DENSITIES AND SUMMABILITY

by

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Densities and Summability

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ABSTRACT

The concept of density of a set of positive integers is introduced along with some of the basic properties. A very general density is defined in terms of a sequence of non-negative regular matrices and two filters. It is shown that most of the known densities, i.e., matrix method densities, 0-1 densities, uniform density and some complete densities are subsumed under the general formulation.

The class of sets of upper density zero are called zeroclasses. Special zeroclasses are studied, in particular zeroclasses consisting of lacunary sets. Some surprising inclusions between some of these are proved.

An R-type summability method (RSM), S, is a regular linear functional on a real sequence space $c_s$ such that $|c_s|^0$, the set of all sequences which are $s$-strongly summable to zero, is a solid subspace of $c_s$. It is shown that $s$ is non-negative and continuous. A Bounded Consistency type theorem for the strong convergence fields of RSMs is proved. RSMs and non-negative regular summabilities are compared and interesting matrix methods are examined. Progress is made regarding the characterization of RSMs in terms of densities and zeroclasses.
DEDICATION

This thesis is dedicated to my father.
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CHAPTER 0

INTRODUCTION

In this thesis we shall discuss three aspects of sequence space theory. The first is general density theory, the second is the concept of zeroclasses consisting of lacunary sets and, lastly the theory of strong summability with respect to R-type summability methods. Most of our work extends that of Freedman and Sember [3], [4], [8].

In Chapter I we define a general set function $d = \Delta_{M,G,F}(\mathcal{A})$ where $\mathcal{A}$ is a subset of $\mathbb{I}$, the set of positive integers. $M$ is a certain type of sequence of infinite matrices and $G$ and $F$ are filters on $\mathbb{I}$. It turns out that $d$ is a density (see Definition 1.2) on $\mathbb{I}$. By judicious choices of $M, G$ and $F$, $d$ can become identical with any of densities considered in the literature: e.g., matrix method densities, 0-1 densities, uniform density and complete densities.

Chapter II comprises combinatorial results concerning classes of lacunary sets. Such classes arise naturally in some sequence space and combinatorial studies. For example, Freedman [5] has shown that the class $L$ of lacunary sets corresponds exactly to the set of 0-1 sequences in the space $bs + c$ where $bs$ is the space of sequences with bounded partial sums and $c$ is the space of all convergent sequences. More recently, Brown and Freedman [2], have shown that the famous conjecture of Erdős (that the set $A \subseteq \mathbb{I}$ has arbitrarily long arithmetic progressions whenever $\Sigma \frac{1}{a} : a \in A = \infty$)
is true if and only if it is true for every lucunary set and that the conjecture does indeed hold when $A$ is an $L_1$-lucunary set (see definition 2.1). In Chapter II we introduce many subclasses of lucunary sets, show which of them are full (see definition 2.4), [8] and relate them to one another by means of set theoretic inclusions of which some are surprising.

In Chapter III we carry on the investigation of $R$-type summability methods (RSMs) introduced by Freedman and Sember [3]. The connection between RSMs and densities is made clear through the use of an analytic definition based on the concept of zero class. Our efforts culminate with a somewhat surprising result which amounts to bounded consistency for RSMs on the associated strong summability field. The result does not require that the RSM be generated by a regular matrix and so is in a sense not comparable to the traditional bounded consistency theorem (BCT). On the other hand, since it applies only to the strong convergence field, it does not require the powerful analytic machinery for its proof as does the traditional BCT.

Finally, Chapter III attempts to add to the knowledge of RSMs that are generated by regular matrices. This, as the reader will see, is a difficult topic.

Many of the propositions are well known but we have not bothered to cite sources.

Our special notation will be introduced as needed. The notation used in set theoretic or sequence space discussions is all standard. A list of symbols and their definitions can be found in the Appendix.
CHAPTER I

GENERAL DENSITIES

In this Chapter, a general concept of density is defined. In particular, density is defined in terms of a sequence of non-negative regular matrices and two filters. Many of the standard densities will be subsumed under our definition. These include ordinary asymptotic density, uniform density, non-negative regular matrix densities and 0-1 densities defined by zero-classes. [3], [9]

Definition 1.1. Two subsets, A and B of I are asymptotic if \( A \Delta B \) is finite. \( A \Delta B \) means the symmetric difference of A and B. In this case we write \( A \sim B \).

Definition 1.2. [3] A function \( \delta: 2^I \to \mathbb{R} \) is called a lower asymptotic density (or just a density) if the following five axioms hold:

(D1) for each \( A \in 2^I \), \( 0 \leq \delta(A) \leq 1 \);

(D2) if \( A \sim B \), then \( \delta(A) = \delta(B) \);

(D3) if \( A \cap B = \emptyset \), then \( \delta(A) + \delta(B) \leq \delta(A \cup B) \);

(D4) for all \( A, B \), \( \delta(A) + \delta(B) \leq 1 + \delta(A \cap B) \);

(D5) \( \delta(I) = 1 \).

Definition 1.3. If \( \delta \) is a density, we define \( \bar{\delta}: 2^I \to \mathbb{R} \), the upper density associated with \( \delta \), by \( \bar{\delta}(A) = 1 - \delta(A^c) \) where \( A^c = I - A \).
At first, we will list some basic properties of $\delta$ and $\overline{\delta}$ omitting most of the proofs since the verifications involve only simple arguments and most appear in [3].

**Proposition 1.4.** Let $\delta$ be a lower asymptotic density and $\overline{\delta}$ its associated upper density. For $A, B \in 2^I$, we have

(i) $A \subset B \Rightarrow \delta(A) \leq \delta(B)$,
(ii) $A \subset B \Rightarrow \overline{\delta}(A) \leq \overline{\delta}(B)$,
(iii) $\delta(\emptyset) = \overline{\delta}(\emptyset) = 0$,
(iv) $A \cap B = \emptyset \Rightarrow \delta(A \cup B) \leq \delta(A) + \overline{\delta}(B)$,
(v) $A \sim B \Rightarrow \delta(A) = \overline{\delta}(B)$,
(vi) For all $A$, $\delta(A) \leq \overline{\delta}(A)$,
(vii) For all $A, B$, $\delta(A) + \overline{\delta}(B) \geq \overline{\delta}(A \cup B)$,
(viii) $A \cup B = I \Rightarrow \delta(A) + \overline{\delta}(B) \geq 1 + \overline{\delta}(A \cap B)$.

**Proof:** (iv) Suppose that $A \cap B = \emptyset$. By (D3) $\delta(A \cup B) + \delta(B^C) \leq 1 + \delta((A \cup B) \cap B^C)$. On the other hand, $A \cap B = \emptyset$ implies $(A \cup B) \cap B^C = A$. Thus $\delta(A \cup B) + \delta(B^C) \leq 1 + \delta(A)$. Therefore we have $\delta(A \cup B) \leq \delta(A) + 1 - \delta(B^C) = \delta(A) + \overline{\delta}(B)$.

(vii) Suppose that $A \cup B = I$ and so we get $A^C \cap B^C = \emptyset$. By (D2) $\delta(A^C) + \delta(B^C) \leq \delta(A^C \cup B^C)$. Therefore $1 - \delta(A^C) + 1 - \delta(B^C) \geq 1 + 1 - \delta(A^C \cup B^C)$. Thus we get $\delta(A) + \overline{\delta}(B) \geq 1 + \overline{\delta}(A \cap B)$.

**Definition 1.5.** Let $\delta$ and $\overline{\delta}$ be associated lower and upper densities. We define
\[ \eta_\delta = \{ A \subset I : \delta(A) = \overline{\delta}(A) \} , \]
\[ \eta_\delta^0 = \{ A \subset I : \overline{\delta}(A) = 0 \} . \]

We say that \( A \subset I \) has natural density (resp. has natural density zero) with respect to \( \delta \) in case \( A \in \eta_\delta \) (resp. \( A \in \eta_\delta^0 \)).

Note that \( A \in \eta_\delta \) and \( \overline{\delta}(A) = \delta(A) = 0 \) if and only if \( \overline{\delta}(A) = 0 \).

The basic facts concerning \( \eta_\delta \) and \( \eta_\delta^0 \) are contained in the following proposition. We omit the proofs.

**Proposition 1.6.** For any \( A,B \in 2^I \).

(i) \( A \sim I = A \notin \eta_\delta \),

(ii) \( A \sim \phi = A \notin \eta_\delta^0 \),

(iii) \( A \in \eta_\delta \) and \( A \sim B = B \notin \eta_\delta \).

**Definition 1.7.** A class \( X \) of subsets of \( I \) will be called a zeroclass \([4]\) if the following conditions hold:

(i) \( A \) is finite \( \Rightarrow A \in X \),

(ii) \( A,B \in X \Rightarrow A \cup B \in X \),

(iii) \( A \subset B \in X \Rightarrow A \in X \),

(iv) \( I \notin X \).

Note that a zeroclass is just a non principal ideal on \( 2^I \).

**Proposition 1.8.** If \( \delta : 2^I \to \mathbb{R} \) is a lower asymptotic density then \( \eta_\delta^0 \) is a zeroclass.
Proof: (i) If \( A \) is finite, then \( A \sim \phi \) and so \( \phi \in \eta_0^0 \).

(ii) Let \( A, B \in \eta_0^0 \) so that \( \bar{\delta}(A) = 0 \) and \( \bar{\delta}(B) = 0 \). By proposition 1.4, (vii) \( \bar{\delta}(A \cup B) \leq \bar{\delta}(A) + \bar{\delta}(B) = 0 \). Therefore \( \bar{\delta}(A \cup B) = 0 \). Hence \( A \cup B \in \eta_0^0 \).

(iii) If \( A \subseteq B \in \eta_0^0 \), then \( 0 \leq \bar{\delta}(A) \leq \bar{\delta}(B) = 0 \). Thus \( \bar{\delta}(A) = 0 \) and so \( A \in \eta_0^0 \).

(iv) Since \( \bar{\delta}(I) = 1 - \bar{\delta}(\phi) = 1 \), we have \( I \notin \eta_0^0 \).

\[ \text{Definition 1.9.} \] A filter on a set \( X \) is a family \( F \) of subsets of \( X \) which has the following properties:

(i) \( A, B \in F \Rightarrow A \cap B \in F \),

(ii) \( A \subset B \) and \( A \in F \Rightarrow B \in F \),

(iii) \( \phi \notin F \).

\[ \text{Definition 1.10.} \] Let \( F_0 = \{ A \in 2^I : A^c \text{ is finite} \} \). Then \( F_0 \) is called the Fréchet filter.

Remark: If \( X \) is a zeroclass then \( F_\chi = \{ A^c : A \in X \} \) is a filter finer than the Fréchet filter, i.e., \( F_\chi \succ F_0 \).

In order to introduce a particular method of constructing densities, we will first present several lemmas. In these lemmas, \( G \) and \( F \) will be filters on \( I \).

\[ \text{Lemma 1.11:} \] For any positive integer \( m, n \) let \( P(m,n) \) and \( Q(m,n) \) be corresponding statements such that for each \( m, n, \)

\[ P(m,n) = Q(m,n). \]
Then we have

(i) for each \( n \)

\[
\{m: \text{P}(m,n) \text{ is true}\} \in G
\]

\[
= \{m: \text{Q}(m,n) \text{ is true}\} \in G .
\]

(ii) \( \{n: \{m: \text{P}(m,n) \text{ is true}\} \in G\} \in F \)

\[
= \{n: \{m: \text{Q}(m,n) \text{ is true}\} \in G\} \in F .
\]

Proof: (i) For each \( n \), by the hypothesis,

\[
\{m: \text{P}(m,n) \text{ is true}\} \subseteq \{m: \text{Q}(m,n) \text{ is true}\} .
\]

Since \( G \) is a filter, we have the result.

(ii) By (i), \( \{n: \{m: \text{P}(m,n) \text{ is true}\} \in G\} \subseteq \{n: \{m: \text{Q}(m,n) \text{ is true}\} \in G\} .
\]

Since \( F \) is a filter, we have the result.

**Corollary 1.12.** For any \( m,n \in I \), let \( S(m,n) \) and \( T(m,n) \) be corresponding real numbers with \( S(m,n) \leq T(m,n) \). Then for any \( \alpha \in \mathbb{R} \), we have

(i) for each \( n \in I \),

\[
\{m: \alpha \leq S(m,n)\} \in G = \{m: \alpha \leq T(m,n)\} \in G .
\]

(ii) \( \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F \)

\[
= \{n: \{m: \alpha \leq T(m,n)\} \in G\} \in F .
\]

Proof: Take \( \text{P}(m,n) \) to mean \( \alpha \leq S(m,n) \) and \( \text{Q}(m,n) \) to mean \( \alpha \leq T(m,n) \). By Lemma 1.11 (i) and (ii) hold.

**Lemma 1.13.** For any \( m,n \in I \), let \( \text{P}(m,n) \), \( \text{Q}(m,n) \) and \( S(m,n) \) be corresponding statements. Suppose that for each \( m,n \in I \),
\[(P(m,n) \text{ and } Q(m,n)) = S(m,n).\]

Then

(i) for each \( n \), if \( \{m: P(m,n) \text{ is true}\} \in G \) and \\( \{m: Q(m,n) \text{ is true}\} \in G \) then \( \{m: S(m,n) \text{ is true}\} \in G \).

(ii) if \( \{n: \{m: P(m,n) \text{ is true}\} \in G\} \in F \) and \\( \{n: \{m: Q(m,n) \text{ is true}\} \in G\} \in F \) then \( \{n: \{m: S(m,n) \text{ is true}\} \in G\} \in F \).

Proof: (i) For each fixed \( n \), by the hypothesis,
\[\{m: P(m,n) \text{ is true}\} \cap \{m: Q(m,n) \text{ is true}\} \subset \{m: S(m,n) \text{ is true}\}.\] (1)

Suppose that \( \{m: P(m,n) \text{ is true}\} \in G \) and \\( \{m: Q(m,n) \text{ is true}\} \in G \). Since \( G \) is a filter \( \{m: P(m,n) \text{ is true}\} \cap \{m: Q(m,n) \text{ is true}\} \subset \{m: S(m,n) \text{ is true}\} \in G \).

(ii) For each \( n \), let
\[P_{1}(n) \equiv \{m: P(m,n) \text{ is true}\} \in G,
\]
\[Q_{1}(n) \equiv \{m: Q(m,n) \text{ is true}\} \in G, \text{ and}
\]
\[S_{1}(n) \equiv \{m: S(m,n) \text{ is true}\} \in G.
\]

By the proof of (i), for each \( n \in I \), \( (P_{1}(n) \text{ and } Q_{1}(n)) = S_{1}(n) \).

Since \( F \) is a filter, we can apply (i). Thus, we have
\[\{n: P_{1}(n) \text{ is true}\} \in F \text{ and } \{n: Q_{1}(n) \text{ is true}\} \in F \]
\[= \{n: S_{1}(n) \text{ is true}\} \in F.\]

Hence the proof of (ii) is completed.
Corollary 1.14. For each \( m, n \in I \), let \( S(m,n) \) and \( T(m,n) \) be corresponding real numbers and \( \alpha \in R \). Then we have

(i) For each \( n \), if \( \{ m: \alpha \leq S(m,n) \} \in G \), and \( \{ m: \beta \leq T(m,n) \} \in G \), then \( \{ m: \alpha + \beta \leq S(m,n) + T(m,n) \} \in G \).

(ii) If \( \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \) and \( \{ n: \{ m: \beta \leq T(m,n) \} \in G \} \in F \), then \( \{ n: \{ m: \alpha + \beta \leq S(m,n) + T(m,n) \} \in G \} \in F \).

Proof: Let \( P(m,n) \equiv \"\alpha \leq S(m,n)\" \), \( Q(m,n) \equiv \"\beta \leq T(m,n)\" \) and \( S(m,n) \equiv \"\alpha + \beta \leq S(m,n) + T(m,n)\" \). By the lemma 1.13, we get the results (i) and (ii).

Lemma 1.15. Let \( F \) be a filter on \( I \) which is finer than the Fréchet filter, \( F_0 \). Then for any \( A \in 2^I \) and for any \( N \in I \), \( A \in F \) if \( A \cap J_N^C \in F \), where \( J_N = \{1,2,3,\ldots,N\} \).

Proof: Since \( F \) is finer than \( F_0 \), for any \( N \in I \), \( J_N^C \in F \). Suppose that \( A \in F \). Since any intersection of two members of a filter is also a member, \( A \cap J_N^C \in F \). Conversely suppose that \( A \cap J_N^C \in F \). Any superset of a member of a filter is also a member. Thus \( A \in F \).

Lemma 1.16. For any \( n,m \in I \), let \( S(m,n) \) and \( T(m,n) \) be corresponding real numbers with the property that there exists \( N \in I \) such that \( n > N \Rightarrow S(m,n) \leq T(m,n) \) for all \( m \). Suppose that \( F \) is a filter finer than \( F_0 \). Then

\[
\sup\{\alpha: \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \} \\
\leq \sup\{\alpha: \{ n: \{ m: \alpha \leq T(m,n) \} \in G \} \in F \}.
\]

Proof: Since for any \( n > N \) and for any \( m \in I \), \( S(m,n) \leq T(m,n) \).
By Corollary 1.12(i), for any \( n > N \) and for any real number \( \alpha \), we have

\[
\{m: \alpha \leq S(m,n)\} \in G \Rightarrow \{m: \alpha \leq T(m,n)\} \in G.
\]

Thus we have

\[
\{n: \{m: \alpha \leq S(m,n)\} \in G\} \cap J_N^c \subset \{n: \{m: \alpha \leq T(m,n)\} \in G\} \cap J_N^c.
\]

Since \( F \) is a filter,

\[
\{n: \{m: \alpha \leq S(m,n)\} \in G\} \cap J_N^c \in F \quad (A)
\]

By Lemma 1.15, since \( F \) is finer than \( F_0 \), (A) is logically equivalent to

\[
\{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F
\]

\[
\Rightarrow \{n: \{m: \alpha \leq T(m,n)\} \in G\} \in F.
\]

Thus we have,

\[
\{\alpha: \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F\}
\]

\[
\subset \{\alpha: \{n: \{m: \alpha \leq T(m,n)\} \in G\} \in F\}.
\]

Hence

\[
\sup\{\alpha: \{n: \{m: \alpha \leq S(m,n)\} \in G\} \in F\}
\]

\[
\leq \sup\{\alpha: \{n: \{m: \alpha \leq T(m,n)\} \in G\} \in F\}.
\]
Lemma 1.17. Let $G$ and $F$ be filters and $t \in \mathbb{R}$ then we have

(i) $\sup \{a: \{n: a \leq t\} \in F\} = t$.

(ii) $\sup \{a: \{m: a \leq t\} \in G\} \in F = t$.

Proof: (i) For each $n$, we have

$$\{n: a \leq t\} = \begin{cases} I & \text{if } a \leq t \\ \emptyset & \text{if } a > t. \end{cases}$$

Thus $\{a: \{n: a \leq t\} \in F\} = \{a: a \leq t\}$. Hence $\sup \{a: \{n: a \leq t\} \in F\} = t$.

(ii) If $\beta \leq t$, then $\{m: \beta \leq t\} = I \in G$ and so,

$$\{n: \{m: a \leq t\} \in G\} = I \in F.$$

If $\beta > t$, then $\{m: \beta \leq t\} = \emptyset \notin G$ and so, $\{n: \{m: \beta \leq t\} \in G\} = \emptyset \notin F$. Therefore $\{a: a \leq t\} = \{a: \{n: \{m: a \leq t\} \in G\} \in F\}$. Thus we have the result (ii).

Definition 1.18. Let $x = (x_n) \in \omega$ (the space of all real sequences) and let $A = (a_{nk})$, $n,k = 1,2,3,...$ be an infinite real matrix. Then the product $Ax$ denote the sequence $(y_i)$, if it exists, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. We denote $(Ax)_i = y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. We also define $c_A = \{x \in \omega: Ax \in c\}$. In Chapter 3, we will write $C_A$ for $c_A$.

Definition 1.19. An infinite matrix $A$ is called regular if $c \subseteq c_A$ and for any $x \in c$, $\lim_{i} x_i = \lim_{i} (Ax)_i$.
Let us state the well-known Silverman-Toeplitz Theorem [6] without proving it. Furthermore

**Proposition 1.20.** An infinite matrix \( A \) is regular if and only if

1. \( \sup \sum_i \left| a_{ij} \right| < \infty \),
2. \( \lim_{i} a_{ij} = 0 \) for \( j = 1, 2, 3, \ldots \),
3. \( \lim_{i} \sum_j a_{ij} = 1 \).

**Definition 1.21.** A matrix \( A = (a_{ij}) \) is called nonnegative if \( a_{ij} \geq 0 \) for any \( i, j = 1, 2, 3, \ldots \).

Note that, for any matrix \( A \), \( c_A \) is a linear subspace of \( \omega \).

For any \( x, y \in c_A \) and \( \alpha \in \mathbb{R} \), \( A(x+y) = Ax + Ay \) and \( A(\alpha x) = \alpha Ax \).

The following proposition introduces our general method of constructing densities.

**Proposition 1.22.** Let \( M = \{M_m\} \) be a sequence of nonnegative regular matrices. Let us denote \( M_m = (a_{ik}^m) \). Suppose that the following uniformity conditions hold.

1. For any \( \varepsilon > 0 \), there exists \( N \) such that \( n > N \) implies
   \[
   1 - \varepsilon \leq \sum_{k=1}^{\infty} a_{nk}^m \leq 1 + \varepsilon \quad \text{for all } m.
   \]
   i.e., \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}^m = 1 \) uniformly in \( m \).
For any \( \varepsilon > 0 \) and for any \( s \in I \), there exists \( N \) such that \( n > N \) implies \( \sum_{n_1}^{m} a_{n_1} + \sum_{n_2}^{m} a_{n_2} + \ldots + \sum_{n_s}^{m} a_{n_s} < \varepsilon \), for all \( m \).

Suppose further that \( F \) is a filter finer than \( F_0 \) and \( G \) is any filter. Let \( d_{M,G,F}(A) = \sup\{\alpha: \{m: \alpha \leq (M A) \in G \in F\} \in F\} \), where \( X_A(n) (=1 \text{ if } n \in A, =0 \text{ otherwise}) \) is the characteristic sequence of \( A \). Then \( d_{M,G,F} \) is a lower asymptotic density.

