ANTIFERROMAGNETISM, CONFINEMENT AND SPIN RESPONSE IN THE QED$_3$ EFFECTIVE THEORY OF HIGH-TEMPERATURE SUPERCONDUCTIVITY

by

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Abstract

In this thesis, we study the effective theory of a phase-fluctuating d-wave superconductor at zero temperature, formulated by quantum electrodynamics in three space-time dimensions (QED$_3$). This theory describes the quantum critical behaviour in underdoped high-temperature superconductors in terms of an emergent gauge field. The gauge field couples minimally to nodal spin degrees of freedom (spinons) at low energies. It is massive in the superconductor but exhibits Maxwell dynamics when superconductivity is destroyed by strong phase fluctuations of the Cooper pairs.

We show that, when dynamical chiral symmetry breaking in QED$_3$ is supplemented by residual interactions, namely, the velocity anisotropy around the nodes, short-range repulsion between electrons, and nonlinear effects of dispersion (all irrelevant for the critical behaviour itself), the loss of superconductivity gives rise to an antiferromagnetic state, in accord with observation.

Then, we turn to the problem of confinement of spinons outside the superconducting phase. We assume that the gauge group is a compact U(1) and, thus, allows for monopole configurations. In the absence of fermions, the interaction between monopoles is Coulombic, monopoles form a free plasma, and static fermionic charge is confined for all values of the gauge coupling by a linear potential mediated by free monopoles. We show that this permanent confinement survives in the presence of dynamical fermionic matter.

This work comprises three separate studies. We first support our claim, for relativistic fermions, by an electrostatic study of the monopole gas. This is backed up by a controlled renormalization group analysis on the equivalent sine-Gordon theory. In the
second study, we extend these findings to the non-relativistic case, with a spinon Fermi surface.

In the last study, we provide a variational approach to the problem, in agreement with our other works. Finally, we focus our attention on the more practical application of the QED$_3$ theory to spin response in the superconductor, relevant for neutron scattering measurements. We show that the theory explains the observed spin gap numerically and the evolution of the response in energy and momenta qualitatively. We study the issue of resonance in these measurements by developing a formalism for exciton bound states.

**Keywords:** High-temperature superconductivity; Antiferromagnetism; Spinons; Spin response; Three-dimensional quantum electrodynamics; Chiral symmetry breaking; Confinement; Duality transformation; renormalization group; Variational methods;
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My Mother

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My Best Friend who is also My Wife
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Chapter 1

Introduction and Basic Philosophy

The discovery of superconductivity in the barium-doped cuprate ceramic La$_2$CuO$_4$ by Bednorz and Müller [1] in 1986 was awarded the 1987 Nobel prize in physics. That such a quick acknowledgment of the importance of their achievement was well deserved is witnessed by the huge field of “high-$T_c$” that their breakthrough has spawned. The critical temperature ($T_c$) of about 30 K found by Bednorz and Müller was pushed up rapidly to around 90 K in less than a year [2], a dramatic increase compared to the highest record of 23 K in 1985.\(^1\)

The high critical temperature was an obvious reason to be enthused by the new superconductors with obvious potential for applications, such as superconducting wires, strong magnets, magnetic levitation, and more, some of which have already been realized. However, from a theoretical perspective, there is a lot more to be excited about than just the applications of cuprates. The occurrence of superconductivity in cuprates is an immediate challenge for the theoretical condensed-matter physicist. Undoped cuprates are strong Mott insulators: even though their highest band is half-filled, the motion of electrons is prohibited by strong repulsive Coulomb interactions. This means that we are faced with a strongly-correlated electronic system. The difficulty of understanding the complex and often puzzling behaviour of such systems can be appreciated by noting the number of competing theories (and the confusion) in the field twenty years since the original discovery. It is hoped that the work presented in this thesis is a step towards a better (and less confused) understanding of the problem of superconductivity in cuprates.

\(^1\)The world record at ambient pressure is currently 138 K, held by the thallium-doped, mercury-based cuprate, (Hg$_{0.6}$Tl$_{0.2}$)Ba$_2$Ca$_2$Cu$_3$O$_{8.33}$ [3].
CHAPTER 1. INTRODUCTION AND BASIC PHILOSOPHY

1.1 Superconductivity in cuprates

As noted above, the band structure of cuprates predicts a half-filled metal for the undoped ("parent") compounds. However, strong Coulomb repulsions inhibit the movement of electrons. Moreover, the parent compounds exhibit antiferromagnetism (AF) below a Néel temperature, $T_N \approx 300 \, K$.

The crystal structures of the parent compound of two families of cuprates are shown in Fig. 1.1. A universal crystalline feature in all cuprate families is the existence of nearly square-lattice copper-oxide planes, highlighted in Fig. 1.1. The number of planes per unit cell in the cuprate families varies from one up to seven. Superconductivity sets in when the parent compound is doped by holes. This is done by substituting some of the out-of-plane atoms with acceptor atoms, for instance, lanthanum with strontium ($La_{2-x}Sr_xCuO_4$), changing the oxygen concentration ($YBa_2Cu_{3-x}O_{7-\delta}$), rearranging oxygen chains, applying pressure, etc. In all these cases, it is believed that what is being tuned is the concentration of holes per site, $x$, in the copper-oxide planes. Superconductivity is achieved at a rather low critical value of $x_c \approx 0.05$.

As doping is increased, the critical temperature first rises steadily in the "underdoped" region of the phase diagram, Fig. 1.2. At around $x_o \approx 0.2$, called "optimal doping," the critical temperature reaches a maximum and then turns down in the "overdoped" region to finally vanish when the sample is about 35% doped with holes.

The overdoped transition seems to be well described by the conventional Bardeen-Cooper-Schrieffer (BCS) theory that explains the opening of the superconducting gap upon cooling, assuming the existence of a pairing potential. The underdoped part of the phase diagram is still a source of great puzzlement.

It was suggested early on (within a year after the original discovery) by Anderson [4] that an appropriate model that captured both the insulating antiferromagnetic state and superconductivity was the two-dimensional Hubbard model [5] on a square lattice,

$$H_{\text{Hubbard}} = - \sum_{ij \sigma} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}.$$

Here, $t_{ij}$ is the hopping amplitude between lattice sites $i$ and $j$, and $U$ is an on-site repulsive interaction. $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$ is the number of electrons with spin $\sigma$ at site $i$.

This proposal bears on the assumption that superconductivity occurs mainly in the
copper-oxide planes. There is very good evidence from, among others, penetration depth measurements \([6, 7]\), in support of this assumption in the underdoped and optimally doped regions. For instance, the ratio of out-of-plane and in-plane penetration depths is as high as 100 in YBa\(_2\)Cu\(_3\)O\(_{7-\delta}\) \([7,8]\).

The nearly half-filled Hubbard model with nearest-neighbour hopping \(t\) and a large \(U \gg t\) reduces \([9, 10]\) to the \(t\)-\(J\) model

\[
H_{tJ} = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + J \sum_{\langle ij \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right).
\]

The spin operators \(S_{ij} = \frac{1}{2} \sum_{\alpha \beta} c_{i\alpha}^\dagger (\sigma_\alpha)_{\alpha \beta} c_{j\beta}\), where \(\{\sigma_\alpha\}\) are Pauli matrices. The total number operator \(n_i = \sum_\sigma n_{i\sigma}\). The perturbative coupling \(J = t^2/U\) provides an antiferromagnetic exchange interaction and, in order to enforce the \(U \gg t\) in the unperturbed Hamiltonian, we must forbid double occupancy of lattice sites. This is usually done by projecting onto the singly occupied subspace using, say, the Gutzwiller projection \([11]\), \(\prod_i (1 - n_{1i} n_{i\uparrow})\).

Thus, at half filling the strongly-coupled Hubbard model with no double-occupancy is transformed to the Heisenberg model and supports an antiferromagnetic ground state. On the other hand, we may write the \(J\)-term as

\[
J \sum_{\langle ij \rangle} b_{ij}^\dagger b_{ij},
\]

where

\[
b_{ij}^\dagger = \frac{1}{\sqrt{2}} \left( c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger \right)
\]

creates a singlet on a pair of neighbouring sites. This form reveals a set of already formed singlet pairs, which, of course, cannot move due to large repulsive interaction \(U\). Furthermore, in Fourier space, the pairing may be decomposed into \(d\)-wave and (extended) \(s\)-wave components \([12]\). Mean-field theory shows that there is a \(d\)-wave superconducting (dSC) ground state for any small doping \([9]\). Other extensive numerical studies seem to validate this conclusion \([13]\). It is not quite clear how one could explain the appearance of superconductivity at finite doping within this approach. Potential candidates range from the usual suspect, i.e., disorder, to competition with the AF order \([13]\).

\(^2\)Pictures created by Prof. Mona Berciu of University of British Columbia, used here with permission.
Other possible ground states of the $t$-$J$ model, such as the resonating valence band (RVB) state proposed by Anderson [4], have been considered as well. Field-theoretical studies of the $t$-$J$ model based on the idea of a RVB state result in descriptions of the system in terms of gauge field theories [14, 15, 16].

Superconductivity in cuprates doped with electrons instead of holes is also an interesting and ongoing subject of research. Though the phase diagram for the electron-doped materials looks somewhat similar to that of the hole-doped ones, there is no particle-hole symmetry. The AF phase is much wider and the dSC phase exists in only a small interval of doping and with lower critical temperatures. In this thesis, we only consider hole-doped cuprates.
1.2 \textit{d-Wave symmetry}

The symmetry of the superconducting gap function was determined experimentally some years after the original discovery of superconductivity in cuprates [6]. Today it has become one of the best established empirical facts [17, 18, 19] about the phase diagram of cuprates besides the antiferromagnetic order at half filling.

The \textit{d}-wave symmetry has very important implications for any theoretical description of cuprates. The \textit{d}-wave gap has nodes on the Fermi surface where the gap function vanishes and, thus, the \textit{dSC} phase has excitations at arbitrarily low energies, in complete contrast to the conventional \textit{s}-wave case in the BCS theory. Accordingly there is a linear density of states around the nodes.

Thus, at low energies there are "nodal quasiparticles" that must be part of any viable theory of cuprates.
1.3 Phase fluctuations

Fluctuations in the phase of the gap function also seem to play an important role in the phase diagram of cuprates [20, 21]. The reason phase fluctuations are significant for cuprates is simple: all families of cuprates are strongly type-II superconductors, with a ratio of penetration depth to coherence length \( \kappa = \lambda_L/\xi_0 \) of about 100 [22, 23, 24]. Emery and Kivelson [25] also argued that phase fluctuations must play an important role in cuprates. Although they themselves apparently abandoned this route, many others [26, 27, 28, 29, 12, 30, 31, 32] have investigated this idea with very important results.

The most important such fluctuations are vortices in the phase configuration of the gap function, where the phase change around a closed loop does not vanish but, instead, equals an integer multiple of \( 2\pi \). Vortices can exist even at \( T = 0 \) due to quantum phase fluctuations. The regular, "spin-wave" fluctuations of the phase cannot disorder the phase enough to destroy superconductivity [33].

The existence of vortices has also been used to explain some aspects of the nonsuperconducting phase above \( T_c \), the so-called "pseudogap" phase, labeled PG in Fig. 1.2. We will now briefly discuss this phase.

1.4 Pseudogap phase

In the phase diagram of Fig. 1.2 we have shown with the dashed line an energy scale, \( T^* \), that seems to separate a mysterious pseudogap region from the rest of the diagram at higher doping. The pseudogap phase is characterized by some unusual responses of the system: a marked reduction in uniform spin susceptibility below \( T^* \) and opening of a partial gap on the Fermi surface in the antinodal directions; however, the frequency-dependent conductivity behaves like that of a metal [34].

Although we will not directly study the pseudogap phase in this thesis (except for the problem of confinement), the ideas we use as the basis of our study have direct application for the understanding of the pseudogap phase [35, 36]. In particular, the relevance of vortices for the pseudogap phenomenology is demonstrated by a series of Nernst effect measurements in and around the pseudogap phase by Ong and colleagues [37, 38, 39, 24, 21].

The Nernst effect measurements detect the transverse voltage across a sample in a tem-
temperature gradient with open circuit boundary conditions. Since the electric current must vanish in an open circuit, the Nernst signal is very small in ordinary metals, where heat is transferred largely by similarly charged excitations. In type-II superconductors like cuprates, however, the Nernst signal is large, since vortex excitations can transfer heat and no charge.

In the pseudogap phase of the cuprates, Ong's measurements detect a giant signal. Similar to the superconducting phase, the current understanding of the Nernst signal relies on the existence of vortices in the pseudogap phase [40, 41].

1.5 Basic philosophy of this thesis

Our discussion of the phase diagram of cuprates in the underdoped region focuses on three fundamental features: (i) quasi-two-dimensionality, (ii) $d$-wave symmetry of the gap function, and (iii) the essential role of vortices.

In this thesis, we take the $d$-wave nature of the superconducting gap as one of our central working assumptions. We make no attempt at deriving this symmetry from other assumptions in contrast to the conventional approaches to the problem, as in the theories starting from the Hubbard or the $t$-$J$ model. Instead, we will derive the consequences of having a $d$-wave gap in the superconducting state. They include, quite amazingly, the AF order as an instability of the $d$SC due to phase-disordering effects of vortices in the underdoped region.

Whether the transition out of $d$SC is immediately into the AF phase or whether there is an intermediate passage through another phase (presumably the $T = 0$ cousin of the pseudogap phase) is a matter of debate. The simplest scenario within the QED$_3$ theory is the immediate, albeit very weak, formation of the AF order. This is shown in Fig. 1.2 by the dashed tail of the AF phase boundary. However, this picture may be changed by the effects of disorder and other interactions. It should be stressed, however, that in our theory there is no coexistence of $d$SC and AF.

In our interpretation, $T^*$ is the energy scale for the onset of pairing without the necessary phase coherence for superconductivity. Pairing is associated with the (maximum) amplitude of the gap function, $\Delta_0$. We take $\Delta_0$ and $T^*$ to have the same physical significance. The gap extracted from heat transport experiments [42] supports this claim. Cooling further down,
phase coherence sets in at $T_c$ and the $d$-wave superconductor is formed.

However, instead of starting from the non-superconducting phase, we will trace the transition from the opposite end. That is, we start from a $d$-wave superconductor, introduce phase fluctuations, and study the fate of such a phase-fluctuating $d$-wave superconductor. Throughout this work, $T = 0$. The advantages of this approach are hard to dismiss: the $dSC$ phase is by far the best understood among the underdoped states, both experimentally and theoretically; and the relevance of phase fluctuations is supported both by experiment and theoretical consideration. It is natural to put them together in a unifying theory.

There is a growing body of work along this path. Some of the topics addressed in this literature include: scanning tunneling spectroscopy [43, 44], heat transport [45], neutron scattering [46], and superfluid density observed in penetration-depth measurements [47, 48, 49]. This thesis sits within this class of investigations.

1.6 Outline

We start our presentation in Chapter 2 by laying out the theoretical framework of our study throughout this thesis. We formulate the effective description of a phase fluctuating $d$-wave superconductor and derive an effective field theory for it with two important ingredients: an emergent gauge field representing the quantum phase fluctuations discussed before and two flavours of four-component Dirac spinors describing the low-energy nodal quasiparticles. The structure of the theory is, therefore, very similar to quantum electrodynamics in three space-time dimensions, QED$_3$.

In Chapter 3, we apply this description to determine the ground state of the system once phase fluctuations have destroyed superconductivity for $x < x_0$. We show [50] that the ground state is indeed given by an AF state with spin density waves.

The next two Chapters (4 and 5) focus on the problem of confinement of spin degrees of freedom outside the superconducting phase. We show in separate studies [51, 52, 53] that in a QED$_3$ formulation with a compact gauge field, as is the case in the gauge theories of the $t$-$J$ model, there is a permanent confinement and, thus, no separation of spin and charge degrees of freedom of the physical electrons.

Chapter 6 is devoted to the study of (dynamical) spin response in the QED$_3$ theory of the $dSC$ phase. We improve on a previous computation of spin response in QED$_3$ by
CHAPTER 1. INTRODUCTION AND BASIC PHILOSOPHY

including, separately, the full spectrum of quasiparticles and the full gauge field propagator. This allows us to study the possibility of particle-hole bound states within QED$_3$ and its connection with the resonance observed in inelastic neutron scattering experiments.

We conclude by giving a general discussion of our findings, possible caveats, and related problems for future study in Chapter 7.
Chapter 2

Theoretical Framework

In this chapter, we will lay out the foundational framework of this thesis. It is expressed as the low-energy effective theory of the phase-fluctuating d-wave superconductor at $T = 0$ and takes the form of quantum electrodynamics in three space-time dimensions. We derive the form of this theory, discuss its symmetries, especially an emergent “chiral” symmetry and its breaking across the superconductor-insulator transition.

The material presented in this chapter were first developed by a number of authors. The phase transformation employed to treat the singular phase fluctuations of the superconducting order parameter was initially suggested by Franz and Tešanović [54]. They also proposed the QED$_3$ theory as a description of the pseudogap phase [30]. The effective theory of a phase-fluctuating superconductor discussed in this chapter was first derived by Herbut, [31, 32] who also pointed out the existence of the chiral symmetry and showed that its dynamical breaking could lead to an antiferromagnetic state [31].

2.1 d-Wave superconductor

Having chosen to take the superconducting state with $d$-wave pairing symmetry as our starting point, we write the action of the system at a finite temperature $T$ as

$$S = S_{\text{BCS}} + S_U, \quad (2.1)$$
where

\[ S_{BCS} = T \sum_{\omega_n} \int \frac{d^2 q}{(2\pi)^2} \left[ \sum_{\sigma} \left( i\omega_n + \xi(q) \right) c_\sigma^\dagger(q, \omega_n) c_\sigma(q, \omega_n) \right. \]
\[ \left. + \Delta(q) c_\sigma^\dagger(q, \omega_n) c_\sigma(q, -\omega_n) + \text{h.c.} \right], \quad (2.2) \]

and

\[ S_U = U \int_0^\beta d\tau \int d^2 r \left( \sum_{\sigma} c_{\sigma}^\dagger(r, \tau) c_\sigma(r, \tau) \right)^2. \quad (2.3) \]

The \( \omega_n \) denote fermionic Matsubara frequencies, \( \beta = T^{-1} \), and \( \Delta(q) \) is the \( d \)-wave gap. In the continuum, \( \Delta(q) = \frac{1}{4} \Delta_0 (k_2^2 - k_1^2) \); on the lattice, \( \Delta(q) = \frac{1}{2} \Delta_0 (\cos k_1 - \cos k_2) \), with the coordinates defined as in Fig. 2.1. In general, \( \Delta_0 = |\Delta_0| e^{i\phi} \) is a complex number. \( \xi(q) \) is the normal-state dispersion measured from the Fermi surface. It may be written, for instance, as a tight-binding dispersion in terms of hopping amplitudes of electrons on the lattice. \( S_U \) represents the short-range repulsion between electrons, with \( U > 0 \).

This is the basic electronic action with which we will be working. It contains the phenomenology that is discussed in this thesis. Our treatment goes beyond the usual BCS mean field in that we will allow for fluctuations in the phase of the gap function \( \Delta(q) \). We will also need to adopt certain representation and approximation schemes, as well as some assumptions about the form of the effective theory that is based on this action, in order to investigate the relevant physics in the underdoped region of the phase diagram of cuprates. This is what is discussed in the rest of this chapter.

The low-energy (long-distance) physics important for the critical behaviour of the system is dominated by the excitations around the four nodes of the \( d \)-wave gap function. At any other point around the Fermi surface, a finite gap in the excitation spectrum makes the role of those excitations in the critical behaviour irrelevant at low-enough energies, or long-enough distances.

A useful representation for describing this low-energy physics is a special, \( 8 \times 8 \), Nambu representation defined by forming a spinor field as follows. We shift the momenta around the nodes as \( q = K_i + q, i = I, II \), and rotate the coordinate frame from \( (k_1, k_2) \) to \( (q_x, q_y) \) as in Fig. 2.1. Now, we define the eight-component Dirac field \( \Psi = (\Psi_I^\dagger, \Psi_{II}^\dagger) \) where for
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Figure 2.1: Four nodes of the \textit{d}-wave order parameter in the Brillouin zone and other definitions. Dashed line is the putative Fermi surface.

At each node $i$,

$$
\Psi_i(q, \omega_n) \equiv \begin{pmatrix}
    c_i(K_i + q, \omega_n) \\
    c_i^\dagger(-K_i - q, -\omega_n) \\
    c_i(K_i + q, \omega_n) \\
    c_i^\dagger(-K_i - q, -\omega_n)
\end{pmatrix}.
$$

(2.4)

At $T = 0$, with $q_0 \equiv \omega$, the action may be then written as

$$
S = \int \frac{d^3q}{(2\pi)^3} \bar{\Psi}(q)\Gamma_0\{i\hbar q + iM(q)\}\Psi(q) + U \int d\tau d^2r \left( \bar{\Psi}(r, \tau)B_U\Psi(r, \tau) \right)^2.
$$

(2.5)

In this expression, $M(q) = \text{diag}(M_1(q_x, q_y), M_{II}(q_y, q_x))$ and $iM_i = \text{diag}(\mathcal{H}_i, \mathcal{H}_i)$ is a $4 \times 4$ matrix defined by

$$
\mathcal{H}_i = \begin{pmatrix}
    \xi(K_i + q) & \Delta(K_i + q) \\
    \Delta^*(K_i + q) & -\xi(-K_i - q)
\end{pmatrix}.
$$

(2.6)

As usual, $\bar{\Psi} = \Psi^\dagger \Gamma_0$ with $\Gamma_0 = \text{diag}(\gamma_0, \gamma_0)$.\footnote{Given the square matrices $T_1, T_2, \ldots, T_n$, we define $\text{diag}(T_1, T_2, \ldots, T_n)$ as the block-diagonal matrix formed by placing $T_1, T_2, \ldots, T_n$ on the diagonal.} We have chosen to work with the following
4 × 4 representation of γ-matrices here (and throughout this thesis):

\begin{align}
\gamma_0 & = \sigma_1 \otimes I, \\
\gamma_1 & = -\sigma_2 \otimes \sigma_3, \\
\gamma_2 & = -\sigma_2 \otimes \sigma_1, \\
\gamma_3 & = -\sigma_2 \otimes \sigma_2.
\end{align}

They satisfy the Clifford algebra \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \). For future use, we also define

\[
\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_3 \otimes I.
\]

Finally, \( B_U = \Gamma_0 A_U \), with \( A_U = \text{diag}(\sigma_3, \sigma_3, \sigma_3, \sigma_3) \). More explicitly, \( B_U = I \otimes (\sigma_1 \otimes \sigma_3) \).

Our manipulations so far have been exact. In order to make progress, however, we need to focus our attention exclusively around the four nodes of the gap function. Around the nodes, we may expand the energy dispersion function in powers of \( q \) and keep the lowest terms. The first non-vanishing term, and also the most important one physically, is linear in \( q \). This is done in the next section.

### 2.2 Dirac spectrum

We now expand \( i M(q) \) in Eq. (2.5) around the nodes of the superconducting order parameter in powers of \( q \). It is convenient to define the following matrices:

\begin{align}
M_1 & = -i \sigma_3 \otimes \sigma_3 = i \gamma_0 \gamma_1, \\
M_2 & = -i \sigma_3 \otimes \sigma_1 = i \gamma_0 \gamma_2.
\end{align}

We note that these definitions are fixed by the original action and our definition of the spinor field in Eq. (2.4) and do not depend on our choice of representation of \( \gamma \)-matrices. Using the symmetry property of the \( d \)-wave gap and the quasiparticle dispersion, \( \Delta(-k) = \Delta(k) \) and \( \xi(-k) = \xi(k) \), we find:

\begin{align}
S_{\text{BCS}} & = S_0 + S_A + S_{\text{NL}}, \\
S_0 & = \int d\tau d^2r \bar{\Psi}(r, \tau) \{ \Gamma_0 \partial_\tau + \Gamma_1 \partial_x + \Gamma_2 \partial_y \} \Psi(r, \tau), \\
S_A & = \int d\tau d^2r \bar{\Psi}(r, \tau) \{ B_1 \partial_x + B_2 \partial_y \} \Psi(r, \tau).
\end{align}
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Our notations in these relations are as follows: $\Gamma_{1,2} = \Gamma_0 A_{1,2}$ and $B_{1,2} = \Gamma_0 \delta A_{1,2}$. The matrices $A_1 = \text{diag}(M_1, M_2)$ and $A_2 = \text{diag}(M_2, M_1)$ are independent of the choice of $\Gamma_0$. The quantities $v_F$ and $v_\Delta$ are the two characteristic velocities of the linearized spectrum along $q_x$ and $q_y$, respectively, in units of some fixed velocity $c$, which we set to $c = 1$. They are defined as the nodal Fermi velocity,

$$v_F = \frac{\partial}{\partial q_x} \xi(K_1 + q)|_{q=0},$$

and the nodal gap slope,

$$v_\Delta = \frac{\partial}{\partial q_y} \Delta(K_1 + q)|_{q=0}.$$  

\(S_0\) is the isotropic piece of the linearized BCS action, for $v_F = v_\Delta$. The variation matrices $\delta A_1 = \text{diag}(\lambda_F M_1, \lambda_\Delta M_2)$ and $\delta A_2 = \text{diag}(\lambda_\Delta M_2, \lambda_F M_1)$ represent the anisotropic piece of the linearized theory through the anisotropy parameters, $\lambda_F = v_F - 1$ and $\lambda_\Delta = v_\Delta - 1$. We will discuss the form of $S_{NL}$ that represents the nonlinear terms arising from the higher order derivative terms in Chapter 3. There we also discuss its effect on the fate of different insulating states, which await the demise of the superconductor due to phase fluctuations.

In this form, the “bare” action, $S_0$, looks like that of a non-interacting Dirac action with an emergent Lorenz invariance. Attention must be paid to the fact that the spinor field here has eight components instead of the usual four in Dirac’s theory. This is equivalent to $N_f = 2$ species, or flavours, of four-component Dirac spinors. This number will play a crucial role in the critical behaviour of $S_0$ as discussed in the next chapter.

This non-interacting Dirac form does not yet fully bring out the important physics of the system. We still need to deal with the fluctuations in the phase of the order parameter $\Delta_0 = |\Delta_0|e^{i\phi}$. As discussed before, we expect these fluctuations to play a crucial role in the critical behaviour of the underdoped cuprates. To this end, we must allow the phase $\phi$ to become a fully dynamical degree of freedom. As it couples to the electronic degrees of freedom through the exponential in the off-diagonal elements of $\mathcal{H}_i$, it is not easy to treat $\phi$ in the common ways of field theory, where couplings are usually with current fields and have (finite) polynomial form.

In order to do so, we need to perform some further mappings and approximations. We turn to this subject in the next section.
2.3 Phase fluctuations

The fluctuations in the phase $\phi$ can be put in two categories: regular and singular. The categorization arises because of the compact nature of the domain of the values that the phase can assume, namely, $0 \leq \phi \leq 2\pi$, with $0$ and $2\pi$ identified.

Smooth fluctuations contain only infinitely small variations over arbitrarily short distances. They give rise to an excitation spectrum that may only be critical at $T = 0$. At finite temperatures and in two spatial dimensions there is no critical behaviour resulting from regular fluctuations, due to the Mermin-Wagner-Hohenberg theorem [55, 56, 57]. For this reason, it is the singular fluctuations that contain the interesting physics for the critical behaviour of our theory. This distinction will be made more clear when we derive the dynamics of the phase fluctuation in the following section.

Singular fluctuations contain regions where the phase changes by a finite amount (equal to $2\pi n$, with an integer $n$) over an arbitrarily short distance. These regions form lines on which the phase is undefined, i.e., branch-cuts. In a sense, all the critical behaviour of the theory derives from the ways the branch-cuts organize themselves into different phases as a relevant control parameter, e.g., temperature, doping concentration or a chemical potential, is changed across a critical value. The end point of a branch-cut is called a vortex. The accumulated change of phase over a closed loop containing the vortex is equal to the finite change of phase across the branch cut, $2\pi n$. $n$ is called the vorticity. A physical picture of the phases of the system can also be given in terms of vortices. In general, the higher the vorticity, the less critical the corresponding vortex configurations. For this reason, we will be considering in the following only the case for $n = +1$ and $-1$, referred to as a vortex and an anti-vortex, respectively.

More precisely, the regular and singular configurations of the phase are recognized by looking at $\nabla \times \nabla \phi = 2\pi \rho_v$, where $\rho_v$ is the density of vortices. For regular fluctuations, the left-hand side is zero, and there are no vortices. If we work on the lattice instead of the continuum, as is always the physical case in condensed matter systems, the shortest distance is set by the lattice spacing. The regular and singular fluctuations are then defined by looking at the phase change over closed lattice loops. The gradient operator $\nabla$ is then also defined on the lattice as a difference operator.

It is common to take phase fluctuations into account by assigning the phase to the elec-
tronic operators. That is, by mapping the BCS pairing term $|\Delta_0|e^{i\phi}c_1^\dagger c_1^\dagger$ to $|\Delta_0|c_1^\dagger c_1^\dagger$. This will reproduce the coupling of the phase to electrons in the more common polynomial form. When doing so for singular fluctuations we must be careful about the single-valuedness of the mapping. For instance, we may define such a mapping by assigning the phase to spin-$\sigma$ operators, $\sigma \in \{\uparrow, \downarrow\}$, as

$$c_\sigma^\dagger = e^{i\omega w}c_\sigma^\dagger,$$

where $\{w_\sigma\}$ is an arbitrary partitioning of unity: $w_\uparrow = w, w_\downarrow = 1 - w, w \in [0, 1]$. However, if we circle a vortex once, we pick up a phase of $2\pi w$ for spin-$\uparrow$ and $2\pi(1 - w)$ for spin-$\downarrow$ from the singular part of the phase. Thus, a single-valued mapping is possible only for $w = 0$ or 1.

