SPECTRAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS

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Hongchang Tian

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Name: Hongchang Tian

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Examining Committee:

Chairman: *Dr. C. Schwarz*

Dr. T. Tang
Senior Supervisor

Dr. M. Trummer

Dr. M. C. Kropinski

Dr. H. Huang
External Examiner

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SPECTRAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS

Author:

(signature)

Hongchang Tian

(name)

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(date)

Abstract

Although spectral methods have attracted much attention in solving differential equations, little experience is available in applying spectral methods to solve Volterra integral equations. In this thesis, we apply the idea of spectral methods to solve the second kind Volterra integral equations. It is found that spectral-type methods using one element are not convergent. Therefore, emphasis is given to the spectral element type methods. Particular attention will be given to accuracy and efficiency of the methods. It is found that our numerical schemes are more accurate than commonly used collocation type methods, particularly when dealing with stiff problems and oscillatory problems. Moreover, the amount of computer time for our methods is comparable with that of collocation methods.

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Dedication

To my wife Ms. Yunqing Wu

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Chapter 1

Introduction

Numerical methods for solving ordinary and partial differential equations have always been important in scientific investigations. With the advent of computers, the use of numerical methods has been popularized, and more importantly, people are now able to attack those problems which are fundamental to our understanding of scientific phenomenon, but were so much more difficult to study in the past.

There are a large number of numerical methods available for use now, and among them, spectral methods have attracted much of the attention in current research on numerical methods. In this thesis, we will study spectral methods for solving the second kind Volterra integral equations. In this chapter, we first introduce the second kind Volterra integral equations and some basic ideas of numerical methods. We also briefly discuss pseudospectral Chebyshev methods. Finally, we give an outline of the thesis plan.

1.1 Volterra Integral Equations and Their Typical Numerical Methods

In the classical theory of integral equations one distinguishes between Fredholm equations and Volterra equations. In a Fredholm equation the region of integration is fixed, whereas in a Volterra equation the region is variable, usually depending in

some simple fashion on the independent variables.

The distinction between Fredholm and Volterra equations is analogous to the distinction between boundary and initial value problems in ODEs. Volterra equations frequently occur in connection with time-dependent or evolutionary systems. Much of the work on the numerical solution of Volterra equations was carried out between 1960 and 1980.

An equation of the form

$$g(t) + \int_a^t K(t, s, f(s))ds = f(t), \qquad a \le t \le T, \tag{1.1}$$

is a Volterra equation of the second kind (see, e.g. [13]). Here the unknown is $f(\cdot)$. The function $g(t)$ and the kernel $K(t, s, \cdot)$ are assumed to be known.

Some conditions are assumed to guarantee (1.1) has a unique continuous solution (see Chapter 2). Suppose that for a given stepsize $h > 0$ we know the solutions at points $t_i = a + ih$, $i = 0, 1, \cdots n-1$. An approximation to $f(t_n)$ can then be computed by replacing the integral on the left-hand side of (1.1) by a numerical integration rule using values of the integrand at t_i, $i = 0, 1, \cdots n$, and by solving the resulting equation for $f(t_n)$. Since $f(t_0) = g(t_0)$, the approximate solution can be computed in this step-by-step fashion.

Let F_n denote the approximate value of $f(t_n)$. We assume that we have an integration rule in (1.1) with the form

$$\int_a^{a+nh} \phi(t)dt \approx h \sum_{i=0}^{n} w_{ni}\phi(t_i). \tag{1.2}$$

Then we are led to consider the numerical method

$$g(t_n) + h \sum_{i=0}^{n} w_{ni}K(t_n, t_i, F_i) = F_n, \qquad n = r, r+1, \cdots, \tag{1.3}$$

where the starting values F_0 to F_{r-1} can be obtained by some other methods. If, as is generally the case, the weights w_{ni} are uniformly bounded, then the equation has a unique solution for sufficiently small h.

1.2 Pseudospectral Chebyshev Methods

Spectral methods are seldom used in solving initial value problems (IVPs), even though they are very popular for boundary value problems (BVPs). In spectral methods, the solution is assumed to be a finite linear combination of some sets of global analytic basis functions, for example, Chebyshev polynomials. The differential equation yields then a system of equation for the coefficients.

These schemes can be very efficient because the rate of convergence or the order of accuracy as the number of modes increases depends only on the smoothness of the solution. In particular, for an analytic solution of the differential equation, the error decays exponentially (see, e.g. [11]). By contrast, for example, in finite difference methods, the order of accuracy is fixed by the scheme.

For pseudospectral Chebyshev methods, the solution is discretized at Chebyshev collocation points. The approximation solution is forced to satisfy the equation only at Chebyshev collocation points. The basic idea of pseudospectral Chebyshev methods consists of replacing exact derivatives by derivatives of interpolating polynomials at Chebyshev collocation points (see, e.g. [11]).

We consider a smooth function $u(x)$ in the domain $[-1,1]$. The Chebyshev collocation points

$$x_i = \cos\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, \cdots, N,$$

are the extrema of the N-th order Chebyshev polynomial

$$T_N = \cos(N \cos^{-1} x).$$

The function $u(x)$ is interpolated by a polynomial $P(x)$, $(P(x_i) = u(x_i) = u_i)$, of degree $\leq N$,

$$P(x) = \sum_{j=0}^{N} u_j L_j(x),$$

where L_j is the polynomial of degree N with

$$L_j(x_k) = \delta_{jk}.$$

It can be shown that

$$L_j(x) = \frac{(-1)^{j+1}(1-x^2)T'_N(x)}{c_j N^2 (x - x_j)}, \qquad j = 0, 1, \cdots, N,$$

where

$$c_j = \begin{cases} 2, & j = 0, N, \\ 1, & j = 1, \cdots, N-1. \end{cases}$$

The derivative of $u(x)$ at the collocation points x_j can be approximated in many different ways. The most obvious way to compute the derivative is via matrix-vector multiplication. The entries of the Chebyshev derivative matrix D_x are computed by taking the analytical derivative of $L_j(x)$ and evaluating it at the collocation points x_k for $j, k = 0, \cdots, N$, i.e., $d_{kj} = L'_j(x_k)$. Then the entries of the matrix are

$$d_{kj} = \frac{c_k}{c_j} \frac{(-1)^{j+k}}{(x_k - x_j)}, \qquad j \neq k,$$

$$d_{kk} = -\frac{1}{2} \frac{x_k}{(1 - x_k^2)}, \qquad k \neq 0, N,$$

$$d_{00} = -d_{NN} = \frac{2N^2 + 1}{6},$$

and now the derivative of $u(x_i)$ becomes

$$u'_i = \sum_{j=0}^{N} d_{ij} u_j, \qquad i = 0, \cdots, N.$$

The other popular method uses FFT (Fast Fourier Transform), and is asymptotically faster ($O(N \log N)$ operations) than a matrix -vector multiplication ($O(N^2)$) (see, e.g. [15]).

1.3 Thesis Plan

In this thesis, we will study spectral methods for solving the second kind Volterra integral equations. The plan of the thesis is as follows. In Chapter 2, we will review some numerical methods for Volterra integral equations. In Chapter 3, we will study spectral methods for solving the second kind Volterra integral equations. First, we briefly discuss pseudospectral Chebyshev method. An application of pseudospectral Chebyshev method to solve BVPs of ODE is explained. In the main part of this chapter, we investigate spectral methods for solving Volterra integral equation in one

element. The integrand $K(t, s, f(s))$ in (1.1) is to be expanded using Chebyshev series. Unfortunately, the schemes are not convergent by increasing the number of collocation points. To avoid this being happened, we divide $[-1, 1]$ into several elements, namely, spectral element techniques are used. Compare with the trapezoidal rule and Simpson's methods, it is found that the spectral element methods have higher order of convergence. One advantage of using the spectral methods is that the integration with arbitrary limits for the basis functions can be obtained analytically. In the last part of this chapter, we will study some stiff and oscillatory problems for solving the second kind equations by the spectral element methods. In Chapter 4, some possible extensions are discussed.

Chapter 2

Trapezoidal Rule and Simpson's Methods

2.1 Volterra Integral Equations

2.1.1 Introduction

An equation of the form

$$g(t) + \int_a^t K(t, s, f(s))ds = f(t), \qquad a \leq t \leq T, \tag{2.1}$$

is a Volterra equation of the second kind. Here the unknown is $f(\cdot)$. The function $g(t)$ and the kernel $K(t, s, \cdot)$ are given. Equation (2.1) is one of several forms in which a Volterra equation can be written. More generally, one might consider the form

$$F(f(t), t, \int_a^t K(t, s, f(s))ds, g(t)) = 0, \qquad a \leq t \leq T.$$

For our purpose we assume that T is finite. In many practical applications, the behavior of the solution on the whole real axis is of interest. In this situation the limiting behavior of the solution is usually found from its behavior for large, but finite T. In numerical computations it is necessary in any case to use a finite T.

Without loss of generality, we choose the range of the independent variable so that

the lower limit is -1 and consider only the equation

$$g(t) + \int_{-1}^{t} K(t, s, f(s))ds = f(t), \qquad -1 \leq t \leq T. \tag{2.2}$$

The above equation is linear if

$$K(t, s, f(s)) = k(t, s)f(s).$$

Actually Volterra equations can be considered as a special case of Fredholm equations. For example

$$g(t) + \int_{-1}^{t} K(t, s, f(s))ds = f(t),$$

can be written as a Fredholm equation

$$g(t) + \int_{-1}^{1} K(t, s, f(s))ds = f(t), \tag{2.3}$$

if we set $K(t, s, \cdot) = 0$ for $s > t$. The classical Fredholm theory therefore also applies to Volterra equations, but loses much of its power because the kernel is not symmetric. A direct study of Volterra equations yields many results which can not be obtained with the Fredholm theory (see, e.g. [2, 3, 13]).

