AN OPERATIONS RESEARCH APPROACH TO
LABOUR HOARDING

by

John William Struthers
B.A., Simon Fraser University, 1971

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS
in the Department
of
Economics and Commerce

C JOhn William Struthers 1973
SIMON FRASER UNIVERSITY
February 1973

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.
APPROVAL

Name: John William Struthers
Degree: Master of Arts
Title of Thesis: An Operations Research Approach to Labour Hoarding

Examiners Committee:

Chairman: Dennis R. Maki

Dennis R. Maki
Senior Supervisor

Parzival Copes

Robert C. Brown
External Examiner
Associate Professor
Department of Geography
Simon Fraser University

Date Approved: ________

11
ABSTRACT

When personnel quit unexpectedly a firm faces both the costs of replacing the workers and the costs of suffering the vacancies created. If the latter costs are high the firm may elect to "hoard" a reserve of workers capable of replacing the employees who quit. Clearly this hoarding involves costs which may offset the savings realized by filling vacancies as they arise. If the firm can derive a probability distribution for the unanticipated quits, the optimum number of persons to hoard may be determined. Personnel may be considered to be analogous to replacement parts held in materials inventory. Then the inventory model of S. Karlin\(^1\) can be adapted to determine optimal hoarding. Consideration of the exogeneous variables in the model indicates that policies designed to convert them to endogenous (decision) variables will permit a further reduction in total costs expected. The model thus provides a rationale for the use of mandatory notice periods, probation periods, and non-vested benefit plans. It is also noted that the actions of rational firms will serve to accentuate conditions of "tightness" or "looseness" of the labour market.

Although only large firms could satisfy the rigid assumptions incorporated in the model, the approach used is of interest for the implications derived and for its possible extensions.

---

TABLE OF CONTENTS

Introduction ............................................. 1
Assumptions ............................................. 2
The model ................................................. 4
Implications .............................................. 14
Bibliography ............................................. 20
Introduction

A firm experiences turnover in its work force as employees retire, are discharged or laid off, or voluntarily quit. Retirement, layoff, and discharge are commonly anticipated accurately by the firm's management, but terminations for other reasons, under the control of the worker involved, cannot be anticipated precisely. It is this latter group of actions that we shall define simply as "quits".

The cost to the firm of each quit is the sum of the cost of replacing that worker plus the cost of suffering the vacancy created by the worker's departure. Each of these component costs varies with the duration of the vacancy. As the duration increases, costs of finding a replacement likely fall\(^1\), and conversely, the costs of suffering a vacancy rise. Where the cost of vacancies is high, or rises rapidly either with time or as the number of vacancies increases, the firm may demand labour to act as a reserve force capable of replacing workers as they quit; i.e. the firm may wish to hoard labour. However, hoarding will itself entail a cost that rises over time. If it is further noted that the acquisition of replacements involves both costs that are fixed and costs that are variable, with respect to the number of replacements, the pattern of trade-offs appears

---

1. See A. Alchian, "Information Costs, Pricing, and Resource Unemployment", in E. Phelps et al, Microeconomic Foundations of Employment and Inflation Theory, p. 41
to be logically analogous to that encountered in materials inventory problems. If we further recognize that a firm might be able to determine the probability distribution of quits over time, the problem of minimizing expected costs is then relevant, and appears to be conceptually analogous to materials replacement or renewal problems encountered in operations research.

The objective of this paper is to investigate both the feasibility of viewing part of the firm's demand for labour from the vantage of inventory or materials replacement theory, and the implications of such an approach.

**Assumptions**

Hoarding of labour will be defined as the acquisition of workers to serve as a reserve capable of replacing employees who might quit in the future. The motive for hoarding is then related only to expected changes in the demand for workers; defence against expected changes in supply conditions is eliminated as a motive. Isolation of the relevant determinants of the demand for labour to be hoarded is then the necessary beginning of a simple model of hoarding.

If it is assumed that the firm operates in an environment with demand for its output known and constant, and with fixed factor input requirements, then it must replace every quit. Such assumptions are only simplifications of reality, and do not alter the feasibility of the approach. Others are
more critical and limiting, and will require explicit justification. For example, the requirement that any one replacement hoarded may effectively replace any one of many individuals who might be expected to quit is of major conceptual importance. This constraint is in fact a requirement that the work force group under consideration be homogeneous with respect to required skills and training.

