A MODEL INTERACTING FERMI GAS AND THE FRACTIONAL QUANTUM HALL EFFECT

by

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B.Sc., Simon Fraser University, 1985

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Title of Thesis/Project/Extended Essay
A Model Interacting Fermi Gas
And The Fractional Quantum Hall Effect

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ABSTRACT

In experimental studies of a high mobility, high density, two dimensional electron gas in a uniform magnetic field of sufficient strength to confine all electrons to the few lowest Landau levels, plateaus have been observed in the Hall resistance when the magnetic field is varied.

Plateaus at Hall resistances corresponding to an integer number of filled Landau levels are referred to as the normal quantum Hall effect. Similar plateaus when the lowest Landau level is fractionally filled are referred to as the fractional quantum Hall effect. Fractional quantum Hall plateaus have only been seen at Hall resistances corresponding to $p/q$ of a full Landau level where $p$ and $q$ are integers and $q$ is odd.

A model two dimensional Fermi gas system in the lowest Landau level is investigated in this thesis. A new Hamiltonian for the interaction of the Fermions is introduced. Systems with up to 12 particles have been simulated on a toroidal surface.

The lowest eigenvalue of the system for a particular combination of momentum, particle density and number of particles is obtained by raising the Hamiltonian operator to a high integer power so that it is an approximate projection operator onto the state with the lowest eigenvalue. The resultant operator is then applied to an arbitrary state to obtain an approximation of the lowest eigenvalue state.

The lowest eigenvalue has been calculated as the momentum, particle density, and number of particles are varied. Downward deviations in the energy per particle as the density is varied suggest the presence of downward cusps in infinite systems at or near densities corresponding to $\frac{1}{2}$ and $\frac{1}{3}$ of a full Landau level.
A downward cusp in the energy at a certain electron density is associated with the existence of a quantum Hall plateau around that density. Therefore, the present model may exhibit the fractional quantum Hall effect.

Calculations of the occupation density of single particle basis states for the ground state suggest that the charged excitations of these systems are localized quasiholes and quasiparticles for particle densities close to $\frac{1}{2}$ or $\frac{1}{3}$ of a full Landau level. These excitations appear to repel each other and to have fractional charges.
DEDICATION

To the memory of my father.
ACKNOWLEDGEMENTS

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CHAPTER 1
INTRODUCTION

1.1 Two-Dimensional Electron Gas

A high-density, high-mobility two-dimensional electron gas has been seen in the inversion layer of metal oxide semiconductor field effect transistors\(^1\) and also in GaAs–Al\(_x\)Ga\(_{1-x}\)As heterostructures.\(^2\) For these systems, it is possible to apply a uniform perpendicular magnetic field of sufficient strength that the density of electrons is less than that of a single filled Landau level. This is the extreme quantum limit, the regime in which the quantum Hall effect occurs.

For densities much less than a full Landau level, the low temperature ground state of a two dimensional electron gas is a Wigner crystal.\(^3\)

Tsui et al.\(^4\) used molecular-beam epitaxy techniques to prepare a sample consisting of layers of undoped GaAs, undoped Al\(_{0.3}\)Ga\(_{0.7}\)As, Si-doped Al\(_{0.3}\)Ga\(_{0.7}\)As, and Si-doped GaAs single crystals. They reported electron densities of up to 1.4\(\times\)10\(^{11}\) cm\(^{-2}\) and mobilities of up to 100000 cm\(^2\)/V-sec. By continuously varying the magnetic field up to 20 T they obtained electron densities down to 1/4 of a full Landau level.

In the experiment of Chang et al.\(^5\) a structure consisting of a layer of undoped GaAs, a layer of undoped AlGaAs, and a layer of Si-doped AlGaAs was used. The

two-dimensional electron gas is established at the GaAs and AlGaAs interface. They reported electron densities of up to \(2.1 \times 10^{11} \text{ cm}^{-2}\) with a mobility of \(1.5 \times 10^6 \text{ cm}^2/\text{V-sec}\). Electrical contacts were added in order to measure the Hall resistance and the longitudinal resistance. Different electron densities were obtained by varying the backgate voltage.

In the classical Hall effect, the density of carriers in a conducting medium is measured by applying a uniform current density with \(x\)-component \(j\) and a uniform magnetic field with \(z\)-component \(B\), and measuring the electric field, \(E\), in the \(y\)-direction. Throughout this thesis, equations are indicated in MKS units. If the current carriers have charge \(q\), the carrier density per unit volume, \(N\), is given by

\[
N = \frac{jB}{qE}. \quad (1.1.1)
\]

Consider a rectangular solid sample with edges parallel to the \(x\), \(y\), and \(z\) axes. The carrier density per unit area in the \(x-y\) plane is \(n\). Suppose that the current density is uniform and parallel to the \(x\)-axis with total current \(I_x\). The total voltage drops along the \(x\) and \(y\) axes are \(V_x\) and \(V_y\) respectively. Then \(n\) is equal to \(I_x B / q V_y\). The Hall resistance, \(R_H\), is defined to be \(V_y / I_x\), and is equal to \(B / n q\).

In the presence of the magnetic field, the resistance of the sample is no longer a scalar, but becomes a 2 by 2 matrix. The diagonal elements of the matrix are both \(R_L\) which is the longitudinal resistance. One off diagonal matrix element is \(R_H\) and the other is \(-R_H\). The voltage vector, \((V_xV_y)\), is obtained by applying this matrix to the current vector \((I_xI_y)\).

In a similar way, the conductance in the \(x-y\) plane is a 2 by 2 matrix. The diagonal elements are both \(\sigma_{xx}\) in an isotropic material. The off diagonal elements are \(\sigma_H\) and \(-\sigma_H\). The current vector, \((I_xI_y)\), is obtained by applying this matrix to the voltage vector \((V_xV_y)\). Thus, the conductance matrix is the inverse of the resistance matrix.
In the situation under consideration, \( I_y = 0 \), so that \( R_L \) is \( V_x/I_x \). The longitudinal conductance, \( \sigma_{xx} \), is obtained by inverting the resistance matrix:

\[
\sigma_{xx} = \frac{R_L}{(R_H^2 + R_L^2)}.
\]  

(1.1.2)

It is of note that if \( R_H \) is nonzero and \( R_L \) is zero then \( \sigma_{xx} \) is also zero.

1.2 Plateaus In The Hall Resistance

In the two dimensional electron gas at the inversion layer of a metal oxide semiconductor field effect transistor, von Klitzing et al.\(^6\) first observed that plateaus were present in the Hall resistance as the gate voltage controlling the electron density was varied. On each plateau, the reciprocal of the Hall resistance was quantized to an integer multiple of \( e^2/h \), where \( h/e^2 \) is about 25813 \( \Omega \). This characteristic resistance was relatively insensitive to sample geometry and impurity concentration.

This Hall resistance would occur if some Landau levels were completely filled and all other Landau levels were empty. The number of electron states per unit area in a single Landau level is \( B/\Phi_0 \) where \( \Phi_0 = h/e \) is the "quantum of magnetic flux." If the first \( j \) Landau levels are filled then the areal density of electrons is \( jB/\Phi_0 \). This yields a Hall resistance of \( h/je^2 \).

The density of electrons per unit area may also be expressed in terms of the "filling factor", \( \nu \), which is related to \( n \) by

\[
\nu = n\Phi_0/B.
\]  

(1.2.1)

The filling factor, \( \nu \), reflects how many Landau levels are filled. It is not restricted to integer values since a Landau level may be partially filled.

Tsui et al.\(^7\) first observed quantization of the Hall conductance at values other than integer multiples of \(e^2/h\). They observed a Hall plateau corresponding to a filling factor of one third of a Landau level. Fractional quantizations of the Hall resistance have since been observed\(^8\) at \(\nu = 2/7, 1/3, 2/5, 3/5, 2/3, 4/5, 4/3\) and \(5/3\). All of these fractions have odd denominators.

In addition, all plateaus in the Hall resistance are associated with a drop in the longitudinal conductance and resistance. In the limit of low temperature, both of these quantities apparently go to zero. The variation of conductance with temperature is consistent with thermal activation.\(^9\)

1.3 The Integer Quantum Hall Effect

A simple argument to account for the integer quantum Hall effect goes as follows: In the absence of impurities and electron-electron interactions, the energy levels of the two-dimensional electron gas are highly degenerate Landau levels. The Landau levels come in two sets, allowing for both values of the spin. In each set, the Landau levels are uniformly spaced by an energy \(\hbar \omega_c\), where \(\omega_c\) is the "cyclotron frequency." Given \(m^*\), the electron effective mass, the cyclotron frequency is given in MKS units by

\[
\omega_c = \frac{eB}{m^*}.
\]  

Suppose that the Fermi level is chosen to be any value other than one of the Landau levels. Then an integer number, \(j\), of Landau levels below the Fermi level will be completely filled and all those above the Fermi level will be empty. Therefore, the electron

density will be $j$ full Landau levels, so that the Hall conductance will be $je^2/h$.