2.1.2 Some Applications

Before setting out on a detailed investigation of Volterra equations, we briefly look at some actual applications where such equations arise.

Volterra equations arise most naturally in certain types of time-dependent problems whose behavior at time t depends not only on the state at that time, but also on the states at previous times. The solution of an ordinary differential equation

$$y'(t) = f(t, y(t)) \tag{2.4}$$

is completely determined for $t > t_0$ if $y(t_0)$ is known. This is true for any t_0. Information prior to $t = t_0$ is irrelevant to the solution after t_0. There are, however, situations where knowledge of the current state alone is not enough, and where it is necessary to know how the state $y(t_0)$ is arrived at in order to predict the future.

Such models are sometimes called *history-dependent* or *systems with memory*. If the history dependence can be represented by a term

$$\int_0^t K(t, s, y(s))ds, \tag{2.5}$$

then the modeling equation is of Volterra type.

One of the best known examples of this type is the so-called *renewal equation*. Consider a component of some machine which is subject to failure as time passes. In general, the failure time is a random variable characterized by a probability density $\rho(t)$ such that, in the small interval $(t, t+\Delta t)$ the probability of failure of a component which is new at t' is

$$\rho(t - t')\Delta t. \tag{2.6}$$

If every component eventually fails then

$$\int_{t'}^{\infty} \rho(t - t')dt = 1, \tag{2.7}$$

for every t', so that $\rho(t)$ must satisfy

$$\int_0^{\infty} \rho(t)dt = 1. \tag{2.8}$$

Assume now that as soon as the component fails it is replaced by a new one. This new component will be replaced when it fails, and so on. Of practical interest is the *renewal density* $h(t)$ which measures the probability for the need of a replacement. It is defined so that the probability that a renewal has to be made in the interval $(t, \Delta t)$ is given by $h(t)\Delta t$. The probability for a needed replacement is the sum of (a) the probability that the first failure occurs in $(t, \Delta t)$, and (b) the probability that a renewal was made at time t', followed by another failure after $t - t'$ time units. If all contributions are added and the limit $\Delta t \to 0$ taken, we get the equation

$$h(t) = \rho(t) + \int_0^t h(t')\rho(t - t')dt'. \tag{2.9}$$

This is the renewal equation. It is a Volterra equation of the second kind of a particularly simple form.

Integral equations also find their use in applications where the more obvious model is a differential equation. There may be several advantages to reducing (when possible) to a differential equation. From a theoretical point of view, integral operators are more easily dealt with than differential operators, and properties of the solution may be more readily inferred from the integral form. The simplest and best known example, which will be mentioned in the following, is the reduction of an ordinary differential equation to an integral form. There may also be some practical advantages. In some cases the integral equation reduces the dimensionality. For example, certain partial differential equations in two variables can be shown to be equivalent to integral equations in one variable, thereby considerably simplifying the numerical computations. (see, e.g. [9, 13]).

Finally, integral equations arise in some situations where experimental observations yield not the variable of interest but rather some integrals thereof. To compute the actual variable then requires the solution of integral equations.

2.1.3 Volterra Equations and Initial Value Problems

Integral equations are used extensively in the study of the properties of differential equations. The most elementary observation is that the differential equation

$$y'(t) = F(t, y(t)), \qquad t \geq -1, \tag{2.10}$$

$$y(-1) = y_0$$

can be converted by integration into the Volterra equation

$$y(t) = y_0 + \int_{-1}^{t} F(s, y(s))ds, \tag{2.11}$$

this is often the starting point for the exploration of the qualitative properties of the solution of (2.10).

Consider the equation (2.2) under the conditions

(1) $g(t)$ is a continuous function in $-1 \leq t \leq T$,

(2) the kernel $K(t, s, y)$ is continuous in $-1 \leq s \leq t, \ -\infty < y < \infty$,

(3) the kernel satisfies the Lipschitz condition

$$|K(t,s,y_1) - K(t,s,y_2)| \leq L|y_1 - y_2|, \tag{2.12}$$

for all $-1 \leq s \leq t \leq T$, and all y_1, y_2.

As shown in [13], these conditions are sufficient to guarantee that (2.1) has a unique continuous solution. The analysis of the numerical methods in this thesis will utilize these assumptions and, strictly speaking, hold only when they are satisfied. The algorithms can usually be applied to other equations as well, although not all conclusions are necessarily valid. For certain types of kernels it may actually be necessary to modify the procedures, as well.

2.2 Trapezoidal Rule

Suppose that for a given stepsize $h > 0$ we know the solutions at points $t_i = -1 + ih$, $i = 0, 1, \cdots n-1$. An approximation to $f(t_n)$ can then be computed by replacing the integral on the left-hand side of (2.2) by a numerical integration rule using values of the integrand at t_i, $i = 0, 1, \cdots, n$, and by solving the resulting equation for $f(t_n)$. Since $f(t_0) = g(t_0)$, the approximate solution can be computed in this step-by-step fashion.

Let F_n denote the approximate value of $f(t_n)$. We can compute F_n by

$$F_n = g(t_n) + h\Big\{\frac{1}{2}K(t_n,t_0,F_0) + \sum_{i=1}^{n-1} K(t_n,t_i,F_i) + \frac{1}{2}K(t_n,t_n,F_n)\Big\}, \tag{2.13}$$

with $F_0 = g(t_0)$, for $1 \leq n \leq N$. In the linear case we can of course solve it directly for F_n; in the nonlinear case we would normally use some iterative techniques to solve for F_n within a desired accuracy. The method (2.13) is called the trapezoidal rule. The computation shows that the trapezoidal rule has a second order convergence (see, e.g. [13]). The following is an example.

Example 1. The equation

$$f(t) = 2 - e^{t+1} + \int_{-1}^{t} e^{t-s} f(s)ds \tag{2.14}$$

has exact solution

$$f(t) = 1. \tag{2.15}$$

The errors with the trapezoidal rule are shown in Table 2.1. As expected, the method has second order accuracy.

Table 2.1: Example 1, observed errors $|F_i - f(t_i)|$ at even points with the trapezoidal rule

t	$h = 0.1$	$h = 0.05$	$h = 0.025$
-1.0	0.	0.	0.
-0.8	$1.94D-4$	$5.05D-5$	$1.28D-5$
-0.6	$4.95D-4$	$1.27D-4$	$3.18D-5$
-0.4	$9.43D-4$	$2.40D-4$	$6.03D-5$
-0.2	$1.61D-3$	$4.09D-4$	$1.03D-5$
0.	$2.61D-3$	$6.62D-4$	$1.66D-4$
0.2	$4.10D-3$	$1.04D-3$	$2.61D-4$
0.4	$6.32D-3$	$1.60D-3$	$4.02D-4$
0.6	$9.64D-3$	$2.44D-3$	$6.12D-4$
0.8	$1.46D-2$	$3.69D-3$	$9.26D-4$
1.0	$2.20D-2$	$5.56D-3$	$1.39D-3$

2.3 Simpson's Methods

Assume that we have an integration rule of the form

$$\int_{-1}^{-1+nh} \phi(t)dt \approx h\sum_{i=0}^{n} w_{ni}\phi(t_i). \tag{2.16}$$

Using this to replace the integral in (2.2), we are led to consider the numerical method

$$g(t_n) + h\sum_{i=0}^{n} w_{ni}K(t_n, t_i, F_i) = F_n, \qquad n = r, r+1, \cdots, \tag{2.17}$$

where r is a fixed integer and the starting values F_0 to F_{r-1} can be obtained by some other methods. This reflects the fact that higher order integration rules require a

minimum number of points. If, as is generally the case, the weights w_{ni} are uniformly bounded, then the equation has a unique solution for all sufficiently small h.

Simpson's rule can be applied only when n is even. For odd n some adjustments have to be made. One way is to apply the so-called three-eighths rule over four adjacent points and the standard Simpson's method is used at the rest points on the interval. If the three-eighths rule is used on the ponts t_0, t_1, t_2, t_3, one gets the weights (for $n \geq 2$)

$$
\begin{aligned}
n \text{ is even:} \quad & w_{n0} = w_{nn} = \tfrac{1}{3}, \\
& w_{n,2i} = \tfrac{2}{3}, && i = 0, 1, \cdots, n/2 - 1, \\
& w_{n,2i+1} = \tfrac{4}{3}, && i = 0, 1, \cdots, n/2 - 1, \\
n \text{ is odd:} \quad & w_{n0} = \tfrac{3}{8}, \\
& w_{n1} = w_{n2} = \tfrac{9}{8}, \\
& w_{n3} = \tfrac{17}{24} - \tfrac{1}{3}\delta_{n3}, \\
& w_{n,2i} = \tfrac{4}{3}, && i = 2, 3, \cdots, (n-1)/2, \\
& w_{n,2i+1} = \tfrac{2}{3}, && i = 2, 3, \cdots, (n-3)/2, \\
& w_{nn} = \tfrac{1}{3}, && n \geq 5.
\end{aligned}
$$

In these formulas, δ_{ij} denotes the Kronecker delta

$\delta_{ii} = 1$ and $\delta_{ij} = 0$, $i \neq j$.

Numerical scheme (2.17) with the above weights is called Simpson's method I.

We again consider Example 1. Table 2.2 shows that Simpson's method I has the fourth order convergence.