This constraint might be satisfied, or at least be adequately approximated in assembly line type production processes where each of many workers performs a simple repetitive task. In such a case, training might involve only basic orientation and possibly development of limited manual skills. However, this constraint need not limit the application of the model to lower hierarchical levels. The skill requirements need not be simplistic; an M.B.A. or an engineering degree may be the measure of required skills, and some training peculiar to the firm might be a supplement. Then the analysis is applicable to many levels, as long as external entry is permitted, and as long as many persons with the same basic skills are employed. The homogeneity constraint thus limits the analysis to firms employing large groups of similar personnel, but surely does not render the concept trivial.

1. The analysis is also applicable to the special case of a small group with high quit rates, and homogeneous skill requirements.
A second restraint of major importance is that the probability distribution of quits be known. It seems probable that such information would be available to the management of large firms. Without explicit data to work from, it seems most reasonable to assume that the probability distribution of quits is described by a Poisson distribution. This assumption is justified if the job life of each worker is a random variable with an exponential distribution, and if the work force is made up of personnel with differing job experience. The first criterion seems reasonable if one considers that a worker is less likely to quit his job the longer he has held it. Non-vested fringe benefits and seniority systems strengthen this case. The criterion of differing job experience eliminates from consideration only those firms which have just hired a new, inexperienced work force; a trivial restraint. Thus the Poisson distribution is appropriate for nearly all cases that satisfy the homogeneity constraint.

The model

"An Inventory Model With Poisson Input" is developed by S. Karlin. Karlin's model rests on assumptions similar to those specified above. If this model is followed, but

---

reinterpreted to apply to labour hoarding as defined, rather than to storage of spare parts for a machine as specified, certain implications regarding this part of a firm's demand for labour should become evident.

The modified Karlin model will consider a cyclic recruitment-training-use program designed to minimize total expected costs of replacing quits. A reserve force is recruited, trained, and stored or hoarded in anticipation of future quits. Toward the end of each cycle an order is placed for the recruiting and training of a new reserve. As this recruiting and training takes time (assumed for simplicity to be a fixed interval $'a'$), some employees in the work group may be expected to quit during this interval. Thus the firm cannot run its reserve down to zero before reordering. The model is for this reason based on a policy of ordering the recruitment of $'T'$ replacements each time the reserve force falls to a "trigger level" of $'R'$ persons. Both $'R'$ and $'T'$ are integers, with $T > R$. The problem then becomes one of choosing optimal values of $'R'$ and $'T'$ so as to minimize expected costs over the recruiting-training-use cycle.

The assumption that the recruiting-training interval is fixed need not be too limiting. Karlin's model may be modified to consider a variable recruiting-training interval\(^1\) with-

---

1. Karlin's notation is used throughout.
2. Karlin, op. cit., p.288
out changing the basic technique of analysis. Mention will be made of changes in alternatives available to the firm when 'a' is a variable. A fixed recruiting-training period might be encountered when a firm hires through an external employment agency and then provides some basic orientation peculiar to the firm. However, this might imply that the cycle is very short, and hoarding unnecessary. So it is likely more relevant to apply this model to firms which must submit recruits to an extensive training program of fixed minimum duration. Then, to justify consideration of the fixed interval 'a' we need only assume that recruiting time is fixed or is insignificant compared to the training time.

This constraint probably eliminates the assembly line work force from consideration, and emphasizes use of the model for groups at higher hierarchical levels. Management trainees may form a group that satisfies all the constraints mentioned. In the special case of very tight labour markets, which could be typified by a training interval that is insignificant compared to the recruiting interval, the model will also be applicable. It must be emphasized that the model can be altered to consider a variable interval 'a'.