**Proof:** In this proof, we denote \( d_{M,G,F}(A) \) by \( d(A) \). For any \( A \in 2^I \), \( (M A) n = \sum_{j=1}^{\infty} a_{n j} X_A(j) \leq \sum_{j=1}^{\infty} a_{n j} \). Since \( M_m \) is a non-negative regular matrix \( \sum_{j=1}^{\infty} a_{n j} \) converges, thus \( (M A) n = \sum_{j=1}^{\infty} a_{n j} X_A(j) \) is a real number. Hence our definition of \( d(A) \) is well defined.

Suppose that \( A \subset B \). Then \( X_A(j) \leq X_B(j) \) for all \( j \). For each \( m, n, (M A) n = \sum_{n=1}^{\infty} a_{n j} X_A(j) \leq \sum_{j=1}^{\infty} a_{n j} X_B(j) = (M B) n \). By lemma 1.6 we have

\[
\sup\{\alpha: \{m: \alpha \leq (M A) \in G \in F\} \in F\} \\
\leq \sup\{\alpha: \{m: \alpha \leq (M B) \in G \in F\} \in F\}.
\]

Hence \( d(A) \leq d(B) \).

Since \( X_\phi = 0 \), we have, for any \( m, n, (M \phi) n = 0 \). Thus by Lemma 1.17, with \( t = 0 \), \( d(\phi) = \sup\{\alpha: \{m: \alpha \leq (M \phi) \in G \in F\} \in F\} = 0 \).

Next we will show \( d(I) = 1 \). Since \( X_I = (1,1,1,\ldots) = e \),

\[
(M X_a) n = \sum_{j=1}^{\infty} a_{n j} \].

By the assumption (i) there exists \( N \) such that
\[ n > N \text{ implies } 1 - \varepsilon \leq \sum_{j=1}^{\infty} a_{nj}^m \leq 1 + \varepsilon \text{ for all } m. \]

Let \( S(m,n) = 1 - \varepsilon, \ T(m,m) = (M \chi_m) \) and \( V(m,n) = 1 + \varepsilon \). By Lemma 1.16

\[ \sup\{a: \{n: \{m: a \leq 1 - \varepsilon\} \in G\} \in F\} \]
\[ \leq \sup\{a: \{n: \{m: a \leq (M \chi_m)\} \in G\} \in F\} \]
\[ \leq \sup\{a; \{n: \{m: a \leq 1 + \varepsilon\} \in G\} \in F\}. \]

By Lemma 1.17 and the definition of \( d \), \( 1 - \varepsilon \leq d(I) \leq 1 + \varepsilon \). Since \( \varepsilon \) is arbitrary \( d(I) = 1 \). Let \( A \in 2^I \), since \( \emptyset \subset A \subset I \) we have, \( 0 = d(\emptyset) \leq d(A) \leq d(I) = 1 \).

Let \( A \in 2^I \), and \( L \) be a positive integer. Then

\[ M^{\chi_{A \cup J_L}} \leq M^{(\chi_A + \chi_{J_L})} = M^{\chi_A} + M^{\chi_{J_L}}. \]
For any \( \varepsilon > 0 \), by the assumption (ii) of the proposition there exists a positive integer \( N \) such that \( n > N = (M \chi_{J_L}) \in \sum_{j=1}^{L} a_{nj}^m < \varepsilon \) for all \( m \). Let

\[ T(m,n) = (M \chi_m) + \varepsilon, \text{ then by Lemma 1.16, we have} \]

\[ \sup\{a: \{n: \{m: a \leq (M \chi_{A \cup J_L})\} \in G\} \in F\} \]
\[ \leq \sup\{a: \{n: \{m: a \leq (M \chi_m) + \varepsilon\} \in G\} \in F\} \]
\[ = \sup\{a: \{n: \{m: a - \varepsilon \leq (M \chi_m)\} \in G\} \in F\} \]
\[ = \sup\{\beta + \varepsilon: \{n: \{m: \beta \leq (M \chi_m)\} \in G\} \in F\} \]
\[ = \varepsilon + \sup\{\beta: \{n: \{m: \beta \leq (M \chi_m)\} \in G\} \in F\}. \]
Hence \( d(A \cup J_L) \leq \varepsilon + d(A) \). Since \( \varepsilon > 0 \) is arbitrary \( d(A \cup J_L) \leq d(A) \).

Also \( d(A) \leq d(A \cup J_L) \) whence \( d(A) = d(A \cup J_L) \).

If \( A \sim B \), then \( A \cup J_L = B \cup J_L \) for some \( L \in I \).

Hence \( d(A) = d(A \cup J_L) = d(B \cup J_L) = d(B) \).

Next we prove that:

\[ A \cap B = \emptyset \Rightarrow d(A) + d(B) \leq d(A \cup B). \]

By the definition of \( d \), for any \( \varepsilon > 0 \), there exist real numbers \( \alpha \) and \( \beta \) such that

\[ d(A) - \varepsilon < \alpha, \quad d(A) - \varepsilon < \beta, \]

\[ \{n: \{m: \alpha \leq (M \chi_A)_n \} \in G\} \in F \]

and

\[ \{n: \{m: \beta \leq (M \chi_B)_n \} \in G\} \in F. \]

Since \( A \cap B = \emptyset \), we have \( \chi_A \cup B = \chi_A + \chi_B \) and so \( M \chi(A \cup B) = M \chi_A + M \chi_B \). Therefore \( (M \chi_A)_n + (M \chi_B)_n = (M \chi(A \cup B))_n \)

By Corollary 1.14

\[ \{n: \{m: \alpha + \beta \leq (M \chi_A) + (M \chi_B)_n \} \in G\} \in F \]

equivalently,

\[ \{n: \{m: \alpha + \beta \leq (M \chi(A \cup B))_n \} \in G\} \in F. \]

Therefore \( \alpha + \beta \leq d(A \cup B) \) so we have \( d(A) - \varepsilon + d(B) - \varepsilon < \alpha + \beta \leq d(A \cup B) \).

Hence \( d(A) + d(B) \leq d(A \cup B) + 2\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary

\[ d(A) + d(B) \leq d(A \cup B). \]

Finally we want to show that \( d(A) + d(B) \leq 1 + d(A \cap B) \).

For any \( \varepsilon > 0 \), there exist \( \alpha \) and \( \beta \) such that

\[
d(A) - \varepsilon \leq \alpha, \quad d(B) - \varepsilon \leq \beta,
\]

\[
\{n : \{m : \alpha \leq (M \chi_n^A)_m \} \in G\} \in F \quad \text{and}
\]

\[
\{n : \{m : \beta \leq (M \chi_n^B)_m \} \in G\} \in F.
\]

By Corollary 1.14,

\[
\{n : \{m : \alpha + \beta \leq (M \chi_n^A)_m + (M \chi_n^B)_m \} \in G\} \in F.
\]

On the other hand \( \chi_A \chi_B = \chi_A \cup B + \chi_A \cap B \leq \chi_I + \chi_A \cap B \). So that

\[
(M \chi_n^A)_n + (M \chi_n^B)_n = (M \chi_n^A + M \chi_n^B)_n = (M (\chi_n^A + \chi_n^B))_n \leq (M (\chi_n^I + \chi_n(A \cap B)))_n =
\]

\[
(M \chi_n^I)_n + (M \chi_n(A \cap B))_n. \quad \text{By Corollary 1.12(ii), we have}
\]

\[
\{n : \{m : \alpha + \beta \leq (M \chi_n^A)_m + (M \chi_n(A \cap B))_m \} \in G\} \in F.
\]

By the hypothesis (i) for any \( \varepsilon > 0 \), there exists \( N \in I \) such that

\( n > N \) implies

\[
(M \chi_n^I)_n = \sum_{j=1}^{\infty} a_n^m < 1 + \varepsilon.
\]

By Lemma 1.16, we have
\[ \alpha + \beta \leq \sup \{ \delta : \{ n : \delta \leq (M^m \chi)_n + (M^m \chi_A \cap B)_n \} \in G \} \in F \]

\[ \leq \sup \{ \delta : \{ n : \delta \leq 1 + \varepsilon + (M^m \chi_A \cap B)_n \} \in G \} \in F \]

\[ = \sup \{ \delta : \{ n : \delta - 1 + \varepsilon \leq (M^m \chi_A \cap B)_n \} \in G \} \in F \]

\[ = \sup \{ \gamma + 1 + \varepsilon : n \in \{ n : \gamma \leq (M^m \chi_A \cap B)_n \} \in G \} \in F \]

\[ = 1 + \varepsilon + \sup \{ \gamma : \{ n : \gamma \leq (M^m \chi_A \cap B)_n \} \in G \} \in F \]

\[ = 1 + \varepsilon + d(A \cap B) . \]

Hence \( d(A) - \varepsilon + d(B) - \varepsilon < \alpha + \beta \leq 1 + \varepsilon + d(A \cap B) \). Since \( \varepsilon > 0 \)
is arbitrary we get \( d(A) + d(B) \leq 1 + d(A \cap B) \).

**Proposition 1.23.** Let \( M \) be a nonnegative regular matrix
and \( M = \{ M_m \} \) where \( M_m = M \) for each \( m = 1, 2, 3, \ldots \). Let
\( G = \{ A \in 2^I : 1 \in A \} \) and \( F \) be a filter finer than \( F_0 \), then

\[ d_{M, G, F}(A) = \sup \{ \alpha : n \in \{ \alpha \leq (M \chi_A)_n \} \in F \} \]

for each \( A \in 2^I \). In this case, we write

\[ d_{M, F} = \sup \{ \alpha : n \in \{ \alpha \leq (M \chi_A)_n \} \in F \} . \]

**Proof:** Clearly conditions (i) and (ii) of Proposition 1.22 are
satisfied. Hence $d_{M,G,F}$ is a density. Since for each $A \in 2^I$, 

$$\{m: \alpha \leq (M^mA_n) \} \in G \Rightarrow 1 \in \{m: \alpha \leq (M^mA_n) \}$$

$$\Rightarrow \alpha \leq (M^A_n)$$

Hence 

$$d_{M,G,F}(A) = \sup \{\alpha: \{n: \alpha \leq (M^mA_n) \} \in G \} \in F$$

$$= \sup \{\alpha: \{n: \alpha \leq (M^A_n) \} \in F\}.$$ 

**Example 1.24.** Let $F_0$ be the Fréchet filter and $M$ a non-negative regular matrix, then $d_{M,F_0}(A) = \lim \inf_n (M^A_n)$. In this case we write $d_M(A) = \lim \inf_n (M^A_n)$, which is called a matrix method density. ([9] Definition 3.5).

Proof: Let 

$$r = d_{M,F_0}(A) = \sup \{\alpha: \{n: \alpha \leq (M^A_n) \} \in F_0 \}.$$ 

Then for any $\varepsilon > 0$, there exists $\alpha$ such that 

$$r - \varepsilon < \alpha \quad \text{and} \quad \{n: \alpha \leq (M^A_n) \} \in F_0.$$ 

Since $F_0$ is the Fréchet filter, there exists $N$ such that $n > N$ implies $\alpha \leq (M^A_n)$. Hence 

$$r - \varepsilon < \alpha \leq \lim \inf_n (M^A_n)$$

and
\[ r \leq \lim \inf_{n} (Mx_{A})_{n}. \]

Also
\[ \{n: r + \epsilon \leq (Mx_{A})_{n} \} \notin F_{0}, \]

and since \( F_{0} \) is the Fréchet filter, for infinitely many \( n \),
\[ r + \epsilon > (Mx_{A})_{n}. \]

Hence
\[ r + \epsilon \geq \lim \inf_{n} (Mx_{A})_{n}. \]

Therefore we have
\[ r = \lim \inf_{n} (Mx_{A})_{n}. \]

Example 1.25. Let
\[ M = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1/2 & 1/2 & 0 & 0 & \ldots \\ 1/3 & 1/3 & 1/3 & 0 & \ldots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \]

be the Cesáro matrix. Then
\[ d_{M}(A) = \lim \inf_{n} (Mx_{A})_{n} \]

\[ = \lim \inf_{n} \sum_{i=1}^{n} a_{ni}x_{A}(i) \]

\[ = \lim \inf_{n} \frac{x_{A}(1) + x_{A}(2) + \ldots + x_{A}(n)}{n} \]

\[ = \lim \inf_{n} \frac{A(n)}{n} \]
where \( A(n) \) is the cardinality of the set \( A \cap \{1,2,\ldots,n\} \). We write
\[
d(A) = \lim \inf_n \frac{A(n)}{n},
\]
and say it is the ordinary asymptotic density \([9],[3]\).

**Example 1.26.** Let \( M \) be the Cesáro matrix and let

\[
N = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

then \( MN^{m-1} = (a^n_m) \), where

\[
a^n_m = \begin{cases}
\frac{1}{n} & \text{if } m \leq i \leq m+n-1 \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( M_m = MN^{m-1} \), \( G = \{I\} \) and \( F_0 \) be the Fréchet filter.

Then \( d_{M,G,F_0}(A) = \lim \inf_n \min_{A[m+1,m+n]} \frac{A[m+1,m+n]}{n} \) where \( A[m+1,m+n] \) is the cardinality of the set \( A \cap \{m+1,m+2,\ldots,m+n\} \). In this case we write

\[
u(A) = \lim \inf_n \min_m A[m+1,m+n] \geq 0 \frac{A[m+1,m+n]}{n}
\]

and say that it is the uniform density \([9]\).

**Proof:** By the definition of \( d_{M,G,F_0}(A) \)
\[ d_{M,G,F}(A) = \sup \{ \alpha : \{ n : \{ m : \alpha \leq (M^n_A)_n \} \in G \} \in F_0 \} \]

\[ = \sup \{ \alpha : \{ n : \{ m : \alpha \leq (M^n_A)_n \} = I \} \in F_0 \} \]

\[ = \sup \{ \alpha : \{ m : \alpha \leq (M^n_A)_n \}, \text{ for all } m \in I \} \in F_0 \} \]

\[ = \sup \{ \alpha : \{ n : \alpha \leq \inf_{m \geq 1} (M^n_A)_n \} \in F_0 \}. \]

By the same method of proof as in the previous example

\[ d_{M,G,F} = \lim \inf \inf_{n \geq 1} \inf_{m \geq 1} (M^n_A)_n \]

\[ = \lim \inf \inf_{n \geq 1} \inf_{m \geq 1} \sum_{i=1}^{\infty} \frac{a^m_{ni} \chi_A(i)}{n} \]

\[ = \lim \inf \inf_{n \geq 1} \sum_{i=m}^{m+n-1} \frac{1}{n} \chi_A(i) \]

\[ = \lim \inf \inf_{n \geq 1} \frac{A[m,m+n-1]}{n}. \]

Since for \( m \geq 1 \), \( \frac{A[m,m+n-1]}{n} \in \{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{m} \} \),

\[ \inf_{m \geq 1} \frac{A[m,m+n-1]}{n} = \min_{m \geq 1} \frac{A[m,m+n-1]}{n} \]

\[ = \min_{m \geq 0} \frac{A[m+1,m+n]}{n}. \]

Therefore we have
\[ \frac{d_{M,G,F}(A)}{d_{M,G,F}(A)} = \lim_{n \to \infty} \inf_{m \geq 0} \min_{n} A[m+1,m+n]. \]

The last two examples show that two of the important densities which have fundamental differences (see [9]) are subsumed under our general density in Proposition 1.22.

Example 1.27. Let \( X \) be a zero class and \( F = \{ I - A \mid A \in X \} \). Note that \( F \) is finer than \( F_0 \). Let \( M_m \) be the identity matrix for all \( m \). Let \( G = \{ I \} \). Then

\[ d_{M,G,F}(A) = \begin{cases} 1 & \text{if } A^C \in X \\ 0 & \text{otherwise.} \end{cases} \]

We write \( d_{M,G,F}(A) = d_{X}(A) \). (If \( X_0 \) is the set of all finite sets of \( I \), then \( d_{X_0} \) is called the discrete density.)

Proof. Since \( (M_m X_n) = X_A(n) \), we have

\[ \{ m: \alpha \leq (M_m X_n) \} = \begin{cases} I & \text{if } \alpha \leq X_A(n) \\ \phi & \text{if } \alpha > X_A(n). \end{cases} \]

Thus \( \{ n: \{ m: \alpha \leq (M_m X_n) \} \in G \} = \{ n: \alpha \leq X_A(n) \} \). Therefore
we have \( d_{M, G, F}(A) = \sup\{a: \{n: a \leq \chi_A(n)\} \in F\} \). Since

\[
\{n: a \leq \chi_A(n)\} = \begin{cases} 
\phi & \text{if } 1 < a \\
\{A\} & \text{if } 0 < a \leq 1 \\
I & \text{if } a \leq 0,
\end{cases}
\]

it follows that

\[
\{a: \{n: a \leq \chi_A(n)\} \in F\} = \begin{cases} 
\{a: a \leq 1\} & \text{if } A \in F \\
\{a: a \leq 0\} & \text{if } A \notin F
\end{cases}
\]

Hence

\[
d_{M, G, F}(A) = \begin{cases} 
1 & \text{if } A \in F \\
0 & \text{if } A \notin F
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } I - A \in X \\
0 & \text{otherwise}
\end{cases}
\]

Now we want to express \( \overline{d}_{M, G, F}(A) \) by a formula similar to the definition of \( d \).

**Proposition 1.28.** Suppose that \( M, G \) and \( F \) are defined as in Proposition 1.22. Then we have

\[
\overline{d}_{M, G, F}(A) = \inf\{a: \{m: a \geq (M_m \chi_A)_n\} \in G\} \in F\}.
\]
Proof: Consider $d_{M,G,F}(A^c)$ first.

$$d_{M,G,F}(A^c) = \sup\{ \alpha: \{ m: \alpha \leq (M_m \chi_n^c) \} \in G \in F\}$$

$$= \sup\{ \alpha: \{ m: \alpha \leq (M_m (\chi_I^G - \chi_A^G)) \} \in G \in F\}$$

$$\leq \sup\{ \alpha: \{ m: \alpha \leq (M_m \chi_I^G) - (M_m \chi_A^G) \} \in G \in F\}$$

By the condition (ii) in Proposition 1.22 and Lemma 1.16, it follows that for any $\varepsilon > 0$ ,

$$\sup\{ \alpha: \{ m: \alpha \leq 1 - \varepsilon - (M_m \chi_A^G) \} \in G \in F\}$$

$$\leq \sup\{ \alpha: \{ m: \alpha \leq (M_m \chi_I^G) - (M_m \chi_A^G) \} \in G \in F\}$$

$$\leq \sup\{ \alpha: \{ m: \alpha \leq 1 + \varepsilon - (M_m \chi_A^G) \} \in G \in F\} .$$

Thus

$$1 - \varepsilon + \sup\{ \alpha: \{ m: \alpha \leq -(M_m \chi_A^G) \} \in G \in F\}$$

$$\leq d_{M,G,F}(A^c)$$

$$\leq 1 + \varepsilon + \sup\{ \alpha: \{ m: \alpha \leq -(M_m \chi_A^G) \} \in G \in F\} .$$

Hence
1 - \varepsilon - \inf \{ \alpha : \{ m: \alpha \geq (M_n \chi_n) \} \in G \} \in F

\leq d_{M,G,F}(A^C)

\leq 1 + \varepsilon - \inf \{ \alpha : \{ m: \alpha \geq (M_n \chi_n) \} \in G \} \in F).

Since \varepsilon > 0 \ is arbitrary,

\[ d_{M,G,F}(A^C) = 1 - \inf \{ \alpha : \{ m: \alpha \geq (M_n \chi_n) \} \in G \} \in F. \]

Consequently,

\[ \bar{d}_{M,G,F}(A) = \inf \{ \alpha : \{ m: \alpha \geq (M_n \chi_n) \} \in G \} \in F. \]

Lemma 1.29. For any \( m, n \in I \), let \( S(m,n) \) be the corresponding real number. Let \( G, F \) be filters. Then we have

\[ \sup \{ \alpha : \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \} \]

\[ = \sup \{ \alpha : \{ n: \{ m: \alpha < S(m,n) \} \in G \} \in F \}. \]

Proof: For any \( \alpha \in R, \alpha < S(m,n) = \alpha \leq S(m,n) \). By Lemma 1.11(ii),

\[ \{ n: \{ m: \alpha < S(m,n) \} \in G \} \in F = \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F. \]

Therefore we have

\[ \{ \alpha : \{ n: \{ m: \alpha < S(m,n) \} \in G \} \in F \} \subset \{ \alpha : \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \}. \]
Thus

\[ \sup \{ \alpha : \{ n: \{ m: \alpha < S(m,n) \} \in G \} \in F \} \]

\[ \leq \sup \{ \alpha : \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \} . \]

Conversely, let \( \varepsilon > 0 \) be fixed, then for any \( \alpha \in \mathbb{R} \), it follows that \( \alpha \leq S(m,n) = \alpha - \varepsilon < S(m,n) \). By Lemma 1.11 (ii), we have

\[ \sup \{ \alpha : \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \} \]

\[ \leq \sup \{ \alpha : \{ n: \{ m: \alpha - \varepsilon < S(m,n) \} \in G \} \in F \} \]

\[ = \sup \{ \beta + \varepsilon : \{ n: \{ m: \beta < S(m,n) \} \in G \} \in F \} \]

\[ = \varepsilon + \sup \{ \beta : \{ n: \{ m: \beta < S(m,n) \} \in G \} \in F \} . \]

Since \( \varepsilon \) is arbitrary,

\[ \sup \{ \alpha : \{ n: \{ m: \alpha \leq S(m,n) \} \in G \} \in F \} \]

\[ \leq \sup \{ \alpha : \{ n: \{ m: \alpha < S(m,n) \} \in G \} \in F \} . \]

Thus we have the result.

**Remark 1.30.** By Lemma 1.29, we have

\[ d_{M,G,F}(A) = \sup \{ \alpha : \{ n: \{ m: \alpha < (M \chi_{A})_n \} \in G \} \in F \} . \]

\[ \overline{d}_{M,G,F}(A) = \inf \{ \alpha : \{ n: \{ m: \alpha > (M \chi_{A})_n \} \in G \} \in F \} . \]
Definition 1.31. Let \( d: 2^I \to \mathbb{R} \) be a lower asymptotic density, we say that \( d \) is complete if, for any \( A \in 2^I \), \( d(A) = \overline{d}(A) \). That is, every set in \( I \) has natural density with respect to \( d \) or \( 2^I = \eta_d \).