Table 2.2: Example 1, observed errors $|F_i - f(t_i)|$ at even points with Simpson's method I

t	$h = 0.1$	$h = 0.05$	$h = 0.025$
-1.0	0.	0.	0.
-0.8	$1.27D-7$	$8.89D-9$	$5.56D-10$
-0.6	$3.65D-7$	$2.30D-8$	$1.40D-9$
-0.4	$7.24D-7$	$4.43D-8$	$2.67D-9$
-0.2	$1.27D-6$	$7.63D-8$	$4.57D-9$
0.	$2.09D-6$	$1.24D-7$	$7.42D-9$
0.2	$3.31D-6$	$1.96D-7$	$1.17D-8$
0.4	$5.15D-6$	$3.03D-7$	$1.80D-8$
0.6	$7.91D-6$	$4.64D-7$	$2.75D-8$
0.8	$1.20D-5$	$7.04D-7$	$4.16D-8$
1.0	$1.82D-5$	$1.06D-6$	$6.28D-8$

If we use the three-eighths rule at the upper end, namely, on the points t_{n-3}, t_{n-2}, t_{n-1}, t_n, we obtain the weights

n is even: as in Simpson's method I,

n is odd:

$$
\begin{aligned}
&w_{n0} = \tfrac{1}{3}, && n \geq 5,\\
&w_{n,2i} = \tfrac{2}{3}, && i = 1, 2, \cdots, (n-5)/2,\\
&w_{n,2i+1} = \tfrac{4}{3}, && i = 0, 1, \cdots, (n-5)/2,\\
&w_{n,n-3} = \tfrac{17}{24} - \tfrac{1}{3}\delta_{n3},\\
&w_{n,n-1} = w_{n,n-2} = \tfrac{9}{8},\\
&w_{nn} = \tfrac{3}{8}.
\end{aligned}
$$

Numerical scheme (2.17) with the above weights is called Simpson's Method II.

In Table 2.3 we list pointwise errors with Simpson's method II for Example 1. It is found that the convergence order is the same as that for Simpson's method I.

Table 2.3: Example 1, observed errors $|F_i - f(t_i)|$ at even points with Simpson's method II

t	$h = 0.1$	$h = 0.05$	$h = 0.025$
-1.0	0.	0.	0.
-0.8	$1.27D-7$	$8.89D-9$	$5.55D-10$
-0.6	$3.65D-7$	$2.29D-8$	$1.40D-9$
-0.4	$7.18D-7$	$4.38D-8$	$2.65D-9$
-0.2	$1.24D-6$	$7.51D-8$	$4.52D-9$
0.	$2.03D-6$	$1.22D-7$	$7.31D-9$
0.2	$3.20D-6$	$1.91D-7$	$1.14D-8$
0.4	$4.95D-6$	$2.95D-7$	$1.77D-8$
0.6	$7.56D-6$	$4.49D-7$	$2.70D-8$
0.8	$1.15D-5$	$6.80D-7$	$4.08D-8$
1.0	$1.73D-5$	$1.02D-6$	$6.14D-8$

Chapter 3

The Pseudospectral Chebyshev Methods

3.1 Introduction

Spectral methods have now been widely used for solving differential equations. The basic idea of spectral methods is to replace exact derivatives by derivatives of interpolating global polynomials, for example, Chebyshev polynomials. The differential equation yields then a system of equations for the coefficients. These methods are very efficient because the rate of convergence or the order of accuracy as the number of modes increases depends only on the smoothness of the solution. In particular, for an analytic solution, the error decays exponentially (see, e.g. [11, 18]). By contrast, for example, in finite difference methods, the order of accuracy is fixed by the scheme.

For pseudospectral Chebyshev methods, the solution is discretized at Chebyshev collocation points. The approximate solution is forced to satisfy the equation only at the Chebyshev collocation points. This method has the advantage of being able to deal easily with nonlinear terms.

3.1.1 Chebyshev Polynomials

First, we review several results from approximation theory (see, e.g. [7, 11]). The Chebyshev polynomial of degree k ($k = 0, 1 \cdots$) on $[-1, 1]$ is defined by the formula

$$T_k = \cos(k \cos^{-1} x).$$

Clearly, $|T_k(x)| \leq 1$ for $x \in [-1, 1]$. T_k are indeed polynomials in x. For example,

$$T_0(x) = 1, \qquad T_1(x) = x,$$

by definition, and using elementary trigonometric identities, we can obtain the recursion

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \qquad k \geq 1. \tag{3.1}$$

Some properties of the Chebyshev polynomials are

$$|T_k'(x)| \leq k^2,$$

$$T_k(\pm 1) = (\pm 1)^k,$$

$$2T_k(x) = \frac{1}{k+1}T_{k+1}'(x) - \frac{1}{k-1}T_{k-1}'(x), \tag{3.2}$$

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x), \qquad m \geq n,$$

$$\int_{-1}^{1} \frac{T_k^2(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2}c_k,$$

where, $c_k = 2$ if $k = 0$ and $c_k = 1$ if $k \geq 1$.

Let $L^2_{[-1,1]}$ be the space of square integrable functions defined on $[-1, 1]$. Then the functions T_k constitute an orthogonal basis with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-\frac{1}{2}}dx$$

in $L^2_{[-1,1]}$. The convergence theory of Chebyshev polynomial expansion is very similar to that of Fourier cosine series. In fact, suppose for $f(x) \in L^2_{[-1,1]}$, we write formally

$$f(x) = \sum_{k=0}^{\infty} a_k T_k, \tag{3.3}$$

the so-called Chebyshev series associate with $f(x)$, where a_k is to be determined, then $G(\theta) := g(\cos\theta)$ is the Fourier cosin series of $F(\theta) := f(\cos\theta)$ for $0 \leq \theta \leq \pi$. This result follows from the definition of T_k, because $T_k(\cos\theta) = \cos(k\theta)$, $G(\theta) = g(\cos\theta) = \sum_{k=0}^{\infty} a_k \cos(k\theta)$. Thus,

$$a_k = \frac{2}{\pi c_k}\int_0^{\pi} f(\cos\theta)\cos(k\theta)d\theta = \frac{2}{\pi c_k}\int_{-1}^{1} f(x)T_k(x)(1-x^2)^{-\frac{1}{2}}dx.$$

It follows from this close relation between Chebyshev series and Fourier series that if $f(x)$ is piecewise continuous and if $f(x)$ is of bounded total variation for $-1 \leq x \leq 1$ then $g(x) = \frac{1}{2}[f(x+)+f(x-)]$ for each x $(-1 < x < 1)$ and $g(1) = f(1-)$, $g(-1) = f(-1+)$. Also, if $f^{(p)}(x)$ is continuous for all $\mid x \mid \leq 1$ for $p = 0, 1, \cdots\cdots, n-1$, and $f^{(n)}(x)$ is integrable, then

$$a_k = O(k^{-n}).$$

Since $|T_k(x)| \leq 1$, it follows that the remainder after k terms of the Chebyshev series (3.3) is asymptotically much smaller than $k^{-(n-1)}$ as $k \to \infty$. If $f(x)$ is infinitely differentiable for $\mid x \mid \leq 1$, the error in the Chebyshev series goes to zero more rapidly than any finite power of k^{-1} as $k \to \infty$ (see, e.g. [15]).

The most important feature of Chebyshev series is that their convergence properties are not affected by the values of $f(x)$ or its derivatives at the boundaries $x = \pm 1$, but only by the smoothness of $f(x)$ and its derivatives throughout $-1 < x < 1$. In contrast, the Gibbs phenomenon shows that the rate of convergence of Fourier series depends on the value of $f(x)$ and its derivatives at the boundaries in addition to the smoothness of $f(x)$ and its derivative in the interior of the interval. The reason for the absence of the Gibbs phenomenon for the Chebyshev series of $f(x)$ at $x = \pm 1$ is due to the fact that $F(\theta) = f(\cos\theta)$ satisfies $F^{2p+1}(0) = F^{2p+1}(\pi) = 0$ provided only that all derivatives of $f(x)$ of order at most $2p+1$ exist at $x = \pm 1$ (see, [7, 15, 11]).

3.1.2 A Pseudospectral Method for Boundary Value Problems of ODE

In this subsection, we consider a pseudospectral Chebyshev method (i.e. Chebyshev collocation method) for a boundary value problem of the ordinary differential equation,

$$U''(x) + p(x)U'(x) + q(x)U(x) = f(x), \qquad U(1) = \alpha, \qquad U(-1) = \beta, \tag{3.4}$$

where p, q and f are given functions. Suppose that the problem is well-posed and in particular that has a unique solution. The Chebyshev interpolation polynomial can be written as

$$\overline{U}(x) = \sum_{j=0}^{N} \overline{U_j} L_j(x), \tag{3.5}$$

where $x_j = \cos(\frac{j\pi}{N})$, $0 \le j \le N$ are the interpolation points (also called the collocation points), $\{\overline{U}_j\}_{j=1}^{N-1}$ are the unknown coefficients to be determined, and $L_j(x)$ is the Lagrange interpolation polynomial associated with x_j. Since $L_j(x_k) = \delta_{jk}$, it follows that $\overline{U}(x_j) = \overline{U_j}$.