Given the fixed recruiting-training interval 'a' and a Poisson distribution of quits over this interval, the expected length of time between quits is \( \frac{1}{\lambda} \) where \( \lambda \) is the mean number of quits per time period. The cycle to be considered is that of use-recruiting-training with the moment that the 'T'
recruits complete training as the start of the cycle. Further, it is assumed that one of the 'T' trainees is immediately used to replace a quit. Then the expected number of unproductive man-time units suffered as the reserve falls from its level at the beginning of the cycle to the "trigger level" 'R' is:
\[(T+U-1)\frac{1}{\lambda} + (T+U-2)\frac{1}{\lambda} + \ldots + (R)\frac{1}{\lambda}\]
where:  
- \(T\) = number of persons recruited and trained for the reserve,
- \(U\) = number of trainees from the previous cycle that are left at the beginning of the current cycle;  
- \(0 < U < R\),
- \(R\) = number of persons in the reserve when the recruiting-training program is initiated.

Note that both \(R\) and \(T\) are integer constants, to be determined by management policy. Then, if the series is summed it is seen that the expected man-time units suffered as the firm holds personnel in an unproductive reserve are equal to:
\[
\frac{(T+U-R)(R+T+U-1)}{2\lambda}
\]
(1)

Following Karlin's model and notation, if the distribution of quits over the interval 'a' is Poisson, the probability distribution of 'U' for cases where the reserve force of the previous cycle was not completely depleted over the interval 'a' is:

\[
Pr\{U=k\} = e^{-\lambda a} \frac{(\lambda a)^k}{(R-k)!} \quad 1 \leq k \leq R
\]
(2)

(2) is derived from the general model of the Poisson where
\[ f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{with} \quad x = R-k \quad \text{and} \quad \lambda = \lambda a. \] Similarly, the probability that the reserve did not last over the interval 'a' (and therefore that 'U' will be zero) is:

\[ \Pr \{ U = 0 \} = e^{-\lambda a} \sum_{L=R}^{\infty} \frac{(\lambda a)^L}{L!} \quad (3) \]

That is, the probability of carrying reserves over from the end of one cycle to the beginning of the next depends upon the number of quits during the interval 'a'. If the number of quits (or losses) 'L' is less than or equal to R-1, 'U' will be a positive integer. In this model it is assumed that over the interval 'a' the quits follow the Poisson distribution.

i.e. \( (L) = e^{-\lambda a} \frac{(\lambda a)^L}{L!} \)

With knowledge of the distribution of quits and of the derived distribution of 'U', the expected direct costs of holding the T+U persons in reserve over the interval as T+U diminishes to R may be calculated. Karlin sets R-U = V (i.e. V is the number of replacements at the beginning of the cycle which is required to fill positions vacated by quits in excess of R-1 over the previous interval 'a') and assumes that the direct costs of hoarding are linear (h dollars per man-time unit). Then the expected costs of hoarding as T+U diminishes to R are:

\[ h \left\{ R-1 \sum_{V=0}^{(T-V)(2R+T-V-1)} \frac{\varphi(V)}{2 \lambda} + \frac{(T-R)(R+T-1)}{2 \lambda} \sum_{V=0}^{\infty} \varphi(V) \right\} \quad (4) \]
Multiplication of the first term within the large brackets in (4) by \( h \) expresses the cost if, in the previous cycle, all \( R \) replacements are not used over the interval \( 'a' \). (i.e. if \( 'U' \) is positive.) Similarly, the second term is the cost if all \( R \) were used. As each term is multiplied by the probability of its occurrence, the sum of the two is the expected cost of hoarding (as \( T+U \) diminishes to \( R \)).

A similar approach is used to derive the expected costs of hoarding over the interval \( 'a' \). As quits are assumed to be characterized by the Poisson distribution over the interval \( 'a' \), they may be taken as uniformly distributed over that interval. Then, if the reserve was depleted over that interval, \( L\&R \) quits must have happened. Thus if the interval \( 'a' \) is subdivided into \( L+1 \) intervals of length \( \frac{a}{L+1} \) it is clear that the man-time periods of hoarding are:

\[
(R-1) \frac{a}{L+1} + (R-2) \frac{a}{L+1} + \ldots \ldots + \frac{a}{L+1}
\]

which sums to:

\[
\frac{R(R-1)}{2} \frac{a}{L+1}
\]

As in consideration of the interval where \( T+U \) diminishes to \( R \), this time measure must be multiplied by the cost of hoarding per man-time period \( (h) \) and by the probability of \( L\&R \) quits during the interval \( 'a' \) to get the expected cost of storage:

\[
\frac{R(R-1)}{2} \lambda h \sum_{V=R+1}^{\infty} \varphi(V).
\]
Similarly, if \( L < R \), the cost is:

\[
ha \sum_{V=0}^{R-1} (R-1-V) \varphi(V).
\]

As before, the total expected cost of hoarding the 'R' reserves over the recruiting-training interval 'a' is simply the sum of these two expressions.