If, however, the electron density is chosen arbitrarily, the Fermi level will almost certainly lie at one of the Landau levels, where the above argument breaks down. Because of the requirements of charge neutrality, the total number of electrons in the vicinity of the conducting layer is dependent only on the gate voltage, which is directly controlled when the experiment is done. It seems unlikely that the electron density could remain unchanged over a significant variation in the backgate voltage.

An adequate explanation\(^\text{10}\) for the integer quantum Hall effect takes into account the effects of impurities in the conducting layer. Impurities tend to broaden each Landau level into a band of conducting states surrounded by upper and lower "tails" of localized states.\(^\text{11}\) When the Fermi level is in the tail of localized states around a Landau level, it can be increased by a finite amount without changing the density of electrons in conducting (i.e. non-localized) states. In this way, plateaus can be present in the Hall resistance over certain finite ranges of electron density.

The discussion up to this point accounts only for the flatness of the Hall resistance plateau, and not for the quantization of the Hall resistance. It might well be expected that a moderate impurity potential would tend to change the number of non-localized states in a Landau level, and consequently alter the Hall resistance. Yet in experiments, the Hall resistance at each plateau is independent of moderate impurity concentrations. Laughlin's argument of gauge invariance\(^\text{12}\) shows that whenever the Fermi level lies in a gap or a region of localized states, the longitudinal conductivity is zero and the Hall conductance is quantized at a value corresponding to an integer number of filled Landau levels in the


absence of impurities. Thus, the plateaus of Hall resistance always correspond exactly to an integer number of filled Landau levels.

1.4 The Fractional Quantum Hall Effect

The fractional quantum Hall effect (FQHE) is associated with the energetic favorability of certain filling factors. Consider the Hall resistance plateau at filling factor $\frac{1}{3}$. Suppose that the ground state is translationally invariant so that it can slide freely when a field is applied. Then, when $\nu$ is precisely $\frac{1}{3}$ the Hall resistance will have the correct value.

The crucial assumption is that for values of $\nu$ slightly less than $\frac{1}{3}$, the ground state consists of an underlying $\nu = \frac{1}{3}$ state interspersed with a small density of charged "quasiholes". Impurities tend to localize the quasiholes so that only the underlying $\frac{1}{3}$ state will contribute to the Hall resistance. In this way, the Hall resistance for a range of $\nu$ from somewhat less than $\frac{1}{3}$ to exactly $\frac{1}{3}$ will be constant.

Similarly, for $\nu$ a bit more than $\frac{1}{3}$ a small density of charged "quasiparticles" will be superposed on an underlying $\nu = \frac{1}{3}$ state. This leads to a plateau in the Hall resistance on both sides of $\nu = \frac{1}{3}$.

Yoshioka, Halperin and Lee\textsuperscript{13} first proposed this picture of the fractional quantum Hall effect. However, the key ingredient was supplied by Laughlin\textsuperscript{14} who proposed translationally invariant ground states for all filling factors of the form $\nu = 1/q$ for odd positive integers $q$. In addition, he proposed a form for fractionally charged quasiholes and quasielectrons.


Haldane\textsuperscript{15} has proposed an extension of Laughlin’s wavefunctions which gives translationally-invariant ground states for all filling factors of the form $p/q$ where $q$ is odd. This accounts for all observed fractional Hall plateaus.

1.5 Laughlin States

Laughlin\textsuperscript{16} has obtained trial wavefunctions close to the ground states of a two-dimensional electron gas with filling factors $\nu = 1/q$ for positive odd integers $q$.

Consider a system of $N$ electrons. The position of the $j$th electron in the $x$-$y$ plane is specified by the complex coordinate $z_j$. For each $j$,

$$z_j = x_j - iy_j$$

(1.5.1)

where $i$ is the square root of $-1$.

The Larmour length, $a_0$, corresponds to the radius of the classical orbit of a particle in a magnetic field with angular momentum $\hbar$. Its value in MKS units is

$$a_0 = \frac{\hbar}{eB}.$$  

(1.5.2)

The magnetic field, $B = \nabla \times A$, comes from the symmetric gauge vector potential, which is

$$A = \frac{B}{2} ( -y, x, 0 ) .$$  

(1.5.3)

In this gauge, an orthogonal basis of (un-normalized) single particle wavefunctions in the lowest Landau level labelled by the non-negative integer index $\mu$ are given for a single


electron with coordinate $z$ by

$$
\phi_\mu(z) = z^\mu \, e^{-|z|^2/4a_o^2}.
$$

(1.5.4)

For any positive odd integer $q$, the wavefunction of the $N$ electron Laughlin state with filling factor $\nu = 1/q$ is

$$
\psi(z_1, \ldots, z_N) = \left[ \prod_{j<k} (z_j - z_k)^q \right] \left[ \prod_j e^{-\frac{|z_j|^2}{4a_o^2}} \right].
$$

(1.5.5)

Numerical calculations of systems of three electrons interacting with Coulomb repulsion showed that the overlap of the Laughlin states on the ground states was better than 99%.

The charged excitations of the Laughlin states are particles with charge $\pm 1/q$. In order to create a quasihole at some point in the $x$-$y$ plane, $z_0$, a Laughlin state is multiplied by the quasihole creation operator

$$
\Pi_j \left( z_j - z_0 \right).
$$

(1.5.6)

Recalling equation (1.5.4), it is apparent that the effect of multiplying a single particle state by $z$ is to scatter each $\phi_\mu$ into $\phi_{\mu+1}$. The effect of multiplying a multi-electron state by $\Pi z_j$ is to shift the occupation of single particle states so that state $\phi_0$ is empty. The result is a depression in the average density of electrons near the origin.

In a similar way, a quasiparticle creation operator may be defined which tends to increase the density of electrons near a given point.

Laughlin obtained his wavefunctions by considering multiparticle states formed from an arbitrary function $f(z)$ in the following way:

$$
\psi(z_1, \ldots, z_N) = \left[ \prod_{j<k} f(z_j - z_k) \right] \left[ \prod_j e^{-\frac{|z_j|^2}{4a_o^2}} \right].
$$

(1.5.7)
By varying \( f(z) \), Laughlin found that for \( \nu = 1/q \), \( f(z) = z^2 \) minimized the energy. In this way he obtained the wavefunctions of equation (1.5.5).

1.6 Small System Simulations

It is possible to calculate the exact ground state of a Coulomb-interacting system of \( N \) electrons in the lowest Landau level when \( N \) is sufficiently small. The results of such calculations may be extrapolated to an infinite system.

In order to avoid the edge effects associated with systems of finite numbers of particles in an infinite plane, a finite subset of the plane is taken in such a way that no edges exist. One approach is to place the particles on the surface of a sphere and have the magnetic field coming out from a monopole in the center. Another is to place the particles on the surface of a torus. In each case, the properties of the finite systems approach those of the infinite system as systems with larger and larger numbers of particles are simulated.

Haldane et al.\(^{17}\) have simulated systems of up to 8 electrons in the lowest Landau level on the surface of a sphere. The electrons interact due to Coulomb repulsion. The projection of Laughlin's state on the actual ground state is over 99%. The spectrum of neutral excitations for \( \nu = \frac{1}{3} \) shows a discernible gap. Quasiholes and quasiparticles are generated in these calculations in the following way: For some number, \( N \) of electrons, the strength of the magnetic field is chosen so that there are \( 3N+1 \) single particle states. When the \( N \) electron ground state for this configuration is calculated, the particle density is approximately \( \frac{1}{3} \) except in a localized region. The numerically obtained quasiholes and quasiparticles are similar in their charge density profiles to the corresponding excitations predicted by Laughlin.

Yoshioka has simulated systems of up to 8 electrons in the lowest Landau level on a toroidal surface. The torus is obtained by applying periodic boundary conditions to a rectangular region. The ground state is found to be a liquid-like state, and agrees very well with Laughlin's wavefunction. The energy per particle is plotted as a function of $\nu$ and there is evidence of downward cusps at $\nu = \frac{1}{3}, \frac{2}{5}$ and $\frac{1}{2}$.

