HOMOMORPHISM PROPERTIES OF GRAPH PRODUCTS

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Homomorphisms Properties of Graph Products

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This thesis consists of two main parts.

In the first part (Chapters 2, 3, and 4), we continue the investigation, initiated by R. Häggkvist, P. Hell, D. J. Miller and N. Lara, of graph multiplicativity. A graph $W$ is called multiplicative if the class of graphs that do not admit a homomorphism to $W$ is closed under (categorical) products. Two natural weaker versions of multiplicativity arise as auxiliary notions in this investigation, and we analyse their relationship to multiplicativity and to each other. The main research contributions arising from this part are: (1) a purely combinatorial proof of the multiplicativity of certain directed cycles, a result conjectured by J. Nešetřil and A. Pultr and first proved by R. Häggkvist et. al using a result from homotopy theory; (2) a complete characterization of multiplicative oriented paths; and (3) partial results on the multiplicativity of oriented cycles, which imply, in particular, that almost all oriented cycles are non-multiplicative. We also have partial results on the weaker versions of multiplicativity for oriented paths and cycles, and on the multiplicativity of general digraphs.

In the second part of this thesis (Chapter 5), we study homomorphism properties (in this case, just colorability properties) of Cartesian products of undirected graphs. The chromatic difference sequence of a graph essentially encodes the proportions of the largest $i$-colorable subgraphs, where $i$ is a positive integer. We obtain
four main results. They concern the chromatic difference sequences of products of bipartite graphs, of products of graphs with special chromatic difference sequences, the behaviour of the chromatic difference sequence on powers of graphs, and particularly on powers of circulant graphs.
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Chapter 1  Introduction

1.1 Definitions and Notation

For the definitions not given here see [12]. In particular, all graphs considered here are simple and finite. Furthermore, graphs $G$, $H$, etc., could be graphs or digraphs; similarly, the edge $gg'$, could mean the undirected edge $\{g,g'\}$ or the directed arc $\overrightarrow{gg'}$. The vertex set of $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$. The product $G \times H$ (also known as the categorical product [14], [19], conjunction [12], cardinal product, Kronecker product [5], [24], or weak direct product) has the vertex set $V(G) \times V(H)$ and the edges $(g,h)(g',h')$ where $gg' \in E(G)$ and $hh' \in E(H)$. The product $G \Box H$ (also known as Cartesian product [23]) has the vertex set $V(G) \times V(H)$ and the edges $(g,h)(g',h')$ where either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

An $n$-colouring $\varphi$ of $G$ is a map of $V(G)$ to $\{1, 2, \ldots, n\}$ such that $gg' \in E(G)$ implies $\varphi(g) \neq \varphi(g')$. We say that $G$ is $n$-chromatic and write $\chi(G) = n$ if $n$ is the minimum $k$ for which there is a $k$-colouring of $G$. A homomorphism $f: G \rightarrow H$ is a mapping $f: V(G) \rightarrow V(H)$ for which $f(g)f(g') \in E(H)$ whenever $gg' \in E(G)$. The existence, respectively non-existence, of a homomorphism $f: G \rightarrow H$ will be denoted by $G \rightarrow H$, respectively $G \not\rightarrow H$. Note that there is an $n$-colouring of $G$ just if $G \rightarrow K_n$ where $K_n$ is a complete undirected graph of $n$ vertices. Two graphs $G$ and $H$ are homomorphically equivalent,
denoted by $G \leftrightarrow H$, if both $G \to H$ and $H \to G$. We write $G' \subseteq G$ if $G'$ is a subgraph of $G$, or $G'$ is isomorphic to a subgraph of $G$. We write $G' \not\subseteq G$ if no subgraph of $G$ is isomorphic to $G'$. A subgraph $G'$ of $G$ is a retract of $G$ [15,16], if there is a homomorphism, called a retraction $r: G \to G'$ such that $r(g) = g$ for each $g \in V(G')$. We write $G' \not\in G$ to mean that there is a subgraph $G''$ of $G$ isomorphic to $G'$ such that $G''$ is a retract of $G$. We write $G' \not\in G$ to mean that there is a subgraph of $G$ isomorphic to $G'$, but no subgraph of $G$ isomorphic to $G'$ is a retract of $G$. We emphasize that both $G' \not\subseteq G$ and $G' \not\in G$ imply that $G' \subseteq G$.

Obviously, if $G'$ is a retract of $G$, or if $G' \not\subseteq G$, then $G' \leftrightarrow G$. A graph is called a core (or minimal graph, see [6]) if it has no proper retracts. Equivalently a core is a graph $H$ with $H \to G$ for any proper subgraph $G$ of $H$; i.e., a graph $H$ in which each homomorphism $H \to H$ is an isomorphism (see Lemma 1.3.6). Specifically, if a subgraph $W$ of a graph $G$ is a core, then $G \to W$ if and only if $W$ is a retract of $G$ (Lemma 1.3.7). In this case any subgraph of $G$ isomorphic to $W$ is a retract of $G$. A graph $G$ is called a core of another graph $H$, if $G$ is a core and $G \not\subseteq H$. Cores were studied in [6, 17].

A directed (respectively undirected) connected graph $W$ is multiplicative [11] if $G \leftrightarrow W$ and $H \leftrightarrow W$ imply $G \times H \leftrightarrow W$ for all directed (respectively undirected) graphs $G$ and $H$. It is important to note here that the graphs $G$ and $H$ are taken to be undirected graphs if $W$ is undirected, and directed graphs if $W$ is directed. (We shall omit all such remarks in the future.) A connected graph $W$ is weakly multiplicative if $W \not\in G$ and $W \not\in H$ imply $W \not\in G \times H$ for all connected graphs $G$ and $H$. A connected graph $W$ is very weakly
multiplicative if there do not exist two connected graphs G and H with $G \not\leftrightarrow H$, $H \not\leftrightarrow G$ such that $W \leftrightarrow G \times H$. We shall use $\mathcal{M}$, $\mathcal{W}$ and $\mathcal{V}$ to denote the classes of multiplicative graphs, weakly multiplicative graphs and very weakly multiplicative graphs respectively. We shall use $\overline{\mathcal{M}}$, $\overline{\mathcal{W}}$ and $\overline{\mathcal{V}}$ to denote the classes of non-multiplicative, non-weakly-multiplicative and non-very-weakly-multiplicative connected graphs respectively.

All graphs in the classes $\mathcal{M}$, $\mathcal{W}$, $\mathcal{V}$, $\overline{\mathcal{M}}$, $\overline{\mathcal{W}}$ and $\overline{\mathcal{V}}$ are q-connected. When we consider the multiplicative properties of a graph $W$ we always assume that $W$ is connected, and $W \neq K_1$. We shall not mention this assumption from now on.

Let $K_n$ denote the complete undirected graph with $n$ vertices. A complete digraph with $n$ vertices, denoted by $\vec{K}_n$, is a digraph with $n$ vertices and an arc between every ordered pair of vertices. Thus each $\vec{K}_n$ has $n(n-1)$ arcs.

In a digraph $G$, the directed edge (or arc) from a vertex $v_1 \in V(G)$ to a vertex $v_2 \in V(G)$ is denoted by $\overrightarrow{v_1v_2}$. The directed path $\vec{P}_n$ has a sequence of vertices $v_0$, $v_1$, $\ldots$, $v_n$ and arcs $\overrightarrow{v_0v_1}$, $\overrightarrow{v_1v_2}$, $\ldots$, $\overrightarrow{v_{n-1}v_n}$. The directed cycle $\vec{C}_n$ has a sequence of vertices $v_0$, $v_1$, $\ldots$, $v_{n-1}$ and arcs $\overrightarrow{v_0v_1}$, $\overrightarrow{v_1v_2}$, $\ldots$, $\overrightarrow{v_{n-2}v_{n-1}}$, $\overrightarrow{v_{n-1}v_0}$. An oriented path (cycle) is a digraph obtained from an undirected path (cycle) by choosing one direction for each edge. The net length of an oriented path (cycle) is the absolute value of the difference between the number of edges directed forward, and the number of edges directed
backward, with respect to any particular traversal order of the path (cycle).