One way to avoid complications of multi-valued mappings is to require only vortices with double vorticity [58]. However, the critical behaviour of the vortices with unit vorticity sets in before that of the ones with higher vorticity, i.e., at lower critical temperatures. Such double vortices have not been observed experimentally [59]. We will, then, require that the most important contribution to the critical behaviour, even at $T = 0$, comes from the former, and will not follow this path.

The observation made below Eq. (2.19) was made useful by Franz and Tešanović [54], who proposed to divide the vortices into two groups $A$ and $B$ as shown in Fig 2.2. By separating the regular and singular part of the phase as $\phi = \phi_r + \phi_s$ so that $\nabla \times \nabla \phi_r = 0$
and $\nabla \times \nabla \phi_s = \nabla \times \nabla \phi$, we proceed by defining
\begin{align*}
\phi_A &= w_1 \phi_r + \phi_{s,A}, \\
\phi_B &= w_1 \phi_r + \phi_{s,B}.
\end{align*}
\hspace{1cm} (2.20)
\hspace{1cm} (2.21)

$\phi_{s,A/B}$ are the singular parts coming from vortices in group $A$ and $B$, respectively. Now, $w$ need not be 0 or 1, since it appears only in the regular part of the phase. Finally, the Franz-Tešanović transformation is given by
\begin{align*}
ce_1^+ &= e^{i\phi_A} c_1^+, \\
ce_1^- &= e^{i\phi_B} c_1^-.
\end{align*}
\hspace{1cm} (2.22)
\hspace{1cm} (2.23)

On the spinor field in Eq. (2.4) this transformation takes the form,
$$\psi(r) = R(r) \Psi(r),$$
\hspace{1cm} (2.24)

with $R = \text{diag}(e^{i\phi_A}, e^{-i\phi_B}, e^{i\phi_A}, e^{-i\phi_B})$. Now, by circling any vortex, say, in group $A$, we are guaranteed to pick up only a phase of $2\pi$ for spin-$\uparrow$ and 0 for spin-$\downarrow$ and so the mapping is single-valued.

By applying this transformation to $S_0$ in Eq. (2.15), we uncover the following gauge fields
\begin{align*}
a_\mu &= \frac{1}{2} \partial_\mu (\phi_A - \phi_B), \\
v_\mu &= \frac{1}{2} \partial_\mu (\phi_A + \phi_B).
\end{align*}
\hspace{1cm} (2.25)
\hspace{1cm} (2.26)

The transformed action takes the form
$$S_0 + S_A = \int d^3r \left[ \bar{\psi}(r) \{ \Gamma_0 (\partial_r + ia_r) + (\Gamma_1 + B_1)(\partial_z + ia_z) \right.
\hspace{1cm} + \left. (\Gamma_2 + B_2)(\partial_y + ia_y) \} \psi(r) + i v_\mu J_\mu \} \right].$$
\hspace{1cm} (2.27)

$J_\mu$ is the electronic current,
$$J_\mu = \bar{\psi} \left( B_U, -\frac{v_F}{2} (I + \sigma_3) \otimes (\sigma_2 \otimes I), -\frac{v_F}{2} (I - \sigma_3) \otimes (\sigma_2 \otimes I) \right) \psi.$$  
\hspace{1cm} (2.28)

$B_U$ is related to the total electronic density. It appeared in Eq. (2.5) and was given to be $I \otimes (\sigma_1 \otimes \sigma_3)$ below Eq. (2.11).
Manifestly, $a$ couples \textit{minimally} to the transformed spinor field $\psi$. On the other hand, $v$ couples directly to the physical current, the same way as the external magnetic gauge field. Therefore, $\psi$ is electromagnetically neutral; instead, it carries a new \textit{vortex} charge via its minimal coupling to $a$. Being a Dirac spinor, it also carries a spin of $\frac{1}{2}$. For these reasons, we dub the excitations represented by $\psi$ "spinons."

The action in Eq. (2.27) reflects a sort of spin-charge separation. This is the sort that is expected to happen \textit{inside} a superconductor, reflecting the fact that charge is not a good quantum number any more, while spin is. The new structure uncovered by the Franz-Tešanović transformation is the minimal coupling of spinons to the gauge field $a$ representing the \textit{singular} phase fluctuations.

Finally, we note that there is a true gauge freedom for $a$. This is effected by both the regular and the singular part of the phase. For the regular part, a change $w \rightarrow w + \delta w$ will send $a_\mu \rightarrow a_\mu + \delta w \partial_\mu \phi_r$. So the choice of $w$ is equivalent to fixing a regular gauge for $a$. Since the regular part does not contain any critical dynamics, we simply choose a spin-symmetric gauge by setting $w = \frac{1}{2}$. In this gauge $a$ contains only the singular part of the phase. Of course, there remains a larger regular gauge freedom produced by $\phi_A \rightarrow \phi_A + \chi_r$ and $\phi_B \rightarrow \phi_B - \chi_r$; we fix $\chi_r = 0$ as well.

The singular gauge freedom is related to the freedom of choosing the division groups $A$ and $B$. As shown in Fig. 2.2, a different division of vortices corresponds to a change, $\phi_A \rightarrow \phi_A + \chi_s$ and $\phi_B \rightarrow \phi_B - \chi_s$, which send $a_\mu \rightarrow a_\mu + \partial_\mu \chi_s$. This corresponds to a singular gauge transformation for $a$. In contrast to the regular gauge, which we fixed, it is important to make the theory fully gauge-invariant with respect to the singular gauge. To this end, we sum over all possible ways of dividing the vortex configuration into groups $A$ and $B$, with the weight determined by the combinatorial factor equal to the number of ways a given division may be performed in identical ways. This is explicitly implemented in the next section.

We have now identified spinons as an important degree of freedom of the theory and incorporated their interaction with singular phase fluctuations by the minimal coupling to the gauge field $a$. The final step in deriving the effective theory we set out to find, is to include the dynamics of the gauge fields $a$ and $v$. This we take up in the next section.
2.4 QED\textsubscript{3} of a phase-fluctuating \textit{d}-wave superconductor

The dynamics of \(a\) and \(v\) are governed by the phase \(\phi\). As noted before, we work at zero temperature, where fluctuations are quantum ones. In principle, these fluctuations could be taken into account by including the additional terms in the BCS action, Eq. (2.2), that give the dynamics of the gap function. However, at the level of our effective description, it suffices to model the phase fluctuations by a three-dimensional (two spatial plus one temporal) XY (3DXY) model. This is expected to correctly describe the critical behaviour observed in underdoped cuprates due to vortex proliferation.

At finite temperatures, the critical behaviour is essentially two-dimensional and of the Kosterlitz-Thouless (KT) type \cite{60, 61}. In the phase-coherent state, vortices and anti-vortices are bound into dipoles. By tuning their condensation energy to a finite value, vortices and anti-vortices undergo a KT unbinding transition, rendering the superconducting phase incoherent.

At zero temperature, the 3DXY model also supports two phases. But now, singular phase structures extend in the temporal direction to form closed vortex loops in place of finite-temperature vortices and anti-vortices. In the phase-coherent state, vortex loops are finite in size, whereas in the phase-incoherent state they are infinite.

The action for the 3DXY model on the (square) lattice is

\[
Z_{3\text{DXY}} = \int \mathcal{D}\phi \exp \left[ -K \sum_{\vec{r}, \vec{\mu}} \cos(\Delta_{\vec{\mu}} \phi_{\vec{r}}) \right].
\]  

(2.29)

The lattice sites are labeled with \(\vec{r} = (\tau, x, y)\) and the nearest-neighbour directions of the lattice, with \(\vec{\mu} = (\hat{\tau}, \hat{x}, \hat{y})\). The lattice gradient is defined as \(\Delta_{\vec{\mu}} \phi_{\vec{r}} \equiv \phi_{\vec{r}+\vec{\mu}} - \phi_{\vec{r}}\). This definition could be generalized to include lattice divergence and curl operations in obvious ways.

In order to derive the gauge field dynamics from this action, we will need to implement the Franz-Tešanović division of vortices into groups \(A\) and \(B\) and, so, we need to work with vortex density variables, instead of the phase itself. To this end, we take advantage of duality transformations \cite{62, 63}, which are useful in this respect. The procedure is, however, rather involved and we will need to introduce various variables along the way. Even though the final result may be motivated based on its symmetry properties and phase structure, it is worth going through the derivation with some rigour.
The first step is to rewrite the cosine function using a Villain approximation [64] as

\[
Z_{3DXY} \rightarrow \int \text{d}[\phi] \sum_{[n]} \exp \left[ -\frac{K}{2} \sum_{r,\bar{\mu}} (\Delta_{\bar{\mu}} \phi_r - 2\pi n_{r,\bar{\mu}})^2 \right].
\]  

(2.30)

The arrow indicates that we do not keep track of unimportant factors in our transformation. The \(n_{i,\bar{\mu}}\) are integer fields (on the bonds of the lattice), which reproduce the periodic minima of the cosine. In other words, they represent the finite jump (in units of \(2\pi\)) in the gradient of phase over the branch cuts. Therefore, based on our discussion in the previous section, \(m_r = \Delta \times n_r\) must be identified with the vortex density, \(\rho_v\).

In the next step, we derive the current representation of the 3DXY in terms of \(m_r\) using standard Hubbard-Stratonovich decoupling with a field \(b_r\) and integration over \(\phi\):

\[
Z_{3DXY} \rightarrow \int \text{d}[b_r, \phi] \sum_{[n]} \exp \left[ -\sum_{r,\bar{\mu}} \left\{ \frac{1}{2K} b_{r,\bar{\mu}}^2 + i b_{r,\bar{\mu}} (\Delta_{\bar{\mu}} \phi_r - 2\pi n_{r,\bar{\mu}}) \right\} \right]
\]

\[
\quad \rightarrow \int \text{d}[s_r] \sum_{[n]} \exp \left[ -\sum_{r} \left\{ \frac{1}{2K} (\Delta \times s_r)^2 + 2\pi i (\Delta \times s_r) \cdot n_r \right\} \right]
\]

\[
\quad \rightarrow \int \text{d}[s_r] \sum_{[m]} \exp \left[ -\sum_{r} \left\{ \frac{1}{2K} (\Delta \times s_r)^2 - 2\pi i s_r \cdot m_r \right\} \right].
\]  

(2.31)

In the second line we have used \(b_r = \Delta \times s_r\) to enforce \(\Delta \cdot b_r = 0\) arising from the integration over \(\phi\) in the first line. In the third line, we have used our definition of the vortex density variables, \(m_r = \Delta \times n_r\). Note that by definition \(\Delta \cdot m_r = 0\) and the vortex lines form closed loops, the aforementioned vortex loops. The prime over the sum is to indicate this constraint.

Now we are in a position to proceed with dividing the vortices into groups \(A\) and \(B\): \(m_r = m_{A,r} + m_{B,r}\). To enforce the singular gauge invariance discussed in the previous section, we will sum over all integer values of \(m_{A,r}\) and \(m_{B,r}\). Then, \(a_r\) and \(v_r\) are defined to satisfy

\[
\Delta \times a_r = 2\pi (m_{A,r} - m_{B,r})/2,
\]

\[
\Delta \times v_r = 2\pi (m_{A,r} + m_{B,r})/2.
\]  

(2.32)  

(2.33)
We make further manipulations in the following way,

\[
Z_{3DXY} \rightarrow \int \mathcal{D}[S, \zeta_A, \zeta_B, v, a] \sum_{[m_A, m_B]} \exp \left[ -\sum_r \left\{ \frac{1}{2K} (\Delta \times S_r)^2 ight. ight.
\]

\[
- 2\pi i S_r \cdot (m_{A,r} + m_{B,r}) + i \zeta_{A,r} \cdot (\Delta \times a_r + \Delta \times a_r - 2\pi m_{A,r}) 
\]

\[
+ i \zeta_{B,r} \cdot (\Delta \times v_r - \Delta \times a_r - 2\pi m_{B,r}) \right\} \right]
\]

\[
\rightarrow \int \mathcal{D}[S, v, a] \sum_{[k_A, k_B]} \exp \left[ -\sum_r \left\{ \frac{1}{2K} (\Delta \times S_r)^2 ight. ight.
\]

\[
+ i (k_{A,r} + k_{B,r} - 2S_r) \cdot (\Delta \times v_r) + i (k_{A,r} - k_{B,r}) \cdot (\Delta \times a_r) \right\} \right].
\]

The first line introduces \( a \) and \( v \) though Lagrange multipliers \( \zeta_A \) and \( \zeta_B \). Summing over \( m_A \) and \( m_B \) enforces \( S + \zeta_A = k_A \) and \( S + \zeta_B = k_B \) to be integers, as written in the second line. We may now integrate over \( S \) to find,

\[
Z_{3DXY} \rightarrow \int \mathcal{D}[v, a] \sum_{[k_A, k_B]} \exp \left\{ -\sum_r \left[ 2K v_r^2 + i (k_{A,r} + k_{A,r}) \cdot \Delta \times v_r 
\right.
\]

\[
+ i (k_{A,r} - k_{B,r}) \cdot \Delta \times a_r \right\}. \quad (2.35)
\]

Let us pause here to note that \( v \) has now acquired a finite gap, from the \( 2K v_r^2 \) term, irrespective of the state of the system. This is what we meant, in the previous section, by stating that the regular phase fluctuations present in \( v \) do not contribute to the critical behaviour we wish to address. In the following we keep \( v \) for completeness but, in the end, we are only interested in \( a \) and will drop \( v \) for the rest of this thesis. Even though \( v \) is not important for the critical behaviour of the system, it plays an important role in the charge sector of the theory and has been studied in some detail elsewhere [48, 32, 47].

At this stage, we will turn the integer fields \( k_A \) and \( k_B \) into real fields \( \frac{1}{2} (A_+ + A_-) \) and \( \frac{1}{2} (A_+ - A_-) \) respectively:

\[
Z_{3DXY} \rightarrow \int \mathcal{D}[v, a, A_+, A_-] \sum_{[l_A, l_B]} \exp \left[ -\sum_r \left\{ 2K v_r^2 + i A_{+,r} \cdot \Delta \times v_r + i A_{-,r} \cdot \Delta \times a_r 
\right.
\]

\[
+ i \pi (A_{+,r} + A_{-,r}) \cdot l_{A,r} + i \pi (A_{+,r} - A_{-,r}) \cdot l_{B,r} \right\].
\]
We can write this action in a more convenient form if we introduce a chemical potential \( x \rightarrow 0 \) for bond variables \( l_+ \) and \( l_- \) in the form \( \sum_r (x/2)(l_{A,r}^2 + l_{B,r}^2) \). Then we can trace back from this action through a Villain approximation to find the (almost final!) form

\[
Z_{3DXY} \rightarrow \lim_{x \rightarrow 0} \int d[v, a, A_+, A_-, \theta_A, \theta_B] \exp \left[ -\sum_r \left\{ \frac{1}{x} \cos(\Delta \theta_{A,r} - \pi A_{+,r} - \pi A_{-,r}) \\
+ 2K v_r^2 + i A_{+,r} \cdot \Delta \times v_r + \frac{1}{x} \cos(\Delta \theta_{B,r} - \pi A_{+,r} + \pi A_{-,r}) + i A_{-,r} \cdot \Delta \times a_r \right\} \right].
\]

We have, thus, a description of the dynamics of \( v \) and \( a \) in terms of two three-dimensional frozen \((x \rightarrow 0)\) lattice superconductors \([65, 62]\) for the dual angular fields \( \theta_A \) and \( \theta_B \), and the gauge fields \( \frac{1}{2}(A_+ + A_-) \) and \( \frac{1}{2}(A_+ - A_-) \).

We can now safely take \( x \) to be finite without affecting the critical behaviour of the system \([66, 67]\) and pass to the continuum limit \([68, \text{Section II.5.2}]\) to write the full low-energy action as

\[
S = S_0 + S_A + S_g,
\]

with

\[
S_g = \int d^3 r \left[ \left| (\nabla - i A_1) \varphi_1 \right|^2 + \left| (\nabla - i A_2) \varphi_2 \right|^2 + \mu_0^2 \left( |\varphi_1|^2 + |\varphi_2|^2 \right) \\
+ \frac{i}{2\pi} a \cdot \nabla \times (A_1 - A_2) + 2K v^2 + \frac{i}{2\pi} v \cdot \nabla \times (A_1 + A_2) \\
+ \frac{\lambda_1}{2} (|\varphi_1|^4 + |\varphi_2|^4) + \frac{\lambda_2}{2} (|\varphi_1|^2 + |\varphi_2|^2)^2 \right];
\]

the other terms have been described before in Eq. (2.27). We have rescaled the gauge field by a factor of \( \pi \). In this form, \( \varphi_{1,2} \), which we call the “vortex fields,” and \( A_{1,2} \) are the corresponding bosonic, continuum complex order parameters and the gauge fields, respectively, of the lattice superconductors formulated by the phase field \( \theta_{A,B} \) and the gauge field \( A_{\pm} \) in Eq. (2.36) \([68]\). The vortex condensate is given by \( |\langle \varphi_1 \rangle| = |\langle \varphi_2 \rangle| = \varphi_0 \), assuming that the symmetry \( \varphi_1 \leftrightarrow \varphi_2 \) of the action is not broken. At the mean-field level, this is true if \( \lambda_1 > 0 \). At low energies we will drop \( v \) hereafter. This completes our derivation of the QED\(_3\) effective theory of a phase-fluctuating \( d \)-wave superconductor.

In the continuum formulation we may identify the phases of the system with the value of the order parameter, \( \varphi_0 \):
1. \( \varphi_0 = 0 \) for \( \mu^2 > 0 \). \( \mu^2 = \mu_0^2 + O(\lambda) \) is the renormalized mass of the vortex field. This is the original d-wave superconductor with phase coherence and sharp d-wave quasi-particles. The vortex loops are finite and the vortex condensate is zero. We may integrate out \( \varphi_1 \) and \( \varphi_2 \) in the one-loop approximation to find

\[
S_g - \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} A_\mu(-p) \Pi_{AA}(p, \mu)(\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu) A_\nu(p) + \int d^3 r \frac{i}{2\pi} a \cdot \nabla \times A,
\]

where \( A = A_1 - A_2 \). The polarization \( \Pi_{AA}(p, \mu) = (|p|/8\pi) F(\mu/|p|) \),

\[
F(z) = (4z^2 + 1) \tan^{-1} \frac{1}{2z} - 2z + O(\lambda).
\] (2.39)

We may then integrate over \( A \) to find

\[
S_g - \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} a_\mu(-p) D^{-1}(p)(\delta_{\mu\nu} - \hat{p}_\mu \hat{p}_\nu) a_\nu(p),
\] (2.40)

\[
D(p) = \frac{\pi}{4|p|} F\left(\frac{\mu}{|p|}\right).
\] (2.41)

In the infrared, i.e., for \( |p| \ll \mu \), we find a gap for the gauge field, \( D^{-1} \propto m \). Thus, \( a \) is irrelevant for the low-energy physics, as should be the case in the superconductor. In the ultraviolet, \( D^{-1} \propto |p| \). This ultraviolet behaviour is an exact result [69]. As \( \mu \to 0 \), at the transition, \( D^{-1} \propto |p| \) over all energies.

2. \( \varphi_0 = \sqrt{-\mu^2/(2\lambda_2 + \lambda_1)} \neq 0 \) for \( \mu^2 < 0 \). This corresponds to the phase-incoherent state where vortex loops have become infinite and superconductivity must be lost. \( \varphi_0 \) is the finite vortex condensate. The gauge fields \( A_1 \) and \( A_2 \) acquire a gap through spontaneous breaking of gauge symmetry (Higgs mechanism) [70, 71]. Again writing \( A = A_1 - A_2 \), we have:

\[
S_g - \int d^3 r \left( \frac{1}{2} \varphi_0^2 A^2 + \frac{i}{2\pi} a \cdot \nabla \times A \right).
\]

Upon integration over \( A \), the gauge field \( a \) now takes on a Maxwell dynamics,

\[
S_g - \int d^3 r \frac{(\nabla \times a)^2}{8\pi^2 \varphi_0^2},
\] (2.42)

with the charge being proportional to the vortex condensate, \( \varphi_0 \). So, we find \( D^{-1} \propto p^2/\varphi_0^2 \) for the infrared behaviour. In the ultraviolet we go back to the universal linear dependence, \( D^{-1} \propto |p| \).
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Figure 2.3: The inverse gauge propagator, from top to bottom: in the phase-coherent \( d \)-wave superconductor \( \mu^2 > 0 \), at the transition \( \mu^2 = 0 \), and in the phase-incoherent insulator \( \mu^2 < 0 \).

The above classification of phases of the phase-fluctuating \( d \)-wave superconductor is the central result of this section. The various forms of the inverse gauge propagator are shown in Fig. (2.3).

2.5 Symmetries and chiral symmetry breaking

The action we have derived for the effective low-energy theory of a phase-fluctuating \( d \)-wave superconductor has some important symmetries. The familiar ones are global spatial and spin rotations; parity; and time reversal. Our treatment of phase fluctuations by introducing the gauge fields \( a \) and \( v \) also adds two local gauge symmetries in addition to the physical magnetic gauge symmetry. There is also an additional global "chiral" symmetry due to the fact that our Dirac spinons are massless; that is, there is no term \( \sim \bar{\psi} \psi \) in action (2.27).
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If we work with the isotropic action $S_0$, a global chiral rotation is given by

$$\psi \rightarrow e^{i \sum_{i=1}^{16} \theta_i J_i} \psi,$$

with the $J_i, i = 1 \ldots 16$, generators of the $U_c(4)$ group; the index refers to the chiral origin. We classify the generators by their commutation or anti-commutation with the set of $8 \times 8 \Gamma$-matrices, $\{\Gamma_0, \Gamma_1, \Gamma_2\}$.

- Commuting:

$$J_1 = I \otimes 1; \quad J_2 = \sigma_3 \otimes 1;$$

$$J_3 = I \otimes i\gamma_3\gamma_5; \quad J_4 = \sigma_3 \otimes i\gamma_3\gamma_5;$$

$$J_5 = \sigma_1 \otimes \frac{i}{\sqrt{2}}(\gamma_1 - \gamma_2)\gamma_5; \quad J_6 = \sigma_2 \otimes \frac{i}{\sqrt{2}}(\gamma_1 - \gamma_2)\gamma_5;$$

$$J_7 = \sigma_1 \otimes \frac{i}{\sqrt{2}}(\gamma_1 - \gamma_2)\gamma_5; \quad J_8 = \sigma_2 \otimes \frac{i}{\sqrt{2}}(\gamma_1 - \gamma_2)\gamma_5.$$ (2.44, 2.45, 2.46, 2.47)

- Anti-commuting:

$$J_9 = I \otimes \gamma_3; \quad J_{10} = \sigma_3 \otimes \gamma_3;$$

$$J_{11} = I \otimes \gamma_5; \quad J_{12} = \sigma_3 \otimes \gamma_5;$$

$$J_{13} = \sigma_1 \otimes \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2); \quad J_{14} = \sigma_2 \otimes \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2);$$

$$J_{15} = \sigma_1 \otimes \frac{i}{\sqrt{2}}\gamma_0(\gamma_1 + \gamma_2); \quad J_{16} = \sigma_2 \otimes \frac{i}{\sqrt{2}}\gamma_0(\gamma_1 + \gamma_2).$$ (2.48, 2.49, 2.50, 2.51)

If we include the anisotropic piece $S_A$, for $v_F \gg v_\Delta$, the $U_c(4)$ is broken down to $U_c(2) \times U_c(2)$. Each $U_c(2)$ is associated with a diagonal pair of nodes, and is generated by the set $\{1, \gamma_3, \gamma_5, i\gamma_3\gamma_5\}$. $1$ (the four-dimensional identity matrix) generates a trivial $U(1)$ and the rest generate a chiral $SU_c(2)$ group.

The chiral symmetry may be dynamically broken in the presence of strong gauge field fluctuations and the average $\langle \bar{\psi}(r)\psi(r) \rangle$ becomes non-zero [72, 73, 74]. The chiral $SU_c(2)$ is then further broken down to $U_c(1)$ generated by $i\gamma_3\gamma_5$, which still commutes with $\gamma_0$. The dynamical symmetry breaking is a non-perturbative effect: at any finite order of perturbation theory, the average $\langle \bar{\psi}(r)\psi(r) \rangle$ remains zero. It is usually studied by solving the Schwinger-Dyson equation, shown diagrammatically in Fig. 2.4. The effect is driven by the
gauge field polarization by the spinons, which add a linear term to the inverse gauge field propagator,

$$\Pi_{aa}(p) = \frac{N_f}{8} |p|. \quad (2.52)$$

These studies show that for a Maxwell gauge field dynamics, the chiral symmetry is broken below a critical number of spinon flavours, \(N_f^c \approx 3\) [75, 76, 77, 78].

In the superconductor the gauge field is gapped and there is no effect coming from the polarization, \(\Pi_{aa}\). However, outside the superconductor, the low-energy dynamics of the gauge field is governed by a Maxwell term and, for the physical case, \(N_f = 2 < N_f^c\), the chiral symmetry is dynamically broken.

The nature of the order that sets in for a non-zero value of \(\langle \bar{\psi}\psi \rangle\) in terms of the (original) electronic system is studied in the next chapter. There we will show that this is in fact antiferromagnetism, in complete accord with reality.

\[\text{There is also a suggestion that } N_f^c \leq 3/2 \text{ [79].}\]
Chapter 3

Antiferromagnetism

In Chapter 2 we laid out the effective description of the phase-fluctuating $d$-wave superconductor in terms of the QED$_3$ theory and the emergent, degenerate chiral manifold spanned by the elements of the chiral group, $U_c(4)$. In this chapter we will show how this degeneracy is lifted, once the spinon gap is developed through dynamical breaking of the chiral symmetry, by residual interactions that are irrelevant for the occurrence of the transition itself. Specifically, we will show that a state with incommensurate spin density waves (SDW) is the one with the lowest energy. Hence, connecting this SDW insulator continuously to the Mott-insulating antiferromagnet near half filling, we find that antiferromagnetism arises naturally once the $d$-wave superconductivity is killed via phase fluctuating effects of vortices. The content of this chapter was originally published in Ref. 50.

We first discuss, in Section 3.1, the structure of the chiral manifold and specify the prominent physical states in it. In Section 3.2 we lay out our procedure for finding the effects of residual interactions on the chiral manifold. In the remaining sections, we present the actual computations for velocity anisotropy, short-range repulsion, and nonlinear dispersion, and the cross terms between them. In Section 3.7, we collect our results for the fine structure of the chiral manifold in Fig. 3.2.

3.1 Degenerate chiral manifold

Our choice of $\Gamma$-matrices in the previous chapter is not unique. It is easy to see why, especially if we focus our attention on just one pair of nodes and work with the $4 \times 4$
representation. Because the matrices $M_1 = i\gamma_0\gamma_1$ and $M_2 = i\gamma_0\gamma_2$ in Eqs. (2.12)-(2.13) are fixed by the original electronic action, the choice of $\gamma_0$ is set by the solution, $\gamma_x$, to the Clifford algebra,

$$\gamma_x^2 = 1; \{\gamma_x, \gamma_0\gamma_1\} = \{\gamma_x, \gamma_0\gamma_2\} = 0.$$  

(3.1)

We find the following solutions

$$\gamma_x^{c-SDW} = \gamma_0;$$  

(3.2)

$$\gamma_x^{dip} = i\gamma_0\gamma_3;$$  

(3.3)

$$\gamma_x^{s-SDW} = i\gamma_0\gamma_5.$$  

(3.4)

The meaning of the superscripts is made clear in the following. There is a fourth solution as well, $i\gamma_1\gamma_2$, which does not, however, produce an independent four-dimensional Clifford algebra; it reduces to a two-dimensional algebra generated by Pauli matrices. We also note that those among the generators of the chiral $SU_c(2)\{\gamma_3, \gamma_5, i\gamma_3\gamma_5\}$ introduced in the previous chapter that are broken by the particular choice of $\gamma_x$ rotate these solutions among each other,

$$e^{-i\theta\gamma_3}\gamma_x^{c-SDW}e^{i\theta\gamma_3} = \cos(2\theta)\gamma_x^{c-SDW} + \sin(2\theta)\gamma_x^{dip};$$  

(3.5)

$$e^{-i\theta\gamma_5}\gamma_x^{c-SDW}e^{i\theta\gamma_5} = \cos(2\theta)\gamma_x^{c-SDW} + \sin(2\theta)\gamma_x^{s-SDW};$$  

(3.6)

$$e^{-i\theta(i\gamma_3\gamma_5)}\gamma_x^{dip}e^{i\theta(i\gamma_3\gamma_5)} = \cos(2\theta)\gamma_x^{dip} + \sin(2\theta)\gamma_x^{s-SDW}. $$  

(3.7)

When the chiral symmetry is dynamically broken outside the superconductor, the average $\langle\psi^\dagger\gamma_x\psi\rangle$ becomes non-zero. So, the first question to answer is, what sort of order does this non-zero average signify? We may find the answer by rewriting the composite field in terms of the original electronic operators. This is easily done:

- For $\gamma_x^{c-SDW} = \gamma_0$ we find an incommensurate spin density wave (SDW) order, cosine-modulated by the wave-vectors $2K_1$ and $2K_{II}$ spanning between diagonal pairs of nodes:

$$\sum_{i=I,II}\psi_i^\dagger(r)\gamma_0\psi_i(r) = 4\sum_{i,\alpha}\sigma c_i^\dagger(r,\tau)c_{\alpha}(r,\tau)\cos(2K \cdot r).$$  

(3.8)

- For $\gamma_x^{dip} = i\gamma_0\gamma_3$ we find an additional particle-particle pairing interaction that changes
Figure 3.1: The states corresponding to the solutions to the Clifford algebra (3.1) are shown at the vertices of the triangle. The broken generators of the $SU_c(2)$, shown on the edges, rotate the chosen state to the others, according to Eqs. (3.5)–(3.7).