Combining the expected costs of hoarding over the interval as \( T+U \) diminishes to \( R \) and those over the interval 'a' yields the direct costs of hoarding over the cycle. Karlin reduced the combined costs to:

\[
h\left(\frac{T^2}{2\lambda} - \frac{T}{2\lambda} + \frac{RT}{\lambda} \sum_{V=0}^{R-1} \varphi(V) - aT \sum_{V=0}^{R-2} \varphi(V)\right) = C_1
\]

(6)

To this point the only costs considered are those incurred by keeping men in reserve to replace expected quits. The costs resulting from inaction must also be considered; i.e., costs of shortages must be evaluated. The probability that a vacancy is suffered is \( \sum_{L=R}^{\infty} \varphi(L) \) and the expected time per cycle that the firm suffers at least one vacancy is then:

\[
a \sum_{L=R}^{\infty} \left(1 - \frac{R}{L+1}\right) \varphi(L) = a \sum_{L=R}^{\infty} \varphi(L) - \frac{R}{\lambda} \sum_{L=R+1}^{\infty} \varphi(L)
\]

(7)

If one vacancy is sufficient to cause the work process to stop or if the penalty or opportunity costs of vacancies do not increase with the number of vacancies suffered, and if the penalty cost is linear with respect to time, we may follow Karlin's model further. He weights the expected.
duration of the shutdown by a penalty cost per time period (designated 'p') to get the expected cost of shutdown:

\[ \text{pa} \sum_{L=R}^{\infty} \varphi(L) - \frac{\text{RD}}{\lambda} \sum_{L=R+1}^{\infty} \varphi(L) = C_2 \]  

In a more realistic model, or in one which seems to better satisfy the constraints already encountered, costs would likely increase rapidly as the number of vacancies rises. Weighting of the expected duration associated with each expected vacancy with increasing penalty costs would then be required. However, for simplicity we will abstract from this possibility and continue with Karlin's basic model.

Karlin's model results in total costs of hoarding of:

\[ h \left\{ \frac{T^2}{2\lambda} - \frac{T}{2\lambda} - \frac{RT^{R-1}}{\lambda} \sum_{V=0}^{\infty} \varphi(V) - \frac{R^2}{\lambda} \sum_{V=0}^{\infty} \varphi(V) \right\} + \]

\[ \text{pa} \sum_{L=R}^{\infty} \varphi(L) - \frac{\text{RD}}{\lambda} \sum_{L=R+1}^{\infty} \varphi(L) = C_1 + C_2 \]  

Karlin notes several criteria that may be constraints, and suggests methods for finding the optimum values of T and of R to minimize the total expected cost over the cycle\(^1\).

Karlin's model does not however, explicitly consider another cost commonly included in inventory models: the cost of re-ordering. (i.e. the cost of recruiting-training in the context of this model.) This cost is relevant to the consideration of the firm's demand for labour, as each time the

---

1. Karlin, S., op. cit., pp. 287, 288
reserve force falls to the "trigger level" \( R \), administrative costs associated with recruiting and training are encountered, and are affected by the choice of 'T'. We will assume for simplicity that reordering (recruiting-training) involves certain costs which are fixed with respect to the number of workers recruited and trained, as well as other costs 'r' which are linearly related to the time interval 'a' and to the number recruited and trained. This relationship might be described as:

\[ raT + k = C_3 \]  

(10)

The problem faced is the minimization of total cost per cycle; i.e. minimize \( C_1 + C_2 + C_3 \).

