© Karen E. Reed 1996

SIMON FRASER UNIVERSITY
August 1996

All rights reserved. This work may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.
The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-612-17072-1
PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis, project or extended essay (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this work for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this work for financial gain shall not be allowed without my written permission.

Title of Thesis/Project/Extended Essay

Grade 11 Students' Understanding of Circle Geometry in a Computer Environment

Author:

(Signature)  

Karen Elizabeth Reed

(Name)  

August 16, 1996

(Date)
APPROVAL

NAME
Karen Elizabeth Reed

DEGREE
Master of Science

TITLE
Grade 11 Students' Understanding of Circle Geometry in a Computer Environment

EXAMINING COMMITTEE:

Chair
Michael Roth

Rina Zazkis, Associate Professor
Senior Supervisor

A. J. (Sandy) Dawson, Professor
Member

Dr. Tom O'Shea, Faculty of Education, SFU
External Examiner

Date: 8/16/1996
ABSTRACT

This research explored grade 11 students’ understanding of circle geometry working in a computer environment. The questions I wished to investigate were:

- To what extent do students discover properties of circle geometry working in a computer environment?
- How can the van Hiele model be used to describe students’ learning of circle geometry working in a computer setting?
- What role do computers play in helping students to visualize concepts in circle geometry?

These questions were investigated by designing and conducting an exploratory study unit that included five geometric properties of a circle. (For example: A tangent line to a circle is perpendicular to the radius at the point of intersection). Rather than presenting students with theorems to be proved, the theorems were posed to students as problems for investigation and discovery. These properties (theorems) were part of the grade 11 mathematics curriculum in British Columbia. To my knowledge, there has been no prior study investigating students’ understanding of circle geometry in a computer environment.

The participants were students enrolled in a grade 11 mathematics course. The work of four students, working in pairs, exploring properties of a circle with a computer, was videotaped and transcribed. Their understanding was analyzed according to the van Hiele levels of geometric thinking. I provided detailed descriptions of the students’
investigation of each of the five properties. Additional data were drawn from observations and students' written work.

The results of the study indicated that students can successfully discover for themselves geometric properties in a computer environment. Students, in this study, showed progression of thinking from van Hiele level 0 to level 2. The levels, however, appeared to be dynamic rather than static and of a more continuous nature contradicting the notion that the levels are discontinuous, discrete steps. Furthermore, the results indicated that students can operate at different levels of thinking depending on the geometry concept.

The results of this study also indicated that computers help students to visualize concepts in geometry. In this study, students were more apt to choose visualization as their method of analysis and reasoning.
ACKNOWLEDGEMENTS

I would like to express my thanks and gratitude to all the people who contributed to this thesis, directly or indirectly, but especially to Dr. Rina Zazkis, my senior supervisor, for her thoughtful guidance and unending patience and assistance. She always had an encouraging word to give and lifted my spirits when needed.

I am especially grateful to my parents, Eleanor and Denys Reed; for so generously providing continuous support and encouragement. This thesis would not have been possible without their endless caring and support.

I would also like to express my thanks to the students who participated in the study. They devoted a lot of their time and effort and to this I am truly grateful.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>APPROVAL</td>
<td>II</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>III</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>V</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>VI</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>X</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>XI</td>
</tr>
<tr>
<td>CHAPTER I</td>
<td>1</td>
</tr>
<tr>
<td>THE NATURE AND PURPOSE OF THE STUDY</td>
<td>1</td>
</tr>
<tr>
<td>Rationale</td>
<td>3</td>
</tr>
<tr>
<td>Purpose of the Study</td>
<td>4</td>
</tr>
<tr>
<td>Overview of the Study</td>
<td>5</td>
</tr>
<tr>
<td>Organization of the Thesis</td>
<td>6</td>
</tr>
<tr>
<td>CHAPTER II</td>
<td>8</td>
</tr>
<tr>
<td>LITERATURE REVIEW</td>
<td>8</td>
</tr>
<tr>
<td>Why Use Technology?</td>
<td>8</td>
</tr>
<tr>
<td>Benefits</td>
<td>9</td>
</tr>
<tr>
<td>Why Geometry?</td>
<td>11</td>
</tr>
<tr>
<td>General Problems with Geometry</td>
<td>12</td>
</tr>
<tr>
<td>Benefits of Technology in Geometry</td>
<td>14</td>
</tr>
<tr>
<td>Issues Concerning Technology</td>
<td>16</td>
</tr>
<tr>
<td>The Changing Roles of Teacher and Student</td>
<td>17</td>
</tr>
</tbody>
</table>
Mathematical Visualization

The Reluctance to Visualize in Mathematics.
The Importance of Visualization

CHAPTER III

METHODOLOGY

The School
The Mathematics Department
Mathematics 11 Course
Participants
My Role as Teacher/Researcher
Lesson Objectives

Geometer's Sketchpad
Data Collection Procedures
Obstacles in Carrying Out This Research

Chapter Summary

CHAPTER IV

THE VAN HIELE MODEL

Background

The Levels
Level 0: Visualization
Level 1: Analysis
Level 2: Informal Deduction
Level 3: Formal Deduction
Level 4: Rigor
Reduction of Level

Phases of Learning

A Different View of the van Hiele Model

CHAPTER V

DATA ANALYSIS
LIST OF TABLES

TABLE 1: HOME LANGUAGE 27
TABLE 2: TIMETABLE AT GLADSTONE SECONDARY SCHOOL 28
TABLE 3: SUMMARY OF GEOMETRIC TOPICS 34
LIST OF FIGURES

FIGURE 1: PARALLEL LINES USING GEOMETER’S SKETCHPAD 35
FIGURE 2: THE EFFECTS OF THE MEASUREMENTS FROM MOVING POINT B 37
FIGURE 3: A CONSTRUCTION EXPLORED BY KATHY AND JULIE 54
FIGURE 4: A CONSTRUCTION OF A CIRCLE AND A RADIUS BY JANRAJ AND SANDEEP 56
FIGURE 5: PARALLEL TANGENT LINES 58
FIGURE 6: AN ATTEMPT BY KATHY AND JULIE TO CONSTRUCT TWO TANGENT LINES 59
FIGURE 7: THE POINT OF INTERSECTION OCCURRING OFF THE COMPUTER SCREEN 60
FIGURE 8: KATHY’S OBSERVATION OF THE MOVING EXTERIOR POINT 62
FIGURE 9: THE RESULT OF MOVING POINT ‘A’ AND ‘B’ AWAY FROM EACH OTHER 64
FIGURE 10: THE DIAGRAM INVESTIGATED BY JANRAJ AND SANDEEP 66
FIGURE 11: THE PROCESS AND RESULT OF MOVING POINT ‘A’ AWAY FROM POINT ‘B’ 68
FIGURE 12: TWO INSCRIBED ANGLES SUBTENDED BY THE CHORD AB 72
FIGURE 13: INSCRIBED AND CENTRAL ANGLES SUBTENDED BY CHORD AB 76
FIGURE 14: CONSTRUCTION BY JANRAJ AND SANDEEP OF INSCRIBED AND CENTRAL ANGLES 80
FIGURE 15: AN EXAMPLE OF INEXACT ANGLE MEASUREMENTS 82
FIGURE 16: CYCLIC QUADRILATERAL 86
FIGURE 17: CYCLIC QUADRILATERAL 89
FIGURE 18: TANGENT LINES IN STANDARD POSITION 92
FIGURE 19: INSCRIBED AND CENTRAL ANGLES IN AN UPRIGHT POSITION 93
FIGURE 20: AN EXAMPLE OF A “MONSTER” CASE 100
FIGURE 21: A TAME CASE 109
FIGURE 22: A “MONSTER” CASE 109
CHAPTER I
THE NATURE AND PURPOSE OF THE STUDY

In recent years, the discussion and procedures surrounding the use of computers in the classroom has increased dramatically. Elementary and secondary school teachers are witnessing a growing number of classrooms equipped with technology such as computers and calculators. The NCTM (National Council of Teachers of Mathematics) Professional Standards for Teaching Mathematics states that the teacher of mathematics, in order to enhance discourse, should encourage and accept the use of computers, calculators, and other technology (NCTM, 1991). With the advancement of technology, today's mathematics teachers and their students have the opportunity to experience and explore mathematics in ways that were not possible a decade ago.

In my five years of teaching mathematics, I have noticed that students thoroughly enjoy working with technology when exploring mathematical concepts. Graphic calculators are an example of a technology becoming widely used in the mathematics classroom. I use graphic calculators in grades ten through twelve for exploring linear, quadratic and trigonometric functions. The students appear highly motivated and extremely excited when exploring these functions on graphic calculators. They, on their own, try to discover methods for translating the functions left, right, up, and down. They can readily explore relationships between the function and its derivative. Through my brief experience using graphic calculators as a tool for instruction, I believe that students
understand the concepts more thoroughly and grasp the concepts in a relatively short time span.

Geometry is another area of mathematics where technology may be utilized. Geometry is a core topic in the British Columbia curriculum and, yet, many students have trouble with the concepts (Usiskin, 1982). Geometry continues to be an important aspect in our lives. In professions such as surveying or construction, geometric figures and measurement techniques are essential for designing the work and describing the results. In science, geometry provides the ideas for modeling astronomical phenomena, the action of physical forces, and the forms of living objects. In the arts, geometric concepts of symmetry and projection have been fundamental in creations such as Renaissance paintings (Fey, 1984). Geometry, therefore, is an extremely important and versatile subject. As educators, we need to develop teaching methods so our students can enjoy, experience, understand, and appreciate this fascinating subject.

The NCTM Curriculum and Evaluation Standards articulate that instruction should increase attention on visualization of two and three dimensional figures and that CAD (computer-assisted design) software is of particular value (NCTM, 1989). The graphing capabilities of computers offer impressive tools for teaching geometric concepts and principles (Fey, 1984). Moreover, computer graphics software that allow students to create and manipulate shapes provides an exciting environment in which they can make conjectures and test their attempts at two-dimensional visualization (NCTM, 1989). I believe that computers offer a positive and efficient learning environment to study geometry concepts.
Rationale

Unfortunately, geometry is commonly taught in traditional ways in many secondary schools. Students’ participation in these geometry classes consists of simply answering textbook questions. The theorem or the rule is provided by the teacher and the student absorbs the information. An example of a geometric concept is presented by the teacher and the student simply answers questions that resemble the teacher’s example. This method of instruction does not readily lend itself to student input or student creativity. Students are not given the opportunity to discover and explore theorems or properties on their own. They are merely told about them and are asked, in many cases, to memorize them.

To make geometry class more meaningful and exciting for students, the instruction must become more student-centered. Students need to play a more active part in the instruction of geometry. They need to discover, on their own, theorems or properties and have time to explore and reflect these new ideas. The NCTM Professional Standards for Teaching Mathematics emphasizes the discovery approach. Students should be encouraged to make conjectures, to invent, and to problem solve (NCTM, 1991). Much cognitive psychology and mathematics education research indicates that learning occurs as students construct their own meanings (Case and Bereiter 1984; Cobb and Steffe 1984; Davis 1984; Lampert 1985; Schoenfeld 1987).

Computers provide an ideal environment for student exploration and discovery. Drawing objects such as line segments and circles is effortlessly accomplished using a
Moreover, computers can instantly calculate a variety of measurements such as lengths of line segments, degree of angles, circumferences, and areas. I have found that many students struggle with various concepts in geometry. Grade 11 mathematics students seem to experience difficulty understanding and visualizing various properties in circle geometry. For this reason, I wanted to investigate students’ understanding of circle geometry learning in a computer environment.

**Purpose of the Study**

The objective of this study was to investigate grade 11 students’ understanding of circle geometry situated in a computer environment. The questions I wished to investigate were:

- To what extent do students discover properties of circle geometry working in a computer environment?
- How can the van Hiele model be used to describe students’ learning of circle geometry working in a computer setting?
- What role do computers play in helping students visualize concepts in circle geometry?
Overview of the Study

The study was conducted in an inner-city Secondary School in Vancouver, British Columbia. The participants, two girls and two boys, were chosen from a grade 11 mathematics course. They demonstrated average mathematical ability and had adequate communication skills, the criteria for participating in this study.

Throughout the months of October to December of 1995, the participants worked on a study unit which consisted of five lessons. Each lesson explored a different property of circle geometry using the computer software package, Geometer's Sketchpad (the software is presented in Chapter 3, section 7). The participants were asked to construct various diagrams and make conjectures based on their findings. Transcriptions of the sessions were made from video tape and served as the main source of my data. In addition, I collected any artifacts that the students produced for the lessons. The data were analysed using a contemporary theory on learning geometry, namely the van Hiele Model (the Model is described in detail in Chapter 4).

As a result of this study, I feel that students learn and understand circle geometry effectively when working in a computer environment. I realize that students can be at different levels of understanding for different geometric concepts. Furthermore, I realize the importance that instructional materials can have to encourage and promote higher levels of student thinking.
Organization of the Thesis

Chapter 2, presents a review of the literature relevant to this study. I start Chapter 2 by presenting the benefits of technology in the mathematics classroom. The next section begins by making a case for including geometry in general education. It, then, addresses various problems students encounter when studying geometry and concludes with numerous benefits technology brings to geometry. It further addresses issues concerning technology and concludes with a discussion concerning visualization in mathematics.

A detailed description of the methodology for the study is presented in Chapter 3. This chapter starts with the description of the learning environment in which the study took place: it describes the school, the mathematics department, and the mathematics 11 course. It further introduces the participants and presents my role as teacher/researcher. The lesson objectives are outlined as well as a description of Geometer's Sketchpad. The following sections present the procedures of data collection and relate the obstacles encountered in carrying out this research.

Chapter 4 presents the van Hiele model which was used to determine students' understanding of circle geometry. The chapter begins with a brief history of the model and a portrayal of its creators. It provides a description of the five levels of geometric thinking and discusses the five phases of learning found in each of the five levels. The chapter concludes with my interpretative view of the van Hiele model.

Chapter 5 presents the data collected throughout the study. The procedure used for analyzing the data is outlined in the first section. The chapter details the students'
responses and understanding of five concepts pertinent to circle geometry. The students’
development of understanding was interpreted according to the van Hiele model. The
chapter concludes with personal observations that are relevant to the study but not
implicit in the data.

Sections in the last chapter constitute a discussion about the student responses and my
conclusions. The chapter provides a summary of students’ levels of understanding for
each of the five lessons. Factors affecting the performance of the students on the lessons
are detailed followed by a discussion of the effects of learning geometry in a computer
environment. It further provides a description of the conclusions drawn from this study.
In this chapter, I also address limitations of the study, suggestions for further research and
personal benefits resulting from conducting this research.
CHAPTER II

LITERATURE REVIEW

In this chapter, I review a broad variety of literature that is relevant to my study. The first section addresses the benefits technology brings to the mathematics classroom. This section mentions several problematic issues concerning the use of technology. The second section reviews the importance of geometry in the high school curriculum. Finally, the third section discusses the concept of mathematical visualization and its role in learning mathematics.

Why Use Technology?

The National Council of Teachers of Mathematics (NCTM) Curriculum and Evaluation Standards (1989) promote increased attention to the use of calculators and computers as tools for learning and doing mathematics (p. 129). It further states that calculators and computers, with appropriate software, transform the mathematics classroom into a laboratory where students use technology to investigate, conjecture, and verify their findings (p. 128). This suggests that technology aids the transition from the traditional mathematics classroom to an exploratory, exciting, rewarding student-centered classroom. Below is a discussion of the benefits that technology brings to the mathematics classroom and to the learners. The various issues surrounding the use of technology will also be discussed.
Benefits

Many educators strongly believe students will learn better if they discover relationships for themselves rather than from traditional ways of learning mathematics. Technology offers the opportunity for students to investigate and explore various mathematical relationships. Ruthven (1992) utilizes graphic calculators for the exploration of sine and cosine graphs.

Ruthven asks students to explore, on the graphic calculators, the consequences when sine and cosine expressions are combined to give expressions such as $\cos x + \sin x$, $\cos 2x + \sin x$, and $2\cos x + \sin x$. Ruthven claims that the advantage of using technology when teaching mathematics is that the depth of students’ conceptual understanding is greater and they ask higher level questions. For example: What is the relation about? Under what conditions does it hold? Why does it hold? He claims that technology helps students to formulate the nature of the relation more clearly and fully, and it promotes the search for an appropriate generalization. Furthermore, technology is particularly helpful in making ideas more accessible to students who would otherwise struggle with an approach dependent on algebraic methods. In other words, technology can enable students to visualize and grasp mathematical concepts that they may not have understood. Ruthven concludes that, with technology, students play a more active role in developing and evaluating mathematical ideas. Technology can help students to not only grasp new ideas but also to develop the capacity to tackle novel situations.
Logo-based programming computer environments also offer students the opportunity to explore and examine mathematical concepts. Edwards (1992) conducted a study in which ten to twelve year olds learned transformation geometry through a computer-based learning environment using the computer language of Logo. She developed an introductory curriculum in transformation geometry and presented it to the students. The students (using Logo) worked on the curriculum, exploring several concepts in transformation geometry.

