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A SURVEY OF 1-FACTORIZATIONS

by

Teresa Raymond

B.Sc., Simon Fraser University, 1979

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
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of
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A Survey of 1-Factorizations

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ABSTRACT

A survey of results on the existence of \( l \)-factorizations or colourings of graphs is given. The first chapter deals with the existence of \( l \)-factorizations of certain graphs. These graphs include complete graphs, bipartite graphs, circulants, line graphs of some graphs and products of some graphs. The latter includes cartesian, lexicographic, tensor and strong products of graphs.

The second chapter deals with the existence of \( l \)-factorizations with certain properties. Perfect \( l \)-factorizations, Kotzig factorizations and graphs with certain \( Q \)-indices are studied.
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The definition of a graph which is used is that of an
undirected graph with no loops or multiple edges.

**Definition 0.1.** A graph \( G \) is a set of vertices \( V(G) \) and a set of
edges \( E(G) \) which are unordered pairs of elements of \( V(G) \) such that
if \( v_i, v_j \in E(G) \), then \( v_i, v_j \in V(G) \) and \( i \neq j \).

A 1-factor or perfect matching and a 1-factorization or a
colouring are defined as follows.

**Definition 0.2.** A 1-factor \( F_1 \) of a graph \( G \) has \( F_1 \subseteq E(G) \) such that
each vertex of \( V(G) \) has degree 1 in \( F_1 \). A 1-factorization
\( F = \{ F_0, F_1, \ldots, F_n \} \) of a graph \( G \) is a partitioning of \( E(G) \) into
1-factors \( F_i \) where \( i \in \{0, 1, \ldots, n\} \).

Another way to view a 1-factorization of a graph \( G \) is as a
colouring of the edges of \( G \) so that each vertex is incident with
exactly one edge of each colour. In 1879, the problem of the
existence of colourings of graphs was mentioned by Kempe [13]. The
concept of factorizations of graphs was dealt with by König in a book
on graph theory [14] which was published in 1936.

Note that necessary conditions for the existence of a
1-factorization of a graph \( G \) are that \( G \) be regular and \(|V(G)|\)
be even.

A near 1-factorization is defined on the complete graph \( K_n \)
where \( n \) is odd.
Definition 0.3. A near 1-factor $F_i$ of a graph $G$ has $F_i \subseteq E(G)$ such that each vertex of $V(G) \setminus \{v_i\}$ has degree 1 and $v_i$ has degree 0 in $F_i$. A near 1-factorization $F = \{F_0, F_1, \ldots, F_{n-1}\}$ of $K_n$ is a partition of $E(K_n)$ into near 1-factors $F_i$, where $i \in \{0, 1, \ldots, n-1\}$, for $K_n$ a complete graph on the vertices $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$.

The first chapter deals with the existence of 1-factorizations of certain graphs including complete graphs, bipartite graphs, line graphs and certain products. The second chapter deals with the existence of 1-factorizations having certain properties including perfect 1-factorizations, a generalization called a $Q$-index and Kotzig factorizations.
CHAPTER 1

THE EXISTENCE OF 1-FACTORIZATIONS OF GRAPHS

Section 1. Basic Results

Although 1-factorizations of $K_2$, $K_4$ and $K_6$ are isomorphic, all other complete graphs having an even number of vertices have more than one non-isomorphic 1-factorization. For each complete graph $K_{2n}$, one of these 1-factorizations is the pyramidal 1-factorization described in definition 2.1.2 using the abelian group $\mathbb{Z}_{2n-1}$ with generator 1 and another is the bipyramidal 1-factorization described in definition 2.1.3 using the abelian group $\mathbb{Z}_{2n-3}$ with generator 1.

A 1-factorization of every complete bipartite graph $K_{n,n}$ exists.

Definition 1.1.1. A bipartite graph is a graph having a vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ such that each edge is of the form $u_i v_j$ for some $i$ in $\{1, 2, \ldots, n\}$ and $j$ in $\{1, 2, \ldots, m\}$. A complete bipartite graph $K_{n,m}$ is a graph on the vertices $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\}$ with edges $\{u_i v_j | i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}\}$.

Note that the existence of a 1-factorization of $K_{n,m}$ implies that $n = m$. For $n \geq 4$ there is more than one 1-factorization of $K_{n,n}$. One of these is $F = \{F_1, F_2, \ldots, F_n\}$ where $F_i = \{v_{1+i}, v_{2+i}, \ldots, v_{n+i}\}$.
A result of Stern and Lenz [5] leads to the existence of 1-factorizations of some circulants. The proof of this result uses Vizing's theorem [27].

Definition 1.1.2. A circulant is a graph $G(n, S)$ on $n$ vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ with symbol $S$ such that $S \subseteq \{1, 2, \ldots, n-1\}$, if $i \in S$, then $n-i \in S$, and $(i-j) \mod n \in S$ if and only if $v_i v_j \in E(G(n, S))$.

Theorem 1.1.1. (Vizing [27]). The edges of a graph $G$ with maximum degree $k$ can be coloured in $k$ or $k+1$ colours so that no two distinct edges incident with a vertex have the same colour.

Theorem 1.1.2. (Bolletino [5]). If a circulant $G(n, S)$ has an $s \in S$ such that the order of the subgroup of $\mathbb{Z}_n$ generated by $s$ is even, then a 1-factorization of $G(n, S)$ exists.

Proof: If $s, s' \in S$, with the order of the subgroup generated by each of $s$ and $s'$ even, then $G(n, \{s, -s\})$ consists of even length cycles and forms two 1-factors unless $s = -s$ and then $G(n, \{s\})$ itself is a 1-factor. Thus if $G(n, S \setminus \{s, -s\})$ has a 1-factorization then $G(n, S)$ has a 1-factorization. By induction, this leaves the case where $S$ contains only one $s$ which generates an even order subgroup of $\mathbb{Z}_n$.

Suppose there is only one $s \in S$ such that the order of the subgroup of $\mathbb{Z}_n$ generated by $s$ is even. As above the subgroup $G(n, \{s, -s\})$ consists of even length cycles and forms two 1-factors $F_0$ and $F'_0$ unless $s = -s$ and then $G(n, \{s\})$ is a 1-factor $F_0$. The
remaining edges $G(n,S)\backslash\{s,-s\}$ form two vertex disjoint isomorphic subgraphs on $\frac{n}{2}$ vertices. By Vizing's theorem stated in Theorem 1.1.1, each of these subgraphs can be coloured in $\frac{|S\backslash\{s,-s\}|}{2}$ or $\frac{|S\backslash\{s,-s\}|}{2} + 1$ colours. Colour with corresponding colours in each subgraph so that vertices joined by edges $F_0$ have the same colour edges incident with them. If $\frac{|S\backslash\{s,-s\}|}{2}$ colours are used then a 1-factorization of $G(n,S)$ is formed. If $\frac{|S\backslash\{s,-s\}|}{2} + 1$ colours are used then each pair of corresponding vertices is incident with edges of all but one colour. Colour the edges of $F_0$ with the corresponding missing colours. A 1-factorization of $G(n,S)$ is formed.

This leaves circulants $G(n,S)$ where $G(n,\{s,-s\})$ consists of odd length cycles for each $s \in S$. Note that if $n$ is even, say $n = 2^k n'$ where $n'$ is odd, then $2^k$ divides each $s \in S$ and the components of $G(n,S)$ each contain an odd number of vertices. Therefore $G(n,S)$ does not have a 1-factorization if each $s \in S$ generates an odd order subgroup of $\mathbb{Z}_n$. This leads to the following corollary.

**Corollary 1.1.3.** A circulant $G(n,S)$ has a 1-factorization if and only if there exists an $s \in S$ such that $n/gcf(n,s)$ is even.

A Tait colouring is a 1-factorization of a regular graph of degree 3. Tait conjectured that other than a few specified exceptions all regular graphs of degree 3 have Tait colourings. Mark Watkins [29] and Castagna and Prins [7] prove the existence of Tait colourings for a class of regular graphs of degree 3 with one exception. The class of graphs is called generalized Petersen graphs and the exception is the Petersen graph.
Definition 1.1.3. The generalized Petersen graph $GP(n,k)$ is the graph on $2n$ vertices $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ with

$$E(GP(n,k)) = \{u_1v_1, u_2v_2, \ldots, u_nv_n, u_1u_2, u_2u_3, \ldots, u_{n-1}u_1, v_1v_{k+1}, v_{k+2}v_n, \ldots, v_nv_k\}.$$

Theorem 1.1.4, (Castagna and Prins [7]). A 1-factorization exists for every generalized Petersen graph other than the Petersen graph $GP(5,2)$. $\Box$
Section 2. Line Graphs

A class of graphs where some results are known on the existence of 1-factorizations is line graphs of regular graphs.

Definition 1.2.1. A line graph $L(G)$ of a graph $G$ has vertices $E(G)$ and edges $E(L(G)) = \{e_i e_j | e_i, e_j \in E(G) \text{ and } e_i \text{ and } e_j \text{ are adjacent in } G \}$.

Two theorems of Jaeger are useful in proving that the line graphs of certain regular graphs have 1-factorizations.

Theorem 1.2.1. (F. Jaeger [11]). Given a connected, regular graph $G$ with a 1-factorization and $|E(G)|$ even, then $L(G)$ has a 1-factorization.

Theorem 1.2.2. (F. Jaeger [12]). Given a regular graph $G$ with $|E(G)|$ even, then there exists a 1-factorization of $L(G)$ if there exists a partition of $E(G)$ into Hamiltonian cycles.

Proof. Suppose there are an odd number of Hamiltonian cycles in the partition of $E(G)$ then since $|E(G)|$ is even, each Hamiltonian cycle has even length and $G$ has a 1-factorization. By theorem 1.2.1 $L(G)$ has a 1-factorization.

Suppose $E(G)$ is partitioned into an even number of Hamiltonian cycles $H_1, \ldots, H_{2k}$. Note that if $|V(G)|$ is even then the proof can be done as above. This is not the case if $|V(G)|$ is odd. Now $E(L(G))$ can be partitioned into $2k$ cycles each of length
and corresponding to one of the cycles $H_i$ with a 4-factor between each pair of cycles. In the original graph the edge $v_i v_j$ is adjacent to two edges in the same Hamiltonian cycle and is adjacent to two edges at $v_i$ and two edges at $v_j$ in every other Hamiltonian cycle.