Let $W$ be a fixed graph. A set $\theta \subseteq \{G: G \rightarrow W\}$ is called a complete set of obstructions for $W$ [11], if

1. for each $G$ with $G \rightarrow W$ there is an $X \in \theta$ such that $X \rightarrow G$ and

2. for each $X, X' \in \theta$ there is an $X^* \in \theta$ such that $X^* \rightarrow X$ and $X^* \rightarrow X'$.

Let $W$ be a fixed graph and $G$ an arbitrary graph. The map graph $W(G)$ is defined as follows: the vertices of $W(G)$ are the mappings $f: V(G) \rightarrow V(W)$ and the edges of $W(G)$ are just those $ff'$ for which $f$ and $f'$ are two mappings of $V(G)$ to $V(W)$ and $f(v)f'(v')$ is an edge of $W$ whenever $vv'$ is an edge of $G$. Note that $W(G)$ is directed or undirected depending on $G$ and $W$ being directed or undirected. Although normally our graphs have no loops, the map graph may have loops - cf. Lemma 1.3.5 (a).

The chromatic difference sequence $\text{cds}(G) = (a(1), \ldots, a(n))$ of an undirected graph $G$ of chromatic number $n$ is defined by

$$\sum_{j=1}^{t} a(j) = \text{maximum number of vertices in an induced } t\text{-colorable subgraph of } G \text{ (for } t = 1, 2, \ldots, n).$$

The normalized chromatic difference sequence of a graph $G$ is defined by

$$\text{ncds}(G) = \frac{\text{cds}(G)}{|V(G)|}.$$
The n-term sequence \((x_k)\) is said to dominate the n-term sequence \((y_k)\), written \((x_k) \geq^* (y_k)\), or \((y_k) \leq^* (x_k)\), if

\[
(1) \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k ;
\]

\[
(2) \sum_{k=1}^{p} x_k \geq \sum_{k=1}^{p} y_k \quad \text{for } p = 1, 2, \ldots, n-1.
\]

The chromatic difference sequence of a graph \(G\) of chromatic number \(n\), \(cds(G) = (r_1, r_2, \ldots, r_n)\), is said to be achievable if there exists an \(n\)-coloring of \(V(G)\): \(V_1, \ldots, V_n\) (i.e., \(V(G) = V_1 \cup V_2 \cup \ldots \cup V_n\), \(V_i \cap V_j = \emptyset\) (\(i \neq j\)) and each \(V_i\) is an independent set) such that \(r_i = |V_i|, i = 1, 2, \ldots, n\). A finite sequence of integers is said to be flat [1] if it is non-increasing and the largest and smallest nonzero terms differ by at most one. A finite sequence is called a non-drop flat (ND-flat) sequence if all terms are equal. A finite sequence of integers is called a first-drop flat (FD-flat) sequence if all terms are equal, except for the first term, which is greater by one.

Let \(N\) be a subset of \(\{1, 2, \ldots, p-1\}\) with the property that \(i \in N\) implies \(p-i \in N\). The circulant graph \(G(p, N)\) is the graph with vertices \(0, 1, \ldots, p-1\) and an edge joining \(i\) and \(j\) if and only if \(j-i \in N\), where we take \(j-i\) modulo \(p\). We call \(N\) the symbol of \(G(p, N)\).

We recursively define \(G^1 = G\), \(G^k = G^{k-1} \Box G\), and call \(G^k\) the \(k\)-th power of graph \(G\).
1.2 Overview

Given a graph $G$, a coloring $\varphi$ of $G$ and any graph $H$, there is a natural induced coloring $\varphi'$ of $G \times H$, namely $\varphi'(g,h) = \varphi(g)$. Thus we see that $\chi(G \times H) \leq \min(\chi(G), \chi(H))$. Hedetniemi [13] conjectured that equality holds for all graphs $G$ and $H$. An equivalent formulation of this conjecture is the following:

(i) For all positive integers $n+1$, $\chi(G) = \chi(H) = n+1$ implies $\chi(G \times H) = n+1$ for all graphs $G$ and $H$.

Or using the general terminology introduced in 1.1:

(ii) $K_n$ is multiplicative.

Hedetniemi's conjecture has been verified only in very few special cases. (The cases $n=1, 2$ are easy – cf., [13] – and the proof for $n = 3$ was a major breakthrough, [4].) It has been suggested in [11] that one should study multiplicativity of graphs in general, in order to gain insights relevant for the eventual proof of multiplicativity, or non-multiplicativity, of $K_n$. We continue this task in our thesis. In [11], the authors established the first infinite family of multiplicative (undirected) cores. In particular, they proved that each undirected odd cycle is multiplicative. As a consequence they also derived the multiplicativity of a large class of undirected graphs defined by Gerards [8]. (Cf. Section 2.3 for the exact definition of this class.) In [11] and [20], the authors also derived the multiplicativity of several families of directed graphs – namely of all transitive tournaments, all directed paths, and all directed cycles of prime power length. This was proved for prime lengths and conjectured for prime power lengths in [20], and the conjecture was proved in [11]; however, that
proof depends on a deep result from homotopy theory. Several related concepts, questions, and conjectures have also arisen, cf. [3] and [25].

There are in the literature several examples of (undirected or directed) graphs which are not multiplicative. However, in most instances they were constructed by taking two connected graphs $G$ and $H$ with $G \leftrightarrow H$, $H \leftrightarrow G$, and by taking $W = G \times H$. (For instance $\overrightarrow{C_p} \times \overrightarrow{C_q}$ is non-multiplicative for this reason when $p$ and $q$ are relatively prime, [11].) It will follow from Lemma 1.3.1 that this construction - equivalent to our definition of non-very-weak-multiplicativity of $W$ - assures that $W$ is non-multiplicative. On the other hand, Duffus et. al [3] and independently Welzl [25], discovered that $K_n$ is (in our terminology) weakly multiplicative. Because of these connections, we have investigated multiplicativity in conjunction with the auxiliary notions of weak and very weak multiplicativity.

It turns out that, for cores, weak multiplicativity and very weak multiplicativity coincide (Corollary 2.2.4), and are implied by multiplicativity (Claim 1.4.3). However, the converse is not true, and we show that the core $\overrightarrow{K_n}$ is weakly multiplicative but not multiplicative. This is then the first known example where a non-multiplicative graph $W$ is not constructed to be (homomorphically equivalent to) some $W = G \times H$; i.e., by the method explained above. Since Duffus et. al. [3] and Welzl [25] have proved the weak multiplicativity of $K_n$, this result suggests that the multiplicativity of
$K_n$ may well turn out to fail for some $n$. We obtain other connections among the various concepts of multiplicativity, cf. Sections 1.4 and 2.2.

In the direction of proving multiplicativity, we present a direct proof of the multiplicativity of directed paths. (Other proofs of their multiplicativity have been given in [11] and [20].) More importantly, we give a purely combinatorial proof of the multiplicativity of directed cycles of prime power length. As mentioned above, this was conjectured in [20], and previously [11], only a proof based on Lefschetz Duality Theory was known. Our proof is detailed in Section 4.2. Two other classes of multiplicative graphs are constructed in Sections 2.3, and 2.4. The one given in Section 2.4 is based on a construction of Komárek [18].