Edwards' main focus was to design and investigate an interactive computer environment in which students could explore geometric transformations. She found that the students were successful in using the microworld and her curriculum "to build an initial and generally correct understanding of Euclidean transformations" (Edwards, 1992, p.140). The students were also successful in applying this new understanding to various problems. Edwards administered a final exam and found that the students who participated in the study scored above the average on ten of the twelve items.

In summary, NCTM supports the use of technology in teaching mathematics. Research shows that technology aids in students' learning and understanding of various ideas in mathematics. Below, I discuss the importance of geometry and general problems students encounter when learning geometry. Following the discussion is a review of how technology may alleviate some of the problems.
**Why Geometry?**

Recently, there have been several new efforts to reform curricula to make geometry less worthy than its counterparts such as algebra, data analysis, and number theory (Goldenberg, Cuoco, and Mark, in press). Goldenberg, Cuoco, and Mark disagree strongly with this reform and make two claims why geometry should play a key role in general education. Their first claim is that geometry can help students connect with mathematics. They state that geometry can connect well with students, and they with it. There are many hooks in geometry that capture the student in much the same way as art, physical science, imagination, biology, curiosity, mechanical design, and play. These hooks include the physical aspects that students experience every day. Furthermore, geometry also connects richly with the rest of mathematics, including such topics and themes in discrete and continuous mathematics as combinatorics, algorithmic thinking, geometric series, optimization, functions, limits, trigonometry, and more. Geometric approaches can be a route into new mathematical ideas, or can provide new insights into mathematics that one already knows.

Their second claim is that geometry can be an ideal vehicle for developing different ways of thinking. It’s an ideal intellectual territory within which to perform experiments, develop visually-based reasoning styles, learn to search for invariants, and use these and other reasoning styles to spawn constructive arguments as well. NCTM supports the importance of studying geometry, stating that it helps students represent and make sense of the world in which they live (NCTM, 1989).
Kapadia (1980) also supports geometry as a key subject in general education. Kapadia makes a case for teaching Euclidean geometry in secondary schools. He refers to an article written by Professor Krygowska who argues for a reformed geometry curriculum. Kapadia disagrees with many of her points. One point deals with the axioms of geometry while the other point concerns application of Euclidean geometry.

Krygowska makes reference to the classical axioms of geometry. She claims that the axioms are complicated and difficult to express simply and concisely. Kapadia disagrees with this statement. He claims that the classical axioms of geometry are natural since they idealize everyday experiences.

Another argument against geometry that Krygowska makes is that The Elements has no applications in the mainstream of mathematics, nor is it of any practical use. Kapadia responds by recalling that many new mathematical concepts are inspired by geometrical ideas. Furthermore, geometry began from practical problems. Kapadia claims that many professions use geometrical ideas and that physics masters complain about the lack of geometry in modern curriculum.

Geometry is fundamental to the learning of mathematics. It helps students to connect with mathematics and to develop various methods of thinking. Unfortunately, many students experience extreme difficulty in geometry.

General Problems with Geometry

Geometry contributes to many areas of human activity.
In practical tasks, like surveying or construction, geometric figures and measurement techniques are essential for the design of work and the description of results. In science, geometry provides the ideas for modeling astronomical phenomena, the action of physical forces, and the forms of living objects. In the arts, geometric concepts of symmetry and projection have been fundamental in creations like the mosaics in Moorish palaces or the perspective in Renaissance paintings. For scholars in many disciplines, the formulation of geometry as an axiomatic system has served as the model of hypothetical-deductive reasoning.

Source: Fey, 1984, p.31

Geometry is essential to various applications and real-world problems. Yet, geometry is the most troubled and controversial topic in school mathematics (Fey, 1984). Fey (1984) discusses the concerns surrounding geometry taught in high schools. He categorizes the concerns into two main areas. First, he mentions that many teachers feel that geometry teaches students to reason logically. However, he continues, other teachers doubt the validity of this claim.

The doubting teachers’ reason is based on the fact that students are introduced to formal geometry at the wrong age. Fey states that “research has suggested that many students at the age when formal geometry is usually studied are incapable of the formal and abstract thinking required” (p. 32). Therefore, students stumble through a year-long course with a few facts and a vague sense of axioms, theorems and definitions. Perhaps if geometry was taught at a different age, students would have a greater appreciation of geometry. This is Fey’s first concern.

Fey’s second concern and criticism of high school geometry is that the course remains stagnant at Euclid’s first synthesis. High school geometry does not include fascinating
topics such as projective and affine geometries, differential geometry, or non-Euclidean geometries. These topics have not found a place in mainstream high school mathematics.

As mentioned earlier, geometry is an essential topic in school mathematics. It is important, then, for geometry to be presented in a non-traditional matter. However, geometry must be presented in such a way to captivate students' interest. The following discusses some of the benefits technology brings to the geometry classroom.

**Benefits of Technology in Geometry**

Fey (1984) discusses some of the advantages technology brings to the geometry classroom. First, computers offer a rich new mathematical environment in which geometry students can experiment with shapes and relations. Fey claims that this exploration produces strong geometric intuition that is essential for problem solving and theory building.

Fey addresses the issue concerning geometry being the right way to teach reasoning skills. Many teachers feel that geometry instruction trains students to think logically. Fey suggests that computer programming and computer graphics might be a more practical, interesting, and effective way to teach logical reasoning.

Computers address another concern about high school geometry which was mentioned earlier. The concern was based on high school geometry focusing solely on topics introduced by Euclid long ago. However, there have been recent proposals to include transformations, co-ordinate, and vector topics into high school geometry. Fey claims that the reluctance of schools to teach a transformation approach to geometry was partly
due to the difficulty of projecting motion. Drawings on the chalkboard seemed to be inadequate. However, "computers make the transformation approach to geometry attractive because they illustrate the central ideas so effectively" (p. 44).

Clements and Battista (1994) further support technology in the mathematics classroom. They claim that computers can perform certain functions that other teaching tools cannot easily duplicate. Clements and Battista discuss several research studies that describe such functions and evaluate the functions’ contribution to students’ learning of geometry. One such function is the capability of computers to aid students to higher levels of thinking.

Research done by Hoyles and Noss (1992) found that computer environments facilitate students’ progression to higher levels of geometric thinking. To measure students’ geometric levels, they used the van Hiele theory of levels for geometric thinking. The van Hieles’ theory claims that students achieve higher levels through working on suitable geometric problems, not by direct teacher instruction. They further found that students’ work in Logo (computer graphics program) relates closely to their level of geometric thinking.

Clements and Battista also found that computer environments “help students construct more viable knowledge because students are constantly evaluating a graphical manifestation of their thinking” (p.188). In addition, computer environments offer a window into students’ geometric thinking. Students’ difficulties and misconceptions that were masked by traditional teaching methods are easily visible to researchers and teachers.
They conclude that computer graphics programs can help students develop more abstract patterns for geometric concepts. Clements and Battista state that "the objects on the computer screen become manipulable representations...of the students thinking" (p.187). Therefore, the students can make conjectures, evaluate their constructions, and reformulate their thoughts.

Research suggests that students who have used computers to learn geometry perform better than their non-computer counterparts on geometry exams (Yerushalmy, Chazan, & Gordon, 1988; McCoy, 1991). McCoy (1991) compared the geometry achievement of a class which used the Geometric Supposer periodically during one school year and a similar class which did not use the software. McCoy found that the group which worked with the Supposer scored significantly higher on the final examination in geometry than the control group.

This section addressed the importance and the difficulties of geometry in general education. It discussed how technology might alleviate some of the problems and how technology might support student learning in geometry. The next section reviews some issues surrounding the use of technology in education.

**Issues Concerning Technology**

Many educators welcome the opportunity to utilize technology in their classrooms. However, documentation of implementing technology into the current curriculum is vague. Beginning September 1997, British Columbia schools will be starting two new
grade 11 courses. September 1998 will introduce two corresponding grade 12 courses. The new curriculum guides for these courses offer various ways for implementing technology into a vast number of mathematical topics. However, despite the fact that the new curriculum guides provide invaluable information on implementing technology, it will remain with the educators to implement technology into their classrooms.

The immersion of technology into the mathematics classroom brings up several interesting issues. This section focuses on two important issues: the first is the acquisition of changing roles on the part of teacher and student. The second concerns types of software systems currently available for use in the mathematics classroom.

The Changing Roles of Teacher and Student

An important issue surrounding the use of technology in the classroom is the changing role on the part of student and teacher. Traditionally, the teacher’s role in the classroom has been that of possessing knowledge and giving the knowledge to the students. The students' role is that of receiving the acquired knowledge from the teacher. Technology changes these roles drastically. Echoing this statement, NCTM’s Curriculum and Evaluation Standards (1989) state that the use of technology during instruction alters both the teaching and the learning of mathematics.

The role of the teacher shifts from dispensing information to facilitating learning: “From that of director to that of catalyst and coach” (NCTM, 1989, p.128). This environment encourages students to explore, formulate and test conjectures, prove generalizations and discuss and apply the results of their investigations (p. 128). The
teacher encourages experimentation and provides opportunities for students to summarize ideas and establish connections with previously studied topics (NCTM, 1989).

Goldenberg, Cuoco, and Mark (1995) conceptualize the new role for teachers as one of "senior research partner".

The role of the student transforms from that of receiver to that of doer. The classroom’s main focus will be on increasing student independence and encouraging students to become self-directed learners. The students will routinely engage in “constructing, symbolizing, applying, and generalizing mathematical ideas” (NCTM, 1989, p. 128). NCTM further states that “the most fundamental consequence of changes in patterns of instruction in response to technology-rich classroom environments is the emergence of a new classroom dynamic in which teachers and students become natural partners in developing mathematical problems” (p. 128).

Software Systems

Another issue that needs to be addressed is that of software systems. There are currently two major types of software that can be purchased for the mathematics classroom. One type is on-line with traditional methods of teaching (teacher-directed instruction), while the other type of software system is more progressive, more student-centered. Stoddart and Niederhauser (1992/93) discuss the two types of software systems they call the integrated learning system (ILS) and the tool-based system (TBS).

Stoddart and Niederhauser argue that the ILS, currently the most commonly used computer-based instructional system being used in the public schools, is essentially
conservative and fits in easily with traditional teaching methods. An example of the ILS is the tutorial program, *First Words*, which was written for very young children. The program presents ten pictures within a particular category. The child is prompt to name the object in the picture. If the child is correct, “the blob turns a somersault” (p. 8). If the child answers incorrectly the program waits for a second response. If the second response is incorrect, the object is shown alone in the middle of the screen. In the ILS programs, students work through computer activities and the computer provides feedback regarding whether the question was answered correctly. In addition, the source of knowledge is external to the learner. Knowledge is within the computer and the student must memorize the information. The authors claim that the ILS is consistent with traditional pedagogy where the focus is on knowledge and memorizing rather than conceptual understanding.

The second type of software systems that Stoddart and Niederhauser describe is a tool-based system (TBS) which is based on constructivist views. This type of software provides students with the experiences that allow them to discover concepts on their own. The source of knowledge with TBS is within the learner. Students use the computer as a tool to develop their problem solutions and become active learners. I feel that the computer software used in this study, *Geometer's Sketchpad*, is a tool-based system. *Geometer's Sketchpad* is discussed in greater detail later in chapter 3.

This section provided an outline on the two software systems currently used in mathematics education. The first system, called integrated learning systems, is classified as conservative. This software is seen as consistent with traditional teaching
methodology. The second software, known as a tool-based system, mimics a more progressive pedagogy.
The term “visualization” is unfamiliar to some in the context of mathematics, and its connotations may not be obvious. The term differs somewhat from common usage in everyday speech and in psychological studies which focus on the subject’s ability to form and manipulate mental images. From the perspective of mathematical visualization, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualization what we are interested in is precisely the student’s ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagram to achieve understanding, and as an aid in problem solving. In mathematics, visualization is not an end in itself but a means toward an end, which is understanding.


In the common curricula, mathematics is rarely presented in a visual manner. In fact, geometry represents the only visually oriented mathematics that students encounter (Goldenberg, Cuoco, and Mark, in press). For some students, a visual approach may be essential for their success and to sustain their interest in mathematics (Krutetskii, 1976). This section discusses the reluctance on part of students, teachers and mathematicians to visualize in mathematics. In addition, the section focuses on the importance and relevance of visualizing in mathematics.
The Reluctance to Visualize in Mathematics.

Research finds that students have a tendency to think algebraically rather than visually (Mundy, 1987; Vinner, 1989) even though a visual approach would alleviate many learning difficulties. Eisenberg and Dreyfus (1991) provide three reasons as to why students are reluctant to accept the benefits of visualizing mathematical concepts.

They found that many mathematicians and teachers consider visual methods to be of limited value. This attitude is easily picked up by their students who, in turn, regard visual methods as low-level mathematics. Eisenberg and Dreyfus also claim that students tend to avoid visual methods because of how they are taught. Mathematicians’ knowledge is very intricate and contains many links and connections. However, school mathematics is presented in a very sequential and linear matter. It is because of this method of presentation that students choose analytic rather than visual procedures.

Another reason Eisenberg and Dreyfus list for students choosing analytic methods over visual methods rests with cognitive demands on visual thinking. They claim that thinking visually demands higher cognitive thought than thinking analytically. Therefore, students tend to steer clear of visual methods.

Tall (1991) claims that the role of visualization in mathematics has been both advantageous and misleading. As an example of visualization being misleading, he states that “[f]or two thousand years Euclidean geometry was held as the archetypal theory of logical deduction until it was found...that implicit visual clues had insinuated themselves without logical foundation” (p.105). In light of this, visual mathematics has taken a back seat to mathematics that can be proved by formal means.
However, Tall sees visual mathematics as advantageous. Indeed, he claims that “to
deny visualization is to deny the roots of our most profound mathematical ideas” (p.105).
Tall presents various topics in calculus using drawn diagrams and a graph-plotting
program. He feels that to strengthen students’ understanding of calculus topics, the
presented ideas must be not simple but, rather, complex. Difficult ideas, Tall argues,
appeal to the visual side of mathematical thinking. This, in turn, “give[s] a much broader
picture of the ways in which concepts may be realized, thus giving much more powerful
intuitions than in a traditional approach” (p. 118).

The Importance of Visualization

Dreyfus (1994) makes a case for the importance of visual reasoning. He claims that
there is tremendous potential for visual approaches in learning mathematics.
Furthermore, he argues that computerized learning environments open an avenue to
realizing this potential.

Dreyfus acknowledges that visualization is commonly seen only as an aid or a step on
the way to the real mathematics. This attitude on the part of mathematics educators
influences their students. Students would be less likely to use visual arguments if their
teacher deemed visualization as unimportant. This attitude, Dreyfus concluded, has
unfortunate effects. First, “it eliminates a versatile tool of mathematical reasoning for all
students” (p.108). Also, the reluctance to use visualization may prevent some students
from successful problem solving.
Dreyfus feels in order to give students the opportunity to appreciate visualization, mathematics educators need to upgrade the status of visual reasoning. "In our own mathematical thinking, we need to generate visual arguments, to learn how to examine their validity and to accord them the same weight which we accord to verbal and formal arguments" (p. 116). Dreyfus mentions that in the mathematics classroom, teachers need to give students many opportunities to discuss valid and invalid visual arguments. Moreover, teachers need to give students full credit for correct visual arguments.

In addition to the importance of visualization in mathematics, Dreyfus discusses visual mathematics in a computer environment. He claims that "computers make it possible to represent visual mathematics with an amount of structure not offered by any other medium" (p. 118). In a computer environment, Dreyfus claims that it is possible to address and overcome some of the problems associated with visualization (problems related to lack of flexibility in the student's thinking). Also, computer environments transfer the control over the mathematical actions to the student.

Klotz (1991) also discusses the importance of visualization. He focuses on visual geometry. Klotz claims that it is essential to several branches of mathematics and to several occupations, including "biochemists, surgeons, aviators, mechanical shovel operators, sculptors, choreographers, and architects" (Klotz, 1991, p. 95). He set up the Visual Geometry Project in response to the need for three-dimensional visual materials in the schools. The project provided high-quality computer generated visual images with which one could interactively explore the two-dimensional implications of three-dimensional work. The result was *The Geometer's Sketchpad*. *The Geometer's Sketchpad* "allows the user to make accurate geometric drawings and measurements to
replicate a series of ruler-and-compass constructions on different figures, to build figures by specifying parts and properties, and to continuously...transform figures” (p. 99).