To pair off these cycles a 1-factorization $F = \{F_1, F_2, \ldots, F_{2k-1}\}$ of $K_{2k}$ on the vertices $\{u_1, u_2, \ldots, u_{2k}\}$ is used. If $u_i u_j$ is an edge of $K_{2k}$ this corresponds to the pairing of $H_i$ with $H_j$. $F_1$ corresponds to pairs of Hamiltonian cycles and the 4-factor between those pairs including the edges of the cycles. $F_2, F_3, \ldots, F_{2k-1}$ each pair off the cycles and correspond to the 4-factor between each pairing, not including edges of the cycles. Now $F_2, \ldots, F_{2k-1}$ each correspond to regular bipartite graphs of degree 4 each having a 1-factorization. This leaves the edges corresponding to $F_1$ which is a graph isomorphic to $L(H_i \cup H_j)$ where $H_i$ and $H_j$ are Hamiltonian cycles.

Now $L(H_i \cup H_j)$ is two cycles of length $|V(G)|$ with a 4-factor in between. $L(H_i \cup H_j)$ can be partitioned into three Hamiltonian cycles each of even length, giving a 1-factorization of $L(H_i \cup H_j)$. To do this the 4-factor is partitioned into two Hamiltonian cycles and then the cycles of length $|V(G)|$ and one of the Hamiltonian cycles are partitioned into two Hamiltonian cycles.

To partition the 4-factor into two Hamiltonian cycles direct the edges of $H_i$ into a directed cycle and then define the following
Hamiltonian cycles $B^+$ and $B^-$ in $L(G)$.

Let $B^+ = \{ e, e_j | e_i \in H_i, e_j \in H_j \text{ and } v_i \notin e_i \cap e_j \text{ where } v_i \text{ is the out-vertex of } e_i \}$

and $B^- = \{ e, e_j | e_i \in H_i, e_j \in H_j \text{ and } v_i \notin e_i \cap e_j \text{ where } v_i \text{ is the in-vertex of } e_i \}$.

Looking at $B^+$, suppose $e_j, e_j', e_j''$ are consecutive edges in $H_j$ with $e_j$ incident with and directed away from $e_j'$, $e_j''$ incident with and directed away from $e_j$ and $e_j'$, and $e_j''$ incident with and directed away from $e_j'$ and $e_j''$ where $e_i', e_i'\prime, e_i'' \in H_i$.

A cycle is formed in $L(H_i \cup H_j)$ around the edges $e_j, e_i', e_i'\prime, e_j''$.

$e_j, e_i', e_i'\prime, e_j''$ around to $e_j, e_j'$ going around the edges of $H_j$. Note that each edge of each of the two cycles occurs twice, thus a Hamiltonian cycle is formed.

The reasoning is the same for $B^-$.

Now take $B^+$ and the two cycles of length $|V(G)|$ in $L(H_i \cup H_j)$ coming from $H_i$ and $H_j$. Name these two cycles $A_i$ and $A_j$. Choose an edge in $H_j$ and label it $e_0$ with end vertices labelled $v_1$ and $v_2$. Label the edges of $H_j$ adjacent at $v_1$ and $v_2$ by $e_1$ and $e_2$ and the edges of $H_i$ coming from $v_1$ and $v_2$ by $e_1'$ and $e_2'$. 
In $A_i$ there are two paths between $e_1'$ and $e_2'$. Colour the vertices and edges of one of these with one colour and of the other with the other colour leaving the vertices $e_1'$ and $e_2'$ not coloured.

If $e_k'$ and $e'_j$ in $A_i$ are adjacent to $e_i \neq e_1'$ or $e_2'$, then colour $e_1'e_k'$ and $e_1'e_j'$ with the colour with which $e_1$ is not coloured and colour $e_k'e_j'$ with the same colour as $e_1$. Note that $e_k'$ and $e_j'$ are adjacent to the same outgoing edge of $H_i$ and are adjacent in $H_j$. Thus $e_k'e_j'$ is an edge of $A_j$.

In each colour there are two vertex disjoint chains. One in $A_i$ ends at $e_1'e_2'$ and one in $A_j \cup B'$ ends at $e_1'e_2'$. By colouring $e_1'e_1',e_2'e_2',e_0'e_0'$ in one colour and $e_2'e_2',e_0'e_0',e_0'e_1'$ in the other colour, two Hamiltonian cycles are formed.

These Hamiltonian cycles and $B'$ are of even length.

Therefore $L(H_i \cup H_j)$ has a 1-factorization for any Hamiltonian cycles $H_i$ and $H_j$.

Therefore, given that $G$ can be partitioned into Hamiltonian cycles, $L(G)$ has a 1-factorization.

The technique of this proof can be used to show that $L(G(n,\{1,k,n-k,n-1\}))$ for any $n$ and $k \in \{2,\ldots,\lceil \frac{n}{2} \rceil\}$ has a 1-factorization using a decomposition into one Hamiltonian cycle and one 2-factor. This result can also be proved as a corollary of Theorem 1.2.2 and Theorem 1.1.4.

**Corollary 1.2.3.** Given any $n$ and $k \in \{2,3,\ldots,\lceil \frac{n}{2} \rceil\}$ there exists a 1-factorization of the circulant $L(G(n,\{1,k,n-k,n-1\}))$. 
Proof. The circulant $L(G(n,\{1,k,n-k,n-1\}))$ can be partitioned into a copy of $GP(n,k)$ and three 1-factors. By Theorem 1.1.4 $GP(n,k)$ has a 1-factorization unless $n=5$ and $k=2$. Thus if $G(n,\{1,k,n-k,n-1\}) \neq G(5,\{1,2,3,4\})$ then $L(G(n,\{1,k,n-k,n-1\}))$ has a 1-factorization. By Theorem 1.2.2, $L(G(5,\{1,2,3,4\}))$ has a 1-factorization as $G(5,\{1,2,3,4\})$ can be partitioned into two Hamiltonian cycles."

Theorem 1.2.4. (B. Alspach [1]). A 1-factorization of $L(K_n)$ exists if and only if $n \equiv 0$ or $1 \pmod 4$.

Proof. The number of vertices of $L(K_n)$, $|V(L(K_n))| = \frac{n(n-1)}{2}$ is odd for $n \equiv 2$ or $3 \pmod 4$ and a 1-factorization cannot exist in these cases.

Now the number of vertices of $L(K_n)$ is even for $n \equiv 0$ or $1 \pmod 4$. For all $n \equiv 0 \pmod 4$, $K_n$ has a 1-factorization. Therefore by Theorem 1.2.1 $L(K_n)$ has a 1-factorization for $n \equiv 0 \pmod 4$. For $n \equiv 1 \pmod 4$ a partitioning of the edges of $K_n$ into Hamiltonian cycles exists. Therefore by Theorem 1.2.2 a 1-factorization of $L(K_n)$ exists for $n \equiv 1 \pmod 4$."

Corollary 1.2.5. Given $n$ and $S$ so that every component of $G(n,S)$ has an even number of vertices, then $L(G(n,S))$ has a 1-factorization.

Proof. By Corollary 1.1.3 $G(n,S)$ has a 1-factorization. Thus by Theorem 1.2.1 a 1-factorization of $L(G(n,S))$ exists, since the number of edges in $G(n,S)$ is even."
Corollary 1.2.6. The line graph of any generalized Petersen graph
with an even number of edges has a 1-factorization.

Proof. This is a direct result of Theorem 1.2.1 and Theorem 1.1.4.

Corollary 1.2.7. The line graph of the complete bipartite graph
\( K_{n,n} \) has a 1-factorization if and only if \( n \) is even.

Proof. If \( n \) is odd the number of edges is odd and \( L(K_{n,n}) \) does
not have a 1-factorization. If \( n \) is even then the number of edges
is even and since \( K_{n,n} \) has a 1-factorization, \( L(K_{n,n}) \) has a
1-factorization by Theorem 1.2.1.

Monar, Pisanski and Shawe-Taylor have two results dealing
with line graphs of biregular graphs.

Definition 1.2.2. A biregular graph \( G \) with degrees \( \ell \) and \( n \) is
a bipartite graph with all vertices of degree \( \ell \) or degree \( n \); if
\( v_i, v_j \in V(G) \) have the same degree then \( v_i, v_j \notin E(G) \).

Theorem 1.2.8. (Monar, Pisanski and Shawe-Taylor [23]). Let \( G \) be a
biregular graph with degrees \( 2\ell \) and \( 2n \). Then a 1-factorization
of \( L(G) \) exists.

The second result uses subdivision graphs.

Definition 1.2.3. Let \( G \) be a graph. Then a subdivision graph of \( G \),
denoted \( S(G) \), replaces each edge \( e = uv \) of \( G \) with a path \( u_x \rightarrow v \) where
\( x \) has degree 2.
Note that if $G$ is a regular graph of degree $d$, then $S(G)$ is a biregular graph with degrees 2 and $d$. Also if $G$ is a biregular graph of degrees 2 and $d$, then a regular graph $H$ of degree $d$ exists such that $G = S(H)$.

Theorem 1.2.8. (Monar, Pisanski and Shawe-Taylor [23]). Let $G$ be a biregular graph with degrees 2 and $d$ where $d$ is odd. Then a 1-factorization of $L(G)$ exists if and only if a 1-factorization of $H$ exists where $G = S(H)$. □
Section 3. Products of Graphs

In this section the problem of the existence of 1-factorizations of cartesian, lexicographic, tensor and strong products of graphs is considered.

Definition 1.3.1. For graphs $G_1$ and $G_2$ the **cartesian product**

$G_1 \times G_2$ has $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = 
\{(u_1,v_1)(u_2,v_2) | u_1 = u_2 \text{ and } v_1v_2 \in E(G_1) \text{ or } v_1 = v_2 \text{ and } 
\}
\}
$.

Definition 1.3.2. For graphs $G_1$ and $G_2$ the **lexicographic product**

(wreath product) $G_1 \circ G_2$ ($G_1[G_2]$) has $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \circ G_2) = 
\{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } 
$v_1v_2 \in E(G_2) \}$ .

Definition 1.3.3. For a graph $G$ and positive integer $m$, let

$G(m) = G \circ K_m$ .

Definition 1.3.4. For graphs $G_1$ and $G_2$ the **tensor product**

$G_1 \otimes G_2$ has $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = 
\{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2) \}$ .

Definition 1.3.5. For a graph $G$ and positive integer $m$, let

$G[m] = G \otimes K_m$ .

Definition 1.3.6. For graphs $G_1$ and $G_2$ the **strong product** $G_1 * G_2$
has $V(G_1 \ast G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \ast G_2) = E(G_1 \times G_2) \cup E(G_1 \otimes G_2)$.