Thus, the pairing symmetry of this term can be characterized as $p$-wave and the full symmetry is then, $d + ip$. We should remember, however, that the superconducting phase is incoherent and so, we are in an insulating state.

- For $\gamma_x^{a-SDW} = i\gamma_0\gamma_5$ we find an incommensurate SDW, but now sine-modulated by the same wave-vectors as above.

These relations now justify why we chose the particular superscripts to distinguish our different solutions. They are put together in Fig. 3.1.

When we consider the complete $8 \times 8$ representation, for $v_F = v_\Delta$, we find more solutions to the Clifford algebra, similar to Eq. (3.1), that one must solve to find $\Gamma_0$. These solutions are also rotated among themselves by the broken generators of the chiral $U_c(4)$. For a particular choice of $\Gamma_x$ we can then write,

$$\Gamma_x = e^{i\frac{\pi}{3} J_x} \Gamma_0 e^{-i\frac{\pi}{3} J_x}. \quad (3.10)$$

Naturally, they include what we have already found in the $4 \times 4$ representation. Referring to the generators of $U_c(4)$ defined in Eqs. (2.44)–(2.51) we have:
1. $J_x = J_1$, obviously, gives back the (cosine-modulated) SDW:

$$\Gamma_x^{\text{SDW}} = \Gamma_0 = I \otimes \gamma_0.$$  \hfill (3.11)

2. $J_x = J_5$ produces the $d + ip$ insulator:

$$\Gamma_x^{\text{dip}} = I \otimes (i\gamma_0\gamma_3).$$  \hfill (3.12)

3. $J_x = J_{13} - J_{15}$ yields a $d + is$, similarly defined with the $d + ip$, insulator:

$$\Gamma_x^{\text{dis}} = I \otimes (i\gamma_1\gamma_2).$$  \hfill (3.13)

4. $J_x = J_{14}$ gives charge density waves (CDW) with the periodicity set by the wavevector $K_l + K_{II}$:

$$\Gamma_x^{\text{CDW}} = \frac{i}{\sqrt{2}}\sigma_2 \otimes (\gamma_1 - \gamma_2)\gamma_0.$$  \hfill (3.14)

In this case, the mass term is expressed as

$$\psi^{\dagger}(r)\Gamma_x^{\text{CDW}}\psi(r) = 4 \sum_\sigma c_\sigma^{\dagger}(r, \tau)c_\sigma(r, \tau)\sin((K_l - K_{II}) \cdot r).$$  \hfill (3.15)

Of course, there is a continuum of other chiral rotations that are possible and lead to various linear combinations, or rotated and/or phase-shifted modulations, of the four fundamental states defined above.

In the case of the maximal $U_c(4)$ chiral symmetry, all the above insulating states and their various combinations are equally likely outcomes of the dynamical symmetry breaking. In the rest of this chapter, we will see how this degeneracy is lifted in favour of the SDW state.

### 3.2 Fine structure of the chiral manifold

Chiral symmetry is, in fact, not an exact symmetry of the $d$-wave superconducting state. It arises only in the low-energy limit of the standard quasiparticle action, Eq. (2.2). At low energies one is allowed to linearize the spectrum near the nodes of the $d$-wave order parameter and to neglect the nonlinear terms from higher-order derivatives and the short-range interactions between quasiparticles. These terms are both linearly irrelevant in the
renormalization group sense by power counting. The anisotropy, $v_F \gg v_\Delta$, is marginally irrelevant [80, 81] too and may be relaxed for the low-energy description.

However, this degeneracy is in reality removed by the symmetry breaking perturbations, the most prominent of which have already been outlined. For example, it was shown in Ref. 82 that the short-range repulsion between quasiparticles favours the SDW over the $d + ip$ insulator and, moreover, enhances the SDW order deeper in the insulating state.

In the remainder of this chapter, we study the effects of weak velocity anisotropy, short-range repulsive interaction, and nonlinear terms on the pattern of chiral symmetry breaking in the QED$_3$ theory of the phase fluctuating $d$-wave superconductors. To do so, we assume that superconductivity is already lost due to phase fluctuations and take the whole effect of the gauge field to be the breaking of the chiral symmetry, thus generating a mass term, $m\bar{\psi}\psi$, in the action. We will drop the gauge field from our calculations except for this effect.

All of these perturbations are irrelevant at low energies. We show that their effects on the energies of various states with weakly broken chiral symmetry can be ordered according to their engineering dimensionality: velocity anisotropy, being only marginally irrelevant, provides then the dominant perturbation. Weak repulsion and the second order derivatives are both equally (linearly) irrelevant, but to the first order it is only the repulsion that affects the energies of the insulating states.

### 3.3 Anisotropy

We set $U = 0$ to study the effect of weak anisotropy $(\lambda_F, \lambda_\Delta \ll 1)$ represented by $S_A$ in Eq. (2.16), on the energy of the degenerate ground states of the chiral symmetry broken, isotropic QED$_3$.

Since the action $S_{\text{DCS}}$ is symmetric under the exchange $v_F \leftrightarrow v_\Delta$, the energies of various states should be invariant under the same exchange. Moreover, if $v_F = v_\Delta$, both velocities can be rescaled out of the problem. The energy splittings between the states must therefore be proportional to $(\lambda_F - \lambda_\Delta)^2$ to the lowest order.

Indeed, if the three dimensional volume is defined as

$$V = \int d\tau d^2 r,$$
the first order correction of the energies per volume due to the velocity anisotropy is

\[
\Delta E_A^{(1)} = \frac{1}{V} \langle S_A \rangle_0
\]

\[
= \frac{1}{V} \int \frac{d^3q}{(2\pi)^3} \langle \tilde{\psi} \{iq_x B_1 + iq_y B_2 \} \psi \rangle_0
\]

\[
= -i \int \frac{d^3q}{(2\pi)^3} \text{tr} [(q_x B_1 + q_y B_2)G_0(q)]
\]

\[
= -m^3 \sum_i \text{tr} (B_i \Gamma_i) I_0
\]

\[
= 8m^3(\lambda_F + \lambda_\Delta) I_0,
\]

independent of the choice of \( \Gamma_x \). We have defined in Eq. (3.16) the spinon propagator,

\[
(\tilde{\psi}(k)\psi(q))_0 \equiv (2\pi)^3 \delta(k - q)G_0(q) = (2\pi)^3 \delta(k - q) \frac{-i\Gamma_\mu q_\mu + \Sigma(q)}{q^2 + \Sigma^2(q)},
\]

where \( \Sigma(q) \) is the self-energy of the spinons. Since we are interested in the chiral symmetry-broken phase, we have assumed a finite "mass," \( m \equiv \Sigma(0) \). \( I_0 \) is a positive, dimensionless integral

\[
I_0 = \frac{1}{3} \int^{\Lambda/m} \frac{d^3x}{(2\pi)^3} \frac{x^2}{\sigma^2(x) + x^2}.
\]

Here, \( x = q/m \) and \( \sigma(x) = \Sigma(x)/m \) are the the rescaled momentum and spinon self-energy, respectively. \( \Lambda \) denotes the upper cut-off. We can see that the result in Eq. (3.18) is independent of the choice of \( \Gamma_x \) by recalling the definition of the matrices \( B_i \) and \( \Gamma_i \) in Eqs. (2.15) and (2.16):

\[
-\text{tr}(B_i \Gamma_i) = -\text{tr}(\Gamma_x \delta A_i \Gamma_i)
\]

\[
= -\text{tr}(\delta A_i \Gamma_i \Gamma_x)
\]

\[
= +\text{tr}(\delta A_i \Gamma_x \Gamma_i)
\]

\[
= +\text{tr}(\delta A_i A_i)
\]

\[
= 8(\lambda_F + \lambda_\Delta),
\]

where we have used the symmetry of trace and the fact that \( \Gamma_x \) and \( \Gamma_i \) anti-commute by construction. Therefore, \( \Delta E_A^{(1)} \), provides only an overall energy shift, the same for all states.
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To the second order, however, we find

\[ \Delta E_A^{(2)} = -\frac{1}{2V} \left( \langle S_A^2 \rangle_0 - \langle S_A \rangle_0^2 \right) \]
\[ = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \sum_{ij} q_{ix}q_{xj} \text{tr} [B_iG_0(q)B_jG_0(q)] \]
\[ = -\frac{1}{2} m^3 \sum_{ij} [\text{tr}(B_iB_j)I_{ij} - \text{tr}(B_i\Gamma_\alpha B_j\Gamma_\beta)I_{ij,\alpha\beta}] . \tag{3.21} \]

Again, we have defined the following dimensionless integrals,

\[ I_{ij} = \int_{\Lambda/m}^{\Lambda} \frac{d^3x}{(2\pi)^3} \frac{\sigma^2(x)x_ix_j}{(\sigma^2(x) + x^2)^2} , \tag{3.22} \]
\[ I_{ij,\alpha\beta} = \int_{\Lambda/m}^{\Lambda} \frac{d^3x}{(2\pi)^3} \frac{x_ix_j\sigma}{{\sigma}^2(x) + x^2)^2} . \tag{3.23} \]

Since the rescaled self-energy \( \sigma(x) \) in the QED3 falls off rapidly for \( x \gg 1 \) [75], we will set \( \Lambda = m \) in \( I_{ij} \) and approximate \( \sigma(x) = 1 \) under the integral. This approximation seems not to be appropriate for the integral \( I_{ij,\alpha\beta} \), which is without \( \sigma(x) \) in the numerator of the integrand. Luckily, however, it multiplies \( \text{tr}(B_i\Gamma_\alpha B_j\Gamma_\beta) \), which is independent of the choice of \( \Gamma_0 \). Therefore, the term with \( I_{ij,\alpha\beta} \) does not contribute to the energy differences, but only to the overall shift of the energies. This will be the generic situation in all further calculations in this chapter. Thus, we will hereafter take \( \Lambda = m \) and \( \sigma(x) = 1 \) in all the integrals of this section.

Similarly calculating the traces appearing in Eq. (3.21) by substituting the defining relations for \( B_i \) and \( \Gamma_i \) and using the forms given before for the specific \( \Gamma_x \), we find that the lowest (second) order effect of the velocity anisotropy is to increase the energy of the CDW relative to the SDW, \( d + ip \) and \( d + is \), which remain degenerate:

\[ \Delta E_{A,\text{CDW}} - \Delta E_{A,\text{other}} = 4m^3(v_F - v_\Delta)^2 I > 0 , \tag{3.24} \]
\[ I = \frac{1}{3} \int_1^1 \frac{d^3x}{(2\pi)^3} \frac{x^2}{(1 + x^2)^2} = \frac{10 - 3\pi}{48\pi^2} . \]

The energy splitting vanishes when \( v_F = v_\Delta \), as expected.

The fact that \( d + ip \) and \( d + is \) insulators have the same energy in the presence of velocity anisotropy can be shown to be generally true to any order of the perturbation theory. To see
this, note from Eqs. (3.12) and (3.13) that $d + ip$ and $d + is$ states are represented by, respectively,

$$
\Gamma_{x}^{dip} = \text{diag}(\sigma_2, -\sigma_2, \sigma_2, -\sigma_2),
$$

(3.25)

$$
\Gamma_{x}^{dis} = \text{diag}(\sigma_2, \sigma_2, \sigma_2, \sigma_2).
$$

(3.26)

When written in the $2 \times 2$ form, the only difference between the two is in the signs of some terms, which always may be changed by a unitary transformation. Put differently, the choice of $\Gamma_0$ enters the energy calculation only through the combination $B_i = \Gamma_0 \delta A_i, i = 1, 2$. Matrices $B_i$, on the other hand, have to appear in even numbers in our calculation, as in Eq. (3.21), otherwise the accompanying integrals will be odd in some momentum component and vanish. Since the $A_i$ are block-diagonal, the sign of the block-diagonal elements in the $B_i$, then, cannot matter: $d + ip$ and $d + is$ states remain degenerate to all orders in weak anisotropy.

### 3.4 Repulsion

Now, we set $v_F = v_\Delta = 1$ to work out the first finite, energy contribution of the short-range repulsion $S_U$ to the degenerate ground states of the isotropic action. It is found that

$$
\Delta E_{U}^{(1)} = \frac{1}{V} \langle S_U \rangle_0
$$

$$
= U \int \frac{d^3r}{(2\pi)^3} \langle \left( (\bar{\psi}(r) B_U \psi(r)) \right)^2 \rangle_0
$$

$$
= U \int \frac{d^3r}{(2\pi)^3} \left[ \text{tr}^2(B_U G_0(r)) - \text{tr}(B_U G_0(r)B_U G_0(r)) \right]
$$

$$
= m^4 U J^2 \left[ \text{tr}^2(B_U) - \text{tr}(B_U^2) \right].
$$

(3.27)

The dimensionless integral is

$$
J = \int_1^1 \frac{d^3x}{(2\pi)^3} \frac{1}{1 + x^2} = \frac{4 - \pi}{8\pi^2}.
$$

The first term in Eq. (3.27) vanishes identically for all states; the second term also vanishes for the CDW; but, it increases the energy of the $d + ip$ and the $d + is$, while decreasing the
energy of the SDW:

\[ \Delta E_{U,d+ip}^{(1)} = \Delta E_{U,d+is}^{(1)} = +8m^4UJ^2, \]  

(3.28)

\[ \Delta E_{U,SDW}^{(1)} = -8m^4UJ^2, \]  

(3.29)

\[ \Delta E_{U,CDW}^{(1)} = 0. \]  

(3.30)

Note that \( d + ip \) and \( d + is \) remain degenerate in the presence of repulsive interactions, as well.

Here we only considered the first-order contribution of the repulsive interaction to the free energy. There may be other contributions, in the second order, from the cross terms involving just a single power of \( S_U \). These terms are examined in Section 3.6 and found not to affect the energy levels.

### 3.5 Non-linear terms

We may expand \( i M(q) \) beyond the linear terms that led to the anisotropic action, Eqs. (2.15) and (2.16). For instance, if we define

\[ M_\xi = I \otimes \sigma_3 = i\gamma_2\gamma_3, \]

(3.31)

\[ M_\Delta = I \otimes \sigma_1 = -i\gamma_1\gamma_3, \]

(3.32)

the second-derivative term is given by

\[ S_{NL} = -\int d\tau d^2 r \psi_\dagger \{ M_\xi \xi''(\partial/\partial r) + M_\Delta \Delta''(\partial/\partial r) \} \psi + \{ I \rightarrow II, x \leftrightarrow y \} + \ldots, \]

(3.33)

where \( \xi''(z_1, z_2) = \sum_{ij} z_i z_j \frac{\partial^2}{\partial x_i \partial x_j} \xi(x_1, x_2) |_{x=0} \) and similarly for \( \Delta'' \). The ellipsis indicates higher-than-second-derivative terms. We have dropped the phase derivatives (gauge field) that would arise from performing the Franz-Tešanović transformation on the original electronic operators to map them onto the \( \psi \)-spinor. These derivatives will be irrelevant in the renormalization group sense and, as we have assumed the whole effect of the phase fluctuations to be captured by the dynamical mass \( m \) of the spinons, we neglect their other dynamical effects.

It seems, then, that we need to specify the dispersion relation and the gap function to determine the effect of the higher-order derivative terms on the energies of the degenerate
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ground states. Interestingly, it turns out that to the first non-vanishing (second) order of perturbation, the qualitative effect of these terms is to raise the energy of the SDW, lower that of \( d + ip \) and \( d + is \), and keep the energy of the CDW unchanged, independently of the functional form of the dispersion and the gap function.

For definiteness, we present here the results for the tight-binding model on a square lattice and defer the proof of the general statement to Appendix A. The dispersion and the \( d \)-wave gap are then given by

\[
\begin{align*}
\xi(k_1, k_2) &= -2t(\cos k_1 \ell + \cos k_2 \ell) + 4t \cos(k_F \ell) \\
\Delta(k_1, k_2) &= \frac{1}{2} \Delta_0 (\cos k_1 \ell - \cos k_2 \ell). 
\end{align*}
\]

(3.34)

The nearest-neighbor hopping matrix element is denoted by \( t \); \( \ell \) is the lattice spacing; \( k_F \) is the Fermi wave vector at the node, \( K = (k_F, k_F) \); and \( \Delta_0 \) is the amplitude of the \( d \)-wave superconducting order parameter. We find,

\[
v_F = 2\sqrt{2} t \sin(k_F \ell); \quad v_\Delta = \frac{\sqrt{2}}{2} \Delta_0 \sin(k_F \ell).
\]

In this case, the components defined in Appendix A are

\[
\begin{align*}
\xi_{+-}(q) &= 0, \\
\xi_{++}(q) &= \frac{\cot(k_F \ell)}{\sqrt{2}} v_F \left[ \cos \left( \frac{q_1 \ell + q_2 \ell}{\sqrt{2}} \right) + \cos \left( \frac{q_1 \ell - q_2 \ell}{\sqrt{2}} \right) - 2 \right], \\
\Delta_{++}(q) &= \frac{\cot(k_F \ell)}{\sqrt{2}} v_\Delta \left[ \cos \left( \frac{q_1 \ell + q_2 \ell}{\sqrt{2}} \right) - \cos \left( \frac{q_1 \ell - q_2 \ell}{\sqrt{2}} \right) \right].
\end{align*}
\]

(3.35) (3.36) (3.37)

Then, from Eqs. (A.11) and (A.12), the result is

\[
\begin{align*}
\Delta E_{NL, d+ip}^{(2)} &= \Delta E_{NL, d+is}^{(2)} = -8m^3(m\ell)^2 L, \\
\Delta E_{NL, SDW}^{(2)} &= +8m^3(m\ell)^2 L, \\
\Delta E_{NL, CDW}^{(2)} &= 0.
\end{align*}
\]

(3.38) (3.39) (3.40)

\( L \) is a positive, dimensionless integral, given by

\[
L = \frac{\cot^2(k_F \ell)}{4(m\ell)^4} \int_1^1 \frac{d^2x}{(2\pi)^2} \frac{1}{(1 + x^2)^2} \left\{ [v_F(\cos(m\ell x_1) + \cos(m\ell x_2) - 2)]^2 \\
+ [v_\Delta(\cos(m\ell x_1) - \cos(m\ell x_2))]^2 \right\},
\]

(3.41)
where \( x_{1,2} = (q_x \pm q_y)/\sqrt{2}m \). Assuming that near the transition, \( m\ell \ll 1 \), to zeroth order in \( m\ell \) we find

\[
L = \frac{1}{16} \cot^2(k_F\ell) \int \frac{d^3x}{(2\pi)^3} \frac{1}{(1 + x^2)^2} \left[ v_F^2(x_1^2 + x_2^2)^2 + v_A^2(x_1^2 - x_2^2)^2 \right] + O(m^2\ell^2)
\]
\[
= \frac{15\pi - 46}{2880} \cot^2(k_F\ell)(2v_F^2 + v_A^2) + O(m^2\ell^2).
\]

### 3.6 Cross terms

Looking back at the contributions to the free energy from each of the perturbations considered here, we see that \( \Delta E_A \sim m^3 \), \( \Delta E_U \sim m^4 \), and \( \Delta E_{NL} \sim m^5 \). Since we are performing a second-order perturbation theory, we must also consider the cross terms arising from these perturbations to \( O(m^5) \). There are three such cross terms. We will show in this section that they do not contribute to the energy splittings.

First, we will consider the \( SU-SA \) cross term. We expect this to be \( O(m^4) \). Defining the connected average,

\[
\langle S_US_A \rangle_{\text{oc}} = \langle S_US_A \rangle_0 - \langle S_U \rangle_0 \langle S_A \rangle_0
\]

We have,

\[
\langle S_US_A \rangle_{\text{oc}} = \int d^3rd^3r' \sum_i \left\langle \left( \bar{\psi}(r)B_U\psi(r') \right)^2 \bar{\psi}(r')\frac{\partial}{\partial r_i}\psi(r') \right\rangle_{\text{oc}}
\]
\[
= -2 \int d^3rd^3r' \sum_i \text{tr} \left[ B_UG_0(0)B_UG_0(r-r')B_i\frac{\partial}{\partial r_i}G_0(r'-r) \right]
\]
\[
= 2Vm^4IJ \sum_i \text{tr} \left[ B_U^2(\Gamma_iB_i + B_i\Gamma_i) \right]
\]
\[
= 0.
\]

Integrals \( I \) and \( J \) are the same as those defined in Eqs. (3.25) and (3.28), respectively. The last line follows because, interestingly, these seemingly nontrivial traces vanish identically for all the states considered here.

Second, we turn to the \( S_{NL}-S_A \) cross term, which must be \( O(m^4) \), as well. Similarly to
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We do not need the form of the integrals $K^{(1)}$ and $K^{(2)}$ since, upon inspection, all the traces appearing in the final expression vanish for the main states of the chiral manifold.

The vanishing of the traces extends to the case of the $S_U-S_{NL}$ cross term, which is $O(m^5)$. We only state the result of the calculation, which is similar to that for the previous cases:

$$\langle S_{NL} S_A \rangle_{0c} = -2 V m^5 U J \left\{ v_F K^{(3)}_{\mu} \text{tr} \left( B_0^2 \Gamma_\mu B_\xi \Gamma_\mu \right) - v_F K^{(4)} \text{tr} \left( B_0^2 B_\xi \right) + v_\Delta K^{(6)} \text{tr} \left[ B_0^2 \Gamma_1 \left( B_0 B_\xi + B_\xi B_0 \right) \right] \right\}$$

$$= 0.$$  \hspace{1cm} (3.45)

### 3.7 Antiferromagnetic ground state

Collecting our results together, we find that the shift in energy of a given state may be written as

$$\Delta E = 8 m^3 \left\{ \frac{1}{2} \theta_A (v_F - v_\Delta)^2 I + \theta_U (m U) J^2 + \theta_{NL} (m l)^2 L \right\}.$$  \hspace{1cm} (3.46)

Each $\theta_{A,U,NL} = 0$ or $\pm 1$, depending on the change in energy of the state in consideration. In the approximation we employed for the self-energy of the spinons, namely, that $\Sigma(p) \approx 0$ for $p > m$ and thus $\Lambda = m$, all three integrals $I$, $J$ and $L$ are mass independent positive constants. Therefore, Eq. (3.46) represents an expansion of the energies of the insulating states in powers of the dynamically generated mass, $m$. 

\[
\langle S_{NL} S_A \rangle_{0c} = \int d^3 r \int d^3 r' \sum_i \left\langle \bar{\psi}(r) B_i \frac{\partial}{\partial r_i} \psi(r') \left[ v_F B_\xi (\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}) + 2 v_\Delta B_\Delta \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \right] \psi(r') \right\rangle_{0c} \\
= V \int d^3 r \sum_i \text{tr} \left[ B_i \frac{\partial}{\partial r_i} G_0(r) \left( v_F B_\xi (\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}) + 2 v_\Delta B_\Delta \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) G_0(r) \right] \\
= -V m^4 \sum_{i\mu} \left\{ K_{i\mu}^{(1)} \text{tr} \left[ B_i (\Gamma_\mu B_\xi + B_\xi \Gamma_\mu) \right] + K_{i\mu}^{(2)} \text{tr} \left[ B_i (\Gamma_\mu B_\Delta + B_\Delta \Gamma_\mu) \right] \right\} \\
= 0. \hspace{1cm} (3.44)
\]
According to Eq. (3.46), there is a hierarchy of the perturbation terms for weak chiral symmetry breaking, in which each symmetry-breaking perturbation assumes a place according to the degree of its irrelevancy. The velocity anisotropy, being the marginal perturbation at the bare level, provides the dominant contribution, while the repulsive interaction, is the subdominant one. Nonlinear terms arising from second order derivatives, although equally irrelevant as the repulsion, contribute to the energies of the insulating states only to the second order and, therefore, are the least important perturbation. As a result of this hierarchy, the degeneracy between different insulators is broken into ever finer structures. Thus, we obtain a “fine structure” of energies due to each perturbation, which is depicted schematically in Fig. 3.2.

The main conclusion of our discussion in this chapter is that, to the leading order, the degeneracy of the chiral manifold is broken in favor of the incommensurate antiferromagnetic insulator represented by the SDW state. It is the lowest-energy state when the weak perturbations to the maximally chirally symmetric QED$_3$ action are taken into account. The translationally symmetric $d + ip$ and $d + is$ insulators remain degenerate, as one would expect, since the translational symmetry remains intact even in the presence of all three perturbations.
Assuming that the $d$-wave superconductor-insulator transition is continuous or, possibly, weakly first order implies then that the insulating state is the SDW. Although in real systems none of the perturbations considered here is truly weak, we believe our result provides a useful qualitative guide. In particular, it agrees with the standard picture of underdoped cuprates, upon identification of the SDW insulator as being continuously connected with the Mott insulating antiferromagnet near half filling. If the transition is strongly first order, the chiral mass $m$ increases and it is conceivable that there could be some level crossings in our fine structure of the chiral manifold.
Chapter 4

Confinement of Spinons

In the following two chapters we will focus on the problem of confinement of spinons outside the superconductor. We assume that the gauge field $a$ belongs to a compact $U(1)$ gauge group and, thus, allows for monopole configurations. We briefly discuss the basis of this assumption in the following section.

The problem is best studied in a dual sine-Gordon theory, or equivalently a monopole gas. We present the dual mapping in Section 4.3. In the absence of fermions the interaction between monopoles is Coulomb. Thus, monopoles form a free plasma. Polyakov showed [83] many years ago that as a result, static fermionic charge is confined for all values of the gauge coupling by a linear potential mediated by free monopoles. We reproduce Polyakov's argument using our formulation in Sections 4.4 and 5.1.

In the rest of these two chapters, we argue that Polyakov's permanent confinement survives in the presence of dynamical fermionic matter. We first support our claim, for relativistic fermions, by an elementary electrostatic study of the monopole gas in Section 4.5. This is then backed up by a controlled renormalization group analysis on the equivalent sine-Gordon theory in Section 4.6. In Appendix B we provide the details for the computation of screening in various phases of the monopole gas. These results were first published in Ref. 51. We then extend these findings to the non-relativistic case, with a spinon Fermi surface, in Section 4.8 and Appendix C. In the next chapter, we take a variational approach to the problem, which lends further support to our claim above, in agreement with our other studies. This part appeared in Ref. 52.
4.1 Compact gauge field

The question of confinement of spinons takes a central position when the gauge group is compact, such as a compact $U(1)$ or $SU(2)$. In this case, $a$ is called a compact gauge field. For the compact $U(1)$ group, the gauge field $a$ takes on the significance of an angular variable, and the gauge field dynamics remain invariant upon shifting $a \rightarrow a + 2\pi n$. Then, the associated "magnetic field," $\nabla \times a$, admits magnetic monopoles \[84, 85, 86\], whose density is given by,

$$\nabla \cdot \nabla \times a = 2\pi \rho.$$  \hspace{1cm} (4.1)

Looking back at the derivation of the gauge field dynamics in our QED$_3$ theory, we see from Eq. (2.35) that the action is invariant under such integer shifts of $a$. This is not the case for the other gauge field, $v$, which is gapped. This may suggest that $a$ is a compact gauge field. However, things are a bit more subtle. The dynamics of $a$ derives from the vortices of the 3DXY model, in which they form closed loops. This means that the magnetic flux carried by $a$ cannot terminate at a point, and hence there are no monopoles. More precisely, we see from the current representation of the 3DXY model on the lattice, Eq. (2.31), that $\Delta \cdot m = 0$. Since dividing the vortices into groups $A$ and $B$ should not affect this property locally, it is natural to assume that this constraint is inherited by $m_A$ and $m_B$ as well. Then from the definitions (2.32–2.33), we see that

$$\Delta \cdot \Delta \times a = \Delta \cdot \Delta \times v = 0.$$

Thus, $a$ and $v$ both appear to be non-compact.

A compact gauge field appears naturally in other theoretical descriptions of high temperature superconductivity, e.g. gauge theories of the $t$-$J$ model, and spin liquid theories. It is also conceivable that starting from a different microscopic theory for phase fluctuations would allow for monopole configurations in the same 3DXY universality class of the superconductor-insulator transition. On the superconducting side, $a$ would be gapped and therefore, monopoles are irrelevant for the transition. The effect of the monopoles would then only be important for the fate of spinons outside the superconductor.

In the rest of this chapter, we will consider a compact QED$_3$ (cQED$_3$) characterized by a compact $U(1)$ gauge group. Somewhat more involved theories with a $SU(2)$ gauge group have been considered in the literature \[16, 87\]. The $SU(2)$ gauge theory also allows
monopole configurations similar to the $U(1)$ case, with similar interactions between them, and so we expect our results to apply there as well.

We take the dynamics of the gauge field, $\theta$, on the three-dimensional lattice to be given by,

$$S_g[\theta] = -\frac{1}{g_e} \sum_r \cos(\Delta \times \theta_r). \quad (4.2)$$

The sites of the lattice are indexed by $r = (\tau, x, y)$ and $1/g_e$ is the gauge field coupling.