Koltz feels that computer-generated images with some appropriate hands-on materials coupled with good interactive computer programs is worthwhile. However, he claims one major limitation to the development of visualization media is money.

In summary, this section discussed the issues surrounding visualization in mathematics. The reluctance to visualize in mathematics on part of students, teachers and mathematicians was reviewed. Quick to follow was a discussion of the importance and benefits associated with visualizing in mathematics. The section concludes with Klotz's discussion on computer generated visual mathematics.
CHAPTER III
METHODOLOGY

This chapter describes issues related to the planning, preparation and carrying out of the study. It describes the learning environment in which the study was conducted and the methods of data collection. The chapter includes information about the school in which the study took place, the participants, the subject-matter of the study, the study unit that was developed and administered, and various student activities through which the data was gathered.

The School

The study was conducted at Gladstone Secondary School in Vancouver, British Columbia. This inner-city school consisting of 1400 students and 90 full-time teaching staff (September, 1995), has a culturally diverse student population. The languages used at home by the students are numerous. There are 35 different languages spoken and Table 1 offers a breakdown of the language diversity.
Table 1: Home Language

<table>
<thead>
<tr>
<th>Language Group</th>
<th>Student Count</th>
<th>Percentage of School Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cambodian</td>
<td>6</td>
<td>0.4%</td>
</tr>
<tr>
<td>Chinese</td>
<td>516</td>
<td>38.6%</td>
</tr>
<tr>
<td>Croatian</td>
<td>8</td>
<td>0.6%</td>
</tr>
<tr>
<td>English</td>
<td>338</td>
<td>25.3%</td>
</tr>
<tr>
<td>Greek</td>
<td>15</td>
<td>1.1%</td>
</tr>
<tr>
<td>Hindi</td>
<td>44</td>
<td>3.3%</td>
</tr>
<tr>
<td>Japanese</td>
<td>6</td>
<td>0.4%</td>
</tr>
<tr>
<td>Korean</td>
<td>7</td>
<td>0.5%</td>
</tr>
<tr>
<td>Portuguese</td>
<td>35</td>
<td>2.6%</td>
</tr>
<tr>
<td>Punjabi</td>
<td>69</td>
<td>5.2%</td>
</tr>
<tr>
<td>Spanish</td>
<td>45</td>
<td>3.4%</td>
</tr>
<tr>
<td>Tagalog</td>
<td>34</td>
<td>2.5%</td>
</tr>
<tr>
<td>Vietnamese</td>
<td>156</td>
<td>11.7%</td>
</tr>
<tr>
<td>others</td>
<td>42</td>
<td>3.1%</td>
</tr>
</tbody>
</table>

(Source: Vancouver School Board Information Systems; report date November 24, 1994)

Of special note is the fact that the proportion of Gladstone students who speak English at home is lower than that of any other Vancouver secondary school (Vancouver School Board, 1994).

The school is situated in the Kensington-Cedar Cottage neighbourhood of East Vancouver. The residents of the neighbourhood contribute a rich range of cultural, ethnic, and religious influences to Vancouver. Similar to the city-wide trend, the number of people in Kensington-Cedar Cottage who list English as their mother tongue has decreased since 1971. In 1991, 43% of residents listed English (as compared to 60% city-wide), while Chinese (30%) was the next most common language group in the
neighbourhood (18% city-wide). Those residents listing Chinese as their mother tongue jumped from 8% to 30% between 1971 and 1991.

The median 1991 household income in Kensington-Cedar Cottage of $35 396 was 3.5% greater than the city median of $34 174. This neighbourhood consists mostly of single-family dwellings (81% of all dwellings). The total number of people in the area that rent their dwellings is 43%. Kensington-Cedar Cottage’s 22 government-assisted housing projects contain 639 dwelling units for low- and moderate-income households.¹

Gladstone Secondary School is currently operating on a four by two timetable. The students attend four 75 minute classes each day on a Day 1 and Day 2 rotation (see table 2).

Table 2: Timetable at Gladstone Secondary School

<table>
<thead>
<tr>
<th>DAY 1</th>
<th>DAY 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block A</td>
<td>Block E</td>
</tr>
<tr>
<td>10 minute break</td>
<td>10 minute break</td>
</tr>
<tr>
<td>Block B</td>
<td>Block F</td>
</tr>
<tr>
<td>Lunch</td>
<td>Lunch</td>
</tr>
<tr>
<td>Block C</td>
<td>Block G</td>
</tr>
<tr>
<td>Block D</td>
<td>Block H</td>
</tr>
</tbody>
</table>

The school day starts at 8:40 a.m. and ends at 3:10 p.m. with a 40 minute lunch break.

¹ (Source: City of Vancouver Planning Department, Kensington-Cedar Cottage: A Community Profile, 1994)
The Mathematics Department

The Mathematics Department at Gladstone Secondary School consists of five full-time Mathematics teachers with many junior classes being taught by members of other departments. The mathematics department offers the usual range of courses to the students, including Math 8, Math 9 and 9A, Math 10 and 10A, Math 11 and 11A, Introductory Math 11, and Math 12. Approximately 17% of the students in Grade 9 elect the ‘A’ course, about 18% in Grade 10, and 37% of the Grade 11 students elect the Math 11A or Intro Math 11 courses. The department further offers Advanced Placement Calculus (a first year University course that enrolls about 14% of the Grade 12’s), Enriched Math 8, and Accelerated Math 9 through to Math 11. The accelerated program offers the students the opportunity to complete Math 9, 10, 11 and 12 in three years. In the fourth year, their grade 12 year, the accelerated students have the opportunity to enrol in the Advance Placement Calculus course. Enrolment for the accelerated Mathematics courses usually runs about 13% of the student population.

Mathematics 11 Course

The Mathematics 11 course covers topics such as algebra, trigonometry, data analysis, and geometry. Euclidean geometry is the geometry that is primarily taught in the British Columbia secondary school system. In geometry at the Math 11 level, the focus is circle geometry, where tangent lines, inscribed and central angles are explored in great detail.

The material of circle geometry is regularly taught using traditional methods of teaching. Theorems and properties of geometry are presented by the teacher and the
students proceed to answer textbook questions. Technology, however, offers a new
dimension to teaching and learning mathematics. The *Geometer's Sketchpad* software
program lends itself nicely to the concepts of circle geometry. *Geometer's Sketchpad* is
later discussed in this chapter.

**Participants**

With a high percentage of ESL (English as a second language) students attending this
school, it was imperative to select students with adequate English speaking skills. The
nature of this study calls for students to communicate to one another freely. They must
be able to express their thoughts without limitations due to language difficulty. It is
frequently difficult to express thoughts and ideas in one's first language. This can be
compounded greatly if it is being done in one's second language. Therefore, the students
who would participate in this study had to have reasonably good English communication
skills.

Along with adequate English skills, the participants were required to be enrolled in a
Math 11 course. The content of this study deals with concepts in the Math 11 British
Columbia curriculum. The participants should be of good standing (achieving a passing
grade) in Math 11. This study is interested not in mathematical giftedness nor in slow
learners but, rather, "average" students for the purpose of this research. Based on these
criteria, the participants were chosen.

Four grade eleven students were selected to participate in the study. They each had
adequate communication skills in English and they were achieving anywhere between C,
C+ or B standing in Math 11. The students worked in pairs in front of a Macintosh LC630 computer. One pair consisted of two East Indian boys, and the other pair, two Chinese girls. To protect the identities of the participants, their real names were not used in this study.

My Role as Teacher/Researcher

I am currently teaching Math 11 at Gladstone Secondary School and, at the time the study was conducted, had been teaching the participants for two months. My behaviour in class is that of a helper and an observer.

I interrupted students' work when I felt they were becoming frustrated with their progress. The source of the frustration usually stemmed from procedural trouble with the software. Particularly at the beginning of the study, the students had some difficulties with labelling points and deleting mistakes. I would intervene so they could continue on with their work.

In addition, my role was to clarify their procedures and their interpretations. Often, the students would arrive at a conclusion about their diagram and I would ask them to describe it to me. My reason for asking their description was two-fold: First, I wanted to be certain that I understood the students correctly and, second, I wanted to be assured that the students arrived at the geometric properties found in the Math 11 curriculum.
Lesson Objectives

I designed the study unit with the purpose of conducting this research and teaching this unit in subsequent years. The study covers material that relates to the Intended Learning Outcome (ILO) 11.29 (b to g) in the British Columbia Mathematics Curriculum Guide (1988) for Mathematics 11. The ILO 11.29 is as follows:

b) the measure of the central angle is equal to twice the measure of the inscribed angle subtended by the same arc.

c) inscribed angles subtended by the same arc are congruent.

d) an inscribed angle in a semicircle is a right angle.

e) opposite angles of a cyclic quadrilateral are supplementary.

f) a tangent to a circle is perpendicular to the radius at the point of tangency.

g) tangent segments to a circle from an external point are congruent.

The lessons prepared for the study covered the above ILO but not necessarily in the order that they appear in the curriculum guide. The content was organized into 6 distinct lessons. The objectives of these lessons are described below.

The first lesson, the student pairs were asked to explore the different tools offered in the software Geometer's Sketchpad. The students discovered methods to draw circles, line segments and points. Furthermore, they studied the menus located at the top of the screen. At the end of the introduction, students were asked to create pictures (a funny face and a castle) using the newly acquired tools. This lesson was deemed necessary
since the students had never used Geometer's Sketchpad and, moreover, had limited computer experience.

The second lesson dealt exclusively with circle geometry. The theorem that the student pairs explored was the following: If a line is perpendicular to a radius of a circle at a point of intersection with the circle, then the line is a tangent to the circle at that point. The student pairs were provided with detailed instructions relating to the construction of a circle and a tangent line. At no time were the students made aware that the radius is perpendicular to the tangent line at the point of intersection. They were, however, asked to make a conjecture about their drawings.

Lessons three, four, five, and six used the same approach where instructions were provided and the students examined their drawings. Lesson three focused on the principle of the length of two tangent lines to a circle from an exterior point. The students were asked to construct a circle and two tangent lines and were asked to observe where the two lines intersect. They labelled the point of intersection ‘C’ and were instructed that this point is called the exterior point of the circle. While moving the tangent lines around the circle, the students examined the phenomenon of the moving point ‘C’. Finally, the students calculated the different angles, lengths and distances and were asked to come up with a conjecture about the tangent lines through an exterior point.

Lesson four examined the measurements of the inscribed angles subtended by the same chord. The students constructed a circle and a chord AB. They proceeded to construct two inscribed angles. They were asked to measure the inscribed angles and to move the chord so that it resembles a diameter. The students were asked to comment on their findings.
Lesson five explored the relationship between inscribed angles and central angles subtended by the same chord. Again, the students constructed the appropriate angles and commented on their discoveries. Lesson six examined the angles of a cyclic quadrilateral, and the students made conjectures based on their diagrams. The students were given limited instructional content in the last two lessons. The students had become quite competent with Geometer's Sketchpad once they reached the last two lessons. In all the lessons, the student pairs were asked to write down their conjectures.

The table below offers a summary of the topics explored with the corresponding lesson number.

Table 3: Summary of Geometric Topics

<table>
<thead>
<tr>
<th>Lesson Number</th>
<th>Topics Explored by the Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction to Geometer's Sketchpad</td>
</tr>
<tr>
<td>2</td>
<td>The radius is perpendicular to the tangent line at the point of intersection.</td>
</tr>
<tr>
<td>3</td>
<td>From an exterior point outside a circle there are exactly two tangent lines which can be drawn. Furthermore, these two tangent lines are of equal length from the exterior point to the circle.</td>
</tr>
<tr>
<td>4</td>
<td>If two inscribed angles are subtended by the same chord of a circle, then the angles are congruent.</td>
</tr>
<tr>
<td>5</td>
<td>An inscribed angle is half the measurement of a central angle when both angles are subtended by the same chord.</td>
</tr>
<tr>
<td>6</td>
<td>If a quadrilateral is cyclic, then its opposite angles are supplementary.</td>
</tr>
</tbody>
</table>
**Geometer’s Sketchpad**

The *Geometer’s Sketchpad* is a computer software program recommended for use as a supplement in high school geometry courses. This software is a tool-based system which allows the user to rediscover or reinvent geometry concepts, as opposed to the more common drill and practice computer-assisted instruction software. Below is a description of some of the many features *Geometer’s Sketchpad* offers. Implementation of *Geometer’s Sketchpad* into the mathematics’ classroom will also be reviewed.

*Geometer’s Sketchpad* allows the user to explore various topics in geometry dynamically. In other words, the user can construct geometric figures and easily manipulate the size and shape of the figure while preserving the mathematical relationships. For example, the user can construct a pair of parallel lines and can move the lines around the computer screen. Although the lines move to different positions, they always remain parallel. (see Figure 1).

![Figure 1: Parallel Lines Using Geometer’s Sketchpad](image)
The user can effortlessly change a construction by clicking onto an object in the construction. In the case of a quadrilateral, the student can use the mouse to click on one of the vertices and "drag" the point anywhere on the screen. As the vertex moves, the figure remains intact; it remains a quadrilateral. The "drag" feature is an outstanding tool in the Geometer’s Sketchpad.

Another feature of this software is the ability to measure and display the measurements of areas, lengths, and angles of any construction. Moreover, as the construction changes (when the user implements the "drag" feature), this feature calculates the measurements and continuously displays accurate measurements of the changing diagram. Figure 2 attempts to show measurements changing based on quadrilateral, ABCD changing as vertex B is moved to the left.

Distance(A to B) = 4.45 cm
Angle(ABC) = 99 °
Angle(ADC) = 137 °
Area(Polygon 2) = 10.28 square cm
Perimeter(Polygon 2) = 13.89 cm
Distance(A to B) = 5.11 cm
Angle(ABC) = 59°
Angle(ADC) = 137°
Area(Polygon 2) = 17.56 square cm
Perimeter(Polygon 2) = 17.89 cm

Figure 2: The Effects of the Measurements From Moving Point B

As the user changes and distorts the diagram, the measurements change accordingly.

Key Curriculum Press (1993) describes some of the features *Geometer's Sketchpad* offers. They claim that *Geometer's Sketchpad* is in line with the research done by the Dutch mathematics educators Pierre van Hiele and Dina van Hiele-Geldof. The van Hieles found that students pass through a series of levels of geometric thinking: visualization, analysis, informal deduction, formal deduction, and rigor. The van Hieles claim that many geometry text books expect students to use formal deduction from the beginning. They conclude little is done to enable students to visualize or to encourage students to make conjectures (p. 1). The *Geometer's Sketchpad* assists students through the first three levels, encouraging a process of discovery.

*Geometer's Sketchpad* can be used in various classroom settings according to Key Curriculum Press. In a classroom with one computer, small groups of students can take turns using the computer. Each group can investigate or confirm their conjectures. One computer classroom can also make use of projecting the computer screen on the overhead
for all to view. Students can also work in pairs in a computer lab. The advantage of working in pairs is that students can better stimulate ideas and lend help to one another.

NCTM Standards stress that students need to communicate mathematically. *Geometer's Sketchpad* has many features which make it ideally suited for teacher and student presentations. This software has text capabilities, action buttons, and, with appropriate system software, video and sound capabilities.

*Geometer's Sketchpad* is a dynamic geometry construction and exploration device. With *Sketchpad*, the user can design figures with freehand drawing and precise construction techniques. The user can then transform, manipulate, and distort the figure interactively, while preserving all of its geometric relationships. Dynamic interaction gives the user the power to explore, analyze, and understand geometry (King, 1996).

**Data Collection Procedures**

The participants were videotaped in pairs as they worked on the six lessons (each lesson was approximately one hour in length). Videotaping proved to be a more complete and accurate method for gathering data. "Move it over here" is an example of the type of statements students commonly made. It would have been difficult to know what "it" is and where "here" is through audio-taping. However, by way of videotaping, I was able to witness, observe, and understand what the students meant by these types of statements.

The videotaping was conducted after school hours in the Tutorial Center. During this time, the participants were not present in regular class sections. The Tutorial Center is
used as a tutorial classroom by many students of the school. Students would go to this classroom to seek extra academic help. This study was conducted on the days when the Tutorial Center was closed or had a small number of students working in the classroom.