Kotzig's result, dealing with cartesian products of regular graphs, is given first.

Theorem 1.3.1. (Kotzig [15]). Let $G_1, G_2, \ldots, G_n$ be regular graphs such that $G_i$ has a 1-factorization for some $i \in \{1, 2, \ldots, n\}$ or $G_i$ and $G_j$ each have a 1-factor for some $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, then a 1-factorization of $G_1 \times G_2 \times \ldots \times G_n$ exists.

Proof. If there is a $G_i \in \{G_1, G_2, \ldots, G_n\}$ such that $G_i$ has a 1-factorization $F = \{F_1, F_2, \ldots, F_k\}$, then colour the edges of the cartesian product $G = G_1 \times G_2 \times \ldots \times G_{i-1} \times G_{i+1} \times \ldots \times G_n$ with $d+1$ colours where the degree of each vertex in $G$ is $d$. This existence is a direct result of Vizing's Theorem [27]. Then a $d+1$ colouring or a 1-factorization of $F_1 \times G$ is formed, colouring the new edges with the missing colour at the corresponding vertices of $G$.

Add a new colour for each $F_i, i \geq 2$, colouring the edges of the form $(f, g)(f', g')$ with colour $i$ for $f, f' \in F_i, i \geq 2$.

If there are $G_i$ and $G_j \in \{G_1, G_2, \ldots, G_n\}$ such that $F_i$ and $F_j$ are 1-factors of $G_i$ and $G_j$, respectively, then partition the edges of $G_i \times G_j$ into $F_i \times H_j$ and $H_i \times F_j$ where $H_i$ and $H_j$ are graphs on the vertex sets of $G_i$ and $G_j$, respectively such that $E(G_i) = E(F_i) \cup E(H_i)$ and $E(G_j) = E(F_j) \cup E(H_j)$. Now $F_i \times H_j$ and $H_i \times F_j$ are made up of disjoint graphs which form two copies of $H_j$ or $H_i$ with corresponding vertices joined by a 1-factor.

Vizing's Theorem [27] is used again as above giving a 1-factorization of
To show that the conditions given in the above theorem are not necessary, Kotzig goes on to prove the following result.

**Theorem 1.3.2.** (Kotzig [15]). For $C$ a cycle of length greater than 3 and $G$ a 3-regular graph, a 1-factorization of $C \times G$ exists.\(\Box\)

The existence of 1-factorizations of lexicographic, tensor and strong products have been studied by Pisanski, Shawe-Taylor and Monar. A certain lexicographic product is looked at first.

**Theorem 1.3.3.** (Monar and Pisanski [22]). Let $G$ be a regular graph. Then a 1-factorization of $G(m)$ exists in each of the following cases:

a) A 1-factorization of $G$ exists,

b) $G$ is of even degree and $m$ is even,

c) $m \equiv 0 \pmod{4}$,

d) $G$ has a 1-factor and $m$ is even,

e) $G$ is cubic and $m$ is even, and

f) $G$ is bipartite.\(\Box\)

Laskar and Hare [18] show that a 1-factorization of $K_n(m)$ exists if and only if $mn$ is even. Parker [25] shows that if $G$ is a cycle on $n$ vertices, a 1-factorization of $G(m)$ exists if and only if $mn$ is even.

Now a certain tensor product is looked at.
Theorem 1.3.4. (Pisanski, Shawe-Taylor and Monar [26]). Given a regular graph \( G \) and a positive integer \( m \), then a 1-factorization of \( G^{(2m)} \) exists. \( \square \)

Theorem 1.3.5. (Monar, Pisanski and Shawe-Taylor [23]). Let \( G_1, G_2, \ldots, G_n \) be regular graphs such that \( G_i \) has a 1-factorization for some \( i \in \{1, 2, \ldots, n\} \) or \( G_i \) and \( G_j \) each have a 1-factor for some \( i, j \in \{1, 2, \ldots, n\}, i \neq j \). Then a 1-factorization of the lexicographic product \( G_1 \circ G_2 \circ \ldots \circ G_n \) exists.

Proof. Their proof is reduced to proving the existence of a 1-factorization of \( G \circ H \) where the edges of \( G \circ H \) can be partitioned into \( G \times H \) and \( G(\mid V(H)\mid) \). The proof is completed using Theorem 1.3.1 and 1.3.4 where \( H \) has a 1-factorization or a 1-factor since \( \mid V(H)\mid \) is even. This leaves the case where a 1-factorization of \( G, F = \{F_1, F_2, \ldots, F_m\} \) exists, then \( G(\mid V(H)\mid) = F_1(\mid V(H)\mid) \oplus F_2(\mid V(H)\mid) \oplus \ldots \oplus F_m(\mid V(H)\mid) \). Since each \( F_i(\mid V(H)\mid) \) for \( i \in \{1, \ldots, m\} \) can be reduced to copies of regular bipartite graphs each of which is 1-factorable, a 1-factorization exists. This completes the proof. \( \square \)

To prove that the conditions given in the above Theorem are not necessary they go on to prove the following result.

Theorem 1.3.6. (Pisanski, Shawe-Taylor and Monar [26]). For \( C \) a cycle of length greater than 3 and \( G \) a 3-regular graph, a 1-factorization of \( C[G] \) exists.
Theorem 1.3.7. (Monar, Pisanski and Shawe-Taylor [23]). Let

\begin{equation*}
G_1, G_2, \ldots, G_n \text{ be regular graphs such that } G_i \text{ has a 1-factorization for some } i \in \{1, 2, \ldots, n\}. \text{ Then a 1-factorization of } G_1 \otimes G_2 \otimes \ldots \otimes G_n \text{ exists.}
\end{equation*}

Proof. Since the tensor product is commutative, assume that \( G_1 \) has the 1-factorization \( F = \{F_1, F_2, \ldots, F_k\} \). Looking at the graph \( G_1 \otimes G_2 \), the set of edges \( E(G_1 \otimes G_2) \) is made up of copies of \( F_i \otimes G_2 \) for \( i \in \{1, 2, \ldots, k\} \). Each of these can be partitioned into vertex disjoint copies of \( H(2) \). By Theorem 1.3.4, the proof is complete. \( \Box \)

Theorem 1.3.8. (Monar, Pisanski and Shawe-Taylor [23]). Let

\begin{equation*}
G_1, G_2, \ldots, G_n \text{ be regular graphs such that } G_i \text{ has a 1-factorization for some } i \in \{1, 2, \ldots, n\}. \text{ Then a 1-factorization of the strong product } G_1 * G_2 * \ldots * G_n \text{ exists.}
\end{equation*}

Proof. Now \( E(G * H) \) can be partitioned into \( G \times H \) and \( G \otimes H \).

Theorems 1.3.1 and 1.3.7 complete the proof. \( \Box \)
Chapter 2

The Existence of 1-Factorizations with Certain Properties

Section 1. Perfect 1-Factorizations

Definition 2.1.1. A perfect 1-factorization is a 1-factorization $F = \{F_1, F_2, \ldots, F_n\}$ such that for every $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $E(F_i \cup F_j)$ is a Hamiltonian cycle.

A 1-factorization which is formed from a 1-factor by fixing a vertex and performing a cycle permutation on the other vertices is called a pyramidal 1-factorization by Mendelsohn and Rosa in [21].

Definition 2.1.2. Let $V(K_{2n}) = \{v_0, v_1, v_2, \ldots, v_{2n-2}\}$, where $0, a_1, a_2, \ldots, a_{2n-2}$ are the elements of an abelian group $G$ of order $2n-1$. Additive notation is used and for $a_i \in G$, $= a_i = \infty$. Let $F_0$ be a 1-factor of $K_{2n}$. Then $F = \{v_{a_i + a_j}, v_{a_i + a_j + a_k}, \ldots, v_{a_j + a_k}\}$, where $v_{a_i + a_j}$ is a 1-factor of $K_{2n}$. If the collection of 1-factors $F_0, F_1, \ldots, F_{2n-2}$ is a 1-factorization of $K_{2n}$, then it is called a pyramidal 1-factorization.

A pyramidal 1-factorization is used to construct a perfect 1-factorization of $K_{p+1}, K_{16}, K_{28}, K_{244}$ and $K_{344}$. A 1-factorization which gives a different perfect 1-factorization of $K_{24}$ is called a bipyramidal 1-factorization and is defined as follows.
Definition 2.1.3. Let \( P = \{ P_i \mid i \in Z_{2n-1} \} \) be the pyramidal 1-factorization of \( K_{2n} \) described in definition 2.1.2. Let

\[
V(K_{2n+2}) = \{ u_0, u_1, \ldots, u_{2n-1}, u_v, u_v' \}, \quad \tau(x) =
\begin{cases} 
  x & \text{if } x < \frac{3n}{2} \\
  x+1 & \text{if } x \geq \frac{3n}{2}
\end{cases}
\]

\[ \tau(\infty) = \infty. \]

Additive notation in the group \( Z_{2n} \) is used and for \( a \in Z_{2n}, a + \infty = \infty \). The 1-factors

\[ P_i = \{ u_\tau(x)+i, u_\tau(y)+j \mid x, y \in F_0 \} \cup \{ u_i, u_i' \} \text{ for } i \in Z_{2n} \text{ and } \]

\[ P^* = \{ u_{i+n} \mid i = 0, 1, \ldots, n-1 \} \cup \{ u_v, u_v' \} \text{ form a 1-factorization } F' \text{ of } K_{2n+2}. \]

This 1-factorization is called a bipyramidal 1-factorization.

The following lemma reduces the number of subgraphs to be checked in proving that a pyramidal 1-factorization of \( K_{2n} \) is perfect, leaving \( n-1 \) cases to be checked.

Lemma 2.1.1. (B.A. Anderson [2]). Let \( G \) be an additive group of order \( 2n-1 \) generated by \( a_1 \), with \( a_1 = a_{i-1} + a_1 \). Let the vertices of \( K_{2n} \) be labelled \( v_0 \) and \( v_{a_i} \) where \( a_i \in G \). Let \( F = \{ F_{a_i} \mid a_i \in G \} \)

be a pyramidal 1-factorization of \( K_{2n} \) with \( v_a v_{a_i} \in F_{a_i} \) for \( a_i \in G \)

such that \( F_0 \cup F_k \) is a Hamiltonian cycle for \( k \) in \( \{1, \ldots, n-1\} \).

Then \( F \) is a perfect 1-factorization.
Proof: All arithmetic is done in \( G \) with \( a_k + \infty = \infty \) for \( a_k \in G \).