The Chebyshev method substitutes the interpolation polynomial (3.5) into (3.4), and then replaces $\overline{U}(x_0)$ by α and $\overline{U}(x_N)$ by β. Doing this, we have

$$\overline{U}''(x) + p(x)\overline{U}'(x) + q(x)\overline{U}(x) = f(x), \qquad \overline{U}(1) = \alpha, \qquad \overline{U}(-1) = \beta. \tag{3.6}$$

Using collocation points $\{x_j\}_{j=1}^{N-1}$, we have collocation equations

$$\overline{U}''(x_j) + p(x_j)\overline{U}'(x_j) + q(x_j)\overline{U}(x_j) = f(x_j), \qquad 1 \le j \le N-1. \tag{3.7}$$

Donating by $D_x = (d_{ij})$ the matrix of the first derivatives $L_i'(x_j)$ $(i, j = 0, 1, \cdots, N)$, and letting the vector $W' = [\overline{U}'(x_0), \overline{U}'(x_1), \cdots, \overline{U}'(x_N)]^T$, $W = [\overline{U}(x_0), \overline{U}(x_1), \cdots, \overline{U}(x_N)]^T$ gives

$$W' = D_x W. \tag{3.8}$$

In view of the fact that the matrix of the second derivatives $L_i''(x_j)$ is the square of D_x, we obtain the system of linear equation

$$\sum_{j=1}^{N-1} [(D_x^2)_{ij} + p(x_i)(D_x)_{ij} + q(x_i)\delta_{ij}]\overline{U}(x_j) \tag{3.9}$$

$$= f(x_i) - [(D_x^2)_{i0} + p(x_i)(D_x)_{i0}]\,\alpha - [(D_x^2)_{iN} + p(x_i)(D_x)_{iN}]\,\beta$$

for $\overline{U}(x_i)$, $1 \le i \le N-1$. For simplicity, we write equation (3.9) in matrix form

$$A\overline{U} = b, \qquad \overline{U} := [\overline{U}(x_1), \overline{U}(x_2), \cdots, \overline{U}(x_{N-1})]^T, \tag{3.10}$$

where $A = (a_{ij})$ is an $(N-1)\times(N-1)$ matrix with

$$a_{ij} = (D_x^2)_{ij} + p(x_i)(D_x)_{ij} + q(x_i)\delta_{ij}, \qquad 1 \le i, j, \le N-1,$$

and $b = [b_1, b_2, \cdots, b_{n-1}]$ with

$$b_i = f(x_i) - [(D_x^2)_{i0} + p(x_i)(D_x)_{i0}]\,\alpha - [(D_x^2)_{iN} + p(x_i)(D_x)_{iN}]\,\beta.$$

For a fixed integer N, solving system (3.10) for $\overline{U}$ and substituting $\overline{U}$, α and β into (3.5) will lead to the numerical solution $\overline{U}$ to the BVP (3.4).

3.2 Spectral Methods for Volterra Integral Equations

3.2.1 Spectral Methods on a Single Interval: explicit method

In this section, we will investigate spectral methods for the equation (2.2),

$$g(t_n) + \int_{-1}^{t_n} K(t_n, s, f(s))ds = f(t_n), \qquad n = 0, 1, \cdots, N, \tag{3.11}$$

with a smooth function $K(t, s, f(s))$ in the domain $-1 \le s < t \le 1$, where

$$t_n = -\cos\left(\frac{\pi n}{N}\right), \qquad n = 0, 1, \cdots, N, \tag{3.12}$$

are the extrema of the N-th order Chebyshev polynomial

$$T_N(x) = \cos(N \cos^{-1} x). \tag{3.13}$$

The points $\{t_n\}$ are called Chebyshev-Gauss-Labatto points.

Let us approximate the integrand in equation (3.11) by letting

$$K(t_n, s, f(s)) = \sum_{j=0}^{n-1} T_j(s) w_{nj}, \qquad s = t_0, t_1, \cdots, t_{n-1}, \qquad n = 1, 2, \cdots, N, \quad (3.14)$$

where $T_j(s)$, $j = 0, 1, \cdots, n-1$, are the basis functions.

- **How to solve for the weights w_{nj}**

In principle, w_{nj}, $0 \le j \le n-1$ can be obtained by solving a matrix equation

$$\begin{bmatrix} T_0(s_0) & T_1(s_0) & \cdots & T_{n-1}(s_0) \\ T_0(s_1) & T_1(s_1) & \cdots & T_{n-1}(s_1) \\ \vdots & \vdots & & \vdots \\ T_0(s_{n-1}) & T_1(s_{n-1}) & \cdots & T_{n-1}(s_{n-1}) \end{bmatrix} \begin{bmatrix} w_{n0} \\ w_{n1} \\ \vdots \\ w_{n(n-1)} \end{bmatrix} = \begin{bmatrix} K(t_n, s_0, F_0) \\ K(t_n, s_1, F_1) \\ \vdots \\ K(t_n, s_{n-1}, F_{n-1}) \end{bmatrix}, \quad (3.15)$$

where $s_j = t_j$, $j = 0, 1, \cdots, n-1$, are Chebyshev collocation points defined in (3.12). We can obtain the right-hand side vector by using the numerical solutions $F_0, F_1, \cdots, F_{n-1}$ obtained in previous steps.

For simplicity, we rewrite matrix equation (3.15) as

$$AX = b, \quad (3.16)$$

where A is the $n \times n$ matrix with

$$A = A_{n \times n} = \begin{bmatrix} A_{(n-1)\times(n-1)} & c \\ d^T & e \end{bmatrix}. \quad (3.17)$$

Here $c = [c_0, c_1, \cdots, c_{n-1}]^T$ with $c_j = T_{n-1}(s_j)$, $d = [d_0, d_1, \cdots, d_{n-1}]^T$ with $d_j = T_j(s_{n-1})$, and $e = T_{n-1}(s_{n-1})$.

Direct solution of (3.16) requires $O(n^3)$ operations. We will use the special form of (3.17) to find the inverse of A using $O(n^2)$ operations. This will reduce $O(n^3)$ operations for solving (3.16) to $O(n^2)$ operations.

Assume that A^{-1} at $(n-1)$-th step is given, i.e. $A^{-1}_{(n-1)\times(n-1)}$ is found. We construct the $n \times n$ matrix with

$$A^* = \begin{bmatrix} A^{-1}_{(n-1)\times(n-1)} & 0 \\ 0^T & 1 \end{bmatrix},$$

where 0 is the zero vector. Hence

$$\begin{bmatrix} A^{-1}_{(n-1)\times(n-1)} & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} A_{(n-1)\times(n-1)} & c \\ d^T & e \end{bmatrix} = \begin{bmatrix} E & A^{-1}_{(n-1)\times(n-1)}c \\ d^T & e \end{bmatrix}, \tag{3.18}$$

where E is the identity matrix. For simplicity, we let $c^* = A^{-1}_{(n-1)\times(n-1)}c$. Observe

$$\begin{bmatrix} E & 0 \\ -d^T & 1 \end{bmatrix} \begin{bmatrix} E & c^* \\ d^T & e \end{bmatrix} = \begin{bmatrix} E & c^* \\ 0^T & \bar{e} \end{bmatrix}, \tag{3.19}$$

where $\bar{e} = e - d^T c^*$.

Moreover, we observe that

$$\begin{bmatrix} E & -\bar{e}^{-1}c^* \\ 0^T & \bar{e}^{-1} \end{bmatrix} \begin{bmatrix} E & c^* \\ 0^T & \bar{e} \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0^T & 1 \end{bmatrix}. \tag{3.20}$$

Therefore, from (3.18) to (3.20), we obtain

$$A^{-1} = \begin{bmatrix} \bar{A} & -\bar{e}^{-1}c^* \\ -\bar{e}^{-1}d^T A^{-1}_{(n-1)\times(n-1)} & \bar{e}^{-1} \end{bmatrix}, \tag{3.21}$$

where $\bar{A} = A^{-1}_{(n-1)\times(n-1)} + \bar{e}^{-1}c^* d^T A^{-1}_{(n-1)\times(n-1)}$. It is easily shown that this procedure for finding A^{-1} needs $O(n^2)$ operations.

Further, it follows from (3.14) that

$$\int_{-1}^{t_n} K(t_n, s, f(s))ds \approx \sum_{j=0}^{n-1} \int_{-1}^{t_n} T_j(s)ds \; w_{nj} = \sum_{j=0}^{n-1} a_{nj} w_{nj}, \tag{3.22}$$

where a_{nj}, $0 \le j \le n-1$, $n = 1, 2, \cdots, N$, can be obtained analytically.

- **How to obtain a_{nj} analytically**

It follows from (3.2) that

$$T_j(s) = \frac{1}{2(j+1)}T'_{j+1}(s) - \frac{1}{2(j-1)}T'_{j-1}(s). \tag{3.23}$$

Integrating both sides of (3.23), we obtain

$$a_{nj} = \frac{1}{2(j+1)}\int_{-1}^{t_n} T'_{j+1}(s)ds - \frac{1}{2(j-1)}\int_{-1}^{t_n} T'_{j-1}(s)ds, \tag{3.24}$$

which gives

$$a_{nj} = \frac{1}{2(j+1)}\Big(T_{j+1}(t_n) - (-1)^{j+1}\Big) - \frac{1}{2(j-1)}\Big(T_{j-1}(t_n) - (-1)^{j-1}\Big). \tag{3.25}$$

Consequently, the equation (3.11) becomes

$$g(t_n) + \sum_{j=0}^{n-1} a_{nj}w_{nj} = F_n, \qquad n = 1, 2, \cdots, N. \tag{3.26}$$

Since the left-hand side of (3.26) is independent of the unknown F_n, it is an explicit scheme. Therefore, the method is called the explicit spectral method.

Example 1 Revisited. Table 3.1 shows the average errors for Example 1 with the explicit spectral method. It is seen that when N, the number of the Chebyshev collocation points, is getting large, the explicit spectral method apparently diverges. The smallest errors with the explicit method for this example are obtained with $N \approx 10$.