Although videotaping was a primary source of collecting the data for this study, it was by far not the only source. In pairs, the students wrote their conclusions and conjectures for each topic of circle geometry covered in this study. They also printed out their computer diagrams. The print-out material and the students' written comments contribute greatly to the collection of data. Written tests and classroom observation completed the data collection.

The data of this study was analyzed according to the van Hiele model of geometric thinking. A description of this model is presented in chapter 4. The procedures used for analyzing the data will be discussed in chapter 5.

**Obstacles in Carrying Out This Research**

A large number of students working in the Tutorial Center proved to be extremely distracting for the participants. Also, the high noise level generated by a large group of people in this classroom affects the quality of the sound on the videotape. On occasion, people would enter the Tutorial Center conversing quietly to one another. Even with quiet talking, I found it difficult to hear the participants' conversation on the videotape.

Periodically, the printer beside the computer started to print as the videotaping was taking place. The noise was extremely loud on tape but did not seem to bother the
participants. The participants continued their discussion but, unfortunately, the videotape could only pick up the printing noise and not the dialogue.

In addition, students would occasionally move in front of the computer screen and, unknowingly, obstruct the camera view. Another difficulty was the availability of the Tutorial Centre. If the computers were in use, I had to postpone the videotaping. This caused delay in gathering data. I usually tried to arrange times when the Tutorial Centre was closed or, at the very least, only a few individuals were scheduled to use the Centre.

It is important to note that the videotaping was conducted after school between the months of October and December. I mention this because the participants (mostly the female students) wanted to arrive home before dark. This resulted in a time constraint on the lessons. On occasion, it would get dark before we had finished a lesson. We would, therefore, continue the lesson on a separate day. This proved to be time consuming. I would have to schedule the Tutorial Center when it was convenient for me and the students. Furthermore, once we were back in the Center, the students would have to again reconstruct the diagram and rethink the problem.

Chapter Summary

This chapter discussed the methodology used in this study. A description of the school, the mathematics department and the Math 11 course was detailed. In addition, the chapter described the participants and my role as teacher/researcher. It provided a description of the learning environment in which the study was conducted and the
methods of data collection. The software, *Geometer's Sketchpad*, was described and the obstacles in carrying out this research were listed.
CHAPTER IV

THE VAN HIELE MODEL

This chapter presents the theory used, namely the van Hiele model, to analyze the data of this study. The first section provides a brief description of the model and an explanation of how and why the model was developed. In addition, it includes a short description of the two creators responsible for the development of the model. The second section names and describes the five levels of geometric thinking. Within each level, the van Hieles claim that there are five phases of learning. These phases are highlighted in the third section. The chapter concludes with my interpretative view of the van Hiele model.

Background

Two Dutch educators, P. M. van Hiele and his wife Dina van Hiele-Geldof were greatly concerned about the difficulties their students encountered with secondary school geometry. They believed that secondary school geometry involves thinking at a relatively high “level” and students did not have sufficient experiences in thinking at prerequisite lower “levels”. Their research work focused on levels of thinking in geometry and the role of instruction in helping students move from one level to the next (Fuys, Geddes and Tischler, 1988).

The van Hiele model of geometric thinking emerged from the doctoral works of Dina van Hiele-Geldof and Pierre van Hiele. These works were completed simultaneously at
the University of Utrecht (Crowley, 1987). Dina van Hiele-Geldof died shortly after finishing her dissertation. Her husband continued to advance, amend, and clarify the theory. In the Soviet Union, the geometry curriculum was revised in the 1960s to conform to the van Hiele model. Only since the 1970s has there been increased interest in the van Hiele model in North America.

According to the van Hieles, the learner passes through five levels. The levels are labelled as “visualization”, “analysis”, “informal deduction”, “formal deduction”, and “rigor” (Crowley, 1987). The model asserts that the learner moves sequentially from the initial level (visualization) to the highest level (rigor). The learner cannot achieve one level of thinking successfully without having passed through the previous levels.

The Levels

This section provides an overview of the five levels of geometric thinking. The van Hieles discussed the likelihood of students “reducing to a lower thinking level”. An explanation of this concept, concerning a reduction of thinking level, concludes this section.

Level 0: Visualization

In the initial stage, student identifies, names, compares and operates on geometric figures according to their appearance. Geometric concepts are viewed as total entities rather than as having components or attributes (Crowley, 1987). A student at this level
can learn geometric vocabulary and can identify specified shapes. They can say triangle, square, cube, and so forth, but they do not explicitly identify properties of figures (Hoffer, 1983).

Level 1: Analysis

At this level, an analysis of geometric concepts begins. The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (Fuys, Geddes, and Tischler, 1988). For example, a student at this level recognizes that rectangles have equal diagonals and a rhombus has all sides equal (Hoffer, 1983). However, a person at this stage would not recognize the relationships between properties. Interrelationships between figures are still not seen and definitions are not yet fully understood.

Level 2: Informal Deduction

Students can establish the interrelationships of properties both within figures and among figures (Crowley, 1987). They realize, for example, that every square is a rectangle. Definitions are meaningful and informal arguments can be followed. The student at this stage, however, does not comprehend the role of axioms.
**Level 3: Formal Deduction**

The student proves theorems deductively and establishes interrelationships among networks of theorems. A person can construct proofs and recognizes the possibility of developing a proof in different ways (Crowley, 1987). For example, a student can show how the parallel postulate implies that the angle sum of a triangle is equal to 180 degrees (Hoffer, 1983).

**Level 4: Rigor**

The student establishes theorems in different axiomatic systems and analyzes/compares these systems (Fuys, Geddes and Tischler, 1988).

**Reduction of Level**

The van Hieles discussed the possibility of students reducing to a lower level of thinking. According to the van Hieles, it is feasible to present material to students above their actual thinking level. For example, students can be given properties for inscribed angles and are asked to memorize them rather than discovering the properties themselves (level 1), “or students just copy a proof rather than creating it themselves or at least supplying reasons in the proof (level 2)” (Fuys, Geddes, and Tischler, 1988, p. 7). The process of learning a principle without understanding it, is referred to as a “reduction” of the subject matter to a lower level.
A reduction of thinking level occurs when a student is at one level and the instruction is at a different level. In particular, if the content, instructional materials, vocabulary, and so on, are at a higher level than the learner, the desired learning and progression will not occur (Crowley, 1987).

**Phases of Learning**

The van Hieles noted that learning is a discontinuous process and that there are jumps in the learning curve which reveal the presence of “levels” (Fuys, Geddes, Tischler, 1988). The van Hieles proposed that at each level the learner progresses through five sequential phases: information, guided orientation, explicitation, free orientation, and integration. They claim that instruction, developed according to this sequence of phases, promotes the acquisition of a level (Crowley, 1987). Below is an overview of the five phases.

Phase 1: Information: The student becomes acquainted with the working domain (e.g., examines examples and non-examples).

Phase 2: Guided Orientation: The student becomes conscious of the relations, tries to express them in words, and learns technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures).

Phase 3: Free Orientation: The student learns, by doing more complex tasks, to find his/her own way in the network of relations (e.g., knowing properties of one kind of shape helps students to investigate properties for a new shape).
Phase 5: Integration: The student summarizes all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g., properties of a figure are summarized).


At the end of the fifth phase, students have attained a new level of thinking. The new domain of thinking replaces the old, and students are ready to repeat the phases of learning at the next level.

Research has supported the accuracy of the van Hiele model for assessing student understanding of geometry (Burger, 1985; Burger and Shaughnessy 1986; Geddes, Fuys and Tischler 1988; Usiskin 1982). The next section explores the possibility of the van Hiele model being used in disciplines other than geometry. In addition, the next section considers the idea of the van Hiele model being implemented into various topics within geometry for assessing student understanding.

**A Different View of the van Hiele Model**

The van Hieles felt that students in secondary school geometry classes should be thinking at levels 3 and 4. Levels 0, 1 and 2 are thought to be attained in elementary school. Hoffer (1983) extended the van Hiele levels to topics other than geometry. He claims that this extension is possible since the levels provide a scheme for organizing material for students to learn. Moreover, goals and mathematical expectations for learning are inherent in the levels. Thus, Hoffer believes that the level structure can be
used in various topics of mathematics or any other structured discipline. In fact, in the Netherlands, “levels have been used to structure courses in chemistry and economics” (Hoffer, 1983, p.220).

If we ponder for a moment the idea of the van Hiele thinking levels being implemented into disciplines such as econonics and chemistry, then could these levels be utilized in various topics within geometry? I believe that the van Hiele levels of thinking can be used effectively in assessing student understandings of various topics within the discipline of geometry. For example, the van Hiele levels can offer insight into students’ understandings of transformational geometry, projective geometry, and, the focus of this study, circle geometry.

In British Columbia, students are first introduced to circle geometry in grade 11. It is assumed that most students entering grade 11 mathematics have not had prior opportunity to explore concepts in circle geometry. I wanted to investigate at what levels students understand circle geometry and I believed that the van Hiele levels could be applied to assess students’ understandings.

The van Hiele model can be seen as deterministic, that is, as specifying a student's level of geometrical thinking at a certain time. However, circle geometry offers new terminology (exterior point, inscribed angles, central angles, etc.), to be explored by the students entering grade 11. Therefore, the students would be entering circle geometry at level 0 or 1. This led me to an assumption that students can be at different thinking levels depending on the geometry topic. One individual may be thinking at a high level when studying parallel lines but might be at a low level when studying spherical geometry. Therefore I had to develop an extension of the van Hiele model, that takes into account this assumption. My extension of the van Hiele model consisted of identifiers for levels that are specific to the content of circle geometry. With this view of the van Hiele model, the data of this study was analyzed.
The data of this study was divided into 5 lessons. The van Hiele levels of geometric thinking were used to analyze the students' understandings of the 5 lessons in geometry through a circle. The van Hiele levels 0, 1 and 2 were used. Levels 3 and 4 focus on proving theorems deductively and establishing theorems in different postulational systems and, therefore, were outside the scope of this analysis.
CHAPTER V

DATA ANALYSIS

In this chapter, I present the data collected throughout the study. The first section acquaints the reader with the analysis procedure used in this chapter. The bulk of this chapter consists of five sections each of which presents the students' understanding of a particular property in circle geometry. Students' development of understanding was interpreted according to the van Hiele model. The chapter concludes with various observations that are relevant to the study but not implicit in the data.

Procedure for Analysis

The data for this study is divided into five sections. These sections (which I refer to as 'lessons') each represent a specific property established in circle geometry. Furthermore, the properties are emphasized in the British Columbia grade 11 mathematics curriculum. Within each lesson, a detailed account of the students' development and discovery of the property is provided. In addition, students' progression through the van Hiele thinking levels is summarized for each lesson.

Descriptors of the Levels

My initial version of the van Hiele model was relatively simplistic and based on Fuy's, Geddes' and Tischler's (1988) and Crowley's (1987) characterization of the levels.
While this version provided an adequate starting point, it lacked sufficient detail to be an operational model for the assessment of a students' level of thinking of circle geometry. Therefore, I needed to develop a skeletal version. The descriptors for levels 0, 1 and 2 are described below.

Level 0: At the base level, students identify components of the diagram by its appearance. Students are able to identify a circle, radius, tangent line, and so on. Some students may not be able to identify all components of a diagram possibly due to lack of experience in geometry or to incomplete prior learning. Such students will be described as “level 0” since there is no level below level 0 in the van Hiele model.

Level 1: The primary indicator of level 1 thinking is student engagement in analyzing their diagram. I looked for strategies employed by the students to indicate they were analyzing. One method of analysis is to identify and test relationships among components while another method is to classify components into groups. The following were used as level 1 descriptors:

- Students discover components of their diagram.
- Students use and recall appropriate vocabulary.
- Students formulate properties empirically and generalize properties.
- Students tend to talk about angles, lengths, etc., by attaching values to each measurement rather than discussing them in general.
Level 2: At this level, students provide informal arguments and are able to explain their analysis or conjectures. The following were used as level 2 descriptors:

- Students interrelate previously learned properties.
- Students are able to provide more than one explanation to support a concept.
- Students seem capable of discussing angle or distance measurements in general rather than attaching values.

Lesson 1: Tangent Line Perpendicular to the Radius

The first lesson investigates the property of a tangent line perpendicular to the radius of the circle at the point of intersection. The student pairs followed detailed instructions to construct an appropriate diagram. They measured various line segments, distances and angles and were asked to make a conjecture based on their findings.

In this lesson, I was interested in students’ discovery of the property; a tangent line is perpendicular to the radius. I looked at their basic understanding of geometric components pertinent to this lesson (level 0). The terms consisted of circle, radius and tangent line. I focused on their analysis of the construction and looked for elements they deemed important and relevant in their diagram. I was interested in their strategy they employed for analysis (for example, classifying elements of the construction into similar groups). Evidence of students analyzing their construction indicated thinking at level 1. Statements made by the students, such as, “the tangent line never goes off the circle” or “this angle measurement changes but this one remains constant” indicated that the
students were analysing their diagrams and, therefore, were thinking at level 1.

Furthermore, student self-discovery that the tangent line is always perpendicular to the radius revealed level 1 thinking. The transition into level 2 occurred when students were able to provide an explanation or an informal argument for their property.

**Kathy and Julie**

Kathy and Julie constructed the diagram in figure 3. First they constructed the circle followed by labelling the center ‘A’ and a point on the circle ‘O’. They were asked to construct and name the line segment from point ‘O’ to point ‘A’.

```
Julie: (reading instructions) What have you constructed?
Kathy: A line.
Teacher: Yes, but that line is a special line. It’s called something in particular.
Kathy: A segment...a line segment.
Julie: I thought it was called... It’s like... it’s starts with a ‘r’.
Kathy: Radius?
Julie: Ya, radius.
```

Kathy simply answered the question with “a line” then a “line segment”. Julie, however, thought that it started with the letter “r” and Kathy responded “radius”. In the above excerpt, level 0 thinking is apparent. At this base level, students recognize the figure by its appearance. The pair recognized that this line was called the “radius” of the circle.

After they successfully constructed a tangent line to the circle, they measured the length of the radius, circumference, and they measured the angles. They moved several points around the diagram, manipulating the construction. They focused on the
measurements that changed versus measurements that did not change when one element of the diagram moved. For example, Julie clicked on point B and slid it up and down the tangent line (figure 3). They have previously calculated the measurements of the radius and angles BOA and OBA. Some measurements continuously changed as Julie moved point B, while other measurements remained static. Julie commented on the measurements.

![Diagram of a circle with labeled angles BOA and OBA]

**Figure 3: A Construction Explored by Kathy and Julie**

Julie: OK, so when you move point B, the angle OBA changes but angle AOB doesn’t. It is this one that always changes.

Julie exhibited level 1 thinking because she initiated analyzing several elements in the diagram. She explained the difference between the two angles by classifying one angle as having measurements that change and the other angle as remaining constant.

At this point, I was hoping that the pair would realize that the angle BOA always remains 90 degrees. Even with heavy prompting, they struggled with this concept.
Teacher: Can you make a property? What always holds true with the radius and the tangent line?
Julie: Everything stays the same.
Teacher: More than that. What do you mean ‘everything’?
Julie: The circle...
Kathy: The measurements.
Teacher: Which measurements?
Julie: The radius.
Kathy: The angles.
Teacher: Which angles?
Julie: Both of them.

The pair made these observations while moving only the tangent line repeatedly around the circle. I reminded them that their conclusion must be based on all possible movements. They still needed more prompting.

Teacher: Only one of those measurements remain the same.
Kathy: Angle AOB?
Teacher: OK, and what did it always remain?
Kathy: 90 degrees.

With heavy prompting, Kathy noticed that the angle between the tangent line and the radius equals 90 degrees. However, the pair did not seem to realize that this was a property.

In this lesson, the pair acknowledged various elements of their construction (circle, radius tangent line). This was a descriptor for level 0. They analysed their construction by classifying measurements into two groups: those that change and those that remained static (level 1). Kathy and Julie were not able to discover on their own the property that the tangent line is always perpendicular to the radius.
Janraj and Sandeep constructed a similar diagram to that of Kathy and Julie. They constructed a circle with point ‘A’ at the center and point ‘O’ on the circumference (see figure 4). They constructed a line connecting ‘A’ and ‘O’ and were asked to identify it.

![Figure 4: A Construction of a Circle and a Radius by Janraj and Sandeep](image)

Sandeep: (reading instructions) What have you constructed?
Sandeep: (laughs) What kind of line?
Janraj: The radii (laughs)...Oh, we constructed the radius.
Sandeep: The radius. Thanks Janraj. The radius of the circle.

As with Julie and Kathy, this pair successfully identified the line segment as the radius of the circle (level 0).