Let \( \sigma \) be a permutation of the vertices of \( K_{2n} \) defined as
\[
\sigma(v_{a_k}) = v_{a_k + a_1} \quad \text{for} \quad v_{a_k} \in V(K_{2n}), \quad \text{with the corresponding permutation}
\]
on the edges \( \sigma(v_{a_i}a_j) = \sigma(v_{a_i})\sigma(v_{a_j}) = v_{a_i}a_j \).

Given \( i, j \in \{1, 2, \ldots, 2n-2\} \), \( i < j \) then \( j-i \leq n-1 \) or \( i-j = k (\text{mod } 2n-1) < n-1 \). If \( j-i \leq n-1 \), then
\[
\sigma^{-1}(F_{a_i}) = F_{a_0} \quad \text{and} \quad \sigma^{-1}(F_{a_i} \cup F_{a_j}) = F_{a_0} \cup F_{a(j-i)}.
\]
If \( i-j = k (\text{mod } 2n-1) < n-1 \), then
\[
\sigma^{-j}(F_{a_i}) = F_{a_k} \quad \text{and} \quad \sigma^{-j}(F_{a_i} \cup F_{a_j}) = F_{a_0} \cup F_{a(j-i)}
\]
and \( \sigma^{-j}(F_{a_i} \cup F_{a_j}) = F_{a_0} \cup F_{a_k} \) for \( k < n-1 \). Thus for \( i, j \) in
\[
\{1, 2, \ldots, 2n-2\} \quad F_{a_i} \cup F_{a_j} \equiv F_{a_0} \cup F_{a_k}
\]
for some \( k \) in \( \{1, 2, \ldots, n-1\} \).

Therefore, if \( F \cup F_{a_k} \) is a Hamiltonian cycle for \( k \) in
\[
\{1, 2, \ldots, n-1\}, \quad \text{then } F \text{ is a perfect } 1\text{-factorization.} \]

**Theorem 2.1.2.** (Kotzig [15]). For any odd prime \( p \) a perfect
1-factorization of \( K_{p+1} \) exists.

**Proof:** Using Lemma 2.1.1, with the group \( \mathbb{Z}_p \) and generator \( 1 \) with
addition modulo \( p \) on residues \( 0, 1, \ldots, p-1 \), a pyramidal
1-factorization of \( K_{p+1} \) is shown to be a perfect 1-factorization.

Let the vertex set of \( K_{p+1} \) be \( \{v_0, v_1, \ldots, v_{p-1}, v_\infty\} \).

Consider the 1-factor \( F_0 \).
Let \( F = \{F_0, F_1, \ldots, F_{p-1}\} \) be the set of 1-factors described in Lemma 2.1.1 using the permutation \( \sigma \). The edge \( v_i v_i \) for \( i \) in \( \{0, 1, \ldots, p-1\} \) occurs in exactly one 1-factor of \( F \), namely \( F_1 \).

Note that for \( j, k \in \{1, \ldots, p-1\}, j \neq k, v_j v_k \in E(F_0) \) if and only if \( k + j = 0 \) and thus, in general, \( v_j v_k \in E(F_1) \) if and only if \( k + j - 2i = 0 \). Since there is exactly one \( k \) such that \( i + j = k \), the edge \( v_i v_j \) occurs in exactly one 1-factor, namely \( F_k \).

Therefore, \( F \) is a 1-factorization of \( K_{p+1} \) and by definition a pyramidal 1-factorization.

Given \( k \in \{1, 2, \ldots, p-1\} \), note that the edges of \( F_0 \) are in the form \( v_0 v_0 \) for \( \alpha \in \{1, 2, \ldots, \frac{p-1}{2}\} \) and \( v_0 v_\alpha \), while the edges of \( F_k \) are in the form \( v_k v_{k+0} v_{k-\alpha} \) for \( \alpha \in \{1, \ldots, \frac{p-1}{2}\} \) and \( v_k v_\alpha \). Thus \( F_0 \cup F_k \) contains the cycle with the sequence of vertices.
A perfect \(1\)-factorization of a graph is a partition of its edges into \(1\)-factors. In this context, we are considering a specific type of \(1\)-factorization for a graph \(G\) with a certain structure. For any \(i, j \in \{0, 1, \ldots, \frac{p-1}{2}\}\), any of \(2ik = 2jk, -2ik = -2jk\) or \(2ik = 2jk\) give \(i = j\) or \(i = -j\) since \(k \neq 0\). If \(j\) is in \(\{0, 1, \ldots, \frac{p-1}{2}\}\), then \(-j\) is not in \(\{0, 1, \ldots, \frac{p-1}{2}\}\) giving \(i \neq -j\).

Thus, a cycle of length \(p + 1\) is formed, and \(\mathcal{F}_0 \cup \mathcal{F}_k\) forms a Hamiltonian cycle for \(k \in \{1, \ldots, p-1\}\).

Therefore, by Lemma 2.1.1, \(\mathcal{F}\) is a perfect \(1\)-factorization.

**Theorem 2.1.3.** (B.A. Anderson [2]). A perfect \(1\)-factorization of \(K_{16}\) exists.

**Proof:** Using Lemma 2.1.1 with the group \(\mathbb{Z}_{15}\) and generator 1 with addition modulo 15 on residues 0, 1, \ldots, 14, a pyramidal \(1\)-factorization is shown to be a perfect \(1\)-factorization.

Let the vertex set of \(K_{16}\) be \(\{v_0, v_1, \ldots, v_{14}, v_{\infty}\}\). Consider the \(1\)-factor \(\mathcal{F}_0\).

![Figure 2](image-url)
Let $F = \{F_0, F_1, \ldots, F_{14}\}$ be the set of $1$-factors described in Lemma 2.1.1 using the permutation $\sigma$. The edge $v_i \cdot v_j$ for $i \in \{0, 1, \ldots, 14\}$ occurs in exactly one $1$-factor of $F$, namely $F_i$.

Note that $\{\pm(i \pm j)\mid v_i v_j \in E(F_0), i, j \neq \infty\} = \{1, \ldots, 14\}$. Thus $F$ is a $1$-factorization of $K_{16}$ and by definition a pyramidal $1$-factorization.

By checking that $F_0 \cup F_k$ forms a Hamiltonian cycle for $k \in \{1, \ldots, 7\}$, by Lemma 2.1.1, it can be shown that $F$ is a perfect $1$-factorization.

The following construction of Mullin and Nemeth for Room Squares gives a $1$-factorization of $K^p_{p+1}$ where $p$ is an odd prime, $p^m > 3$ and $p^m \equiv 3 \pmod{4}$. This construction is used to prove the existence of a perfect $1$-factorization of $K^8_{28}$ and $K^4_{344}$. The added structure allows for checking the union of only one pair of $1$-factors to prove that the $1$-factorization is perfect.

**Definition 2.1.4.** For an odd prime $p$ and an integer $m$ such that $p^m > 3$, $p^m \equiv 3 \pmod{4}$, let $x$ be a generator of the multiplicative subgroup of order $p^m - 1$ in $GF(p)$. Let $V(K^m_p) = \{v_0, v_1, \ldots, v_{p^m - 1}\}$. Let $F_0$ be defined by $E(F_0) = \{v_0 v_m, v_1 v_{x^2}, v_2 v_{x^3}, \ldots, v_{r^3} v_{x^m}, v_{r^2} v_{x^m-3}, v_{r^2} v_{x^m-2}\}$ and $E(F_k) = \sigma^k(F_0)$ where $\sigma$
is as defined in Lemma 2.1.1. The resulting collection of 1-factors is shown to be a pyramidal 1-factorization in the next result and is called a \textit{Mullin-Nemeth 1-factorization}.

\textbf{Lemma 2.1.4.} (Mullin and Nemeth [24]). Let \( F \) be a Mullin-Nemeth 1-factorization, using the multiplicative generator \( x \), of \( K_{p^m+1} \) where \( p \) is an odd prime and \( m \) is an integer such that \( p^m > 3 \),

\[ p^m \equiv 3 \pmod{4}, \]

then \( F \) is a pyramidal 1-factorization of \( K_{p^m+1} \).

\textbf{Proof:} All arithmetic is done in \( GF[p^m] \). Since \( x \) is a generator of the multiplicative subgroup of \( GF[p^m] \), each of the vertices of \( K_{p^m+1} \) occurs in \( F \) exactly once.

Now look at the set \( S = \{tx^0(1-x), tx^2(1-x), \ldots, tx^{p^m-3}(1-x)\} \).

Note that \( 1-x \in GF[p^m] \) and \( 1-x \neq 0 \) since \( p^m > 3 \). If

\[ x^{2\alpha}(1-x) = x^{2\beta}(1-x) \quad \text{for some } \alpha, \beta \text{ with } 0 \leq \alpha, \beta \leq \frac{p^m-3}{2}, \]

then

\[ x^{2\alpha} = x^{2\beta}, \quad 2\alpha \equiv 2\beta \pmod{p^m-1} \]

since \( x \) generates \( GF[p^m] \), and \( \alpha = \beta \) since \( 0 \leq \alpha, \beta \leq \frac{p^m-3}{2} \). If

\[ x^{2\alpha}(1-x) = -x^{2\beta}(1-x) \quad \text{for } \alpha, \beta \text{ such that } 0 \leq \alpha, \beta \leq \frac{p^m-1}{2}, \]

then

\[ x^{2\alpha} + x^{2\beta} = 0. \]

If \( \alpha = \beta \) then \( 2x^{\alpha} = 0 \) and \( GF[p^m] \) has characteristic 2 contradicting \( p \) an odd prime. If \( \alpha \neq \beta \), say \( \alpha < \beta \), then \( x^{2\alpha}(1 + x^{2\beta-2\alpha}) = 0 \) and since \( x^{2\alpha} \neq 0 \), \( x^{2\beta-2\alpha} = -1 \).
Thus \( x^{2\beta - 2\alpha} = -1 \) since \( 0 < 2\beta - 2\alpha \leq p^{m-1} \) which gives

\[
2\beta - 2\alpha = \frac{p^{m-1}}{2}.
\]

Thus \( p^{m-1} = 4(j-1) \) and \( p^m \equiv 1 \pmod{4} \)

contradicting \( p^m \equiv 3 \pmod{4} \).

Therefore \( S = GF(p^m) \backslash \{0\} \) and \( F \) is a pyramidal

1-factorization.