Table 3.1: Example 1, observed average errors in a single interval with the explicit spectral method

N	Average Error
6	$3.10D-2$
12	$8.20D-3$
18	$2.43D-1$
24	$54.28D+0$

3.2.2 Spectral Methods on a Single Interval: implicit method

The numerical scheme (3.26) is an explicit method in the sense that the left-hand side of (3.26) depends only on the numerical values at previous steps. A more stable and more accurate method is based on an implicit approach. More precisely, we modify (3.14) to include $s = t_n$:

$$K(t_n, s, f(s)) = \sum_{j=0}^{n} T_j(s) w_{nj}, \qquad s = t_0, t_1, \cdots, t_n, \qquad n = 1, 2, \cdots, N. \tag{3.27}$$

Again, we solve the weights w_{nj} by using the numerical values at discrete points $s = t_0, t_1, \cdots, t_n$. This gives the following system of equation

$$\begin{bmatrix} T_0(s_0) & T_1(s_0) & \cdots & T_n(s_0) \\ T_0(s_1) & T_1(s_1) & \cdots & T_n(s_1) \\ \vdots & \vdots & & \vdots \\ T_0(s_{n-1}) & T_1(s_{n-1}) & \cdots & T_n(s_n) \end{bmatrix} \begin{bmatrix} w_{n0} \\ w_{n1} \\ \vdots \\ w_{nn} \end{bmatrix} = \begin{bmatrix} K(t_n, s_0, F_0) \\ K(t_n, s_1, F_1) \\ \vdots \\ K(t_n, s_n, F_n) \end{bmatrix}. \tag{3.28}$$

It can be written in a form $BX = b$, where B is the above matrix and b is the right-hand side vector. Similar to the last subsection, we can use $O(n^2)$ operations to find the inverse of B, $B^{-1} = (\gamma_{ij})$. Therefore, we have

$$w_{nj} = \sum_{k=0}^{n} \gamma_{jk} K(t_n, s_k, F_k). \tag{3.29}$$

It follows from (3.11), (3.27) and (3.29) that

$$g(t_n) + \sum_{j=0}^{n} a_{nj} \sum_{k=0}^{n} \gamma_{jk} K(t_n, s_k, F_k) = F_n, \tag{3.30}$$

where a_{nj} are given by (3.25).

Consequently, we have

$$F_n - \sum_{j=0}^{n} a_{nj}\gamma_{jn} K(t_n, s_n, F_n) = g(t_n) + \sum_{j=0}^{n} \sum_{k=0}^{n-1} a_{nj}\gamma_{jk} K(t_n, s_k, F_k). \tag{3.31}$$

At the n-th step, the right-hand side of (3.31) is known. If K is a nonlinear function with respect to its third argument, then some iterative methods, say Newton's iteration , have to be applied. The scheme (3.31) is called implicit spectral method.

Example 1 Revisited. The average errors for Example 1 with the implicit spectral method are shown in Table 3.2. The results indicate that the implicit spectral method on a single interval also diverges, although it is more accurate than the explicit spectral method. The smallest errors with the implicit method for this example are obtained with $N \approx 20$.

Table 3.2: Example 1, observed average errors in a single interval with the implicit spectral method

N	Average Error
6	$9.84D-4$
12	$4.35D-6$
18	$1.41D-6$
24	$2.02D-6$
30	$9.30D-4$
36	$1.81D-1$

3.3 Spectral Element Methods

In order to improve the implementation of spectral methods we split the interval $[-1,1]$ into several elements $\cup_{j=0}^{M-1}(\bar{t}_j, \bar{t}_{j+1})$, where

$$\bar{t}_j = -1 + \frac{2j}{M}, \qquad j = 0, 1, \cdots, M. \tag{3.32}$$

It is easy to verify that the transformation

$$t = w_j(\zeta) = \frac{\bar{t}_{j+1} + \bar{t}_j}{2} + \frac{\bar{t}_{j+1} - \bar{t}_j}{2}\zeta, \qquad -1 \le \zeta \le 1, \tag{3.33}$$

maps $t \in [\bar{t}_j, \bar{t}_{j+1}]$ to $\zeta \in [-1,1]$. Using this transformation, we can rewrite (3.11) in an equivalent form

$$g(w_j(\zeta)) + \sum_{k=0}^{j-1} \frac{\bar{t}_{k+1} - \bar{t}_k}{2} \int_{-1}^{1} K(w_j(\zeta), w_k(s), f(w_k(s)))ds \tag{3.34}$$
$$+ \frac{\bar{t}_{j+1} - \bar{t}_j}{2} \int_{-1}^{\zeta} K(w_j(\zeta), w_j(s), f(w_j(s)))ds = f(w_j(\zeta)).$$

Let ζ_n, $0 \le n \le N$, be the Chebyshev-Gauss-Lobatto points, namely

$$\zeta_n = -\cos\left(\frac{\pi n}{N}\right), \qquad n = 0, \cdots, N. \tag{3.35}$$

From (3.33) we have the collocation points in each subinterval $[\bar{t}_j, \bar{t}_{j+1}]$,

$$t_{nj} = \frac{\bar{t}_{j+1} + \bar{t}_j}{2} + \frac{\bar{t}_{j+1} - \bar{t}_j}{2}\zeta_n, \qquad n = 0, \cdots, N.$$

Assuming that (3.34) is satisfied at the Chebyshev-Gauss-Lobatto points ζ_n, we obtain

$$g(t_{nj}) + \sum_{k=0}^{j-1} \frac{\bar{t}_{k+1} - \bar{t}_k}{2} \int_{-1}^{1} K(t_{nj}, w_k(s), f(w_k(s)))ds \tag{3.36}$$
$$+ \frac{\bar{t}_{j+1} - \bar{t}_j}{2} \int_{-1}^{\zeta_n} K(t_{nj}, w_j(s), f(w_j(s)))ds = f(t_{nj}).$$

Consider a smooth function $\omega(\zeta)$, $-1 \le \zeta \le 1$. If $\omega(\zeta_n)$, $0 \le n \le N$, are given, then pseudospectral methods will lead to

$$\int_{-1}^{1} \omega(\zeta)d\zeta \approx \sum_{l=0}^{N} c_l \omega(\zeta_l), \tag{3.37}$$

where the coefficients c_l are independent of the function ω and can be found analytically. Using the above result, we replace the integral in the second term of (3.36) by

$$\int_{-1}^{1} K(t_{nj}, w_k(s), f(w_k(s)))ds \approx \sum_{l=0}^{N} c_l K(t_{nj}, t_{kl}, F_{kl}), \tag{3.38}$$

where $F_{kl} \approx f(t_{kl})$, $0 \le k \le j-1$, $0 \le l \le N$, are approximation values in the previous elements and have been computed. Observe that the weights c_l are independent of k in the formula (3.38). Therefore, the cost of computer time for evaluating the integral in (3.11) over $[-1, t_j]$ is of the same order as that with Simpson's methods.

We will treat the integral in the third term of (3.36) using the same techniques introduced in the last two sections. If the integral is approximated by the explicit (implicit) spectral method as introduced in subsection 3.2.1 (3.2.2), then our scheme is called explicit (implicit) spectral element method.

Example 2. The equation

$$f(t) = \sin(\omega t) + \cos(\omega t)/\omega - \cos(\omega)/\omega + \int_{-1}^{t} f(s)ds \tag{3.39}$$

has exact solution

$$f(t) = \sin(\omega t), \tag{3.40}$$

where ω is a given constant.

Example 3. The equation

$$f(t) = -t^8/56 + t^6 - t/7 - 1/8 + \int_{-1}^{t} (t-s)f(s)ds \tag{3.41}$$

has exact solution

$$f(t) = t^6. \tag{3.42}$$

In Tables 3.3 to 3.5 we list average errors obtained by using the explicit and implicit spectral element methods, the trapezoidal rule and Simpson's methods for Examples 1, 2 and 3, respectively. For all of the examples, we used two elements and 12 mesh points in each element. From these tables we observe that both the explicit and implicit spectral element methods have more accurate results than the typical metheds. Furthermore, we note that the most accurate scheme is the implicit spectral element method and the least accurate one is the trapezoidal rule. These facts are also reflected in Figures 3.1 to 3.3, where we have plotted pointwise errors for different methods. All of the figures suggest that both the implicit spectral element method (imspe) and Simpson's method (simp) are convergent. It is also observed that the errors with the implicit spectral element method are much smaller than those with all of the other methods at the endpoints of each element.

In Tables 3.6 and 3.7 we fix the number of Chebyshev-Gauss-Labatto points ($N = 10$) but vary the number of elements. In the tables we list average errors for Example 1 with the explicit and implicit spectral element methods using $M = 1, 2, 4, 5, 6, 8,$ and 10. It is shown that when N is fixed both the explicit and implicit spectral element methods converge rapidly.

Table 3.3: Example 1, observed average errors in 2 elements with four different methods, $N = 12$

Method	Average Error
Explicit spectral	$1.18D-4$
Implicit spectral	$3.33D-8$
Trapezoidal	$3.42D-3$
Simpson's I	$1.81D-6$
Simpson's II	$1.71D-6$

Table 3.4: Example 2, observed average errors in 2 elements with four different methods, $N = 12$

Method	Average Error
Explicit spectral	$4.43D-5$
Implicit spectral	$1.09D-8$
Trapezoidal	$4.72D-4$
Simpson's I	$2.71D-7$
Simpson's II	$2.31D-7$

Table 3.5: Example 3, observed average errors in 2 elements with four different methods, $N = 12$

Method	Average Error
Explicit spectral	$1.14D-3$
Implicit spectral	$2.78D-6$
Trapezoidal	$5.18D-3$
Simpson's I	$9.35D-5$
Simpson's II	$6.92D-5$

Table 3.6: Example 1, the average errors with different M for the explicit spectral element method, $N = 10$

M	Average Error
1	$6.34D-3$
2	$9.63D-5$
4	$2.43D-6$
5	$8.06D-7$
6	$3.41D-7$
8	$8.73D-8$
10	$3.03D-8$

Table 3.7: Example 1, the average errors with different M for the implicit spectral element method, $N = 10$

M	Average Error
1	$1.62D-5$
2	$1.03D-7$
4	$1.58D-9$
5	$4.50D-10$
6	$1.63D-10$
8	$3.46D-11$
10	$1.06D-11$

Table 3.8: Example 1, Simpson's method II with different number of collocation points

N	Average Error	Cpu-Time
10	$7.49D-5$	0.03
20	$3.52D-6$	0.04
40	$1.83D-7$	0.06
60	$3.38D-8$	0.09
80	$1.03D-8$	0.12
100	$4.13D-9$	0.16

Table 3.9: Example 1, the implicit spectral element method with different M, $N = 10$

M	Average Error	Cpu-Time
1	$1.62D-5$	0.04
2	$1.03D-7$	0.05
4	$1.58D-9$	0.08
6	$1.63D-10$	0.12
8	$3.46D-11$	0.17
10	$1.06D-11$	0.23

In Tables 3.8 and 3.9 we list average errors and cpu-times (in SPARC 10) for Example 1 with different number of collocation points by using the implicit spectral element method and Simpson's method II, respectively. It is shown that the amount of computer time for our methods is comparable with that of collocation methods.