Our main contribution here is in developing a new method for proving the non-multiplicativity of directed graphs, in particular of oriented paths and cycles. We prove that each oriented path which is not homomorphically equivalent to a directed path is non-multiplicative (each oriented path which is homomorphically equivalent to a directed path is multiplicative). These constructions are quite interesting and are obtained by playing against each other various properties and parameters of oriented paths (Sections 3.2, 3.3 and 3.4). We also obtain partial classifications of oriented paths with respect to weak and very weak multiplicativity (Theorems 3.2.14 and 3.2.19). We do show that all oriented paths which are not cores are not weakly multiplicative except two examples. It seems likely that only the directed paths are weakly multiplicative.
We were able to derive, from the properties of oriented paths, the non-multiplicativity of a large class of arbitrary digraphs. For oriented cycles, these results are strong enough to imply that almost all oriented cycles are non-multiplicative. As mentioned earlier, all directed cycles of prime power length are multiplicative (and those of non-prime-power length are not multiplicative). Another class of multiplicative oriented cycles is the class of cycles $C_{k,k}$ (cf. Section 4.3) – each of these is homomorphically equivalent to the multiplicative graph $F_k$. However, we shall show that $C_{k,m}$ is non-multiplicative as long as $|k-m| \geq 2$ (Section 4.3). Conceivably, all oriented cycles (other than directed cycles of prime-power length) which are cores could be non-multiplicative. Partial results about weak and very weak multiplicativity of oriented cycles can also be found in Chapters 3 and 4.

In the second part of this thesis we study the behavior of Cartesian products with respect to another homomorphism concept, namely the chromatic difference sequence defined in 1.1.

In 1976, Greene and Kleitman studied the cds of comparability graphs of a poset [9, 10]. But they did not use the name cds. Then in 1980, Albertson and Berman proposed the concept of cds for general graphs and stated a conjecture characterizing when an n-term sequence is the cds of some n-colorable graph [1], and proved it when $n \leq 4$.

Recently Albertsen and Collins proved a "no-homomorphism" lemma [2]:

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If there exists a homomorphism from $G$ to $H$ and $H$ is vertex transitive, then $\text{ncds}(G)$ dominates $\text{ncds}(H)$.

Let $P$ denote the Petersen graph. They further studied the Cartesian powers of $P$ and proved that

$$\text{ncds}(P^k) = \frac{1}{3} \left( 1 + \frac{2}{10^k}, 1 - \frac{1}{10^k}, 1 - \frac{1}{10^k} \right)$$

These results motivate our research in the second part of this thesis.

We prove the following main results.

Assume that $\text{cds}(G) = (r_1, r_2, \ldots, r_n)$ is achievable and

$$|V(G)| = r = \sum_{i=1}^{n} r_i.$$ 

If $\text{cds}(H)$ is a first drop flat sequence $(s+1, s, \ldots, s)$ with $n$ terms, then

$$\text{cds}(G \Box H) = (rs+r_1, rs+r_2, \ldots, rs+r_n).$$

If $\text{cds}(H)$ is a non-drop flat sequence $(s, s, \ldots, s)$ with $n$ terms, then

$$\text{cds}(G \Box H) = (rs, rs, \ldots, rs).$$

Since $\text{cds}(G)$ is achievable if $\text{cds}(G)$ is a flat sequence, we obtain the formula for $\text{cds}(G^k)$ if $\text{cds}(G)$ is a first-drop or non-drop flat sequence.

The "non-increasing" theorem states that

$$\text{ncds}(G \Box H) \leq \min(\text{ncds}(G), \text{ncds}(H))$$

It has two important applications:

1. $\lim_{k \to \infty} \text{ncds}(G^k)$ exists.

2. If $G$ is a circulant graph, then $\text{ncds}(G^k) = \text{ncds}(G)$ for all $k$.

In the proof of (2) we use the no-homomorphism lemma.

We also prove the following results:
For bipartite graphs, if \( \text{cds}(G) = (r_1, r_2) \) and \( \text{cds}(H) = (s_1, s_2) \) are both achievable, then \( \text{cds}(G \square H) = (r_1 s_1 + r_2 s_2, r_1 s_2 + r_2 s_1) \). A formula for \( \text{cds}(G^k) \) is also obtained in that case. For circulant graphs we prove that if \( \text{cds}(G) \) is achievable, then \( \text{cds}(G^k) \) is achievable for all \( k \).

We propose two conjectures:

1. \( \text{cds}(G) \) is achievable for any circulant graph \( G \).
2. For any graph \( G \) of chromatic number \( n \), \( \lim_{k \to \infty} ncds(G^k) \) is either equal to \( \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \) or to \( ncds(G) \).

1.3 Preliminary Results

**Lemma 1.3.1** [11]

(a) \( G \times H \rightarrow G \) and \( G \times H \rightarrow H \).

(b) If \( X \rightarrow G \) and \( X \rightarrow H \), then \( X \rightarrow G \times H \).

(c) \( G \rightarrow G \times H \) if and only if \( G \rightarrow H \). \( \square \)

**Lemma 1.3.2** Suppose \( W' \leftrightarrow W \). Then

\[ W \in \mathcal{M}, \text{ if and only if } W' \in \mathcal{M}; \]

\[ W \in \mathcal{U} \mathcal{W} \mathcal{M}, \text{ if and only if } W' \in \mathcal{U} \mathcal{W} \mathcal{M}. \]

**Proof** From the definitions. \( \square \)

Remark: It is not true that \( W' \leftrightarrow W \) implies that \( W \in \mathcal{U} \mathcal{W} \mathcal{M} \) if and only if \( W' \in \mathcal{U} \mathcal{W} \mathcal{M} \). For instance, \( \mathcal{P}_2 \subseteq \mathcal{U} \mathcal{W} \mathcal{M} \) (Theorem 3.2.19), and \( \mathcal{P}_1 \subseteq \mathcal{U} \mathcal{W} \mathcal{M} \) (Corollary 3.2.14).
Corollary 1.3.3

(a) Assume that $W' \triangleleft W$. Then $W \in \mathcal{M}$ if and only if $W' \in \mathcal{M}$; $W \in \cup \cup \mathcal{M}$ if and only if $W' \in \cup \cup \mathcal{M}$.

(b) If $G \rightarrow H$ and $H \rightarrow G$, then $G \times H \in \mathcal{M} \cap \cup \cup \mathcal{M}$.

Recall that when we consider the multiplicative properties of a graph we always assume that the graph is connected. Therefore in Corollary 1.3.3 (b) we implicitly assume that $G \times H$ is connected. Thus both $G$ and $H$ must be connected.

If undirected graphs $G$ and $H$ are connected, then $G \times H$ is connected if and only if either $G$ or $H$ contains an odd cycle [24].

For directed graphs $G$ and $H$, different kinds of connectivity of $G \times H$ are discussed in [7].

Lemma 1.3.4 [11] $W \in \mathcal{M}$ if and only if there is a complete set of obstructions for $W$.

Lemma 1.3.5 [11]

(a) $W(G)$ has loops if and only if $G \rightarrow W$.

(b) $G \times W(G) \rightarrow W$.

(c) $G \rightarrow W(W(G))$ by an one to one homomorphism.

(d) $W \rightarrow W(G)$ by an isomorphism onto an induced subgraph of $W(G)$.

(e) $G \times H \rightarrow W$ if and only if $H \rightarrow W(G)$.

(f) $W \in \mathcal{M}$ if and only if $W(G) \rightarrow W$ whenever $G \rightarrow W$.