One way to implement fermionic degrees of freedom, in our case spinons, on the lattice, is the so-called staggered fermion formulation. With $N_f$ (even) flavours of four-component spinors, $\chi_\alpha$, it takes the following form [88]

$$S_f[\chi, \theta] = \frac{1}{2} \sum_{r, \hat{\mu}} \sum_{\alpha=1}^{N_f/2} \eta_{r, \hat{\mu}} \left[ \bar{\chi}_{r, \alpha} e^{i\theta_{r, \hat{\mu}}} \chi_{r+\hat{\mu}, \alpha} - \bar{\chi}_{r+\hat{\mu}, \alpha} e^{-i\theta_{r, \hat{\mu}}} \chi_{r, \alpha} \right], \quad (4.3)$$

where $\hat{\mu} = (\hat{\tau}, \hat{x}, \hat{y})$ are the three nearest-neighbour directions of the square lattice, and the factors $\eta_{r, \hat{\tau}} = 1$, $\eta_{r, \hat{x}} = (-1)^x$ and $\eta_{r, \hat{y}} = (-1)^{x+y}$. The complete action is then

$$S_{cQED}[\chi, \theta] = S_f[\chi, \theta] + S_g[\theta]. \quad (4.4)$$

In the continuum limit,

$$S_g = \int d^3 r \frac{1}{2e^2} \left( \nabla \times a \right)^2,$$

where, $a = \ell^{-1} \theta$ is the continuum gauge field, and $e^2 = \ell^{-1} g_e$ the continuum charge. $\ell$ is the lattice constant. The continuum limit is found by keeping $\alpha$ and $e$ constant while sending $\ell \to 0$. So, $g_e = 0$ should naively correspond to the continuum QED$_3$ in our analysis.

### 4.2 Coupling to spinons

In order to proceed with our study of monopole effects in cQED$_3$, we will integrate out the gapless spinon fermionic fields. This is a common step in most studies of compact gauge theories coupled to matter fields [89, 90]. As pointed out already in Eq. (2.52), in the continuum limit and for large $N_f$, the inverse gauge field propagator is renormalized by spinons to acquire a linear piece in momentum space. Outside the superconductor, where
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the gauge field is massless this piece dominates the infrared regime. We take this behaviour in the continuum as a guide for the form of the action on the lattice upon integrating out the spinon fields, $\chi_\alpha$.

Combining the above with a Villain approximation to Eq. (4.2), we may then write

$$S_{\text{cQED}} \approx \frac{1}{2} \sum_{rr',\mu} (\Delta \times \theta_r - 2\pi n_r)_{\mu} u_{rr'} (\Delta \times \theta_{r'} - 2\pi n_{r'})_{\mu},$$  \hspace{2cm} (4.5)

with

$$u_{rr'} = \left( \frac{1}{g_e} + \frac{N_f}{8|\Delta_r|} \right) \delta_{rr'}.$$  \hspace{2cm} (4.6)

The term proportional to $N_f$ is the lattice version of the continuum renormalization of the gauge field propagator mentioned above. $1/|\Delta_r|$ should be understood as an inverse of the square root of the lattice gradient squared. The integer field $n$ is introduced by the Villain approximation and represents the monopole configuration in the “magnetic field,” $\Delta \times \theta$; so, by Eq. (4.1) we may write

$$\Delta \cdot n = \rho.$$  \hspace{2cm} (4.7)

The action in Eq. (4.5) is a quadratic approximation to the the original cQED$_3$ action (4.4). For our purposes in this thesis, we will take them to be in the same universality class. As noted above, for $N_f \neq 0$ and at large distances, the gauge coupling $1/g_e$ is irrelevant by power counting.

4.3 Duality mapping

In this section, we apply standard duality transformations to the action in Eq. (4.5) to derive equivalent actions that allow a more systematic study of the monopole effects. More specifically, we derive first a description of the monopole system in terms of a “monopole gas” with pairwise interactions determined by the kernel $u_{rr'}$ in (4.6), suitable for our electrostatic treatment in Section 4.5. We then derive an equivalent continuum sine-Gordon action, which lets us perform a controlled momentum-shell renormalization group analysis of the phases of the monopole system.
To this end, we decouple the quadratic form of Eq. (4.5) by introducing a Hubbard-Stratonovich field $b_{r,\mu}$,

$$S_{\text{cQED}} = \frac{1}{2} \sum_{rr',\mu} b_{r,\mu} u_{rr'}^{-1} b_{r',\mu} + i \sum_r (\Delta \times \theta_r - 2\pi n_r) \cdot b_r. \quad (4.8)$$

Integrating over the gauge field $\theta$ constrains the $b$-field to be curl-free, so we can take it to be a gradient on the lattice $b = \Delta \varphi$. Performing the lattice version of integration by parts and integrating over $\varphi$ yields

$$S_{\text{cQED}} \to \frac{1}{2} \sum_{rr'} \Delta_{\mu} \varphi_r u_{rr'}^{-1} \Delta_{\mu} \varphi_{r'} + 2\pi i \sum_r \varphi_r \Delta \cdot n_r \quad (4.9)$$

$$\to \frac{1}{2} \sum_{rr'} \rho_r v(r, r') \rho_{r'}, \quad (4.10)$$

where

$$v^{-1}(r - r') = -\frac{1}{4\pi^2} \Delta_{\mu} u_{rr'}^{-1} \Delta_{\nu}, \quad (4.11)$$

is the inverse of the potential and $\rho = \Delta_{\mu} n_\mu$ is the density of magnetic monopoles as in Eq. (4.7). Thus, for a system of $N$ monopoles with a density $\rho_r = \sum_{a=1}^N g_a \delta(r - r_a)$, we obtain

$$S_{\text{cQED}} \to S_{\text{mon}} = \frac{1}{2} \sum_{a,b} g_a g_b v(r_a - r_b). \quad (4.12)$$

This is our "monopole gas" system advertised in the beginning of the section, with the pairwise interaction $v(r - r')$.

The pairwise interaction takes a simple form in two limits:

1. $N_f = 0$ (pure gauge cQED$_3$). In this limit,

$$v(r - r') = \frac{4\pi^2}{g_e} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i k \cdot (r - r')}}{k^2} \equiv \frac{\pi}{g_e} V_C(r - r'), \quad (4.13)$$

$$V_C(r) = \frac{1}{|r|}. \quad (4.14)$$

That is, we find a Coulomb potential and a Coulomb gas of monopoles. We note that the effective temperature of the gas is given by $\tilde{T} = g_e / \pi$. 

2. $N_f \neq 0$ and low-energies. In this limit, as discussed before, we neglect the gauge coupling $1/g_c$ and find

$$v(r - r') = \frac{\pi^2 N_f}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(r-r')}}{|k|^3} \equiv \frac{N_f}{4} V_L(r - r'),$$

(4.15)

$$V_L(r) = -\ln(\Lambda|r|) + \text{const.}$$

(4.16)

$\Lambda \approx 1/\ell$ is the ultraviolet cut-off necessary to regularize the integral.\(^1\) Thus, we find a logarithmic pairwise potential for the monopole gas in this limit. Again, we note that there is an effective temperature for the monopole gas, given by $\tilde{T} = 4/N_f$.

We now proceed to show that (4.12) with $q_a = \pm 1$ is equivalent to a sine-Gordon action. Higher charges are irrelevant for small enough $\tilde{T}$ (large enough $N_f$). Let us introduce the bare action

$$S_b[\phi] = \frac{1}{2} \int d^3r d^3r' \phi(r) v^{-1}(r - r') \phi(r'),$$

(4.17)

so that $\langle \phi(r) \phi(r') \rangle_b = v(r - r')$. Showing the fugacity of the monopoles by $y = e^{E_c/\tilde{T}}$, where $E_c$ is the chemical potential for adding a monopole to the system, we may write the the grand-canonical partition function of the monopole gas as

$$Z_{\text{mon}} = \sum_N \frac{y^N}{N!} \int \prod_{a=1}^N d^3r_a \sum_{[q_a = \pm 1]} \exp \left\{ -\frac{1}{2} \sum_{a,b} q_a q_b \langle \phi(r_a) \phi(r_b) \rangle_b \right\}$$

$$= \sum_N \frac{y^N}{N!} \int \prod_{a} d^3r_a \sum_{[q_a = \pm 1]} \exp \left\{ i \sum_a q_a \phi(r_a) \right\}_b$$

(4.18)

$$= \left\langle \exp \left\{ 2y \int d^3r \cos \phi(r) \right\} \right\>_b$$

$$= Z_b^{-1} Z_{sG},$$

(4.19)

where $Z_b = \int d[\phi] e^{-S_b[\phi]}$ is an unimportant factor and $Z_{sG} = \int d[\phi] e^{-S_{sG}[\phi]}$ is the partition function for the sine-Gordon action,

$$S_{sG} = \frac{1}{2} \int d^3r d^3r' \phi(r) v^{-1}(r - r') \phi(r') - 2y \int d^3r \cos \phi(r),$$

(4.20)

\(^1\)We also need to introduce an infrared cut-off, which is buried in the constant term. See Appendix B, Eqs. (B.5)-(B.7)
advertised above. From Eq. (4.18), we can see that the physical significance of the sine-Gordon field $\phi$ is provided by the relation

$$\langle \phi(r) \rangle = i \sum_a q_a \langle \nu(r - r_a) \rangle_{\text{mon}},$$

which is, up to a factor $i$, the average potential of the monopole gas.

### 4.4 Polyakov’s permanent confinement

In the absence of dynamical fermion fields, $N_f = 0$, also known as the pure gauge cQED$_3$, Polyakov showed [83] that the charge carried by the fermions is confined into dipoles for all values of the gauge coupling $1/g_e$ by a linear potential. This permanent confinement of “electric” charge is in direct relationship with the plasma phase of magnetic monopoles. The linearly confining potential is mediated by free monopoles between electric test charges.

The dual correspondence between the phase of the monopole gas and the fermion charge is expounded using a gauge-invariant quantity called the Wilson loop. It is defined over a loop $C$ as

$$W_C = \left\langle \exp \left\{ i \oint_C a_\mu \, d\tau^\mu \right\} \right\rangle. \quad (4.22)$$

If we take the loop $C$ to have a side, $T$, along the (imaginary) time axis and a spatial side $R$, then $W_C$ is related to the probability amplitude of creating a pair of static (non-dynamical) “test charges” at a distance $R$ and destroying them (by pair annihilation) at a time $T$ later in the presence of the magnetic field generated by the gauge field $a$. We expect that, for large enough $C$, the Wilson loop is

$$W_C \approx e^{-TE(R)}, \quad (4.23)$$

where $E(R)$ is the potential energy between the static pair. On the other hand, over large loops, one may write

$$W_C \approx e^{i\Phi_C}. \quad (4.24)$$

$\Phi_C$ is the flux of the monopoles through $C$. Polyakov showed that for $N_f = 0$, this flux is,

$$\Phi_C \propto i \sqrt{g_R} \times \text{Area of } C, \quad (4.25)$$
where $y_R$ is the renormalized fugacity of the monopoles. This relation is called the "area law."

When the monopoles are free and in a plasma phase, there is a finite renormalized fugacity, $y_R$, and we find, from Eqs. (4.23)-(4.25), the linearly confining potential

$$E(R) \propto R.$$ 

If the monopoles are bound in a dipole phase, $y_R = 0$ and the fermion charge would be deconfined. Based on this duality, we can obtain Polyakov's permanent confinement, by noting that the Coulomb gas of monopoles, found in Eq. (4.13), is always in the plasma phase with a finite $y_R$. We will give a proof of this statement in the renormalization group treatment, Section 4.6, and also in the variational approach presented in the next chapter, Section 5.1

In the rest of this and the following chapter, we will study the problem of confinement for $N_f \neq 0$ described by the logarithmic monopole gas of Eq. (4.15). We will, however, assume that the above duality between the fermion charge and the monopole gas remains valid, and focus on the phase structure of the latter.

### 4.5 Dipole screening

In the logarithmic monopole gas at the effective temperature $\tilde{T} = 4/N_f$, the excess of free energy due to a single isolated monopole in the sample of a linear size $L$ is

$$\Delta F = \frac{1}{2} \ln L - \tilde{T} \ln L^3,$$

so that for $\tilde{T} < 1/6$ monopoles would be expected to be bound in pairs. This corresponds to a "critical" number of fermion components $N_f = 24$ found in the literature before [91, 92, 90, 93]. So it appears that there is a confinement-deconfinement transition similar to the two-dimensional Kosterlitz-Thouless (KT) transition, due to the competition between the logarithmic potential and the entropy [61].

However, this argument neglects the effect of other dipoles. In the standard KT transition the presence of other dipoles does not change the critical temperature obtained by the simple energy-entropy argument, since the KT fixed point lies at zero fugacity [60].
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The effect of other dipoles is simply to renormalize the dielectric constant. The situation for a logarithmic potential in three dimensions is, on the other hand, completely different. The screening of the potential in the dipole phase is not simply a renormalization of the dielectric constant, but instead, a change of the form of the potential. In the following we will demonstrate this statement by an elementary electrostatic argument, before turning to a more detailed renormalization group study in the next section.

Let us assume a distribution of monopoles \( \rho(r) \) interacting via the logarithmic interaction \( V(r) = -\ln |r| \) in 3D, located in a region of a finite size \( R \). At a distance \( r \gg R \) we can write the interaction as

\[
V(r) = \int \! d^3 r' \rho(r') \left( -\ln |r| + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \ldots \right) \tag{4.27}
\]

in the spirit of multipole expansion [94, Chapter 4]. Thus, for a medium with a monopole density \( \rho(r) \) and a dipole moment density \( P(r) = r \rho(r) \), the potential is given by

\[
V(r) = \int \! d^3 r' [\rho(r') + \nabla' \cdot P(r') + \ldots] (-\ln |r - r'|). \tag{4.28}
\]

In a weak uniform external field \( E \), the energy of a dipole of size \( \mathbf{r} \) is \(-E \cdot \mathbf{r} - 2E_c\), where \( E_c \) is the chemical potential for creating a single monopole. So, the thermal-averaged induced dipole moment can be expanded as

\[
\langle P \rangle = y^2 \int \! d^3 r \frac{r e^{E r / T}}{\int \! d^3 r e^{E r / T}} \quad \approx \frac{y^2 \langle r^2 \rangle}{3T} E + O(E^2). \tag{4.29}
\]

Thus the electric susceptibility is given by \( \chi = y^2 \langle r^2 \rangle / 3T \).

Writing Eq. (4.28) in momentum space, and combining Eq. (4.30) with \( E = \nabla V \) as \( P(q) = -i \chi q V(q) \), we obtain the interaction due to an external charge \( Q \) in the presence of a finite density of dipoles, in momentum space,

\[
V(q) = \frac{Q}{|q|^3 + \chi q^2}. \tag{4.31}
\]

In real space,

\[
V(r) = \frac{Q}{4\pi \chi |r|} + O\left( \frac{1}{r^2} \right). \tag{4.32}
\]
Therefore, dipole screening renormalizes the form of the potential from logarithmic to Coulomb. This implies that in the presence of other dipoles the energy term in Eq. (4.26) scales as $\sim 1/L$, so that the entropy term always dominates. Thus, the largest dipoles should dissociate into free monopoles. The dielectric phase of logarithmically interacting monopoles in 3D is thus, unlike in 2D, destabilized by dipole screening, as anticipated in Ref. 95. A similar conclusion has also been drawn before in the study of instantons in quantum antiferromagnets [96].

4.6 Renormalization group study

The renormalization group (RG) treatment is applied to the equivalent sine-Gordon theory formulated by Eq. (4.20). It closely parallels the electrostatic considerations in the previous section. It is set up as a momentum-shell RG, in which the modes of $\phi$ in Eq. (4.20) with momenta $\Lambda/b < |q| < \Lambda$ (fast modes) are integrated out. $\Lambda$ is the ultraviolet cut-off and $b$ is the RG parameter.

From Eq. (4.15) we write the potential $v^{-1}$ in momentum space as,

$$v^{-1}(q) = T|q|^\sigma + aq^2,$$  

where $T = \tilde{T}/2\pi^2$ and $\tilde{T}$ is the effective temperature. This form covers both the pure gauge cQED$_3$ and the full cQED$_3$. As discussed in Section 4.3, $\sigma = 2$, $\tilde{T} = g_e/\pi$ (and $a = 0$) in the former, and $\sigma = 3$, $\tilde{T} = 4/N_f$ in the latter. We will also perform our calculations in a general dimension $D$. The quadratic term is absent initially, but as we will see, even for the full cQED$_3$, it gets generated to the second-order in $y$. Formally this is just what one expects, since there is no particular symmetry in the Eq. (4.20) that would prohibit its appearance. Physically, as we have seen, this simply expresses the screening effect of smaller dipoles onto the interaction between the monopoles in a larger dipole. This term was effectively overlooked in the previous work on cQED$_3$ [91, 90, 93, 89, 92], where the authors consequently arrived at a different final result. This is the central point of our RG analysis.

In order to integrate out the fast modes, we make a decomposition into slow and fast
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modes,

\[ \phi(r) = \phi_<(r) + \phi_>(r), \quad \text{Eq. (4.34)} \]
\[ \phi_<(r) = \int_{0<|q|<\Lambda/b} \frac{d^Dq}{(2\pi)^D} e^{iqr} \phi(r), \quad \text{Eq. (4.35)} \]
\[ \phi_>(r) = \int_{\Lambda/b<|q|<\Lambda} \frac{d^Dq}{(2\pi)^D} e^{iqr} \phi(r). \quad \text{Eq. (4.36)} \]

Then,

\[ Z_{SG} = \int d[\phi_<]d[\phi_> \exp \left\{ -S_b[\phi_<] - S_b[\phi_> + 2y \int d^D r \cos (\phi_<(r) + \phi_>(r)) \right\} \]
\[ = \int d[\phi_<]e^{-S_b[\phi_<]}Z_> \left\{ \exp \left\{ 2y \int d^D r \cos (\phi_< + \phi_> \right\} \right\}. \quad \text{Eq. (4.37)} \]

\( S_b \) is defined in Eq. (4.17), and the average is with respect to \( S_b[\phi_>] \). The slow and fast modes decouple because \( S_b \) is diagonal in momentum space and there the overlap between the two modes vanishes.

The rest of the computation follows straightforwardly by expanding the exponential in the average up to the second order in \( y \),

\[ \left\langle \exp \left\{ 2y \int d^D r \cos (\phi_< + \phi_>) \right\} \right\rangle_\rangle = 1 + 2y \int d^D r \left\langle \cos (\phi_<(r) + \phi_>(r)) \right\rangle_\rangle \]
\[ + 2y^2 \int d^D r d^D r' \left\langle \cos (\phi_<(r) + \phi_>(r)) \cos (\phi_< (r') + \phi_>(r')) \right\rangle_\rangle + O(y^3), \]

and then re-exponentiate back to find:

\[ \exp \left\{ 2ye^{-\frac{1}{2}G_>(0)} \int d^D r \cos \phi_<(r) + y^2 e^{-G_>(0)} \int d^D r d^D r' \left[ \cos (\phi_<(r) + \phi_< (r')) \right. \right. \]
\[ \left. \left. \times (e^{-G_>(r-r')} - 1) + \cos (\phi_< (r) - \phi_< (r')) (e^{G_>(r-r')} - 1) + O(y^3) \right] \right\}, \]

where the correlation function

\[ G_>(r) = \int_{\Lambda/b<|q|<\Lambda} \frac{d^Dq}{(2\pi)^D} \frac{e^{iqr}}{|q|^\sigma + aq^2}. \quad \text{Eq. (4.38)} \]

Since \( G_> \) is a restricted Fourier transform, with \( \Lambda/b < |q| < \Lambda \), we expect \( G_>(r) \) to quickly drop off to zero for \( |r| \gtrsim /\Lambda \). We will make this claim more precise shortly.
Assuming it to be true for the moment, we see that the factors, $e^{\pm G_\sigma(r-r')} - 1$, are negligible for large $|r-r'|$. So, we may approximate the cosine functions as
\[
\cos (\phi_<(r) + \phi_<(r')) \approx \cos (2\phi_<(r)) \\
\cos (\phi_<(r) - \phi_<(r')) \approx \sum_{\mu\nu} (r-r')_\mu (r-r')_\nu \partial_\mu \phi_<(r) \partial_\nu \phi_<(r) + \text{const.}
\] (4.39) (4.40)

The term, $\cos(2\phi_\sigma)$, generated above is related to monopoles with double the unit charge, which are expected to be relevant only at higher temperatures. We will disregard it for this reason. Therefore, the exponent $\sim y^2$ leads to the renormalization of the quadratic term in the potential as expected. Defining
\[
I(b) = \frac{1}{D} \int d^D r \, r^2 (e^{G_\sigma(r)} - 1),
\] (4.41)
we find that the original couplings are renormalized as
\[
T(b) = b^{D-\sigma} T + O(y^3), \\
y(b) = b^D y e^{-\frac{1}{2}G_\sigma(0)} + O(y^3), \\
a(b) = b^{D-2} [a + y^2 e^{-G_\sigma(0)} I(b)] + O(y^3),
\] (4.42) (4.43) (4.44)

Evaluating $G_\sigma$ requires some care. First, we focus on the case for $\sigma = 3, D = 3$, which describes the full cQED$_3$. Assuming a low cut-off, or effective temperatures $T \ll y/\Lambda^3$, we may rewrite it as
\[
G_\sigma(r) = \frac{1}{T\Lambda + a} \int \frac{d^3 q}{(2\pi)^3} \frac{e^{iqr}}{q^2} \left[ \frac{q^2}{q^2 + (\Lambda/b)^2} - \frac{q^2}{q^2 + \Lambda^2} \right],
\] (4.45)
where the integration is now unconstrained, but we have introduced a smoothing function $q^2/(q^2 + \Lambda^2)$ instead of the sharp cut-off [97]. With this regularization we find
\[
G_\sigma(r) = \frac{1}{4\pi T + a/\Lambda} \ln(b) - e^{-\Lambda r} + O(\ln^2(b)),
\] (4.46)
and
\[
I(b) = \frac{8}{\Lambda^5} \frac{\ln(b)}{T + a/\Lambda}.
\] (4.47)
Eq. (4.46) proves our claim above that $G_\sigma(r)$ decays quickly for $r \gtrsim 1/\Lambda$. 
We can also check that when applied to the standard sine-Gordon problem \((\sigma = 2)\) in \(D = 2\), our smooth cut-off regularization scheme gives the correct anomalous dimension at the KT transition. In this case, we find
\[
G_\gamma(r) = \frac{|r| \ln(b)}{2\pi T/\Lambda} \frac{dK_0(z)}{dz}_{z = \Lambda |r|},
\]
so that \(G_\gamma(0) = \ln(b)/2\pi T\). Using Eq. (4.43), this gives a critical temperature \(T_c = 1/8\pi\) which yields the correct universal anomalous dimension, \(\eta_{KT} = 2\pi T_c = 1/4\) [98].

Finally, we could now put our calculations together, to find
\[
\begin{align*}
\beta_T &= 0 + O(\hat{y}^2), \\
\beta_y &= 3 \left(1 - \frac{T_{\text{cross}}}{T + \hat{a}}\right) \hat{y} + O(\hat{y}^3), \\
\beta_\hat{a} &= \hat{a} + \frac{8}{T + \hat{a}} \hat{y}^2 + O(\hat{y}^3),
\end{align*}
\]
where \(\hat{y} = y/\Lambda^3\) and \(\hat{a} = a/\Lambda\) are the dimensionless couplings, and the RG \(\beta\)-functions are defined, as usual, by
\[
\beta_z = \left. \frac{d\hat{z}(b)}{d\ln(b)} \right|_{b = 1}.
\]
\(T_{\text{cross}} = 1/24\pi\) corresponds to a cross-over value for the number of spinon flavours, \(N_{\text{cross}} = 48/\pi\).

Two important features of the flow equations (4.48)-(4.50) should be noted: (i) the relevant coupling \(a\), even if absent initially, becomes generated at lower cutoffs, and (ii) \(T\) is marginal to \(O(y^3)\). We suspect \(T\) to be an exactly marginal coupling, since the action for slow modes should be analytic in \((|q|/\Lambda)^2\), so the coefficient of the non-analytic \(|q|^3\) term can not get renormalized to any order [99].

These \(\beta\)-functions admit two fixed points in general:
\[
\begin{align*}
\text{FP1} : \quad &\hat{a}_1^* = 0, \quad \hat{y}_1^* = 0; \\
\text{FP2} : \quad &\hat{a}_2^* = T_{\text{cross}} - T, \quad \hat{y}_2^* = -\frac{1}{8} T_{\text{cross}}(T_{\text{cross}} - T).
\end{align*}
\]

As we change \(T \propto 1/N_f\), the character of the flow around these fixed points changes. For \(T > T_{\text{cross}}\), FP1 is unstable, and FP2 is critical. For \(T < T_{\text{cross}}\) FP2 is unstable and FP1 is critical. The flow in the \(\hat{y}^2-\hat{a}\) plane for \(N_f < N_{\text{cross}}\) and \(N_f > N_{\text{cross}}\), is depicted in Fig. 4.1.
Figure 4.1: The flow diagram in the $\hat{y}^2\hat{a}$ plane for (a) $N_f < N_{\text{cross}}$ where fugacity always increases and (b) $N_f > N_{\text{cross}}$ where fugacity first decreases before increasing. $\hat{a}$ is always increasing in the physical region ($\hat{a} > 0$). The nontrivial fixed point in (a) has moved to the lower right quadrant (not shown) in (b).

In our theory, the physical region is restricted to $\hat{a} > 0$ (and $\hat{y}^2 > 0$), i.e. the upper right quadrant in Fig. 4.1. Starting from $\hat{a} = 0, \hat{y} = \hat{y}_0$, the fugacity for $N_f < N_{\text{cross}}$ monotonically increases as $b \to \infty$. For $N_f > N_{\text{cross}}$, however, it begins to increase only for $b > b^*(\hat{y}_0)$ corresponding to the minima located at $\hat{a}_2^* = T_{\text{cross}} - T$.

We may estimate $b^*$ for, say, $T \approx 0$ ($N_f \gg 1$) by writing

$$\beta_a(T \approx 0, \hat{a} \approx 0) = \frac{8\hat{y}_0^2}{\hat{a}} \implies \hat{a}(b \approx 1) = 4\hat{y}_0\sqrt{\ln(b)}.$$ 

On the other hand, for $b \lesssim b^*$, we find $\hat{y}(b) < \hat{y}_0$ and

$$\beta_a(T \approx 0, \hat{a} \approx T_{\text{cross}}) = \hat{a} + O(\hat{y}_0^2/T_{\text{cross}}) \implies \hat{a}(b \lesssim b^*) = \hat{a}_1 b + O(\hat{y}_0^2/T_{\text{cross}}).$$
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We can sew these boundary solutions into the following expression

\[
\hat{a}(b < b^*) \approx 4 \bar{y}_0 \frac{b - 1}{\sqrt{\ln(b)}}.
\]

Finally, setting \( \hat{a}(b^*) = T_{\text{cross}} - T = T_{\text{cm}} \), we find

\[
b^* \approx \frac{T_{\text{cross}}}{4 \bar{y}_0} \sqrt{\ln \frac{T_{\text{cross}}}{4 \bar{y}_0}} \approx \frac{1}{96\pi \bar{y}_0}.
\]

So, the linear size (number of lateral sites) of a finite system in which the cross-over behaviour may be seen scales with \( 1/\bar{y}_0 \).

### 4.7 Phase diagram of cQED\(_3\)

Since \( \bar{y} = 0 \) is an invariant line under the RG flow, and the flow trajectories can not intersect except at a fixed point, fugacity must increase at large enough \( b \) for any initial value. We interpret this upward flow of the fugacity as an indication that monopoles are always in the plasma phase. This agrees with the simple energy-entropy argument once the screening effect of the finite density of dipoles is included.

Interestingly, massless fermions still make themselves felt in such a plasma, since the nonanalytic \( |q|^3 \) term in Eq. (4.33) \( (\sigma = 3) \) translates into a power-law behavior of the screened interaction at long distances. Instead of the expected Debye-Hückel exponential decay in the quadratic approximation to (4.20) we find

\[
V_{\text{scr}}(r) \approx -\frac{24}{\pi^4 \bar{y}^2 N_f |r|^6},
\]

at large \( |r| \). (See Appendix B.) We note a similar phenomenon in metals, the Friedel oscillations, where a non-analyticity due to the Fermi surface produces a power-law behavior of the screened interaction as well. In the present case the sign of the above power-law term indicates over-screening, so that now like charges attract. This is presumably due to the extremely long range of the bare (logarithmic) interaction between monopoles. One should be careful about the origin of the power-law screening: it is the free monopoles, not the dipoles. This could be seen by considering the high-temperature phase of the logarithmic monopole gas, where monopoles are expected to be free and in the plasma phase. The
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Figure 4.2: The proposed phase diagram of $c\text{QED}_3$. A true phase transition between the chirally symmetric (CS) and chiral symmetry broken (CSB) phases of spinons occurs only in the continuum limit ($g_e \to 0$). Monopoles are in the plasma phase at all $g_e > 0$. The dashed line marks a crossover that corresponds to $N_{\text{cross}}(g_e)$, as discussed in Section 4.6.

screening in this case is the same power-law, as derived in Appendix B. In the dipole phase, with no free monopoles, the screening leads to a Coulomb potential at large distances as demonstrated in Section 4.5.