Lemma 2.1.5. (B.A. Anderson [3]). Let \( F = \{F_0, \ldots, F_{p-1}\} \) be a

Mullin and Nemeth 1-factorization of \( K^m \) using the generator \( x \)

where \( p \) is an odd prime, \( p^m > 3, p^m \equiv 3 \pmod{4} \), then \( F \) has the

property that for \( i, j, i', j' \in GF(p^m), i \neq j, i \neq j' \),

\[
F_i \cup F_j \cong F_i \cup F_{j'},
\]

Proof: All arithmetic is done in \( GF(p^m) \).

For \( \alpha \in \{x^0, x^1, \ldots, x^{p^m-3}\} \) define the permutation of the vertices

of \( K^m \) for \( v_i \in V(K^m) \). The corresponding

permutation of the edges of \( K^m \) is defined by \( \tau_{\alpha}(v_i, v_j) = v_i^\alpha v_j^\alpha \)

for \( v_i, v_j \in E(K^m) \). Note that \( \tau_{\alpha}(0) = F_0 \).

Given \( k \in GF(p^m) \backslash \{0\} \) then either \( k^{-1} \in \{x^0, x^1, \ldots, x^{p^m-3}\} \)
or \( -(k^{-1}) \in \{x^0, x^1, \ldots, x^{p^m-3}\} \) since \(-k = k(x^{p^m-1}) \) and

\[
\frac{p^{m-1}}{2} \equiv 1 \pmod{4}.
\]

Let \( k' \) be \( k^{-1} \) or \(-k^{-1}\) such that

\[
k' \in \{x^0, x^1, \ldots, x^{p^m-1}\}.
\]

Now \( \tau_{k'}(F_0) = F_0 \) and
For \( k \in GF[p^m] \), define the permutation \( \sigma_k \) as before on the vertices of \( K_{p^m+1} \). Then \( \sigma^{-k}_k(F_k) = F_0 \) for \( k \in GF[p^m] \). Thus for any \( i,j \in GF[p^m] \), \( \sigma^{-k}_i(F_i \cup F_j) = F_0 \cup F_{j-i} \). If

\[
(j^{-1})^{-1} \in \{ x^0, x^2, \ldots, x^{p^m-1} \}
\]

then \( (j^{-1})^{-1}(\sigma^{-k}_i(F_i \cup F_j)) = F_0 \cup F_1 \).

If \( (j^{-1})^{-1} \not\in \{ x^0, x^2, \ldots, x^{p^m-1} \} \) and

\[
\sigma^{-1}_1((j^{-1})^{-1}(\sigma^{-k}_i(F_i \cup F_j))) = F_0 \cup F_1.
\]

Therefore \( F_i \cup F_j \not\subseteq F_i, \cup F_j \) for \( i,i', j,j' \in GF[p^m] \).

Theorem 2.1.6. (B.A. Anderson [3]). A perfect \( 1 \)-factorization of \( K_{28} \) exists.

Proof: The polynomial \( y^3 + 2y^2 + 1 \) is irreducible over \( GF[3] \). Thus

the root \( x \) of \( y^3 + 2y^2 + 1 \) is a generator of \( GF[3^3] \).

Using the Mullin-Nemeth \( 1 \)-factorization let

\[
F_0 = \{ v_0, v_{\infty}, v_1, v_x, v_{2x}, v_{x+2}, v_{x+2x}, v_{2x+2}, v_{2x+2}, v_{2x} \}
\]

where

\[
\begin{align*}
F_0 &= \{ v_0, v_{\infty}, v_1, v_x, v_{2x}, v_{x+2}, v_{x+2x}, v_{2x+2}, v_{2x+2}, v_{2x} \} \\
&= \{ v_{x^2}, v_{2x^2+2}, v_{x^2+2}, v_{2x^2+2x}, v_{2x^2+2x+2}, v_{2x^2+2x+2} \} \\
&= \{ v_{x^2}, v_{2x^2+2}, v_{x^2+2}, v_{2x^2+2x}, v_{2x^2+2x+2}, v_{2x^2+2x+2} \}
\end{align*}
\]


Perfect 1-factorizations of $K_{244}$ and $K_{344}$ exist.

**Proof:** For $K_{244}$, $244 = 3^5 + 1$, the polynomial $y^5 + 2y + 1$ is irreducible over GF[3]. The Mullin-Hemeth 1-factorization using $x^3$ as a generator gives a perfect 1-factorization of $K_{244}$.

For $K_{344}$, $344 = 7^3 + 1$, the polynomial $y^3 + 6y + 2$ is irreducible over GF[7]. The Mullin-Hemeth 1-factorization using $x^6$ as a generator gives a perfect 1-factorization of $K_{344}$. 

By Lemma 2.1.5, $P$ is a perfect 1-factorization, $\Box$
A construction for a perfect 1-factorization of $K_{2p}$, where $p$ is an odd prime, uses a cyclic permutation of the vertices with corresponding permutation of the edges to partition the edges $v_i v_j$, where $|i-j|$ is even or $|i-j| = p$, into 1-factors. This leaves the edges of the circulants $G(2p,\{k,-k\})$ for odd $k \in \{0, 1, \ldots, p-1\}$. Now for odd $k \in \{0, 1, \ldots, p-1\}$, $G(2p,\{k,-k\})$ forms a Hamiltonian cycle on an even number of vertices, which has the obvious pair of 1-factors.

**Theorem 2.1.8. (Kotzig [16]).** For any prime $p$ a perfect 1-factorization of $K_{2p}$ exists.

**Proof:** All arithmetic is done modulo $2p$ on the residues $\{0, \ldots, 2p-1\}$. Let the vertex set of $K_{2p}$ be $\{v_0, v_1, \ldots, v_{2p-1}\}$.

Consider the 1-factor $F_0$.

![Diagram](https://via.placeholder.com/150)

**Figure 3**

Rotate this configuration through $p-1$ rotations using the permutation $\rho = (v_0, v_1, \ldots, v_{2p-1})$. The corresponding permutation on
the edges is \( \rho(v_i, v_j) = v_{i+1}v_{j+1} \). Thus \( p \)-factors \( F_i \) for 

\[ i \in \{0, ..., p-1\} \text{ are formed where } E(F_i) = \rho_i E(F_0). \]

Note that 

for \( j, k \in \{1, ..., 2p-1\}, j \neq k, v_jv_k \in E(F_0) \) if and only if 

\[ k+j \equiv 0 \pmod{2p}. \]

Thus, \( v_jv_k \in E(F_i) \) if and only if 

\[ \frac{k+j}{2} = i \pmod{2p}. \]

For \( i \in \{0, 1, ..., p-1\} \) the \( 1 \)-factor \( F_i \) also

includes the edge \( v_i v_{i+p} \).

This leaves the edges \( v_i v_j \) where \( |i-j| \in \{1, 3, ..., p-2\} \)

which are exactly the edges of the circulants \( G(2p, \{k, -k\}) \) for 

\[ k \in \{1, 3, ..., p-2\} \], each of which is a Hamiltonian cycle. For 

\[ k \in \{1, 3, ..., p-2\}, \text{ let } E(F'_k) = \bigcup_{\alpha \in \{1, 3, ..., 2p-1\}} \{v_\alpha k^{(\alpha+1)k}\} \]

and \( E(F''_k) = \bigcup_{\alpha \in \{0, 2, ..., 2p-2\}} \{v_\alpha k^{(\alpha+1)k}\} \).

Note that 

\[ F'_i \cup F''_i \cong G(2p, \{k, -k\}). \]

Thus, \( F = \bigcup_{i \in \{0, ..., p-1\}} F_i \cup (\bigcup_{i \in \{1, 3, ..., p-2\}} F'_i) \cup (\bigcup_{i \in \{1, 3, ..., p-2\}} F''_i) \) is a \( 1 \)-factorization of \( K_{2p} \).

\( F \) is a perfect \( 1 \)-factorization of \( K_{2p} \) if, for each 

\[ i, i' \in \{0, 1, ..., p-1\} \text{ and } j, j', k, k' \in \{1, 3, ..., p-2\}, \]

\( F_i \cup F_{i'}, F'_j \cup F'_{j'}, F''_k \cup F''_{k'}, F_j \cup F'_{j'}, F_i \cup F''_i \) and \( F'_j \cup F''_j \) form 

Hamiltonian cycles for \( i \neq i', j \neq j', k \neq k' \).
**Case 1:** A proof that for \(i,i' \in \{0,1,\ldots,p-1\}, i \neq i'\), \(F_i \cup F_{i'}\) is a Hamiltonian cycle is given.

Note that for \(i > i'\) there is an \(a \in \{1,\ldots,p-1\}\) such that \(a = i-i'\), \(\rho^a(F_a) = F_i\) and \(\rho^a(F_{i'}) = F_{i'}\). Thus

\[
F_i \cup F_{i'} = F_a \cup F_{i'}.
\]

The edges of \(F_0\) are in the form \(v_\beta v^-\beta\) for \(\beta \in \{1,\ldots,p-1\}\) and \(v_0 v_p\). The edges of \(F_a\) are in the form \(v_{a+\beta} v_{a-\beta}\) for \(\beta \in \{1,\ldots,p-1\}\) and \(v_a v_{a+p}\). Thus, for each

\(a \in \{1,\ldots,p-1\}\), \(F_0 \cup F_a\) contains the path given by the sequence of vertices \(v_p, v_0, v_{2a}, v_{-2a}, v_{4a}, v_{-4a}, \ldots, v_{(p-1)a}, v_{-(p-1)a}\). If \(a\) is odd, then \(-(p-1)a = p+a \pmod{2p}\) and \(pa \equiv p \pmod{2p}\). Since \(v_{p+a} v_a\) is an edge of \(F_a\), the path continues with the sequence of vertices \(v_a, v_{-a}, v_{3a}, v_{-3a}, \ldots, v_{pa}\) forming a cycle with \(2p\) vertices. If \(a\) is even, then \(-(p-1)a = a \pmod{2p}\) and \(p+pa \equiv p \pmod{2p}\). Since \(v_a v_{p+a}\) is an edge of \(F_a\), the path continues with the sequence of vertices \(v_{p+a}, v_{p-a}, v_{p+3a}, v_{p-3a}, \ldots, v_{pa}\) forming a cycle with \(2p\) vertices.

Thus, for all \(a \in \{1,\ldots,p-1\}\), \(F_0 \cup F_a\) is a Hamiltonian cycle. Therefore, for \(i,i' \in \{0,\ldots,p-1\}, i \neq i'\), \(F_i \cup F_{i'}\) is a Hamiltonian cycle.
Case 2: A proof that for $j, j' \in \{1, 3, \ldots, p-2\}$, $j \neq j'$, $P'_j \cup P'_j'$ forms a Hamiltonian cycle is given.