(g) $G \rightarrow G'$ implies $W(G') \rightarrow W(G)$ for every $W$.

(h) $W \rightarrow W'$ implies $W(G) \rightarrow W'(G)$ for every $G$. 

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Lemma 1.3.6  The graph $H$ is a core if and only if $H \rightarrow G$ for any proper subgraph $G$ of $H$.

Proof  If a graph $H$ is not a core, then there exists a proper subgraph $G$ of $H$, which is the retract of $H$ and hence $H \rightarrow G$.

Conversely, if there exists a proper subgraph $G$ of $H$ such that $H \rightarrow G$, let $G^*$ be a subgraph of $H$ with the minimum number of vertices such that there exists a homomorphism $f: H \rightarrow G^*$. Then $\alpha = f|_{G^*}$ maps $G^*$ onto $G^*$ and therefore is one-to-one. Recall that we only consider finite graphs. Thus $\alpha^{-1}$ exists, $\alpha^{-1}$ is an homomorphism and the mapping $g = \alpha^{-1} f$ is a retraction of $H$ to $G$. Hence $H$ is not a core.

An argument similar to the one above shows the following result.

Lemma 1.3.7  Assume that a core $W$ is a subgraph of $G$. Then $G \rightarrow W$ if and only if $W$ is a retract of $G$.

Lemma 1.3.8  Suppose that $W$ is a core. Then
(a) if $W \subseteq G$, then $W \triangleleft G \times W(G)$; and
(b) if $W \in \mathcal{W}$, and both $W$ and $G$ are connected, then $W \not\triangleleft G$ implies $W \triangleleft H$ for any component $H$ of $W(G)$, $W \subseteq H$.

Proof  (a) By Lemma 1.3.5(d) $W \subseteq W(G)$, thus $W \subseteq G \times W(G)$.

From Lemma 1.3.5(b) and the fact that $W$ is a core, we obtain $W \triangleleft G \times W(G)$ by Lemma 1.3.7.

(b) Suppose a core $W \in \mathcal{W}$. Suppose that $W \not\triangleleft G$ and for some component $H$ of $W(G)$ we have $W \not\triangleleft H$. Then $W \not\triangleleft G \times H$. But
Lemma 1.3.9 [11]

(a) \( P \rightarrow \overrightarrow{P_n} \) for any oriented path \( P \) of net length \( k, k \geq n+1 \).

(b) \( D \rightarrow \overrightarrow{P_n} \) if and only if \( P \rightarrow D \) for some oriented path \( P \) of net length \( k = n+1 \).

(c) \( C \rightarrow \overrightarrow{C_n} \) for any oriented cycle \( C \) of net length \( k, k \neq 0 \) (mod \( n \)).

(d) \( D \rightarrow \overrightarrow{C_n} \) if and only if \( C \rightarrow D \) for some oriented cycle \( C \) of net length \( k \neq 0 \) (mod \( n \)).

(e) \( D \rightarrow \overrightarrow{C_n} \) if and only if \( D \) contains some oriented cycle \( C \) of net length \( k, k \neq 0 \) (mod \( n \)).

\[ \square \]

1.4 The Role of Connectivity in \( \mathcal{M}_1, \mathcal{W}_1 \) and \( \mathcal{V}_1 \mathcal{W}_1 \)

Here we analyse what happens to the definitions of \( \mathcal{M}_1, \mathcal{W}_1 \) and \( \mathcal{V}_1 \mathcal{W}_1 \) when we add the requirement of connectivity on \( G \) and \( H \) in \( \mathcal{M}_1, \mathcal{W}_1 \) and \( \mathcal{V}_1 \mathcal{W}_1 \), or remove the requirement of connectivity on \( G \) and \( H \) in \( \mathcal{M}_1, \mathcal{W}_1 \) and \( \mathcal{V}_1 \mathcal{W}_1 \).

Let \( \mathcal{M}_1, \mathcal{W}_1 \) and \( \mathcal{V}_1 \mathcal{W}_1 \) be defined as in Section 1.1.

Let \( \mathcal{M}_1^* \) be the class of connected graphs \( W \) for which \( G \rightarrow W \) and \( H \rightarrow W \) imply \( G \times H \rightarrow W \) for any two connected graphs \( G \) and \( H \).

Let \( \mathcal{W}_1 \mathcal{M}_1^* \) be the class of connected graphs \( W \) for which \( W \not\in \mathcal{M}_1 \) and \( W \not\in \mathcal{W}_1 \) imply \( W \not\in \mathcal{M}_1 \times H \) for any two graphs \( G \) and \( H \).
Let $\mathfrak{m} \subseteq \mathfrak{m}_*$ be the class of connected graphs $W$ for which there do not exist two graphs $G$ and $H$ with $G \rightarrowtail H$ and $H \rightarrowtail G$ such that $W \nrightarrow G \times H$.

**Claim 1.4.1** $\mathfrak{m} = \mathfrak{m}^*$.

**Proof** $\mathfrak{m} \subseteq \mathfrak{m}^*$ is obvious. Now suppose $W \in \mathfrak{m}^*$. Consider two graphs $G$ and $H$ with $G \rightarrowtail W$ and $H \rightarrowtail W$. Let $G = G_1 \cup \ldots \cup G_k$, $H = H_1 \cup \ldots \cup H_n$, where each $G_i$ ($1 \leq i \leq k$) is a connected component of $G$, and each $H_j$ ($1 \leq j \leq n$) a connected component of $H$. Then at least one $G_i$, ($1 \leq i \leq k$), say $G_1$, satisfies $G_1 \rightarrowtail W$, and at least one $H_j$ ($1 \leq i \leq n$), say $H_1$, satisfies $H_1 \rightarrowtail W$. Since $G_1$ and $H_1$ are connected and $W \in \mathfrak{m}^*$, we have $G_1 \times H_1 \rightarrowtail W$ by the definition of $\mathfrak{m}^*$. Therefore $G \times H \rightarrowtail W$, as $G_1 \times H_1$ is a component of $G \times H$. Hence $W \in \mathfrak{m}$.

The proof is obvious.

**Claim 1.4.2** $\mathfrak{m}_* \subseteq \mathfrak{m}_* \cup \mathfrak{m}$; $\mathfrak{m}_* \subseteq \mathfrak{m} \cup \mathfrak{m}_*$.

The proof is obvious.

**Claim 1.4.3** If a core $W \in \mathfrak{m}_*$, then $W \in \mathfrak{m}_*$ and $W \in \mathfrak{m}$.

**Proof** Assuming on the contrary that a core $W \notin \mathfrak{m}_* \cup \mathfrak{m}$ we have (connected) graphs $G$ and $H$ such that $W \nrightarrow G$, $W \nrightarrow H$ and $W \nrightarrow G \times H$. Then $G \rightarrowtail W$, $H \rightarrowtail W$ and $G \times H \rightarrowtail W$ by Lemma 1.3.7. Hence $W \notin \mathfrak{m}$.

The proof is obvious.

**Claim 1.4.4** $\mathfrak{m} \subseteq \mathfrak{m}_* \cup \mathfrak{m}_*$ and $\mathfrak{m} \subseteq \mathfrak{m} \cup \mathfrak{m}_*$.
Proof If a graph \( W \in \mathcal{M}_* (\mathcal{M} \cup \mathcal{M}_*) \), we have (connected) graphs \( G \) and \( H \) with \( G \leftrightarrow H \) and \( H \leftrightarrow G \) such that \( W \leftrightarrow G \times H \). Thus \( G \leftrightarrow W \), for otherwise \( G \rightarrow W \rightarrow G \times H \rightarrow H \) by Lemma 1.3.1(a), a contradiction. Similarly \( H \leftrightarrow W \). Hence \( W \notin \mathcal{M} \). \( \square \)

Claim 1.4.5 \( \mathcal{M}_* \subseteq \mathcal{M} \) and \( \mathcal{M} \cup \mathcal{M}_* \subseteq \mathcal{M} \).