Figure 3.1: Example 1, four different methods, $M = 2$, $N = 15$

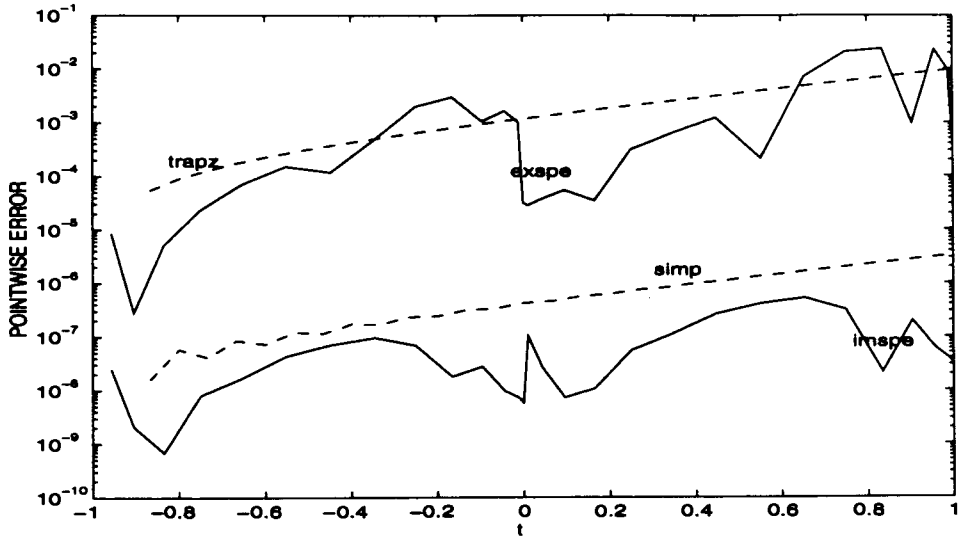


Figure 3.2: Example 2, four different methods, $M = 2$, $N = 15$

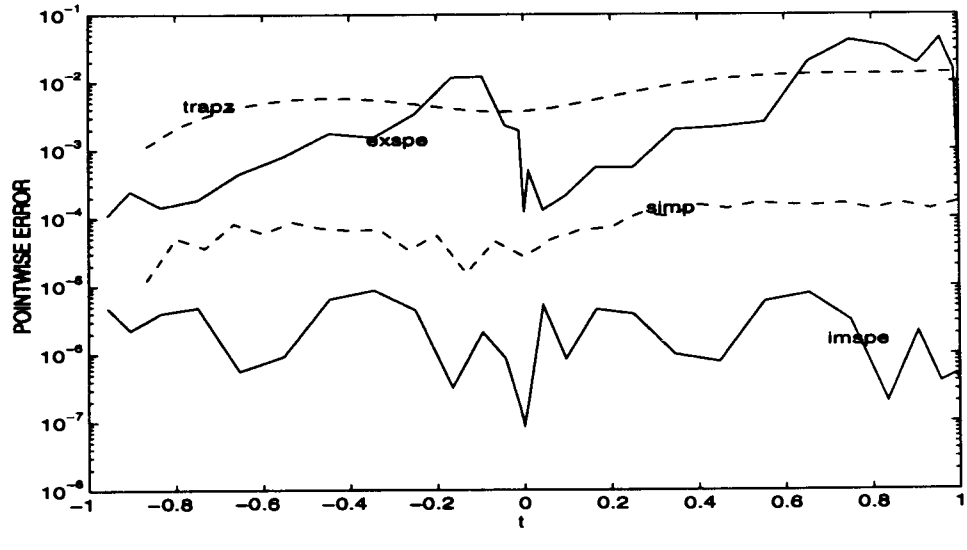


Figure 3.3: Example 3, four different methods, $M = 2$, $N = 15$

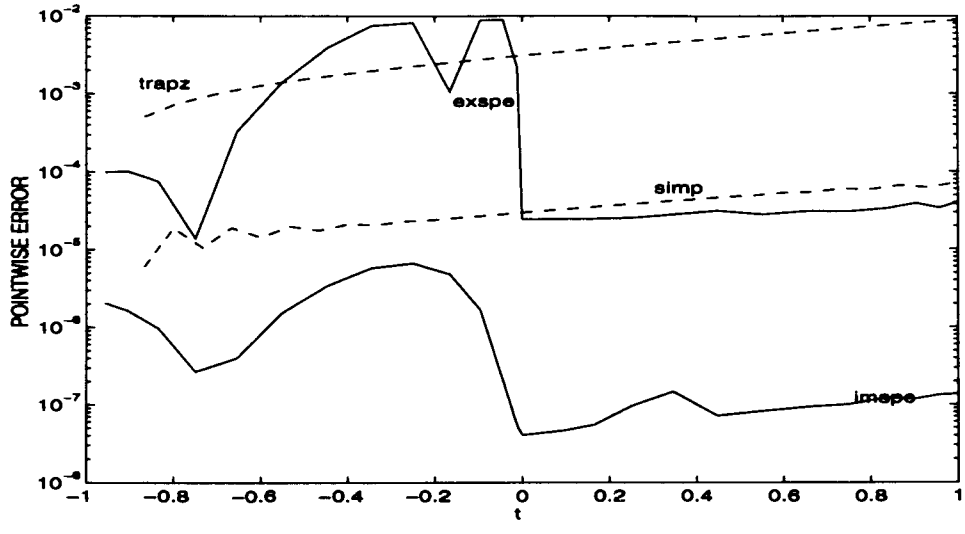


Figure 3.4: Example 1, the error with the explicit spectral element method and the trapezoidal rule, $N = 10$

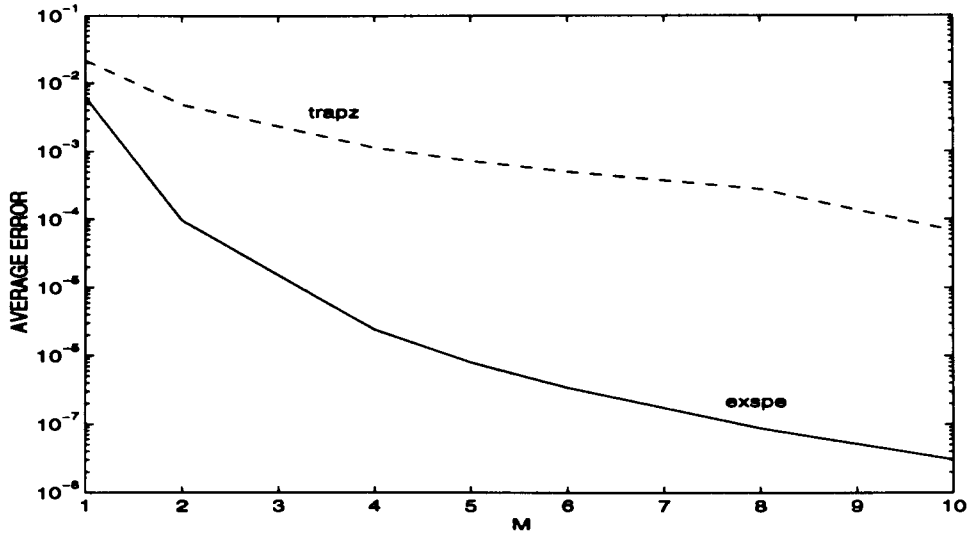
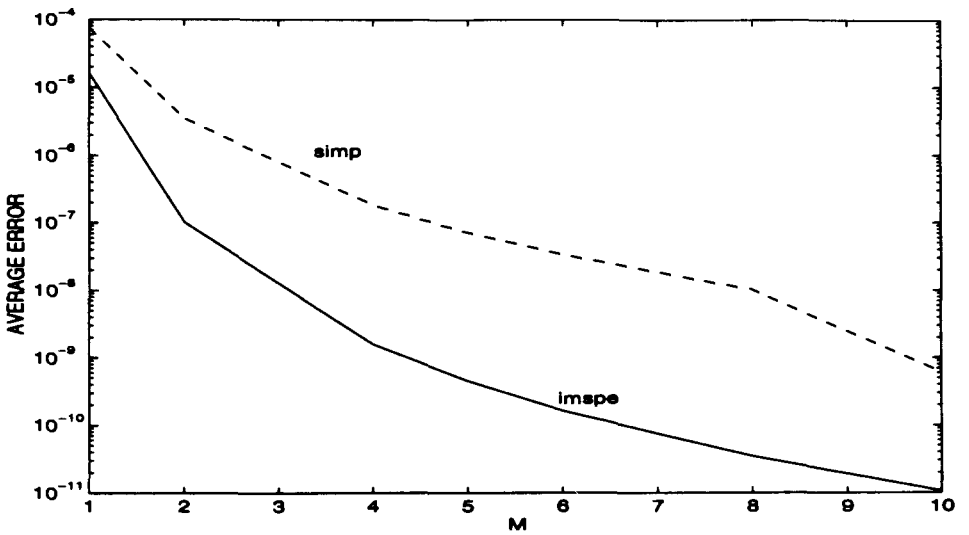


Figure 3.5: Example 1, the error with the implicit spectral element method and Simpson' method II, $N = 10$



3.4 Spectral Element Methods for Stiff Problems

In this section, we will solve some stiff problems in Volterra integral equations using the implicit spectral element method and Simpson's method II. These two schemes are more accurate than the explicit spectral element method and the trapezoidal rule, respectively.