In the 1-factor $P'_j$, the vertex $v_{aj}$ is a vertex of the edge $v_{aj}(a-1)j$ if $a$ is even and of the edge $v_{aj}(a+1)j$ if $a$ is odd. Thus, for each $i, i' \in \{1, 3, \ldots, p-2\}$, $i \neq i'$, $P'_{i} \cup P'_{i'}$ contains the path represented by the sequence of vertices

$v_0', v_{-1}', v_{-i}', v_{i'-2i}', v_{2i'-2i}', \ldots, v_{(p-1)i'-i}', v_{-i}', v_0$ since $|i'-i|$ even ($i, i'$ are odd).

Suppose $\alpha(i'-i) = \beta(i'-i)$ then $\alpha = \beta$ since $i \neq i'$. Now $\alpha(i'-i)$ is even and $\beta(i'-i)-i$ is odd so that $\alpha(i'-i) \neq \beta(i'-i)-i$. Thus there are $2p$ distinct vertices in the above path and $P'_i \cup P'_i'$ forms a Hamiltonian cycle.

Case 3: A proof that for $k, k' \in \{1, 3, \ldots, p-2\}$, $k \neq k'$, $P'_k \cup P'_k'$ forms a Hamiltonian cycle is given.

Note that $\partial(P'_k) = P''_k$ and $\partial(P'_k') = P''_k$. Thus $P''_k \cup P''_k' = P'_k \cup P'_k'$.

Therefore, $P'_k \cup P'_k'$ forms a Hamiltonian cycle.

Case 4: A proof that for $i \in \{0, 1, \ldots, p-1\}$ and $j \in \{1, 3, \ldots, p-2\}$, $P'_1 \cup P'_1'$ and $P'_3 \cup P'_3'$ form Hamiltonian cycles is given.
Thus \( F_i \cup F'_j \) and \( F_i \cup F''_j \) are each isomorphic to one of \( F'_0 \cup F'_j \) and \( F''_0 \cup F''_j \).

The edges of \( F'_0 \) are of the form \( v_k v_k' \) for \( k \in \{1, \ldots, p-1\} \) and \( v_0 v_p \). In the 1-factor \( F''_j \), the vertex \( v_{aj} \) is a vertex of the edge \( v_{aj}(2-1) \) if \( a \) is even and of the edge \( v_{aj}(3-1) \) if \( a \) is odd. In the 1-factor \( F''_j \), the vertex \( v_{aj} \) is a vertex of the edge \( v_{aj}(2-1) \) if \( a \) is even and of the edge \( v_{aj}(3-1) \) if \( a \) is odd.

For each \( j \in \{1, 2, \ldots, p-2\} \), \( F'_0 \cup F'_j \) contains the path represented by the sequence of vertices \( v_p, v_0, v_j, v_j, v_{j+1}, v_{j-1}, \ldots, v_{(p-1)/2}, v_{(p-1)/2}, v_{(p-1)/2}, v_{(p-1)/2}, \ldots, v_0, v_p \). For each \( j \in \{1, 2, \ldots, p-2\} \), \( F''_0 \cup F''_j \) contains the path represented by the sequence of vertices \( v_p, v_0, v_j, v_j, v_{j+1}, v_{j-1}, \ldots, v_{(p-1)/2}, v_{(p-1)/2}, v_{(p-1)/2}, \ldots, v_0, v_p \).

As an earlier case it is easy to verify that the vertices are distinct so that the cycles are indeed Hamiltonian.

**Case 5:** A proof that for \( j, k \in \{1, 3, \ldots, p-2\} \), \( F'_j \cup F'_k \) forms a Hamiltonian cycle is given.

In the 1-factor \( F'_j \), the vertex \( v_{aj} \) is a vertex of the edge \( v_{aj}(3-1) \) if \( a \) is even and of the edge \( v_{aj}(2-1) \) if \( a \) is odd.
is odd and in the $k$-factor $F_k^\alpha$ the vertex $v_{ak}$ is a vertex of the
edge, $v_{ak} v_{(a+1)k}$ if $a$ is even and of the edge $v_{ak} v_{(a-1)k}$ if $a$
is odd. For each $j,k \in \{1,3,\ldots, p-2\}$, $F_j \cup F_k^\alpha$ contains the path
represented by the sequence of vertices $v_0, v_k, v_{k+j}, v_{2k+j}, \ldots,$
$v_{pk+(p-1)j} v_0$.

As in earlier cases it is easy to verify that the vertices
are distinct so that the cycles are indeed Hamiltonian.

Therefore, $F$ is a perfect $k$-factorization.

Another class of graphs which has been studied to determine
the existence of perfect $k$-factorizations is complete bipartite
graphs $K_{n,m}$. Note that the existence of a $k$-factorization of a
bipartite graph requires that $n = m$. The following result of
Kotzig implies that for the existence of a perfect $k$-factorization of
$K_{n,n}$, $n$ must be odd.

Theorem 2.1.9. (Kotzig [15]). If $G$ is a bipartite graph, regular
of degree greater than 2 with a perfect $k$-factorization then
$v(G) = 2 \pmod{4}$.

P.J. Laufer has proved the following result giving the
existence of perfect $k$-factorizations of complete bipartite graphs
$K_{2n-1,2n-1}$ depending on the existence of a perfect $k$-factorization
of $K_{2n}$.
Theorem 2.1.10. (P. Laufer [19]). If a perfect \( l \)-factorization of \( K_{2n} \) exists, then a perfect \( l \)-factorization of \( K_{2n-1,2n-1} \) exists.\( \Box \)
Section 2. Q-indices.

This section deals with a property of a 1-factorization $F$ of a complete graph called a Q-index of $F$.

Definition 2.2.1. Given an integer $n$, let $F = \{F_1, \ldots, F_{2n-1}\}$ be a 1-factorization of $K_{2n}$ and $Q$ be a class of regular graphs of degree 2. The Q-index of $F$, denoted $Q(F)$, is the largest integer $m$ such that there exists a partition of the 1-factors of $F$ into classes $F^{(1)}, \ldots, F^{(r)}$ with $|F^{(i)}| \geq m$ for $i = 1, 2, \ldots, r$ and if $F_i, F_j \in F^{(k)}$ then there is a graph $G \in Q$ such that $F_i \cup F_j \subseteq G$.

If $Q$ is the class of graphs which are Hamiltonian cycles, then the 1-factorization $F$ of $K_{2n}$ is a perfect 1-factorization if $Q(F) = 2n-1$.

Definition 2.2.2. If $Q$ is a class of graphs with at most one graph on $2n$ vertices, then $Q_{2n}$ is the graph on $2n$ vertices.

Theorem 2.2.1. (E. Mendelsohn and A. Rosa [20]). Let $Q$ and $Q'$ be classes of graphs of degree 2 such that for each $n$, $Q$ or $Q'$ has at most one graph on $2n$ vertices and $Q_{2n} \neq Q'_{2n}$. Then for any 1-factorization $F$ of $K_{2n}$, $Q(F) > (2n-1)/(2k+1)$ implies that $2'(F) \leq 2k-1$ for $k$ in $\{1, \ldots, n-1\}$.

Proof: Suppose $Q(F) > (2n-1)/(2k+1)$. By definition of the Q-index, there exists a partition of the 1-factors of $F$ into classes $F^{(1)}, i$ in $\{1, \ldots, r\}$, such that $F_i \cup F_j \subseteq Q_{2n}$ for $F_i, F_j$ in $F^{(i)}$. 
\[
\text{and } |F^{(i)}| \geq Q(F) > \frac{(2n-1)/(2k+1)}{2n} \text{ for each } i \text{ in } \{1,2,\ldots,r\}. \text{ Thus } r < 2k+1.
\]

To find \(Q'(F)\), look at any partition of the 1-factors of \(F\) into sets \(G^{(1)}, \ldots, G^{(s)}\), such that \(G \cup G' \neq Q'_{2n}\) for \(G, G' \in G\). Note that for any two 1-factors \(F\) and \(F'\) of \(F^{(i)}\), \(F\) and \(F'\) must belong to distinct \(G^{(j)}\)'s. For any two 1-factors \(G\) and \(G'\) of \(G^{(j)}\), \(G\) and \(G'\) must belong to distinct \(F^{(i)}\)'s. Thus
\[
|G^{(j)}| < r < 2k+1.
\]

Therefore \(Q'(F) < 2k+1. \Box\)

In [21] E. Mendelsohn and A. Rosa give two results concerning the existence of 1-factorizations with certain \(Q\)-indices where \(Q\) is a certain class of regular graphs of degree 2.

The first of these requires a result on Steiner loops.

**Definition 2.2.2.** A **Steiner loop** \(G\) with the binary operation \(\ast\) is defined by the following properties.

1. For any \(a, b\) in \(G\) the equations \(a \ast b = x\), \(a \ast x = b\) and \(x \ast a = b\) each have a unique solution.

2. There exists an element \(1\) in \(G\) such that \(a \ast 1 = a = 1 \ast a\) for every \(a\) in \(G\).

3. \(a \ast a = 1\) for all \(a\) in \(G\).

4. \(a \ast b = b \ast a\) for all \(a, b\) in \(G\).

5. \(a \ast (a \ast b) = b\) for all \(a, b\) in \(G\).
Lemma 2.2.2. (R. Bruck [6]). A Steiner loop of order \( n+1 \) exists if and only if a Steiner triple system of order \( n \) exists.

Proof: Suppose a Steiner triple system \( T \) of order \( n \) exists. If \( abc \) is any block of \( T \), let \( a \cdot b = c \), \( b \cdot c = a \) and \( a \cdot c = b \).

Since \( T \) is a Steiner triple system, any pair of elements occurs exactly once and each of \( a \cdot b = x \), \( a \cdot x = c \) and \( x \cdot b = c \) would have a unique solution for \( x \). Note that \( a \cdot b = c = b \cdot a \).

If \( a \cdot b = c \) then \( a \cdot (a \cdot b) = a \cdot c = b \) for any \( a, b, c \) in \( T \).

Add an element \( 1 \) to the set of elements of \( T \) and define \( 1 \cdot a = a \cdot 1 = a \) and \( a \cdot a = 1 \) for all \( a \) in \( T \). Thus, the elements of \( T \) with \( 1 \) and the above operation form a loop of order \( n+1 \). Therefore, there exists a Steiner loop of order \( n+1 \).