Next, we turn to possible implications of our findings for spinons. First, as pointed out earlier, $g_e \to 0$ should correspond to the continuum QED$_3$. In this theory chiral symmetry for spinons is expected to be broken, and the condensate $\langle \chi \chi \rangle = 0$, for $N < N_f^c$, where $N_f^c$ is of order unity [100, 79, 101].

On the other hand, for $g_e > 0$ we argued that once spinons are integrated out, the theory has no phase transition. If this is correct, it implies that, had we proceeded in reverse and integrated out everything but the spinons, we would have to find them in a single phase at any finite $g_e$. Relying on numerical evidence that free monopoles seem to enhance chiral
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symmetry breaking [102], we propose that it is the phase with broken chiral symmetry that survives a finite $g_e$. We also note that, conversely, if the spinons were massive to begin with, as in the chirally broken phase, integrating them out would lead to a Coulomb gas of monopoles and the classic result of Polyakov for permanent confinement would follow.

It was found previously [77] that the critical $N_f$ for chiral symmetry breaking at $g_e = \infty$ is significantly larger than the $N^c_f$ in the continuum limit. We expect that for $g_e = \infty$, just like for any $g_e > 0$, there is no true phase transition in the cQED$_3$, but just the cross-over occurring at $N_f \approx N_{\text{cross}}$. Making the simplest assumption that $1/g_e$ always remains irrelevant, the same is true for all $g_e > 0$, but with the cross-over line shifted, according to Eq. (4.6), to

$$N_{\text{cross}}(g_e) = N_{\text{cross}} - \frac{8\Lambda}{g_e} + O(\Lambda^2/g_e^2).$$

The conjectured phase diagram that summarizes this discussion is already presented in Fig. 4.2. $\langle \chi \chi \rangle \neq 0$ everywhere, except on the line $g_e = 0$, $N > N^c_f$.

4.8 Absence of $U(1)$ spin liquids

We have so far focused our attention on spinons with relativistic Dirac dispersion at the nodes of the $d$-wave gap function on the electronic Fermi surface. This is the content of the QED$_3$ effective theory of high-temperature superconductivity we derived in Chapter 2. However, the non-relativistic case, where spinons form a spinon Fermi surface, is also of great interest.

The anomalous normal state of the cuprate superconductors has focused a great deal of attention on exotic states with electron fractionalization. One model, which has enjoyed a great deal of popularity has the electron spin reside on neutral $S = 1/2$ fermionic spinons which form a Fermi surface. The microscopic theory of the formation of spinons shows that each spinon carries unit charge of a compact gauge group, which is usually $U(1)$ [103, 104, 15, 87, 92]. The compact gauge group then allows monopole events to exist. Following essentially the same reasoning as in the previous sections of this chapter, we show that the screening between monopoles renders the spinon Fermi surface generically unstable in two spatial dimensions at zero temperature.

A fundamental characteristic of the proposed spin liquid state is the overdamped nature
of the transverse gauge field propagator. This damping arises from the low energy fluctuations of the spinon Fermi surface. After integrating out the fermionic spinon fields, the spin liquid action, $S_{SL}$, for the gauge field $a_\mu$ at small wavevectors ($k$) and small frequencies ($\omega$) has the following singular form [104, 15],

$$S_{SL} = \int \frac{d^3k d\omega}{(2\pi)^3} \left[ \frac{1}{2} a_i a_j \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \left( |\omega| \sigma_s(k) + \chi_d k^2 \right) \right. \left. + \frac{\chi_s}{2} \left( 1 + \frac{\gamma |\omega|}{k} \right) \left( a_0 - \frac{\omega}{k^2} k_i a_i \right)^2 \right].$$ (4.55)

Here $\sigma_s(k)$ is the spinon "conductivity", $\chi_d$ is the spinon diamagnetic susceptibility, $\chi_s$ is the spinon compressibility, and $\gamma$ is a damping coefficient characteristic of the spinon Fermi surface. All of the terms in $S_{SL}$ are characteristic of a fermionic spinon system with a finite density of states at the Fermi level. Note that for the relativistic case with a Dirac spinon spectrum discussed so far, there is a vanishing density of states at the Fermi level, which makes the effective gauge field action quite different. For $\sigma(k)$, it is conventional to assume [15, 104]

$$\sigma_s(k) \sim \begin{cases} 
  l & \text{for } kl < 1 \\
  1/k & \text{for } kl > 1 
\end{cases}$$ (4.56)

where $l$ is the mean-free path associated with scattering off static impurities. The regions $kl < 1$ and $kl > 1$ distinguish, respectively, the dirty and the clean limit.

We may perform duality transformations on $S_{SL}$ similar to those presented in Section 4.3 to map it to an equivalent sine-Gordon action. We leave the details of the mapping in this case to Appendix C.1, and quote the result as,

$$S_{SL_{\text{SG}}} = \frac{1}{2} \int \frac{d^3k d\omega}{(2\pi)^3} \left( \frac{\omega^2}{|\omega| \sigma_s(k)/k^2 + \chi_d + 1/e^2 + e^2 k^2} \right) |\phi(k, \omega)|^2 - 2y \int d^2rd\tau \cos(2\pi \phi(r, \tau)).$$ (4.57)

In the calculations leading to Eq. (4.57) we have also included a regular $(\nabla \times a)^2/2e^2$ term in the action (4.55) for the gauge field; such a term will invariably be generated by integrating out high energy fluctuations of matter fields at short scales.

We may now perform a similar renormalization group study of the sine-Gordon action, $S_{SL_{\text{SG}}}$. Here we will make a few simplifying assumption. To be specific, we assume that in the sine-Gordon theory the wavevectors $k < \Lambda$, with $\Lambda l < 1$ (dirty limit), and frequencies
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\[ |\omega| < \Omega. \]

We also take \( \Omega \gg (\chi_d + e^{-2})\Lambda^2 / l \), so one can write the sine-Gordon field propagator for low momenta and frequencies as

\[
G_{\text{SLG}}^{-1}(k, \omega) = \frac{|\omega|k^2}{l} + a_k k^2 + a_\omega \omega^2,
\]

where the bare (unrenormalized) coefficients are \( a_k = e^2 \), and \( a_\omega = 0 \). Our methods below are easily generalized to the clean limit case where \( \sigma_s (k) \sim 1/k \), and the results are very similar.

Again, we leave the details of deriving the flow equations to Appendix C.2. The results are

\[
\beta_{a_k} = \frac{8\hat{y}^2}{\hat{a}_k} + O(\hat{y}^3),
\]

\[
\beta_{a_\omega} = 2\hat{a}_\omega + \frac{\pi \hat{y}^2}{\hat{a}_k} + O(\hat{y}^3),
\]

where we have introduced the dimensionless combinations \( \hat{y} = y/\Lambda^2 \Omega \), \( \hat{a}_k = a_k/\Omega \), and \( \hat{a}_\omega = a_\omega \Omega / \Lambda^2 \).

A crucial feature of the above \( \beta \)-functions is that although initially the coefficient \( \hat{a}_\omega = 0 \), for \( b > 1 \) we have \( \hat{a}_\omega (b) > 0 \). This expresses the simple physical effect discussed in Section 4.5 that the interaction between two distant monopoles in a medium with a finite polarizability is always Coulomb \( \sim 1/\sqrt{|r|^2 + \tau^2} \) in three space-time dimensions. Thus, \( a_\omega \) becomes generated to the second order in fugacity, and then becomes a relevant coupling. With this term included the fugacity flows according to

\[
\beta_y = \left( 2 - \frac{1}{8\pi \sqrt{\hat{a}_k \hat{a}_\omega - 1/4l^2}} \right) \hat{y} + O(\hat{y}^3),
\]

and always becomes relevant at long length scales. In Eq. (4.61) we assumed \( \hat{a}_k \hat{a}_\omega \gg 1/4l^2 \); other limits and more complete expressions are presented in Appendix C.2. We interpret this as an instability of the deconfined phase in the original theory, since the monopoles are always in the plasma phase, and the interaction between two well-separated monopoles is screened as before.
Chapter 5

Variational Approach to Confinement

In this chapter we look at the problem of confinement from yet another angle. We will set up a variational framework to include the effects of dipole screening discussed in the previous chapter. A first-order calculation would suggest, much like the first-order RG analysis, that a deconfinement transition exists for fermion flavour number less than \( N_f = 24 \). However, we show the screening effects that can only be captured in the second-order calculation make the deconfined state unstable. As a consequence, the system only admits a plasma phase for monopoles, or a confined phase for spinons. These findings were published in Ref. 53.

In Section 5.1, we discuss the Bogoliubov-Feynman (BF) inequality and its application to the sine-Gordon action, which forms the basis of our variational method. We also derive Polyakov's permanent confinement using the BF inequality. In Section 5.2 we repeat this first-order calculation for the logarithmic monopole gas and point out its shortcomings. In Section 5.3 we extend the variational method to higher orders. We use this formulation to perform a second-order variational calculation for the logarithmic monopole gas and confirm the existence of a confining solution.

A clarification of notation in this chapter is due at this point. We shall use \( \text{Tr} \) to denote the path integral over the field \( \phi \). This should not be confused with the matrix trace, \( \text{tr} \), over \( \gamma \)-matrix indices used in other chapters. For later use, we note that in this notation, the ensemble average of a functional \( A[\phi] \) with an arbitrary action \( S \) is written as

\[
\langle A[\phi] \rangle = \frac{\text{Tr}(e^{-S[\phi]}A[\phi])}{\text{Tr}(e^{-S[\phi]})} = e^{F} \text{Tr}(e^{-S[\phi]}A[\phi]),
\]

(5.1)
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where $F$ is the free energy.

5.1 Bogoliubov-Feynman inequality

The Bogoliubov-Feynman inequality imposes a strict bound on the true free energy, $F$, of the system through the relation

$$F \leq F_{\text{var}} \equiv F_0 + \langle S - S_0 \rangle_0. \quad (5.2)$$

The action $S$ represents the full action of the system. For our purposes this is the same as $S_{\text{SG}}$ defined in Eqs. (4.20) and (4.33). $S_0$ is a trial action chosen to approximate $S$; $F_0$ is the free energy associated with $S_0$ and $\langle \cdots \rangle_0$ denotes averaging within this ensemble.

The BF inequality follows readily from the convexity of the exponential function, which yields

$$e^{-f(X)} \leq e^{-\langle f(X) \rangle},$$

where $\langle \cdots \rangle$ indicates an arbitrary averaging over the stochastic variable $X$, and $f$ is an arbitrary function. Taking $X$ to be the field $[\phi]$, $f = S - S_0$, and the average $\langle \cdots \rangle = \langle \cdots \rangle_0$, we find

$$e^{-\langle S - S_0 \rangle_0} \leq \langle e^{S_0 - S} \rangle_0 = e^{F_0} \text{Tr}(e^{-S}) = e^{F_0 - F},$$

from which the BF inequality (5.2) follows immediately.

We may use the BF inequality as a variational principle: choosing a class of “bare” actions $S_0$ we may minimize the variational free energy $F_{\text{var}}$ in this class in order to estimate the true free energy $F$. This procedure would also allow us to investigate the phases of the system by establishing the phases of the minimized bare action.

As a simple application of this variational principle, we consider the sine-Gordon theory given in Eq. (4.20), which we rewrite partially in Fourier space as

$$S_{\text{SG}}[\phi] = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \phi(-q)\nu^{-1}(q)\phi(q) - 2y \int d^3r \cos \phi(r). \quad (5.3)$$
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We choose a Gaussian bare action, written in Fourier space as

$$S_0[\phi] = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \phi(-q)G_0^{-1}(q)\phi(q).$$  \hspace{1cm} (5.4)

The propagator $G_0^{-1}$ is at this point an arbitrary function. The variational free energy can be then computed (up to a constant) as

$$\frac{F_{\text{var}}[G_0]}{V} = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ -\ln G_0(q) + G_0(q)v^{-1}(q) \right] - 2y \exp \left[ -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} G_0(q) \right].$$ \hspace{1cm} (5.5)

$V$ is the volume of the system. We will next minimize $F_{\text{var}}[G_0]$ with respect to $G_0$ to find the optimal Gaussian theory that approximates $F_{\text{SG}}$:

$$\frac{\delta F_{\text{var}}[G_0]}{\delta G_0} = 0 \implies G_0^{-1}(q) = v^{-1}(q) + \rho,$$  \hspace{1cm} (5.6)

with the ‘mass’ $\rho$ determined self-consistently through

$$\rho = 2y \exp \left[ -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{v^{-1}(q) + \rho} \right].$$ \hspace{1cm} (5.7)

We recall that $y = e^{E_c/T}$, where $E_c$ is the chemical potential for creating a monopole and, $T$ is the effective temperature of the monopole gas. Then the physical meaning of $\rho$ is made clear if we notice that the right-hand side of Eq. (5.7) is nothing but

$$\frac{\rho}{V} = \frac{\rho}{V} \frac{\partial F_{\text{var}}}{\partial y} = -\frac{\tilde{T}}{V} \frac{\partial F_{\text{var}}}{\partial E_c},$$

or, the density of monopoles, hence the notation. Thus, $\rho$ plays the role of the order parameter for a confinement-deconfinement transition. If $\rho = 0$ we have no free monopoles, just dipoles and the spinons would be free. For $\rho \neq 0$ we find a plasma of free monopoles, and the spinons would be confined.

Using Eq. (5.7), we may now reproduce Polyakov’s permanent confinement in the pure gauge $c\text{QED}_3$. As was shown in chapter 4, this theory is equivalent to the conventional sine-Gordon theory with $v^{-1}(q) = Tq^2$. Assuming an ultraviolet cut-off, $\Lambda$, we find

$$\rho = 2y \exp \left[ \frac{\Lambda}{4\pi^2T} \left( 1 - \sqrt{\frac{T\Lambda^2}{\rho}} \tan^{-1} \sqrt{\frac{T\Lambda^2}{\rho}} \right) \right] \hspace{1cm} (5.8)$$

$$\rho = \begin{cases} 2ye^{\Lambda/(4\pi^2T)}, & \rho \to 0, \\ 2y, & \rho \to \infty. \end{cases} \hspace{1cm} (5.9)$$
Since the right-hand side of this equation is bounded from below and above, it only admits a finite solution $\rho^* \neq 0$. Therefore, the monopoles in pure gauge $cQED_3$ are always in the plasma phase and the spinons always confined.

### 5.2 Deconfinement transition

If we now consider $cQED_3$ with the gauge field coupled to relativistic spinons, the propagator of the sine-Gordon theory takes the form $v^{-1}(q) = T|q|^3$. The same variational equation (5.7) now reads,

$$\rho = 2y \left( 1 + \frac{T \Lambda^3}{\rho} \right)^{-T^*/T}.$$  \hspace{1cm} (5.10)

$T^* \equiv 1/(12\pi^2)$. Determining the solutions of Eqn. (5.10) amounts to identifying the roots of the function

$$f(\rho) = \rho - 2y \left( 1 + \frac{T \Lambda^3}{\rho} \right)^{-T^*/T}.$$  \hspace{1cm} (5.11)

It is evident that $\rho = 0$ is one such root for all values of $T$. We next demonstrate that a finite solution exists for $T > T^*$. In the limit of small $\rho$, $f(\rho)$ has the form

$$f(\rho \ll \Lambda^3) = \begin{cases} 
\rho, & T < T^* \\
-\rho^{T^*/T}, & T > T^*
\end{cases} \hspace{1cm} (5.12)$$

while for large $\rho$

$$f(\rho \gg \Lambda^3) = \rho, \; \forall T.$$  \hspace{1cm} (5.13)

For $T > T^*$, $f(\rho)$ changes sign and thus has a root with $\rho > 0$, while only the $\rho = 0$ solution exists for $T < T^*$.

The instability of the $\rho = 0$ solution for $T > T^*$ can be established from the variational free energy (5.5) with the solution (5.6) for $G_{0}^{-1}$. Evaluating the free energy we get

$$\frac{1}{V}F_{\text{var}}(\rho) = T^* \Lambda^3 \ln \left( \rho + T \Lambda^3 \right) - 2y \left( 1 + \frac{T \Lambda^3}{\rho} \right)^{-T^*/T}.$$  \hspace{1cm} (5.14)

Then

$$\frac{1}{V} \left[ F_{\text{var}}(\rho) - F_{\text{var}}(0) \right] \approx \frac{\rho(T^* - T)}{T} + O(\rho^2),$$ \hspace{1cm} (5.15)

\footnote{Provided $y$ is small enough. For larger values of $y$, the system may undergo a first-order transition [105].}
so that for \( T > T^* \) any solution with \( p > 0 \) is of lower free energy than with \( p = 0 \). That is, the stable solution at \( T > T^* \) has finite \( p \). The simple variational calculation would therefore suggest that monopoles undergo a binding-unbinding transition at \( T = T^* \) in exact analogy with the equivalent calculation one can perform for the standard KT transition. The value of \( T^* \) (or \( N^*_f = 2/\pi^2 T^* = 24 \)) also agrees with the simple energy-entropy argument that was constructed for an isolated vortex in Section 4.5.

An obvious objection to this simple calculation is that minimization of the variational free energy (5.5) by construction cannot yield any momentum dependence of the self-energy, but can only determine its constant part, the "mass" \( p \). As we saw in the previous chapter, the renormalization group treatment of the sine-Gordon theory suffers from the same problem to the lowest order in fugacity, and would likewise naïvely suggest the KT transition. The same holds for the straightforward perturbative evaluation of the sine-Gordon self-energy.

However, it is easy to check that the self-energy does become momentum dependent to the second order in fugacity, with the leading analytic term \( \sim q^2 \) at low momenta. This is just what one would expect based on the simple electrostatic analysis of Section 4.5, where this term translates into the Coulombic interaction in real space. The presence of such a term would drastically alter our present considerations. Indeed, if we add by hand a term \( Qq^2 \) with \( Q \neq 0 \) to \( v^{-1}(q) \), and thus, in the denominator of the integrand in the self-consistent equation (5.10), we find

\[
f(\rho) = \begin{cases} 
-2y \left( 1 + \frac{\Lambda T}{Q} \right)^{-3T^*/T}, & \rho \ll \Lambda^3 \\
\rho, & \rho \gg \Lambda^3 
\end{cases}
\]

for all \( T \). Hence, the non-trivial solution would exists for all temperatures, exactly as in the Polyakov's original treatment of the pure gauge theory. This is natural since \( Q \neq 0 \) means that the original logarithmic interaction between monopoles is, in the presence of other dipole pairs, screened into the Coulomb interaction for which the standard argument for the confined phase readily applies.

In the next section we propose a modified self-consistent calculation which provides a systematic perturbative approximation to the free energy and which reduces to the BF method to the lowest order. As we will see in Section 5.4, such an approach has the advantage of including the screening effects in a self-consistent way, therefore overcoming the
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limitations of the purely variational theory discussed in this section.

5.3 Self-consistent perturbation theory

There are many ways in which one may generalize the variational method of the previous sections. For instance, one may add a second-order term \(-\frac{1}{2}\langle (S_s - S_0)^2 \rangle_0 + \frac{1}{2}\langle S_s - S_0 \rangle_0^2\) to \(F_{\text{var}}\) and extremize the new energy functional [106]. Such a second-order extension, however, has little variational justification. For a more systematic generalization, we go back to the BF inequality (5.2) and exchange \(S\) with \(S_0\) to find

\[ F_\prec \equiv F_0 + \langle S - S_0 \rangle \leq F. \]  

(5.17)

Extremizing \(F_\prec\) with respect to a quadratic action \(S_0\) yields

\[ \langle \phi(-q)\phi(q) \rangle_0 = \langle \phi(-q)\phi(q) \rangle, \]  

(5.18)

which is nothing but the equation for the exact propagator in the full theory, \(S\). The right-hand side (RHS) of the equation, on the other hand, may be rewritten as

\[ \langle \phi(-q)\phi(q) \rangle = \frac{\langle \phi(-q)\phi(q)e^{-\Delta S} \rangle_0}{\langle e^{-\Delta S} \rangle_0}, \]  

(5.19)

with \(\Delta S \equiv S - S_0\). Eq. (5.18) in this form may be understood as a self-consistent equation for the action \(S_0\), which we may attempt to solve by expanding the RHS in powers of \(\Delta S\), for example. To the first order in \(\Delta S\) this becomes

\[ \langle \phi(-q)\phi(q)\Delta S \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0\langle \Delta S \rangle_0 = 0, \]  

(5.20)

which is precisely the relation one would obtain from extremizing \(F_{\text{var}}\) with respect to \(S_0\). That is, the first order approximation to Eq. (5.18) reproduces the BF result from the previous section.

Eq. (5.18) forms the basis of our modified variational approximation to \(F_{SG}\) when we take \(S = S_{SG}\) in the next section. To the first order in \(\Delta S\) it reduces to the BF equation of the previous sections, and when solved self-consistently to all orders gives the best variational lower bound to the free energy, provided by \(F_\prec\) in (5.17). In addition, consider the
expansion of Eq. (5.18) to order \((\Delta S)^n\). We will show below that the resulting expression is the same as the one that would arise from extremizing the function

\[
F^{(n)}_{\text{var}} \equiv \frac{F^{(1)} + F^{(2)} + \cdots + F^{(n)}}{n}.
\]  

Here \(F^{(n)}\) stands for the expansion of the true free energy of the system, \(F\), in powers of \(\Delta S\), truncated at \((\Delta S)^n\). Similarly denoting by \(F^{(n)}_{\varphi}\) the truncated expansion of \(F^{(n)}\) in Eq. (5.17), we will also show that

\[
F^{(n)}_{\varphi} = F^{(n)} + \frac{F^{(n)} - F^{(n)}_{\varphi}}{n}.
\]  

It is then clear that the sequence \(\{F^{(n)}_{\varphi}\}\) converges to \(F\) for any \(S_0\). Therefore, the \(S_0\) determined self-consistently from Eq. (5.18) also yields the variational sequence that best approximates \(F\) from above within the family \(\{F^{(n)}_{\text{var}}[S_0]\}\) with arbitrary \(S_0\).

Let us first show that \(F^{(n)}_{\varphi}\) indeed satisfies Eq. (5.22). To this end, we denote \(\Delta F^{(n)} \equiv F^{(n+1)} - F^{(n)}\). We may then equivalently show that

\[
F^{(n)}_{\varphi} = F_0 + \Delta F^{(1)} + 2\Delta F^{(2)} + \cdots + n\Delta F^{(n)}.
\]

We define, for a real variable \(t\),

\[\mathcal{F}(t) \equiv -\ln \text{Tr} \left( e^{-S_0} e^{-t\Delta S} \right).\]  

Then, \(\mathcal{F}(1) = -\ln \text{Tr} \ exp(-S) = F\) and

\[
\left. \frac{d\mathcal{F}(t)}{dt} \right|_{t=1} = \frac{\text{Tr} \ (\Delta S e^{-S})}{\text{Tr} \ (e^{-S})} = \langle \Delta S \rangle.
\]

On the other hand, We may expand the RHS of Eq. (5.24) in powers of \(\Delta S\) as

\[
\mathcal{F}(t) = F_0 + \sum_{i=1}^{\infty} \Delta \mathcal{F}^{(i)}(t)
\]

where \(\Delta \mathcal{F}^{(i)}(t) = t^i \Delta \mathcal{F}^{(i)}(1) = t^i \Delta F^{(i)}\). Thus

\[
\langle \Delta S \rangle = \left. \frac{d\mathcal{F}(t)}{dt} \right|_{t=1} = \sum_{i=1}^{\infty} i \Delta F^{(i)}.
\]
Upon insertion of Eq. (5.27) into the definition of $F_<$ in Eq. (5.17) and truncating the expansion at $i = n$ we find (5.23).

Next, we will give the proof for our claim that the extremum of $F_{\text{var}}^{(n)}$ as defined in (5.21) is given by the expansion of Eq. (5.18) to order $(\Delta S)^n$, i.e.

$$\frac{\delta F_{\text{var}}^{(n)}}{\delta G_0(q)} = 0 \iff \langle \phi(-q)\phi(q) \rangle_0 = \langle \phi(-q)\phi(q) \rangle^{(n)}.$$  \hspace{1cm} (5.28)

The calculations are, for general $n$, cumbersome and not very instructive so we will first present the case for $n = 2$ which is also the one with which we are concerned in Section 5.4.

To the second order Eq. (5.18) reads

$$\langle \phi(-q)\phi(q)\Delta S \rangle_{0c} - \frac{1}{2} \langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_{0c} = 0,$$  \hspace{1cm} (5.29)

where both terms are connected averages given by:

$$\langle \phi(-q)\phi(q)\Delta S \rangle_{0c} = \langle \phi(-q)\phi(q)\Delta S \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0 \langle \Delta S \rangle_0,$$  \hspace{1cm} (5.30)

$$\langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_{0c} = \langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0 \langle (\Delta S)^2 \rangle_0$$

$$-2\langle \phi(-q)\phi(q)\Delta S \rangle_0 \langle \Delta S \rangle_0 + 2\langle \phi(-q)\phi(q) \rangle_0 \langle \Delta S \rangle_0^2.$$  \hspace{1cm} (5.31)

Thus, we see that Eq. (5.29) is readily found by an expansion of the RHS of Eq. (5.19). To show that the same result arises from extremizing $F_{\text{var}}^{(n=2)}$, it is first useful to note

$$\frac{\delta F_0}{\delta G_0(q)} = \left\langle \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0 = \frac{-1}{2(2\pi)^3[G_0(q)]^2} \langle \phi(-q)\phi(q) \rangle_0,$$  \hspace{1cm} (5.32)

$$\frac{\delta (g)}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \langle g \rangle_0 + \left\langle \frac{\delta g}{\delta G_0(q)} - g \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0,$$  \hspace{1cm} (5.33)

where $g = g(S_0)$ is an arbitrary function of $S_0$. By choosing appropriate forms of $g$ as $F^{(1)} = F_0 + \langle \Delta S \rangle_0$ and $F^{(2)} = F^{(1)} - \frac{1}{2} \langle (\Delta S)^2 \rangle_0 + \frac{1}{2} \langle (\Delta S)^2 \rangle_0^2$ we can establish

$$\frac{\delta F^{(1)}}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \langle \Delta S \rangle_0 - \left\langle \Delta S \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0,$$  \hspace{1cm} (5.34)

$$\frac{\delta F^{(2)}}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \left[ -\frac{1}{2} \langle \Delta S^2 \rangle_0 + \langle \Delta S \rangle_0^2 \right]$$

$$+ \frac{1}{2} \left\langle \frac{\delta S_0}{\delta G_0(q)} (\Delta S)^2 \right\rangle_0 - \left\langle \frac{\delta S_0}{\delta G_0(q)} (\Delta S) \right\rangle_0 \langle \Delta S \rangle_0.$$  \hspace{1cm} (5.35)
Inserting Eq. (5.32) into Eqs. (5.34), (5.35) and adding them we find that the restriction
\[ \delta F^{(2)}_{\text{var}} / \delta G_0(q) = 0 \]
leads to the same equation as Eq. (5.29).

The proof for arbitrary \( n \) goes along essentially the same steps as above. Various truncated expansions we have defined can be read off the Taylor expansion identity
\[
- \ln \text{Tr} \left( e^{-S_b - V} \right) = F_b + \sum_{i=1}^{\infty} \sum_{l=1}^{i} \frac{(-1)^{i+l}}{l!} \sum_{[k_a]} \delta_{k_1! \cdots k_l!} (V^k)_b \cdot \cdots \cdot (V^k)_b, \tag{5.36}
\]
by setting \( i \) to the desired order. In (5.36) \( S_b \) and \( V \) give an arbitrary splitting of the action into a bare and potential part respectively and
\[
\sum_{[k_a]}' \equiv \sum_{k_1=1}^{i} \cdots \sum_{k_l=1}^{i} \delta_{k_1! \cdots \cdot k_l!}. \]

Notice that in Eqn. (5.34) and (5.35) all the terms are to the same order of \( \Delta S \), which is also the largest in the corresponding expansion of the free energy. By choosing \( S_b = S_0 \) and \( V = \Delta S \) and setting \( i = n \) in (5.36) one can see, after some lengthy algebra, that the same is true for arbitrary \( n \):
\[
\frac{\delta F^{(n)}}{\delta G_0(q)} = \sum_{i=1}^{n} (-1)^{n+i} \sum_{[k_a]} \left( \langle (\Delta S)^{k_1} \rangle_0 \cdots \langle (\Delta S)^{k_l} \rangle_0 / k_1! \cdots k_l! \right) \frac{\delta F_0}{\delta G_0(q)}
- \sum_{i=1}^{n} (-1)^{n+i} \sum_{[k_a]} \left( \frac{\delta S_0}{\delta G_0(q)} (\Delta S)^{k_1} / k_1! k_2! \cdots k_l! \right) \langle (\Delta S)^{k_2} \rangle_0 \cdots \langle (\Delta S)^{k_l} \rangle_0. \tag{5.37}
\]

Let us now define, for a real variable \( t \),
\[
G(q, t) \equiv - \ln \text{Tr} \left\{ e^{-S_b - \Delta S - t \phi(-q) \phi(q)} \right\}, \tag{5.38}
\]
so that \( \partial G(q, t)/\partial t \big|_{t=0} = \langle \phi(-q) \phi(q) \rangle \). Then, taking \( S_b = S_0 \) and \( V = \Delta S + t \phi(-q) \phi(q) \) in Eq. (5.36) to compute this derivative, it can be shown through additional tedious but straightforward algebra that
\[
\langle \phi(-q) \phi(q) \rangle^{(n)} - \langle \phi(-q) \phi(q) \rangle_0 = -2n(2\pi)^3[G_0(q)]^2 \frac{\delta F^{(n)}_{\text{var}}}{\delta G_0(q)}, \tag{5.39}
\]
where we have also made use of Eq. (5.37). Thus, the requirement that \( F^{(n)}_{\text{var}} \) be an extremum implies Eq. (5.18) truncated at \( n \)th order, and vice versa, proving our claim (5.28).
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Having set our framework, we will discuss the results of the second-order self-consistent Eq. (5.29) for the sine-Gordon model with cubic propagator in the next section. In particular, we will show that the density of free monopoles is finite at all \( T > 0 \), and that charge should consequently be permanently confined in cQED3.