Proof Suppose \( W \in \mathcal{M} \) (note that we always assume \( W \) is connected). Then there exist graphs \( G \) and \( H \) such that \( G \leftrightarrow W \), \( H \leftrightarrow W \) and \( G \times H \rightarrow W \). Thus \( G \leftrightarrow H \) for otherwise \( G \rightarrow G \times H \rightarrow W \) a contradiction. Similarly \( H \leftrightarrow G \). Now, letting \( A = G \cup W \) and \( B = H \cup W \), we have \( W \not\subset A \), \( W \not\subset B \), \( A \leftrightarrow B \), \( B \leftrightarrow A \) and \( W \not\subset A \times B = G \times H \cup G \times W \cup W \times H \cup W \times W \) (because \( W \not\subset W \times W \) and \( G \times H \rightarrow W \), \( G \times W \rightarrow W \), \( W \times H \rightarrow W \)). Therefore \( W \not\leftrightarrow A \times B \), and \( W \notin \mathcal{M}_* \), \( W \notin \mathcal{M} \cup \mathcal{M}_* \). \( \square \)

Claim 1.4.6 \( \mathcal{M} \not\subset \mathcal{M} \) and \( \mathcal{M} \cup \mathcal{M}_* \not\subset \mathcal{M} \).

Proof The digraph \( \overrightarrow{K}_n \) belongs to \( \mathcal{M} \) by Theorem 2.2.5, and also belongs to \( \mathcal{M} \cup \mathcal{M}_* \) by Theorem 2.2.3 as \( \overrightarrow{K}_n \) is a core. But \( \overrightarrow{K}_n \notin \mathcal{M} \) for \( n > 2 \) by [21]. \( \square \)

Claim 1.4.7 \( \mathcal{M} \not\subset \mathcal{M}_* \) and \( \mathcal{M} \cup \mathcal{M}_* \not\subset \mathcal{M}_* \).

Proof By Claims 1.4.5 and 1.4.6. \( \square \)

By the above analysis, we don't need to include the connectivity condition on \( G \) and \( H \) in the definition of multiplicativity. We must however include the connectivity of \( G \) and \( H \) in the definitions of
weak and very weak multiplicativity. Otherwise very weak multiplicativity would be equivalent to multiplicativity (Claims 1.4.4 and 1.4.5); weak multiplicativity would imply multiplicativity (Claim 1.4.5); and weak multiplicativity for cores would also be equivalent to multiplicativity (Claims 1.4.3 and 1.4.5).
Chapter 2 Multiplicativity, Weak Multiplicativity, And Very Weak Multiplicativity

2.1 Introduction

As outlined in Chapter 1 this chapter is devoted to the relationships among multiplicativity, weak multiplicativity and very weak multiplicativity. These are summarized in Figure 2.1.

Figure 2.1 Relationships among multiplicativity, weak multiplicativity, and very weak multiplicativity

For a core $W$, multiplicativity implies weak multiplicativity and weak multiplicativity is equivalent to very weak multiplicativity. Weak multiplicativity is strictly weaker, even for cores: $K_n$ is weakly multiplicative for any positive integer $n$, but not multiplicative for
n > 2. ($\overline{K}_n$ is the only such example we have found so far.)

Interestingly enough, as we found in Chapter 1 (Claim 1.4.4), regardless of whether a graph W is a core or not, multiplicativity implies very weak multiplicativity.

If $W \Leftrightarrow W^*$, then the multiplicativity of W is equivalent to the multiplicativity of $W^*$. If W is a weakly multiplicative core, and if moreover $W^*$ has a special form (W being a "strong retract of $W^*$"), then $W^*$ is also weakly multiplicative.

Finally we use a construction due to Komarek to give a class of multiplicative digraphs. That construction was originally used to illustrate a certain "good characterization".

2.2 Relationship Between Multiplicativity, Weak Multiplicativity, And Very Weak Multiplicativity

**Theorem 2.2.1** Let W be a core. If $W \in \mathcal{U} \mathcal{W} \mathcal{M}$, then $W \in \mathcal{W} \mathcal{M}$.

**Proof** Suppose otherwise that a core $W \notin \mathcal{W} \mathcal{M}$. Then there exist connected graphs G and H such that $W \not\mathcal{G}, W \not\mathcal{H}$, and $W \not\mathcal{G} \times \mathcal{H}$. Therefore $W \Leftrightarrow G \times H$, and since W is a core, $G \not\mathcal{W}$ by Lemma 1.3.7. Hence $G \not\mathcal{H}$ for otherwise $G \not\mathcal{G} \times H \not\mathcal{W}$ by Lemma 1.3.1, a contradiction. The same argument shows that $H \not\mathcal{G}$. Thus $W \notin \mathcal{U} \mathcal{W} \mathcal{M}$. \qed
Lemma 2.2.2  For any graphs $G$, $H$, $A$ and $B$, if $G$ is a retract of $A$ and $H$ is a retract of $B$, then $G \times H$ is a retract of $A \times B$.

Proof  Retractions $r: A \rightarrow G$ and $s: B \rightarrow H$ yield $f: A \times B \rightarrow G \times H$, 
$$f(x, y) = (r(x), s(y)) \quad \text{for} \quad (x, y) \in A \times B.$$ 
This is a retraction. \hfill \Box

Theorem 2.2.3  Let $W$ be a core. If $W \in \wp \wr \mathcal{M}$, then $W \in \wp \wp \mathcal{M}$.

Proof  Suppose that a core $W \not\in \wp \wp \mathcal{M}$. Then there exist two connected graphs $G$ and $H$ such that $G \leftrightarrow H$, $H \leftrightarrow G$, and $W \leftrightarrow G \times H$. Since $G \times H \rightarrow G$, $G \times H \rightarrow H$, we have $W \rightarrow G$, $W \rightarrow H$. Let $\varphi: W \rightarrow G$, $\psi: W \rightarrow H$ be two homomorphisms. Take an arbitrary $w_1 \in V(W)$ and choose $g_1 \in V(G)$ so that 
$$\{g_1, \varphi(w_1)\} \in E(G).$$  Such a vertex exists because $w_1$ is incident to some edge of $W$. Construct a graph $A$ as follows: 
$$V(A) = V(G) \cup V(W)$$  and 
$$E(A) = E(G) \cup E(W) \cup \{e_1\},$$  where $e_1 = \{g_1, w_1\}$. Take an arbitrary $w_2 \in V(W)$ and choose $h_2 \in V(H)$ so that $\{h_2, \psi(w_2)\} \in E(H)$. Such a vertex exists for the same reason as above. Construct a graph $B$ as follows: 
$$V(B) = V(H) \cup V(W)$$  and 
$$E(B) = E(H) \cup E(W) \cup \{e_2\},$$  where $e_2 = \{h_2, w_2\}$. Obviously both $A$ and $B$ are connected, $G$ is a retract of $A$, and $H$ is a retract of $B$. From Lemma 2.2.2, $G \times H$ is a retract of $A \times B$. Since $G \times H \rightarrow W$, we have $A \times B \rightarrow W$. By the construction of $A$ and $B$, $W \subseteq A \times B$. Therefore $W \not\rightarrow A \times B$ by Lemma 1.3.7. Also we have $G \not\rightarrow W$ and $H \not\rightarrow W$, for otherwise if, say, $G \rightarrow W$, then 
$$G \rightarrow W \rightarrow G \times H \rightarrow H$$  by Lemma 1.3.1, a contradiction. Therefore $W$ is neither a retract of $A$ nor a retract of $B$. Hence $W \not\in \wp \wp \mathcal{M}$. \hfill \Box
If a graph $W$ is not a core, then $W \subseteq \cup \psi \mathfrak{m}$ will not imply $W \in \psi \mathfrak{m}$. (See Theorems 3.2.14 and 3.2.19.) We don't know if $W \in \psi \mathfrak{m}$ implies $W \in \cup \psi \mathfrak{m}$ or not.