Suppose a Steiner loop \( T \) with operation \( \cdot \) and identity \( 1 \) of order \( n+1 \) exists. Let \( a, b, c \) be in \( T \) and \( a, b, c \neq 1 \). Let the block of a block design be \( a \cdot b \cdot c \) when \( a \cdot b = c \). If \( a \cdot b = c \), then \( a \cdot (a \cdot b) = a \cdot c = b \), \( c \cdot a = b \), \( b \cdot a = c \), \( b \cdot (b \cdot a) = b \cdot c = a \) and \( c \cdot b = a \). Since \( a \cdot b = x \), \( a \cdot x = c \) and \( x \cdot b = c \) each have a unique solution for \( x \), each pair of non-identity elements will occur together in exactly one block. Therefore a Steiner triple system of order \( n \) is formed.

Theorem 2.2.3. (E. Mendelsohn and A. Rosa [21]). Let \( Q \) be a class of regular graphs of degree 2, so that for each \( n \), \( Q \) contains at most one graph on \( 2n \) vertices, \( Q_{2n} \). Then for any \( n \geq 4 \) there is a 1-factorization \( F \) of \( K_{2n} \) such that \( Q(F) = 1 \).
Proof: Let \( n \geq 4 \) be fixed.

**Case 1:** Let \( Q_{2n} \) be disconnected.

Look at the bipyramidal 1-factorization of \( K_{2n} \) described in Definition 2.1.3 coming from the pyramidal 1-factorization with \( F_0 \) as described in Theorem 2.1.2. The union of \( F^* \) and \( F'_0 \) is the Hamiltonian cycle \( (u_{2n}, u_0, u_{n-1}, u_{n-2}, u_{2n-3}, u_1, u_n, u_{n-3}, u_{2n-4}) \).

\[
\begin{align*}
U_2, \ldots, U_{\left\lfloor \frac{3(n-1)}{2} \right\rfloor +1}, U_{\left\lfloor \frac{3(n-1)}{2} \right\rfloor +n-1}, U_{\left\lfloor \frac{3(n-1)}{2} \right\rfloor }, U_0, U_0). 
\end{align*}
\]

Now \( F^* \) is the same if a cyclic permutation is applied to the vertices \( u_0, u_1', \ldots, u_{2n-3} \). Thus for any 1-factor \( F'_1, F^* \cup F'_1 \) forms a Hamiltonian cycle. Therefore, since \( Q_{2n} \) is disconnected, \( Q(F') = 1 \).

**Case 2:** \( Q_{2n} \) is connected.

Let \( n \equiv 1 \) or 2 \((\text{mod } 3)\). By Lemma 2.3.2, since a Steiner triple system of order \( n' \) exists if and only if \( n' \equiv 1 \) or 3 \((\text{mod } 6)\) there is a Steiner loop \( T \) of order \( n'' \), if and only if \( n'' \equiv 2 \) or 4 \((\text{mod } 6)\). Thus there exists a Steiner loop of order \( 2n \).

Set up a 1-factorization \( F \) of \( K_{2n} \) as follows. For \( a, b, a \neq b \), in \( T \) if \( a \cdot b = c \) is in \( T \) let the edge \( a \cdot b \) be in the 1-factor \( F_c \). From property 1 of a Steiner loop each edge is in one 1-factor and each vertex is an endpoint of an edge in each 1-factor exactly once. Let \( b, c \) be in \( T \), then there is an \( a \) in \( T \) such that \( a \cdot b = c \). Thus \( 1c \) and \( a \cdot b \) are edges in 1-factor \( F_c \) and \( 1b \) and \( a \cdot c \) are edges in 1-factor \( F_b \). Thus \( F_c \cup F_b \) has a component which is a 4-cycle \((1b, a \cdot c, 1l)\) and at least one more component since \( n \geq 4 \).
Let $n \equiv 3 \pmod{6}$. Label the vertices of $K_{2n} v_1, u_1$

where $i \in S, i \neq 1$, and $S$ is a Steiner loop of order

$n + 1 \equiv 4 \pmod{6}$ with elements $\{1, 2, \ldots, n+1\}$. Form near

1-factorizations $F^v = \{F^v_1, F^v_2, \ldots, F^v_{n+1}\}$ and $F^u = \{F^u_1, \ldots, F^u_{n+1}\}$

on $K_n$ on the vertex sets $\{v_2, \ldots, v_{n+1}\}$ and $\{u_2, \ldots, u_{n+1}\}$ as

follows. For $a, b, a \neq b$ and $a \neq 1, b \neq 1$ then if $a = b = c$

let the edge $v_a v_b \in F^v_c$ and the edge $u_a u_b \in F^u_c$. From property 1

of a Steiner loop, each edge is in one near 1-factor and each vertex

is in each near 1-factor $F_i^v, F_i^u$ as an end vertex except $v_i$ or $u_i$

which stand alone. For the 1-factor $F_i^v$, take the two near

1-factorizations $F_i^v, F_i^u$ and the edge $v_i u_i$. To complete the

1-factorization form any 1-factorization of the edges between the two

sets of vertices deleting the 1-factor already used. By the same

argument as above any two 1-factors of the first type will have at

least three components, a 4-cycle in each set of vertices and at least

one other component since $2n \geq 18$. There are $n$ 1-factors of the

first type and $n-1$ of the 1-factors formed from cross edges. None

of the first can occur together. Therefore $Q(F) = 1$.

Let $n \equiv 0 \pmod{6}$. Label the vertices

$c_1, c_2, \ldots, c_n, x_1, x_2, \ldots, x_n$. Take a 1-factorization $H = \{H_1, \ldots, H_n\}$

of $K_{n,n}$ on the sets $\{c_1, c_2, \ldots, c_n\}$ and $\{x_1, x_2, \ldots, x_n\}$

corresponding to a unipotent (the element 1 down the main diagonal)

latin square $C = \{c_{ij}\}$ formed from the latin square $A$ with object

set 1 through $\frac{n}{2}$ which is unipotent and the latin square $B$ on
the object set $\frac{n}{2} + 1, \ldots, n$. Let the object set of $A$ be relabelled so that there are all 1's down the main diagonal. Label the columns of $C, c_1, c_2, \ldots, c_n$ and the rows of $C, r_1, r_2, \ldots, r_n$. If $c_{ij} = k$, then let the edge $r_i c_j$ be in 1-factor $k, H_k$. Note that for each $k \in \{2, 3, \ldots, n\}$ there is a proper subsquare containing $k$ and 1. If $k$ in $\{1, 2, \ldots, \frac{n}{2}\}$, then $k \in A$, a proper subsquare.

If $k$ in $\{\frac{n}{2} + 1, \ldots, n\}$ then there is a subsquare of order 2 containing $k$ and 1. Thus the union of $H_l$ and $H_j$ forms a disconnected graph for $j$ in $\{2, 3, \ldots, n\}$. To complete the 1-factorization $F$, take a 1-factorization on $n$ vertices and take two copies of it; one on the vertices $c_i$ and one on the vertices $r_i$ such that when one 1-factor is taken from each set to form a 1-factor of $K_{2n}, c_1 c_j$ will be in the 1-factor if and only if $r_i r_j$ is. Thus $H_1$ with any of these new 1-factors will form many 4-cycles.

Therefore $H_1$ must occur by itself in a partition used in finding the $Q$-index of $F$. Therefore $Q(F) = 1$.

Another result of E. Mendelsohn and A. Rosa deals with $Q$ being a class of regular graphs of degree two where $Q_{2n}$ is made up of 4-cycles with possibly one 6-cycle if $n$ is odd. Here the $Q$-index is called the tightness index.
Definition 2.2.3. Let $Q$ be a class of regular graphs of degree 2 such that for every integer $n \geq 3$, $Q_{2n}$ is made up of 4-cycles with possibly one 6-cycle if $n$ is odd. For any 1-factorization $F$, the $Q$-index of $F$, $Q(F)$ is called the tightness index of $F$, $TI(F)$.

Theorem 2.2.4. If $n \equiv 0 \pmod{2}$, then there is a 1-factorization $F$ of $K_{2n}$ such that $TI(F) \geq 2$.

Proof: Let $n = 2k$. Label the vertices of $K_{2k}$, $u_i$ and $v_i$ for $i = 1, 2, \ldots, k$. Let $F$ be a 1-factorization of $K_{2k}$ with $F = \{F_1, F_2, \ldots, F_{2k-1}\}$ where $F_1 = \{u_{i}v_{i}: i = 1, 2, \ldots, k\}$. Label the vertices of the $k$-partite graph $K_{4, 4, \ldots, 4}$, $u_{i}, v_{i}, u'_{i}, v'_{i}$ for $i$ in $\{1, 2, \ldots, k\}$. Construct a partitioning of $K_{4, 4, \ldots, 4}$ into 4-cycles as follows. Let the 4-cycles $(u'_{i}, u'_{m}, v'_{i}, v'_{m})$ or $(v_{i}, u'_{i}, v'_{i}, v'_{m})$ be in $G_j$, a 2-factor of $K_{4, 4, \ldots, 4}$, if and only if $u_{i}v'_{m}, u'_{i}v_{m}$ or $v_{i}v'_{m}$ is in $F_{j}$ for $j$ in $\{2, 3, \ldots, 2k-1\}$. Partition each $G_j$ into two 1-factors $G_j'$ and $G_j''$ for $j$ in $\{2, 3, \ldots, 2k-1\}$. By definition $G_j' \cup G_j''$ forms a graph whose components are 4-cycles.

The above 1-factors leave $k$ disjoint copies of $K_4$ to partition into 1-factors. Let $G_1'$ be the set of edges $v'_{i}v'_{1}$ and $u'_{i}u'_{1}$ for $i$ in $\{1, 2, \ldots, k\}$, let $G''$ be the set of edges $v'_{i}u'_{1}$ and $v''_{i}u''_{1}$ for $i$ in $\{1, 2, \ldots, k\}$ and let $G_1'''$ be the set of edges...
\( v_i^1 u_i^1 \) and \( v_i^2 u_i^1 \) for \( i \) in \( \{1, 2, \ldots, k\} \). Note that the union of any pair of these last three 1-factors forms a graph whose components are 4-cycles.

The 1-factors in \( \{G_i', G_i'', G_i''' | i = 1, 2, \ldots, 2k-1\} \) form a 1-factorization, \( F \) of \( K_{4k} \). The partition of the 1-factors \( F^{(1)} = \{G_1', G_2'', G_3'''\} \), and \( F^{(i)} = \{G_i', G_i''\} \) for \( i \) in \( \{2, 3, \ldots, 2k-1\} \)

shows that \( TI(F) \geq 2.0 \)

Another \( Q \)-index defined for a particular class of graphs is called the Dundas index.