1.7 Object Of Research

The object of this thesis is to investigate the properties of a gas of fermions in the lowest Landau level interacting through a model two-body interaction, and to attempt to relate the properties of this system to the fractional quantum Hall effect.

The innovation of this calculation is the use of a new two-body interaction different from the Coulomb interaction. The purpose of using an interaction other than the Coulomb interaction is to facilitate the study of larger systems.

Systems with up to 12 particles have been simulated in a toroidal geometry. The results of these calculations are extrapolated to an infinite system.

Downward cusps in the energy in the infinite system are suggested by the presence of downward deviations in the energy per particle as the filling factor is varied. The results obtained suggest cusps at filling factors of $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$. Another feature which may also be associated with a cusp is at filling factor $\frac{2}{5}$.

Downward cusps in the energy are necessary at filling factors which have a plateau in the Hall resistance.

---

Systems with filling factors close to $\frac{1}{3}$ apparently contain quasiholes and quasiparticles superposed on an underlying $\frac{1}{3}$ state. The profile of these excitations suggest that quasiholes and quasiparticles in the infinite system have a finite size. The same phenomenon is observed at filling factors close to $\frac{1}{2}$.

Localized charged excitations at filling factors close to $\frac{1}{3}$ are an important prediction of the theory of the fractional quantum Hall effect.

The spectrum of neutral excitations is also calculated for filling factors $\frac{1}{3}$ and $\frac{1}{2}$. Unfortunately, it was impossible to obtain significant results for large enough systems for any conclusions to be drawn.

The results of these calculations suggest that for systems of many particles, the present model exhibits the fractional quantum Hall effect. However, the present model also seems to predict plateaus in the Hall resistance at filling factors where they are not seen experimentally, such as $\nu = \frac{1}{2}$. 
2.1 Model Interacting Fermi Gas

In this thesis, a new model interaction for particles in the lowest Landau level is introduced. The purpose of introducing this model interaction is to create a simpler system which will exhibit behavior similar to the fractional quantum Hall effect. The advantage of this interaction is that somewhat larger systems may be simulated, thus reducing finite system effects. It is hoped that such a system may be a useful phenomenology for the fractional quantum Hall effect in the two dimensional electron gas.

In the present model, a set of fermions are constrained to move in the $x$-$y$ plane. A uniform magnetic field pierces the plane with $z$-component $B$. The innovation of this model is the two particle Hamiltonian through which the fermions interact. This Hamiltonian is described in section 2.4. All of the particles lie in the lowest Landau level, and the Hamiltonian is chosen so as not to scatter particles into higher Landau levels.

In order to extrapolate the properties of infinite systems, calculations of systems with up to 12 particles have been performed. For finite systems, it is convenient to place the particles on the surface of a torus rather than on the plane. In this way, systems with uniform density may be simulated using only finitely many particles.

2.2 Single Particle States

For finite systems, the particles are constrained to lie in a rectangular region with
dimensions $L_x$ by $L_y$. The region, $R$, consists of the set of points

$$R = \{ (x,y) \mid 0 \leq x \leq L_x, 0 \leq y \leq L_y \} . \quad (2.2.1)$$

For each $y$, the point $(0,y)$ on the left edge of $R$ is defined to be connected to the point $(L_x,y)$ on the right edge of $R$. Similarly, for each $x$, $(x,0)$ is connected to $(x,L_y)$. In this way, the region is periodic with toroidal topology.

Yoshioka et al.\(^1\) have considered such a system. They choose the vector potential in $R$ to be the Landau gauge,

$$A = (0, xB, 0) . \quad (2.2.2)$$

This vector potential is continuous except at the points $(0,y)$ and $(L_x,y)$ for each $y$. At these points $A$ is discontinuous so that the magnetic field, $B = \nabla \times A$, is not defined everywhere. As a result of this, the system may not be made periodic and quantized unless for some integer $M$,

$$L_x L_y = 2 \pi M a_o^2 . \quad (2.2.3)$$

In this case, there are $M$ orthogonal single particle states in the lowest Landau level. The single particle basis states are numbered from 1 to $M$ with the index $\mu$. The wavefunctions of the single particle basis states are

$$\phi_{\mu}(x,y) = e^{\sum_{m=-\infty}^{\infty} \frac{i y (2\pi a_o^2 \mu / L_y + mL_x)/2}{a_o^2} e^{-(2\pi \mu a_o^2 / L_y + mL_x - x)^2 / 2a_o^2} . \quad (2.2.4)$$

For systems with $L_x$ much greater than $a_o$, all terms in equation (2.2.4) except $m = 0, \pm 1$ may be neglected.

The normalized single particle basis states are denoted as $|1\rangle, |2\rangle \ldots |\mu\rangle \ldots |M\rangle$. The normalization factor for the wavefunctions in equation (2.2.4) has been omitted for clarity. However, it is the same for all the basis states.

For notational convenience in what follows, define arithmetic operators "+" and "\alpha" on the set of integers from 1 to $M$, corresponding to arithmetic modulo $M$. For integers $a$, $b$, and $c$ between 1 and $M$, $a + b = c$ if and only if $c = a + b + mM$ for some integer $m$. Also, $c \alpha b = a$ if and only if $c = a + b$.

2.3 Multi Particle States

The quantum mechanical momentum operator in the presence of a vector potential, $A$, for particles with charge $q$ is, in MKS units,

$$P = -i\hbar \nabla + qA .$$

(2.3.1)

The expectation values of the $x$ and $y$-components of momentum are both zero for any state in the lowest Landau level. Furthermore, the lowest Landau level contains no eigenstates of $x$ or $y$ momentum. Thus, it is not possible to choose a basis of momentum eigenvectors. However, the single particle basis state index $\mu$ has a physical interpretation which is very similar to mechanical momentum.

To help interpret the significance of the single particle basis state index, $\mu$, consider the matrix elements (for each integer $m$) of an external potential

$$V_m(y) = e^{2\pi iym/L_y} .$$

(2.3.2)

The matrix element $\langle \mu_1 | V_m | \mu_2 \rangle$ is zero unless $\mu_2 = \mu_1 + m$. This suggests that $\mu \hbar /L_y$ may be interpreted as being the $y$-component of momentum for the single particle state. The result of a scattering process between an external potential and a particle in the
The lowest Landau level seems paradoxical: A particle with zero average momentum absorbs momentum from the external potential and is scattered to a new state which also has zero average momentum. This happens because momentum is not conserved in a magnetic field. Nevertheless, the total amount of γ-momentum absorbed by a particle is reflected in the quantum number μ.

Now, a set of N-particle basis states is constructed for this system. In the nontrivial case, 1 ≤ N ≤ M−1. In each basis state the N particles are distributed among the M single particle basis states. Imposing Fermi statistics, there are \( \binom{N}{M} \) different basis states. Each multi particle basis state is characterized by a set of N distinct \( \{\mu_j\} \) which specify the occupied single particle basis states.

This notation \( |\mu_1\rangle |\mu_2\rangle \ldots |\mu_N\rangle \) is used to denote such a state.

Each multi particle basis state may be assigned a μ according to the sum of the \( \{\mu_j\} \) as follows

\[
\mu = \mu_1 \mp \mu_2 \mp \ldots \mp \mu_N .
\] (2.3.3)

It is helpful to interpret \( \mu \hbar /L_y \) as a "γ-momentum" of the state, even though the actual expectation value of the γ-momentum is zero. The only nonzero matrix elements of the potential (2.3.2) between multi particle basis states are between states whose μ differs by \( m \).

Now a rotation operator R is defined whose effect is to rotate a multi particle basis state by one single particle basis state. The effect of R on a multi particle basis state is

\[
R |\mu_1\rangle |\mu_2\rangle \ldots |\mu_N\rangle = |\mu_1\pm1\rangle |\mu_2\pm1\rangle \ldots |\mu_N\pm1\rangle .
\] (2.3.4)

In a similar way, define a reflection operator M which reflects each basis state around single particle basis state number M. The effect of M on a basis state is
\[ M | \mu_1 \rangle | \mu_2 \rangle \ldots | \mu_N \rangle = | M^a \mu_1 \rangle | M^a \mu_2 \rangle \ldots | M^a \mu_N \rangle. \tag{2.3.5} \]

Thus, both \( R \) and \( M \) are unitary in the space of multi particle states. Physically, \( R \) corresponds to a translation parallel to the \( x \)-axis, while \( M \) corresponds to a reflection in a line parallel to the \( y \)-axis.

The purpose of introducing these operators is to assist in labelling the multi particle basis states. Two basis states are congruent if it is true both that one state can be transformed into the other by successively applying \( R \) and \( M \) any number of times, and that the two states have the same total \( \mu \). The set of basis states may then be divided into congruence classes, such that any two basis states in a given congruence class are congruent to each other.