Karlin implicitly assumes the cost of reordering (or recruiting-training in this context) is fixed, by using the expected cycle time as a restraint on ordering; i.e. ordering may occur only once per cycle. Thus the problem becomes: "if \( (R, T) \) satisfy \[ a + \frac{T}{a} - \frac{R}{a} \sum_{R}^{0} \varphi(L) - \sum_{R}^{a} \varphi(L) = \beta, \]

where \( \beta \) is the prescribed length of time of a cycle, minimize

\[
\mathcal{L} = h \left\{ \frac{T^2}{2a} - \frac{T}{2a} + \frac{RT^{R-1}}{V=0} \varphi(V) - aT \sum_{V=0}^{R-2} \varphi(V) \right\} + \]

\[
pa \sum_{L=R}^{\infty} \varphi(L) - \sum_{L=R+1}^{\infty} \varphi(L)
\]

Since \( \mathcal{L} \) is a quadratic in \( T \), we could solve for \( T \) in terms of \( R \) and use numerical methods to approximate the optimum...
value of $R$.$^1$

If we consider the recruiting-training cost as a variable which increases with $T$, and note that hoarding costs behave similarly, whereas expected costs of shortages fall with increases in $T$, it is clear that the problem faced is conceptually that of simple economic order quantity inventory models. Minimum total cost will occur where the curve representing the recruiting-training costs plus hoarding costs cuts the cost of shortages curve. (Costs are plotted against the size of the reserve force ($T$) on the abscissa.) Thus minimization of costs is achieved when $C_1 + C_3 = C_2$ or when:

$$h\left(\frac{T^2}{2\lambda} - \frac{T}{2\lambda} + \frac{RT}{\lambda} \sum_{V=0}^{R-1} \varphi(V) - aT \sum_{V=0}^{R-2} \varphi(V)\right) + raT + k =
$$

$$pa \sum_{L=R}^{\infty} \varphi(L) - \frac{RP}{\lambda} \sum_{L=R+1}^{\infty} \varphi(L)$$

(11)

The main decision variables over which management has control are $T$ and $R$. However, in the long run the firm's management may also be able to effect changes in $\lambda$ (and thereby in the distribution of vacancies, $\varphi(V)$ and $\varphi(L)$, associated with any $T$ and $R$.) Changes in $\lambda$ may be effected by more careful screening of employees (and thereby an increase in $r$) or possibly through improved personnel policies.

1. Karlin, S., op. cit., p. 288
or increased non-vested benefits to the employees (and thereby likely an increase in h). For this simple model 'a' is given and fixed, and it will be assumed that h, r, and p are imposed upon the firm by outside groups and are therefore not management decision variables.

Implications

The model supports several intuitively obvious policy actions. For example, if the cost of hoarding (h) rises, one logical response available to the cost minimizing firm is the reduction of the size of the reserve force trained (T). Conversely, if the penalty cost (p) rises the firm may increase the size of the reserve force trained. Both of these reactions are indicated by this simple model. Also, it supports the obvious conclusion that if the variable costs of training and recruiting (r) rise, the firm may respond by reducing T to minimize costs.

A less obvious response to changes in 'h' or 'p' is available to the firm which is satisfactorily depicted by this model. As noted by Karlin, the "trigger level" force (R) that is held to replace quits over the interval 'a' is a major decision variable related to 'T'. Thus if the cost of hoarding (h) rises, the firm may elect to reduce R. This will reduce the term \( \frac{RT}{n} \) in equation (11) and should also reduce the probability of a positive U (i.e. \( \sum_{V=0}^{R-1} \varphi(V) \) will also fall) thereby producing a reduction in \( C_1 \) that offsets.
the exogeneous increase caused by 'h' changing. However, by altering 'R' the firm is also altering the expected costs of shortages, $C_2$. A reduction in 'R' will increase the probability that 'L' will exceed 'R', and thereby increase the expected penalty cost. It must be noted that the magnitude of this increase (which restricts the use of 'R' as a policy or decision variable) depends critically on the distribution of 'L' (i.e. on $\varphi(L)$). As this distribution is Poisson, $\lambda$, the mean number of quits per time period, is the critical factor. The larger is $\lambda$ (and therefore the less is the skewness of the distribution) the smaller will be the effect on the expected penalty costs of changing 'R' one unit from the previous optimum. However, if $\lambda$ is small (and therefore the distribution is very skewed) a minor change in 'R' away from the original optimum will have a large effect on expected penalty costs, as the likelihood that 'L' will exceed 'R' will then increase rapidly with reductions in 'R'. Thus the use of 'R' as a decision variable when 'a' is fixed is applicable only to firms with large turnover (in the homogeneous group under consideration) or with insignificant penalty costs. The latter restriction is trivial as small penalty costs would imply little need to hoard a reserve force.