Definition 1.32. An ultrafilter \( F \) on a set \( X \) is a filter such that there is no filter on \( X \) which is strictly finer than \( F \).

Proposition 1.33. Let \( M, G, F \) and \( d_{M,G,F} \) be defined as in Proposition 1.12. If \( G, F \) are ultrafilters then \( d_{M,G,F} \) is a complete density.

Proof. For \( A \in 2^I \), let

\[
K = \{ \alpha: \{ n: \{ m: \alpha \leq (M \chi_n) \} \in G \} \in F \}. 
\]

By Lemma 1.11, if \( \alpha_1 \in K \) and \( \alpha_2 \leq \alpha_1 \), then \( \alpha_2 \in K \). Since \( K \) is bounded above by \( d_{M,G,F}(A) = r \leq 1 \), it follows that

\[
K = \{ x \in \mathbb{R}: x < r \} \quad \text{or} \quad K = \{ x \in \mathbb{R}: x \leq r \}. 
\]

Thus

\[
K^C = \{ x \in \mathbb{R}: x \geq r \} \quad \text{or} \quad K^C = \{ x \in \mathbb{R}: x > r \}. 
\]

Hence

\[
r = \inf K^C
\]

\[
= \inf \{ \alpha: \{ n: \{ m: \alpha \leq (M \chi_n) \} \in G \} \in F \}^C
\]

\[
= \inf \{ \alpha: \{ n: \{ m: \alpha \leq (M \chi_n) \} \in G \} \notin F \}.
\]

Since \( F \) is an ultrafilter, for any \( D \in 2^I \), \( D \notin F \Rightarrow D^C \in F \). Thus we have

\[
r = \inf \{ \alpha: \{ n: \{ m: \alpha \leq (M \chi_n) \} \in G \}^C \in F \}
\]

\[
= \inf \{ \alpha: \{ n: \{ m: \alpha \leq (M \chi_n) \} \notin G \} \in F \}.
\]
G is also an ultrafilter, so that we have

\[ r = \inf \{ \alpha : \{ n : \alpha \leq (M \chi_A)_n \}^c \in G \} \in \mathcal{F} \]

\[ = \inf \{ \alpha : \{ n : (M \chi_A)_n \} \in G \} \in \mathcal{F} \]

By Remark 2.30, \( r = d_{M,G,F}(A) \), and so \( d_{M,G,F}(A) = d_{M,G,F}(A) \).

**Corollary 1.34.** (1) If \( M \) is regular matrix and \( F \) is an ultrafilter then \( d_{M,F}(A) = \sup \{ \alpha : \{ n : \alpha \leq (M \chi_A)_n \} \in F \} \) is complete.

(2) Let \( X \) be a zeroclass, \( G = \{ I - A : A \in X \} \). If \( G \) is ultrafilter, then

\[
d_X(A) = \begin{cases} 
1 & \text{if } I - A \in X \\
0 & \text{otherwise}
\end{cases}
\]

is complete.

**Proof:** By Proposition 1.23, it is obvious that (1) is true. By example 1.27 it is obvious that (2) is true.

**Example 1.35.** Let \( M \) be defined as in Example 1.27. Let \( G = \{ I \} \), and let \( F \) be any filter finer than the Fréchet filter. Then \( d_{M,G,F} \) is not complete.

**Proof:** By Example 1.27 we have

\[
d_{M,G,F}(A) = \sup \{ \alpha : \{ n : \alpha \leq \min_{m \geq 0} A[m+1,m+n] \in F \} \},
\]
\[
\tilde{d}_{M,G,F}(A) = \inf\{\alpha: \{n: \alpha \geq \max_{m \geq 0} \frac{A[m+1,m+n]}{n} \in F\} \}.
\]

Let \( A = \bigcup_{n=2^{2n},2^{2n+1}} \). Since for each \( n \),

\[
\min_{m \geq 0} \frac{A[m+1,m+n]}{n} = 0 \quad \text{and} \quad \max_{m \geq 0} \frac{A[m+1,m+n]}{n} = 1,
\]

it follows that \( \tilde{d}_{M,G,F}(A) = 0 < 1 = \tilde{d}_{M,G,F}(A) \). Hence \( d \) is not complete. Since \( F \) may be taken to be an ultrafilter, this example shows that, in general we do not get a complete density if \( G \) is not also an ultrafilter.
CHAPTER II

LACUNARY SETS

As mentioned in the Introduction, the family of lacunary sets arises naturally in sequence space and combinatorial studies. In this Chapter we introduce several natural subclasses of the lacunary sets, show their inter-relationships and consider their "fullness".

First we introduce several types of lacunary sets.

Definition 2.1. Let the elements of a set \( A = \{a_i\} \in 2^I \) be represented by an increasing sequence \( (a_n) \) and let \( (d_n) \) be the difference sequence, that is, \( d_n = a_{n+1} - a_n \). Then we define the following:

1. \( L = \{A \in 2^I : \lim d_n = \infty \} \cup \chi_0 \), where \( \chi_0 \) is the class of all finite subsets of \( I \).
2. \( L_1 = \{A \in L | d_n \leq d_{n+1}, \text{ for each } n \} \cup \chi_0 \),
3. \( L_2 = (L_1 \cap \{A \in 2^I | \sum_{a \in A} \frac{1}{a} = \infty \}) \cup \chi_0 \),
4. \( L_3 = \{A \in L | d_n < d_{n+1}, \text{ for each } n \} \cup \chi_0 \),
5. \( L_{M_1} = \{A \in L | d_m \leq d_n + i, \text{ for } m \leq n \} \cup \chi_0 \).

A set \( A \in 2^I \) is called a lacunary (resp. \( L_T \) lacunary) set if \( A \) is a finite union of members of \( L \) (resp. \( L_T \)). Note also that \( L_{M_0} = L_1 \).
Definition 2.2. Suppose that $A$ is a family of sets. Let
\[ \hat{A} = \{ B \mid B \subseteq A, \text{ for some } A \in A \} \]
the hereditary closure of $A$, and let $[\hat{A}]$ be the family of all finite unions of all members of $A$.

We prove a simple proposition.

Proposition 2.3. For any family $A$ of sets, $[\hat{A}] = [\hat{A}]$.

Proof: For any $B \in [\hat{A}]$, $B = B_1 \cup B_2 \cup \ldots \cup B_n$ for some $B_i \in \hat{A}$, $i = 1, 2, \ldots, n$.

For each $i \in \{1, 2, 3, \ldots, n\}$, there exists $A_i \in A$ such that $A_i \supseteq B_i$. Then $B = B_1 \cup B_2 \cup \ldots \cup B_n \subseteq A_1 \cup A_2 \cup \ldots \cup A_n$.

Since $A_1 \cup A_2 \cup \ldots \cup A_n \in [A]$, $B \in [\hat{A}]$. Hence $[\hat{A}] \subseteq [\hat{A}]$.

Conversely suppose that $B \in [\hat{A}]$, then $B \subseteq A$ where $A \in [\hat{A}]$.

Then $A = A_1 \cup A_2 \cup \ldots \cup A_n$ where $A_i \in A$. Let $B_i = B \cap A_i$, for $i \in \{1, 2, \ldots, n\}$, then $B_i \subseteq A_i$ and $B_i \in \hat{A}$ for $i \in \{1, 2, \ldots, n\}$.

Hence $B = B_1 \cup B_2 \cup \ldots \cup B_n \in [\hat{A}]$. Therefore $[\hat{A}] \subseteq [\hat{A}]$.

Now we proceed to investigate the fullness of the various classes of lacunary sets just defined. We begin with a definition:

Definition 2.4. A class $\Phi \subseteq 2^I$ is full if,

(a) $\bigcup \{ S : S \in \Phi \} = I$ (covering);

(b) $S \in \Phi$ whenever $S \subseteq T \in \Phi$ for some $T$ (hereditary); and

(c) if $(t_k)$ is a sequence of real numbers and $\Sigma_{k \in S} |t_k| < \infty$ for each $S \in \Phi$, then $\Sigma_{k=1}^{\infty} |t_k| < \infty$. 

Proposition 2.5. If a class $A \subset 2^I$ is full and $[A] \neq 2^I$, then $[A]$ is a full zeroclass (see definition 1.7.).

Proof: Suppose that $A$ is a full class, for each $a \in I$, since $\bigcup A = I$, there exists $S \in A$ such that $a \in S$. Thus $\{a\} \subset S$. By the hereditary property of full classes, $\{a\} \in A$, since $A$ contains any singleton, $[A]$ contains all finite sets. Clearly $[A]$ is closed under finite unions. By the hereditary property of full classes, we have $A = \hat{A}$. Thus $[A] = [\hat{A}] = [\hat{A}]$. Hence $[A]$ also has hereditary property. By the hypothesis, $[A] \neq 2^I$ implies $I \notin [A]$. Hence $[A]$ is a zeroclass.

Proposition 2.6. $L, \hat{L}_1, \hat{L}_2$ are full classes. (Note that $L = \hat{L}$).

Proof: Since $\hat{L}_2 \subset \hat{L}_1 \subset L$, and $L, \hat{L}_1$ and $\hat{L}_3$ are hereditary, $\hat{L}_2$ full would imply that $L$ and $\hat{L}_1$ are also full. Hence we prove only that $\hat{L}_2$ is full. That $\hat{L}_2$ is covering is obvious. We show $\hat{L}_2$ has property (c) of definition 2.4. Let $(t_k)$ be a sequence of real numbers for which $\sum_{k=1}^{\infty} |t_k| = \infty$. For each $n \in I$, there exists $b_n \in I$ such that $\sum_{k=1}^{\infty} |t_k| = \infty$.

We will construct two sequences $(M_n)_{n \geq 2}$ and $(N_n)_{n \geq 1}$ in $I$ with the following properties:

1. $N_n < M_{n+1} < N_{n+1}$ (n \geq 1)
2. $M_n \equiv b_n \mod 2^n$ (n \geq 2)
3. $N_n \equiv M_n \mod 2^n$ (n \geq 2)
where

\[ B^S_{[a,b]} = \{a, a+s, a+2s, \ldots , a+\lfloor \frac{b-a}{s} \rfloor s\} . \]

Take \( N_1 = b_1 \) and suppose that we have constructed two sequences \( M_n^{m-1} \) and \( N_n^{m-1} \) such that (1) and (4) are true for \( n = 1, 2, \ldots , m-2 \) and further (2), (3), (5), (6), (7) are true for \( n = 2, 3, 4, \ldots , m-1 \). Since \( 2^m \) and \( 2^m+1 \) are relatively prime, by the Chinese remainder theorem there exists \( x_0 \in I \) such that

\[ x_0 \equiv b_m \mod 2^m \]

(*)

\[ x_0 \equiv N_{m-1} \mod (2^m + 1) . \]

As long as \( x \equiv x_0 \mod 2^m(2^m + 1) \), \( x \) is also a solution of the system (*) Therefore we can take \( M_m \in I \) such that

\[ M_m \equiv b_m \mod 2^m , \]

\[ M_m \equiv N_{m-1} \mod (2^m + 1) , \]
\[ M_m > b_m \quad \text{and} \quad M_m > N_{m-1}. \]

Since \( \Sigma_{k=1}^{\infty} \frac{1}{m} \) is \( \infty \) and \( M_m = b_m \mod 2^m \) we have

\[ \Sigma_{k=1}^{\infty} \frac{1}{(M_m + 2^m k)} = \infty. \]

By the integral test, \( \Sigma_{k=1}^{\infty} \frac{1}{M_m + 2^m k} = \infty. \)

Now we can take \( N_m \in I \) such that

\[ N_m \equiv M_m \mod 2^m, \]

\[ \Sigma a \in B_{[M_m, N_m]}^{2^m} \]

and

\[ \Sigma a \in B_{[M_m, N_m]}^{2^m} \]

This completes inductive definitions of \( (M_m) \) and \( (N_n) \). Let

\[ A = \bigcup_{k=1}^{\infty} \left( B_{[N_k, M_{k+1}]}^{2^k+1} \cup B_{[M_{k+1}, N_{k+1}]}^{2^{(k+1)}} \right). \]

Clearly \( A \in L_2 \) and \( \Sigma_{a \in A} |t_a| = \infty. \)

**Proposition 2.7.** The class \( \hat{L}_3 \) is not full.

**Proof:** For the real sequence \( \left( \frac{1}{k} \right)_{k=1}^{\infty}, \Sigma_{k=1}^{\infty} \frac{1}{k} = \infty. \) For any \( A \in \hat{L}_3 \), then there exists \( B \in L_3 \) such that \( A \subset B \). Suppose that
B is expressed as a sequence \( (b_n) \), and let \( d_n = b_{n+1} - b_n \) for \( n = 1, 2, \ldots \). For \( n \geq 2 \)

\[
  b_n = b_1 + d_1 + d_2 + \cdots + d_{n-1} > 1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}.
\]

So that we have

\[
  \sum_{a \in A} \frac{1}{a} \leq \sum_{b \in B} \frac{1}{b} \leq \frac{2}{p_1} + \sum_{n=2}^{\infty} \frac{2}{n(n-1)} < \infty.
\]

Hence \( \hat{L}_3 \) is not full.

We next show that if we perform the hereditary closure on \( L_{M_1} \) we get all the lacunary sets.

**Proposition 2.8.** \( \hat{L}_{M_1} = L \).

**Proof:** Let \( A = \{a_i\} \in L \). Let \( N_0 = 1 \). Since \( \lim_{n} d_n = \infty \), for any \( k \geq 1 \), there exists \( N_k > N_{k-1} \) such that \( d_n > k^2 \) whenever \( n > N_k \). For each \( n \) with \( N_k \leq n \leq N_{k+1} \),

\[
  d_n = q_n k + r_n, \quad \text{where} \quad 0 \leq r_n < k.
\]

So that \( q_n k = d_n - r_n > k^2 - k = (k-1)k \). Hence \( q_n > k-1 \) and

\[
  d_n = (q_n - r_n)k + (k+1)r_n. \quad \text{Let} \quad \alpha_n = (\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nq_n}) \quad \text{be the finite sequence} \quad (k, k, \ldots, k, k+1, \ldots, k+1) \quad \text{such that there are} \quad (q_n - r_n) \quad \text{many} \quad k \quad \text{'s in the first part and} \quad r_n \quad \text{many} \quad (k+1) \quad \text{'s in the second part.}
\]

Let

\[
  (e_m) = (\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_{11}, \ldots, \alpha_{1q_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2q_2}, \ldots, \alpha_{n1}, \ldots, \alpha_{nq_n}, \ldots).
\]
Note that \( e_i \leq e_j + 1 \) for \( i \leq j \) but, if
\[
N_k \leq n_1 \leq n_2 \leq N_{k+1}
\]
then \( e_i = e_j - 1 \) when \( e_i \) is the last term of \( \alpha_{n_1}^\alpha \) and \( e_j \) is the first term of \( \alpha_{n_2+1}^\alpha \). Also, clearly
\[
d_n = \alpha_{n_1}^\alpha + \alpha_{n_2}^\alpha + \cdots + \alpha_{n_q}^\alpha.
\]
Letting \( b_1 = a_1 \) and
\[
b_m = a_1 + e_1 + e_2 + \cdots + e_{m-1}
\]
where \( m > 1 \), we have
\[
B = \{b_m : m \in I\} \in L_{M_1}
\]
(and in general \( B \notin L_{M_0} \)).

For any \( a_n \in A \),
\[
a_n = a_1 + d_1 + d_2 + \cdots + d_{n-1}
\]
\[
= a_1 + \sum_{i=1}^{q_1} \alpha_{1i} + \sum_{i=1}^{q_2} \alpha_{2i} + \cdots + \sum_{i=1}^{q_n} \alpha_{ni}
\]
\[
= b_m \in B
\]
where \( m = 1 + q_1 + q_2 + \cdots + q_n \). Hence \( A \subseteq B \). Therefore
\[
L \subseteq \hat{L}_{M_1}.
\]

Conversely since \( L = \hat{L} \) and \( L_{M_1} \subseteq L \) we have \( \hat{L}_{M_1} \subseteq \hat{L} = L \).

Note that for \( i \geq 2 \), \( L_{M_1} \subseteq L_{M_1} \subseteq L \), thus we conclude that
for any \( i \geq 1 \), \( \hat{L}_{M_1} = L \).

We now proceed to show that \( \hat{L}_1 = \hat{L}_{M_0} \neq L \). Actually, this
result is included in the stronger Proposition 2.28 below. We present
it here, however, in order to see the method of proof which is different.

Let's introduce some definitions and lemmas before we prove \( \hat{L}_1 \notin L \).

**Definition 2.9.** (1) Let \( a, x_1, x_2, \ldots, x_n \) be positive integers
with \( a = x_1 + x_2 + \ldots + x_n \) and \( x_1 \leq x_2 \leq \ldots \leq x_n \). Then \((x_1, x_2, \ldots, x_n)\) is called a partition of \( a \) and \( n \) is called the length of the partition \((x_1, x_2, \ldots, x_n)\).

(2) Let \((a_1, a_2, \ldots, a_n)\) be any finite sequence of positive integers and let \((y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, \ldots, y_{n1}, \ldots, y_{nk_n})\) be a nondecreasing sequence such that \((y_{i1}, y_{i2}, \ldots, y_{ik_i})\) is a partition of \( a_i \).

Then \((y_{11}, y_{12}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})\) is called a (monotone) partition of the sequence \((a_1, a_2, \ldots, a_n)\).

**Definition 2.10.** Let \((x)_{n=1}^{\infty}\) be a sequence. Then the finite subsequence \((x_s, x_{s+1}, \ldots, x_t)\) is called a part of the sequence \((x)_{n=1}^{\infty}\).

**Lemma 2.11.** Let \((a_1, a_2, \ldots, a_n)\) be a strictly decreasing sequence of prime numbers with \( n \geq a_n \). Then there does not exist a partition of \((a_1, a_2, \ldots, a_n)\) such that its first term is larger than 1.

**Proof:** Suppose that there exists a partition 

\[(y_{11}, y_{12}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})\]

of \((a_1, a_2, \ldots, a_n)\) with \( y_{11} > 1 \).

Assume that \( k_i = 1 \) for some \( i < n \). Since \( y_{(i+1)1} \) is a member of the partition of \( a_{i+1} \), we have \( y_{(i+1)1} \leq a_{i+1} \). Since 

\[(y_{11}, \ldots, y_{1k_1}, \ldots, y_{n1}, \ldots, y_{nk_n})\]

is nondecreasing, 

\[a_i = y_{i1} \leq y_{(i+1)1} \leq a_{i+1}\]. This contradicts that \((a_1, a_2, \ldots, a_n)\) is a strictly decreasing sequence. Therefore, for any \( i = 1, 2, \ldots, n-1, k_i > 1 \).
Clearly $y_{i1} < y_{ik_i}$ for each $i = 1, 2, \ldots, n - 1$, since otherwise, $a_i = y_{i1} + y_{i2} + \ldots + y_{ik_i} = k_i y_{i1}$, which contradicts that $a_i$ is a prime number. Hence $y_{i1} < y_{ik_i} \leq y_{(i+1)l}$ for $i = 1, 2, \ldots, n - 1$. Therefore $1 < y_{i1} < y_{21} < \ldots < y_{nl} \leq a_n$ and thus $n < y_{nl} \leq a_n$, which contradicts the hypothesis $a_n \leq n$.

**Proposition 2.12.** $L_1 \not\subseteq L$.

**Proof:** Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the sequence of prime numbers. Let $\{a_n\}$ be a sequence of natural numbers whose difference sequence $(d_n = a_{n+1} - a_n)$ is given by

$$
\{d_n\} = \left\{ \frac{p_1}{k_1}, \ldots, \frac{p_2}{k_2}, \ldots, \frac{p_1}{k_2}, \ldots, \frac{p_2}{k_3}, \ldots, \frac{p_1}{k_3}, \ldots, \frac{p_2}{k_4}, \ldots, \frac{p_3}{k_4}, \ldots, \frac{p_1}{k_4}, \ldots, \frac{p_2}{k_5}, \ldots, \frac{p_3}{k_5}, \ldots, \frac{p_1}{k_5}, \ldots, \frac{p_2}{k_6}, \ldots \right\}
$$

where $k_\ell - 1 > p_\ell$, for each $\ell \geq 1$.

By the construction $\lim_{n \to \infty} d_n = \infty$. Thus $A \in L$. We want to show that $A \notin \hat{L}_1$.

Suppose that $A \in \hat{L}_1$ and thus $A \subseteq B$ for some $B \in L_1$.

Let $\{e_m\}$ be the difference sequence of $B = \{b_m\}$ so that $e_i \leq e_{i+1}$ for all $i$. Since $A \subseteq B$, for any $n$, $a_n = b_{f(n)}$ for some function $f$ on $I$ and $d_n = a_{n+1} - a_n = b_{f(n+1)} - b_{f(n)} = (b_1 + e_1 + \ldots + e_{f(n+1)}) - (b_1 + e_1 + \ldots + e_{f(n)}) = e_{f(n)+1} + e_{f(n)+2} + \ldots + e_{f(n+1)}$. So that $(e_{f(n)+1}, e_{f(n)+2}, \ldots, e_{f(n+1)})$ is a partition of $d_n$. Since
\[ \lim_{m \to \infty} e = \infty, \quad e_f(n) > 1 \text{ for some } n. \] Suppose that
\[(p_{k+1}, p_{k+1}, \ldots, p_{k+1}, p_{k+1}) \text{ is the part of the sequence } \{d_n\} \text{ such that }\]
\[e_f(m) = (d_m, d_{m+1}, \ldots, d_{m+(k-\ell)}) \text{ and } e_f(m) > 1.\]
Then \((p_{k+1}, p_{k+1}, \ldots, p_{k+1}, p_{k+1}) = (d_m, d_{m+1}, \ldots, d_{m+(k-\ell)})\) is
partitioned into \((e_f(m)+1, \ldots, e_f(m+1), \ldots, e_f(m+k-\ell)+1, \ldots, e_f(m+k-\ell+1))\)
which is a contradiction by previous lemma. Therefore \(A \notin \hat{L}_1.\)

Next let's prove that \(\hat{L}_2 \neq \hat{L}_1.\) First we will prove some
lemmas.