Let the collection of all congruence classes with a given \( \mu \) be labelled by the index \( \kappa \). Then, within each congruence class, the basis states are labelled with the parameters \( \Xi \) and \( \Lambda \). The basis states are denoted as \( | \mu; \kappa; \Xi; \Lambda \rangle \).

The parameters \( \Xi \) and \( \Lambda \) are assigned as follows. Within each congruence class, a basis state is chosen arbitrarily to be \( | \mu; \kappa; 0; 0 \rangle \). If for any \( \Xi \) from 1 to \( M \) the basis state \( R^\Xi | \mu; \kappa; 0; 0 \rangle \) is in the same congruence class, then it is labelled \( | \mu; \kappa; \Xi; 0 \rangle \). When there is ambiguity, the lowest possible \( \Xi \) is chosen. A state is labelled \( | \mu; \kappa; \Xi; 1 \rangle \) if it has not been previously labelled, and if it is equal to \( R^\Xi M | \mu; \kappa; 0; 0 \rangle \).

In general, not all combinations of \( \Xi \) and \( \Lambda \) will be used to label the members of a given congruence class, since some combinations correspond to basis states with different \( \mu \). Now, consider which combinations will be represented. Let the filling factor \( \nu = N/M \) be \( p/q \), where \( p \) and \( q \) are integers with no common factor. Then \( N/p \) is an integer. Applying \( R^\Xi \) to any basis state leaves \( \mu \) unchanged if and only if \( \Xi \) is an integer multiple of \( N/p \). Thus, for a given congruence class, the allowed values of \( \Xi \) with \( \Lambda = 0 \) are integer
multiples of \(N/p\) less than \(M\). If the congruence class has any states with \(\Lambda = 1\), then the values of \(\Xi\) for these states are evenly spaced with differences of \(N/p\), but do not necessarily start at zero. For some congruence classes, the number of values of \(\Xi\) corresponding to distinct basis states is less than \(N/p\). For these classes, the lowest possible values of \(\Xi\) are used until all states are enumerated.

Now, among the basis vectors within each congruence class, a transformation is made to a new multi particle basis. The quantum numbers \(\Xi\) and \(\Lambda\) will be replaced by new quantum numbers \(\xi\) and \(\lambda\). For a particular congruence class, if \(\Xi\) takes \(n\) different values then \(\xi\) takes on all integer multiples of \(M/n\) between 1 and \(M\). For a congruence class in which \(\Lambda\) is only zero, \(\lambda\) is only zero. For a congruence class in which \(\Lambda\) can be zero or one, \(\lambda\) is zero or one. The transformation is defined as follows:

\[
|\mu,\kappa,\xi,\lambda\rangle = (\sum_{\Xi,\Lambda})^{-0.5} \sum_{\Xi} \sum_{\Lambda} (1-2\Lambda\lambda) e^{2\pi i \Xi \xi / M} R^\Xi M^\Lambda |\mu;\kappa;0;0\rangle . \tag{2.3.6}
\]

The summations in (2.3.6) are carried out only over combinations of \(\Xi\) and \(\Lambda\) that correspond to elements of the given congruence class. The prefactor is the normalization coefficient.

Consider the matrix elements (for each integer \(m\)) of an external potential

\[
V_m(x) = e^{2\pi imx/L_x} . \tag{2.3.7}
\]

First, note the relation

\[
V_m R^\Xi = e^{2\pi im\Xi / M} R^\Xi V_m . \tag{2.3.8}
\]
Then, using (2.3.6) and (2.3.8) it is true that

\[
\langle \mu', \kappa'; \xi', \lambda' | V_m | \mu, \kappa, \xi, \lambda \rangle = \\
(\sum_{\Xi} \sum_{\Lambda} \sum_{\Xi'} \sum_{\Lambda'} \sum_{\Xi''} \sum_{\Lambda''} (1-2\Delta' \lambda')(1-2\Delta \lambda) e^{2\pi i(\Xi \xi - \Xi' \xi'')/M}) e^{2\pi i m \Xi'/M} \times \\
\times \langle \mu'; \kappa'; 0; 0 | M^\Lambda R^\Xi - E^\Xi' V_m M^\Lambda | \mu; \kappa; 0 \rangle 
\] (2.3.9)

Carrying out the summations shows that the matrix element is zero unless \( \xi' = \xi + m \). In addition, this potential also conserves \( \mu \). Therefore, it is helpful to interpret \( \xi \hbar / L_x \) as the \( x \)-component of the momentum of each basis state. Once again, this "\( x \)-momentum" has nothing to do with the mechanical momentum of the state. It only reflects the amount of momentum absorbed in scattering processes.

2.4 Model Hamiltonian

The Hamiltonian for the system under consideration includes the kinetic energy, which is the same for all states in the lowest Landau level, and a two particle interaction. The integer quantum Hall effect depends on the presence of a strong impurity potential in order to have a large number of localized states. By contrast, the essential features of the ground state that lead to the fractional quantum Hall effect may be studied in the absence of an impurity potential. Analysis of these states will then show that in the presence of impurities, the localization of the quasiholes and quasiparticles will lead to the Hall resistance plateaus.

For purposes of comparison, the Hamiltonian for the Coulomb interaction is given by Yoshioka et al.\(^2\) Their expression for the matrix elements is not in closed form.

The operators \( c_j^+ \) and \( c_j \) create and annihilate a particle in single particle basis state \( j \). These obey the usual commutation relations \( c_j^+ c_k^+ + c_k^+ c_j = 0 \), \( c_j^+ c_k + c_k^+ c_j = 0 \), and

\[ c_j^c c_k + c_k^c c_j^c = \delta_{ij}. \]

In terms of these operators, the following Hamiltonian is introduced for the system under consideration, which acts on the space of multi-particle states:

\[
H = -\sum_{\mu} c_{\mu + 1} c_{\mu + 2} c_{\mu + 3} + c_{\mu + 3} c_{\mu + 3} c_{\mu + 4} + c_{\mu + 4} c_{\mu + 3} c_{\mu + 4}.
\]

Qualitatively, this Hamiltonian has two kinds of terms. When particles are present in single particle states \( \mu + 1 \) and \( \mu + 2 \), the first group of terms scatters them into single particle states \( \mu \) and \( \mu + 3 \), and back again. The second group scatters particles in single particle states \( \mu + 1 \) and \( \mu + 3 \) into \( \mu \) and \( \mu + 4 \), and back again. Thus, an interaction between particles takes place when they are separated by 0, 1, 2 or 3 single particle states, but not for larger separations.

Several observations may immediately be made regarding the symmetries of this Hamiltonian. First of all, \( H \) commutes with the translation and reflection operators:

\[
[H, R] = [H, M] = 0.
\]

Furthermore, each term of \( H \) has nonzero matrix elements only between basis states with the same \( \mu \).

Using the commutation relations of (2.4.2), and the transformation of (2.3.6) it is true that

\[
\langle \mu', \kappa', \lambda' \rangle | H | \mu, \kappa, \lambda \rangle =
\]

\[
\sum_{\Sigma \Sigma} (\Sigma \Sigma \sum) \left( -0.5 \sum_{\Sigma} (1 - 2\Lambda)(1 - 2\Lambda') \right) e^{2\pi i(\Xi - \Xi')/M} x \]

\[
\sum_{\Sigma} \sum_{\Sigma} \sum_{\Lambda} \sum_{\Lambda} \langle \mu' ; \kappa'; 0 ; 0 | M^\Lambda R^{\Xi - \Xi'} M^\Lambda H | \mu ; \kappa; 0 ; 0 \rangle.
\]
Carrying out the summations over $\Xi$ and $\Xi'$, the matrix element is seen to be zero unless $\xi = \xi'$. Also, if $\xi = \xi' = 0$, the matrix element is zero unless $\lambda = \lambda'$. In addition, the detailed conservation of $\mu$ implies that $\mu' = \mu$.

The motivation for choosing the particular Hamiltonian of (2.4.1) was to define an interaction which yields behavior similar to the fractional quantum Hall effect while at the same time making the calculations possible for larger systems. The model Hamiltonian has far fewer nonzero matrix elements than the Coulomb Hamiltonian. This fact is exploited to allow the approximation of eigenvalues for larger systems than is possible with the Coulomb interaction.