Janraj and Sandeep measured various lengths, distances and angles. They were asked to come up with a conjecture based on their findings.

Sandeep: The tangent line and circle... isn’t it always...
Janraj: Never goes off the circle and angle BOA always stay the same.
The pair began to analyze their construction (level 1). They have noticed two components of their diagram: (1) the tangent line remains on the circle and, (2) angle BOA “remains the same”.

In the following excerpt, the students continued their analysis and developed a conjecture.

Sandeep: OK, (trying to write something down) never goes off the circle... what else?
Janraj: The...
Sandeep: This is the tangent line (points to it on the screen) right? Never goes off the circle and... what else?
(pause)
Sandeep: It always remains 90 degrees with the (points to the radius on the screen) right?
Janraj: Ya. The tangent line and radius line always remain 90 degrees.

In this excerpt, Janraj and Sandeep aligned their two elements (tangent line remains on the circle and angle BOA remains the same) and arrived at a general conclusion. They concluded that “the tangent line and the radius line always remain 90 degrees”. The pair did not provide an argument nor an explanation for their analysis.

**Lesson 2: Tangent Lines from an Exterior Point are Equal**

In this lesson, the property concerning tangent lines from an exterior point is explored. Students were asked to construct two tangent lines to a circle and to explore if the two tangent lines always intersect. The students were then asked to make a conjecture based
on their findings. I was interested in how students perceive the tangent lines and the exterior point when the two tangent lines were parallel (see figure 5).

Figure 5: Parallel Tangent Lines

I was interested also in how the computer helped the students to visualize the movement of the exterior point as the tangent lines became parallel.

At level 0, students were able to recognize the tangent lines, the radius, and the exterior point. At level 1, students began to analyze their diagram. I looked for statements from students that indicated they were analyzing their construction such as “the exterior point moves closer to (or farther from) the circle when I move this point” and “the tangent lines become parallel when the exterior point is really far from the circle”. Furthermore, I wanted to observe the conjectures the students developed. Conjectures may include, “the exterior point is always equal distance from point ‘A’ and point ‘B’” and “the exterior point is equal distance from the circle”. At level 2, students were able to provide an informal argument for their analysis or conclusions.
Kathy and Julie

Kathy and Julie experienced difficulty constructing a tangent line to the circle. Figure 6 shows the diagram they constructed. They were clearly confused about what a tangent line should look like and how to construct it. This is an example of a reduction of level. I believe that this is a direct result of the difficulty they experienced in the first lesson in which the pair did not discover, on their own, that the tangent line remains perpendicular to the radius at the point of intersection. With my help, the pair successfully constructed two tangent lines.

![Figure 6: An Attempt by Kathy and Julie to Construct Two Tangent Lines](image-url)
The instructions asked them to mark the point of intersection of the two tangent lines. Figure 7 illustrates their diagram. Their tangents lines were drawn such that the point of intersection occurred somewhere off the screen.

![Diagram of a circle with tangents at points A and B, intersecting off-screen.]

Figure 7: The Point of Intersection Occurring Off the Computer Screen

In the following excerpt, Julie seemed confused about the location of the point of intersection. Ignoring Julie, Kathy continued reading the instructions.

Kathy: (Reading instructions) Notice that your tangent lines intersect at a point.
Julie: There's no point there that they intersect. (Point of intersection is off the screen)
Kathy: (Ignoring Julie's comment, she continues reading) Select the two tangent lines and go to the construct menu... click on point at intersection.

Julie followed the instruction to produce a point of intersection but seemed confused since nothing appeared to happen on the screen. I decided to intervene to see if Julie could visualize where the point of intersection might occur.
Julie: Nothing happened, did it?
Teacher: Where do you think your two lines intersect?
Julie: Down here somewhere? (Points off screen)
Teacher: Ya.

Julie was able to correctly indicate where the point of intersection was located. She was able to visualize the two tangent lines merging closer together and she was able to anticipate the lines crossing somewhere off the screen. Julie then manipulated the diagram such that the point of intersection was visible on the screen.

The pair were asked if the two tangent lines always intersect. First, they examined the result when point ‘A’ and ‘B’ moved close together. The following excerpt reveals the pair arriving at a conclusion that the tangent lines always intersect.

Kathy: (Reading instructions) Pick point A. Move the point around the circle and notice what happens with the tangent lines. Do they intersect?
Julie: (Moves the points very close together) No, not always. No, they don‘t intersect... no, I don’t know.
Kathy: I think they do.
Julie: Ya, they do.
Kathy: It says to explain.
Julie: I can’t explain that. Because... Why?... Explain it?
Kathy: Is it because point B moves?
Julie: Cause point A never moves off the circle and um....
Kathy: Try to get (the two points) close....
Julie: I’m trying...
Kathy: The intersection when point A and B are together. What happens when A and B are close together and what happens when they are farther apart?
Julie: When A and B come close together C comes closer together.
Kathy: OK.

Julie and Kathy both studied their diagram carefully. They were very attentive watching point ‘C’ move as a result of moving point ‘A’ or ‘B’. The pair began to analyze the movement of the exterior point ‘C’ (level 1). They reasoned that the tangent lines always
intersect because the exterior point moves closer to ‘A’ and ‘B’ as ‘A’ and ‘B’ move close together.

In the following excerpt, Kathy and Julie continued their analysis of the exterior point’s movement by comparing and observing the changes in the distances between the three points ‘A’, ‘B’ and ‘C’.

Kathy: When they are close together, C is ...point C moves... Point C moves along the (tangent) line.
Teacher: Moves along a line?
Kathy: Point C moves...it moves towards B.
Teacher: OK.

Kathy made the observation that, as a result of moving point ‘A’ toward point ‘B’, the exterior point ‘C’ travelled along the tangent line BA toward point ‘B’. Figure 8 attempts to reproduce Kathy’s observation. That is, point ‘A’ moving toward point ‘B’ and, as a result, point ‘C’ moving accordingly.

Figure 8: Kathy’s Observation of the Moving Exterior Point
Julie offered a different view of the same motion presented in figure 8. She watched point ’C’ get closer to point ’B’ and remarked the following about the tangent line CA.

Julie: It (CA) becomes perpendicular to the radius OB.

For me, this was an exciting statement. Julie tried to explain the motion (see figure 8) using the property from lesson 1 (tangent line perpendicular to the radius). Through her examination of the tangent lines in this lesson, Julie develops understanding of the property from lesson 1.

At this point, both students have indicated level 0 thinking, where they recognized and identified the tangent lines and the exterior point. Level 1 thinking had also appeared when they began to analyze the movement of the exterior point. Two observations were noted: first, the exterior point “moves along the (tangent) line” and second, the tangent line (CA) “becomes perpendicular to the radius OB”.

The pair then examined the result of moving points ‘A’ and ‘B’ further apart (figure 9). The following dialogue transpired.

Teacher: What happens when A and B are far apart?
Julie: Like, how far apart.... this way? (Constructs the diagram shown in figure 9)
Kathy: Hm, point C is...moves further apart.
Teacher: OK, what do you know about the tangent lines?
Julie: They are parallel.
Teacher: They are parallel. So, what do you know about point C?
Julie: That it will never come.... never intersect. When it is parallel,...when it’s parallel, point C will never intersect.
Kathy and Julie were able to clearly anticipate what happens to point ‘C’ when the tangent lines become parallel. They were able to visualize point ‘C’ moving further away from the circle as points ‘A’ and ‘B’ moved further away from each other. They came to the realization that the lines would become parallel and the point of intersection would not exist (level 1).

One object of this lesson was to observe the conjectures produced by the students. The pair gathered all of their previous observations concerning this lesson and began to formulate a conjecture.

| Kathy: | These two are the same distances. |
| Teacher: | What two? |
| Kathy: | C and B and... |
| Julie: | A and C. |
| Kathy: | Ya. |
| Julie: | Are they always the same distance? |
| Kathy: | Ya. |
Julie: OK, so, when you move A around point A to C is B to C. It’s basically the same distances. So, when you move A and B, point A to C and B to C stay the same distance.

Kathy and Julie concluded that the exterior point remains equal distance to point ‘A’ as to point ‘B’. The students were able to connect their earlier observations together and, as a result, produced a conjecture.

In summary, Julie and Kathy acknowledged the elements (tangent lines, exterior point, radius) required in this lesson (level 0). They examined the movement of the exterior point ‘C’ as points ‘A’ and ‘B’ moved towards and away from each other. They indicated that the exterior point “move(d) along the tangent line” and that the tangent line (CA) became “perpendicular to the radius”. They further stated that the tangent lines can be parallel and, therefore, the exterior point would not exist. These observations made by Kathy and Julie indicated thinking at level 1. The pair, from their observations, developed a conjecture stating the exterior point always remains equal distance from point ‘A’ and point ‘B’.

Janraj and Sandeep

Sandeep and Janraj constructed a similar diagram to Kathy’s and Julie’s. They were asked if the tangent lines always intersect. The pair examined their diagram while moving several points and lines, changing the shape and size of their diagram.

Sandeep: Do they (the two tangent lines) always intersect?  
Janraj: Yup.
Sandeep: Ay?
Janraj: Oh, no they don’t. See when it’s up here they don’t intersect. (Tangent lines are parallel)
Sandeep: OK, explain.
Janraj: Ahhhhh (laughs).
(Both trying to explain at the same time).
Janraj: They don’t always intersect because...
Sandeep: This thing (the exterior point) goes off...
Janraj: The tangent lines are ... parallel. Do you think they are parallel?

In the above excerpt, Janraj concluded that the tangent lines do not always intersect.

Janraj indicated that the tangent lines become parallel when point ‘A’ and point ‘B’ are at opposite poles of the circle. Figure 10 offers a glimpse of their diagram.

![Diagram](image)

**Figure 10: The Diagram Investigated by Janraj and Sandeep**

Sandeep, however, was not as quick to assume that the two lines were parallel. Below, he indicated that he wanted to examine the same movement using the other point.
Sandeep: Well, you can't... Let's do the same thing with the other point. Let's try that.

Janraj continued to examine the possibility of the tangent lines becoming parallel. He moved the tangent lines around the circle and tried to explain what he sees to Sandeep.

Janraj: OK, OK. It's intersecting (moving the tangent lines), and it is still intersecting, right, (point C is off the screen), because the lines...
Sandeep: Ya. It always intersects, we just don't see it (the exterior point), right?
Janraj: Ya, OK, they are always intersecting except at the parallel point. See, now they (the tangent lines) are parallel to each other.

In the above excerpt, Janraj moved point ‘A’ slowly away from point ‘B’. During the movement, he explained to Sandeep that the two tangent lines were still intersecting even though they could not see point ‘C’ on the screen. Janraj continued moving the tangent line and concluded that the tangent line ceased to intersect because they were parallel.

Janraj reiterated his notion concerning the intersection of the tangent lines. He concluded that the tangent lines always intersects except when points ‘A’ and ‘B’ are “directly opposite”.

Janraj: Ya, OK... We got it. Eventually they intersect (moving the tangents), and eventually they intersect (moving the tangents further apart) except right here, when A and B are directly opposite.

Figure 11 demonstrates the procedure Janraj explained above.
Figure 11: The Process and Result of Moving Point ‘A’ Away from Point ‘B’
Janraj’s analysis of the intersection point and the tangent lines demonstrated level 1 thinking. He observed that as points ‘A’ and ‘B’ move farther from each other, the intersection point moves away from the circle. He also stated that the tangent lines always intersect “except at the parallel point”. The pair continued to analyze their construction but, this time, moving point ‘A’ and ‘B’ together.

Similar to above, Janraj moved the points closer together at a very slow rate. Both of them studied the result on the screen. Janraj was the first to articulate the movement on the screen.

Sandeep: When they (the points) are close together, what happens?
Janraj: They’re...
Sandeep: Move it closer, closer...
Janraj: The exterior point is always, in the.... always in the middle.
(Sandeep looks closer) Ya, see it is always in the middle. It is in the middle.

Janraj stated that the exterior point “is always in the middle” as points ‘A’ and ‘B’ move towards each other. It was unclear what Janraj meant by “in the middle”.

In the following excerpt, Janraj clarified his analysis.

Sandeep: What happens when they are far apart?
Janraj: Can you say they have equal distance between each point?
Sandeep: OK.
Janraj: The distance always remains equal to each point.
Sandeep: What happens to the point, though?
Janraj: What’s your point...when?
Sandeep: When they are closer or far apart?
Janraj: What do you mean... it moves. Obviously. (laughs)
Sandeep: The point of intersection C.
Janraj: And is always equal ... has always equal distance to both points.
(pause)
Sandeep: The distance between... between the two points... is that what you are saying?
Janraj: Ya, equal distance.

Janraj clarified his earlier notion about the exterior point is “always in the middle”. This acknowledgement was an important step to develop a conjecture for this lesson. Janraj made an assumption that the intersection point ‘C’ looks the same distance from point ‘A’ as it is from point ‘B’. The pair decided to measure the distances from CA and CB to check their assumption. They arrived at the conclusion that “the point of intersection moves and there’s always an equal distance between the two points”. Their conjecture reflected the observations made earlier. They included in their conjecture such observations as (1) the intersection point moved as ‘A’ and ‘B’ moved and (2) “the exterior point is always in the middle”.

Janraj provided a visual argument to support his assumption that two tangent lines do not always intersect. He utilized the graphics on the computer screen to illustrate the many cases where the exterior point (the point of intersection) exists. He showed the exterior point moving further from the circle and eventually the point will not exist because the tangent lines become parallel. His argument to “prove” that the tangent lines do not always intersect indicated a progression towards level 2.

Janraj and Sandeep seemed to understand the terms (exterior point, parallel, tangent lines) in this lesson (level 0). They analyzed their diagram, first, as points ‘A’ and ‘B’ moved away from each other and, second, as points ‘A’ and ‘B’ moved close together (level 1). The pair developed a conjecture stating that the intersection point ‘C’ is always equal distance from point ‘A’ as it is from point ‘B’.
Lesson 3: Inscribed Angles Subtended by the Same Chord

In this lesson, the students constructed a circle with chord AB. Also, they constructed two inscribed angles each subtended by chord AB and were asked to comment on inscribed angles.

At level 1, students begin to analyze the construction. I looked for strategies they employed to aid them in their analysis. The most likely strategy in this lesson was to measure and compare the inscribed angles. As a result, I used, as an indicator of level 1, the acknowledgement that the inscribed angles have equal measurements. In addition, I watched for the realization that when the chord changed in length, the measurements of the inscribed angles changed accordingly, yet, preserving its equality. Within level 1, students are able to recognize that the inscribed angles can be constructed at any point on the circumference of the circle and continue to have equal measurements (as long as the inscribed angles are located on the same side of the chord).

An indicator of the progression towards level 2 is students developing an informal argument for their conjecture. Another indicator is students' capability of discussing elements of their conjecture in general.

Kathy and Julie

The pair constructed a circle with two inscribed angles subtended by the chord AB (figure 12). They were asked to make a conjecture about the inscribed angles. The
following excerpt reveals the pair concluding that the inscribed angles have equal measurements.

![Figure 12: Two Inscribed Angles Subtended by the Chord AB](image)

Julie: (reading) Can you make a conjecture about the inscribed angles?
Kathy: They are the same. The angle measurements are the same.

Kathy seemed to arrive at this conclusion by examining the measurements of the angles. Recognizing the measurements were the same was an indicator of level 1 thinking.

The pair manipulated the diagram to check if their conjecture remains valid.

Kathy: Ya, they are the same. So, it doesn’t matter where the chord is or the inscribed angles. They will be the same.

Kathy tried to generalize her conjecture addressing inscribed angles. Earlier, Kathy recognized that the two inscribed angles had equal measurements. She incorporated this earlier observation to obtain a conclusion. She stated that the inscribed angles remain equal to each other even as the chord changes length.
Kathy and Julie began this lesson by measuring the inscribed angles. This strategy lead them to the conclusion that the inscribed angles had equal measurements. Through manipulating the construction (changing the length of the chord and moving the location of the inscribed angles), the pair concluded that the inscribed angles remain equal to each other regardless of their location. They also concluded that the measurements of the inscribed angles were directly affected by the length of the chord.