**Corollary 2.2.4** Assume that $W$ is a core. Then $W \in \psi \mathfrak{m}$ if and only if $W \in \cup \psi \mathfrak{m}$. □

Our principal example of a weakly multiplicative graph which is not multiplicative is complete digraph $\vec{K}_n$. Its non-multiplicativity was proved in [21]. Here we employ a technique similar to that used in [3] and [25] to show its weak multiplicativity.

**Theorem 2.2.5** $\vec{K}_n \in \psi \mathfrak{m}$.

**Proof** It is obvious that for a digraph $H$, $H \xrightarrow{\psi} \vec{K}_n$ if and only if the underlying graph of $H$ is $n$-colorable.

**Claim:** If $\vec{K}_n \subseteq H$ and $H \not\xrightarrow{\psi} \vec{K}_n$ for a connected digraph $H$, then there is an unique $n$-coloring for the underlying graph of $\vec{K}_n \times H$.

Write
\[
V(\vec{K}_n \times H) = \{x_1\} \times V(H) \cup \{x_2\} \times V(H) \cup \ldots \cup \{x_n\} \times V(H),
\]
where $\{x_1, ..., x_n\}$ is the vertex set of $\vec{K}_n$. Firstly, the partition of $V(\vec{K}_n \times H)$ into parts
\[
\{x_1\} \times V(H), \{x_2\} \times V(H), ..., \{x_n\} \times V(H)
\]
is indeed an $n$-coloring of the underlying graph of $\vec{K}_n \times H$. If $f$ is another $n$-coloring of the underlying graph of $\vec{K}_n \times H$, then there is no $y \in H$ such that
\[
\{f((x_1, y)), ..., f((x_n, y))\} = \{1, 2, ..., n\};
\]
otherwise for any \( z \in H \) with \( \overrightarrow{yz} \in E(H) \) or \( \overrightarrow{zy} \in E(H) \), we shall have 
\[ f((x_i,y)) = f((x_i,z)) \] (for \( i = 1, 2, \ldots, n \)). Since the underlying graph \( H \) is connected, we have 
\[ f((x_i,y)) = f((x_i,z)) \] (for all \( z \in V(H) \) and \( i = 1, 2, \ldots, n \)). This is just the \( n \)-coloring we have given at the beginning. Hence for every \( y \in V(H) \), some of the colors \( f((x_1,y)), \ldots, f((x_n,y)) \) coincide. Colour \( y \in V(H) \) by \( g(y) = j \) if two of \( f((x_1,y)), \ldots, f((x_n,y)) \) are equal to \( j \). If \( \overrightarrow{yz} \in E(H) \), \( g(y) = i \) and \( g(z) = j \), then two of \( f((x_1,y)), \ldots, f((x_n,y)) \) are equal to \( i \) and two of \( f((x_1,z)), \ldots, f((x_n,z)) \) are equal to \( j \). Hence there must be an arc of \( K_n \times H \) joining two such vertices with \( i \neq j \). Thus \( g \) is a proper colouring of \( H \), a contradiction.

Now we shall continue with the proof of the theorem. Let \( G \) and \( H \) be connected digraphs such that \( \text{ff}), G, \text{PH}. Then \( H \rightarrow K_n \), \( G \rightarrow K_n \) by Lemma 1.3.7. Suppose \( G \times H \rightarrow K_n \). Then the underlying graph of \( G \times H \) is \( n \)-colorable. By the above claim the underlying graphs of the subgraphs \( K_n \times H \) and \( G \times K_n \) are both uniquely \( n \)-colorable. Moreover, the proof of the claim implies that any \( n \)-coloring of the underlying graph of \( G \times H \) has to assign the same color to all vertices of \( \{x_i\} \times V(H) \) for each \( x_i \in V(K_n) \subseteq V(G) \) and also to all vertices \( V(G) \times \{y_i\} \) for each \( y_i \in V(K_n) \subseteq V(H) \). This is impossible.

2.3 Strong Retracts

In general, we have found that if \( W \leftrightarrow W^* \), where \( W \) is a core, then the multiplicativity of \( W \) will imply both the weak multiplicativity of \( W \) (Corollary 1.4.3) and the multiplicativity of \( W^* \) (Lemma 1.3.2). However, if \( W^* \) is not a core, then the multiplicativity
of the core $W$ will not imply the weak multiplicativity of $W^*$. For instance, if $W$ is a directed path and $W^*$ is an oriented path with $W$ being its proper core, then we have $W \in \mathcal{P}, W \in \mathcal{W} \mathcal{P}, W^* \in \mathcal{W}^{-1} \mathcal{P}$ (see Theorem 3.2.13, Corollary 3.2.14 and Theorem 3.2.19). Nevertheless, if a multiplicative core $W$ is a strong retract of $W^*$ as defined below, then $W^*$ will be weakly multiplicative.

A subgraph $W$ of $W^*$ is said to be a **strong retract** of $W^*$ if there exists a retraction $r: W^* \rightarrow W$ such that for $x \in V(W)$ and $y \in V(W^*)$, $xr(y) \in E(W)$ if and only if $xy \in E(W^*)$. In such a case $r$ is called a strong retraction. We write $W \ll W^*$ to mean that there is a subgraph $W'$ of $W^*$ isomorphic to $W$ such that $W'$ is a strong retract of $W^*$.

We can now prove that if $W \ll W^*$ and $W$ is a multiplicative core graph, then $W^*$ is weakly multiplicative. We shall first prove some lemmas.

Define $N_w(x) = \{v \in V(W) : xv \in E(W)\}$.

**Lemma 2.3.1** Let $W$ be a core. Then for any $x, y \in V(W)$, $x \neq y$, neither $N_w(x) \subseteq N_w(y)$ nor $N_w(y) \subseteq N_w(x)$.

**Proof** If, for example, $N_w(x) \subseteq N_w(y)$, then we can define a mapping $\varphi: V(W) \rightarrow V(W)$ by $\varphi(x) = y$ and $\varphi(v) = v$ for $v \in V(W) \setminus \{x\}$. Clearly $\varphi$ is a retraction onto $W \setminus \{x\}$, which contradicts the fact that $W$ is a core.

\[\square\]
Lemma 2.3.2  If $W$ is a core and a strong retract of $W^*$, then there is a unique retraction of $W^*$ to $W$.

Proof  Suppose $r: W^* \rightarrow W$ is a strong retraction and $f$ is any other retraction $W^* \rightarrow W$. Then there exist $x \in W$ and $u \in r^{-1}(x)$ such that $f(u) = y \neq x$. Let $N_w(x) = \{x_1, \ldots, x_k\}$. Then $ux_i \in E(W^*)$ for $i = 1, \ldots, k$ by the definition of strong retraction. Now since $f(u) = y$, $x_iy \in E(W)$ for $i = 1, 2, \ldots, k$; i.e., $N_w(x) \subseteq N_w(y)$. By Lemma 2.3.1, $W$ is not a core, a contradiction. \hfill \Box

Lemma 2.3.3  Let $W$ be a core and a strong retract of $W^*$ and let $W^*$ be a subgraph of $G$. Then $W^*$ is a retract of $G$ if and only if $W$ is a retract of $G$.