The Hamiltonian was made to conserve $\xi$ and $\mu$ in an analogy to conservation of $x$ and $y$ mechanical momentum. The matrix elements are designed to effect an interaction only between particles whose $x$-coordinates are very close. However, this Hamiltonian is not invariant under a rotation in the $x$-$y$ plane. The interaction is apparently long ranged along the $y$-axis. For this reason, this Hamiltonian makes the system effectively one dimensional.

The matrix elements are chosen to be real to avoid complex number arithmetic when $\xi = M$. Since all the two particle matrix elements are the same, there are no continuously adjustable parameters in this model.

Since this Hamiltonian was not derived from a classical two particle potential, (e.g. the Coulomb potential) it is difficult to describe what the interaction is like apart from its definition.
2.5 **Eigenstates**

The purpose of this section is to establish certain facts about the model Hamiltonian that will make the calculations easier, or at least justify the correctness of the results.

Consider the matrix elements of the Hamiltonian relative to the basis of states \( |\mu;\kappa;\Xi;\Lambda\rangle \). In this basis, all matrix elements are either zero or negative one. Next, consider the restriction of \( H \) to the space of states with a given \( \mu \). In this space, let the eigenvectors of \( H \) be labelled \( |e_j\rangle \) where for each \( j \), \( H|e_j\rangle = e_j|e_j\rangle \). For any integer \( n \),

\[
\text{Tr}(H^n) = \sum_j e_j^n . \tag{2.5.1}
\]

Evaluating \( \text{Tr}(H^n) \) in the first basis for odd \( n \) must give a nonpositive result, since any matrix element in \( H^n \) is a product of \( n \) nonpositive numbers. Therefore, for any odd \( n \) it is true that

\[
\sum_j e_j^n \leq 0 . \tag{2.5.2}
\]

Let \( e^+ \) be the most positive eigenvalue and \( e^- \) be the most negative eigenvalue of \( H \) with the given value of \( \mu \). Then the sum of these two is nonpositive, for otherwise (2.5.2) would not hold for very large odd \( n \).

This proves that within any subspace with a given \( \mu \), the lowest eigenvalue has a magnitude not exceeded by any other eigenvalue. In addition, studies of actual systems suggest that when the system contains more than a few particles the summation of equation (2.5.2) is negative.

Now, in the space of states with some \( \mu \), consider the Hamiltonian \(-H\). All matrix elements are nonnegative in the basis \( |\mu;\kappa;\Xi;\Lambda\rangle \). Choose some \( |\mu;\kappa;\Xi;\Lambda\rangle \) with a nonzero projection on the lowest energy state of \( H \) with this \( \mu \). Then, apply \(-H\) repeatedly so as to
project out a linear combination of the lowest energy state and a state with the opposite eigenvalue. Let this state be denoted as $\Psi^+ + \Psi^-$. All the basis state coefficients of this state are nonnegative. If the lowest eigenvalue is $-\epsilon$, then applying the Hamiltonian once results in the state $\epsilon(\Psi^+ - \Psi^-)$, which has all nonpositive basis state coefficients. A linear combination of these states results in a representation of the lowest energy state in which all the coefficients of the basis states are nonnegative. Therefore, for any $\mu$, the lowest energy state of $H$ may be chosen to have all nonnegative real coefficients.

Define $|\mu;\kappa;\Xi;\Lambda\rangle$ and $|\mu;\kappa;\Xi;\Lambda\rangle$ to be "directly connected by $H$" if the matrix element of $H$ between the two states is nonzero. In addition, they are "connected by $H$" if the matrix element of some higher power of $H$ between the two is nonzero. Studies of actual systems show that in practice $H$ connects all but a few of the basis states with a given $\mu$ when the filling factor is $\frac{1}{3}$. This is probably true for filling factors between $\frac{1}{3}$ and $\frac{1}{2}$, but breaks down for filling factors like $\frac{1}{4}$. When $\nu = \frac{1}{4}$ only a fraction of the basis states are connected to any particular basis state.

Consider the lowest energy state in the subspace of states with a given $\mu$. Suppose that in this low energy state the coefficient of $|\mu;\kappa;\Xi;\Lambda\rangle$ is zero. It will be proved that all basis states connected to this basis state must also have no projection on the low energy state.

Suppose that $|\mu';\kappa';\Xi';\Lambda'\rangle$ is directly connected to $|\mu;\kappa;\Xi;\Lambda\rangle$ and has a nonzero projection on the low energy state. All of the matrix elements of $H$ are nonpositive in this basis. Furthermore, the coefficient of all basis states can be chosen to be simultaneously nonnegative. Thus, applying $H$ once more to itself would result in a nonzero coefficient of $|\mu;\kappa;\Xi;\Lambda\rangle$. This would mean that the low energy state was not an eigenstate. By contradiction, any basis state directly connected to a basis state with zero projection on the low energy state also has a zero projection on the low energy state. By induction, any basis
state connected to such a basis state must have zero projection on the low energy state.

Since in practice most basis states are connected, except if the filling factor is like $\frac{1}{4}$, this means that the lowest energy state with a given $\mu$ may be obtained by applying the Hamiltonian many times to almost any initial state $|\mu;\kappa;\Xi;\Lambda\rangle$.

Finally, since $H$ conserves $\xi$ the lowest energy state for any $\mu$ must be an eigenstate of $\xi$. Since it is possible to choose the phase of the lowest energy state so that every $|\mu;\kappa;\Xi;\Lambda\rangle$ is nonnegative, the lowest energy state for any $\mu$ must have $\xi = M$ and $\lambda = 0$.

2.6 Approximation Of Ground State

Two properties of the lowest energy states within each subspace have been studied. First, the lowest eigenvalue has been approximated. Second, the occupation probability of each single particle basis state has been calculated.

For a particular combination of $N$, $M$ and $\mu$, the ground eigenvalue (with $\xi = 0$) is estimated by choosing some basis state $|\mu;\kappa,0,0\rangle$ and then evaluating a sequence of numbers for integer $n$ given by

$$a_n = \langle \mu;\kappa,0,0 | H^n | \mu;\kappa,0,0 \rangle .$$  \hspace{1cm} (2.6.1)

If the basis state has a nonzero projection on the ground state, and assuming the ground eigenvalue has the largest magnitude, then the ground eigenvalue is given by

$$e = \lim_{n \to \infty} a_{n+1} / a_n .$$  \hspace{1cm} (2.6.2)

In practice, only the first few terms of the sequence are calculated, so that $e$ must be estimated in some way. The rate of convergence of the sequence is strongly affected by the
choice of basis state. From experience, the best basis state seems to be the one in which the particles are distributed as uniformly as possible among the single particle basis states.

The rate of convergence to the ground eigenvalue varies strongly with μ. For this reason, each sequence had to be examined separately to ensure accuracy of the results.

The lowest energy states with ξ not equal to zero cannot be calculated in exactly the same way. It is not true in general that the ground state has the largest magnitude eigenvalue. Thus, a constant must be added to the Hamiltonian to ensure that the ground state is projected out. In addition, the lowest energy states with ξ not equal zero are not in general eigenstates of λ.

In practice, the rate of convergence of the series is much slower when ξ is not zero. For this reason, the results which have been calculated are limited to small systems. The reason for the slow convergence is probably due to the fact that the basis state coefficients are complex, with different phases.

The occupation probability for single particle state k is the expectation value of the operator $c_k^*c_k$. This was calculated approximately by obtaining $H^n|\mu,\kappa,0,0\rangle$ for some large n. The expectation value of $c_k^*c_k$ was then calculated for this state.

2.7 Limiting Factors Of The Calculation

The coefficients $a_n$ were calculated by directly applying the matrix elements of the Hamiltonian relative to the basis states. The principal limitation was in the time taken to evaluate the matrix element, and not in the storage space needed to store all the intermediate results.
The procedure for the calculation was as follows. For each \( n \), \( H^{n+1} | \mu, \kappa, 0, 0 \rangle \) was calculated by applying \( H \) to \( H^n | \mu, \kappa, 0, 0 \rangle \). This made it necessary to store the coefficients of many different basis states. In principle, this limits \( n \) since storage space is limited. In practice, however, the running time became too long before storage space became a limitation.

Conceivably, the \( a_n \) could have been calculated by Monte Carlo methods. Such techniques would remove the absolute limitation on \( n \) imposed by space limitations. However, it is not clear that such methods would reduce the running time.

A disadvantage of the present method is that the number of intermediate results which must be stored at a given time are too large to store in files. Therefore, a run may not be spaced over several sessions. Monte Carlo methods would not have this limitation.

In writing the computer program to calculate these matrix elements, much effort has been made to reduce the running time. Better techniques for estimating the low energy states would be very useful.