**Lemma 2.13.** Let \(p > 2\) be a prime number and let \((a_1, a_2, \ldots, a_p)\)
be the sequence with \(a_i = p, \) for all \(i = 1, 2, \ldots, p.\) Let
\[(y_{l_1}, y_{l_2}, \ldots, y_{l_k}, y_{2l_1}, \ldots, y_{2l_2}, \ldots, y_{l_p}, y_{l_p}, \ldots, y_{l_p})\]
be a monotone partition of \((a_1, a_2, \ldots, a_p)\) with \(y_{l_1} > 1.\) Then \(k_p = 1\) and
\(y_{l_p} = p.\)

**Proof:** Suppose that \(k_p > 1.\) Then \(y_{l_p} < p\) and, since \(p\) is
a prime, \(y_{l_p} < y_{l_p}.\) It follows that \(k_i > 1\) for all \(i < p\) since
if \(k_i = 1,\) then \(y_{l_i} = p > y_{l_p},\) which is a contradiction. Furthermore,
\(y_{l_1} < y_{l_i}k_i\) since \(a_i = p\) is a prime. Therefore
\(1 < y_{l_1} < y_{l_2} < \ldots < y_{l_p} < p\) which a contradiction.
Lemma 2.14.

\[ \lim_{n \to \infty} \frac{p_{n+1}^2}{p_1^2 + p_2^2 + \cdots + p_n^2} = 0 \]

where, of course, \( p_n \) is the \( n \)-th prime number.

**Proof:** By the prime number theorem

\[ \lim_{n \to \infty} \frac{p_n}{n \log n} = 1. \]

So that \( \lim_{n \to \infty} \frac{p_n}{p_{n+1}} = 1 \). It follows, for any \( k \in \mathbb{I} \), that \( \lim_{n \to \infty} \frac{p_n}{p_{n+k}} = 1 \).

Hence

\[ \lim_{n \to \infty} \frac{p_{n+1}^2}{p_1^2 + p_2^2 + \cdots + p_n^2} \leq \lim_{n \to \infty} \frac{p_{n-k}^2}{(k+1) p_{n-k}} \]

since \( k \) is arbitrary the lemma follows.

Lemma 2.15.

\[ \sum_{m=1}^{\infty} \frac{1}{p_m} \sum_{n=1}^{p_m} \frac{2(p_1^2 + p_2^2 + \cdots + p_m^2)}{2(p_1^2 + p_2^2 + \cdots + p_{m-1}^2) + p_m^2} < \infty \]
Proof: The proof of (1) and (2) are similar. We prove (2) only.

We know \(\lim_{x \to 0^+} \left(\frac{\ln(1+x)}{x}\right) = 1\). Now using this and lemma 2.14, we have

\[
S = \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \left( \frac{2(p_1^2 + p_2^2 + \ldots + p_m^2) + p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)} \right)
\]

\[
= \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \left( 1 + \frac{p_{m+1}^2}{2(p_1^2 + p_2^2 + \ldots + p_m^2)} \right)
\]

\[
= 0 \left( \sum_{m=1}^{\infty} \frac{1}{p_m} \frac{p_{m+1}}{2(p_1^2 + p_2^2 + \ldots + p_m^2)} \right).
\]

Since \(\lim_{m \to \infty} \frac{p_{m+1}}{p_m} = 1\)

\[
S = 0 \left( \sum_{m=1}^{\infty} \frac{\frac{p_m}{p_1^2 + p_2^2 + \ldots + p_m^2}}{p_1^2 + p_2^2 + \ldots + p_m^2} \right)
\]

\[
= 0 \left( \sum_{m=10}^{\infty} \frac{\ln m}{1^2 + 1^2 + \ldots + m^2} \right)
\]

\[
= 0 \left( \sum_{m=10}^{\infty} \frac{\ln m}{m^2} \right).
\]

Since \(\sum_{m=10}^{\infty} \frac{(\ln m)}{m^2}\) converges, \(S\) is finite.
Proposition 2.16. \( \hat{L}_2 \not\subseteq \hat{L}_1 \).

Proof: We construct \( A \in \hat{L}_1 \) such that \( A \notin \hat{L}_2 \). Let \( \{p_1 < p_2 < \ldots < p_m < \ldots\} \) be the set of all prime numbers. Let \( d_m = (p_m, p_m, \ldots, p_m) \) be the sequence of \( 2p_m \) repetitions of \( p_m \).

Let \( \{d_n\} = (D_1, \ldots, D_m, \ldots) \) and finally let \( A = \{a_m\} \subset I \) be the sequence such that \( a_n = a_1 + d_1 + \ldots + d_{n-1} \) for \( n \geq 2 \) where \( a_1 = 1 \).

Clearly \( A \in \hat{L}_1 \subseteq \hat{L}_1 \). Suppose that \( A \in \hat{L}_2 \) so that \( A \subset B = \{b_u\} \), where \( B \in \hat{L}_2 \). Let \( \{e_u\} \) be the difference sequence of \( B \). Since \( B \in \hat{L}_1 \), there exists \( N \) such that for any \( k \geq N \), \( e_k > 1 \). Let us consider \( D_m \), the \( m \)-th part of \( \{d_n\} \) such that \( b_N \leq a_{\alpha m} \).

Consider the following diagram:

```
Clearly \( \alpha m = 1 + 2(p_1 + p_2 + \ldots + p_{m-1}) \) and \( \beta m = \alpha m + p_m \).

Since \( A \subset B = \{b_u\} \), some part of \( \{e_u\} \) is a partition of the \( m \)-th part \( D_m \) of \( \{d_n\} \). (See the proof of Lemma 2.11). Suppose that \( (e_s, e_{s+1}, \ldots, e_t) \) is the partition of \( D_m \). Then \( b_N \leq a_{\alpha m} = b_s \).

Thus \( N \leq s \) and \( e_s > 1 \). By Lemma 2.13, if \( a_{\beta m} \leq b_u < b_{u+1} < a_{\alpha (m+1)} \) then \( e_u = p_m \). (3)
```
Since \( \{e_u\} \) is a nondecreasing sequence if
\[
a_\alpha(m+1) \leq b_u \leq a_\beta(m+1) \quad \text{then} \quad p_m \leq e_u \leq p_{m+1}.
\] (4)

Let

\[
B_m = \{x \in B: a_{\alpha_m} < x \leq a_{\beta_m}\}
\]

\[
B^*_m = \{x \in B: a_{\beta_m} < x \leq a_{\alpha(m+1)}\}
\]

for \( m = 1, 2, 3, \ldots \).

Then from (3), we have

\[
\sum_{b \in B^*_m} \frac{1}{b} = \sum_{k=1}^{p_m} \frac{1}{a_{\beta_m} + kp_m}
\]

\[
< \int_0^1 \frac{1}{a_{\beta_m} + xp_m} \, dx
\]

\[
= \frac{1}{p_m} \ln \frac{a_{\alpha(m+1)}}{a_{\beta_m}}.
\] (5)

From (4) we have

\[
\sum_{b \in B_{m+1}} \frac{1}{b} \leq \sum_{k=1}^{p_m} \frac{1}{a_{\alpha(m+1)} + kp_m}
\]

\[
< \int_0^1 \frac{1}{a_{\alpha(m+1)} + xp_m} \, dx
\]

\[
= \frac{1}{p_m} \ln \frac{a_{\beta(m+1)}}{a_{\alpha(m+1)}}.
\] (6)
Now we have
\[ \sum_{b \in B} \frac{1}{b} = \sum_{b \in S} \frac{1}{b} + \sum_{b \in T} \frac{1}{b} \]
where \( S = \bigcup_{m=1}^{\infty} B^m \) and \( T = \bigcup_{m=1}^{\infty} B^m \). Let \( \alpha_m \geq b_N \) and \( \alpha_m \geq b_N \).

Then \( S - S_1 \) and \( T - T_1 \) are finite. Thus for some real \( M \)
\[ \sum_{b \in B} \frac{1}{b} = M + \sum_{b \in S_1} \frac{1}{b} + \sum_{b \in T_1} \frac{1}{b} \]
\[ = M + \sum_{\alpha_m \geq N} \sum_{b \in B^m+1} \frac{1}{b} + \sum_{\alpha_m \geq N} \sum_{b \in B^m} \frac{1}{b} . \]

By (5) and (6)
\[ \sum_{b \in B} \frac{1}{b} \leq M + \sum_{\alpha_m \geq b_N} \frac{1}{p_m} \ln \frac{\alpha(m+1)}{\beta m} + \sum_{\alpha_m \geq b_N} \frac{1}{p_m} \ln \frac{\beta(m+1)}{\alpha m} \]
\[ \leq M + \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{\alpha(m+1)}{\beta m} + \sum_{m=1}^{\infty} \frac{1}{p_m} \ln \frac{\beta(m+1)}{\alpha m} . \]

By the construction of \( \alpha_m, \beta_m \), we have
\[ \alpha_m = 1 + 2 \left( p_1^2 + p_2^2 + \ldots + p_{m-1}^2 \right) , \]
\[ \beta_m = 1 + 2 \left( p_1^2 + p_2^2 + \ldots + p_{m-1}^2 \right) + p_m . \]
Therefore by Lemma 2.15, we have

\[
\sum_{b \in B} \frac{1}{b} \leq M + \sum_{m=1}^{\infty} \frac{1}{P_m} \ln \frac{1 + 2(p_1^2 + p_2^2 + \ldots + p_m^2)}{1 + 2(p_1^2 + p_2^2 + \ldots + p_{m-1}^2) + p_m^2} + \sum_{m=1}^{\infty} \frac{1}{P_m} \ln \frac{1 + 2(p_1^2 + p_2^2 + \ldots + p_m^2)}{1 + 2(p_1^2 + p_2^2 + \ldots + p_{m-1}^2) + p_m^2} < \infty.
\]

This contradicts \( B \in L_2 \).

**Proposition 2.17.** \([\hat{L}_3] \nsubseteq [\hat{L}_1]\).

**Proof:** Obviously \([\hat{L}_3] \subset [\hat{L}_1]\). Let \( D = \{ A \subset I: \sum_{a \in A} \frac{1}{a} < \infty \} \). Then obviously \( D = [\hat{D}] \). Since \( L_3 \subset D \) (see the proof of Proposition 2.7), it follows that \([\hat{L}_3] \subset [\hat{D}] = D\). Since \( L_2 \) is full (Proposition 2.6) there exists \( x \in L_1 \) with \( \sum_{x \in x} \frac{1}{x} = \infty \). Therefore \( x \notin D \) and so \( x \notin [\hat{L}_3] \). Thus \([\hat{L}_3] \nsubseteq [\hat{L}_1]\).

The following lemma will be used to prove that \([L_{M_1}] \nsubseteq [L_{M_j}]\) for \( 0 \leq i < j \). (Compare the remark following Proposition 2.7).

**Lemma 2.18.** Suppose that \( d, k, s, t, u, v, i \) and \( j \) are integers such that \( d > k^2 + k \), \( 0 \leq i < j < k \), \( 0 \leq s \leq k \), \( 1 \leq v \leq k \) and \( 1 \leq t \leq k \), then
(1) \( v(d+j) \leq t(d+j)+i \Rightarrow v \leq t \) and \( v(d+j) \leq t(d+j) \),

(2) \( vd \leq td+i \Rightarrow v \leq t \) and \( vd \leq td \),

(3) \( v(d+j) \leq s(d+j)+td+i \Rightarrow v < s+t \) and \( v(d+j) < s(d+j)+td \),

(4) \( v(d+j)+sd \leq td+i \Rightarrow v+s < t \) and \( v(d+j)+sd < td \),

(5) \( vd \leq sd+t(d+j)+i \Rightarrow v \leq s+t \) and \( vd < sd+t(d+j) \),

(6) \( vd+s(d+j) \leq t(d+j)+i \Rightarrow v+s \leq t \) and \( vd+s(d+j) < t(d+j) \).

Proof: The proofs of (1) and (2) are similar so we prove (1) only: If \( v > t \) then \( v(d+j) \geq t(d+j)+d+j > t(d+j)+i \) contrary to hypothesis, thus \( v \leq t \) and clearly (1) holds.

The proofs of (3) and (4) are similar, so we prove (3) only:
If \( v \leq s \) then the conclusion clearly holds. Assume \( v > s \) then \( t > 0 \). If \( v \geq s+t \) then \( v(d+j) \geq s(d+j)+t(d+j) > s(d+j)+td+i \) contrary to hypothesis. Hence \( v < s+t \) and

\[
v(d+j) \leq s(d+j)+t(d+j)-(d+j)
= s(d+j)+td+(t-1)j-d
< s(d+j)+td. \quad (\text{Since } (t-1)j-d < 0).
\]

The proofs of (5) and (6) are similar and so we prove (5) only. Since \( vd \leq sd+t(d+j)+i \) is equivalent to \(-i-tj \leq (s+t-v)d\), we have \(-d < -k-\frac{k^2}{2} < -i-tj \leq (s+t-v)d\). Thus we get \(-l < (s+t-v)\) or, equivalently, \( v \leq s+t \).
If \( vd \geq sd + t(d+j) \) then we have \((v-s)d \geq t(d+j)\). Since \( l \leq t, l \leq d \) and \( l \leq j \) we have \( v-s \geq t \) which is a contradiction. Therefore (5) holds.

**Proposition 2.19.** If \( 0 \leq i < j \), then \( [L_{m_i}] \subseteq [L_{M_j}] \).

**Proof:** Let

\[
L_m = (m^3+j, m^3+j, \ldots, m^3+j), \text{ } m \text{ repetitions of } m^3+j,
\]

\[
R_m = (m^3, m^3, \ldots, m^3), \text{ } m \text{ repetitions of } m^3,
\]

and

\[
B_m = (L_m, R_m, L_m, R_m, \ldots, L_m, R_m), \text{ } m \text{ repetitions of } L_m, R_m.
\]

Let \( \{d_n\} = (B_1, B_2, \ldots, B_m, \ldots) \). Let \( A = \{a_n \mid n \in I\} \) such that

\[
a_n = l + d_1 + \ldots + d_{n-1} \text{ for } n \geq 1.
\]

Let \( [a_m, a_n] = \{a_r \in A \mid m \leq r \leq n\} \)

and

\[
(a_m, a_n) = \{a_r \in A \mid m < r < n\} = [a_{m+1}, a_{n-1}].
\]

Clearly \( A \in L_{M_j} \).

Let us consider the \( m \)-th part of \( A \) corresponding to \( B_m \)
(diagram is actually illustrated below).
Here $\alpha(m,t) = 1 + 2(1^2 + 2^2 + \ldots + (m-1)^2) + 2(t-1)m$ for $1 \leq m$ and $1 \leq t \leq m+1$ and $\beta(m,t) = \alpha(m,t) + m$. Note that $\alpha(m+1,1) = \alpha(m,m+1)$.

Let

$$A_{Lmt} = [a_{\alpha(m,t)}, a_{\beta(m,t)}], \quad A_{Lmt}^0 = (a_{\alpha(m,t)}, a_{\beta(m,t)})$$

$$A_{Rmt} = [a_{\beta(m,t)}, a_{\alpha(m,t+1)}], \quad A_{Rmt}^0 = (a_{\beta(m,t)}, a_{\alpha(m,t+1)})$$

Suppose that $X = \{x_q \in L_{M_i} \}$ and $X \subset A$. We want to show that if $m^3 > m^2 + m$ and $j < m$ then

$$|X \cap A_{Rmn}| \leq 2. \quad (*)$$

Let $\{y_q \}$ be the difference sequence of $\{x_q \}$ and $f$ be the function on $I$ such that $x_q = a_f(q)$. Then $f(s+1) - f(s)$ equals the number of $d_n$'s in the sum $y_s = d_f(s) + d_f(s+1) + \ldots + d_f(s+1) - 1$. At first we will consider the following six cases which will be used in proving $|X \cap A_{Rmn}| \leq 2$.

1. If $a_{\alpha(m,t)} \leq x_q < x_{q+1} < x_{q+2} \leq a_{\beta(m,t)}$, i.e., three consecutive elements of $X$ are in $A_{Lmt}$, then since $X \in L_{M_i}$ and $y_q \leq y_{q+1} + i$, we have

$$x_{q+1} - x_q \leq x_{q+2} - x_{q+1} + i$$

$$a_{f(q+1)} - a_{f(q)} \leq a_{f(q+2)} - a_{f(q+1)} + i$$

$$(f(q+1) - f(q))(d+j) \leq (f(q+2) - f(q+1))(d+j) + i$$

where $d = m^3$. 

By the previous lemma, case (1), we conclude that
\[ f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q \leq y_{q+1}. \]

(2) Similarly, if \( x_q < x_{q+1} < x_{q+2} \) are in the interval \( A_{Rmt} \), then we apply the previous lemma case (2) and we get
\[ f(q+1) - f(q) \leq f(q+2) - f(q+1) \text{ and } y_q \leq y_{q+1}. \]

(3) If \( a_\alpha(m,t) \leq x_q < x_{q+1} \leq a_\beta(m,t) < x_{q+2} \leq a_\alpha(m,t+1) \) that is,
\[ x_q, x_{q+1} \text{ are in } A_{Lmt} \text{ and } x_{q+1} \text{ is in } A_{Rmt}, \]
then, since
\[ x_{q+1} - x_q \leq x_{q+2} - x_{q+1} + i, \]
it follows that
\[
\begin{align*}
af(q+1) - af(q) & \leq af(q+2) - af(q+1) + i \\
& = a_\beta(m,t) - af(q+1) + qf(q+2) - a_\beta(m,t) + i,
\end{align*}
\]
which is equivalent to
\[ (f(q+1) - f(q))(d+j) \leq (\beta(m,t) - f(q+1))(d+j) + (f(q+2) - \beta(m,t))d + i. \]

Now we apply the previous lemma case (3) and we get
\[ f(q+1) - f(q) < f(q+2) - f(q+1) \text{ and so } y_q < y_{q+1}. \]

(4) Similarly if
\[ a_\alpha(m,t) \leq x_q < a_\beta(m,t) \leq x_{q+1} < x_q \leq a_\alpha(m,t+1), \]
that is, \( x_q \) is in \( A_{Lmt} \) and \( x_{q+1}, x_{q+2} \) are in \( A_{Rmt} \), then we can apply the previous lemma case (4) and get
\[ f(q+1) - f(q) < f(q+2) - f(q+1) \quad \text{and} \quad y_q < y_{q+1}. \]

(5) If

\[ a_\beta(m, t) \leq x_q < x_{q+1} \leq a_\alpha(m, t+1) < x_{q+2} \leq a_\beta(m, t-1) \]

where \( t \leq m \), that is, \( x_q \) and \( x_{q+1} \) are in \( A_{Rmt} \) and \( x_{q+2} \) is in \( A_{Lm}(t+1) \), then we have

\[ a_f(q+1) - a_f(q) \leq a_f(q+2) - a_f(q+1) + i \]

\[ = a_\alpha(m, t+1) - a_f(q+1) + a_f(q+2) - a_\alpha(m, t+1) + i \]

equivalently

\[ (f(q+1) - f(q))d \leq (\alpha(m, t+1) - f(q+1))d + (f(q+2) - \alpha(m, t+1))(d+j) + i. \]

We apply the previous lemma case (5) and we get

\[ f(q+1) - f(q) \leq f(q+2) - f(q+1) \quad \text{and} \quad y_q < y_{q+1}. \]

(6) Again, if

\[ a_\beta(m, t) \leq x_q < a_\alpha(m, t+1) \leq x_{q+1} < x_{q+2} \leq a_\beta(m, t+1) \]

then we can apply the previous lemma case (6) and obtain

\[ f(q+1) - f(q) \leq f(q+2) - f(q+1) \quad \text{and} \quad y_q < y_{q+1}. \]

Now assume that \( |X \cap A_{Rnn}| \geq 3 \) and so there exist three consecutive elements \( x_w, x_{w+1}, x_{w+2} \) of \( X \) in \( A_{Rnn} \). By the case (2)
\[ f(w+1) - f(w) \leq f(w+2) - f(w+1) \]

and so

\[ 2(f(w+1) - f(w)) \leq f(w+1) - f(w) + f(w+2) - f(w+1) \]

\[ = f(w+2) - f(w) \leq m. \]

Thus \( f(w+1) - f(w) \leq \frac{1}{2} m \) and \( y_w = (f(w+1) - f(w))d \leq \frac{1}{2} md = \frac{1}{2} m^4 \),

the half length of \( A_{Rmm} \).

We claim: for any \( u < w \) and \( x_u \geq a_\alpha(m,1) \), we have

\[ y_u \leq y_w. \]

Proof of claim: Since \( X \in L_i \), we have \( y_u \leq y_w + i \). Thus

let \( y_u = t(d+j) + vd \) and \( y_w =qd \). Then we have \( t(d+j) + vd \leq qd+i \).

If \( t > 0 \) (resp. \( t = 0 \)), then we apply the previous lemma

case (4) (resp. case (2)) and so we get \( y_u \leq y_w \).

By the above claim we conclude that for any \( u \leq w \) and

\( x_u \geq a_\alpha(m,1) \) we have \( y_u \leq y_w \leq \frac{1}{2} m^4 = \frac{1}{2} m \) length of \( A_{Rmt} \) \leq \( \frac{1}{2} \)

length of \( A_{Lmt} \) for \( t = 1,2,\ldots,m+1 \).

Hence, for any \( t \), \( A_{Rmt} \) and \( A_{Lmt} \) each contain at least two elements of \( X \).

Therefore we conclude that:

By case (3) and (4), if \( x_q \in A_{Lmt}^{0} \) and \( x_{q+2} \in A_{Rmt}^{0} \),

then \( f(q+1) - f(q) < f(q+2) - f(q+1) \).
By case (5) and (6), if $x_q \in A_{Rmt}^0$ and $x_{q+2} \in A_{Lmt(t+1)}^0$ then $f(q+1) - f(q) \leq f(q+2) - f(q+1)$.

By case (1) and (2) if $x_q, x_{q+1}, x_{q+2} \in A_{Lmt}$ or $x_q, x_{q+1}, x_{q+2} \in A_{Rmt}$, then $f(q+1) - f(q) \leq f(q+2) - f(q+1)$.