Janraj and Sandeep

Janraj and Sandeep constructed a circle with a chord AB and two inscribed angles. They were asked to make a conjecture about the inscribed angles. In the following excerpt, Sandeep immediately commented that the angles “are the same”. Janraj, however, was tentative about the judgement.

| Sandeep: (reading) Can you make a conjecture about the inscribed angles?... They are the same.  
| Janraj: O...K..... I don’t know.  
| Sandeep: They’re the same aren’t they?  
| Janraj: Ya, they are the same...but.....  
| Sandeep: They are the same. |

Sandeep made an assumption that the inscribed angles “are the same”, however, he did not attempt to analyze or test his assumption. Janraj, uneasy about Sandeep’s conclusion, took the initiative to analyze.

| Janraj: Wait, wait, wait. Hold on. (Moves around some points on the diagram) Oh, wait look at this. They (the angles) are exactly the same. |
By moving various points, Janraj recognized that the angle measurements were “exactly the same” even as the chord AB moved to different positions.

Sandeep and Janraj continued to analyze their construction. They moved the chord so that it resembled the diameter of the circle. Sandeep reinforced their previous assumption.

| Sandeep: When you move it (the chord), the angles change, right? |
| Janraj: Ya. |
| Sandeep: But they bot'l remain the same. |
| Janraj: Ya. |

The pair noticed that the measurements of the inscribed angles changed according to the movement of the chord. They also realized that, although the measurements of the angles changed, they continued to equal each other.

Sandeep and Janraj moved the chord to the position of the diameter of the circle. The pair commented on the measurements when the chord represented the diameter.

| Sandeep: Put it (the chord) on the diameter. 90. They are both 90 degrees. Move C. |
| Janraj: It doesn’t change. |
| Sandeep: They still stay the same. |

In the above excerpt, Janraj and Sandeep noticed that the inscribed angles measured 90 degrees when the chord represented the diameter. They also concluded that the inscribed angles remained 90 degrees at any location on the circle.

In summary, Janraj and Sandeep began to analyze inscribed angles by measuring them and noticed the measurements “are the same”. Subsequently, they noticed that the angle measurements remained equal to each other as the chord AB moved to different locations.
They further realized that the inscribed angles remain equal regardless of where they are situated on the circle. Their conclusion describing that the inscribed angles are equal when they are constructed by the same chord and that the angle measurements changed according to the movement of the chord (however, remaining equal to each other) indicated level 1 thinking.

**Lesson 4: Central and Inscribed Angles**

In this lesson, students were asked to explore the relationship between inscribed angles and central angles subtended by the same chord. Their ability to identify and name inscribed and central angles indicated that students recognized these new geometric figures and, therefore, were thinking at the base level (level 0). Analyzing the properties of their diagram by manipulating various elements signified a progression to level 1. I looked for statements such as “when the chord moves, both inscribed and central angle measurements change”, “this angle is always bigger than the other angle”, and “the central angle is twice the measurement of the inscribed angle” to indicate level 1 thinking. In level 2, students are able to justify their analysis.

**Kathy and Julie**

Kathy and Julie constructed a circle with chord AB, an inscribed angle ACB, and a central angle AOB. Their diagram is shown in figure 13.
The pair proceeded to measure several angles in the diagram. The following excerpt reveals the pair's confusion about the location of the inscribed angle and the central angle.

Kathy: (Points to central angle) This one... which one is the central angle?
Julie: Well... A...
Kathy: (trying to make a conjecture) They are different... the angles are different.
Julie: I don't know what I am doing.
Kathy: I think the angles are different, right?
[pause]
Teacher: What does it say, which angles do you have to compare?
Julie: It doesn't. It just says inscribed and central angles.
Teacher: OK, where is your inscribed angle on the screen.
Julie: A, B and C.
Teacher: Is that three angles or one angle?
Julie: Three angles.

Julie perceived the diagram as having 3 separate inscribed angles. Unfortunately, I did not question her concerning how many central angles she perceived the diagram as having. If she answered three, I would conclude that she could identify central angles and
inscribed angles (level 0). Nevertheless, it appeared that Julie was not making the connection that an inscribed angle is subtended by a chord. She observed three angles and made the assumption that all three angles were inscribed angles regardless of the chord AB.

The next excerpt shows Kathy beginning to understand which angle was defined as inscribed and which was defined as central. Julie continued to struggle.

Teacher: So, where’s your chord?
Julie: AB.
Teacher: Where’s the inscribed angle?
Julie: C.
Teacher: Where’s the central angle?
Julie: B...O (points with mouse).
Teacher: OK.
Julie: What?
Kathy: Yep, ACB and AOB. we just need this one and this one (points to correct angles on screen).

Kathy successfully indicated the central angle and the inscribed angle (level 0) in the diagram but Julie remained confused.

Julie felt that four angle measurements were needed to analyze the construction; one central angle plus three inscribed angles. As she began to explain why four measurements were needed, she suddenly realized that there was only one inscribed angle, rather than three, subtended by chord AB.

Julie: Because they are all different... ‘cause it doesn’t ask us which angle to find C, B, or A. So, that is why we put them all. Because... oh, inscribed angle so, so you just need C and O, right?
Unfortunately, I did not question Julie as to why the sudden change from three inscribed angles to one. However, Julie did acknowledge which angle was inscribed and which angle was central; indication of level 0.

The pair were unclear as to what to do next, so I intervened. I reworded the question from explore the relationship between inscribed angles and central angles to compare the two angles. Rewording seemed to help this pair.

Teacher: Maybe another way to word it is to compare the two angles.
Kathy: One is bigger than the other.
Julie: Like, this one has a bigger angle than this one.

In the above excerpt, Kathy and Julie compared the inscribed angle to the central angle. They summarized that the central angle was bigger than the inscribed angle. This indicated that the students had begun to analyze their construction and, therefore, started to progress to level 1 thinking.

In the following excerpt, the pair continued to analyze their construction and Kathy attempted to formulate a conclusion.

Kathy: (pointing to the central angle) This angle is bigger. This is 60 and this is 120, twice as big.
Julie: What do you mean this is twice as big as that one?
Kathy: No. this is twice as big as this one. (pointing to screen)
Julie: It doesn’t add up to.... doesn’t the triangle always add to 180?
Kathy: But we have two triangles.... Move B around.
Julie: I don’t know.
Kathy: Wait, ACB....
Julie: Isn’t it always half? This number is always half of that. (points to screen)
Kathy listed the central angle as being twice as big as the inscribed angle. She asked Julie to move point ‘B’ so, assuming, she could check her assumption. Kathy originally based her assumption (that the central angle is twice as big as the inscribed angle) on one case (60 degrees and 120 degrees). In the previous lessons, I had to ask the pair if their assumptions hold their validity in other cases. In other words, the pair would easily settle on a property/conclusion based on a single case or example. In this lesson, Kathy took the initiative to examine other cases without my prompting. Julie worded her conclusion differently. She focused on the inscribed angle and noticed that it was always half the measurement of the central angle. The progression to level 1 thinking was evident in both conclusions.

In this lesson, Kathy and Julie examined a construction with an inscribed angle and a central angle subtended by the same chord. They were asked to explore the two angles. The pair identified correctly the inscribed angle and the central angle (level 0) although Julie struggled for a period of time with the concept of inscribed angles. The students experienced difficulty with the instructions that asked them to “explore the relationship between inscribed angles and central angles.” I reworded the question from “explore” to “compare”. Rewording the instructions greatly helped the pair.

Their analysis began with the assumption that the central angle is “bigger” than the inscribed angle. Level 1 thinking was evident by this analysis. The students revealed further progress into level 1 when they formulated properties. Kathy concluded that the central angle is always “twice as big” as the inscribed angle while Julie remarked that the inscribed angle “is always half” the central angle. The pair, however, made no attempt to provide a reason for their conclusion.
Sandeep and Janraj

Sandeep and Janraj investigated central and inscribed angles subtended by the same chord (figure 14). This pair immediately identified the inscribed angle and the central angle (level 0). They measured both angles and at first thought that the angles would add to 180 degrees. Sandeep asked Janraj, “What’s 115 and 57?” Janraj responded, “172”. Sandeep commented, “I think we’ve done something wrong”. Sandeep anticipated the sum of the inscribed angle and the central angle to be 180 degrees.

\[
\text{Angle}(AOB) = 115^\circ \\
\text{Angle}(ACB) = 57^\circ
\]

![Diagram of inscribed and central angles](image)

**Figure 14: Construction by Janraj and Sandeep of Inscribed and Central Angles**

This procedure indicated the pair began to analyse their construction; they were entering level 1. They searched for a relationship between the two angles. They began their
analysis with a hunch that the two angles were supplementary. They concluded that this was not the case.

The pair proceeded to measure two more angles, CAB and CBA. Sandeep read the instruction, “explore the relationship between the inscribed angle and the central angle” and asked Janraj if they really need the last two angles. They decided, together, to delete these measurements.

As a strategy for their analysis, Janraj and Sandeep employed several approaches. First, they thought that the sum of the central angle and the inscribed angle should be 180 degrees. When they added the two angles and found a different sum, Sandeep assumed they did something incorrectly. Second, they measured two more angles but this procedure was short lived. The pair realized that they wanted to explore the relationship between the inscribed angle and the central angle and the two new angles were not representatives of inscribed or central.

As Janraj slowly moved a point on the diagram, Sandeep made the following comment.

Sandeep: It’s the double of it.
Teacher: What’s the double of what?
Sandeep: AOB is double ACB but it is not exact, i.e. It is a little off.

In the above excerpt, Sandeep continued to analyze the construction. He made a claim that the central angle is double the inscribed angle but noticed that the measurements were “not exact”.

The issue of rounding needs to be addressed. The measurements on Geometer’s Sketchpad can be displayed to several decimal places. At this point, Sandeep and Janraj
were looking at measurements rounded to the nearest degree. Therefore, it is common for
the software to display measurements where the central angle is “not exact[ly]” double
the inscribed angle. For example, figure 15 illustrates a central angle measuring 121
degrees and the inscribed angle measuring 61 degrees (instead of 60.5 degrees) The
software is rounding to the nearest degree and, therefore, does not display 60.5 degrees as
the measurement for the inscribed angle.

\[
\begin{align*}
\text{Angle(ABO)} &= 121^\circ \\
\text{Angle(ACB)} &= 61^\circ
\end{align*}
\]

Figure 15: An Example of Inexact Angle Measurements

To get around this problem, the pair decided to display the measurements up to 3
decimal places. The following dialogue transpired.

Janraj: (looking at the measurements) They can be more than 180 degrees.
Sandeep: Ya, it can be.
Janraj: Ya, exactly.
Sandeep: It is double but then...
Janraj: but...
Sandeep: it goes like,...there’s no real limit, is there.
Janraj: It’s like Pi. 3.1415......
It was interesting that Janraj referred to an earlier strategy of taking the sum of the two angles. He made note that the angles can add to an amount more than 180 degrees. The pair tried to make sense of the decimals. They discussed the notion of irrational numbers (for example, Pi) where the digits to the right of the decimal never repeat and are infinite. They acknowledged a comparison of the angle measurements and Pi, stating, similar to Pi, the decimals of inscribed and central angles are infinite. The pair continued to focus on a conclusion concerning the two angles.

| Sandeep: The central angle is double...  
| Janraj: Don’t put the central angle, put AOB.  
| [...]  
| Sandeep: No, no, no. Ya, ‘cause it asks us about everything. Like, the difference between central angles and inscribed angles.  
| Janraj: What?  
| Sandeep: It asks, (reading) the relationship between... not just this thing. It is talking about every single inscribed angle and central angle. Not just AOB and whatever.  
| Janraj: OK.  

Janraj wanted to use the specific name for the central angle, namely AOB. However, Sandeep emphasized that specific angles were not general enough; they needed to build a conclusion encompassing all central and inscribed angles. A descriptor of thinking at level 2 is the capability of discussing geometric elements in general rather than by their specific name. Sandeep indicated progress towards level 2 when he acknowledged that the conclusion must embrace all cases of central angles and inscribed angles.

In the following excerpt, the pair continued to define their conclusion including the notion of rounding decimals.

| Sandeep: Central angles... but then it is not just, it is not just... we’ve got to put something about the decimal places too, you know. You know what I am saying?
|
This pair arrived at a conclusion that the central angle is always double the inscribed angle making special note concerning the decimal places. They felt that in order for the central angle to be “exactly” double the inscribed angle, one must take into consideration all of the digits to the right of the decimal.

Janraj and Sandeep began their analysis (and, therefore, their journey into level 1) by exploring the possibility of the central angle and the inscribed angle being supplementary. They found, however, that the angles did not add to 180 degrees. They measured two other angles but decided to delete the measurements. Sandeep then noticed that the central angle was double the inscribed angle even though the measurements were rounded off to the nearest degree. The pair extended the measurements to three decimal places. They made a conclusion that the central angle is double the inscribed angle as long as all
the digits to the right of the decimal were considered. Sandeep exhibited evidence of level 2 thinking when he emphasized discussing all the central angles and inscribed angles rather than specifying angle AOB and ACB. The software rounded measurements and, as a result, became a significant factor in this pair’s conclusion.

**Lesson 5: Cyclic Quadrilateral**

In this final lesson, students were asked to construct a circle with an inscribed quadrilateral. They were instructed to examine the angles of the quadrilateral. In this lesson, I wanted to investigate how students analyzed their construction, and consequently, progressed into level 1. An example of a statement that would indicate level 1 thinking is the following: “The opposite angles of a cyclic quadrilateral change when a point moves but the other two angles do not change”. This statement indicates a classification strategy for analyzing and, therefore, represents thinking at level 1. One group represents angles that change during a specific movement while the second group represents angles that remain fixed during the same movement. Also, I was interested in the property/conclusion that the students developed. For example, “the opposite angles always add to 180 degrees” is representative of a conclusion.

This lesson had the potential for students to incorporate previously learned properties (i.e., inscribed angles subtended by the same chord are equal) into their analysis and conclusions. Exhibiting signs of interrelating previous learned properties is a descriptor for level 2. Another indicator of level 2 thinking is the capability of explaining and providing reasons for one’s conclusion.
Kathy and Julie

The pair were asked to construct a quadrilateral inscribed in a circle (figure 16).

\[ \begin{align*}
\text{Angle}(\text{ABD}) &= 80^\circ \\
\text{Angle}(\text{ACD}) &= 100^\circ \\
\text{Angle}(\text{BDC}) &= 84^\circ \\
\text{Angle}(\text{CAB}) &= 96^\circ 
\end{align*} \]

![Cyclic Quadrilateral]

**Figure 16: Cyclic Quadrilateral**

Julie began to explore the angle measurements from figure 16. She moved point A and tried to summarize the result. The following excerpt reveals Julie’s choice of analysis. She chose to classify the angles into two distinct groups, angles that have changing measurements and angles that remain constant.

Julie: OK, when you move point A, hm, angles DBA and ACD change.
Kathy: Just two angles?
Julie: Yep.
Julie: And when you move angle B, angles BAC and CDB changes. Do you want to do the rest?
Kathy: Ya.
Julie: When you move point C, angles BAC and CDB changes. When you move point D around, angles DBA and ACD changes. I think this is
The dialogue began with Julie stating which angles change as a result of moving a specific point. Her analysis indicated level 1 thinking. Along with her analysis, Julie offered a brief explanation as to why this classification exists. She noted that the angles that have changing measurements are opposite angles of the quadrilateral.

The pair continued to derive a reason why certain angles change while others remain constant.

Julie: When you move point D around the angles changes. [Point D] and A stays the same, right?
Kathy: Ya, the opposite angles stays the same.
Julie: I don’t know why.
Kathy: Because you are only moving this line and this line (CD and BD), right?
Julie: Ya.
Kathy: So only these two angles will be affected.

Kathy tried to explain why only two angle measurements change as a result of moving one point. She claimed that when one point on the diagram moves, it, in turn, causes two line segments to move. Kathy stated that, as a result, “two angles will be affected”. In other words, as point ‘D’ moves, the line segments DC and DB move accordingly.

Consequently, the measurements of angles C and B will vary. Kathy showed signs of progressing towards level 2 by providing an informal argument for the classification.

After listening to Kathy’s explanation, Julie then asked why angle CDB did not change. She noticed that as point ‘D’ moved, the lines constructing angle CDB also moved, however, its measurement remained unchanged. The following excerpt transpired.
Kathy: Oh, I don’t know... Maybe because both lines are moving not just one.
Julie: Ya...
Kathy: But the end points stay at the same spot. It’s kind of like an inscribed angle.

Kathy pondered the notion that if both lines are moving (not just one line) then the angle measurement will remain constant. Kathy tried to relate a previous property about inscribed angles into her reasoning, although a chord was not constructed in this diagram. She mentioned that the moving angle was “kind of like an inscribed angle” because the “end points of the lines stay at the same spot”. Kathy, again, showed signs of entering level 2 by relating previous discovered properties to formulate an argument.

The pair decided to convert the measurements from three decimal places to the nearest degree. They continued with their analysis and the following dialogue transpired.