A listing of the computer program for this calculation in FORTRAN is available from the author of this thesis.
CHAPTER 3
RESULTS

3.1 Energy Versus Filling Factor

The compressibility of a gas as a function of its volume is the ratio of the incremental decrease in volume $\delta V$, to a corresponding increase in pressure, $\delta P$. At zero temperature, $P$ is the derivative of the internal energy, $e$ with respect to changes in the volume. A gas is "incompressible" if its compressibility is zero. This corresponds to a jump discontinuity in the derivative of $e$ with respect to $V$.

Laughlin$^1$ argues that the states of equation 1.5.5 are incompressible. This is because increasing or decreasing the area of the state changes the energy with a derivative dependent on the sign of the area change. In this section, evidence that the present model has such "incompressible" ground states is presented.

Consider a system of $N$ particles among $M$ single particle states. Then, for each $N$ and $M$, let $e(N,M)$ be the lowest eigenvalue of energy. Then the ground state energy per particle is $e/N$.

In figures 1, 2 and 3, the ground state energy per particle is plotted versus the filling factor for systems with 7 to 11 particles. The curve has an overall approximately parabolic shape with a minimum around $\nu = 0.4$. This is in contrast to the same curve calculated with the Coulomb interaction,$^2$ in which case the energy increases monotonically with density. The interaction in the present model favors an average density of $\nu = \frac{2}{5}$.

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The important feature of these curves is the presence of downward deviations from the overall nearly-parabolic behavior at certain filling factors. In figure 1, noticeable deviations are present at \( \nu = \frac{1}{4}, \frac{1}{3}, \text{ and } \frac{1}{2} \). In figure 2, a weaker downward deviation is present at \( \frac{2}{5} \). An interesting possibility is that in the limit of large \( N \), when the curve is completely filled in, discontinuities in the derivative, or "cusps", will be present at these filling factors.

The apparent scatter in the data between \( \frac{1}{4} \) and \( \frac{1}{3} \) is attributed to finite system effects. A similar explanation is proposed for the scatter at \( \frac{1}{2} \), although it is not clear why this filling factor seems to have more scatter due to finite size effects than others.

It is difficult to attribute probable uncertainties to the individual points. However, all the energies shown have an estimated uncertainty of less than \( \pm 0.0005 \).
Figure 1: The ground state energy per particle is plotted as a function of the filling factor. The approximate parabolic behavior is not of primary interest. The relevant features of this curve are downward deviations that may correspond to cusps in infinite systems. These are present at $\nu = \frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$. 
Figure 2: (Detail from figure 1.) Energy per particle versus filling factor. The cusp-like behavior near $\nu = \frac{1}{3}$ may be seen more clearly. Also, the downward deviation at $\nu = \frac{2}{5}$ is suggestive of a downward cusp.
Figure 3: (Detail from figure 1.) The ground state energy per particle is plotted as a function of the filling factor. The cusp like behavior around $\nu = \frac{1}{4}$ is clearly visible.
3.2 Single Particle State Occupation

The quantum Hall effect ground state must be translationally invariant, for otherwise it will be pinned by impurities in the sample. For this reason, the extent to which the low energy states in the present model are "liquidlike" states is investigated. One expected feature of a true liquidlike state is that the particle density will be independent of position.

For any state, the occupation probability of single particle basis state $|\mu_{sp}\rangle$ is the expectation value of the operator $|\mu_{sp}\rangle\langle\mu_{sp}|$. If the state is an eigenstate of $\mu$, then the spatial density may be calculated by multiplying the spatial density due to each single particle basis state by its occupation probability and summing over all single particle basis states.

The contribution of single particle state $|\mu_{sp}\rangle$ to the density varies with $x$ as a Gaussian distribution centered at $x = 2\pi a_0^2 \mu_{sp}/L_y$. The width of the Gaussian is $a_0$. If $L_y$ is much greater than $a_0$, then the distance between the centers of the single particle states will be much less than the width of the Gaussians. Therefore, as an approximation, the occupation probability of single particle basis state $|\mu_{sp}\rangle$ may be thought of as the particle density at $x = 2\pi a_0^2 \mu_{sp}/L_y$. For conciseness, we will define the "density" to be the single particle basis state occupation probability, even though it only corresponds to an actual density of particles as an approximation.

In figures 4 and 5, the density is plotted versus $\mu_{sp}$. Data is shown in each case for different values of the $y$-momentum $\mu$. Only data for $\mu_{sp} = 1$ to 12 is shown since the data is periodic.

In figure 4, $\nu = \frac{1}{3}$ and $N = 10$. There is a modulation in the density with period 3. The strength of the modulation seems to be independent of $\mu$. 

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In figure 5, \( v = \frac{1}{2} \) and \( N = 10 \). There is a modulation in the density with period 2. The modulation is strongest for the ground state (\( \mu = 10 \)) and gradually becomes smaller until it disappears completely for \( \mu = 15 \).

In practice, the large width of the Gaussians compared to the length of the modulation will completely wipe out any modulations in the particle density as a function of position. Therefore, these modulations may not be of importance.
Figure 4: The density is plotted versus $\mu_{sp}$. This data is for a system of 10 particles with filling factor $\frac{1}{3}$. A modulation of period 3 may be seen. The dotted line shows the average occupation probability of $\frac{1}{3}$. 

Single particle basis state index, $\mu_{sp}$
Figure 5: Single particle state occupation probability is plotted versus $\mu_{sp}$ for various values of $\mu$. This data is for a system with 10 particles and filling factor $\frac{1}{2}$. A period 2 modulation is present except when $\mu = 15$. The dotted line shows the average occupation probability of $\frac{1}{2}$. 

Single particle basis state index, $\mu_{sp}$
3.3 Charged Excitations

A crucial feature of the theory of the fractional quantum Hall effect is the tendency of states near \( \nu = \frac{1}{3} \), for example, to have density \( \frac{1}{3} \) everywhere except at localized regions which have higher or lower densities. Such regions are called quasiholes or quasiparticles. In this section, we investigate ground states close to certain simple filling factors to see if such localized accumulations of charge are present.

States with \( \nu \approx \frac{1}{3} \) have a strong modulation in the density with a period of three single particle states. To remove this, the density has been averaged over each triplet of consecutive single particle states. For \( \nu \approx \frac{1}{2} \), the density has been averaged over each pair of consecutive single particle states.

In figures 6 through 12, the averaged density is shown as a function of the single particle state index. In each case, the \( \mu \) of the state is that which gives the lowest possible energy. Therefore, these are the ground states. The single particle basis state index is denoted as "\( \mu_{sp} \)" to distinguish it from total momentum. In the figures, the range of \( \mu_{sp} \) is extended beyond \( M \) so that more than one period is seen, and all parts of the curve may be seen with clarity.

In figure 6, \( N = 9 \) and \( M = 28 \) so that \( \nu = \frac{1}{3} \times 0.012 \). For \( \mu_{sp} = 11 \) to 21 the density is approximately constant and very close to \( \frac{1}{3} \). There is a single large depression, centered at \( \mu_{sp} = 2 \). We speculate that for an infinite system the density will be very close to \( \frac{1}{3} \) except close to the depression. If this is true, then the charge of the depression is \( +\frac{1}{3}e \), where \( -e \) is the charge of the particles in the system. Such a depression might tentatively be described as a localized "quasihole," in analogy to Laughlin's proposed charged excitations.\(^3\)

The following argument may be used to assign a charge to the quasiholes and quasiparticles. First, assume that the averaged density is close to $\frac{1}{3}$ everywhere except in a relatively small region surrounding the quasihole or quasiparticle. Next, subtract a density of $\frac{1}{3}$ away from everywhere, and identify the remaining density with a quasiparticle or quasihole. Finally, add up the remaining densities over all single particle states to obtain a "charge" of the quasihole or quasiparticle.

Suppose that $N/M = \frac{1}{3}$, and that the system contains $N$ particles with charge $-e$. If the system has $M+1$ single particle states then adding $+\frac{1}{3}e$ to each single particle state results in a total "quasihole charge" of $+\frac{1}{3}e$. Similarly, if the system has $M-1$ single particle states then adding $+\frac{1}{3}e$ to each single particle state results in a total "quasiparticle charge" of $-\frac{1}{3}e$. This argument extends to other filling factors.

In figure 7, $N = 8$ and $M = 23$ so that $\nu = \frac{1}{3} + 0.014$. There is a region from $\mu_{sp} = 13$ to 18 where the density is approximately constant. There is a broadened bump centered at $\mu_{sp} = 4$. We tentatively identify the bump as a localized "quasiparticle" with a charge of $-\frac{1}{3}e$.