Now if we let $x_{s_q}$ be the element of $X$ such that $x_{s_q} \in A_{Lm_q}^0$ and $x_{s_{q+2}} \in A_{Rm_q}^0$ for $q = 1, 2, ..., m$. Then we have

for $q = 1, 2, ..., m-1$,

$$f(s_{q+1}) - f(s_q) < f(s_{q+1} + 1) - f(s_{q+1}).$$

Therefore we get the sequence of inequalities

$$1 \leq f(s_{1}+1) - f(s_1) < f(s_{2}+1) - f(s_2) < ... < f(s_{m}+1) - f(s_m)$$

$$\leq f(w+1) - f(w) \leq \frac{1}{2} m.$$ 

Since there are $m-1$ inequalities, we get a contradiction. Therefore

$$|X \cap A_{Rmm}| \geq 2.$$ 

Finally we show that $A \notin [L_{M_1}]$. Suppose that $A \in [L_{M_1}]$ and so $A = X_1 \cup X_2 \cup ... \cup X_n$ where $X_s \in L_{M_1}$. Since

$$A_{Rmm} = \bigcup_{i=1}^{n} (A_{Rmm} \cap X_i),$$

if we take $m^3 > m^2 + m$ and $m > j$, we have

$$m = |A_{Rmm}| \leq \sum_{i=1}^{n} |A_{Rmm} \cap X_i| \leq 2n.$$ 

Since $m$ can be arbitrarily large, we get a contradiction. Hence

$A \notin [L_{M_1}]$. 


Corollary 2.20. For all \( i \geq 0 \), \( [L_{M_i}] \subsetneq [L] \).

In particular \( [L_1] \subsetneq [L] \).

Proof: Obviously, by Proposition 2.19, \( [L_{M_i}] \subsetneq [L_{i+1}] \subset [L] \).

Proposition 2.21. \( [L_2] \subsetneq [L_1] \).

Proof: Obviously we know \( [L_2] \subset [L_1] \). We want to show that \( A = \{n^2\} \in L_1 \) but \( A \notin [L_2] \).

Suppose that \( A \in [L_2] \) then there exists an infinite \( x \in L_2 \) such that \( x \subseteq A \). Then we have \( \sum \frac{1}{x} \leq \sum \frac{1}{n^2} < \infty \), which is a contradiction.

Proposition 2.22. Let \( G = [L_1] \), then \( \hat{L}_1 \subsetneq G \).

Proof: Let \( A = \{n^2\} \) and \( B = \{n^2+1\} \). Then \( A \cup B \in [L_1] \subset [\hat{L}_1] = G \). But \( A \cup B \notin L = \hat{L} \supset \hat{L}_1 \). Therefore \( \hat{L}_1 \subsetneq G \).

We will also prove that \( [L_1] \subsetneq G \). First, let's define some terms and prove a lemma.

Definition 2.23. Let \( \{a_n\} = A \subseteq I \) be a sequence and \( (a_s, a_{s+1}, \ldots, a_{s+r}) \) be a part of \( \{a_n\} \).

If \( (d_s, d_{s+1}, \ldots, d_{s+r-1}) \) is strictly decreasing sequence, where \( d_i = a_{i+1} - a_i \), then we say that \( (a_s, a_{s+1}, \ldots, a_{s+r}) \) is a descending wave of length \( r+1 \) in \( A \). Let \( d_t = a_{t+1} - a_t \) be called the decreasing steps of the wave.
Lemma 2.24. There exists a function $f(n)$ such that, if $A_1, A_2, \ldots, A_n \in L_1$ and $X$ is any descending wave in $A_1 \cup A_2 \cup \ldots \cup A_n$, then the length of $X$ is less than or equal to $f(n)$.

Proof: We take $f(1) = 2$.

Suppose there exists $f(n-1)$ such that for any $A_1, A_2, \ldots$, and $A_{n-1}$ in $L_1$, and any descending wave $X$ in $A_1 \cup A_2 \cup \ldots \cup A_{n-1}$, the length of $X \leq f(n-1)$.

Let $A = A_1 \cup A_2 \cup \ldots \cup A_{n-1}$ and $B = A_n$ where $A_1, A_2, \ldots, A_n \in L_1$, let

$W_u = \{ a \in A \mid b_u < a < b_{u+1} \}$,

$V_u = \{ c \in A \cup B \mid b_u \leq c \leq b_{u+1} \}$.

(1) Suppose that $X$ is a descending wave in $A \cup B$ and $V_u \subseteq X$ and $V_{u+1} \subseteq X$, then we wish to prove that $|W_u| < |W_{u+1}|$.

Let $e_1 > e_2 > \ldots > e_{q+1} > c_1 > c_2 > \ldots > c_{p+1}$ be the decreasing steps of $V_u \cup V_{u+1}$. Since $B \in L_1$,

$(q+1)e_{q+1} \leq e_1 + e_2 + \ldots + e_{q+1} = b_{u+1} - b_u \leq b_{u+2} - b_{u+1} =

c_1 + c_2 + \ldots + c_{p+1} \leq (p+1)c_1 < (p+1)e_{q+1}$. Therefore $q+1 < p+1$

and so $q < p$ where, clearly $q = |W_u|$ and $p = |W_{u+1}|$.

(2) If $X \subseteq A \cup B$ is a descending wave then we show $|X \cap B| \leq f(n-1) + 2$. 
Suppose otherwise so that \( |X \cap B| > f(n-1)+2 \). Let \( \{b_r, b_{r+1}, \ldots, b_s\} \subseteq X \cap B \), where \( s = r + f(n-1) + 2 \). Then \( V_k \subseteq X \) for all \( r \leq k \leq s-1 \). By (1), \( 0 \leq |W_r| < |W_{r+1}| < \ldots < |W_{s-1}| \), thus we have \( |W_{s-1}| = |W_{r+f(n-1)+1}| \geq f(n-1)+1 \) which is a contradiction since \( W_{s-1} \) is a descending wave in \( A \).

Finally let \( X \) be a descending wave of \( A \cup B \). Then by (2) we have \( |X \cap B| \leq f(n-1)+2 \). Write \( X \cap B = \{b_r, b_{r+1}, \ldots, b_s\} \).

Then \( X \subseteq H \cup V_r \cup \ldots \cup V_{s-1} \cup J \) where \( H \) and \( J \) are the (possibly empty) descending waves in \( A \cap X \) which come before \( b_r \) and after \( b_{s-1} \). Thus \( |X| \leq |H| + |J| + \sum_{i=r}^{s-1} |V_i| \leq (f(n-1)+3)(f(n-1)+2) \) and so we can set \( f(n) = (f(n-1)+2)(f(n-1)+3) \).

**Proposition 2.25.** Let \( G = [\hat{L}_1] \), then \( [L_1] \subseteq G \).

**Proof:** Clearly \( [L_1] \subseteq G \), let \( B_n = (n, (n-1)n, (n-2)n, \ldots, 2n, n) \) and \( \{d_n\} = \{B_1, B_2, \ldots, B_q, \ldots\} = (1, 4, 2, 9, 6, 3, 16, 12, 8, 4, \ldots) \). Let \( a_n = 1 + d_1 + \ldots + d_{n-1} \). Let \( W_m = (m, m, \ldots, m) \), with \( m(m+1)/2 \) repetitions of \( m \) and \( \{y_m\} = (w_1, w_2, \ldots, w_p, \ldots) = (1, 2, 2, 3, 3, 3, 3, 3, 4, 4, \ldots) \). Let \( x_m = 1 + y_1 + y_2 + \ldots + y_{m-1} \).

Then \( \{x_m\} \subseteq L_1 \) and \( \{a_n\} \subseteq \{x_m\} \). Thus \( \{a_n\} \subseteq \hat{L}_1 \subseteq [L_1] \). Since \( \{a_n\} \) contains arbitrary long descending waves, by the previous lemma, \( \{a_n\} \notin [L_1] \). Thus \( [L_1] \notin G \).

We have seen (Proposition 2.12) that \( \hat{L}_1 \subseteq L \). It follows
that $\hat{L}_1 \neq [L]$. Proposition 2.25 shows that $[L_1] \not\subset G$ and it follows that $[L_1] \not\subset [L]$. These together, however, do not imply that $[\hat{L}_1] \not\subset [L]$. This strict inclusion is the last goal of the present chapter. We first prove some useful lemmas.

**Lemma 2.26.** Suppose that $H_1, G$ and $B$ are given real numbers. Then the following two methods of defining $H_1, H_2, \ldots, H_k$ and $M_1, M_2, \ldots, M_k$ are equivalent,

1. $M_t = G(H_t + B)$ and $H_{t+1} = H_t + M_t$ for $t = 1, 2, \ldots, k$.
2. $M_1 = G(H_1 + B)$, $M_t = (1+G)^{t-1}M_1$ and $H_{t+1} = H_t + M_t$ for $t = 1, 2, \ldots, k$.

**Proof:** ((1) $\Rightarrow$ (2)). Since $M_t = G(H_t + B)$, we get $M_1 = G(H_1 + B)$ and, for $t \geq 2$,

$$M_t = G(H_t + B) = G(H_{t-1} + M_{t-1} + B)$$

$$= G(M_{t-1} + H_{t-1} + B) = G(M_{t-1} + (1/G)M_{t-1})$$

$$= (G+1)M_{t-1}.$$ 

Thus $M_t = (G+1)^{t-1}M_1$.

((2) $\Rightarrow$ (1)). Now

$$H_t - H_1 = (H_t - H_{t-1}) + (H_{t-1} - H_{t-2}) + \ldots + (H_2 - H_1)$$

$$= M_{t-1} + M_{t-2} + \ldots + M_1.$$
\[ t = (1 + G)^{t-2} M_1 + (1 + G)^{t-3} M_1 + \ldots + M_1 \]

\[ = \frac{(1 + G)^{t-1} - 1}{G} M_1 \]

\[ = ((1 + G)^{t-1} - 1)(H_1 + B) \]

\[ = (1 + G)^{t-1}(H_1 + B) - (H_1 + B). \]

Therefore we have \( H_1 = B = (1 + G)^{t-1}(H_1 + B) = \frac{1}{G} M_t \) which proves the lemma.

**Lemma 2.27.** Let \( x, u, \) and \( v \) be positive integers. Suppose that:

\[ x + (x-1) + \ldots + (x-u+1) = d_1 + d_2 + \ldots + d_\alpha, \]

\[ (x-u) + (x-u-1) + \ldots + (x-u-v+1) = d_{\alpha+1} + \ldots + d_{\alpha+\beta}. \]

\[ d_1 \leq d_2 \leq \ldots \leq d_{\alpha+\beta} \quad \text{and} \quad \frac{1}{2} uv(u+v) < d_1. \]

Then we have \( d_1 < d_{\alpha+\beta}. \)

**Proof:** Suppose that \( d_1 = d_2 = \ldots = d_{\alpha+\beta} = d. \) Then we get

\[ \frac{u(2x - u+1)}{2} = \alpha d \]

and

\[ \frac{v(2x - 2u - v+1)}{2} = \beta d. \]

It follows that
By subtracting the second equation from the first, we get the equation
\[
\frac{1}{2} uv(u+v) = (\alpha v - \beta u)d.
\]
We conclude that \( d \mid \frac{1}{2} uv(u+v) \) so that \( d \leq \frac{1}{2} uv(u+v) \). This is a contradiction to the hypothesis.

**Proposition 2.28.** \([L_1] \nsubseteq [L]\)

**Proof:** Let \( D_m = (m^2 + m - 1, m^2 - m - 2, \ldots, m) \) and

\( (d_n) = (D_1, D_2, \ldots, D_m, \ldots) \). Let \( A = \{a_n\} \in 2^I \) with \( a_1 = 1, a_n = a_{n-1} + d_1 + \ldots + d_{n-1} \). Then obviously \( A \in L \). We claim that

\( A \notin [L_1] \). Assume that \( A \in [L_1] \) so that \( A \subset A_1 \cup A_2 \cup \ldots \cup A_k \)

where \( A_i \in L_1 \) for \( i = 1, 2, \ldots, k \). Let us denote \( A_i = \{a^i_n\} \),

\( d_i = a^i_{n+1} - a^i_n \) for \( i = 1, 2, \ldots, k \). Since \( A_i \) are lacunary sets,

there exists \( N \) such that \( n \geq N \) implies \( d^i_n > (3k)^3 \) for all

\( i = 1, 2, \ldots, k \). Take \( a^* = \max(a^1_N, a^2_N, \ldots, a^k_N) \).

Consider the part \( P_m \) of \( A \) corresponding to \( D_m \) as indicated in the diagram below,
Let $\alpha(m)$ be the function such that $a_{\alpha(m)}$ is the initial element of $P_m$. Then we compute

$$\alpha(m) = 1^2 + 2^2 + \ldots + (m-1)^2 + 1 = \frac{1}{6} (m-1)m(2m-1)+1.$$

Take $m$ such that $a^* \leq a_{\alpha(m)}$. Let $M_0 = 3k$, $B = \frac{1}{2} (3k-1)$, $G = 9k^3$, $M_1 = G(m+3k+B)$ and $M_t = (1+G)^{t-1}M_1$ for $t = 2, 3, \ldots, k$. Then we have

$$M_k + M_{k-1} + \ldots + M_1 + M_0 = \frac{1}{G} ((1+G)^{k-1})M_1 + M_0 = ((1+G)^{k-1})(m+3k+B) + 3k.$$

Since $M_k + \ldots + M_0 = ((1+G)^{k-1})(m+3k+B) + 3k$ is a polynomial of $m$ of degree 1, we can further take $m$ such that $m^2 \geq M_k + M_{k-1} + \ldots + M_0$.

We will partition part of $P_m$ backwards from $a_{\alpha(m+1)}$ so that we get intervals $L_0, L_1, \ldots, L_k$ from right to left where the number of differences of $A$ in the interval $L_t$ is $M_t$ for $t = 0, 1, 2, \ldots, k$.

Let $H_t$ be the smallest $d_n$ in the interval $L_t$ at the right hand end of $L_t$.

---

**Diagram**

- $a_{\alpha(m)}$ at the left end.
- $a_{\alpha(m+1)}$ at the right end.
- $M_k$ differences between $L_k$ and $H_k$.
- $M_0$ differences between $L_0$ and $H_0$.
Then clearly $H_{t+1} = H_t + M_t$, $t \geq 0$. Since $H_1 = m+3k,$

\[ M_1 = G(H_1 + B), \quad B = \frac{1}{2}(3k-1), \quad \text{and} \quad M_t = (1+G)^t M_1, \]

we apply lemma 2.26 and we get $M_t = G(H_t + B)$ for $t = 1, 2, \ldots, k$.

At first we partition $L_t$ by 3k differences in $A$ from left to right and we get intervals $I^t_1, I^t_2, \ldots, I^t_M$ of $A$.

![Diagram of intervals and differences]

For each $I^t_j$, the number of elements of $A$ in the interval is 3k+1.

Since $I^t_j \subseteq A_1 \cup A_2 \cup \ldots \cup A_k$, we get

\[ 3k+1 = |I^t_j| = |(I^t_j \cap A_1) \cup (I^t_j \cap A_2) \cup \ldots \cup (I^t_j \cap A_k)| \]

\[ \leq |I^t_j \cap A_1| + |I^t_j \cap A_2| + \ldots + |I^t_j \cap A_k|. \]

Thus there exists $A_i$ such that $|I^t_j \cap A_i| \geq 3$.

![Diagram of intervals and differences]

Let $a_{p+1} = a^i_\delta + 1$, $a_{q+1} = a^i_\delta + \alpha + 1$, and $a_{r+1} = a^i_\delta + \alpha + \beta + 1$ be in $I^t_j \cap A_i$.

Then we get the equations
where \( x = d_{\beta} \), \( u = q-p \) and \( v = r-q \). Of course \( d^i_s \leq d^i_{s+1} \) and recall \( d^i_\beta > (3k)^3 \), so we can apply lemma 2.27 and get \( d^i_\delta < d^i_{\delta+\alpha+\beta} \).

Therefore we conclude that, for any \( t \), there exists \( A_i \) such that \( d^i_n \) increases in \( I_j^t \).

At first for the interval \( L_k \), there are \( M_k/3k \) \( I_j^k \)'s.

Thus there are at least \( \frac{M_k}{3k} \) increases in the \( d^i_n \)'s among \( A_1, A_2, \ldots, \) and \( A_k \). Thus there exists \( A_i \) such that there are at least \( \frac{M_k}{3k^2} \) increases of \( d^i_n \)'s in \( A_i \cap [\min L_k, \max L_k] \). Let \( d^i_{n_k} \) be the largest difference of \( A_i \) in the interval \( [\min L_k, \max L_k] \), then \( d^i_{n_k} > \frac{M_k}{3k^2} \). On the other hand \( M_k/G = M_k/9k^3 = (H_k+B) \) and so

\[
M_k/3k^2 = 3k(H_k+B) = 3k(2H_k+3k-1)/2
\]

is the length of the interval \( I_j^k \) as indicated in the diagram below.

Since \( D \) is a decreasing sequence, length of \( \frac{I_j^k}{M_k/3k} > \) length of \( I_j^t \) whenever \( k > t \). Therefore we get \( d^i_{n_k} > \) length of \( I_j^t \) whenever \( k > t \).
Without loss of generality let us assume that $A_1 = A_1$. Then for any $t < k$, $|A_1 \cap I^t_j| \leq 1$. For $I^k_j$ with $1 \leq j \leq \frac{M_{k-1}}{3k}$, there exists as before $A_i$ such that $|I_j^{k-1} \cap A_i| \geq 3$ and clearly $A_i \neq A_1$.

Thus we know that there are at least $M_k/3k$ increases of the $d_i$'s among $A_2, \ldots, A_k$ in $[\min L_{k-1}, \max L_{k-1}]$. So there exists $A_i \neq A_1$ such that there are at least $M_{k-1}/3k^2$ increases of $d_i$'s in the interval $[\min L_{k-1}, \max L_{k-1}]$. Without loss of generality we assume that $A_i = A_2$. Then $d_n^{2(k-1)} > M_{k-1}/3k^2$ where $d_n^{2(k-1)}$ is the largest difference of $A_2$ in the interval $[\min L_{k-1}, \max L_{k-1}]$.

Since $M_{k-1}/3k^2 = 3k(M_{k-1} + G) = 3k(H_{k-1} + B) = 3k(2H_{k-1} + 3k-1)/2$ is the length of the last interval $I^{k-1}_j$. Thus $A_2$ cannot appear more than once in $I_j^{k-1}$'s where $h \leq k-1$. Thus there exists $A_3$ such that there are at least $M_{k-2}/3k^2$ increases in the differences of $A_3$ in the interval $[\min L_{k-2}, \max L_{k-2}]$.

By repeating this process $k$ times, for all $d_i$ are larger than the length of $I^{M_k}_{3k}$. Thus $|A_i \cap I_0| \leq 1$, for

$$i = 1, 2, \ldots, k.$$  Hence

$$|I_0 \cap (A_1 \cup A_2 \cup \ldots \cup A_k)|$$

$$\leq |I_0 \cap A_1| + |I_0 \cap A_2| + \ldots + |I_0 \cap A_k|$$

$$\leq k < 3k+1 = |I_0|.$$
Therefore $I_0$ is not covered by $A_1 \cup A_2 \cup \ldots \cup A_k$, which is a contradiction. This completes the proof.

**Summary for Chapter II.**

In Chapter II, we have shown that:

1. $L, \hat{L}_1, \hat{L}_2$ and $\hat{L}_M$ (for $i \geq 1$) are full. (Propositions 2.6 and 2.8).
2. $\hat{L}_M = L$ (for $i \geq 1$). (Proposition 2.8).
3. $\hat{L}_M \supsetneq \hat{L}_1$ and $\hat{L}_2 \supsetneq \hat{L}_1$. (Propositions 2.12 and 2.16).
4. $[L_3] \supsetneq [\hat{L}_1], [L_1] \supsetneq [L], [L_M] \supsetneq [L_M]$ if $0 \leq i < j$, and $[L_2] \subsetneq [L_1]$. (Propositions 2.17, 2.20, 2.19 and 2.21).
5. $[L_1] \subsetneq [\hat{L}_1]$ and $\hat{L}_1 \subsetneq [\hat{L}_1]$ (Propositions 2.25 and 2.22).
6. $[\hat{L}_1] \subsetneq [L]$. (Proposition 2.28).

Freedman has found the relation:

$A \in [L]$ if and only if $\chi_A \in \text{bs} + c_0$, where

$\text{bs} = \{x \in w : \sup_n \sum_{i=1}^{n} x_i < \infty\}$ and $c$ is the space of convergent sequences.

We have tried to find a sequence space $V_1$ with $A \in [\hat{L}_1]$ if and only if $\chi_A \in V_1$. This is especially important because, as we have seen, $[\hat{L}_1]$ turns out to be unequal to $[L]$. In particular we wanted to construct a $V_1$ which can be defined by an analytic
expression as the space $bs$ is defined by $\sup_{m+n} \left| \sum_{i=1}^{n} x_i \right| < \infty$, or

$bs + c$ is defined by $\sup \limsup_{n} \left| \sum_{i=m+1}^{m+n} (x_i - r) \right| < \infty$ (5). The existence of such an analytic formulation for $V_1$ is still an open question. However, for any zero class $Z$, if we take

$$V = \{ x \in w : \text{ for any } \alpha > 0, \{ i \in I : |x_i| > \alpha \} \in Z \}$$

then we have $A \in X$ if and only if $\chi_A \in V^0$ (Proposition 3.29).

This is the main motivation of our study in Chapter III.
CHAPTER III

R-TYPE SUMMABILITY METHODS

The concept of an R-type summability method (RSM) is introduced and studied to some depth. Each RSM is defined on a subspace of \( \omega \), the space of all real sequences. We topologize \( \omega \) with the topology induced by uniform convergence (this is somewhat unorthodox). It turns out that an RSM is regular, non-negative and continuous with respect to this topology.

We will build on the results of Freedman and Sember [3] and ultimately obtain a bounded consistency type theorem for RSMs on their strong summability fields. When the RSM is induced by a regular matrix, our result is implied by the standard Bounded Consistency Theorem [7] although our proof does not require the same degree of depth. There are however RSMs which are not generated by any matrix. (See Proposition 3.46).

The thesis finishes with an attempt to understand those RSMs which are generated by regular matrices.

Most of the notation employed in this chapter can be found in the appendix.