The terms stiff and stiffness are defined in many different ways in the literature and the concept is often at best a fuzzy one. Stiffness is not necessarily a property of a differential equation, rather it is determined by a number of factors including the initial value, the norm and error control used.

To investigate the concept of stiffness, one usually refers to the ODE system of the form

$$\mathbf{y}' = \mathbf{f}(\mathbf{x}, \mathbf{y})$$

which is defined on the interval $[a, b]$. It is said to be stiff in a neighborhood of a solution $\mathbf{y}$ if there exists a component of $\mathbf{y}$ whose variation is large compared to

$|b-a|^{-1}$.

Usually, we use a numerical scheme to solve this kind of ODE, but very small mesh size is required. We can show this through the initial value problem

$$\mathbf{y}' = -\mathbf{A}\mathbf{y}, \qquad 0 \le t \le t^*, \tag{3.43}$$

$$\mathbf{y}(0) = \mathbf{y}_0.$$

Here $\mathbf{y}$ is an m-vector and $\mathbf{A}$ is a constant $m \times m$ matrix. Corresponding to the increment $h > 0$, we introduce the mesh points $t_n = nh$, $n = 0, 1, \cdots$. The solution

$$\mathbf{y}_n = \mathbf{y}(t_n),$$

of (3.43) obeys the recurrence relation,

$$\mathbf{y}_{n+1} = e^{-\mathbf{A}h}\mathbf{y}_n. \tag{3.44}$$

For convenience we introduce the function $S(z) = e^{-z}$, and we rewrite (3.44) as

$$\mathbf{y}_{n+1} = S(\mathbf{A}h)\mathbf{y}_n. \tag{3.45}$$

The simplest numerical procedure for determining an approximation $\mathbf{u}_n$ to $\mathbf{y}_n$, $n = 1, 2, \cdots$, is furnished by Euler's method,

$$\mathbf{u}_{n+1} - \mathbf{u}_n = -h\mathbf{A}\mathbf{u}_n, \qquad n = 1, 2, \cdots, \tag{3.46}$$

$$\mathbf{u}_0 = \mathbf{y}_0.$$

Using the function $K(z) = 1 - z$, we may rewrite (3.46) as

$$\mathbf{u}_{n+1} = K(\mathbf{A}h)\mathbf{u}_n. \tag{3.47}$$

Assume λ_j, $j = 1, \cdots, m$ are the distinct eigenvalues of matrix $\mathbf{A}$, then for Euler's method we obtain stability (see, e.g. [16]) if

$$|1 - h\lambda_j| \le 1, \qquad j = 1, \cdots, m. \tag{3.48}$$

Clearly, it is the stiff problem if $|\lambda_{max}|$ is enormous, so that either the stability or the error bound or both can only be assured by unreasonable restriction on h (i.e.,

an expressively small h requiring too many steps to solve the initial value problem). By the enormous means, it is meant enormous relative to a scale (which here is $1/t^*$) (see, e.g. [1, 16, 17]).

In a similar way, we can define the stiffness of Volterra integral equations. A Volterra integral equation defined on the interval $[a, b]$ is said to be stiff in a neighborhood of a solution y if the absolute value of y' is large compared to $|b-a|^{-1}$.

Example 4. The equation

$$f(t) = e^{\omega t} + (t+1)/\omega e^{\omega} - (e^{\omega t} - e^{-\omega})/\omega^2 + \int_{-1}^{t} (t-s) f(s) ds$$

has exact solution

$$f(t) = e^{\omega t}, \tag{3.49}$$

where ω is a given constant.

In Example 4, $f'(t) = \omega e^{\omega t}$, which indicates that when ω is large the solution is to be very stiff, especially at the right end point $t = 1$.

In Figures 3.6 to 3.8 we plotted the pointwise errors with the implicit spectral element method (the solid lines) and Simpson's methods (the dotted lines) using $M = 2$, $N = 20$ for different values of ω ($\omega = 1$, 6, and 10). It is obvious that in the case of $\omega = 1$ the implicit spectral element method is more accurate than Simpson's method II. It is also observed that when ω is large ($\omega = 6$ and $\omega = 10$) the implicit spectral element method is much more accurate than Simpson's method II. The figures also indicate that the implicit spectral element method is much more accurate than Simpson's method II at the endpoints of each element ($t = 0$, $t = 1$).

Figure 3.6: Example 4, Example 4 with Simpson's method II and the implicit spectral element method, $M = 2$, $N = 20$, $\omega = 1$

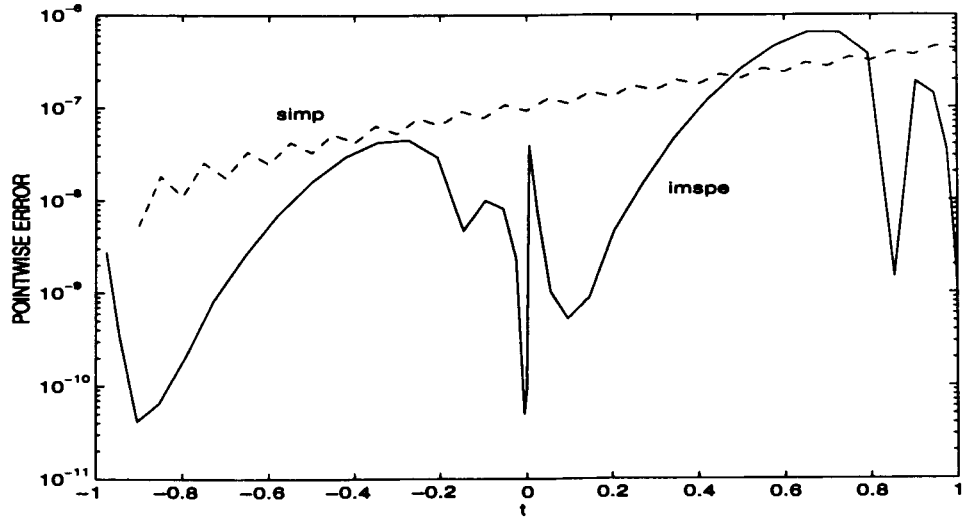


Figure 3.7: Example 4, Example 4 with Simpson's method II and the implicit spectral element method, $M = 2$, $N = 20$, $\omega = 6$

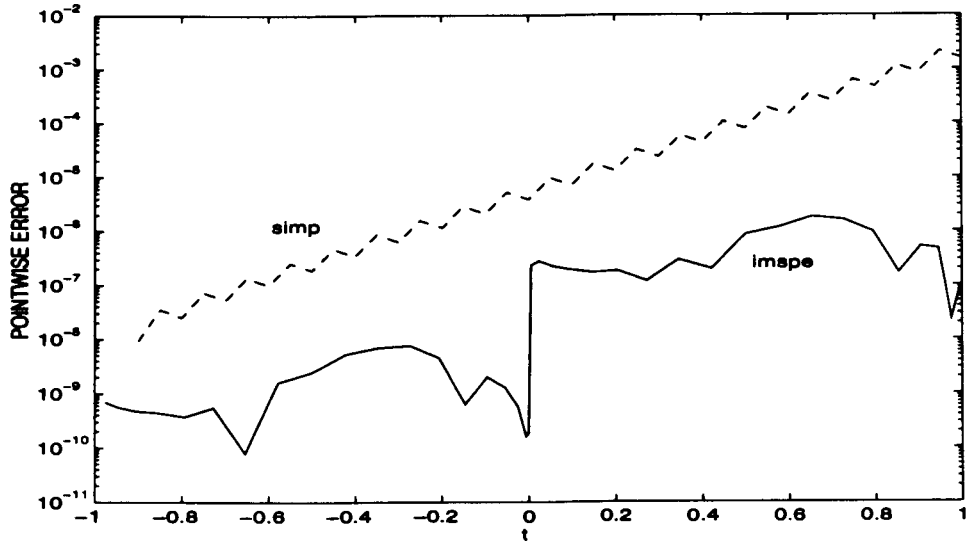
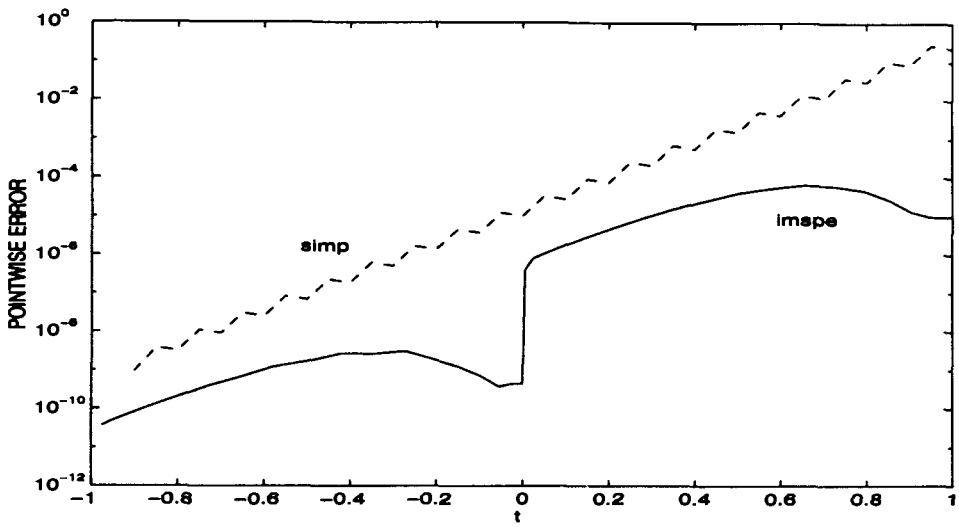


Figure 3.8: Example 4, Example 4 with Simpson's method II and the implicit spectral element method, $M = 2$, $N = 20$, $\omega = 10$



3.5 Spectral Element Methods for Highly Oscillatory Problems

In this section, we will discuss some highly oscillatory problems in Volterra integral equations. Usually this kind of problems is referred to a trigonometry form (see, e.g. [10, 16, 17]).