In figure 8, $N = 8$ and $M = 26$ so that $\nu = \frac{1}{3} - 0.026$. There are two large depressions, centered at $\mu_{sp} = 1$ and 14. In this case, the density does not stay close to $\frac{1}{3}$ in the region between the depressions. However, we qualitatively interpret this system as having two quasiholes with charge $+\frac{1}{3}e$. It is of interest that this system has two quasiholes rather than one large quasihole with charge $+2/3e$. This suggests that the quasiholes repel each other.

In figure 9, $N = 8$ and $M = 22$ so that $\nu = \frac{1}{3} + 0.030$. It is no longer possible to identify anything similar to a localized charged excitation.
In figure 10, \( N = 10 \) and \( M = 21 \) so that \( \nu = \frac{1}{2} - 0.024 \). There is a single broad depression centered at \( \mu_{sp} = 19 \). The density away from the depression stays fairly close to \( \frac{1}{2} \). We qualitatively interpret the depression as a quasihole with charge \( +\frac{1}{2}e \). Because of the exact particle-hole symmetry of the Hamiltonian, the density of a system with \( N = 11 \) and \( M = 21 \) can be seen by looking at this figure upside-down. This corresponds to a quasiparticle with charge \( \frac{1}{2}e \).

In figure 11, \( N = 8 \) and \( M = 21 \) so that \( \nu = \frac{2}{5} - 0.019 \). There is a single broad depression centered at \( \mu_{sp} = 14 \). In the region from \( \mu_{sp} = 1 \) to 5, the density stays close to \( \frac{2}{5} \). We interpret this depression as a quasihole with charge \( +\frac{2}{5}e \).

In figure 12, \( N = 8 \) and \( M = 19 \) so that \( \nu = \frac{2}{5} + 0.021 \). There is a broad bump centered between \( \mu_{sp} = 5 \) and 6. The density is close to \( \frac{2}{5} \) in the region between 12 and 18. We interpret the bump as a quasiparticle with charge \( -\frac{2}{5}e \).
Figure 6: The density, averaged over each triplet of three consecutive single particle states, is plotted versus the single particle state index. The results shown are for a system of 9 particles distributed among 28 single particle states. More than one period is shown for clarity. The dotted line shows the average density of $\frac{1}{3}$. 

Single particle basis state index, $\mu_{sp}$
Figure 7: The single particle state occupation probability, averaged over each triplet of three consecutive single particle states, is plotted versus the single particle state index. The results shown are for a system of 8 particles distributed among 23 single particle states. More than one period is shown for clarity. The dotted line shows the average density of \( \frac{1}{3} \).
Figure 8: The single particle state occupation probability, averaged over each triplet of three consecutive single particle states, is plotted versus the single particle state index. The results shown are for a system of 8 particles distributed among 26 single particle states. More than one period is shown for clarity. The dotted line shows the average density of $\frac{1}{3}$. 
Figure 9: The single particle state occupation probability, averaged over each triplet of three consecutive single particle states, is plotted versus the single particle state index. The results shown are for a system of 8 particles distributed among 22 single particle states. The dotted line shows the average density of $\frac{1}{3}$.
Figure 10: The single particle state occupation probability, averaged over each pair of two consecutive single particle states, is plotted versus the single particle state index. The results shown are for a system of 10 particles distributed among 21 single particle states. More than one period is shown for clarity. The dotted line shows the average density of $\frac{1}{2}$. 
Figure 11: The single particle state occupation probability, averaged over each quintuplet of 5 consecutive states, is plotted versus $\mu_{sp}$. This data corresponds to a system with 8 particles among 21 single particle states with $\mu = 7$. The dotted line shows the average density of $\frac{2}{5}$. 

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Figure 12: The single particle basis state occupation probability, averaged over each quintuplet of 5 consecutive single particle states, is plotted versus $\mu_{\text{sp}}$. This data corresponds to a system of 8 particles among 19 single particle states with $\mu = 6$. The dotted line shows the average density of $\frac{2}{5}$. 
3.4 Neutral Excitations

The occurrence of the fractional quantum Hall effect has been associated with the presence of a gap in the spectrum of neutral excitations. In this section, we look at the spectrum of the neutral excitations of certain ground states.

For various filling factors, the lowest eigenvalue in the subspace has been calculated as the $x$-momentum and $y$-momentum are varied. In all cases we plot the difference between the energy of the state and the energy of the ground state for the particular $N$ and $M$. This energy difference is called the "excitation energy."

In figures 13 and 14, the excitation energy is shown as a function of $\mu$ while the $x$-momentum is kept fixed at zero. In order to present data for different values of $N$ on the same graph, all data is plotted versus $\mu_{\text{rel}}$. For each $N$, $\mu_{\text{rel}}$ differs from $\mu$ by a constant such that the ground state occurs when $\mu_{\text{rel}}$ is zero.

Figure 13 corresponds to systems of 6, 9 and 12 particles with $\nu = \frac{1}{3}$. For each system, the excitation energy is periodic in $\mu$ with a period of $N$. Furthermore, each system has zero excitation energy at $\mu_{\text{rel}} = 0$.

Figure 14 corresponds to a system of 10 particles with $\nu = \frac{1}{2}$. The energy is periodic in $\mu$ with period $N$, and each system has zero energy when $\mu_{\text{rel}} = 0$.

In figures 15 and 16, the excitation energy is shown as $\xi$ is varied. The quantity $\frac{\xi \hbar}{L_x}$ is the $x$-momentum of each state. We assume that the size of the system is varied by changing $L_x$ only. In this case, at constant $\nu$ the $x$-momentum is proportional to $\xi / N$. The $x$-axis shows the quantity $2\xi / N$. In each case, $\mu$ is fixed at the same value as that of the ground state. For these figures, the uncertainties of the energies are less than $\pm 0.005$.

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For each $N$, the excitation energy is zero when the $x$-momentum is zero.

In figure 15, $\nu = \frac{1}{3}$. Data is shown for systems with 2 to 5 particles.

In figure 16, $\nu = \frac{1}{2}$. Data is shown for systems of 4 to 10 particles. The behavior near zero $x$-momentum suggests a linear dependence of the excitation energy on $x$-momentum at the origin.

The limitation on system size prevents significant conclusions from being drawn. However, there is no evidence of a gap for this system, and some of the data suggests a linear dependence of energy on momentum.
Figure 13: The energy of neutral excited states minus the ground state energy is plotted versus $\mu_{\text{rel}}$ for systems with $N = 6, 8, 9$ and 12 particles. In each case the filling factor is $\frac{1}{3}$. The curve is symmetric about the point $\mu_{\text{rel}} = 0$. 
Figure 14: The energy of neutral excited states minus the ground state energy is plotted versus the \( \gamma \)-momentum for systems of \( N = 8, 9, \) and 10 particles. In each case the filling factor is \( \frac{1}{2} \). Each curve is symmetric about the points \( \mu_{\text{rel}} = 0 \).
Figure 15: Excitation energy is plotted versus x-momentum when $\nu = \frac{1}{3}$. The $y$-momentum is kept fixed at the ground state value. The curve is symmetric around the points 0 and 1 on the x-axis, and all systems have zero energy at zero x-momentum by definition.
Figure 16: Excitation energy is plotted versus the $x$-momentum when $\nu = \frac{1}{2}$. All curves have zero energy at zero $x$-momentum. The $y$-momentum is kept fixed at the ground state value. The curve is symmetric about the points 0 and 1 on the $x$-axis.
CHAPTER 4
DISCUSSION

4.1 Summary Of The Present Model

A system of \( N \) particles in a rectangular region with toroidal periodic boundary conditions, subject to a uniform perpendicular magnetic field \( B \) has been considered. The magnetic field strength is chosen so that the system has exactly \( M \) single particles states in the lowest Landau level, and the particles are constrained to lie among these states.

The innovation of this model is to apply a new two particle interaction among the particles. This two particle interaction is the sole term in the Hamiltonian, except for the kinetic energy which is a constant in the lowest Landau level. The interaction is not obtained from a classical two body potential but is defined in terms of a small set of two particle matrix elements. Therefore, the interaction may have no classical analogy.

This Hamiltonian is block diagonalized in terms of two momentumlike quantum numbers. Then, the lowest energy state in each block is found by iteration of the Hamiltonian. By raising the Hamiltonian to high integer powers, an approximate projection operator on the lowest energy state is obtained. Application of this operator to an initial state yields an approximation of the lowest energy state.