Definition 3.1. Let \( C_S \) be a subspace of \( \omega \). Let \( S: C_S \to \mathbb{R} \) be a linear functional. We let
\[ C_S^0 = \{ x \in C_S : S(x) = 0 \}; \]
\[ |C_S|^0 = \{ x \in \omega : |x| \in C_S^0 \}; \]
\[ |C_S| = \{ x \in \omega : \exists r \in R \text{ such that } x-r \in |C_S|^0 \}. \]

We say that \( S \) is a summability method, \( C_S \) is the convergence field associated with \( S \), \( |C_S| \) is the strong convergence field associated with \( S \).

Remark: Since \( S \) is a linear functional \( C_S^0 \) is the kernel of \( S \) and so \( C_S^0 \) is a linear space.

We introduce several types of summability methods.

**Definition 3.2.** Let \( S: C_S \rightarrow R \) be a summability method. We say that

1. \( S \) is regular, in case \( c \subset C_S \) and for any \( x \in c \), \( S(x) = \lim x \).
2. \( S \) is nonnegative if \( x \in C_S \), \( x \geq 0 \) (i.e., \( x_i \geq 0 \) for each \( i \)) then \( S(x) \geq 0 \).
3. \( S \) is an R-type summability method (RSM) in case \( m|C_S|^0 = |C_S|^0 \) and \( S \) is regular.

Note: For matrix methods "regular" has the usual meaning but "nonnegative" is somewhat different here. Definition (1) and (3) are from [3].

**Proposition 3.3.** For any summability method \( S: C_S \rightarrow R \), the condition \( m|C_S|^0 \subset C_S \) is equivalent to the condition \( x \in |C_S|^0 \) and \( |y| \leq |x| \) implies \( y \in C_S \).

**Proof.** Assume that \( m|C_S|^0 \subset C_S \). Suppose that \( x \in |C_S|^0 \) and
and \(|y| \leq |x|\). Let \(z \in \omega\) with

\[
z_i = \begin{cases} 
1 & \text{if } x_i = 0 , \\
\frac{y_i}{x_i} & \text{if } x_i \neq 0 .
\end{cases}
\]

Then \(|z| \leq e = (1,1,...)\) and \(y = zx\). By the assumption

\[
m|C_s| \subseteq C_s , \ y \in C_s .
\]

Assume that \(x \in |C_s| \) and \(|y| \leq |x|\) implies \(y \in C_s\). Suppose \(a \in m, x \in |C_s| \). Let \(|a_i| \leq M\) for each \(i = 1,2,...\). Then \(S(|Mx|) = MS(|x|) = MO = 0\). Therefore \(Mx \in |C_s|\). Since \(|ax| \leq |Mx|\) and \(Mx \in |C_s|\), by the assumption, \(ax \in C_s\).

**Proposition 3.4.** Let \(S: C_s \rightarrow R\) be a regular summability method. Then \(S\) is an RSM if and only if

\[
x \in |C_s| \quad \text{and} \quad |y| \leq |x| \quad \text{implies} \quad y \in |C_s|
\]

**Proof:** Assume that \(m|C_s| = |C_s|\). Suppose that \(x \in |C_s|\)

and \(|y| \leq |x|\). Let \(z \in \omega\) with

\[
z_i = \begin{cases} 
1 & \text{if } x_i = 0 , \\
\frac{y_i}{x_i} & \text{if } x_i \neq 1 .
\end{cases}
\]

As in the proof of the previous proposition \(z \in m\) and \(y = zx\). By

the assumption \(y \in |C_s|\).
Assume that, if $x \in |C_s|^0$ and $|y| \leq x$, then $y \in |C_s|^0$.

Since $|C_s|^0 \subset m|C_s|^0$, it is enough to show $m|C_s|^0 \subset |C_s|^0$. The proof is similar to that of the previous proposition and is omitted.

**Proposition 3.5.** ([3]. Proposition 4.8). If $S: C_s \to R$ is an RSM, then

1. $|C_s|^0 \subset C_s^0$,
2. $m|C_s|^0 \subset C_s^0$,
3. $|C_s| \subset C_s$.

**Proof:** We omit the proof since readers can find the proof in the reference.

**Proposition 3.6.** ([3]. Proposition 4.9). If $S$ is an RSM, then $|C_s|$ and $|C_s|^0$ are subspaces of $C_s$ and $C_s^0$, respectively. Furthermore, $c \subset |C_s|$ and $c_0 \subset |C_s|^0$.

**Proof:** We omit the proof.

Now we proceed to investigate further properties of an RSM.

**Proposition 3.7.** If $S$ is an RSM then $S$ is nonnegative.

**Proof:** Suppose that $x \in C_s$ and $x \geq 0$. We want to show that $S(x) \geq 0$.

Assume that $S(x) = -r$, where $r > 0$. Then $|x+r| = x+r$ and $S(x+r) = S(x) + S(re) = S(x)r = -r+r = 0$. Hence $x+r \in |C_s|^0$. By
proposition 3.4, and since \( re \leq x+r, re \in |C_s|^0 \). Thus \( S(re) = 0 \),
which contradicts that \( S \) is regular. Thus \( S(x) \geq 0 \).

**Proposition 3.8.** If \( S: C_s \rightarrow R \) is a nonnegative summability
method then for any \( x, y \) in \( C_s \), \( x \leq y \) implies \( S(x) \leq S(y) \).

**Proof:** This is a standard result.

**Proposition 3.9.** If \( S: C_s \rightarrow R \) is a nonnegative and regular
summability method then \( S: C_s \rightarrow R \) is a continuous function where
\( C_s \) is given the topology of uniform convergence (write \( T_\infty \) for this
topology).

**Proof:** For \( x \in m \), we denote \( ||x||_\infty = \sup_n |x_n| \). It is sufficient
to show that, for any \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that
\[ |S(x) - S(y)| \leq \varepsilon \]
whenever \( x, y \in C_s \), \( x-y \in m \) and \( ||x-y||_\infty < \delta \).

Take \( \delta = \varepsilon \). If \( x, y \in C_s \), \( x-y \in m \) and \( ||x-y||_\infty < \delta \), then
\[ -\delta e \leq x-y \leq \delta e \]. Thus, by Proposition 3.8, \( S(-\delta e) \leq S(x-y) \leq S(\delta e) \).
Since \( S \) is regular and linear, \(-\delta \leq S(x) - S(y) \leq \delta \).

**Corollary 3.10.** If \( S: C_s \rightarrow R \) is an RSM then \( S \) is continuous,
where \( C_s \) has the topology \( T_\infty \).

**Proof:** If \( S: C_s \rightarrow R \) is an RSM then \( S \) is a nonnegative and
regular by Proposition 3.7. It follows that, by Proposition 3.9, \( S \)
is continuous.

**Proposition 3.11.** If \( S: C_s \rightarrow R \) is an RSM then for any \( x \in C_s \),
\[ \liminf x_n \leq S(x) \leq \limsup x_n, \]

**Proof:** Let \( x \in C_s \). If \( \liminf x_n = -\infty \) then obviously \( \liminf x_n \leq S(x) \). Suppose that \( \liminf x_n > -\infty \). Let \( x \in C_s \) and \( y_n = \inf_{k \geq n} x_k \in \mathbb{R} \). Then \( y = (y_1, y_2, \ldots, y_n, \ldots) \leq x \) and \( \lim y = \liminf x_n \). Consider the eventually constant sequences \( z^n \), where

\[
 z^n = \begin{cases} 
 y_i & \text{if } i \leq n, \\
 y_n & \text{if } i > n. 
\end{cases}
\]

Then \( z^n \in C_s \) and \( \lim z^n = y_n \) and \( z^n \leq x \). Since \( S \) is an RSM, \( z^n \in C_s \) and \( S(z^n) = \lim z^n = y_n \). Therefore \( \liminf x_n = \lim y \leq S(x) \).

Finally, \( \liminf (-x_n) \leq S(-x) \). Thus \( S(x) \leq \limsup x_n \).

**Corollary 3.12.** If \( S: C_s \to \mathbb{R} \) is an RSM then \( C_s \neq \omega \) (the space of all sequences).

**Proof:** By the previous proposition \( \liminf x_n \leq S(x) < \infty \) and \( \limsup x_n > -\infty \). Hence no sequence which diverges to \( \infty \) or \( -\infty \) can be in \( C_s \).

**Proposition 3.13.** Suppose that \( S: C_s \to \mathbb{R} \) is a regular summability method, \( m|C_s|_0 \subseteq C_s \) and \( m \notin C_s \). Then \( S \) is an RSM.
Proof: By Proposition 3.4, it is sufficient to show, $x \in |C_s|^0$ and \[ |y| \leq |x| \] implies $y \in |C_s|^0$. Suppose that $x \in |C_s|^0$ and \[ |y| \leq |x| \]. By Proposition 3.3, $|y| \in C_s$.

Case (1). Assume that $S(|y|) = -r < 0$. Then $|y| + r > 0$ and $S(|y| + r) = S(|y|) + S(re) = -r + r = 0$. By the definition of $|C_s|^0$, \[ |y| + r \in |C_s|^0 \].

Let $z \in m$ with $||z||_{\infty} = b$. Then, for each $i$, \[ \frac{|z_i|}{|z_i| + r} \leq \frac{1}{r} |z_i| \leq \frac{b}{r}. \]

Thus \[ \frac{z}{|y| + r} \in m \] and so \[ z = \frac{z}{|y| + r} (|y| + r) \in m|C_s|^0 \subset C_s \]
by the hypothesis. Therefore $m \subset C_s$, which contradicts the hypothesis.
Hence $S(|y|) \geq 0$.

Case (2). Assume that $S(|y|) = r > 0$. Take $w = |x| - |y|$. Then $|w| = w \leq |x|$. By Proposition 3.3 and the condition $m|C_s|^0 \subset C_s$, \[ |w| \in C_s \]. Therefore \[ S(|w|) = S(|x| - |y|) = S(|x|) - S(|y|) = 0 - r < 0 \].
We can apply the argument of Case (1) to $w$ and get $S(|w|) \geq 0$, a contradiction. Thus $S(|y|) = 0$ and $y \in |C_s|^0$. 
Example 3.14. Let $B$ be a Hamel basis for $c$ and $B \cup D$ be a Hamel basis for $\omega$ where $B \cap D = \emptyset$. For any $x \in \omega$, we can express uniquely

$$x = \sum_{b \in B} \alpha_b b + \sum_{d \in D} \beta_d d$$

where $\alpha_b, \beta_d$ are all zero except for finitely many. Let us write

$$x_B = \sum_{b \in B} \alpha_b b, \quad x_D = \sum_{d \in D} \beta_d d.$$  

We define linear functional $S: \omega \to R$ such that $S(x) = \lim x_B$ for any $x \in \omega$. Then $S: \mathcal{C}_s = \omega \to R$ is a regular summability method such that $m|\mathcal{C}_s|^0 \subset \mathcal{C}_s$, $m \subset \mathcal{C}_s$ but $S$ is not an RSM.

Proof: Obviously $m|\mathcal{C}_s|^0 \subset \mathcal{C}_s$ and $m \subset \mathcal{C}_s$. By Corollary 3.12, since $\mathcal{C}_s = \omega$, $S$ cannot be an RSM.

If $S: \mathcal{C}_s \to R$ is a regular summability method and $\mathcal{C}_s$ is small, that is, $m \not\subset \mathcal{C}_s$, then we can replace the condition $m|\mathcal{C}_s|^0 = |\mathcal{C}_s|^0$ to $m|\mathcal{C}_s|^0 \subset \mathcal{C}_s$ for $S$ being an RSM. For example, $AC, \omega_\delta$ where $\delta$ is ordinary density (see Definition 3.22), $V_X$ when $X$ is not an ultra zeroclass (see Definition 3.27 and Proposition 3.45).

We have studied some properties of RSMs. Next we will illustrate some examples which show the difference between nonnegative regular summabilities and RSMs.

Example 3.15. Let $A \in 2^I$ be an infinite set with $\delta(A) = 0$, where $\delta$ is the ordinary density. Let $\mathcal{C}_s = c \Theta < x_A >$ where $< x_A >$
denotes the linear subspace of $\omega$ spanned by $\chi_A$. Let $S: C_s \to \mathbb{R}$ be defined by $S(\chi s + t) = \lim x$, where $x \in c$ and $t \in < \chi_A >$. Then $S$ is regular and nonnegative, $C_s$ is a closed subset of $(\omega, T_w)$ but $m|C_s| 0 \not\subset C_s$.

**Proof:** $S$ is nonnegative and regular immediately from the definition. Next we want to show: $c \oplus < \chi_A >$ is a closed subset of $(\omega, T_w)$.

Suppose that $x^n \in c$ and $x^n + t_n \chi_A + z \in \omega$ with respect to $T_w$, that is, $\|x^n + t_n \chi_A - z\|_\omega \to 0$ as $n \to \infty$. Suppose that $\lim x^n = r_n$ for each $n$. First let us show $\{r_n\}$ is a Cauchy sequence. For any $\varepsilon > 0$, there exists $N \in I$ such that if $n, m > N$ then $\|x^n + t_n \chi_A - (x^m + t_m \chi_A)\|_\omega < \varepsilon$. Then

$$|r_n - r_m| = |r_n - x^n_i + x^n_i - x^m_i + x^m_i - r_m|$$

$$\leq |x^n_i - r_n| + |x^m_i - x^n_i| + |x^m_i - r_m|.$$ 

Since $I - A$ is infinite, for each $i \in I - A$, we get

$$|r_n - r_m| \leq |x^n_i - r_n| + |x^n_i - t_n \chi_A(i) - (x^m_i - t_m \chi_A(i))| + |x^m_i - r_m|$$

$$\leq |x^n_i - r_n| + |x^m_i - r_m| + \|(x^n_i - t_n \chi_A - (x^m_i - t_m \chi_A)\|_\omega$$

$$\leq |x^n_i - r_n| + |x^m_i - r_m| + \varepsilon.$$
Thus, since $x_i^n \rightarrow r_i$ and $x_i^m \rightarrow r_m$, we have

$$|r_n - r_m| \leq \inf_{i \in I - A} (|x_i^n - r_i| + |x_i^m - r_m|) + \varepsilon$$

$$= 0 + \varepsilon.$$

Therefore $\{r_n\}$ is a Cauchy sequence. Further we know that for any $\varepsilon > 0$ there exists $N \in I$ such that $n, m > N$ implies

$$\|(x^n - t_n x_A) - (x^m - t_m x_A)\|_{\infty} < \varepsilon \quad \text{and} \quad |r_n - r_m| < \varepsilon.$$

Suppose $n, m > N$ and $i \in A$, then we have

$$|t_n - t_m| = |t_n x_A(i) - t_m x_A(i)|$$

$$= |t_n x_A(i) - x_i^n + x_i^n - r_n + r_n - r_m + r_m - x_i^m + x_i^m - t_m x_A(i)|$$

$$\leq |t_n x_A(i) - x_i^n + x_i^m - t_m x_A(i)| + |x_i^n - r_n| + |r_n - r_m| + |r_m - x_i^m|$$

$$\leq \|(x^n - t_n x_A) - (x^m - t_m x_A)\|_{\infty} + |r_m - r_n| + |x_i^n - r_n| + |x_i^m - r_m|$$

$$\leq 2\varepsilon + |x_i^n - r_n| + |x_i^m - r_m|.$$

Since $x_i^m \rightarrow r_m$, $x_i^n \rightarrow r_n$ and $A$ is given to be infinite, we have

$$\inf_{i \in A} (|x_i^n - r_n| + |x_i^m - r_m|) = 0. \quad \text{Therefore} \quad |t_n - t_m| \leq 2\varepsilon. \quad \text{Hence}$$

$\{t_n\}$ is a Cauchy sequence. Let $\lim_{n} t_n = t$. Then $t_n x_A$ uniformly converges to $t x_A$, and so $x_n = (x^n - t_n x_A) + t_n x_A$ uniformly converges to $z - t x_A$. Since $c$ is a closed subset of $(\omega, T_\infty)$, it follows that $z - t x_A \in c$ and so $z \in c \cup < x_A >$. Consequently $c \cup < x_A >$ is a closed subset of $(\omega, T_\infty)$. 


Finally we prove that \( m|_{C_s}^0 \not\subset C_s \). Let \( B \subset A \) such that \( B \) and \( A - B \) are infinite. Then \( \chi_B \in m|_{C_s}^0 \) since \( \chi_A \not\in |_{C_s}^0 \).

We show that \( \chi_B \not\in C_s \).

Suppose otherwise and assume that \( \chi_B = x + r\chi_A \) where \( x \in C \) and \( r \in \mathbb{R} \). We get

\[
0 = \lim_{i \to \infty} \chi_B(i) = \lim_{i \to \infty} (x_i + r\chi_A(i)) = \lim_{i \to \infty} x_i = \lim_{i \to \infty} x.
\]

Again,

\[
0 = \lim_{i \to \infty} \chi_B(i) = \lim_{i \to \infty} (x_i + r\chi_A(i)) = \lim_{i \to \infty} r\chi_A(i) = r.
\]

Consequently \( \chi_B = x + 0\chi_A = x \). But \( \chi_B \not\in C \), a contradiction.

By this example we can declare the following proposition.

**Proposition 3.17.** \( S \) being regular and nonnegative does not imply \( m|_{C_s}^0 \subset C_s \).

**Example 3.18.** Suppose that \( f: m \to \mathbb{R} \) and \( g: m \to \mathbb{R} \) are continuous regular summabilities from \( (m, T_\omega) \) into \( \mathbb{R} \). e.g., \( f \) and \( g \) can be "Banach limits" \([4]\). Let us define \( h: m \to \mathbb{R} \) such that

\[
h(x) = 2f(x_1, x_3, \ldots, x_{2n+1}, \ldots) - g(x_2, x_4, x_6, \ldots, x_{2n}, \ldots).
\]

Then \( h \) is continuous and regular and \( m|_{C_s}^0 \subset C_s \) but \( h \) is not nonnegative.
Proof: Given $x$, let $y = (x_1, x_3, x_5, \ldots)$ and $z = (x_2, x_4, x_6, \ldots)$.

Then we have

$$|h(x)| = |2f(y) - g(z)|$$

$$\leq 2|f(y)| + |g(z)|$$

$$\leq 2\|f\|\|y\|_{\infty} + \|g\|\|z\|_{\infty}$$

$$\leq (2\|f\| + \|g\|)\|x\|_{\infty}.$$

Thus $h$ is bounded, equivalently $h$ is continuous.

For any $x \in \mathcal{C}$, since $f$ and $g$ are regular,

$$h(x) = 2f(x_1, x_3, \ldots, x_{2n+1}, \ldots) - g(x_2, x_4, \ldots, x_{2n}, \ldots)$$

$$= 2 \lim_{n \to \infty} x_{2n+1} - \lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} x.$$

Thus $h$ is regular. But $h(0, 1, 0, 0, \ldots) = 2f(0, 0, 0, \ldots) - g(1, 1, 1, \ldots) = 2 \cdot 0 - 1 = -1$. Thus $h$ is not nonnegative.

By the above example we conclude that:

**Proposition 3.19.** A summability method $S: \mathbb{C}_s \to \mathbb{R}$ being regular, continuous and satisfying $m|\mathbb{C}_s|^0 \subset \mathbb{C}_s$ does not imply that $S$ is nonnegative (compare example 3.14).
Proposition 3.20. If $S : C_\mathbb{R} \to \mathbb{R}$ is regular and nonnegative and $m|C_s|^0 \subseteq C_s$ then $S$ is an RSM. (Compare Proposition 3.4.).

Proof: Suppose that $x \in |C_s|^0$ and $|y| \leq |x|$. Since $m|C_s|^0 \subseteq C_s$ and $|y| \in m|C_s|^0$, we get $|y| \in C_s$. Since $S$ is nonnegative, $0 \leq S(|y|) \leq S(|x|) = 0$, it follows that $S(|y|) = 0$, equivalently $y \in |C_s|^0$. By Proposition 3.4. $S$ is an RSM.

We have shown that an RSM is nonnegative and continuous under the topology $T_\infty$. Next we will find some relation between densities and RSMs. Freedman and Sember have found a connection between RSMs and densities. ([3], Proposition 4.10). We will extend this result so that we obtain a "Bounded Consistency Theorem" on strong convergence fields.

Definition 3.21. Let $x \in \omega$ and $r \in \mathbb{R}$ and $A \in 2^I$ with $I-A$ infinite. We write $x \rightarrow r$ in case for each $\varepsilon > 0$ there exists $N > 0$ such that $|x_n - r| < \varepsilon$ whenever $n \geq N$, $n \notin A$.

Definition 3.22. For any density $\delta$, let

$\omega_\delta = \{x \in \omega : \exists r \in \mathbb{R} \text{ and } A \subset I \text{ with } \delta(A) = 0 \text{ and } x \rightarrow r \}$. 

For any zeroclass $X$, let $\omega_X = \{x \in \omega : \exists r \in \mathbb{R} \text{ and } A \subset I \text{ with } x \in X \text{ and } x \rightarrow r \}$. We call $\omega_\delta$ the set of ($\delta$-) nearly convergent sequences. We call $\omega_X$ the set of ($X$-) nearly convergent sequences.
Proposition 3.23. For any density $\delta$, $\eta_{\delta}^0 = \{ A \in \omega \mid \delta(A) = 0 \}$ is a zeroclass (definition 1.7.).

Proof: Let $A \in 2^I$ be finite. Then $\delta(A) = \delta(\phi) = 0$ (by Proposition 1.1, V). Thus $A \in \eta_{\delta}^0$. Suppose that $A$ and $B$ are in $\eta_{\delta}^0$. Then $\delta(A \cup B) \leq \delta(A) + \delta(B) = 0 + 0 = 0$ and $A \cup B \in \eta_{\delta}^0$. Let $A \subset B$ and $B \in \eta_{\delta}^0$. Then $\delta(A) \leq \delta(B) = 0$ and $A \in \eta_{\delta}^0$. Finally $\delta(I) = 1$ so that $I \notin \eta_{\delta}^0$.

Proposition 3.24. For any zeroclass $X$ let

$$d_X(A) = \begin{cases} 1 & \text{if } I-A \in X \\ 0 & \text{otherwise.} \end{cases}$$

Then $d_X$ is a density with $\eta_{d_X}^0 = X$.

Proof: By example 1.27 $d_X$ is a density. For any $A \in 2^I$, $A \in X$ if and only if $I-(I-A) \in X$ if and only if $d_X(I-A) = 1$ if and only if we $d_X(A) = 0$ if and only if $A \in \eta_{d_X}^0$. Therefore $X = \eta_{d_X}^0$.