**Definition 2.2.4.** Let \( Q \) be a class of graphs such that \( Q_{2n} \) is a Hamiltonian cycle for each \( n, n > 1 \). For any 1-factorization \( F \) the \( Q \)-index of \( F \) is called the Dundas index of \( F \) and denoted \( DI(F) \).

Note that for \( F \) a 1-factorization of \( K_{2n} \), if \( DI(F) = 2n-1 \) then \( F \) is a perfect 1-factorization.
Section 3. Kotzig Factorizations.

A Kotzig factorization contains both a near 1-factorization and a Hamiltonian decomposition.

**Definition 2.3.1.** A Hamiltonian decomposition $H = \{H_1, \ldots, H_n\}$ is a partitioning of the edge-set of a graph into Hamiltonian cycles.

**Definition 2.3.2.** A Kotzig factorization $K(H,F)$ of $K_{2n+1}$ is a Hamiltonian decomposition $H$ of $K_{2n+1}$ with 1-factorization $F$ of $K_{2n+1}$ such that each Hamiltonian cycle of $H$ intersects each near 1-factor of $F$ in exactly one edge.

A construction of E. Mendelsohn and C. Colbourn exhibits a Kotzig factorization of $K_p$ where $p$ is an odd prime. This construction is used by J. Horton in proving the existence of Kotzig factorization of $K_{2n+1}$ for all integers $n$.

**Theorem 2.3.1.** (E. Mendelsohn and C. Colbourn [20]). A Kotzig factorization of $K_{2n+1}$ exists for $2n+1$ a prime.

**Proof.** Let $V(K_{2n+1}) = \{v_0, v_1, \ldots, v_{2n}\}$, $F_i = \{v_jv_{i-j} \mid j = 0,1,\ldots,2n\}$ for $i \in \{1,2,\ldots,n\}$ and $H_i = \{v_jv_{i+j} \mid j = 0,1,\ldots,2n\}$ for $i$ in $\{1,2,\ldots,n\}$. Now $F = \{F_0,F_1,\ldots,F_{2n}\}$ is a near 1-factorization of $K_{2n+1}$ and $H = \{H_1,H_2,\ldots,H_{2n}\}$ is a Hamiltonian decomposition of $K_{2n+1}$. To prove that $K(H,F)$ is a Kotzig factorization of $K_{2n+1}$, for $\ell \in \{1,\ldots,n\}$ let $v_iv_j \in H_\ell$ and $i-j \equiv \ell \pmod{2n+1}$ where
i + j = k (mod 2n+1). Then \(v_i v_j\) is in \(F_k\). Since

\[
\begin{align*}
  j & = \left\{ \begin{array}{ll}
            \frac{\ell + k}{2} \pmod{2n+1} & \text{for } \ell + k \text{ even} \\
            \frac{(\ell + k)(2n+1)}{2} & \text{for } \ell + k \text{ odd}
          \end{array} \right.
\end{align*}
\]

each edge of \(H_t\) is in a different 1-factor.

Mendelsohn and Colbourn [20] also construct Kotzig factorizations of \(K_{2n+1}\) for \(n\) smaller than 21.

A construction of J. Horton [10] gives Kotzig factorizations of \(K_{2n+1}\) for all \(n\). In this construction strong starters are used.

Definition 2.3.3. When considering abelian groups, additive notation is used. A strong starter of an abelian group \(G\) of order \(k\) is a set \(A\) of unordered pairs of elements from \(G\) with the following properties.

(a) For \(x\) in \(G\), \(x \neq 0\), there exists \(y\) in \(G\), \(y \neq 0\), such that \(\{x, y\}\) in \(A\).

(b) If \(\{x, y\}\) and \(\{x, z\}\) are in \(A\), then \(z = y\).

(c) \((x - y)\} \{x, y\} \text{ in } A = G \setminus \{0\}.

(d) For \(\{x, y\}\) in \(A\), \((x + y) \neq 0\) and for any \(\{x', y'\}\) in \(A\), \(\{x', y'\} \neq \{x, y\}\), then \((x + y) \neq (x' + y')\).

Strong starters in \(GF[p^n]\) are known to exist ([8], [9], and [24]) where \(p\) is any odd prime and \(n\) is an integer except for \(p^n = 3, 5\) or 9. For \(p = 3\) the set \(A = \{(1, 2)\}\) is used and
for $p = 5$ the set $A = \{\{1,2\}, \{2,3\}\}$ is used. In the first case $A$ is not a strong starter since $(1+2) \equiv 0 \pmod{3}$ which does not affect the construction, but in the second case the basic construction must be altered.

**Theorem 2.3.2.** (J. Horton [10]). Suppose a Kotzig factorization of $K_{2n+1}$ exists, then a Kotzig factorization of $K_{p(2n+1)}$ exists where $p$ is an odd prime.

**Proof.** Let $K(H,F)$ be a Kotzig factorization of $K_{2n+1}$ on the vertex set $\{v_0,\ldots,v_{2n}\}$, labelled so that $H = \{v_0,v_1,v_2,\ldots,v_{2n}v_0\}$ and where $F = \{F_0,\ldots,F_{2n}\}$, with $v_m$ having degree 0 in $F_m$.

Let $K(H'',F'')$ be a Kotzig factorization of $K_p$ on the vertex set $\{u_0,\ldots,u_{p-1}\}$ described in Theorem 2.3.1. Let $A$ be a strong starter of $GF[p]$ using the set $\{0,1,2,\ldots,p-1\}$. For $p = 3$, let $A = \{\{1,2\}\}$ and for $p = 5$, let $A = \{\{1,2\}, \{2,3\}\}$. Now relabel the vertices $\{u_0,\ldots,u_{p-1}\}$ so that for $p > 5$,

$A = \{\{1,2\}, \{3,4\}, \ldots, \{p-2,p-1\}\}$. Let $H'$ be the Hamiltonian decomposition corresponding to $H''$ and $F'$ the near 1-factorization corresponding to $F''$.

Using $H$, $p$ Hamiltonian cycles will be formed for each Hamiltonian cycle of $K_{2n+1}$. The edges of one of these Hamiltonian cycles, along with the edges of the $2n+1$ edge disjoint $K_p$'s, are partitioned into $\frac{p+1}{2}$ Hamiltonian cycles. The partitioning into a
near 1-factorization uses the near 1-factorizations $F$ and $F'$. In order to ensure that a Kotzig factorization is formed, in using $F'$ the latter $\frac{n+1}{2}$ Hamiltonian cycles are taken into account.

Let $H_1$ be a Hamiltonian cycle of $K_{2n+1}$ in $H$ with edges $(v_0, v_1, v_2, \ldots, v_{2n}, v_0)$. Then for $l \in \{0,1,\ldots,p-1\}$ define

$$H_l = (v_0, v_1, v_2, \ldots, v_{2n}, v_0)$$

and for $k \in \{0,1\}$

$$H^k = (v_0, v_1, v_2, \ldots, v_{2n}, v_0)$$

Thus partition the edges corresponding to $K_{2n+1}$ into Hamiltonian cycles $H^l$, $l \in \{0,1,\ldots,p-1\}$ for each $H_1$ in $H$.

This leaves the edges interior to $2n+1$ disjoint copies of $K_p$. The edges of $H^0 = (v_0, v_1, v_2, \ldots, v_{2n}, v_0)$...
are used to connect these copies of $K_p$ and to form another $\frac{p+1}{2}$ Hamiltonian cycles.

Now delete the edges $i_kj_k$ for $k \in \{0,1,...,2n\}, \{i,j\} \in A$.

In each copy of $K_p$ this leaves $\frac{p-1}{2}$ Hamiltonian paths from $H'$ since each difference occurs in a different Hamiltonian cycle of $H''$.

Define $H_1''$ to be the Hamiltonian path on the vertices $j_1, j_2, ..., p-1$, determined by the Hamiltonian cycle $H_1''$ from $H'$.

Let $H_1''' = \{a_0a_1, b_1b_2, a_2a_3, ..., b_{2n-1}b_{2n}, a_{2n}b_0 | a_jb_j \}$ be the edge deleted from $H_1'' \cup H_0'' \cup H_1'' \cup ... \cup H_{2n-1}''$ for $i \in \{1,2,...,\frac{p-1}{2}\}$ and $H_0''' = \{b_kb_{k+1} | k \in \{1,2,...,\frac{p-1}{2}\}\}$ for $H_1''$.

Let $H_2''' = \{b_{k+1}b_k | k \in \{1,2,...,\frac{p-1}{2}\}\}$ be the edge deleted from $H_1'' \cup H_0'' \cup H_1'' \cup ... \cup H_{2n-1}''$ for $k \in \{0,1,...,2n\}$. Thus $H''' = \{H_1'', H_2''\}$ for $i \in \{0,1,...,p-1\}$.

$\{1,2,...,p-1\}$ is a partitioning of the edges into Hamiltonian cycles.

To construct a near 1-factorization of $K_p(2n+1)$ the edges not internal to the $K_p$'s are partitioned as follows:

$G_i'' = \{v_kv_m | F_i \}$ where $v_kv_m \in F_i$ for $F_i \in F$ and $\ell \in \{0,1,...,p-1\}$.

The edges internal to the copies of $K_p$ are partitioned into near 1-factors as follows: $G_i'' = \{u_iu_{i+1} | \ell \}$ for each $F_i \in F''$ and $i \in \{0,1,...,2n\}$. At this point, some care must be taken in choosing
the $G_j$ to go with $G_j$ for given $j \in \{0,1,\ldots,2n\}$ and $m \in \{0,1,\ldots,p\}$, since both $G_j$ and $G_j$ may contain an edge of $H_j$. Relabel the l-factors as $G_j = G_j$ if $|E(H_j) \cap E(G)^m| = 1$ and $E(G_j) \cap E(H_j) = \emptyset$ for all $j \in \{1,\ldots,\frac{p-1}{2}\}$ and $G_j = G_j$ if $E(H_j) \cap E(G_j) = \emptyset$ for all $j \in \{1,\ldots,\frac{p-1}{2}\}$ and $G_j$ is not defined for $m$.

Let $G_j = G_j \cup G_j$. Then $G = \{G_j \mid i \in \{0,1,\ldots,p-1\}\}$, $t \in \{0,1,\ldots,2n\}$ is a near l-factorization.

Now, $K(G,H')$ is a Kotzig factorization of $K_p(2n+1)$.
BIBLIOGRAPHY