Example 2 Revisited. The exact solution in Example 2 is $\sin(\omega x)$ which is oscillatory in the domain. We also note that the exact solution is to be highly oscillatory when ω is large.

Figure 3.9: Example 2, the oscillatory problem with Simpson's method II and the implicit spectral element method, $M = 2$, $N = 15$, $\omega = 1$

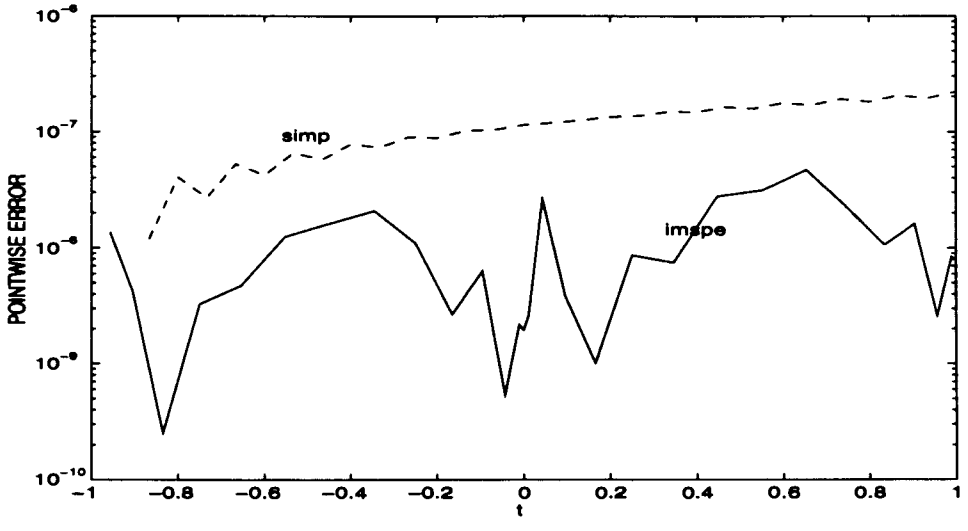


Figure 3.10: Example 2, the oscillatory problem with Simpson's method II and the implicit spectral element method, $M = 3$, $N = 15$, $\omega = 1$

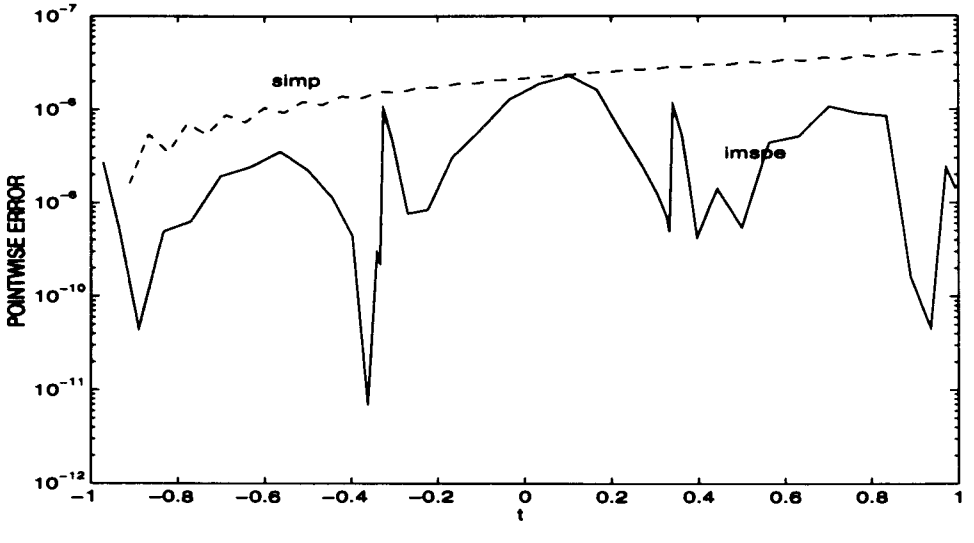


Figure 3.11: Example 2, the oscillatory problem with Simpson's method II and the implicit spectral element method, $M = 2$, $N = 15$, $\omega = 6$

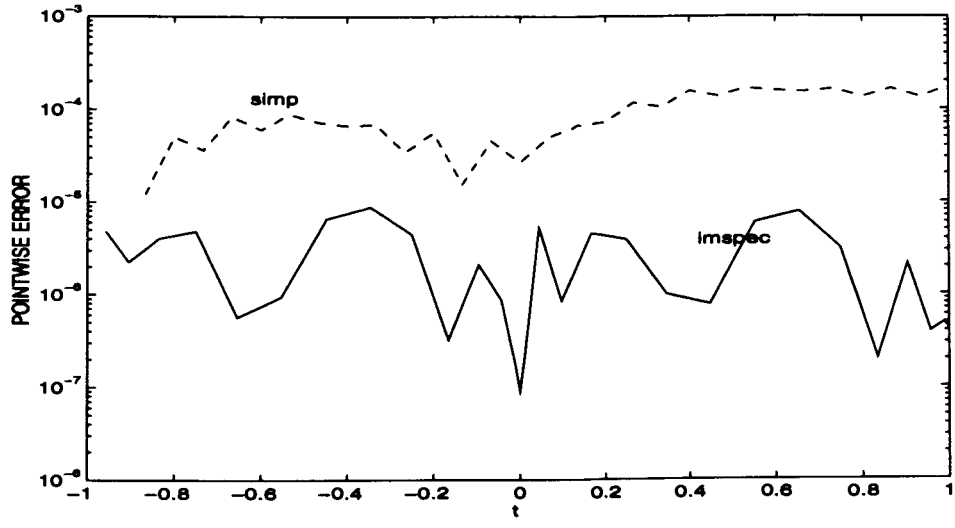
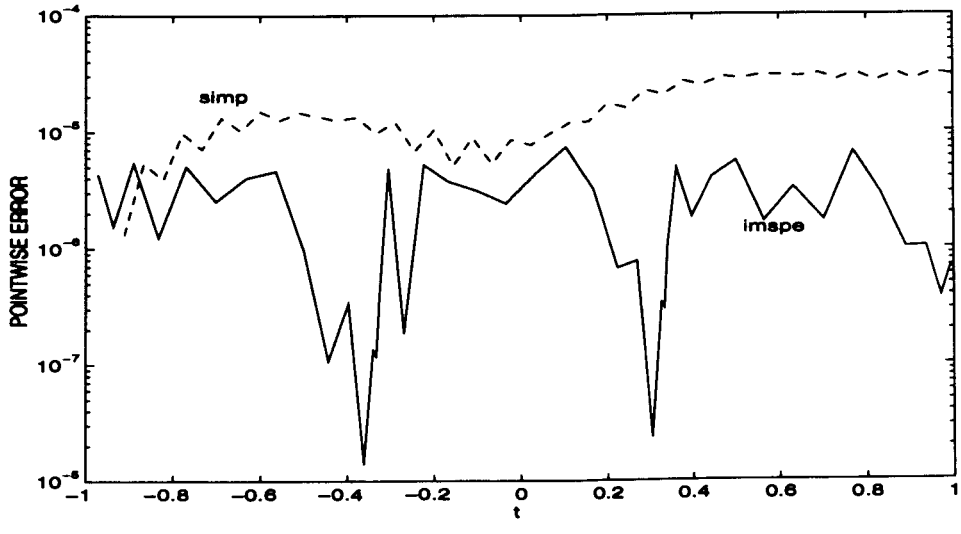


Figure 3.12: Example 2, the oscillatory problem with Simpson's method II and the implicit spectral element method, $M = 3$, $N = 15$, $\omega = 6$



In Figures 3.9 and 3.10 we plotted pointwise errors for Example 2 with the implicit spectral element method (the solid lines) and Simpson's method II (the dotted ines) using $\omega = 1$, $N = 15$ and different values of M ($M = 2$, and 3). In Figures 3.11 and 3.12 the pointwise errors are also plotted for Example 2 with $\omega = 6$, $N = 15$ and different values of M ($M = 2$, and 3). These figures indicate that the implicit spectral element method is more accurate in solving the highly oscillatory problems in Volterra integral equations than Simpson's method II. It is also shown that the implicit spectral element method is much more accurate than Simpson's method II at the endpoints of each element.

Chapter 4

Conclusions

In this chapter we summarize our overall findings obtained during the course of this work. We present some of the difficulties encountered, along with suggested remedies, and some directions towards further research.

In this thesis, we applied the idea of spectral methods to solve the second kind Volterra integral equations on a single interval. The integrand in (2.2) was expanded by Chebyshev series, and the integration of the basis functions of Chebyshev polynomials was found analytically. Furthermore, we derived a reasonably efficient method to obtain the weights used in our spectral methods. It is found that when the number of Chebyshev-Gauss-Labatto points is too large the spectral methods introduced in Chapter 3 diverge, which is not the case for boundary value problems. As mentioned in Chapter 2 Volterra equations are similar to initial value problems and the corresponding numerical methods are not global methods. This suggests that one element is not suitable to obtain accurate solutions.

The explicit and implicit spectral element methods were derived in Chapter 2. We tested four different examples by using the spectral element methods, the trapezoidal rule and Simpson's methods. It is observed that both the explicit and implicit spectral element methods converge. It is also shown that the most accurate scheme is the implicit spectral element method and the least accurate one is the trapezoidal rule. The errors with the implicit spectral element method are much smaller than those with Simpson's methods at the endpoints of each element. Moreover, in dealing with

the stiff and oscillatory problems the implicit spectral element method is much more accurate than Simpson's methods.

Some possible extensions of this research are given below.

- Possible application to Fredholm integral equations.
- Extension to the first kind Volterra integral equations.
- Extension to weakly singular Volterra integral equations.
- Extension to integro-differential equations.

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