Proof  If $W^*$ is a retract of $G$, then $W$ is a retract of $G$. Indeed let $r: G \rightarrow W^*$ and $s: W^* \rightarrow W$ be any retractions. Then $sr: G \rightarrow W$ is also a retraction. Now assume that $W$ is a retract of $G$, and that $f: G \rightarrow W$ is a retraction. Then $r = fl_{W^*}$ is a retraction from $W^*$ to $W$, which is the unique strong retraction by Lemma 2.3.2. Define the mapping $g: V(G) \rightarrow V(W^*)$ as follows:

$$g(x) = \begin{cases} x & \text{if } x \in V(W^*) \\ f(x) & \text{if } x \in V(G) \setminus V(W^*) \end{cases}.$$  

Clearly $g$ keeps the vertices of $W^*$ fixed. Now assuming that $st \in E(G)$, we shall prove that $g(s)g(t) \in E(W^*)$ according to the following 4 cases.

Case 1: $s, t \in V(W^*)$. Trivial.

Case 2: $s \in V(W^*)$, $t \in V(G) \setminus V(W^*)$. Let $f(s) = x$ and $f(t) = y$, then $xy \in E(W)$. Since $f(s) = r(s)$, $r$ is the unique strong retraction, then

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sy ∈ E(W*); i.e., g(s)g(t) ∈ E(W*).

Case 3: s ∈ V(G)\V(W*), t ∈ V(W*). Similar to Case 2.
Case 4: s, t ∈ V(G)\V(W*). Obviously g(s)g(t) = f(s)f(t) ∈ E(W*). □

**Corollary 2.3.4** Let W be a core, W ≪ W* and W* ⊆ G. Then W* ≪ G if and only if W ≪ G.

**Proof** Apply Lemma 2.3.3.

**Theorem 2.3.5** If W ≪ W* and W is a weakly multiplicative core graph, then W* is weakly multiplicative.

**Proof** For any two connected graphs G and H with W* ⊄ G, W* ⊄ H, we have W* ⊆ G×H, since W* ⊆ G, W* ⊆ H. By Corollary 2.3.4, W ⊄ G, W ⊄ H. Therefore W ⊄ G×H, since W is weakly multiplicative. By Corollary 2.3.4 again, we have W* ⊄ G×H. □

**Corollary 2.3.6** If W ≪ W* and W is a multiplicative core graph, then W* is multiplicative as well as weakly multiplicative.

**Proof** The multiplicativity of W* follows from Lemma 1.3.2. The weak multiplicativity of W* follows from Corollary 1.4.3 and Theorem 2.3.5.

If W is a strong retract of W*, then we can describe the structure of W* with respect to W in some detail. In fact, let r: W* → W be a strong retraction. Then

(i) V(W*) = \bigcup_{x \in V(W)} r^{-1}(x), \text{ where } r^{-1}(x) \cap r^{-1}(y) = ∅ \text{ for } x \neq y,

x, y \in V(W),
(ii) if $xy \in E(W)$, then $xv \in E(W^*)$ for any $v \in r^{-1}(y)$ and $uy \in E(W^*)$ for any $u \in r^{-1}(x)$, and

(iii) the only other edges of $W^*$ are of the form $uv$ for some $u \in r^{-1}(x)$, $v \in r^{-1}(y)$, $xy \in E(W)$.

Let $G$ be any graph and $v \in V(G)$. Let $G'$ be the graph obtained from $G$ by duplicating vertex $v$ as follows:

$V(G') = V(G) \cup \{v'\}, \{v'\} \cap V(G) = \emptyset$; and

$E(G') = E(G) \cup \{xv' : xv \in E(G)\}$.

We call the above operation on $G$ an elementary duplication. Any graph $G'$ is called a duplicated $G$ if $G'$ can be obtained from $G$ by consecutively applying finitely many elementary duplications.

Note that if $G'$ is a duplicated $G$, then $G$ is the strong retract of $G'$. Thus we have the following corollary.

**Corollary 2.3.7** (a) If $W$ is a weakly multiplicative core graph, then any duplicated $W$ is weakly multiplicative.

(b) If $W$ is a multiplicative core graph, then any duplicated $W$ is multiplicative as well as weakly multiplicative.

Since the two oriented paths in Figure 2.2 are duplicated $\overrightarrow{P}_1$'s, where $\overrightarrow{P}_1$ is the directed path of length 1, we have the following result.

**Corollary 2.3.8** The oriented paths in Figure 2.2 are multiplicative as well as weakly multiplicative.
We have already mentioned the fact that each undirected odd cycle \((C_{2n+1})\) is a multiplicative core ([4] and [11]). Thus we obtain the following corollary.

**Corollary 2.3.9** Each undirected graph which admits a strong retraction onto an odd cycle is multiplicative as well as weakly multiplicative.

The graphs G in Figure 2.3 are some special cases of Corollary 2.3.9. They have the property that \(C_3 \ll G\). Thus we conclude that the graphs in Figure 2.3 are multiplicative as well as weakly multiplicative.

It has been proved by Gerards [8], that any non-bipartite graph which does not contain an odd-\(K_4\) or an odd-\(K_3^2\) admits a retraction to its shortest odd cycle. Odd-\(K_4\) or odd-\(K_3^2\) are illustrated in Figure 2.4, where wriggled and dotted lines stand for paths. Dotted lines
may have length zero; wriggled lines have positive length. "Odd" means that the corresponding faces have odd cycles around it.

![Diagram of odd-K₄ and odd-K₃²](image)

Odd-K₄ 

Odd-K₃²

Figure 2.4

Based on this, there was the following result [11].

Any graph which does not contain an odd-K₄ or an odd-K₃² is multiplicative.

We can therefore take any non-bipartite core which does not contain an odd-K₄ or an odd-K₃² to duplicate some vertices to obtain many examples of weakly multiplicative (as well as multiplicative) graphs.

We also have examples which contain odd-K₃² and which are weakly multiplicative (as well as multiplicative), cf. Figure 2.3.
2.4 P. Komárek's Construction: A Class of Multiplicative Digraphs.

We begin by relating the terminologies of [11], [18] and [20].

Let \( \mathcal{Q} \) be a class of graphs. We call \( \mathcal{Q} \) *productive* (cf. [18] and [20]) if \( A \in \mathcal{Q} \) and \( B \in \mathcal{Q} \) imply \( A \times B \in \mathcal{Q} \); *subproductive* if \( A \in \mathcal{Q} \) and \( B \in \mathcal{Q} \) imply that there exists a subgraph \( C \) of \( A \times B \) with \( C \in \mathcal{Q} \).

Let \( \mathcal{G} \) be the class of all directed (respectively undirected) graphs and \( \mathcal{W} \subseteq \mathcal{G} \), \( \mathcal{W}^* \subseteq \mathcal{G} \). We define:

\[
\mathcal{W}(\rightarrow \mathcal{W}) = \{ G \in \mathcal{G} : G \rightarrow W \text{ for some } W \in \mathcal{W} \};
\]

\[
\mathcal{W}(\leftrightarrow \mathcal{W}) = \{ G \in \mathcal{G} : G \leftrightarrow W \text{ for any } W \in \mathcal{W} \};
\]

\[
\mathcal{W}(\mathcal{W}^* \rightarrow) = \{ G \in \mathcal{G} : W \rightarrow G \text{ for some } W \in \mathcal{W}^* \};
\]

\[
\mathcal{W}(\mathcal{W}^* \leftrightarrow) = \{ G \in \mathcal{G} : W \leftrightarrow G \text{ for any } W \in \mathcal{W}^* \}.
\]

It is easy to check that the following lemmas hold.