5.4 Confining solution

Equating \( S = S_{SG} \), it is straightforward to calculate the connected averages of Eqs. (5.30) and (5.31). Our second order equation (5.29) then yields a quadratic equation for \( G_0^{-1}(q) \)

\[
[G_0^{-1}(q)]^2 - A[q, G_0]G_0^{-1}(q) + B[q, G_0] = 0, \tag{5.40}
\]

where

\[
A[q, G_0] = \frac{3}{2} T |q|^3 + 3a + ab - 2a^2 \left( c + \sum_{n=0}^{\infty} (-1)^n d_n q^{2n} \right) \tag{5.41}
\]

\[
B[q, G_0] = \frac{1}{2} T^2 q^6 + 2a T |q|^3. \tag{5.42}
\]

In Eqs. (5.41), (5.42), we have defined

\[
a = ye^{-\frac{1}{2} D_0(0)}, \tag{5.43}
\]

\[
b = \int \frac{d^3 q}{(2\pi)^3} \left( \frac{1}{2} T |q|^3 - \frac{1}{2} G_0^{-1}(q) \right) [G_0(q)]^2, \tag{5.44}
\]

\[
c = \int d^3 R [1 - \cosh D_0(R)], \tag{5.45}
\]

\[
d_n = \int d^3 R \frac{[R] \cos \theta)^{2n}}{(2n)!} \sinh D_0(R), \tag{5.46}
\]

and the real-space propagator is \( D_0(R) = \int \frac{d^3 q}{(2\pi)^3} G_0(q) e^{i q \cdot R} \).

We can solve the quadratic Eq. (5.40) and expand in powers of \( |q| \) to yield the result for \( G_0^{-1}(q) \):

\[
G_0^{-1}(q) = m + Q(m) q^2 + T(m) |q|^3 + \cdots \tag{5.47}
\]
where the coefficients are defined as
\begin{align}
m &= \frac{1}{2} \{ A_0 \pm |A_0| \}, \\
Q(m) &= a^2 d_1 \left( 1 \pm \frac{|A_0|}{A_0} \right), \\
T(m) &= \frac{3}{4} T \pm \frac{|A_0|}{A_0} \left( \frac{3}{4} T - \frac{2aT}{A_0} \right); \tag{5.50}
\end{align}
and with \( A_0 \equiv A[q = 0, G_0] \). For these equations, we should choose the solution corresponding to the upper sign in Eqs. (5.48)-(5.50) to ensure that \( m \geq 0 \). Otherwise, there would not exist a plasma phase at all, which is clearly unphysical at high temperatures. In what follows, we neglect terms higher order in \( q \) than \( |q|^3 \) as they should be irrelevant at low momenta.

As announced, the second order result includes additional renormalization of the bare terms as well as the generation of new momentum dependent terms. Most importantly, the leading term proportional to \( q^2 \) has now appeared.

In the analysis of Section 5.2 we found that the bound phase of monopoles corresponded to low \( T \). In what follows we will restrict ourselves to low temperatures by assuming \( T \Lambda \ll Q \), and show that monopoles are unbound even for arbitrarily small temperatures. By continuity this would imply that they are free at all temperatures.

Let us start by examining \( a \):
\begin{align}
a &= y \exp \left\{ -\frac{1}{2} D_0(0) \right\} \\
&\approx y \exp \left\{ -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{Q(m)q^2 + m} \right\} \\
&= y \exp \left\{ -\frac{1}{4\pi^2 Q(m)} \left( \Lambda - \sqrt{\frac{m}{Q(m)}} \tan^{-1} \left( \Lambda \sqrt{\frac{Q(m)}{m}} \right) \right) \right\}, \tag{5.51}
\end{align}
When \( m \to 0 \), we will assume \( m/Q(m) \to 0 \), and justify this assumption \textit{a posteriori}. The coefficient \( a \) now takes the form
\begin{align}
a = y \exp \left\{ -\frac{\Lambda}{4\pi^2 Q(m)} \right\}, \quad m \ll \Lambda^3 + O(T). \tag{5.52}
\end{align}
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Now, we examine the equation for $b$

$$b = \int \frac{d^3q}{(2\pi)^3} \left( \frac{T}{2} |q|^3 - \frac{1}{2} G_0^{-1}(q) \right) [G_0(q)]^2$$

$$= -\frac{1}{2} D_0(0) + O(T).$$

(5.53)

From this we find

$$b = -\frac{\Lambda}{4\pi^2 Q(m)}, \quad m \ll \Lambda^3 + O(T).$$

(5.54)

Next, as the terms $c$ and $d_0$ always appear together, we consider the combination

$$(c + d_0) \approx \int d^3R \left( 1 - \exp \left\{ - \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq \cdot R}}{Q(m)q^2 + m} \right\} \right) + O(T)$$

$$= \int_0^\infty du \frac{ue^{-\sqrt{\frac{m}{Q(m)}}}u}{Q(m)} + O(T).$$

(5.55)

Evaluating this yields

$$(c + d_0) = m^{-1}, \quad m \ll \Lambda^3.$$

(5.56)

A similar analysis applies to the coefficient $d_1$:

$$d_1 = \frac{1}{6} \int_0^\infty du \frac{ue^{-\sqrt{\frac{m}{Q(m)}}}u}{Q(m)} + O(T),$$

(5.57)

which gives

$$d_1 = \frac{Q(m)}{m^2}, \quad m \ll \Lambda^3 + O(T).$$

(5.58)

Evaluating Eq. (5.49) for $Q$ we find

$$Q = 2a^2 d_1$$

$$= 2y^2 \frac{Q}{m^2} \exp \left( -\frac{\Lambda}{2\pi^2 Q} \right) + O(T).$$

(5.59)

Solving this for $Q \neq 0$ yields

$$Q = \frac{\Lambda}{4\pi^2} \left( \ln \frac{\sqrt{2y}}{m} \right)^{-1} + O(T),$$

(5.60)
and we see that \( m/Q(m) \) indeed approaches zero as \( m \to 0 \), thus justifying our earlier assumption. Substituting this solution for \( Q(m) \) into our mass equation (5.48) gives

\[
m = A_0 \\
\approx \frac{m}{\sqrt{2}} \left[ 3 - \sqrt{2} - \ln \frac{\sqrt{2} y}{m} \right],
\]

which can finally be solved for \( m \neq 0 \) to give the finite mass solution

\[
m^* = \sqrt{2} e^{2\sqrt{2} - 3} y.
\]

The corresponding finite value of \( Q \) is

\[
Q^* = \frac{\Lambda}{2\pi^2(3 - 2\sqrt{2})}.
\]

Note that \( m^* \) is proportional to \( y \) so that small fugacity translates to small \( m^* \), in accord with our assumption that \( m \ll \Lambda^3 \).

To show that monopoles are free when \( m \neq 0 \), we calculate the monopole density as in Section 5.2. From Eq. (5.21) we see that the free energy associated with our second-order equation (5.29) is

\[
F_{\mathrm{var}}^{(2)} = F_0 + \langle \Delta S \rangle_0 - \frac{1}{4} (\langle \Delta S \rangle^2)_0 + \frac{1}{4} (\Delta S)_0^2.
\]

From this, the monopole density can be calculated.

\[
\rho^{(2)} = -\frac{y}{V} \frac{\partial F_{\mathrm{var}}^{(2)}}{\partial y} = 2a + ab - 2a^2 c.
\]

For \( m = 0 \), the monopole density vanishes, while for our finite \( m \) solution

\[
\rho^{(2)} = \frac{m^*}{\sqrt{2}} \left( 2\sqrt{2} - 1 + \frac{1}{16\pi} \sqrt{\frac{2m^*}{Q^*}} \right) > 0.
\]

From the free energy (5.64), it is also possible to show that the finite \( m \) solution is the stable solution for all temperatures. In fact, \( F_{\mathrm{var}}^{(2)} \) diverges as \( \log(1/m) \) as \( m \) approaches zero, but has a finite value for finite \( m \). It is then the free phase of monopoles which is favoured at all temperatures.
Thus we have demonstrated that for arbitrarily low $T$ a finite mass solution always exists for the self-consistent equations (5.48)–(5.50). This in turns implies that monopoles are always free at low temperatures, or, in the notation of the original lattice model (4.4), that spinons are confined for any number of flavours, $N_f$.

5.5 Conclusion

In Chapters 4 and 5, we have studied compact $U(1)$ gauge theories coupled to fermionic matter fields (spinons) with both relativistic and non-relativistic dispersion. These theories have been proposed as the effective theory for the anomalous behaviour of the high-temperature superconductors in their pseudogap state.

In the relativistic case, massless relativistic fermions coupled to the compact gauge field result in a logarithmic interaction between magnetic monopoles. One may suspect that this could lead to a KT-like transition where free monopoles bind into monopole-antimonopole pairs at low-enough “effective temperatures,” $T \sim 1/N_f$. Although the simplest mean-field approximation would predict such a transition, we argued that by design this treatment misses the screening effects, expected to be crucial in this problem. To address this issue we developed along three parallel, but ultimately connected, routes: (i) we argued by way of an elementary electrostatic treatment that such a dipole phase is unstable at large distances to the screening effects of smaller dipoles, (ii) we produced a precise renormalization group analysis of the system, which shows that the Coulomb piece of the potential is represented by a relevant parameter, and (iii) we devised a combined variational-perturbative approach which allowed us to include screening self-consistently. The modified theory in all these studies leads to the plasma phase of free monopoles as being stable at all temperatures.

The cQED$_3$ has been studied numerically in Refs. 102 and 107 and more recently in Ref. 108. Our calculation appears to be in agreement with the numerical results of Refs. 107 and 108, where only a single phase was observed.
Chapter 6

Spin Response in the Superconductor

In this chapter we will study the spin response within the QED$_3$ theory of cuprates in the superconducting state. The spin response is observable in neutron scattering experiments, and is a very important probe of the underlying dynamics of the system. It is not only directly related to the magnetic properties of the system, which play an important role in the phenomenology of cuprates, but also provides a rather accurate experimental picture of the excitations of the system and their interactions.

In Section 6.1, we will give a brief and selective overview of the experimental picture and observations relevant for our study and discuss two possible scenarios explored in this chapter. In Section 6.2, we set up the framework of our calculation in terms of the ladder approximation. We will show in Section 6.3 that the simplest low-energy calculation of spin response in QED$_3$ [46] accounts qualitatively for the experiment. In the following sections we go beyond the approximations employed in the low-energy calculation. In Section 6.4, we demonstrate that by taking into account the full spectrum of spinons, we may even obtain a quantitative agreement with experiment. Finally, we study the problem of resonance due to particle-hole bound states in the superconductor. In Section 6.5, we derive a Schrödinger equation describing the formation of such bound states and express the spin response in terms of its eigenfunctions and eigenvalues. We conclude by giving a preliminary discussion of the existence of resonance in the QED$_3$ spin response in Section 6.6.

For simplicity of our expressions in this chapter, we set the lattice spacing $\ell = 1$. It would be easy to recover it, wherever needed, by noting that all momenta are then expressed in units of $\ell^{-1}$.  

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6.1 Overview

The (inelastic) neutron scattering cross section probes the imaginary part of the magnetic susceptibility \[ \chi_{\mu\nu}(p, \omega) = \langle S_\mu(-p,-\omega)S_\nu(p,\omega) \rangle = \delta_{\mu\nu}\chi(p,\omega), \]

\[ \chi(p,\omega) = \langle S_z(-p,-\omega)S_z(p,\omega) \rangle, \]

where rotational invariance in the superconducting plane is assumed and \((p, \omega)\) is the momentum-energy transfer between the incident and scattered neutron.

In the superconducting state of various cuprate families, neutron scattering experiments have revealed a most interesting behaviour [111-129]. Fig. 6.1 outlines the main features of this response in the underdoped regime. Generally, the low-energy response at the commensurate antiferromagnetic ordering wave vector \(Q_{AF} = (\pi, \pi)\) is lost, upon entering the superconductor, into diagonal incommensurate peaks located at the vertices of a square oriented along the nodal directions (±45° with the principal axes of the underlying square lattice). These peaks are shown by the triangles in Fig. 6.1. They do not disperse with variations of frequency \(\omega\). Then, at an energy \(\omega_{res} \approx 41\) meV, a strong signal is found at \(Q_{AF}\) which is rather sharp in \(\omega\). This “41-meV resonance” is shown in Fig. 6.2 for two families of cuprates, YBCO and BSCCO. No appreciable response is found at \(Q_{AF}\) below a “spin gap” of about the same energy, \(\omega_{sg}(Q_{AF}) \approx \omega_{res}\). At higher energies the response shifts to the diagonal incommensurate positions again but, now, the peaks disperse away from the center with increasing \(\omega\).

We take the spin gap here to be the bottom of the particle-hole continuum spectrum. Thus, whether or not the observed signal at \(Q_{AF}\) and \(\omega_{res}\) and/or other peaks are resonances is closely linked to the question of whether or not they occur below the corresponding spin gap. Accordingly, one could think of two possible theoretical scenarios.

In the first scenario, there is no true resonance in the system, that is, no particle-hole bound state, and the peaks are merely maxima of the spin response. At \(Q_{AF}\), the peak would be interpreted as the overlap of the four incommensurate responses at the center. Usually this should also mean that the central peak falls off slowly as \(1/\omega\), rather than being sharp. However, as noted above, the central peak observed in experiment seems to be very sharp, in some cases limited only by the energy resolution of the measurement.
Figure 6.1: Spin response schematics in the Brillouin zone. The center point is $Q_{AF} = (\pi, \pi)$. The triangles show the observed low-energy incommensurate peaks. The solid circles show the parallel peaks at $2K_i$ predicted in $QED_3$ (see text). The Fermi surface at half-filling and in the superconducting state are depicted with dashed and thick solid lines (partial) along with the gap nodes at $(\pm k_F, \pm k_F)$. 
The second scenario is that there is a true resonance below the spin gap, due to the formation of particle-hole bound states. These bound states are referred to as excitons. It would still be important to determine the continuum response so we could decide what effects derive from which source. However, it would be also very important to determine the existence (or lack) of such excitons in the superconductor and their properties.

Herbut and Lee [46] have calculated the QED\textsubscript{3} spin response in a low-energy approximation and found no exciton resonance. We present this calculation in our ladder approximation formulation in Section 6.3. Here, we note that, independent of the approximation scheme, QED\textsubscript{3} spin response at very low energies is concentrated at incommensurate positions, ±2K\textsubscript{i}, spanning diagonally between the nodes. These are at the vertices of a square oriented parallel to the principal axes, shown by small solid circles in Fig. 6.1. The response around these peaks is elongated in diagonal directions due to the velocity anisotropy,
$v_F \gg v_\Delta$, and at slightly higher energies they start to overlap at the triangles in Fig. 6.1. Thus, QED$_3$ predicts the existence of very-low-energy diagonal peaks.

In the rest of this chapter, we first focus our attention on the continuum response and study the numerical value of spin gap in some detail. Then, we will turn to the possibility of excitonic response and determine the necessary and sufficient conditions for their existence within our QED$_3$ theory.

In our notation in this chapter, we usually measure the wave vectors from $2K_i$. So, the response $\chi_i(p, \omega) = \chi(2K_i + p, \omega)$, and $\chi(p)$ is the response at $2K_i + p$. If there is a case for confusion we will specify the origin appropriately. By symmetry, $\chi_i(p) = \chi_i(-p)$

### 6.2 Ladder approximation

The full action of spinons and the gauge field is given by (see Eq. (2.37)):

$$S = \int \frac{d^3r}{(2\pi)^3} \sum_{i=1}^{N_f} \bar{\psi}_i \gamma_\mu (\partial_\mu - ia_\mu) \psi_i + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} a_i^\dagger(-p)D^{-1}(p)a_i^\dagger(p),$$

(6.2)

with $N_f = 2$. Only the transverse components, $a^i$, of the gauge field enter the action. The Dirac spectrum is assumed for spinons. We will also study some aspects of of the full spectrum in Section 6.4. A relation between the spin operator and the spinon fields is given by Eq. (3.8):

$$\bar{\psi}_i(r)\psi_i(r) = 4\cos(2K_i \cdot r)S_z(r).$$

After some straightforward algebra, we find

$$\chi(p) = \frac{1}{16} \int d^3r e^{-ip \cdot r} \sum_{ij} \langle \bar{\psi}_i(r)\psi_i(r)\bar{\psi}_j(0)\psi_j(0) \rangle.$$ 

(6.3)

In order to calculate the spin response in the QED$_3$ action we will adopt a ladder approximation for the four-point correlator, $\chi(p)$. The ladder approximation for $\chi(p)$ is shown diagrammatically in Fig. 6.3. The first diagram represents the “bare” spin response in the absence of gauge field interactions. We denote it by $\chi_0(p)$.

It is standard to reformulate the ladder approximation in terms of corrections to the scalar vertex for the interaction between the external source (neutrons) and spinons arising from the gauge field interactions. These vertex corrections are shown in Fig. 6.4. Defining
the scalar vertex, $\Gamma(k, p)$, as the amputated diagram (without the external legs) on the left-hand side, we have

$$\chi(p) = -\frac{N_f}{16} \int \frac{d^3k}{(2\pi)^3} \text{tr} \left[ G_0(k) \Gamma(k, p) G_0(k + p) \right],$$

(6.4)

where

$$G_0(k) = \frac{-i\gamma_\mu k_\mu}{k^2}$$

(6.5)

is the (bare) four-component spinon propagator in the superconducting state. Also, we see from Fig. 6.4 that in the ladder approximation employed in this chapter, the scalar vertex satisfies the following Bethe-Salpeter equation,

$$\Gamma(k, p) = 1 - \int \frac{d^3q}{(2\pi)^3} \gamma_\mu G_0(q) \Gamma(q, p) G_0(q + p) \gamma_\nu D_{\mu\nu}(k - q).$$

(6.6)

We may now use the symmetries of the QED$_3$ action (discrete symmetries and Lorenz invariance) to write the scalar vertex in general in terms of form factors \[130\],

$$\Gamma(k, p) = 1 F_1(k, p) + k_\mu p_\nu \sigma_{\mu\nu} F_2(k, p) + \gamma_\mu \left[ k_\mu F_3(k, p) + p_\mu F_4(k, p) \right].$$

(6.7)
Here, \( \sigma_{\mu
u} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \). Then, Eq. (6.6) is reduced to

\[
F_i(k, p) = \delta_{i1} + \int \frac{d^3q}{(2\pi)^3} \sum_{j=1}^{4} K_{ij}(k, q, p) F_j(q, p),
\]

(6.8)

with

\[
K_{11}(k, q, p) = 2 \frac{q \cdot (q + p)}{q^2(q + p)^2} D(k - q),
\]

(6.9)

\[
K_{12}(k, q, p) = 2 \frac{q^2p^2 - (q \cdot p)^2}{q^2(q + p)^2} D(k - q),
\]

(6.10)

\[F_3\) and \(F_4\) completely decouple from \(F_1\) and \(F_2\). Also, the spin response in Eq. (6.4) reduces to

\[
\chi(p) = \frac{N_f}{4} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{k \cdot (k + p)}{k^2(k + p)^2} F_1(k, p) + \frac{k^2p^2 - (k \cdot p)^2}{k^2(k + p)^2} F_2(k, p) \right].
\]

(6.11)

That is, \(F_3\) and \(F_4\) do not appear in \(\chi\) and so we may neglect them altogether for the purpose of calculating the spin response.

Within the ladder approximation for the scalar vertex, our manipulations have so far been exact. At this point, we notice that most of the contribution to \(\chi\) must come from \(F_1\) [131], which would start out with a finite value if we were to solve Eq. (6.8) iteratively. We will then drop \(F_2\) at this stage to obtain the set of equations [132]

\[
F_1(k, p) = 1 + \lambda \int \frac{d^3q}{(2\pi)^3} \frac{q \cdot (q + p)}{q^2(q + p)^2} D(k - q) F_1(q, p),
\]

(6.12)

\[
\chi(p) = \frac{N_f}{4} \int \frac{d^3k}{(2\pi)^3} \frac{k \cdot (k + p)}{k^2(k + p)^2} F_1(k, p)
\]

(6.13)

for the spin response, with \(\lambda = 2\). These are our basic equations for this chapter except Section 6.4, where we consider the full spectrum of spinons, thus modifying the kernel of these equations.

### 6.3 Low-energy calculation

We set up a low-energy calculation based on the following two approximations:
(A) Linearized spectrum around the nodes;

(B) Constant (massive) propagator for the vortex gauge field.

Approximation (A) means that we neglect all the higher-derivative terms in the dispersion of quasiparticles. This is a good approximation for \( p < \Delta_0 \approx 40 \text{ meV} \). We keep the velocity anisotropy, \( v_F \gg v_\Delta \), to the lowest order. At this level and for the spin response, we may take into account the effects of the velocity anisotropy by rescaling the momenta in their direction, as \( q_x \to q_x/v_F \) and \( q_y \to q_y/v_\Delta \) at the pair of nodes \((I,\bar{I})\) and with \( x \leftrightarrow y \) for \((II,\bar{II})\). We will restore the velocities at the end of the calculation. With this point in mind, we note that (A) has already been included in deriving Eqs. (6.12) and (6.13).

Approximation (B) is good for \( p \lesssim \Lambda_{\text{Th}} \approx 3\pi\mu/2 \). The "Thirring cutoff," \( \Lambda_{\text{Th}} \), is found by intersecting the linear, ultraviolet asymptote of \( D^{-1}(p) \) and the finite, infrared limit, \( D^{-1}(0) \). As discussed in Ref. 46, we may write \( \Lambda_{\text{Th}} \approx 2.7 T_c \lesssim 30 \text{ meV} \), where the approximate equality is found at optimal doping. Then, we have

\[
D^{-1}(p) \approx D^{-1}(0) = 12\mu/\pi \equiv m, \quad p < \Lambda_{\text{Th}} = \pi^2 m/8. \tag{6.14}
\]

We call (B) the "constant-mass approximation."

We are now in a position to apply the constant-mass approximation for the gauge field propagator. For constant \( D(k - q) = m^{-1} \) we see that the integral on the right-hand side of (6.12) is independent of \( k \), and thus the scalar vertex is independent of its first argument. So, we may easily solve for \( F_1 \) and then \( \chi \) to find the RPA (random phase approximation) form

\[
\chi(p) = \frac{\chi_0(p)}{1 - 4m^{-1}\chi_0(p)}, \tag{6.15}
\]

where we have plugged in the values \( \lambda = 2 \) and \( N_f = 2 \). The bare response, \( \chi_0(p) \), is given by

\[
\chi_0(p) = -\frac{N_f}{16} \int \frac{d^3k}{(2\pi)^3} \text{tr} \left[ G_0(k) G_0(k + p) \right] \tag{6.16}
\]

\[
= \frac{N_f}{4} \int \frac{d^3k}{(2\pi)^3} \frac{k \cdot (k + p)}{k^2(k + p)^2} \tag{6.17}
\]

\[
\Lambda \to \infty \quad \approx \frac{N_f}{8} \left( \frac{\Lambda}{\pi^2} - \frac{|p|}{8} \right). \tag{6.18}
\]
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Figure 6.5: Imaginary part of the spin response at $Q_{AF} = (\pi, \pi)$ for the low-energy calculation. The unit of the vertical axis is arbitrary. $k_F = 0.45\pi$, $v_F = 400$ meV and $m = 50$ meV.

We have introduced a UV cutoff to regularize the integral. In the constant-mass approximation, $\Lambda = \Lambda_{Th}$. We observe that at this lowest level, we have found a linear dependence on momentum. Upon going to real frequencies, $p_0 \rightarrow i\omega + \eta$ ($\eta$ is a positive infinitesimal number), the imaginary part is

$$\Im\chi_0(p, \omega) = \frac{N_f}{64} \sqrt{\omega^2 - p^2} \Theta(\omega^2 - p^2). \tag{6.19}$$

Restoring the velocities, we see that this response exhibits a spin gap,

$$\omega_{sg}(p) = \sqrt{v_F^2 p_x^2 + v_\Delta^2 p_y^2}. \tag{6.20}$$

The full RPA response Eq. (6.15) exhibits the same spin gap, since

$$\Im\chi(p) = \frac{\Im\chi_0(p)}{[1 - 4m^{-1}\Re\chi_0(p)]^2 + [4m^{-1}\Im\chi_0(p)]^2}. \tag{6.21}$$

The (staggered) spin response at $Q_{AF} = (\pi, \pi)$ is plotted in Fig. 6.5. The qualitative behaviour in this plot is in general agreement with experiment. There is an even wider
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qualitative agreement with experiment over a range of momenta and energy as discussed by Herbut and Lee [46]. Here, we concentrate on two important features of the energy plots such as the one in Fig. 6.5, namely, the spin gap below which there is no response, and the peak.

This response fits in the first scenario discussed in Section 6.1. As shown, for realistic values of parameters, we find a spin gap, \( \omega_{sg}(Q_{AF}) \approx 150 - 200 \) meV. This is about a factor of 4 - 5 times bigger than the experimental value. Also the peak appears as a maximum above the spin gap, due to the overlap of the response around the four incommensurate vectors, \( 2K_i \). At higher energies, the response falls off rather slowly as \( \omega^{-1} \).

As we will show in the next section, it is possible to obtain a reduced value for the spin gap close to the experimental values, by considering the full spectrum of the quasiparticles. In this “full-spectrum” calculation, the general qualitative behaviour of the response is not altered; especially, the nature of the peak remains the same. In Section 6.5, we study the effects of the momentum dependence of the gauge field propagator on the spin response, and in particular, the question of whether the observed peak can arise as a particle-hole resonance below the spin gap, in the superconductor.

6.4 Full-spectrum calculation

In this section, we relax approximation (A), i.e. the linear spectrum for quasiparticles, and study the effects of higher-derivative terms on the spin response in the superconductor. We show that it is possible to obtain numerical agreement with observed values of spin gap in experiment for a reasonable range of dispersion parameters. This connects the findings of, say, angle-resolved photoemission spectroscopy (ARPES) and inelastic neutron scattering experiments.

From the full Hamiltonian in Eq. (2.6) (see also Appendix A), we see that retaining the full quasiparticle spectrum amounts to taking the four-component spinon inverse propagator for flavour \( i = 1, \Sigma \) to be given by

\[
G_{i,BCS}^{-1}(k) = i\gamma_{\mu}e_{i\mu}(k) - e_{i\mu}(-k) + i\gamma_{\mu}\gamma_{5}e_{i\mu}(k) + e_{i\mu}(-k) \quad (6.22)
\]
or,

\[ G_{i,\text{BCS}}(k) = i\gamma_\mu \frac{e_i^2(k)e_{i\mu}(-k) - e_i^2(-k)e_{i\mu}(k)}{2e_i^2(k)e_i^2(-k)} + i\gamma_\mu \gamma_5 \frac{e_i^2(k)e_{i\mu}(-k) + e_i^2(-k)e_{i\mu}(k)}{2e_i^2(k)e_i^2(-k)}, \]

(6.23)

with

\[ e_{i\mu}(k_0, k) = [k_0, \xi(K_i + k), \Delta(K_i + k)]. \]

(6.24)

The bare response \( \chi_0 \) is replaced by the corresponding \( \chi_{\text{BCS}} \). Note that,

\[ e_i^2(k_0, k) = k_0^2 + E^2(K_i + k), \]

where \( E(k) = \sqrt{\xi^2(k) + \Delta^2(k)} \) is the full BCS dispersion.

We pause to point out two important issues in this calculation. First, since we are considering general non-linear dispersion, the response around \( 2K_i \) is not symmetric under \( p \leftrightarrow -p \) anymore. So, we need to separately derive \( \chi_i(p, \omega) \) and \( \chi_i(-p, \omega) \). Secondly, in this calculation we are still assuming that spinons are only minimally coupled to the gauge field. Strictly speaking, this is not true, since the gauge field will also appear in higher-derivative terms through the Franz-Tešanović transformation. However, these terms will be important only at much higher energies, and for the purposes of our RPA calculation we are justified in neglecting them.