Julie: Oh, I know why, ‘cause this adds up to 180. When you move a certain point around, the other two angles add to 180. Like, oh, I can’t explain it.
Teacher: Try.
Julie: When you move point D around, B and C stays the same.
Kathy: and the equal to 180.
Julie: The opposite angles always adds up to 180. Ya, that makes sense. ‘Cause this whole thing (the quadrilateral) adds up to 360 degrees.

As soon as the pair changed the display to round the measurements to the nearest degree, Julie noticed that the “changing” angles add to 180 degrees. Kathy also noticed that the other angles (the static angles) also added to 180 degrees. They made a conclusion that the measures of the opposite angles of the cyclic quadrilateral always add to 180 degrees.
Kathy and Julie began their analysis by classifying the angles of the quadrilateral into two groups. The process of classifying components of a diagram indicated progression to level 1. Kathy provided an explanation as to why two angle measurements change and two remain static when any one vertex (point) moved. Her attempt to explain the phenomenon indicated her advancement towards level 2 as did her inclusion of a previous developed property into her argument. Once the measurements were rounded to the nearest degree, Kathy and Julie concluded that the measurements of the opposite angles of a cyclic quadrilateral always sum to 180 degrees.

Sandeep and Janraj

Janraj and Sandeep constructed the cyclic quadrilateral with ease (figure 17). They decided to measure all four angles. In the following excerpt, Sandeep quickly discovers that the opposite angles add to 180 degrees.

\[
\begin{align*}
\text{Angle}(ABC) &= 86 \degree \\
\text{Angle}(BCD) &= 84 \degree \\
\text{Angle}(CDA) &= 94 \degree \\
\text{Angle}(DAB) &= 96 \degree 
\end{align*}
\]

**Figure 17: Cyclic Quadrilateral**
Sandeep: On one side they both equal up to 180.
Sandeep: BCD and BAD are equal to 180 and those two equal 180.
Right?
Janraj: So what do we have to do? Are you saying these two and these two?
Sandeep: Ya. ABC and ... you see, the whole square, right?
Janraj: OK, So you are saying this angle... opposite angles are equal?
Sandeep: No, they add up to 180, they are not equal.

In the above excerpt, Sandeep promptly recognized that the opposite angles of the cyclic quadrilateral always add to 180 degrees. However, he did not offer an explanation for his conclusion.

Sandeep exhibited level 1 thinking by making an analysis of the diagram. He quickly observed that the opposite angles of the quadrilateral had a sum of 180 degrees. However, he neglected to examine other cases to conclude if his assumption continued to remain valid. Janraj did not actively participate in this lesson, perhaps, because Sandeep arrived at a conclusion extremely quickly.

It should be noted that the pair were extremely anxious to leave school and to arrive home because they had made previous plans. Therefore, this might explain their omission to examine other cases and to elaborate on their conclusion.

**General Observations**

In this section, I present some of my observations relevant to the study but, perhaps, not explicit in the data. One observation concerns the speed at which the students progressed through each lesson. As students became more familiar with a specific
subject, they arrived at a general property faster and with relative ease. In addition to being more familiar with a subject area, another factor contributing to the speed is their familiarity with the software. In the earlier lessons, the students spent considerable time constructing diagrams. For example, several attempts were made to construct an appropriate tangent line. As the lessons progressed, the students became acquainted with the software and the ease at which they constructed and manipulated objects was evident.

Another observation concerns the reinforcement of correct mathematical syntax on the part of Geometer’s Sketchpad. For example, Julie intended to calculate the measurement of an angle AOB. To measure an angle, the three points ‘A’, ‘O’, and ‘B’ need to be selected, ensuring point ‘O’ is selected second. Julie selected the three points ordering them ‘O’, ‘A’ and ‘B’. The following excerpt indicates that Julie is unaware that the ordering of the letters matter.

| Teacher: Where’s AOB? |
| Kathy: Oh, we put OAB. |
| Julie: Does it make a difference? |
| Teacher: What do you think? |
| Julie: Probably. |
| Kathy: (measuring the angle) A, O, B, measure. |
| Julie: Oh, shoot. It does make a difference! |

By comparing the measurements of angle AOB and OAB, Julie noticed that the measurements were different and, as a result, realized that ordering is significant.

Janraj and Sandeep experienced a similar dilemma. They measured an angle and anticipated the measurement to be 90 degrees. However, the measurement displayed 45 degrees. They perceived something was wrong and, thus, measured the angle for a second time. One measurement read 45 degrees while the second measurement read 90
degrees. At this point, they realized that ordering was important and commented that the software was “picky” about the ordering of points.

Another observation is that students tended to orient their geometric figures to an upright or standard position (Fuy, Geddes and Tischler, 1988 made a similar observation). In lesson 2, Janraj and Sandeep constructed their tangent lines such that one line was vertical and the other line was horizontal (see figure 18).

![Figure 18: Tangent Lines in Standard Position](image)

The pair examined and manipulated their construction but always returned the tangent lines to the vertical and horizontal positions. On a separate occasion, Sandeep and Janraj constructed a chord $AB$ and an inscribed angle and a central angle each subtended by the chord $AB$. Before they began to measure the angles and analyze their construction, Janraj manoeuvred the diagram in such a way that the chord was horizontal and the angles were upright. Figure 19 represents their diagram.
The tendency for students to prefer their diagrams to be in more customary positions was quite blatant. Perhaps students’ past experiences with similar figures may have been limited to specific orientations (teacher illustrations or textbook examples).
CHAPTER VI

DISCUSSION AND CONCLUSIONS

The purpose of this chapter is to summarize and discuss the findings of the study. The first section provides a summary of students’ van Hiele levels of understanding for each of the five lessons. Factors affecting the students’ learning of the lessons are discussed as well as the effect the computer environment has on students’ learning. The fourth section of this chapter provides a description of the conclusions drawn from this study. Included in this chapter is an account of the study’s limitations as well as suggestions for further research. The chapter concludes with a report of my personal benefits resulting from conducting this study.

Summary of Students’ Levels of Understanding

This section presents an outline of students’ van Hiele levels of understanding for each of the five lessons. Also, a description of students’ conjectures is provided for each lesson.

LESSON 1: Tangent Perpendicular to Radius

In the first lesson, the students investigated the property concerning a tangent line to a circle. Both pairs exhibited level 0 thinking at the outset, where they identified and named various elements of their construction (tangent line, radius, circle). Kathy and
Julie progressed to level 1 by analyzing the construction through classifying measurements; however, the pair did not formulate on their own a conjecture for this lesson.

Sandeep and Janraj used a different strategy from Kathy and Julie for analyzing their construction. While manipulating the construction, they recognized two components of their diagram that remained constant: 1) "the tangent line remains on the circle" and, 2) "angle BOA remains 90 degrees". The pair were able to develop a conjecture relating the two components. Level 1 thinking was evident through their analysis of the construction. However, Sandeep and Janraj did not attempt to provide an explanation to support their conjecture (level 2).

**LESSON 2: Tangent Lines to an Exterior Point**

The students, in this lesson, explored the movement of the exterior point as two tangent lines moved about the circle. Both pairs were able to identify the elements of the construction (exterior point, tangent lines) whereby exhibiting thinking at level 0. They also progressed to level 1 thinking by analyzing their constructions. Janraj made progress toward level 2 by formulating an argument (he attempted to explain why two tangent lines to one circle do not always intersect). Conjectures were produced by both pairs in this lesson.
LESSON 3: Inscribed Angles Subtended by the Same Chord

In this lesson, the students examined two inscribed angles subtended by the same chord. Both pairs identified the elements of this lesson (inscribed angles, chord), thus indicating level 0 thinking. Kathy and Julie continued to level 1 thinking when they analyzed and discovered components of their diagram (inscribed angles remain equal regardless of location, and the measurements of inscribed angles are directly affected by the length of the chord). The pair formulated a conjecture based on their discovered components. Sandeep and Janraj also analyzed their construction and, therefore, progressed to level 1. This pair developed a conjecture based on their analysis. In this lesson, neither pair attempted to provide an explanation to support their conjectures (level 2).

LESSON 4: Inscribed Angles and Central Angles Subtended by the Same Chord

The focus of this lesson was on the relationship between inscribed and central angles subtended by the same chord. Kathy and Julie made progress at level 0 (learning concepts and terms such as inscribed and central angles) and into level 1 (analyzing the concepts). The pair developed a conclusion relating the previously learned concepts. Sandeep and Janraj experienced little difficulty with the concepts and terms (level 0). They analyzed their construction (level 1) and developed a conjecture. Sandeep exhibited level 2 thinking when he emphasized discussing the angle measurements in general (central angles rather than angle AOB).
LESSON 5: Cyclic Quadrilateral

In this lesson students explored properties of a cyclic quadrilateral. Both pairs identified and constructed a cyclic quadrilateral with relative ease (level 0). Kathy and Julie classified the angle measurements into two groups which indicated a form of analysis (level 1). The pair developed a conjecture for this lesson. Kathy progressed toward level 2 by providing explanations and utilizing previously learned properties into her arguments. Sandeep quickly developed a conjecture for this lesson. He indicated thinking at level one when he analyzed the construction. Janraj did not provide much input with the analysis and conjecture. The pair, appearing rushed, did not reveal any explanations for their conclusions (level 2).

Factors Affecting Students’ Performance on the Lessons.

Addressed in this section are three factors that played a role in students’ learning of the five lessons. The section begins with a discussion on the effects of language on students’ performances followed by a description of students’ visual perception of geometric concepts. Students’ views concerning mathematics conclude this section.
Language

One factor that seemed to affect the students' performance on the five lessons was language. Occasionally, the students were confused with my instructions. Right from the start, all four students indicated that they did not understand the word "conjecture", which was used in all five lessons. Sandeep and Janraj experienced no difficulty once the meaning was explained to them. However, Kathy and Julie continued to struggle to fully grasp its meaning. Perhaps, I did not provide a clear explanation. For the first two lessons, this pair thought that "conjecture" signified which measurements change as a result of manipulating the diagram. Julie, indicating her confusion, asked "So, does it (conjecture) ask if anything changes?" In the second lesson, Kathy also inquired about its meaning "Is it (conjecture) asking what doesn't change?" Their lack of understanding of the word "conjecture" hindered the progress for Kathy and Julie in the first two lessons. Julie and Kathy also experienced difficulty with the phrase, "explore the relationship". I reworded the instructions to "compare the angles" which greatly helped this pair.

It was evident from this study that the students were not accustomed to describing mathematical phenomena and speaking mathematically. Many times, students relied on hand gestures to help them with their explanations. For example, in the first lesson, Sandeep used hand gestures to represent perpendicular and, in the second lesson, Janraj motioned with his hands the extension of the lines to explain that the tangent lines intersect off the screen. The hand gestures indicated a need for the students to express their ideas that, perhaps, were not being presented verbally.
Occasionally, students also developed their own terminology to help with an explanation. For example, Janraj tried to explain why two tangent lines do not always intersect. He strained to communicate his ideas and thoughts to Sandeep. He finally stated that the two tangent lines fail to intersect at the “parallel point” (both lines are parallel). Janraj’s term “parallel point” helped him to articulate his impression of the phenomenon occurring on the computer screen. Rubenstein (1996) supports student invention of their own terms and claims that it can enhance student understanding. Nonetheless, she cautions that students may become unfamiliar with the conventional terminology.

In summary, language affected student performance through two means: First, my language in the instructions was periodically misinterpreted and second, the students experienced difficulty expressing their ideas and, as a result, resorted to hand gestures or invented their own terms.

Visual Perception

Another factor was the students’ visual perception of their construction. Burger and Shaughnessy (1986) report that the turning or moving of figures to more customary positions by the students helped them identify various properties. Students’ past experiences with these properties (in textbooks or in teacher illustrations) may have been limited to specific orientations. This study found, as did Fuys, Geddes and Tischler (1988), that students tended to be biased in favour of customary positions. The students in this study manipulated their diagrams so that the figures were in up-right or standard
positions. Janraj and Sandeep preferred their tangent lines to be vertical or horizontal while Kathy and Julie favoured their cyclic quadrilateral to resemble a square.

Furthermore, the students seem to avoid the “monster” cases (a term Goldenberg and Cuoco, 1995, used to describe deformed geometric figures) within their construction. For example, in lesson 4 the students were examining the inscribed and central angles. The students manipulated the chord and observed the angle measurements but at no time did the students explore the result when the chord was moved past the center and situated between the two angles (see figure 20). Moreover, at one point Sandeep told Janraj not to move the chord past the center of the circle.

\[
\begin{align*}
\text{Angle}(FAD) &= 145 ^\circ \\
\text{Angle}(FHD) &= 107 ^\circ 
\end{align*}
\]

Figure 20: An Example of a “Monster” Case

\[\text{To some, this example might not be a typical “disformed” figure but the students in this study did exhibit apprehension about exploring this figure.}\]
This restrictiveness that the students upheld in their construction affected the generality of their conjectures and, perhaps, limited their ability to provide explanations.

Another interesting note was students' perception of a circle. They did not visualize a circle as a locus of points equidistant from a single point but, rather, a solid construction that included all points in the interior of the circle. When instructed to construct a point "on" the circle, the students assumed "inside" the circle.

**Students' Views of Mathematics**

It was evident from the beginning that the students were not accustomed to developing their own conjectures in geometry. When asked to "examine" or "explore", they were very tentative and unsure about moving an element of the diagram. During the earlier lessons, Kathy and Julie constantly checked their procedures with me and appeared afraid to take any risks or any moves before consulting with me. When they made an observation, they immediately checked with me wanting to know if they were right or wrong. Julie and Kathy seemed to be accustomed to a more structured instructional format. They appeared to believe that there was a right way and a wrong way to approach mathematical problems. Therefore, they yearned for reinforcement to indicate they were on the "right" track. At the beginning, their confidence level with respect to exploring and conjecturing seemed extremely low. With the progression of the lessons came an increased level of confidence and a willingness to take control of their investigations.
At the start of the study, Sandeep and Janraj also exhibited an awkwardness towards developing a conjecture but to a much lesser degree. During the first lesson, Janraj stated that he is “not too good (developing) theorems”. This self evaluation indicates that he has not had much experience with conjecturing.

Another factor that affected students’ performance is the limitations of the software. For example, Geometer’s Sketchpad displays measurements of obtuse angles but does not display or calculate measurements of reflex angles. The lack of flexibility in the software can affect or distort students’ understanding of a geometric concept.

**Learning in a Computer Environment**

I found that the computer environment was ideal to utilize the discovery approach to circle geometry. The software makes constructing circles, line segments, and other geometric figures, as well as finding measurements, extremely straightforward and effortless. The learner has control over the creation and manipulation of the construction. Therefore, exploring properties and developing conjectures is an attainable, rewarding activity in a computer environment.

The computer environment may have helped students progress to higher levels of geometric thinking. Students who made progress toward level 2 thinking (which was indicated by their ability to explain a phenomenon) resorted to explaining in visual terms. In the Literature Review Chapter, there was much said on the importance for students to visualize in mathematics. Dreyfus (1994) claims that educators need to acknowledge and
give students full credit for correct visual arguments. The experience of exploring geometric concepts on the computer helped the students to visualize and understand the phenomena and, therefore, enabled them to develop a visual reason and/or explanation.

The computer environment seemed to develop positive attitudes in students (especially noted in Kathy and Julie). Hembree and Dessart (1992) found that with regard to student attitudes, technology helps promote a better attitude toward mathematics and especially a better self-concept in mathematics. As mentioned earlier, Kathy and Julie were accustomed to more rigid instructional procedures. The computer allowed them to take chances and to initiate their own exploration. Cuoco, Goldenberg and Mark (in press) argue that computers put experimental power into the hands of students. As the lessons progressed, Kathy and Julie depended less on the teacher’s confirmation (concerning their achievement) and more on their own reasoning.

Another aspect of the computer environment that was prevalent in this study was the immense enjoyment that the students expressed working on the computer. They claimed that it was “fun” and “didn’t feel like work”. Perhaps their enjoyment was a side effect stemming from their positive attitude. They delighted in the idea of developing their own conjectures rather than being told about theorems and properties. Frequently, the students indicated that they wanted to continue working on the computer after a lesson was completed. My observation supports that of Lieber and Semmel (1987), who found that students are willing to spend more time learning with computers than with paper-and-pencil.
Conclusions

The purpose of this study was to investigate students’ understanding of circle geometry. The questions investigated were:

- To what extent do students discover properties of circle geometry working in a computer environment?
- How can the van Hiele model be used to describe students’ learning of circle geometry working in a computer setting?
- What role do computers play in helping students to visualize concepts in circle geometry?