The sensitivity of the model to the distribution of quits indicates that management might try, at least in the long run, to affect the distribution. As $\lambda$ describes both
the mean and the variance in a Poisson distribution, 
\( \lambda = \mu; \lambda = \sigma^2 \) the firm will try to reduce the mean 
number of quits per time period, and thereby also reduce 
the variance of the distribution. (Skewness will increase.) 
Then, for any choice of 'R' the probability of 'L' exceeding 
'R' will fall. This means that the penalty costs associated 
with expected shortages will fall; a particularly important 
circumstance in cases where the penalty costs rise steeply 
as the number of vacancies suffered increases. As \( \lambda \) is 
reduced, hoarding costs \( (C_1) \) increase, as each person in 
reserve will be expected to remain in reserve longer, but 
the number it is then optimal to hold should fall. 

This model implies that a firm with a work group 
which satisfies the model's assumptions might pay (in direct 
and in indirect, non-vested remuneration) each employee in 
the group more than he would receive if the dynamic effects 
of labour turnover were not considered. That is, the firm 
might increase other costs to achieve a reduction in the 
expected costs of hoarding by reducing \( \lambda \). 

The logical extension of the argument for reducing \( \lambda \) is 
to attempt to remove uncertainty from the system. (i.e. to 
reduce \( \lambda \) to zero.) Perhaps the most common manifestation 
of such an attempt is the use of mandatory notice periods 
and of penalties for failure to do so. If a firm can compel 
an employee to give notice that he intends to quit at a 
specified time, and if the notice period is equal to or
greater than 'a', the firm avoids the need for holding reserves. If the employee can be financially punished for failing to comply to this type of rule, at least part of the costs of the quit can be transferred from the firm to the employee. As long as the costs of holding a dissatisfied employee over the notice period are less than the costs of hoarding, the firm will be willing to utilize the dissatisfied employee.

The firm will likely also seek to reduce costs of hoarding by using the reserve force in productive tasks, rather than holding it idle. From the size of 'T' and the distribution of quits, the firm can ascertain the expected size of the reserve at any time, and may be able to use the force at some job requiring similar skills. The relevant holding cost for the model is then hoarding costs net of productive output.

The firm may elect to reduce hoarding costs by using part-time employees as part of the work force. then quits may be filled by "converting" employees from part to full-time. In this case the additional administrative costs involved would be the major hoarding costs. (Recruiting-training costs would also rise.)

Another method of reducing the cost of a quit and therefore the need for reserves, is to require that the new employee serve a probation period over which vested benefits do not accrue. As the Poisson distribution of quits is
applicable when the individual's likelihood of quitting is described by an exponential distribution, reduction of costs of "infant mortality" in employment is relevant.

The model, though strictly applicable to only a narrow group of very large firms, and only feasible for use by such, seems to provide a rationale for many policies commonly used by a wider array of firms. Also, the model seems amenable to the inclusion of more realistic factors.

One possible extension of the model might incorporate expected product demand with the expected quit rate to get a more realistic measure of reserve requirements. (Note that this necessitates a revision of the definition of hoarding used earlier.) Karlin considered another modification, and altered the basic model to accommodate the case of a variable reordering (recruiting-training) interval (a). ¹

Labour supply considerations might also be incorporated, possibly with implications for cyclic patterns of labour demand. For example, if the labour market is "tight", it seems that the expected number of quits per time period might rise, as would the costs of finding replacements. The firm would then find it necessary to increase the size of its reserve force, thereby reinforcing the excess demand for the particular skill. In a "loose" market the opposite pattern of events would hold, and again hoarding would reinforce the market problem.

¹ Karlin, S., op. cit., p. 288
Thus the model, and the approach of operations research, is of interest for the implications and possible extensions toward greater generality, although feasible in practice for use only by large firms which satisfy the rigid assumptions used.