Separating the two signs of \( \chi_i(\pm p, \omega) \) is not difficult. We note that

\[ \bar{\psi}_i(r) \frac{1 \pm \gamma_5}{2} \psi_i(r) = 2e^{\pm 2ik \cdot r} S_z(r). \]

(6.25)

Thus,

\[ \left\langle \bar{\psi}_i(r) \frac{1 + \gamma_5}{2} \psi_i(r) \bar{\psi}_i(r') \frac{1 - \gamma_5}{2} \psi_i(r') \right\rangle = 4 \int \frac{d^3p}{(2\pi)^3} \chi_i(p, \omega)e^{ip \cdot (r-r')} \]

(6.26)

So, we can calculate

\[ \chi_{i,\text{BCS}}(p) = -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \text{tr} \left[ G_{i,\text{BCS}}(k) \frac{1 - \gamma_5}{2} G_{i,\text{BCS}}(k + p) \frac{1 + \gamma_5}{2} \right] \]

\[ = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e_i(k) \cdot e_i(-k - p)}{e_i^2(k)e_i^2(-k - p)}. \]

(6.27)

Note that the factor \( 1/2 = N_f/4 \). It is immediately seen that this equation reduces to \( \chi_0(p) \) in Eq. (6.17) if we take a linear spectrum around the nodes, \( e_{i\mu}(k) = k_\mu \).
We may now perform the integral over $k_0$ by contour integration, and analytically continue the result to real frequencies by $p_0 \rightarrow i\omega + \eta$:

$$
\chi_{i,\text{BCS}}(p, \omega) = \frac{1}{4} \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{E(E + \omega + \xi' + \Delta')}{E\left[(E + \omega - i\eta)^2 - E'^2\right]} + \frac{E'(E' - \omega + \xi' + \Delta')}{E'\left[(E' - \omega + i\eta)^2 - E'^2\right]} \right\}.
$$

(6.28)

The unprimed variables are calculated at $K_i + k$ and the primed ones at $K_i - k - p$. This is, of course, the well-known BCS result [133, 134]. The full response is again given by the RPA form,

$$
\chi = \frac{\chi_{\text{BCS}}}{1 - 4m^{-1} \chi_{\text{BCS}}}.
$$

For positive frequencies, the imaginary part is found from Eq. (6.28) as

$$
\Im \chi_{i,\text{BCS}}(p, \omega) = \frac{\pi}{8} \int \frac{d^2 k}{(2\pi)^2} \frac{EE' - \xi \xi' - \Delta \Delta'}{EE'} \delta(E + E' - \omega).
$$

(6.29)

Thus, the spin gap is given by the minimum energy of the particle-hole continuum,

$$
\omega_{\text{sg}}(Q) = \min_k \left\{ E(k) + E(k + Q) \right\}.
$$

(6.30)

The vectors in Eq. (6.30) are measured from $(0, 0)$. This is what one expects on physical grounds. It is easy to see that our result (6.20) in the linearized case follows from this equation.

To see if we can get a smaller value of the spin gap at $Q_{\text{AF}}$, we must specify the dispersion $\xi(k)$. The simplest form is the one for a nearest-neighbour tight-binding model on a square lattice, $\xi^{(n)}(k_1, k_2) = -2t(\cos k_1 + \cos k_2) + 4t \cos k_F$, where $t$ is the nearest-neighbour hopping amplitude. It yields a spin gap for a momentum $k$ in Eq. (6.30) along the nodal direction,

$$
\omega_{\text{sg}}^{(n)}(Q_{\text{AF}}) = 8t \cos k_F.
$$

In the linear approximation, we find

$$
\omega_{\text{sg}, \text{linear}}^{(n)}(Q_{\text{AF}}) = 4t(\pi - 2k_F) \sin k_F \leq \omega_{\text{sg}}^{(n)}(Q_{\text{AF}}).
$$

As a result, the nearest-neighbour hopping results in an increased spin gap, and is not adequate for a numerical account of experiment.
We will then add the next-nearest-neighbour hopping, \( t' \), as well. The dispersion now reads

\[
\xi^{(nn)}(k_1, k_2) = -2t(\cos k_1 + \cos k_2) + 4t' \cos k_1 \cos k_2 + 4t \cos k_F - 4t' \cos^2 k_F. \tag{6.31}
\]

The momentum \( k \) in Eq. (6.30), for which the spin gap is found, moves away from the nodal direction due to next-nearest-neighbour hopping.

In a physical sample, \( t' \) is fixed. In order to arrive at a physical picture, we need to set up a self-consistent framework to calculate \( k_F \) and \( \Delta_0 \) and find the spin gap. To this end, we must postulate a microscopic model of the pairing mechanism. The \( t-t'-J \) model is the next simplest model that one can consider \([135, 136]\). A range of values of parameters are used in the literature for modeling different experiments, e.g. ARPES \([137, 138]\). Here, we will use the following values:

\[
t \approx 150 \text{ meV}; \quad t'/t = 0.18; \quad J/t = 0.5.
\]

The self-consistent equations for the \( d \)-wave gap and the doping, \( x \), are written as

\[
J^{-1} = \int \frac{d^2k}{(2\pi)^2} \frac{(\cos k_1 - \cos k_2)^2}{E(k)} \tag{6.32}
\]

\[
x = \int \frac{d^2k}{(2\pi)^2} \frac{\xi(k)}{E(k)}. \tag{6.33}
\]

As noted by Herbut [36], for the relevant values of our parameters, the \( d \)-wave superconductor is in the weakly coupled regime, and the position of the nodes, \( k_F \), is not much affected by setting \( \Delta_0 = 0 \). We then see from Eq. (6.33) that \( x \) decreases with increasing \( k_F \) and vanishes at some \( k_{F0} < \pi/2 \). For our choice of parameters, \( k_{F0} \approx 0.465\pi \). In the right panel of Fig. 6.6 we have plotted \( k_F(x) \) found numerically from Eqs. (6.32)–(6.33). On the other hand, the integrand in Eq. (6.32) vanishes where \( \Delta(k) \) does. So, \( J \) varies strongly with \( \Delta_0 \). Numerical integration shows that at constant \( \Delta_0 \), \( J \) has a minimum at some \( k_{Fm} < k_{F0} \). Putting these two together we find \( \Delta_0(x) \), shown on the left panel of Fig. 6.6. As we noted above, the variation in \( \Delta_0/t \approx 0.3 \) is small, but we observe the presence of a broad maximum in the gap amplitude before turning (rather quickly) down for higher values of doping. This maximum is a consequence of the minimum in \( J(k_F) \) at constant \( \Delta_0 \).
Figure 6.6: The gap amplitude (left) and position of nodes (right) vs. doping for $t'/t = 0.18$ and $J/t = 0.5$. See text for discussion.

Using these solutions, in Fig. 6.7 we compare the values of the full-spectrum and the linear spin gap at $Q_{AF}$ as a function of doping. We see that we get a reduction in the spin gap that increases as doping increases. For example, at $x = 0.12$ in the underdoped region, we have a reduction factor of 2.12. This results in an almost flat spin gap as doping is increased, replacing the steeply increasing behaviour in the low-energy calculation. This is closer to recent experimental results [139], considering the simplicity of our scheme. The spin gap in the full-spectrum calculation is $\omega_{g0}(Q_{AF})/t = 0.6$, which is still higher than the observed value. However, our point is that it is possible to obtain better numerical agreement with experiment by including the high-energy details of the band structure.

By examining our numerical solution for different values of $t'/t$ and $J/t$, we find that, increasing $t'/t$ decreases both $k_{F0}$ and $k_{Fm}$. This pushes the nodes away from ($\pi, \pi$) and further increases the spin gap in the low-energy calculation rather rapidly. Thus, we would need an even bigger reduction in the full-spectrum calculation to match observed values. In fact, we find that the reduction factor also increases with increasing $t'/t$, so this is not necessarily in the wrong direction. A decreasing $k_{Fm}$ pushes the broad maximum in the gap amplitude to higher values of doping, which is harder to reconcile with experiments. Increasing $J/t$ causes an increase in the average value of $\Delta_0/t$. This diminishes the reduction factor of the spin gap. These observations point to the fact that our treatment here needs
Figure 6.7: The spin gap at $Q_{AF} = (\pi, \pi)$ in low-energy and full-spectrum calculations vs. doping for $t'/t = 0.18$ and $J/t = 0.5$. See text for discussion.

to be made even more refined for a more accurate account of experiments. However, on the level of our theory, there is a reasonable agreement both qualitatively and quantitatively between theory and experiments.

The scenario discussed in this section is common to all RPA studies, or the so-called fermiology approaches [140, 141, 142]. What is important for us is the QED$_3$ basis of our RPA calculation. The RPA coupling here is the gauge field mass $m \propto T_c$, and thus it becomes a better approximation around optimal doping. With underdoping, where phase fluctuation effects captured in QED$_3$ become more and more important, the constant-mass approximation necessary to derive the RPA response becomes less reliable, and we may expect to find new physical effects due to higher-energy dependence of the gauge field propagator. In the next two sections, we set up a rigorous way to study this physics in the ladder approximation of Section 6.2.
6.5 Theory of excitons in QED₃

In this section we study the effects of high-energy gauge field interactions on spinons. We relax our approximation (B) for the low-energy calculation.

A distinct possibility considered widely in the literature as an explanation for the observed peak in neutron scattering experiments is the formation of a particle-hole bound state, or spin exciton, which would then yield a resonance peak in the spin response below the continuum spin gap discussed so far. This resonance would be sharp, and would explain the sharp fall-off of the scattering signal in experiment as opposed to the $1/\omega$ tail of the continuum response derived in this chapter. This continuum tail is expected to be independent of the approximations employed in the low-energy calculation, as is seen from, say, the Lehmann spectral representation [143] of the spin response.

Our formulation in terms of ladder diagrams is well suited to study this scenario within the QED₃ theory of phase fluctuating $d$-wave superconductor. In this section we develop the necessary formalism for this purpose. Specifically, we derive a Schrödinger-like equation for the particle-hole bound state, and express the spin response in the ladder approximation in terms of its eigenfunctions and spectrum. This formulation is reminiscent of the theory of excitons in semiconductors [144] and metals [145, 146] where an excited electron forms a bound state with a core hole. There, too, a resonance or edge singularity in the X-ray emission spectra is observed depending on the situation.

Let us define a generalized response

$$\Phi_{\mu\nu}(r, p) = \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}_\mu(\mathbf{k} + p)_\nu}{k^2(\mathbf{k} + p)^2} e^{ik \cdot r} F_1(k, p). \quad (6.34)$$

Then from Eq. (6.13), we see that

$$\chi(p) = \frac{N_f}{4} \Phi_{\mu\mu}(0, p) \equiv \frac{N_f}{4} \Phi(0, p), \quad (6.35)$$

where a summation over repeated greek indices is assumed hereafter. Also, Eq. (6.12) is written as

$$F_1(q, p) = 1 + \lambda \int d^3r' e^{-iq \cdot r'} D(r') \Phi(r', p). \quad (6.36)$$

To proceed from here, we form a matrix response, $\hat{\Phi} = \sigma_\mu \sigma_\nu \Phi_{\mu\nu}$ where $\sigma_\mu = (\sigma_1, \sigma_2, \sigma_3)$
are Pauli matrices (the exact order does not matter). We see that
\[
-e^{-ip \cdot r} \sigma_\mu \partial_\mu e^{ip \cdot r} \sigma_\nu \partial_\nu \hat{\Phi}(r, p) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot r} F_1(k, p) = \delta^{(3)}(r) + \lambda D(r)\Phi(r, p),
\]
where \( \partial_\mu = \partial/\partial r_\mu \) and we have used Eq. (6.36) in the second line. Noting that \( \sigma_\mu \sigma_\nu = \delta_{\mu\nu} + i\epsilon_{\mu\nu\rho} \sigma_\rho \), we find,
\[
-e^{-ip \cdot r} [(ip + p \times \sigma) \cdot \partial + \partial \cdot \partial] \hat{\Phi}(r, p) = \delta^{(3)}(r) + \lambda D(r) \text{tr}\Phi(r, p).
\]
We may decompose the matrix response into its symmetric and anti-symmetric components as
\[
\hat{\Phi} = \mathbb{1}\Phi + i\sigma \cdot \Phi^A,
\]
where \( \Phi^A = \epsilon_{\mu\alpha\beta} \Phi_{\alpha\beta} \). Then, after some algebra we can derive the following four equations for \( \Phi \) and \( \Phi^A \):}

\[
-(ip \cdot \partial + \partial \cdot \partial)\Phi + i(p \times \partial) \cdot \Phi^A = \delta^{(3)}(r) + \lambda D(r)\Phi, \quad (6.40)
\]
\[
ip \times \partial\Phi + (ip \cdot \partial + \partial \cdot \partial)\Phi^A + ip(\partial \cdot \Phi^A) - i\partial(p \cdot \Phi^A) = 0. \quad (6.41)
\]

From Eq. (6.38) it is now clear that we may completely shift away the momentum \( p \), which enters the equation only as a parameter, by a phase transformation of the form,
\[
\hat{\Phi} \rightarrow \hat{\Phi}' = \exp \left[ \frac{1}{2} (ip + p \times \sigma) \cdot \tau \right] \hat{\Phi}.
\]

Denoting the orbital angular momentum of the particle-hole system by \( L = r \times p \), the transformed equation is then
\[
[- \partial \cdot \partial + (p/2)^2] \hat{\Phi}' - \lambda D(r)e^{i\sigma \cdot L} \text{tr}\left[ e^{-i\sigma \cdot L} \hat{\Phi}' \right] = \delta^{(3)}(r). \quad (6.43)
\]
Writing \( \hat{\Phi}' = \Phi' + i\sigma \cdot \Phi^A' \), we see that this transformation is equivalent to
\[
e^{-i\frac{p \cdot \tau}{2}} \Phi' = \cosh |L/2| \Phi + i \sinh |L/2| \hat{L} \cdot \Phi^A, \quad (6.44)
e^{-i\frac{p \cdot \tau}{2}} \Phi^A' = -i \sinh |L/2| \hat{L} \Phi + \cosh |L/2| \Phi^A + i \sinh |L/2| \hat{L} \times \Phi^A. \quad (6.45)
\]

\( \epsilon_{\mu\nu\rho} \) is the totally anti-symmetric tensor in three dimensions.
We also observe that
\[ \Phi'(0, p) = \Phi(0, p). \] (6.46)

By Eq. (6.43) they satisfy
\[
\begin{align*}
\left( (p/2)^2 - \partial \cdot \partial - \lambda \cosh^2 |L/2| D(r) \right) \Phi'(r, p) \\
+ i\lambda \sinh |L/2| \cosh |L/2| D(r) \hat{L} \cdot \Phi'(r, p) &= \delta^{(3)}(r), \quad (6.47) \\
\left( (p/2)^2 - \partial \cdot \partial + \lambda \sinh^2 |L/2| D(r) \hat{L} \hat{L} \right) \Phi'(r, p) \\
+ i\lambda \sinh |L/2| \cosh |L/2| D(r) \hat{L} \Phi'(r, p) &= 0. \quad (6.48)
\end{align*}
\]

These are a set of coupled Schrödinger equations.

Even though we have succeeded in deriving a set of Schrödinger equations for our vertex function, they are still very complicated. It seems that in order to find an analytical solution we need to make further approximations or postulate some ansatz. A numerical solution would also be a useful alternative. In the following section, we analyze an approximate, decoupled Schrödinger equation given by
\[
\left( (p/2)^2 - \partial \cdot \partial - \lambda \cosh^2 |L/2| D(r) \right) \Phi'(r, p) = \delta^{(3)}(r). \quad (6.49)
\]

We expect our results to be valid for small enough \( p \), or an adequately short-ranged interaction \( D(r) \). Both these conditions apply to our situation. Also in the case of constant-mass approximation, where \( D(r) \sim \delta^{(3)}(r) \), we find (6.49) exactly.

With this approximation, we may find \( \chi(p) \) by solving the following three-dimensional Schrödinger equation for the eigenvalue \( e_n(p) \) and the normalized eigenfunction \( \psi_n(r, p) \),
\[
\left[ -\nabla^2 - \lambda \cosh^2 \frac{|r \times p|}{2} D(r) \right] \psi_n(r, p) = e_n(p) \psi_n(r, p). \quad (6.50)
\]

Then,
\[
\Phi'(r, p) = \sum_n \frac{\psi_n(r, p) \psi_n^*(0, p)}{e_n(p) + p^2/4}, \quad (6.51)
\]

and
\[
\chi(p) = \frac{N_f}{4} \Phi'(0, p) = N_f \sum_n \frac{|\psi_n(0, p)|^2}{4e_n(p) + p^2}. \quad (6.52)
\]

By analytically continuing to real frequencies, \( p_0 \rightarrow i\omega + \eta \), we find that if \( \Im e_n(p, i\omega) = 0 \) the imaginary part of the spin response observed in experiment is given by
\[
\Im \chi(p, \omega) = N_f \sum_n |\psi_n(0, p, i\omega)|^2 \text{sgn} \left( \omega - \frac{\partial e_n(p, i\omega)}{2\omega} \right) \delta \left( p^2 + 4e_n(p, i\omega) - \omega^2 \right). \quad (6.53)
\]
For $\Im e_n(p, i\omega) \neq 0$ there is no $\delta$-function, hence no resonance.

From Eq. (6.53), we see that the necessary and sufficient condition for the existence of excitonic resonances at $p$ and $\omega < |p| = \omega_{eg}(p)$ is that our Schrödinger equation (6.50) admits bound state solutions with a real eigenvalue and negative eigenvalues $e_n(p, i\omega)$ that solve the equation

$$\omega^2 = p^2 + 4e_n(p, i\omega). \quad (6.54)$$

A sufficient condition for the existence of such solutions is provided by

$$e_n(p, 0) > -p^2/4 \text{ and } e_n(p, i|p|) < 0.$$

### 6.6 Resonance

In this section we will give a brief discussion of the possibility of particle-hole bound states in the superconductor. Though we have formulated the general conditions to address this issue in terms of the approximate Schrödinger equation (6.50), at this time we do not have a comprehensive study of the bound states of (6.50), partly due to the complicated form of the potential appearing there. Of course, even if there are negative eigenvalues, they have to satisfy Eq. (6.54) for a bound state to exist. Our discussion here is basically only a query on the existence of such negative eigenvalues.

Before we turn to the general case, let us comment on the situation in the constant-mass approximation. As mentioned above, in this case Eq. (6.50) would be an exact result. We have not specified any particular functional dependence for $D(r)$ in our formalism of Section 6.5, and so this would also be a check on its validity.

The RPA response, Eq. (6.15), would exhibit a resonance if the denominator vanished for a frequency $\omega < |p|$, that is, if $1 - 4m^{-1}\Re \chi_0(p, \omega) = 0$. Using Eq. (6.18), we can rewrite this condition as

$$m - \frac{\Lambda}{\pi^2} + \frac{1}{8} \sqrt{p^2 - \omega^2} \geq 0.$$

This is possible if

$$m_R \equiv m - \frac{\Lambda}{\pi^2} \leq 0. \quad (6.55)$$

In the constant-mass approximation, the UV cutoff is $\Lambda = \Lambda_{Th} = \pi^2 m/8$. So, $m_R = 7m/8 > 0$ and the inequality (6.55) is never satisfied. We conclude that there are no bound states, thus, no resonances in the low-energy calculation.
The same conclusion follows from our Schrödinger equation formalism. In the constant-mass approximation, \( D(r) = m^{-1} \delta^{(3)}(r) \). So the Schrödinger equation (6.50) admits a bound state energy \(-|E|\) if it is a solution to

\[
\begin{align*}
\frac{m}{\Lambda - \infty} &= \frac{\Lambda}{\pi^2} - \frac{|E|}{2\pi}.
\end{align*}
\]

We have substituted the value \( \lambda = 2 \) for the final result. This is possible only if \( m_R < 0 \) as in Eq. (6.55).

In the general case, we first need to derive the real-space interaction \( D(r) \). Since there is a gap in the low energy limit, we expect an exponential decay in the long-distance behaviour of \( D(r) \). Also, the \( p^{-1} \) tail in the high-energy limit, should translate to a \( r^{-2} \) singularity at short distances. Let us make these observations exact by computing \( D(r) \) using contour integration techniques similar to those in Appendix B. We start by noting that

\[
D(r) = \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} D(q)
\]

\[
= \frac{\mu}{4\pi r} \int_0^{\Lambda/2\mu} F(1/2z) \sin(2\mu rz) dz
\]

\[
= \tilde{D}(r) - \frac{\mu}{8r},
\]

where \( F \) is the function introduced in Eq. (2.39) and \( \Lambda \) is a UV cutoff. Then,

\[
\tilde{D}(r) = \frac{\mu}{8\pi r} \int_{-\Lambda/2\mu}^{+\Lambda/2\mu} \frac{1 + z^2}{z^2} \tan^{-1} z \sin(2\mu rz) dz.
\]

In order to use contour integration, we need to choose two branch-cuts to define \( \tan^{-1} z \). We take them to be \([+i, +i\infty)\) and \([-i, -i\infty)\), and define

\[
\begin{align*}
z - i &= r_1 e^{i\theta_1}, \quad -\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}, \\
z + i &= r_2 e^{i\theta_2}, \quad -\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}, \\
\tan^{-1} z &= -\frac{i\pi}{2} + \log \frac{r_2}{r_1} + i(\theta_2 - \theta_1).
\end{align*}
\]

We take a contour \( C \) that includes the real axis and closes on itself in the upper-half plane, except that it avoids the upper branch-cut by tracing the contour \( C_1 = -F(0^+) \cup F(0^-) \)
where $F(\varepsilon) = [+i + \varepsilon, +i\infty + \varepsilon]$. So, we can now write

$$\tilde{D}(r) = \frac{\mu}{8\pi r} \Im \left( \int_{C_1}^{C} \left( 1 + \frac{z^2}{2} \arctan z e^{2\mu r z} \right) dz \right)$$

$$= \frac{\mu}{8r} \left[ 1 + \int_{1}^{\infty} \frac{u^2 - 1}{u^2} e^{-2\mu u} du \right]. \tag{6.59}$$

So, we find

$$D(r) = \frac{\mu}{8r} \int_{1}^{\infty} \frac{u^2 - 1}{u^2} e^{-2\mu u} du \tag{6.60}$$

$$= \frac{e^{-2\mu}}{16r^2} - \frac{\mu}{8r} E_2(2\mu r). \tag{6.61}$$

Note that $D(r) > 0$. Here, $E_n(x) = \int_{1}^{\infty} e^{-ux} du/u^n$ is the exponential integral. For $n = 2$ it has the following asymptotic behaviour

$$E_2(x) = \begin{cases} 
    e^{-x} \left( \frac{1}{x} - \frac{2}{x^2} + \cdots \right), & z \to \infty, \\
    1 + (\gamma_E - 1 + \log x)x + \cdots, & z \to 0.
\end{cases}$$

which correspondingly leads to the following asymptotic behaviour for $D(r)$:

$$D(r \to \infty) = \frac{e^{-2\mu}}{64mr^3} + O(r^{-4} e^{-2\mu}), \tag{6.62}$$

$$D(r \to 0) = \frac{1}{16r^2} - \frac{\mu}{4r} - \frac{\mu^2}{4} \log(2\mu r) + \frac{3 - 2\gamma_E}{8} \mu^2 + O(r). \tag{6.63}$$

This potential is closely linked to the inverse-square potential. More specifically, we conjecture here that $-\lambda D(r)$ has a bound state if and only if $-\lambda/16r^2$ does.

The second part ("only if") of the statement is easy to show: from Eq. (6.61), we observe that

$$-\lambda D(r) > -\frac{\lambda}{16r^2}, \tag{6.64}$$

so if the inverse-square potential does not support a bound state, neither does $-\lambda D(r)$.

We have not shown the first ("if") part. Instead, we demonstrate a weaker version of it. First we note that the attractive inverse-square potential, $-\lambda/16r^2$ supports a bound state for $\lambda > \lambda^* = 4$ [147]. We now show that $-\lambda D(r)$ has a bound state for $\lambda > 8$. Taking a
variational trial wave function \( \psi(r) = (8\pi a^3)^{-\frac{1}{2}} e^{-r/2a} \), we find the average energy,

\[
\langle -\nabla^2 - \lambda D(r) \rangle = \frac{1}{2a^3} \left[ \frac{a}{2} - \lambda \int r^2 e^{-r/a} D(r) dr \right] \\
= \frac{1}{2a^3} \left[ \frac{a}{2} - \frac{\mu \lambda}{8} \int_1^{\infty} \frac{u^2 - 1}{u^2} \int r e^{-(2m u + \frac{1}{2})r} dr du \right] \\
= \frac{1}{4a^2} \left\{ 1 - \frac{\lambda}{8\alpha^2} [\alpha(\alpha - 2) + 2 \log(1 + \alpha)] \right\} \rightarrow \frac{1}{4a^2} \left( 1 - \frac{\lambda}{8} \right) < 0 \quad \text{for} \ \lambda > 8,
\]

where \( \alpha = (2\mu a)^{-1} \). The limit in the last line is equal to the average found for \(-\lambda/16r^2\) as expected. This corroborates the expectation that on a more careful examination, our conjecture would be found to be valid.

Lastly, we note that the potential appearing in the Schrödinger equation (6.50) is

\[
-\lambda \cosh^2 \frac{|r \times p|}{2} D(r) < -\lambda D(r),
\]

so if \(-\lambda D(r)\) does support a bound state, so does Eq. (6.50), assuming that \(|p| < 2\mu\). For \(|p| > 2\mu\), the hyperbolic cosine factor overcomes the exponential decay of \(D(r) \sim e^{-2\mu r}\) at large distances, and the potential becomes unbounded from below for a cone of angle \(\sin^{-1}(2\mu/|p|)\) perpendicular to \(\hat{p}\). It is not quite clear whether a bound state can exist in this case.

For \(\lambda = 2\), the above discussion shows that \(-\lambda D(r)\) does not support bound states. However, as our notation has been intentionally chosen, the value of \(\lambda\) is not necessarily fixed at \(\lambda = 2\). The reason is that the scalar vertex \(\Gamma(k, p)\) in the ladder approximation is not gauge invariant. If we introduce a gauge fixing parameter \(\zeta\) by writing

\[
D_{\mu \nu}(p) = D(p) [1 + (\zeta - 1)\hat{p}_\mu \hat{p}_\nu],
\]

we discover that \(\lambda \rightarrow \lambda(\zeta) = 2 + \zeta\). Thus, \(\lambda(\zeta > 2) > \lambda^*\) and our discussion above indicates that for \(\zeta > 2\) there exist negative eigenvalues of the potential, and possibly particle-hole bound states. One way to fix a value for \(\zeta\) is to find such a gauge in which our choice of working with the bare form of the spinon propagator is consistent with gauge invariance. In three space-time dimensions, this gives \(\zeta < 2\) [132, Eq. (B6)].
In conclusion, further work is needed to settle the question of exciton bound states in QED$_3$. We believe the formalism presented here could be helpful to this end. Given the growing trend of experimental evidence on spin response in the superconductor the answer to this question could serve as a stringent test of the QED$_3$ theory of underdoped cuprates.
Chapter 7

Concluding Discussion

In this final chapter we will give a general discussion of our works, the issues surrounding them, and the possible ways they may be improved or superseded. Along the way, we will also comment on interesting problems for the future.

7.1 $\text{QED}_3$

The $\text{QED}_3$ formulation of the phase fluctuating $d$-wave superconductor discussed in this thesis is, of course, closely related to the one commonly and extensively studied in the high-energy physics literature. There, $\text{QED}_3$ has long been recognized as a fertile ground to cultivate and test ideas and theories about such fundamental issues in high-energy physics as confinement and dynamical generation of mass. But, it has only been regarded as a toy model, as it were, to investigate these topics for the more fundamental theory of quantum chromodynamics in four space-time dimensions.

However, as we hope is evident from this thesis, in condensed matter physics $\text{QED}_3$ is a completely realistic theory of actual physical systems, albeit an effective one. Being effective is not a shortcoming or a weakness per se. In one sense, anything beyond individual electrons and quarks (so far) is an effective entity, and so are all condensed matter systems. It is true that $\text{QED}_3$ is effective on a level that does not address some of the questions commonly asked in the field of “high-$T_c$,” such as the origin of superconductivity. But, as our very brief introduction in Chapter 1 sketched, the occurrence of superconductivity is much less of a mystery, in the context of the Hubbard or the $t$-$J$ model, than the rest of the
underdoped phase diagram. These other peculiarities are what QED$_3$ aims to address.

We also hope that this thesis was able to show that, not only does QED$_3$, as a theory of cuprates, have quantitative results and predictions observed in experiment [48, 32, 45], but also it does so rather well, as argued for the spin response in Chapter 6.

As unresolved issue in QED$_3$ is the following question: what is the gauge-invariant electron propagator, $G^\text{inv}(p)$? The imaginary part of this propagator, i.e., the spectral function, is presumably what is measured in ARPES. In fact, the earliest formulation of QED$_3$ [30] was geared toward finding exactly this quantity in the pseudogap phase. There has been quite a bit of confusion and debate [30, 35, 148, 149, 150, 151] about this question because different approximation schemes seem to yield different results for the anomalous dimension $\eta$ found from the low-energy limit $G^\text{inv}(p) \sim p^{-1+\eta}$. Some of these values are actually negative, for which the physical electron propagator would be even more singular at low energy than the bare electron. Intuitively this is an unphysical result, since repulsive interactions mediated by the gauge field are expected to soften such singularities, not enhance them. This question has been pursued in the high-energy community as well [152].

In regard to the work done on this question, it seems unlikely that QED$_3$ would be discredited on this issue. For instance, it is perfectly possible to calculate gauge-invariant four-point correlators with physical anomalous dimensions [153]. Further investigation to resolve this issue is needed.

It is possible to build very similar gauge theories starting from the other end of the phase diagram, at half filling, with a doped Mott insulator. It seems plausible to expect that these similar theories and QED$_3$ converge in the middle, or that they are instances of the same theory in different limits, although the physical interpretations of the ingredients appear to be unconnected at the moment. This unification of some of the existing theories of cuprates would indeed be an important step to a better understanding of high-temperature superconductivity.

### 7.2 Confinement

Our work on confinement presented in this thesis has been met with two classes of criticisms.

The first class criticizes the conclusions drawn from our studies within their own frame-
work. Our work pointed out a shortcoming in previous studies of confinement, which all start, as we do, with integrating out gapless fermions (spinons) and study the effective theory of the gauge field thus found. As discussed in Chapter 4, this effective theory has a cubic inverse propagator (logarithmic potential in real space) between magnetic monopoles. All these studies had overlooked the possibility of the generation of a quadratic piece in the inverse propagator, which screens the real-space logarithmic potential at large distances back to a Coulomb potential. There is no phase transition in a three dimensional Coulomb gas.