Several conclusions can be drawn based on the data.

(1). Working in a computer environment, students were able to develop conjectures that related to the five properties in circle geometry found in the British Columbia grade 11 curriculum. In the first lesson, Kathy and Julie experienced difficulty developing a conjecture. Their difficulty might have stemmed from a misunderstanding concerning the definition of “conjecture” or from a lack of experience with the discovery approach in mathematics. However, all other conjectures were developed and all related to properties in circle geometry found at the grade 11 level.

(2). Students can be at different levels for different concepts. For the most part, students in this study did not exhibit the same level of thinking across all five lessons. This was consistent with the findings from Mayberry (1983). It appears that the van Hiele model,
if taken literally, is not fully applicable. Extension and modification of the model is needed in order to apply the model to various geometric topics or other academic disciplines.

(3). *The students exhibited evidence of thinking at van Hiele level 0 and level 1 in all five properties.* On occasion, some students progressed toward level 2 by interrelating previously learned properties and/or providing reasons for their conclusion. Level 2 thinking began to appear more often during the latter part of the study.

(4). *The students progressed through the levels sequentially.* The students of this study did not skip levels or pass through these levels in a different order. This finding supports the van Hiele claim that the levels are hierarchical in nature.

(5). *The levels appear to be dynamic rather than static and of a more continuous nature.* Periodically, it was difficult to classify students at a particular level. Usiskin (1982) also found that students in transition from one level to the next are difficult to classify. The difficulty in assigning levels can be considered as evidence questioning the discrete nature of the levels, a conclusion shared by Burger and Shaughnessy, (1986).

(6). *As students become more familiar with a specific subject, they arrive at a conjecture faster and with relative ease.* As the lessons progressed, the students developed conjectures swiftly. Students’ increased familiarity with the software may have contributed to the rapidity.
(7). Computers help students to visualize concepts in circle geometry. When students showed signs of progressing toward level 2, they did so by explaining their argument. In each case, students employed a visual argument. For example, Janraj used a visual argument to explain why two tangent lines do not always intersect and Kathy used a visual argument to support why two angle measurements change and the other two remain static.

(8). Computers assist students in developing a flexible image of geometric concepts. A textbook example can provide an illustration of a geometric concept; however, the diagram represents only one case and remains fixed. On the computer, students were able to manipulate their constructions preserving the mathematical relationships. They were able to witness the various shapes and sizes that a particular geometric figure can represent.

Limitations of the Study

The focus of this study was to investigate students' understanding of circle geometry working in a computer environment. The study was specific to grade 11 students and to circle geometry. The scope of this research brings forth various limitations.

Four grade 11 students participated in this study. The small sample size as well as the specific focus on one grade level limits the generalizability of the results. In addition,
the topic focused solely on the properties of circle geometry found in the British Columbia mathematics curriculum. Issues relating to other properties in circle geometry or other topics in geometry were not considered in this study. The confinement to the curriculum is a significant limitation of this study.

In the lessons, I asked the students to develop a conjecture based on their findings from manipulating their diagram. Occasionally, I asked the students to provide an explanation (for example, do the two tangent lines always intersect? Explain.) However, I did not ask for an explanation in all the lessons. The students, periodically, provided explanations when prompted by their partner or myself. More probing during each of the five lessons could have significantly added to the data. Students might have indicated level 2 thinking more often if this instruction was incorporated into all five lessons.

Another source of limitation in this study stems from the procedure in which the data was analyzed. Conversing with colleagues would have created a more accurate tool for data analysis and may have made it easier to incorporate the van Hiele model for determining thinking levels in circle geometry. Naturally, with a more accurate tool combined with a more comprehensive data a more objective analysis could have been obtained.

Suggestions for Further Research

There was a great deal of data collected that was not germane to the focus of my study. The most suggestive of this data was the development of student terminology. Occasionally, the students in the study developed their own mathematical terminology.
Janraj and Sandeep were asked to select a point and the radius. Janraj wanted to clarify the word “radius” and asked “the radius line or the radius point?” He referred to the “radius line” as the actual radius of the circle and the “radius point” as the centre of the circle. On a separate occasion, Janraj was examining the behaviour of the exterior point when the tangent lines move away from each other. He concluded that the tangent lines always intersect except at the “parallel point”. Janraj was referring to when the two tangent lines become parallel. Although there was a hint of student development of mathematical terminology in the data, it was not considered in this study. This issue seems to warrant considerable more attention than I was able to grant it.

Some observations (mentioned in Goldenberg and Cuoco, 1995) suggest that students who are deforming a figure, on the computer screen, tend not to let go of a point that they are dragging when the result would be a “monster” but, instead return points to tame positions before letting go. An example in this study of this type of deforming is as follows: The students were asked to explore the relationship of inscribed angles and central angles. Figure 21 represents a tame case where as figure 22 represents a “monster” case.
Figure 21: A Tame Case

Figure 22: A “Monster” Case

Figure 21 is considered a tame case because the inscribed angle and the central angle are being formed from the same side of the chord AB. Figure 22, in contrast, presents
the inscribed angle on one side of the chord and the central angle on the other side. This is deemed the “monster” case. It seems that students have difficulty “seeing” the central angle in figure 22 when the chord is dragged pass the center of the circle. “Monster” cases are necessary for further investigation because dynamic software may change or alter the way people conceptualize mathematical ideas. The issue of how students perceive the “monster” case is interesting and deserves special attention.

Related to the above issue is the concern of how students interpret the effect of a moving point. Moreover, how do students make sense of the dragging capability of geometry software? Dynamic geometry is relatively new and, there exists limited research on student learning in this environment. This study focused on student understanding of circle geometry in a computer environment. The specific focus concerning students’ perception of the dragging capabilities was not addressed. This is another topic for further research.

Another issue worth investigating concerns the van Hiele levels of understanding. This study investigated student thinking at levels 0, 1, and 2. Levels 3 and 4 were beyond the scope of this study and, therefore, not included in the analysis. Although there has been some research on student progression to levels 3 and 4 working in a Logo computer programming environment (Kieren and Olson, 1983; Ludwig in Olson, Kieren, and Ludwig, 1987), no studies have examined this issue using dynamic software. Student progression toward levels 3 and 4 working with dynamic software needs to be addressed.
Personal Benefits

Studying how students learn circle geometry in a computer environment will be invaluable in my future instructional practice. I must confess to having held the expectation that students can readily visualize a particular mathematical concept that I discuss in class. I have abandoned this expectation. I now realize that students do not easily visualize mathematical concepts and this is especially true if the concepts are new. As mentioned in the Literature Review Chapter, visualizing mathematics is an important step to fully understand a concept. I further recognize that students do not readily receive the opportunity to visualize in mathematics. I would like to investigate and utilize other software that helps students to visualize various topics in mathematics.

I enjoyed witnessing students discover concepts and properties on their own. Now that I know students enjoy and understand geometrical properties by conjecturing and discovering, I plan to utilize the discovery approach frequently to other areas of mathematics. Furthermore, I would like to incorporate this approach with other forms of technology (graphic calculators).

I was very impressed with Geometer’s Sketchpad and the students’ reaction to using it. As a result of conducting this study, I plan to develop exploratory geometry lessons targeting other grades. I would like students in other grades to work with the Geometer’s Sketchpad to develop concepts and properties themselves and make conjectures.

I also witnessed students progressing to higher levels of geometric thinking. I am now more aware that students can be at different geometric thinking levels for various topics. For the students’ benefit, I must be careful not to teach at a higher level and risk them
experiencing a “reduction of level”. This will result in students not fully understanding a concept and, therefore, lead to frustration, anxiety, and a lack of interest toward this subject. I now realize how important my instructional material is to encourage higher levels in students’ thinking.

A rewarding experience was the successful extension of the van Hiele model. I developed identifiers for levels 0, 1, and 2 that are applicable to circle geometry. My extension of the model enables the researcher or educator to measure students’ levels of understanding of geometric concepts in circle geometry. I found, in the literature, support that modification is needed in order to apply the van Hiele model to various geometric topics.

**Summary**

This study sought to investigate the following three questions:

- To what extent do students discover properties of circle geometry working in a computer environment?
- How can the van Hiele model be used to describe students’ learning of circle geometry working in a computer setting?
- What role do computers play in helping students to visualize concepts in circle geometry?

There were several conclusions drawn from this study.

With regard to the first question, this study found that students, working in a computer setting, were able to develop conjectures that related to five circle geometry properties.
These properties are listed in the British Columbia grade 11 curriculum. Furthermore, the students arrived at a conjecture faster and with relative ease as they became more familiar with the subject.

Addressing the second question, the findings of this study indicated that the students develop an understanding of circle geometry by first identifying the terms and concepts of a figure (level 0), then analyzing the figure by manipulating the construction and discovering properties (level 1), and, finally, providing explanations supporting their ideas (level 2). The students progress through the levels sequentially. In addition, the students can be at different levels for different geometric concepts. In this study, the students exhibited evidence of geometric thinking at level 0 and level 1 in all five circle geometry properties and, occasionally, progressed toward level 2. It was evident in this study that the levels appear to be dynamic rather than static and of a more continuous nature.

Regarding the third question, the findings of this study indicated that computers helped students to visualize concepts in circle geometry. In addition, computers assisted students in developing a flexible image of geometric concepts.
APPENDIX A:

WORKSHEET FOR THE GEOMETER'S SKETCHPAD

Day 1
Introduction to Geometer's Sketchpad

1. Explore the tools on the left side of your screen.
   a) The "arrow" tool allows you to select items, the "point" tool to make points, the "circle" tool to make circles, and the "line" tool to make line segments.
   b) The "label" tool (denoted with a hand and the letter 'A') allows you to label points and line segments. The question mark provides the user with information about the drawings.

2. Now, make some points, circles and line segments on your screen.
   a) Click onto the "arrow" tool and move your mouse to a point on your screen.
   b) Click on the point. You will notice that it makes a dark circle around your point. This indicates that you have selected this particular point.
   c) Let's label this point. Move your mouse to the tool with a hand and the letter 'A'. Click on this tool. Your mouse is now in the shape of a hand. Move your mouse close to the selected point. Your mouse hand should turn dark. Click down, and a letter should appear on your screen. Click on the letter and move it around. If you have labelled your point correctly, you will notice that you cannot move the letter away from your point.

3. a) Change your mouse back into the arrow.
   b) Click on the circle. Notice four dots appear on the circle. This indicates that you have selected the circle.
   c) Move the circle around your screen by dragging it with the mouse. To change the size of the circle click on the small open dot on your circle and move your mouse.

4. To delete the circle.
   a) First select it so that the four dark dots appear. Press the delete key on your keyboard.
   b) If you want to delete the other objects on your screen, go up to the EDIT menu and choose "select all". You will notice all the objects on your screen are highlighted. Again, press the delete key on your keyboard. Your screen should now be clear.
5. Can you make a line segment using only the point tool and the CONSTRUCT menu?
   Hint:
   a) Put two points anywhere on your screen. You want to make sure both points
      are selected. To accomplish this, change your mouse back to an arrow and select
      one of the points. While one point is selected, move your mouse to the second
      point. Hold down the shift key and click on the point. Now, both points should
      be selected.
   b) You should now be able to choose “segment” in the CONSTRUCT menu.
   c) Make a third point on your screen and construct a perpendicular line going
      through the third point. Be sure to select both the line and the third point. Go to
      CONSTRUCT menu and choose “perpendicular line”.
   d) Now, let’s make an intersection point. Select both lines and go to the
      CONSTRUCT menu. Choose the “point at intersection” and label the point.
   d) Click on the end point of one of the lines and move it around the screen. If the
      perpendicular lines are constructed correctly, the lines should remain
      perpendicular.

Exercise 1

*Draw a Face*

Can you draw a funny face on your screen? Use the tools on the left side of your screen.

Exercise 2

*Build a Castle*

Try to construct a castle on your screen using as many tools as you’ve learned so far.

**Day 2**

*A line tangent to a circle*

1. Construct a circle. Label the center ‘A’ by choosing the “label” tool (your mouse
   changes to a hand). You might have to double click on the letter if the computer
   produces a letter other than ‘A’.
2. Put a point on the circle. Hint: Select the circle and go into the CONSTRUCT menu.
   Choose “point on object”. Label this point ‘O’.
3. Try to move the point off the circle. If you cannot, the point is drawn correctly.
4. Select the center of your circle and point ‘O’(remember to hold down the shift key and
   click with your mouse so that both points are selected). Go to the CONSTRUCT menu
   and chose “segment”.
   What have you constructed?
5. Draw a tangent line through point ‘O’. (Recall a tangent line is a line that touches the
   circle at exactly one point).
6. Can you move this line off the circle or does this line intersect the circle at 2 points?
   How can we construct a tangent line so that it remains on the circle at point ‘O’?  Hint:
First select both point ‘O’ and the radius. Go to the CONSTRUCT menu and choose “perpendicular line”.

7. Construct a point on the new line by selecting the line and going to the CONSTRUCT menu. Label this point ‘B’. Remember, you should not be able to move point ‘B’ off the line if it is drawn correctly.

8. Calculate the length of the line segments, radius, circumference and angles. Hint: If you want to find the length of a line segment first you must select it and then go to the MEASURE menu. If it is an angle measurement you want, you must select the three points making sure that the vertex is in the middle.

For example, with the above diagram ‘A’ would be selected first followed by ‘O’ and then ‘B’. The vertex ‘O’ must be the middle point selected. Go to the MEASURE menu and choose “angle”.

9. Can you make a conjecture about a tangent line to a circle? Hint: Look at your measurements and examine what remains constant when you move the tangent line (to move the tangent line, click on point ‘O’ and move the point around the circle). Similarly, change the size of the circle and examine the measurements.

Day 3
Tangents to a circle from an exterior point

1. Draw a circle.
2. Draw two points on the circle and label these points ‘A’ and ‘B’.
3. Label the center of the circle ‘O’.
4. Construct two radii to these points. Select both ‘O’ and ‘A’ and go to the CONSTRUCT menu. Repeat it for ‘O’ and ‘B’.
5. Construct two tangent lines through the two points. Recall that the tangent lines are perpendicular to the radius. (Remember to check if you can move your tangent line off the circle. If you cannot, you constructed the tangent lines correctly).
6. Notice that your tangent lines intersect at a point.
7. Select the two tangent lines and go to the CONSTRUCT menu. Click on “point at intersection”. Label the point of intersection ‘C’. This point is the exterior point of the circle.
8. Pick point ‘A’. Move the point around the circle and notice what happens with the
tangent lines. Do they always intersect? Explain.

9. Can you do the same with point ‘B’?

10. What happens to the point of intersection when ‘A’ and ‘B’ are close together and
when they are farther apart?

11. Calculate the different angles, distances, and line segments. Move the points ‘A’ and
‘B’ around the circle. Can you make a conjecture about the tangent lines through an
exterior point?

---

**Day 4**

*Inscribed Angles*

1. Construct a circle.
2. Mark two points on your circle, label them ‘A’ and ‘B’.
3. Construct a line segment between the two points. This line segment is called a
“chord”. Select this segment and go to DISPLAY menu. Select line weight, then
“thick”.
4. Now, construct a third point on the circle and label it ‘C’.
5. Draw the line segments AC and BC.
6. Construct a fourth point ‘D’ on the circle.
7. Draw the line segments AD and BD. The angles <ADB and <ACB are called
“Inscribed angles” and these angles are *subtended* from the same chord AB.
8. Measure the inscribed angles.
9. Can you make a conjecture about the inscribed angles?

10. Move either point ‘A’ and ‘B’ so that the chord resembles the diameter of the circle.
11. Comment on the measurements of the inscribed angles.

---

**Day 5**

*Inscribed and Central Angles*

1. Construct a circle and a chord AB. Make AB a thick line.
2. Choose point ‘C’ on the circle and draw an inscribed angle.
3. Now, mark the center of your circle subtended from the chord AB. It can be done by
selecting the center of the circle and go to the TRANSFORM menu. Choose “mark
center” and label the center ‘O’.
4. Draw line segments AO and BO. The angle \( \angle AOB \) is known as the ‘central angle’. Your diagram should be similar to the following:

![Diagram of a circle with line segments AO and BO](image)

5. Explore the relationship between inscribed angles and central angles.

Day 6

*Cyclic Quadrilateral*

1. Inscribe a quadrilateral ABCD in a circle. It can be done by choosing points ABCD in this order on the circle and constructing segments AB, BC, CD, and DA.
2. Examine the angles of the quadrilateral. What can you conclude?
REFERENCE LIST


Talk presented at the Northwest Conference on Computers in Education, Oregon State University, Corvallis.