Remark: For any density $\delta$, $\eta_{\delta}^0$ is a zeroclass. And for any zeroclass $X$, there exists a density $\delta$ such that $\eta_{\delta}^0 = X$. Therefore we do not have to distinguish between $\omega_{\delta}$ and $\omega_X$. $\omega_{\delta}$ may be considered as $\omega_{\eta_{\delta}^0}$ and $\omega_X$ may be considered as $\omega_{d_X}$. 
Proposition 3.25. For any zeroclass \( X \), \( \omega_X \) is a linear space of sequences with \( c \subseteq \omega_X \). ([3], Proposition 4.3).

Proof: If \( x \in c \) then \( x \xrightarrow{r} r \) for some \( r \in R \). Since \( \phi \in X \), \( x \in \omega_X \).

Let \( x \) and \( y \) be in \( \omega_X \) and let \( r_1, r_2, A, B \) be such that

\[
A, B \in X \quad \text{and} \quad x \xrightarrow{(A)} r_1 \quad \text{and} \quad y \xrightarrow{(B)} r_2, \quad A \cup B \in X \quad \text{and} \quad x + y \xrightarrow{(A \cup B)} r_1 + r_2.
\]

So that \( x + y \in \omega_X \). Suppose that \( k \in R \), then \( kx \xrightarrow{(A)} kr_1 \). Therefore \( kx \in \omega_X \). Hence \( \omega_X \) is a linear space.

Definition 3.26. [3]. A density \( \delta \) (resp. zeroclass \( X \)) and RSM are related if, for each \( A \in 2^I \)

\[
\delta(A) = 0 \quad (\text{resp. } A \in X) \Rightarrow \chi_A \in |C_s|_0^0.
\]

Now, let us introduce a new technique using the zeroclass concept which will pave the way to the bounded consistency theorem on the strong convergence fields.

Definition 3.27. For any zeroclass \( X \), we denote

\[
V_X = \{x \in \omega: \text{for any } \alpha > 0 \{i: \alpha < |x_i| \} \in X\}
\]

\[
V_X = \{x \in \omega: \exists r \in R, x-r \in V_X\}.
\]

Proposition 3.28. For any zeroclass \( X \),
(1) \( V_X \) is a linear space of sequences.

(2) \( V_X^0 \) is a subspace of \( V_X \).

Proof: Suppose that \( x \in V_X \) and \( y \in V_X \) and \( r_1, r_2 \in R \) with \( x-r_1 \in V_X^0 \), \( y-r_2 \in V_X^0 \). Since, for any \( i \),

\[
| x_i + y_i - (r_1 + r_2) | \leq | x_i - r_1 | + | y_i - r_2 | ,
\]

for any \( \alpha > 0 \),

\[
|x_i - r_1| \leq \frac{\alpha}{2} \quad \text{and} \quad |y_i - r_2| \leq \frac{\alpha}{2} \Rightarrow | x_i + y_i - (r_1 + r_2) | \leq \alpha .
\]

Thus

\[
\{ i : | x_i + y_i - (r_1 + r_2) | > \alpha \} \subseteq \{ i : | x_i - r_1 | > \frac{\alpha}{2} \} \cup \{ i : | y_i - r_2 | > \frac{\alpha}{2} \}.
\]

By the definition of \( V_X^0 \) and the properties of zero classes

\[
\{ i : | x_i - r_1 | > \frac{\alpha}{2} \} \cup \{ i : | y_i - r_2 | > \frac{\alpha}{2} \} \subseteq X . \quad \text{Thus}
\]

\[
\{ i : \alpha < | (x_i + y_i) - (r_1 + r_2) | \} \subseteq X . \quad \text{Consequently it follows that}
\]

\( x+y \in V_X \). If \( k \in R \), then for any \( \alpha > 0 \),

\[
\{ i : \alpha < | kx_i - kr_1 | \} = \begin{cases} 
\phi & \text{if } k = 0 \\
\{ i : \frac{\alpha}{|k|} < | x_i - r_1 | \} & \text{if } k \neq 0 .
\end{cases}
\]

Therefore for any \( \alpha > 0 \), \( \{ i : \alpha < | kx_i - kr_1 | \} \subseteq X \), which implies
kx ∈ V_χ. Hence V_χ is a linear space of sequences.

(2) In the proof of (1) we can put r_1 = r_2 = 0. The other steps are all the same. Hence V_χ^0 is a subspace of V_χ.

Proposition 3.29. For any zeroclass X, let T_χ: V_χ → R be the function from V_χ to R defined by

T_χ(x) = r = x - r ∈ V_χ^0.

Then T_χ is an RSM with domain C_{T_χ} = V_χ and |C_{T_χ}| = C_{T_χ} = V_χ
and |C_{T_χ}|^0 = C_{T_χ}^0 = V_χ^0. Further, T_χ is related with the zeroclass X.

Proof: In this proof we will write T_χ as T for convenience.

We first prove T is well defined.

Suppose that x - r_1 ∈ V_χ^0 and x - r_2 ∈ V_χ^0. Since V_χ^0 is a linear space (x - r_1) - (x - r_2) = (r_2 - r_1)e ∈ V_χ^0. From the facts

\[
\{i: α < |(r_2 - r_1)e_i|\} = \begin{cases} \emptyset & \text{if } α ≥ |r_2 - r_1| \\ \{i\} & \text{if } α < |r_2 - r_1| \end{cases}
\]

and \{i: α < |(r_2 - r_1)e_i|\} ∈ X for any α > 0, it follows that r_1 = r_2. Thus T is well defined.

Suppose that x, y ∈ V_χ and T(x) = r_1 and T(y) = r_2.
Then $x-r_1 \in V_X^0$ and $y-r_2 \in V_X^0$. Since $V_X^0$ is a linear space, $x+y-(r_1+r_2) \in V_X^0$ and $kx-kr_1 \in V_X^0$ for any real number $k$.

Therefore $T(x+y) = T(x) + T(y)$ and $T(kx) = kT(x)$. Hence $T$ is a linear functional.

We have

$$c_T^0 = \{ x \in \omega: T(x) = 0 \}$$

$$= \{ x \in \omega: x \in V_X^0 \} = V_X^0$$

$$= \{ x \in \omega: \text{for any } \alpha > 0, \{ i: \alpha < |x| \} \in X \}$$

$$= \{ x \in \omega: |x| \in V_X^0 \}$$

$$= \{ x \in \omega: |x| \in c_T^0 \}$$

$$= |c_T^0|.$$

Therefore

$$c_T^0 = V_X^0 = |c_T^0|.$$

Furthermore,

$$c_T = V_X$$

$$= \{ x \in \omega: \exists r \in R \ x-r \in V_X^0 \}$$

$$= \{ x \in \omega: \exists r \in R \ x-r \in |c_T^0| \}$$

$$= |c_T|.$$
Thus
\[ C_T = V_X = |C_T|. \]

We now use Proposition 3.4 to show that \( T \) is an RSM.

Suppose that \( x \in |C_T| \) and \( |y| \leq |x| \). For any \( i \),
\[ |y_i| \leq |x_i|. \]
Thus \( \alpha < |y_i| = \alpha < |x_i| \). Hence for any \( \alpha > 0 \),
\[ \{i: \alpha < |y_i|\} \subseteq \{i: \alpha < |x_i|\}. \]
On the other hand \( \{i: \alpha < |x_i|\} \in X \) for any \( \alpha > 0 \). Hence, for any \( \alpha > 0 \), \( \{i: \alpha < |y_i|\} \in X \).

Consequently, we have \( y \in |C_T| \). Hence \( T \) is an RSM.

For any \( A \in 2^I \),
\[
\{i: \alpha < x_A(i)\} = \begin{cases} 
A & \text{if } 0 < \alpha < 1 \\
\emptyset & \text{if } 1 \leq \alpha .
\end{cases}
\]

Hence \( x_A \in V_X^0 = |C_T|^0 = A \in X \). Thus \( T \) and \( X \) are related.

**Proposition 3.30.** For any zero class \( X \), \( V_X \) is closed with respect to the topological space \((\omega, \tau)\).

**Proof:** Suppose that \( x \in V_X \) and choose \( \{x^n\} \subseteq V_X \) such that
\[ \|x^n - x\|_\infty = \sup_{i \geq 1} |x^n - x_i| < \frac{1}{n} \quad (n \geq 1). \]
Suppose that \( T_X(x^n) = T(X) = x_n \).

Since \( \{x^n\} \) converges to \( x \), we have, for any \( \varepsilon > 0 \), that there exists \( N \in I \) such that \( n, m \geq N \Rightarrow \|x^n - x^m\|_\infty < \varepsilon \). We want to show that
n,m \geq N \Rightarrow |r_n - r_m| \leq \varepsilon. Suppose that n,m \geq N. For each i \in I

\begin{align*}
|r_n - r_m| &\leq |r_n - x_i^n| + |x_i^n - x_i^m| + |x_i^m - r_m| \\
&< |r_n - x_i^n| + \varepsilon + |x_i^m - r_m|.
\end{align*}

Clearly

I = \{i: |r_n - r_m| - \varepsilon < |r_n - x_i^n| + |x_i^m - r_m|\}

\subseteq \{i: \frac{|r_n - r_m| - \varepsilon}{2} < |r_n - x_i^n|\} \cup \{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^m - r_m|\}.

If

\begin{align*}
|r_n - r_m| > \varepsilon,
\end{align*}

then

\begin{align*}
\{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^n - r_n|\} \in X
\end{align*}

and

\begin{align*}
\{i: \frac{|r_n - r_m| - \varepsilon}{2} < |x_i^m - r_m|\} \in X
\end{align*}

and it follows that I \notin X, which is a contradiction. Therefore

\begin{align*}
|r_n - r_m| \leq \varepsilon.
\end{align*}

Consequently it follows that \{r_m\} is a Cauchy sequence of real numbers. Let \(\lim_{n} r_n = r \in \mathbb{R}\).
Now we claim that \( x \in V_X \). For any \( \alpha > 0 \) there exists \( N \in I \) such that \( n > N = \| x - x^n \|_\infty < \frac{\alpha}{3} \) and \( | r_n - r | < \frac{\alpha}{3} \). For any \( i \in I \),

\[
|x_i - r| \leq |x_i - x^n_i| + |x^n_i - r_n| + |x_n - r| < \frac{2\alpha}{3} + |x^n_i - r_n| .
\]

So that

\[
\{ i : \alpha < |x_i - r| \} \subset \{ i : \frac{\alpha}{3} < |x^n_i - r_n| \} .
\]

Since \( T(x^n) = r_n \),

\[
\{ i : \frac{\alpha}{3} < |x^n_i - r_n| \} \in X .
\]

Therefore \( \{ i : \alpha < |x_i - r| \} \in X \). Hence \( x \in V_X \), and so, it follows that \( \overline{V}_X \subset V_X \).

**Proposition 3.31.** For any zero class \( X \), \( V_X^0 \) is a closed subset of \( (\omega, T_\infty) \).

**Proof:** \( T_X : (V_X, T_\infty) \to R \) is an RSM and so it is continuous. Thus \( T_X^{-1}(0) = V_X^0 \) is a closed subset of \( (V_X, T_\infty) \). Since, by previous proposition, \( V_X \) is a closed subset of \( (\omega, T_\infty) \), \( V_X^0 \) is a closed subset of \( (\omega, T_\infty) \).
Proposition 3.32. For any zero-class \( X \), \( \bar{\omega}_X = V_X \) (where \( \bar{\omega}_X \) denotes the closure of \( \omega_X \) with respect to the topology \( T_\infty \)).

Proof: Suppose that \( x \in \omega_X \) and \( r \in \mathbb{R} \) and \( A \in X \) with \( x \rightarrow r \). Then by the definition of \( x \rightarrow r \), we have, for any \( \alpha > 0 \)

\[
(A) \quad \text{there exists } N \in I \text{ such that } \{i : \alpha < |x_i - r| \} \subset A \cup \{1,2,\ldots,N\}.
\]

Since \( A \in X \) and \( \{1,2,\ldots,N\} \in X \), we have \( A \cup \{1,2,\ldots,N\} \in X \).

Thus for any \( \alpha > 0 \), \( \{i : \alpha < |x_i - r| \} \in X \). So that \( x \in V_X \).

Therefore \( \omega_X \subset V_X \). Since \( V_X \) is closed \( \bar{\omega}_X \subset V_X \).

Suppose that \( x \in V_X \) and \( T(x) = r \). For each \( n \), let \( \{i : \frac{1}{n} < |x_i - r| \} = A_n \). Then \( A_n \in X \). Let us define \( x^n \in \omega \) by

\[
x^n_i = \begin{cases} 
    r & \text{if } i \in I - A_n \\
    x_i & \text{if } i \in A_n.
\end{cases}
\]

Obviously, \( x^n \rightarrow r \) and \( A_n \in X \), thus \( x^n \in \omega_X \). Since

\[
|x^n_i - x_i| = \begin{cases} 
    |r - x_i| & \text{if } i \in I - A_n \\
    0 & \text{if } i \in A_n,
\end{cases}
\]

We get \( ||x^n - x||_\infty \leq \frac{1}{n} \). So that it follows that \( x \in \bar{\omega}_X \). Hence

\( \bar{\omega}_X = V_X \).
Proposition 3.33. For any zero class \( X \), \( \omega_X^0 = V_X^0 \) where

\[
\omega_X^0 = \{ x \in \omega : \exists A \in X \quad x \xrightarrow{(A)} 0 \} .
\]

Proof: In the proof of the previous proposition, we change \( \omega_X \) to \( \omega_X^0 \), \( V_X \) to \( V_X^0 \) and \( r \) to 0. Then we get the proof.

Proposition 3.34. ([3], Proposition 4.10). If \( X \) and \( S \) are related zero class and RSM then

1. \( \omega_X^0 \cap m \subset |c_s| \supset V_X^0 . \)
2. \( \omega_X \cap m \subset |c_s| \supset V_X . \)
3. \( S \) and \( T_X \) have some value on \( |c_s| \), that is,

\[
S |c_s| = T_X |c_s| .
\]

Proof: (1) Let \( x \in \omega_X^0 \cap m \). Then there exist a set \( A \in 2^I \) such that \( A \in X \) and \( x \xrightarrow{(A)} 0 \). Since \( A \in X \), we have \( X_A \in |c_s|^{0} \).

Writing \( x = x X_A + x X_{(I-A)} \) and noting that \( x \in m \), we have, by the definition of RSM, \( x X_A \in |c_s|^{0} \). Further \( x X_{(I-A)} \in c_0 \subset |c_s|^{0} \) by Proposition 3.6, it follows that \( x \in |c_s|^{0} \).

Next we consider any \( x \in |c_s|^{0} \). Then for any \( \alpha > 0 \)

\[
\alpha \chi \{ i : \alpha < |x_i| \} \leq |x| \in |c_s|^{0} . \]

By Proposition 3.4, \( \alpha \chi \{ i : \alpha < |x_i| \} \in |c_s|^{0} \).

Thus \( \chi \{ i : \alpha < |x_i| \} \in |c_s|^{0} \), equivalently \( \{ i : \alpha < |x_i| \} \in X \). Therefore \( x \in V_X^0 \).
(2) Suppose that $x \in \omega_X \cap m$, then there exist $r \in R$ and $A \in X$ such that $x \rightarrow r$. Since $x-r \in \omega^0_X \cap m$ and by (1), $x-r \in |C_s|^0$. Thus we get $x \in |C_s|$. Hence $\omega_X \cap m \subset |C_s|$.

Suppose that $x \in |C_s|$ and $r \in R$ such that $x-r \in |C_s|^0$. Then, by (1), $x-r \in V^0_X$. Therefore $x \in V_X$. Thus $|C_s| \subset V_X$.

(3) Suppose that $x \in |C_s|$ and $r \in R$ with $x-r \in |C_s|^0$. By Proposition 3.5 (1), $x-r \in C^0_s$, so that $S(x-r) = 0$ and $S(x) = r$.

On the other hand $x-r \in |C_s|^0 \subset V^0_X$ by (1). Therefore $T_X(x) = r$. Hence, we get $S(x) = r = T_X(x)$.

**Proposition 3.35.** If $X_1$ and $X_2$ are zero-classes with $X_1 \subset X_2$, then we have

(1) $V^0_{X_1} \subset V^0_{X_2}$,

(2) $V_{X_1} \subset V_{X_2}$,

(3) $T_{X_2}|_{V^0_{X_1}} = T_{X_1}$.

**Proof:** (1) Suppose that $x \in V^0_{X_1}$. Then for any $\alpha > 0$,

{$i: \alpha < |x_i| \in X_1 \subset X_2$. Therefore for any $\alpha > 0$,

{$i: \alpha < |x_i| \in X_2$, equivalently $x \in V^0_{X_2}$. Hence $V^0_{X_1} \subset V^0_{X_2}$.

(2) and (3). For any $x \in V^0_{X_1}$, let $T_{X_1}(x) = r$. Then we have
Thus \( x - r \in V_{X_1}^0 \subseteq V_{X_2}^0 \). Thus \( x - r \in V_{X_2}^0 \) and so \( T_{X_2}(x) = r = T_{X_1}(x) \) and \( x \in V_{X_2} \).

**Proposition 3.36.** (The bounded consistency theorem on strong convergent fields). Let \( S_1: C_{s_1} \to \mathbb{R} \) be an RSM related with a zeroclass \( X_1 \) and \( S_2: C_{s_2} \to \mathbb{R} \) be an RSM related with a zeroclass \( X_2 \).

Suppose that \( X_1 \subseteq X_2 \) and \( C_{s_1} \cap m \subseteq C_{s_2} \). Then:

1. \( |C_{s_1}|^0 \cap m \subseteq |C_{s_2}|^0 \cap m \),

2. \( |C_{s_1}| \cap m \subseteq |C_{s_2}| \cap m \),

3. \( S_1\left(|C_{s_1}| \cap m\right) = S_2\left(|C_{s_1}| \cap m\right) \).

**Proof:** If \( x \in |C_{s_1}|^0 \cap m \). Then \( |x| \in C_{s_1} \cap m \), \( S_1(|x|) = 0 \), \( |x| \in V_{X_1}^0 \) (by Proposition 3.34) and \( T_{X_1}(|x|) = 0 \). Since by hypothesis, \( C_{s_1} \cap m \subseteq C_{s_2} \), we also have \( |x| \in C_{s_2} \) so that \( S_2(|x|) \) is defined. By Propositions 3.34 and 3.35, \( \omega_{X_2} \cap m \subseteq |C_{s_2}| \cap m \subseteq \bar{\omega}_{X_2} \cap m \) and \( |x| \in V_{X_2}^0 \cap m \subseteq \bar{\omega}_{X_2} \cap m \). Thus we can find a sequence \( \{x^n\} \) in \( |C_{s_2}| \cap m \) such that \( x^n \to |x| \) in the sense of \( T_\infty \). Since \( S_2 \) is a RSM and by Proposition 3.9, \( S_2 \) is continuous. Thus \( S_2(x^n) \to S_2(|x|) \).

Since we know that \( x^n \in V_{X_2} \), by Proposition 3.34, \( T_{X_2}(x^n) = S_2(x^n) \).

On the other hand, \( |x| \in V_{X_1}^0 \subseteq V_{X_2}^0 \) and so by Proposition 3.35,

\[ 0 = T_{X_1}(|x|) = T_{X_2}(|x|) \]. Hence we have
\[ 0 = T_{X_2}(|x|) = \lim_{n \to \infty} T_{X_2}(x^n) = \lim_{n \to \infty} S_2(x^n) = S_2(|x|). \]

So that \( x \in |C_{S_2}|^0 \). Consequently, we have

\[ |C_{S_1}|^0 \cap m \subset |C_{S_2}|^0 \cap m. \]

(2) If \( x \in |C_{S_1}| \cap m \), then there exists \( r \in \mathbb{R} \) with \( x-r \in |C_{S_1}|^0 \cap m \).

By (1) \( x-r \in |C_{S_2}|^0 \cap m \). Thus \( x \in |C_{S_2}| \cap m \).

(3) Let \( x \in |C_{S_1}| \cap m \) and \( r \in \mathbb{R} \) with \( x-r \in |C_{S_1}| \cap m \). Then by Proposition 3.34, \( x-r \in V_{X_1}^0 \) and \( T_{X_1}(x) = r \). Again by Proposition 3.35, \( x-r \in V_{X_2}^0 \) and \( T_{X_2}(x) = r \). By Proposition 3.34,

\[ S_1(x) = T_{X_1}(x) = r = T_{X_2}(x) = S_2(x). \]

Hence we have \( S_1|_{|C_{S_1}| \cap m} = S_2|_{|C_{S_2}| \cap m} \).

**Corollary 3.37.** Let \( S_1: C_{S_1} \to \mathbb{R} \) and \( S_2: C_{S_2} \to \mathbb{R} \) be RSMs defined on the same domain \( C_{S_1} = C_{S_2} \) and related with the same zero class \( X \). Then \( |C_{S_1}| \cap m = |C_{S_2}| \cap m \) and

\[ S_1|_{|C_{S_1}| \cap m} = S_2|_{|C_{S_2}| \cap m}. \]

**Proof:** By the previous proposition \( |C_{S_1}| \cap m \subset |C_{S_2}| \cap m \) and \( |C_{S_1}| \cap m \supset |C_{S_2}| \cap m \). Thus \( |C_{S_1}| \cap m = |C_{S_2}| \cap m \) and

\[ S_1|_{|C_{S_1}| \cap m} = S_2|_{|C_{S_2}| \cap m}. \]
Remark: Let $F$ be the collection of all RSMs which are related with zero class $X$. Then $T_X$ is a member of $F$ and for any RSM $S: C \to R$ in $F$, $S$ is $T_X$ on the bounded strong convergence field associated with $S$.

Next, we will study RSMs induced from matrices. Also, we will show that in the matrix case Proposition 3.36 is a special case of the standard Bounded Consistency Theorem on regular matrices. We will also show that Proposition 3.36 is not subsumed under the standard Bounded Consistency Theorem on regular matrices.

Definition 3.38. Let $A$ be a regular matrix. Let $f_A: C_A \to R$ be a function defined by $f_A(x) = \lim_{i} (Ax)_i$ for any $x \in C_A$. Note that $C_A$ is a linear space of sequences and $f_A$ is a linear functional.