Two properties of the low energy states have been investigated. First, the eigenvalue has been obtained by reapplying the Hamiltonian. Second, the occupation probability of each single particle state has been found.
4.2 Evidence Of Cusps In The Energy

In figures 1, 2 and 3, the variation of energy with filling factor (i.e. particle density) is shown for the present model. For infinite $N$, an energy would be defined for each $\nu$, resulting in a continuous curve. Discontinuities in the first derivative of this curve are associated with Hall resistance plateaus. Being restricted to finite $N$, it is possible only to speculate as to the nature of the continuous curve. However, there seem to be structures in the data which are suggestive of downward cusps.

Previous calculations\(^1\) of the variation of energy with density for the Coulomb interaction suggest a cusp at $\nu = \frac{1}{3}$, but are limited to a maximum $N$ of 6 or 7 for the significant parts of the data. The results of the present calculations seem to give clearer indications of possible cusps, since increasing $N$ means that the points on the curve are closer together. However, these results cannot be directly compared since the interactions are different.

In addition to evidence of a cusp at $\nu = \frac{1}{3}$, the results of the present calculations show comparable evidence of cusps at $\nu = \frac{1}{4}$ and $\frac{1}{2}$. The results of previous calculations also show evidence of a downward deviation at $\nu = \frac{1}{2}$, but the data is also consistent with a smooth curve.

In the theory of the fractional quantum Hall effect, cusps in the energy are associated with plateaus in the Hall resistance. Experimentally, plateaus have only been seen at fractions $\nu = p/q$ with odd $q$. Therefore, the present model seems to predict some Hall resistance plateaus at filling factors where they are not seen in experiments.

Finally, there is some evidence for weaker cusps at $\nu = \frac{2}{5}$ and possibly at $\frac{2}{7}$ and $\frac{3}{7}$. Weaker plateaus in the Hall resistance have been seen experimentally at these filling factors.

4.3 Uniform Density Of Ground States

In figures 4 and 5, the occupation probability is plotted for each single particle state. For the ground states at both $\nu = \frac{1}{3}$ and $\frac{1}{2}$ there is a strong modulation of the occupation probability, with periods of three and two single particle states, respectively.

For the $\frac{1}{3}$ ground state, the distance between the maxima in the occupation probability is three times the distance between states, or $6\pi a_o^2/L_y$. The particle density of each single particle state has a Gaussian $x$-dependence with a width of approximately $a_o$. Therefore, for a sample such that $L_y$ is much greater than $a_o$ there will be almost no modulation in the particle density as a function of position.

The modulation of the occupation probability of the single particle states does not lead to modulations in the charge density which would tend to pin the state to impurities. Thus, impurity pinning should not keep these states from sliding when an electric field is applied parallel to the $y$-axis.

4.4 Evidence Of Quasiholes And Quasiparticles

Figures 6 through 12 show the averaged occupation probability plotted for each single particle state. Figure 6 is the best example of a localized charged excitation. One extra single particle state is added to a system with $\nu = \frac{1}{3}$ and the particles redistribute to the new ground state. Rather than uniformly decreasing the density, the density stays close to $\frac{1}{3}$ except in a small region.

The tendency of a system with $\nu$ close to $\frac{1}{3}$ to make up for the deviation from $\frac{1}{3}$ density with a distribution of quasiparticles or quasiholes can be shown to lead to a cusp in the energy versus density curve. This, in turn, leads to quantized Hall resistance plateaus.
The argument proceeds as follows: Consider the ground state with $\nu = \frac{1}{3}$. Then consider the ground state of this system after one more single particle state is added. This corresponds to "adding a quasihole" to the system. The energy of the system is now equal to the energy for a $\frac{1}{3}$ state plus a constant "quasihole energy," which is independent of volume for a sufficiently large system. Assume that if several quasiholes are added to the system, then in the ground state they will be well separated. In this case, the energy of the system should vary linearly with the number of quasiholes added until the density of quasiholes becomes so large that they are no longer far apart.

In a similar way, consider "adding a quasiparticle" to the system by taking away one single particle state. The difference from the energy of the $\frac{1}{3}$ state will equal a "quasiparticle energy" times the number of quasiparticles that have been added, until so many have been added that they are no longer well separated.

If the quasiparticle energy is different from the quasihole energy then there will be a discontinuity in the derivative of energy versus density.

The plateaus in the Hall resistance occur in a similar way. Consider the $\frac{1}{3}$ state, and apply an electric field parallel to the $y$-axis in order to induce a Hall current. Suppose now that one more single particle state is added so that a quasihole appears. If the quasihole moves freely along with the state then the Hall resistance will change. However, if the quasihole is pinned by an impurity so that it cannot move, then the state will carry the same total current as if the quasihole were not present. The resistance will be unchanged. Adding more quasiholes will similarly have no effect until the density of quasiholes grows large. In this way, there will be a plateau in the Hall resistance over a range of filling factors.

Since figure 6 corresponds to a finite system, the quasihole cannot be said to be localized in space. After all, the oscillations extend throughout the sample. However, there is
an approximate localization which suggests true localization in an infinite system. In the remaining figures, the evidence of localization is weaker.

Evidence of localized quasiholes and or quasiparticles is seen for \( \nu = \frac{1}{3}, \frac{1}{2}, \) and \( \frac{2}{5} \). The fact that quasiholes and quasiparticles are present for \( \nu = \frac{1}{2} \) suggests that a Hall resistance plateau is present at this filling factor. The apparent cusps in the energy versus density curve may be related to these quasiparticles and quasiholes, in the way suggested in the previous argument. However, experiments do not show a Hall resistance plateau at \( \nu = \frac{1}{2} \).

4.5 Nature Of Neutral Excitations

Figures 13 through 16 show the dispersion curves for the lowest excited states of the system as a function of momentum. These states are not labelled according to their own mechanical momentum, but according to the momentum necessary to scatter into them from the ground state.

It is not clear from any of the figures what the likely asymptotic behaviour as \( N \) goes to infinity will be. However, it seems unlikely from this data that there is a large energy gap of the type calculated for the Coulomb interacting system.\(^2\)

4.6 Relationship To The Fractional Quantum Hall Effect

The properties of the present model suggest that in the limit of infinite \( N \) it will exhibit behaviour similar to the fractional quantum Hall effect. The apparent cusps in the energy versus density and the apparent localized quasiholes and quasiparticles both relate this model to the current picture of the fractional quantum Hall effect in real systems.

A significant difference seems to lie in the apparent prediction of Hall resistance plateaus at filling factors of $\frac{1}{2}$ and $\frac{1}{4}$. Such plateaus have not been seen experimentally. However, the experimentally seen plateaus are predicted as well.

A further difference is that in the ground states proposed by Laughlin$^3$ all the single particle basis states have the same occupation probability. In the present model, there is a large modulation of the occupation probability of the single particle states. There does not seem to be any way to determine experimentally if such a modulation exists. Therefore, this characteristic of the present model may not be an important difference.

4.7 New Results Of This Thesis

In this thesis, a model system of particles in the lowest Landau level has been presented which has behaviour suggestive of the fractional quantum Hall effect. This in itself is not new, since previous calculations of small systems have shown evidence of cusp in the energy$^4$ and have clearly shown a gap in the spectrum of neutral excitations.$^5$ However, several new results have been found.

First, clearer evidence of a cusp in the energy at $\nu = \frac{1}{3}$ has been seen, as well as evidence of cusps at $\frac{1}{4}$, $\frac{2}{5}$ and $\frac{1}{2}$. This improvement is because of the possibility of simulating larger systems in the present model.

Secondly, evidence of localized charged excitations in systems with filling factors close to $\nu = \frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{5}$ has been found. This was possible because systems with larger numbers of particles could be simulated.


Thirdly, calculations of a system with two quasiholes near $\nu = \frac{1}{3}$ suggest that the quasiholes repel each other.

The apparent prediction of Hall plateaus at $\nu = \frac{1}{2}$ and $\frac{1}{4}$ suggest that the present model has behaviour which is qualitatively different from the experimental system. Some ingredient may be missing from this model which would suppress the plateaus at filling factors with odd denominators.

In conclusion, the present model has behaviour which is similar enough to the fractional quantum Hall effect to make it an interesting phenomenological model. Models like it may be worthy of further investigation.

If systems with even larger numbers of particles could be simulated in the present model, additional information could have been obtained. First, the chemical potential gap associated with each plateau could be measured and compared. The variation of the gap size for different fractions might be compared with experimental data. Second, some conclusions about the asymptotic behaviour of the neutral excitations with large momentum might be drawn. Third, the question of the localization of the quasiholes and quasiparticles might be clarified for a larger variety of filling factors.