After our work was published, two numerical works by Sudbø and colleagues have appeared discussing the phase diagram of the three-dimensional logarithmic gas (3DLG) [154, 155]. Using Monte Carlo simulations, these authors have studied, respectively, the size of the dipoles and the jump of the helicity modulus at the transition for the two- and three-dimensional Coulomb gas (2DCG and 3DCG) and the 3DLG. Their numerical results show a similar behaviour for the 2DCG and 3DLG, while the 3DCG appears to behave differently. These results have been taken to mean that the 3DLG admits a phase transition of the KT type just like the 2DCG. There is no phase transition found for the 3DCG.

Numerical studies of this kind are very subtle affairs. For instance, it usually remains an unfulfilled desire to have been able to perform the numerical methods on a larger system less prone to the limiting factors of the finite size of the system. A more concrete point of caution about the method used in these works is the following. The steps of the Metropolis algorithm defined there constitute an attempted insertion of dipoles of the smallest size (equal to the lattice spacing) that is accepted with a probability equal to the Boltzman factor $p = \exp\left[\frac{E_{\text{old}} - E_{\text{new}}}{T}\right]$. This obviously favours very small dipoles from the outset and aggravates the finite-size restrictions mentioned above. One unwanted effect of this procedure could be that, in a system with a strong bare potential at large distances such as the 3DLG, it would allow larger dipoles only at higher temperatures than physically necessary at the available system sizes, even if no true dipole binding transition exists in the system. In the 3DCG, where the bare potential is weak at large distances, the same unwanted effect is less pronounced. The ultimate way of resolving this issue is either to study larger systems or to modify the Metropolis steps to allow for larger dipoles that are not energetically unfavourable from the outset. Both these paths require much higher memory and computing power; but, we believe that such technical problems are surmountable.

The second, more serious, class of criticisms objects to the very starting point of our
analysis, i.e., the integration out of gapless fermions and the following description in terms of a monopole gas with pairwise interactions. Two papers in this class have appeared recently, one by Hermele et al. [156] and the other by Nogueira and Kleinert [157], on which we will separately comment below.

The question raised about integrating out gapless fermions is a valid concern. Hermele et al. point out that the perturbative expansion of the action in powers of the gauge field around the classical background field \( a_\mu = 0 \) leads to the wrong action, and they suggest, instead, that one must perform the expansion for each classical monopole configuration separately. They believe that non-Gaussian terms would then be impossible to neglect even at large \( N_f \) and, therefore, a monopole gas with pairwise interaction does not describe the physics of the compact \( \text{QED}_3 \) correctly. They also argue, seemingly generally, that the scaling dimension of all monopole operators at the fixed point of the non-compact theory is of order \( N_f \) and, thus, irrelevant at large \( N_f \). In reference to another work [158], they state that there is a one-to-one mapping between monopole operators in \( \text{CP}^N \) model and the quantum states on the surface of a sphere enclosing the magnetic flux of the corresponding monopole. The scaling dimension of the monopole is mapped to the energy of the above state, which is claimed to be bounded below by a number of \( O(N_f) \). Therefore, they conclude that, the non-compact fixed point is stable against monopole events at large \( N_f \) and that, the deconfined phase survives in the presence of monopoles.

The core energy of an instanton with charge \( q \) was calculated by Murthy and Sachdev [159] in the context of the \( \text{CP}^N \) model in \( 2 + 1 \) dimensions. They find that the core energy scales non-linearly with the charge \( q \). This result seems to support the claim that the pairwise interaction is inadequate. However, these authors also pointed out a subtle issue about the order of limits. I believe the same issue could prove important in the analysis of Hermele et al., as well. There are two limits in both problems: \( \xi \to \infty \) and \( N \) or \( N_f \to \infty \), where \( \xi \) is a correlation length characterizing the critical behaviour of the system. In order to analyze the phases of the system one must take the limits as \( \lim_{N \to \infty} \lim_{\xi \to \infty} \), that is, approach the fixed point associated with the phase in question at fixed (and large) \( N \). The opposite order is not enough to establish the phase diagram of the system.

From the way their general proof is constructed, it seems that Hermele et. al. are working in the wrong order of limits. They fix the system at the non-compact fixed point and let \( N_f \to \infty \), the same as Murthy and Sachdev. Ultimately, it seems that the stability
argument put forward by Hermele and coauthors does not apply to a phase of the system but, rather, to a single point. It was, however, mentioned in our published papers that the possibility of deconfined states is not ruled out by our analysis at isolated points where monopole events may be tuned off (perhaps even automatically) by tuning the coefficient of the quadratic piece of the inverse propagator to zero. This appears to be the case for a recent body of work on deconfined criticality, for instance [160, 161]. On this note, I believe the argument presented by these authors does not contradict our work, although their initial point of criticism is not invalid. The way to resolve that issue, however, must be by reconsidering or avoiding integrating out gapless modes on the background of a trivial classical field, in the analysis of the phase diagram of the system.

Nogueira and Kleinert present an elegant RG analysis that is apparently set up as a step in this direction. This is a welcome step, one that may indeed resolve the issue. Their RG argument combines the effects of the gapless matter fields into the RG equations of Polyakov's pure compact theory. The effect they include is the gauge-field renormalization factor, which multiplies the expression for the renormalized charge. Their results indicate that monopole events are irrelevant above a surprisingly small value of fermion flavours, $36/\pi^3 \approx 1.161$. If correct, this means that, in the continuum and for the physically relevant case of $N_f = 2 < N_f^c$, the spinons can be deconfined but massive, hence constituting a deconfined antiferromagnet. Interestingly, they also find that the transition is not of the KT type.

There is also a great industry devoted to simulating gauge theories of fermions on the lattice [88]. Implementing fermions on the lattice is tricky. Also, achieving the chiral limit is a costly computational task. Thus, few reliable studies of compact QED$_3$ exist so far [102]. The most recent one by Hands, Kogut and Lucini [108] seems to support our results.

Our self-consistent perturbation scheme developed in Chapter 5 may have other applications. It would be interesting to investigate them in other systems of interest. It is also interesting to see if there is any relation between our scheme and another variational perturbation theory developed elsewhere [162, 163, 164].
7.3 Spin response

Our study of spin response addresses one of the important probes of the dynamics of cuprates. We have limited ourselves to the superconducting phase, but our methods are more general and may be applied outside the superconductor as well, with careful modification of details. It is also another example of practical applications of QED$_3$ as a theory of cuprates.

In our work, we have neglected any effects of having more than one copper-oxide plane in the system. Such effects are important and expected to explain [165, 166, 142] some of the phenomenology observed in neutron scattering experiments. It remains an interesting problem for the future to study these effects in the spin response of the QED$_3$. Similar effects for the superfluid density within the logic of QED$_3$ have already been studied [32, 48].

Most of the theoretical work on the spin response of cuprates so far [167, 140, 168, 169, 170, 171, 141, 142] rely on an RPA form of spin response, for instance, in the the t-J model [172, 173, 174], or in the Hubbard model [175], or as a working assumption [176]. QED$_3$ also yields a simple RPA form, but only in the lowest-energy calculation. However, unlike the other models, QED$_3$ does not support a particle-hole bound state for its RPA response. The absence of resonance in this approximation is closely related to the absence of coexistence between dSC and AF orders [46]. They both arise because the cutoff introduced in this approximation is not independent of the mass of the gauge field in the superconductor and the renormalized mass $m_R = m - \Lambda/\pi^2 > 0$. This also seems to imply, based on the more general argument presented for the absence of coexistence of dSC and AF orders by Herbut and Lee [46], that there is no resonance in QED$_3$ more generally. Our very preliminary results in Chapter 6 may be interpreted as an indication for this statement.

However, as was noted in Section 6.6, the scalar vertex in our calculation is not gauge-invariant. In fact, it seems possible to obtain resonant response for an infinite range of choices of the gauge-fixing parameter. But it is not quite certain which one of these represent the physical gauge-invariant result. A procedure used in the literature [132] to fix the gauge based on the consistency of the choice of the spinon propagator (in our computation taken to be the bare one) still fails to produce a bound state. Further investigation to clear up this issue is needed at this time.
Our procedure for deriving a Schrödinger equation for the vertex function in QED\textsubscript{3} has not appeared before in the literature to our knowledge, although it is quite similar to the theory of excitons in semiconductors and metals. Since QED\textsubscript{3} has, as mentioned above, found a prominent role in condensed matter as well as high-energy physics, this may be an important calculation. Its results may also be compared to the ones obtained using approximate methods for studying the vertex function in the ladder approximation. In our formalism, the crucial approximation is postponed to the stage of solving the Schrödinger equation, for which, one expects, there should exist a wider array of controlled and accurate solutions, both analytically and also numerically.

Finally, we note that the existence of excitons in conventional superconductors were studied a long time ago [177, 178, 179, 180]. Our treatment is formally more similar to the theory of excitons in semiconductors and metals [181, 146, 145, 144]. But, this formal similarity is not physically important: employing the Nambu spinor-representation of the BCS theory, one would expect to be able to recast the theory of excitons in conventional superconductors in the language used in this chapter.
Appendix A

Effect of general nonlinear terms

In this appendix, we will examine the nonlinear terms discussed in Section 3.5 for a general dispersion relation. For instance, one might wonder how our conclusions would be affected if we considered additional hopping amplitudes (next nearest neighbour, etc.). We will show that our results remain the same even if we assume a general functional form for the dispersion of the normal state on the square lattice.

We start by decomposing the nonlinear pieces of the dispersion and the gap function based on the following symmetry properties: the first $+$ $(-)$ indicates a piece that is even (odd) in $q$ and the second $+$ $(-)$ indicates a piece that is even (odd) upon the reflection $q_x \leftrightarrow q_y$, which sends the node I to II. The four-fold symmetry of the square lattice means that at, say, node I, $\xi(K_1 + (q_x, -q_y)) = \xi(K_1 + (q_x, q_y))$. This means, that at any given node, $\xi$ cannot be odd in both $q_x$ and $q_y$. Thus, $\xi_{++}$ can only be even in both arguments.

In contrast, the $d_{x^2-y^2}$ gap function $\Delta(k_1, k_2) = \frac{1}{2} \Delta_0(\cos k_1 - \cos k_2)$ yields $\Delta_{+-}(q) \equiv 0$, and

$$\Delta_{++}(q_x, q_y) = -\Delta_{++}(-q_x, q_y) = -\Delta_{++}(q_x, -q_y); \quad (A.3)$$

that is, $\Delta_{++}$ is an odd function of each one of its arguments.

Then, we define $B_a = \Gamma_0 A_a$ where $A_a$ is one of the following $8 \times 8$ matrices,

$$A_{\Delta, \Delta} = \mathbf{1} \otimes M_\Delta, \quad (A.4)$$
APPENDIX A. EFFECT OF GENERAL NONLINEAR TERMS

and

\begin{align}
A_{\xi,\xi} &= 1 \otimes M_{\xi}; & A_{\xi,-\xi} &= \sigma_3 \otimes M_{\xi}; & (A.5) \\
A_{1,1} &= 1 \otimes M_1; & A_{1,-1} &= \sigma_3 \otimes M_1; & (A.6) \\
A_{2,2} &= 1 \otimes M_2; & A_{2,-2} &= \sigma_3 \otimes M_2. & (A.7)
\end{align}

With these definition, we can write the nonlinear piece of the action as

\[ S_{NL} = \int \frac{d^3q}{(2\pi)^3} \bar{\psi}(q) \left[ B_{\xi,\xi}\xi_{++}(q) + B_{\xi,-\xi}\xi_{+-}(q) + B_{1,1}\xi_{++}(q) + B_{1,-1}\xi_{+-}(q) \\
+ B_{2,2}\Delta_{++}(q) + B_{2,-2}\Delta_{+-}(q) + B_{2,-2}\Delta_{--}(q) \right] \psi(q). \] (A.8)

In the first order calculation, we have to find the trace of two possible combinations of matrices, \( \text{tr}(B_\alpha) \) and \( \text{tr}(\Gamma_\mu B_\alpha) \). Both are identically zero for all states of the chiral manifold. So the first non-vanishing contribution of the nonlinear terms is generated in the second-order.

In the second order, we have the following possible combinations:

1. \( \text{tr}(\Gamma_\mu B_\alpha B_\beta \Gamma_\nu) = \text{tr}(\Gamma_\mu A_\alpha \Gamma_\mu A_\beta). \)

Noting that

\[ \Gamma_\mu \Gamma_0 = \delta_{\mu0} \mathbb{1} + \delta_{\mu i} A_i, \]

we see that this trace is the same for all states of the chiral manifold independent of the choice of \( \Gamma_0 \).

2. \( \text{tr}(\Gamma_\mu B_\alpha B_\beta) = \text{tr}(\Gamma_0 A_\alpha \Gamma_\mu \Gamma_0 A_\beta) = 0, \)

for all the states of the chiral manifold.

3. The only non-vanishing traces are of the form,

\[ \text{tr}(B_\alpha B_\beta) = \text{tr}(\Gamma_0 A_\alpha \Gamma_0 A_\beta). \] (A.9)

The factors multiplying these traces are expressed as,

\[ \frac{1}{2} V m^5 \int d^3x \frac{m^{-4}}{(2\pi)^3 (1 + x^2)^2} (\xi_\alpha \text{ or } \Delta_\alpha)(mx)(\xi_\beta \text{ or } \Delta_\beta)(mx), \] (A.10)

where \( x = q/m \) is the dimensionless momentum.
Upon further inspection, we discover that most of the terms vanish because the above integral factors are zero. It is easy to see there are no mixing between terms whose \( \pm \) indices do not match, because the integrands will be odd upon one of the transformations, \( q \to -q \) or \( q_x \to q_y \). Also, there is no mixing between \( \xi_{++} \) and \( \Delta_{++} \) because they have different parities as pointed out above.

Moreover, most of the traces vanish too. For instance, \( \text{tr}(B_{1,1}B_{2,2}) \) vanishes for all the states except for CDW. On the other hand, \( \text{tr}(B_{1,1}^2) \) vanishes for CDW and has the same sign for all others. We will neglect these terms as they do not distinguish between \( d + ip / d + is \) and SDW. The energy of the CDW has already been raised above all the rest by the anisotropy, \( S_A \), which is the most important perturbation. We only register how SDW is affected compared to the rest of the states.

This leaves us with the following traces: \( \text{tr}(B_{\xi,\xi}^2) \); \( \text{tr}(B_{\xi,-\xi}^2) \); \( \text{tr}(B_{\Delta,\Delta}^2) \). They have the same value for all the states:

\[
\text{tr}(B^2) = \begin{cases} 
-8 & \text{d + is and d + ip} \\
0 & \text{CDW} \\
+8 & \text{SDW}
\end{cases} \tag{A.11}
\]

The multiplying factor is given by

\[
\frac{1}{2} V m^5 \int_1 d^3 x \frac{m^{-4}}{(2\pi)^3} \frac{1}{(1 + x^2)^2} \left[ \xi_{++}^2(mx) + \xi_{+-}^2(mx) + \Delta_{++}^2(mx) \right], \tag{A.12}
\]

which is positive. Thus, we have shown the general feature that the energy of the SDW is increased, whereas that of the \( d + ip \) and \( d + is \) is decreased, by nonlinear terms, as we set out to do.
Appendix B

Screening in a three-dimensional gas

In this appendix, we investigate the behaviour of the real space propagator of the sine-Gordon action, Eqs. (4.20) and (4.33), in three dimensions ($D=3$) and for the full cQED$_3$ ($\sigma=3$). Since the term $T|q|^3$ is non-analytic at $q=0$, it will lead to some interesting behaviour at large distances. We will also include the fugacity $\gamma$ as a mass term, arising from the quadratic approximation to the cosine function. In the plasma phase $\gamma \neq 0$ and we expect to find the screened potential as a result of our calculations.

Let us write

$$V_{\text{scr}}(r; T, a, \gamma) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq \cdot r}}{T|q|^3 + a|q|^2 + \gamma}$$

$$= \frac{1}{2\pi^2} \int_{\mu |r|}^{\Lambda |r|} \frac{z \sin z}{Tz^3 + a |z|^2 + \gamma |z|^3} dz,$$

where we have introduced the infrared (IR) and ultraviolet (UV) cut-offs $\mu$ and $\Lambda$, respectively, to regularize the integral in various limits as needed.

We will first present some simpler cases before going into the details of the general case. First, consider $T=0$. This corresponds to the regular sine-Gordon action in the plasma phase. Then, using contour integration,

$$V_{\text{scr}}(r; 0, a, \gamma) = \frac{1}{4\pi^2} \int_{C_1} \frac{z e^{iz}}{a |r|^2 + \gamma |r|^3} dz$$

$$= \frac{e^{-\sqrt{\gamma/a} |r|}}{4\pi a |r|}.$$
Contour $C_1$ traces the real axis from $-\infty$ to $\infty$ and closes on itself in the upper-half complex plane. This is the usual Deby-Hückel screening of the Coulomb potential in three dimensions.

Now, consider $a = y = 0$, which corresponds to the putative dipole phase of the logarithmic gas in the absence of any screening. We will need both the IR and UV regularizations in this limit:

$$\gamma_E$$ is the Euler-Mascheroni constant. This result was used in Eq. (4.16).

Next, let us look at a more complicated situation. Taking only $y = 0$, we obtain the dipole screening discussed in Section 4.5. In order to use contour integration again, we need to regularize the non-analytic term $\sim z$ by introducing an infinitesimal parameter: $z = \lim_{\eta \to 0^+}(z^2 + \eta^2)^{1/2}$. Then,

$$V_{\text{sc}}(r; T, a, 0) = \frac{1}{4\pi^2} \lim_{\eta \to 0^+} \int_{-\infty}^{+\infty} \frac{\sin z}{T(z^2 + \eta^2)^{1/2} + a |r|} \frac{dz}{z}$$

In order to make sense of the square root in the complex plane, we need to choose two branch-cuts. We choose them to be $(-i\infty, -i\eta]$ and $[i\eta, +i\infty)$, and define

$$z - i\eta = r_1 e^{i\theta_1}, \quad -\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2},$$
$$z + i\eta = r_2 e^{i\theta_2}, \quad -\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2},$$

so that,

$$(z^2 + \eta^2)^{1/2} = \sqrt{r_1 r_2} e^{i\theta_1 + \theta_2/2}.$$ 

We denote the shifted upper branch-cut by $\varphi(\varepsilon) \equiv [i\eta + \varepsilon, +i\infty + \varepsilon)$ and define a contour $C_2$ that is similar to $C_1$ defined above except that it goes around the branch-cut in the upper-half plane by tracing $C = -\varphi(0^+) \cup \varphi(0^-)$. Having set our notation and integration contours,
we find

\[ V_{\text{sec}}(r; T, a, 0) = \frac{1}{4\pi^2} \lim_{\eta \to 0^+} \Im \left[ \left( \int_{C_2} - \int_C \right) \frac{e^{iz}}{T(z^2 + \eta^2)^{1/2} + a|z|} \, dz \right] \]

\[ = \frac{1}{4\pi^2 a|r|} \frac{T}{3} \int_0^\infty \frac{e^{-u}}{T^2 u^2 + a^2 r^2} \, du \]

\[ \approx \frac{1}{4\pi^2 a|r|}, \]

which is nothing but the Coulomb potential in three dimensions stated in Section 4.5.

Finally, Having dealt with non-analyticity by way of branch-cuts, we are well positioned to attack the general case with finite \( y \), which represents the screening in the plasma phase of the logarithmic monopole gas. It is now easy to see that,

\[ V_{\text{sec}}(r; T, a, y) = \frac{1}{4\pi^2} \lim_{\eta \to 0^+} \Im \left[ \left( \int_{C_2} - \int_C \right) \frac{z e^{iz}}{T(z^2 + \eta^2)^{1/2} + a|z|^2 + y|r|^3} \right] \]

\[ = V_{\text{res}}(r) - \frac{T}{2\pi^2 y^2 |r|} \int_0^\infty \frac{u^4 e^{-u}}{(a|u|^2 - y|r|^3)^2 + T^2 u^6} \, du \]

\[ \approx -\frac{12T}{\pi^2 y^2 |r|^6} + O(aT/y^3 |r|^8). \]

The piece shown by \( V_{\text{res}}(r) \) is the contribution from the residues of the poles of the integrand (B.15) in the upper-half plane. This piece is exponentially decaying and drops out in the \( |r| \to \infty \) limit. Recalling that \( T = \bar{T}/2\pi^2 = 2/(\pi^2 N_f) \) we find the result quoted in Eq. (4.54).
Appendix C

Duality and RG for $U(1)$ spin liquids

In this appendix, we provide the details for the duality and renormalization group analysis of the spin liquid state used in Section 4.8

C.1 Duality

The duality follows closely the one for the relativistic case in Section 4.3. Our subsequent analysis is aided by writing $S_{SL}$ in Eq. (4.55) in a form which is explicitly gauge invariant. We define the electromagnetic field $F_{\mu\nu} = i k_\mu a_\nu - i k_\nu a_\mu$ where $k_0 = \omega$. Then, it is easy to see that

$$S_{SL} = \int \frac{d^3 k d\omega}{(2\pi)^3} \left[ \frac{1}{4} \left( \frac{\omega |\sigma_s(k)|}{k^2} + \chi_d + \frac{1}{e^2} \right) F^2_{ij} + \frac{1}{2} \left\{ \frac{\delta_{ij}}{e^2} + \frac{\chi_s k_i k_j}{k^4} \left( 1 + \frac{\gamma |\omega|}{k} \right) \right\} F_{i0} F_{j0} \right].$$

(C.1)

The expression (C.1) for $S_{SL}$ also includes a regular $F^2_{\mu\nu}/(4e^2)$ term in the action for the gauge field as discussed in the main text below Eq. (4.57).

We may now proceed to dualize the above action for the gauge fields. Before doing so, it is necessary to explicitly account for the compact nature of the $U(1)$ gauge group on the scale of underlying lattice. We do this by discretizing space and time, and writing the
compactified lattice version of the action (C.1) as
\[
S_{SL} = \sum_{r,r',\tau} \left\{ \frac{1}{4} [F_{ij}(r,\tau) - 2\pi n_{ij}(r,\tau)] V_\perp(r - r', \tau - \tau') [F_{ij}(r',\tau') - 2\pi n_{ij}(r',\tau')] \\
+ \frac{1}{2} \Delta_i [F_{i0}(r,\tau) - 2\pi n_{i0}(r,\tau)] V_\parallel(r - r', \tau - \tau') \Delta_j [F_{j0}(r',\tau') - 2\pi n_{j0}(r',\tau')] \right\} \\
+ \frac{1}{2e^2} \sum_{r,\tau} \left[ F_{i0}(r,\tau) - 2\pi n_{i0}(r,\tau) \right]^2,
\]
where the transverse and the longitudinal parts of the interaction in the Fourier space read
\[
V_\perp(k,\omega) = \frac{|\omega|\sigma_s(k)}{k^2} + \chi_d + \frac{1}{e^2}, \quad \text{C.3}
\]
\[
V_\parallel(k,\omega) = \frac{\chi_s}{k^4} \left( 1 + \gamma \frac{|\omega|}{k} \right). \quad \text{C.4}
\]
The integers \( n_{\mu\nu}(r,\tau) \) serve to account for the compact nature of the gauge fields, in the spirit of the Villain approximation. On a lattice, the electromagnetic tensor \( F_{\mu\nu} = \Delta_\mu a_\nu - \Delta_\nu a_\mu \), where \( \Delta_\mu \) is the lattice difference operator as before.

Performing the Hubbard-Stratonovich transformation using fields \( c_{\mu\nu} \) residing on the plaquettes of the lattice, we find
\[
S_{SL} = \sum_{r,r',\tau} \left[ c_{ij}(r,\tau) V_\perp^{-1}(r - r', \tau - \tau') c_{ij}(r',\tau') + \frac{1}{2} e^2 c_{i0}(r,\tau) \left\{ \delta_{ij} \delta_{r,r'} \delta_{\tau,\tau'} \\
+ \frac{1}{|\Delta|^2} \left[ \frac{1}{2e^2} \delta_{r,r'} \delta_{\tau,\tau'} + V_\parallel(r - r', \tau - \tau') |\Delta|^2 \right]^{-1} \right\} \Delta_i \Delta_j \right] c_{j0}(r',\tau') \\
+ i \sum_{r,\tau} c_{\mu\nu}(r,\tau) [F_{\mu\nu}(r,\tau) - 2\pi n_{\mu\nu}(r,\tau)]. \quad \text{C.5}
\]
The integral over the gauge fields can now be performed exactly, and it enforces the constraint
\[
c_{\mu\nu}(r,\tau) = \frac{1}{2} \epsilon_{\mu\nu\alpha} \Delta_\alpha \phi(r,\tau). \quad \text{C.6}
\]
The sum over the integers \( n_{\mu\nu}(r,\tau) \) can also be performed, and it forces the new variable \( \phi(r,\tau) \) to be an integer as well. From the constraint in Eq. (C.6) it immediately follows,
\[
\Delta_i c_{i0}(r,\tau) = 0, \quad \text{C.7}
\]
and the corresponding part in Eq. (C.5), which includes the longitudinal part of the interaction, $V_0$, completely drops out. Relaxing the integer constraint on $\phi$ by introducing a small fugacity $y$ in the usual way, and returning to the continuum limit we find the dual sine-Gordon action for the monopoles in Eq. (4.57).

### C.2 Renormalization group

The renormalization group analysis of Eq. (4.57) also follows along the lines drawn before in Section 4.6. We start by integrating out the short-distance modes with $\Lambda/b < k < \Lambda$, and $|\omega| < \Omega$, with $\ln(b) \ll 1$. Up to $O(y^3)$, the parameters for the remaining long-distance modes are then changed as

\[
\begin{align*}
I(b) &= l, \\
A_k(b) &= a_k + \frac{1}{2} y^2 e^{-G_{>}(0)} \int d^2r d\tau \tau^2 (e^{G_{>}(\tau,\tau)} - 1), \\
A_\omega(b) &= b^2 \left[ a_\omega + y^2 e^{-G_{>}(0)} \int d^2r d\tau \tau^2 (e^{G_{>}(\tau,\tau)} - 1) \right], \\
y(b) &= b^2 y e^{-\frac{1}{2} G_{>}(0)},
\end{align*}
\]

where the correlation function is defined, as before, by

\[
G_{>}(r, \tau) = \int_{\Lambda/b < k < \Lambda} \frac{d^2k}{(2\pi)^2} \int_{\Omega} \frac{d\omega}{2\pi} e^{ik \cdot r + i\omega \tau} G(k, \omega).
\]

and $G$ is the propagator defined in Eq. (4.58).

In order to derive the RG $\beta$-functions, we specialize to computing $G_{>}(r, \tau)$. Before getting into the technical details we note that, if $G(k, \omega)$ were completely independent of frequency, then $G_{>}(r, \tau) \sim \ln(b) e^{-\tau^\Lambda}$, rendering the integral in $a_\omega(b)$ to be exactly zero. This is the only apparent way the generation of this term can be avoided. This unphysical situation would correspond to essentially two dimensional sine-Gordon theory, which would then have the standard Kosterlitz-Thouless transition.

In the physical case, and for the initial values $a_\omega(b = 1) = 0$ and $a_k(b = 1) = e^2$, we
We assume the limit $\Omega/(le^2) \ll 1$, we can neglect the frequency in the denominator of the first integral, and approximate it by

$$\int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi} e^{i\omega r} e^{-\pi(\omega/2\Omega)^2} = \frac{\Omega}{\pi} e^{-\Omega^2 r^2/\pi},$$

where we have also replaced the sharp frequency cut-off by a Gaussian. We will similarly introduce a smooth cut-off for the second integral in Eq. (C.14) as

$$\int_0^{\infty} \frac{dk}{2\pi} k J_0(kr) \left[ \frac{1}{k^2 + (\Lambda/b)^2} - \frac{1}{k^2 + \Lambda^2} \right] = -\frac{\Lambda r}{2\pi} \ln(b) \frac{dK_0(z)}{dz} \bigg|_{z=\Lambda r}.$$

Altogether, we summarize:

$$G_>(r, \tau) \approx -\frac{\Omega \Lambda r}{2\pi^2 a_k} e^{-\Omega^2 r^2/\pi} \frac{dK_0(z)}{dz} \bigg|_{z=\Lambda r}.$$

This form yields $\beta_{ak}$ and $\beta_{aw}$ in Eqs. (4.59) and (4.60).

In order to find $\beta_y$, we assume $a_\omega > 0$. Then, the frequency cut-off $\Omega$ may be taken to infinity in the computation of $G_>(0, 0)$ appearing in the Eq. (C.11). This leads to

$$\beta_y = \left[ 2 - \frac{l}{4\pi \sqrt{D}} \left( 1 - \frac{2}{\pi} \tan^{-1} \frac{1}{\sqrt{D}} \right) \right] \tilde{y} + O(\tilde{y}^3),$$

for $D > 0$, where

$$D = 4\tilde{a}_k \tilde{a}_\omega l^2 - 1.$$

Eq. (C.18) reduces to Eq. (4.61) in the limit $D \gg 1$. For $D < 0$, one finds

$$\beta_y = \left[ 2 - \frac{l}{4\pi^2 \sqrt{|D|}} \ln \frac{1 + \sqrt{|D|}}{1 - \sqrt{|D|}} \right] \tilde{y} + O(\tilde{y}^3).$$
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