Definition 3.39. A matrix $A$ is called an RSM if $f_A: C_A \to R$ is an RSM.

Proposition 3.40. Suppose that $A$ is nonnegative (i.e., $A_{nk} \geq 0$ for all $n,k = 1,2,3,...$) regular matrix. Then $f_A: C_A \to R$ is an RSM.

Proof: Let us write $C_{f_A} = C_A$, $|C_{f_A}|^0 = |C_A|^0$ and $|C_{f_A}| = |C_A|$. (see definition 3.1.). Clearly $f_A$ is regular. Now suppose that $x \in |C_A|^0$ and $|y| \leq |x|$. Since $A$ is nonnegative

$$0 \leq (A|y|)_n = \sum_{k=1}^{\infty} a_{nk}|y_k| \leq \sum_{k=1}^{\infty} a_{nk}|x_k| = (A|x|)_n.$$
Thus

\[
0 \leq \lim_{n \to \infty} (A|x|)_n \leq \lim_{n \to \infty} (A|x|)_n = f_A(|x|) = 0
\]

Hence

\[
\lim_{n \to \infty} (A|y|)_n = 0 \quad \text{and} \quad |y| \in |C_A|^0.
\]

By Proposition 3.4, \( f_A \) is an RSM.

Let us state the Bounded Consistency Theorem (BCT) without proof. [7].

**Proposition 3.41.** If \( A \) and \( B \) are regular matrices such that \( C_A \cap m \subseteq C_B \) then \( f_A(x) = f_B(x) \) for all \( x \in C_A \cap m \).

Remark: The BCT is very general and so this theorem implies Proposition 3.36 when the RSMs \( S_1 \) and \( S_2 \) are induced by matrices.

**Corollary 3.42.** Suppose that \( A \) and \( B \) are RSM matrices and \( f_A \) and \( f_B \) are corresponding RSMs. If \( C_A \cap m \subseteq C_B \) then

1. \( |C_A|^0 \cap m \subseteq |C_B|^0 \cap m \),
2. \( |C_A| \cap m \subseteq |C_B| \cap m \),
3. \( f_A|_{|C_A| \cap m} = f_B|_{|C_B| \cap m} \).

Proof: By the ordinary BCT, \( f_A|(C_A \cap m) = f_B|(C_A \cap m) \).

Hence (3) is true. Let \( x \in |C_A|^0 \cap m \), then \( |x| \in C_A \cap m \) and
\( f_A(\|x\|) = 0 \). By the hypothesis \( C_A \cap m \subseteq C_B \) and by the BCT,
\[ |x| \in C_B \cap m \quad \text{and} \quad f_B(\|x\|) = 0 \]. Therefore \( x \in (C_B)^0 \cap m \). Hence (1) is true.

(2) Follows from (1) as before.

Next we want to show that our Proposition 3.36 is meaningful. In other words Proposition 3.36 is not deduced from the BCT for the non-matrix case. For this purpose, we will construct an RSM
\[ S: C_S + R \] such that for any RSM matrix \( A, |C_S| \cap m \nsubseteq |C_A| \cap m \).

At first let us introduce a definition and some propositions.

**Definition 3.43.** An ultra zeroclass on \( I \) is a zeroclass \( X \) such that there is no zeroclass on \( I \) which is strictly finer than \( X \). (In other words, a maximal element in the ordered family of all zero-classes on \( I \)). This is the same as maximal ideal in \( 2^I \) [1].

**Proposition 3.44.** Let \( X \) be an ultra zeroclass on \( I \). Then for any \( A \in 2^I, A \subseteq X \) or \( I-A \subseteq X \).

**Proof:** Let \( F = \{ A \in 2^I: I-A \subseteq X \} \). Then \( F \) is an ultrafilter. Thus for any \( A \in 2^I, A \subseteq F \) or \( I-A \subseteq F \) (see [1], Chapter I, §6.4, Proposition 5). Hence for any \( A \in 2^I, A \subseteq X \) or \( I-A \subseteq X \).

**Proposition 3.45.** \( X \) is an ultra zeroclass if and only if \( m \subseteq \nu_X \).

**Proof:** Suppose that \( X \) is an ultra zeroclass. Then for any \( A \in 2^I, A \subseteq X \) or \( I-A \subseteq X \), if and only if \( \chi_A \subseteq \nu_X \) or \( \chi_{(I-A)} \subseteq \nu_X \) if and only if \( \chi_A \subseteq \nu_X \) or \( -\chi_A + 1 \subseteq \nu_X \). Thus, since \( \nu_X \) is a
linear space which contains \( c, -X_A + 1 \in V_X \) is equivalent to

\( X_A \in V_X \). Therefore, for any \( A \in 2^I \), \( X_A \in V_X \). Thus

\[ \{X_A: A \in 2^I\} \subset V_X. \]

Since \( V_X \) is a linear space of sequence, \( m_0 \subset V_X \).

Since \( V_X \) is closed in \((\omega, T_\infty), m_0 = m \subset V_X\) (see [10], p. 24, 15 Example).

Suppose that \( X \) is not an ultra zeroclass, then there exists an \( A \in 2^I \) such that \( A \notin X \) and \( I-A \notin X \).

Suppose that \( X_A \in V_X \). Then there is an \( r \in R \) such that

\[ \{i: \alpha < |X_A(i) - r|\} \notin X \text{ for any } \alpha > 0. \]

If \( r = 1 \), then \( \{i: \frac{1}{2} < |X_A(i) - 1|\} = I-A \notin X \).

If \( r = 0 \), then \( \{i: \frac{1}{2} < |X_A(i) - 0|\} = A \notin X \).

If \( r \neq 0 \) and \( r \neq 1 \), take \( 0 < \alpha_0 < \min \{|r|, |r-1|\} \).

Then \( \{i: \alpha_0 < |X_A(i) - r|\} = I \notin X \). And so we have a contradiction.

Hence \( X_A \notin V_X \) since \( X_A \in m \) we have \( m \notin V_X \).

**Proposition 3.46.** If \( X \) is an ultra zeroclass then there does not exist an RSM matrix \( A \) such that \( |V_X| \cap m = C_A \cap m \).

**Proof:** Since \( X \) is an ultra zeroclass \( m \subset V_X \), and so

\[ |V_X| \cap m = V_X \cap m = m. \]

On the other hand, for any regular matrix \( A \), \( C_A \cap m \subset m \) (see [6], p. 187 Theorem 14 (The Steinhaus Theorem)).

Therefore for any RSM matrix \( A \), \( |C_A| \cap m \subset C_A \cap m \subset m = |V_X| \cap m \).
Corollary 3.47. Let $X$ be an ultra zero class and let $C_s = v_X$ and let $S: C_s \to R$ be $T_X$. Then for any RSM matrix $A$,

$$|C_A| \cap m \not\subset |C_s| \cap m.$$ 

**Proof:** By the previous proposition

$$|C_s| \cap m = |v_X| \cap m = m \not\supset C_s \cap m \not\supset |C_A| \cap m.$$ 

Hence our BCT (Proposition 3.36) can be applied in cases unapproached by matrix methods.

In the matrix cases, there are many interesting examples which are not RSMs. For the rest of this chapter we will study regular matrices vis-a-vis RSMs. At first let us define a regular matrix which is not an RSM.

**Example 3.48.** Let $A$ be given by

$$A = \begin{bmatrix}
1 & -\frac{1}{2} & 2 & -\frac{1}{4} & 1 & -\frac{1}{8} & 1 & -\frac{1}{16} & 1 & \cdots \\
0 & 0 & 1 & -\frac{1}{2} & 2 & -\frac{1}{4} & 1 & -\frac{1}{8} & 1 & \cdots \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 2 & -\frac{1}{4} & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 2 & \cdots 
\end{bmatrix}.$$ 

By the Silverman-Toeplitz Theorem, we can easily see that $A$ is regular. Let $x = (1, 2, 1, 2, 1, 2, \ldots)$. Then $Ax = 0$. Hence $x \in |C_A|_0$. But if we take $y = (1, 1, 2, 1, 1, 1, 2, \ldots)$ then $Ay = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \ldots\right)$ and
Also, if we take $x = (0,1,0,1,0,1,...)$ then $Ax = (-1,-1,-1,...)$. Therefore $f_A$ is not nonnegative. This example is generalized in Proposition 3.50. The previous example is also interesting in view of the following.

**Proposition 3.49.** If $A$ is a regular matrix then $f_A: (C_A, T_\infty) \to R$ is continuous.

We omit the proof.

If $A$ is a nonnegative regular matrix then $A$ is an RSM. Thus being a nonnegative regular matrix is a sufficient condition of being an RSM. But it is hard to find nice necessary conditions for a matrix to be an RSM. The following proposition is an attempt to find necessary conditions for being an RSM.

**Proposition 3.50.** Let $A$ be a regular matrix. Suppose that for each column of $A$, all members of that column are either nonnegative or nonpositive and \[ \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}^- = r \] and $r > 0$ where $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$. Then $f_A$ cannot be nonnegative.

**Proof:** Since $A$ is regular and \[ \sum_{k=1}^{\infty} |a_{nk}| = M < \infty, \] the series \[ \sum_{k=1}^{\infty} a_{nk}^- \] converges for each $n \geq 1$.

Now let $x \in \omega$ such that $y \notin C_A$. Thus $m|C_A|^0 \notin C_A$.
\[ x_k^* = \begin{cases} 
0 & \text{if } k\text{-th column of } A \text{ is nonnegative or all zero} \\
1 & \text{if } k\text{-th column of } A \text{ is nonpositive.} 
\end{cases} \]

Then

\[
f_A(x) = \lim_{n \to \infty} (Ax) 
= \lim_n \left( \sum_{k=1}^{\infty} a_{nk}^+ x_k - \sum_{k=1}^{\infty} a_{nk}^- x_k \right) 
= - \lim_n \sum_{k=1}^{\infty} a_{nk}^- 
= -r < 0.
\]

Therefore \( f_A \) is not nonnegative. Hence \( f_A \) cannot be an RSM.

Remark: Thus an RSM matrix cannot have the property stated in Proposition 3.50.

**Example 3.51.** Let a matrix \( A \) be given by:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
2 & -1 & 0 & 0 & 0 & 0 & \ldots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & -1 & 0 & 0 & 0 & \ldots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \ldots \\
0 & 0 & 2 & -1 & 0 & 0 & \ldots 
\end{bmatrix}
\]

Then \( A \) is an RSM.
Proof: We can easily see that $A$ is regular. Suppose that $x \in |c_A|_0$ and so

$$\frac{|x_1| + |x_2| + \ldots + |x_n|}{n} \to 0$$

and

$$2|x_n| - |x_{n+1}| \to 0.$$  

Then we show $|x_n| \to 0$. Assume that $|x_n| \nrightarrow 0$ so that there exists $\epsilon > 0$ such that for infinitely many $n$, $|x_n| > \epsilon$. Since

$$2|x_n| - |x_{n+1}| \to 0,$$

there exists $N$ such that $n > N$ implies

$$2|x_n| - |x_{n+1}| < \frac{\epsilon}{2}.$$  

Take $n_0$ such that $|x_{n_0}| > \epsilon$ and $n_0 > N$. Then we have

$$2\epsilon < 2|x_{n_0}| < \frac{\epsilon}{2} + |x_{n_0+1}|.$$  

Therefore $\frac{3\epsilon}{2} < |x_{n_0+1}|$. If $k > n_0$ and $|x_k| > \epsilon$ then $2\epsilon < 2|x_k| < \frac{\epsilon}{2} + |x_{k+1}|$. Thus $\frac{3\epsilon}{2} < |x_{k+1}|$. Therefore by induction for any $n \geq n_0$, $\epsilon \leq |x_n|$. Thus we have

$$\lim_{n \to \infty} \inf \frac{|x_1| + |x_2| + \ldots + |x_n|}{n} \geq \epsilon,$$

which is a contradiction.

Hence $\lim_{n} |x_n| = 0$ and $|c_s|_0 = c_0$. Since $x \in c_0$ and

$$|y| \leq |x|$$

implies $\lim_{n} y_n = 0$ we have that $A$ is an RSM.

Example 3.52. Let $A$ be given by
that is

\[
\mathbf{A} = \begin{bmatrix}
1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & \cdots \\
\end{bmatrix}
\]

By the Silverman-Toeplitz Theorem, we can easily see that \( \mathbf{A} \) is regular.

Let \( x \in \omega \) be given by

\[ x_i = \begin{cases} 
  j & \text{if } i = 2j + 2 \text{ and } j \in I \\
  0 & \text{otherwise.} 
\end{cases} \]

that is, \( x = (0,0,0,1,0,2,0,3,0,4,0,\ldots) \). Then \((\mathbf{A}x)_n = -1\) for all \( n \). Hence

\[ \lim_{n \to \infty} (\mathbf{A}x)_n = -1 \]

and \( x \in C_\mathbf{A} \) and \( f_\mathbf{A}(x) = -1 \). Since \( x \geq 0 \) and \( f_\mathbf{A}(x) = -1 \), \( f_\mathbf{A} \) is not nonnegative so that \( f_\mathbf{A} \) is not an RSM.

Next let us take \( x \in \omega \) by
that is, $x_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ j & \text{if } i = 2j+2 \text{ and } j \in I \cup \{0\}, \end{cases}$

that is, $x = (1,1,1,2,1,3,1,4,1,5,1,6,\ldots)$. Take $y \in \omega$ by

$y_i = \begin{cases} 2j+1 & \text{if } i = 4j+2 \text{ and } j \in I \\ 1 & \text{otherwise}, \end{cases}$

that is $y = (1,1,1,1,1,1,3,1,1,1,5,1,1,7,\ldots)$. We know that

$(Ax)_n = 0$ for each $n \in I$ so that $x \in \left| \mathcal{C}_A \right|^0$. Since

$(Ay)_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}, \end{cases}$

it follows that $y \notin \mathcal{C}_A$. Obviously $|y| \leq |x|$. Therefore

$m|\mathcal{C}_A|^0 \notin \mathcal{C}_A$.

The previous example is important in the sense that there exists a regular matrix $A$ which is essentially nonnegative and still not an RSM. Essentially, nonnegative matrices were studied by Sonnenschein [9] because of the following proposition.

**Proposition 3.53.** Suppose that $A$ is a regular and essentially nonnegative matrix. Let $d_A(S) = \lim_{n \to \infty} \inf \frac{1}{n} \sum_{k=1}^{\infty} a_{nk} \chi_S(k)$ for any $S \in 2^I$. Then $d_A$ is an asymptotic density.
Proof: (See [9], Page 26, Theorem 3.3).

Definition 3.54. A matrix $A$ is said to be essentially non-negative if

$$\lim_{n,k} a_{nk} = 0.$$  

While we have shown in example 3.52 there exists an essentially non-negative regular matrix $A$ such that $f_A: C_A \to \mathbb{R}$ is not an RSM, if $A$ is an essentially nonnegative regular matrix, then the restriction of $f_A$ to $C_A \cap m$ is an RSM. To prove that, at first, let us prove two lemmas.

Definition 3.55. Let $A$ be a matrix, then we denote

$$A^+ = (a_{ni}^+) \quad \text{and} \quad A^- = (a_{ni}^-).$$

Lemma 3.56. Let $A$ be an essentially nonnegative regular matrix. Then we have

1. $C_A \cap m = C_{A^+} \cap m$,

2. $f_A\big|_{(C_A \cap m)} = f_{A^+}\big|_{(C_A \cap m)}$.

Proof: (1) For any $x \in m$, $x \in C_A$ if and only if $Ax \in C$. Since $A$ is a regular matrix, if $Ax \in C$ then $A^+x$ and $A^-x$ exist and further $Ax = A^+x - A^-x$. Since $A$ is essentially nonnegative and $x \in m$, we have
Thus \( (A^{-1})_i \) = \( \sum_{k=1}^{\infty} a_{ik}x_k \) \( \leq \|x\|_\infty \sum_{k=1}^{\infty} a_{ik} \rightarrow 0 \).

Thus we get \( A^{-1}x \in C_0 \), and so \( Ax \in c \) if and only if \( A^+x \in c \). Hence \( C_A \cap m = C_{A^+} \cap m \).

(2) For any \( x \in C_A \cap m \), \( f_A(x) = \lim (Ax)_i = \lim ((A^+)_i - (A^-)_i) = \lim (A^+_i - \lim (A^-)_i) = \lim (A^+_i - \lim (A^-)_i) = f_{A^+}(x) \). Therefore \( f_A \bigg|_{(C_A \cap m)} = f_{A^+} \bigg|_{(C_A \cap m)} \).

Lemma 3.57. Let \( S: C \rightarrow R \) be an RSM and let \( T = S \big|_{(C \cap m)}: C \cap m \rightarrow R \) and \( C_T = C_s \cap m \). Then

(1) \( |C_T|^0 = |C_s|^0 \cap m \)

(2) \( T: C_T \rightarrow R \) is also an RSM.

Proof: (1) For any \( x \in \omega \).

\( x \in |C_T|^0 \Rightarrow |x| \in C_s^0 \)

\( \Rightarrow |x| \in C_T \text{ and } T(|x|) = 0 \)

\( \Rightarrow |x| \in C_s \cap m \text{ and } S(|x|) = 0 \)

\( \Rightarrow x \in |C_s|^0 \text{ and } x \in m \)

\( \Rightarrow x \in |C_s|^0 \cap m \).

Thus \( |C_T|^0 = |C_s|^0 \cap m \).
Let $x \in \lvert C_T \rvert^0$ and $|y| \leq |x|$. Then $x \in \lvert C_s \rvert^0 \cap m$ and $|y| \leq |x|$. Since $S$ is an RSM, $|y| \in \lvert C_s \rvert^0$. Since $x \in m$ and $|y| \leq |x|$, it follows that $y \in m$. Thus, by (1) $y \in \lvert C_s \rvert^0 \cap m = \lvert C_T \rvert^0$. Therefore $T$ is an RSM.

**Proposition 3.58.** Let $A$ be an essentially nonnegative regular matrix. Let $S = f_A|_{(C_A \cap m)} : (C_A \cap m) \to R$ be the restriction of $f_A$ to $C_A \cap m$. Then $S$ is an RSM.

**Proof:** Since $A^+$ is a nonnegative regular matrix $f_{A^+} : C_{A^+} \to R$ is an RSM. By the previous lemma $f_{A^+}|_{(C_{A^+} \cap m)} : (C_{A^+} \cap m) \to R$ is also an RSM. By Lemma 3.57, $f_{A^+}|_{(C_{A^+} \cap m)} = f_A|_{(C_A \cap m)}$. Thus $S$ is an RSM.
BIBLIOGRAPHY


APPENDIX

List of Notations

R is the set of real numbers.

I is the set of positive integers.

$2^X$ is the power set of a given set $X$.

If $a, b \in \mathbb{R}$, then $\max(a, b)$ (resp. $\min(a, b)$) is the maximum (resp. minimum) of the set $\{a, b\}$.

If $S \subseteq \mathbb{R}$, $\sup S$ (resp. $\inf S$) is the supremum (resp. infimum) of $S$.

If $a \in \mathbb{R}$, $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$.

$\omega$ is the set of all real sequences.

If $x \in \omega$, then $(x_i)$ or $(x_1, x_2, \ldots)$ denote $x$.

$\lim_{i \to \infty} x_i$ or $\lim x$ denote the limit of a real sequence $x$.

$m = \{x \in \omega : \sup_{k} |x_k| < \infty\}$.

$c = \{x \in \omega : \lim x \text{ exists} \}$.

$c_0 = \{x \in \omega : \lim x = 0\}$.

$e, e^n$ are the sequences given by $e_k = 1$ for all $k$ and $e^n_k = 0$ for $k \neq n$, $e^n_n = 1$.

If $A \subseteq \mathcal{P}(\mathbb{I})$, $\chi_A$ is the characteristic sequence of $A$, that is

$(\chi_A)_n = 1$ if $n \in A$, $(\chi_A)_n = 0$ if $n \notin A$.

$m_0$ is the linear span of $\{\chi_A : A \subseteq \mathcal{P}(\mathbb{I})\}$.

$bs = \{x \in \omega : \sup_{n} \left| \sum_{k=1}^{n} x_k \right| < \infty \}$.
$xy = (x_i y_i)$ is the coordinate-wise product of two sequences $x$ and $y$.

For $x \in \omega$, we let $|x| = (|x_i|)$.

For $x \in \omega$ and $r \in \mathbb{R}$, we write $x+r = (x_i+r)$ and $rx = (rx_i)$.

For $A, B \subseteq \omega$, we write

$$AB = \{xy \mid x \in A \text{ and } y \in B\},$$

$$A+B = \{x+y \mid x \in A \text{ and } y \in B\}.$$  

For $x, y \in \omega$, $x \leq y$ means $x_i \leq y_i$ for each $i$.

$M = (a_{nk})$ denotes infinite matrices.

If $M = (a_{nk})$ is an infinite matrix and $(x_i)$ is any sequence, the product $Mx$ denotes the sequence $(y_i)$, if it exists, where

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j.$$  

We also define $c_M = \{x \in \omega : Mx \in c\}$. In Chapter 3, we write $c_M$ for $c_M$.

$0 \in \omega$ denotes zero sequence $(0,0,0,...)$.

Zero matrices $A = (a_{ni})$, where $a_{ni} = 0$ for all $i, j$ is denoted by $0$.

$f: S \to T$ denotes a function from a set $S$ into a set $T$.

If $f: S \to T$ is a function and $U \subseteq S$ then $f|_U: U \to T$ is the restriction of $f$ into $U$.

$a, b \in I$, $[a, b] = \{x \in I : a \leq x \leq b\}$,

$\langle a, b \rangle = \{x \in I : a < x < b\}$,
\(|A|\) is the cardinal number of a given set \(A\).

For two sets \(A\) and \(B\), we let \(A \Delta B = (A-B) \cup (B-A)\) be the symmetric difference of \(A\) and \(B\).

For two sets \(A\) and \(B\), \(A \sim B\) means \(A \Delta B\) is finite.

For \(A \in 2^I\), we let \(A^c = I-A\) be the complement of \(A\).

\(F_0 = \{A \in 2^I: A^c \text{ is finite}\}\) is the Fréchet filter.

\(J_N = \{1,2,3,\ldots,N\}\) when \(N \in I\).

\(\chi_0 = \{A \in 2^I: A\ \text{is finite}\}\).