**Lemma 2.4.1** If a class of graphs \( \mathcal{Q} \) is productive, then \( \mathcal{Q} \) is subproductive.

**Lemma 2.4.2** If \( \mathcal{W}^* = \{ W \} \), then \( \mathcal{W}^* \) is subproductive.

**Lemma 2.4.3** \( \mathcal{W}(\rightarrow \mathcal{W}) = \mathcal{W}(\mathcal{W}^* \rightarrow) \) if and only if \( \mathcal{W}(\rightarrow \mathcal{W}) = \mathcal{W}(\mathcal{W}^* \leftrightarrow) \).

Let \( \text{TT}_n \) be the transitive tournament with \( n \) vertices. Let \( \mathcal{S}_{n+1} \) be the set of oriented paths with net length \( n+1 \). Let \( \mathcal{A}_n \) be the set of oriented cycles with net length \( k \), \( k \equiv 0 \pmod{n} \). Then by Lemma 1.3.9, we have the following examples by Lemma 1.3.9.
Example 2.4.4  
(a) $\mathcal{H}(\rightarrow TT_n) = \mathcal{H}(\rightarrow P_n^2)$ (also by [18, 20])
(b) $\mathcal{H}(\rightarrow P_n^2) = \mathcal{H}(\rightarrow \mathcal{P}_{n+1})$.
(c) $\mathcal{H}(\rightarrow C_n^2) = \mathcal{H}(\rightarrow \mathcal{C}_n)$.

Theorem 2.4.5  Let $W$ be a fixed graph. The following statements are equivalent:

(1) There exists a subproductive family $\mathcal{W}^*$ such that $\mathcal{H}(\rightarrow W) = \mathcal{H}(\mathcal{W}^* \rightarrow)$; and

(2) There exists a complete set of obstructions for $W$.

Proof  (1) implies (2). Suppose that there exists a subproductive family $\mathcal{W}^*$ such that $\mathcal{H}(\rightarrow W) = \mathcal{H}(\mathcal{W}^* \rightarrow)$. Then $\mathcal{W}^* \subseteq \{G: G \rightarrow W\}$, and for any $G \rightarrow W$, there exists $W^* \rightarrow G$ for some $W^* \in \mathcal{W}^*$. For any $W^*_1, W^*_2 \in \mathcal{W}^*$, there exists a $W^* \in \mathcal{W}^*$ such that $W^* \subseteq W^*_1 \times W^*_2$, hence $W^* \rightarrow W^*_1 \times W^*_2$. Thus $\mathcal{W}^*$ is a complete set of obstructions for $W$.

(2) implies (1): Suppose $\theta$ is a complete set of obstructions for $W$. Let $\mathcal{W}^* = \{W^* \in \mathcal{Q} : \text{there exists a } G \in \theta \text{ such that } G \rightarrow W^*\}$. For any $W^*_1, W^*_2 \in \mathcal{W}^*$, there exist $G_1, G_2 \in \theta$ such that $G_i \rightarrow W^*_i$ ($i = 1, 2$). Hence there exists a $G \in \theta, G \rightarrow G_1, G \rightarrow G_2$. By Lemma 1.3.1 (b) $G \rightarrow G_1 \times G_2 \rightarrow W^*_1 \times W^*_2$. Hence $\mathcal{W}^*$ is productive, and also subproductive by Lemma 2.4.1. Now if $H \in \mathcal{H}(\rightarrow W)$ (i.e., $H \rightarrow W$), then there exists $G \in \theta$ such that $G \rightarrow H$, hence $H \in \mathcal{W}^* \subseteq \mathcal{H}(\mathcal{W}^* \rightarrow)$. Conversely if $H \in \mathcal{H}(\mathcal{W}^* \rightarrow)$ (i.e., if there exists a homomorphism of $W^*$ to $H$ for $W^* \in \mathcal{W}^*$), then there exists a $G \in \theta, G \rightarrow W^* \rightarrow H$. Now if $H \in \mathcal{H}(\rightarrow W)$, then $H \rightarrow W$, and $G \rightarrow W$, contradicting the fact that $G \in \theta$. Therefore $\mathcal{H}(\rightarrow W) = \mathcal{H}(\mathcal{W}^* \rightarrow)$. 

\[ \square \]
Corollary 2.4.6 A graph $W$ is multiplicative if and only if there exists a subproductive family $\mathcal{W}^*$ with $\mathcal{H}(\rightarrow W) = \mathcal{H}(\mathcal{W}^* \rightarrow)$.

Proof Apply Theorem 2.4.5 and Lemma 1.3.4.

Corollary 2.4.7 For a graph $W$, if there exists a graph $W^*$ such that $h(\rightarrow W) = \mathcal{H}(W^* \rightarrow)$, then $W$ is multiplicative.

Proof Apply Lemma 2.4.2 and Corollary 2.4.6.

Conditions $\mathcal{H}(\rightarrow W) = \mathcal{H}(W^* \rightarrow)$ or $\mathcal{H}(\rightarrow W) = \mathcal{H}(W^* \leftrightarrow)$ are called good characterizations in [18].

Now we can describe Komárek's construction.

We define the digraph $A_{m,n,p,q} = (V, E)$ $\ (0 \leq p \leq n-2, \ 0 \leq q \leq m-2)$ by

$V = \{0, 1, \ldots, m, m+1, \ldots, m+p, -q', \ldots, -1', 0', 1', \ldots, n' \}$

$E = \{ \overrightarrow{i,i+1} : i = 0, \ldots, m+p-1 \} \cup \{ \overrightarrow{i,i'+1} : i = -q, \ldots, n-1 \} \cup \{ \overrightarrow{0'm} \}$

In Figure 2.5, see the graph $A_{m,n,p,q}$ with $m=5$, $n=3$, $p=1$, $q=2$.

![Figure 2.5](image)

We define the function $\mu(x)$ on the vertex set $V(G)$ of an acyclic
(i.e., without directed cycles) digraph $G$ by $\mu(x) = (d_1, d_2)$ where $d_1$ ($d_2$) is the length of the longest directed path ending (starting, respectively) in the vertex $x$.

We define the digraph $F_r = (V_r, E_r)$ ($r \geq 1$) by

$$V_r = \{ ij: i \text{ and } j \text{ are non-negative integers, } 1 \leq i+j \leq r \};$$

$$E_r = \{ ij, pq: ij, pq \in V_r, i < p, j > q \}.$$

In Figure 2.6, see the graph $F_r$ with $r = 1, 2, 3$.

Let $m, n$ be positive integers with $m+n > 2$ and consider $F_r$ with $r = m+n-2$. Define the two subsets $V_{L(q)}$ and $V_{U(p)}$ of vertices of the graph $F_{m+n-2}$ by

$$V_{L(q)} = \{ ij \in V_{m+n-2}: n \leq j, q \leq i \} \text{ and}$$

$$V_{U(p)} = \{ ij \in V_{m+n-2}: m \leq i, p \leq j \}.$$

Define the set $\mathcal{E}$ by

$$\mathcal{E} = \{ \overrightarrow{v_1, v_2} \in E_{m+n-2}: v_1 \in V_{L(q)}, v_2 \in V_{U(p)} \}.$$
Define now the digraph $B_{m,n,p,q} = (V^-, E^-)$ by $V^- = V_{m+n-2}$, $E^- = E_{m+n-2} \setminus E$. In Figure 2.7, we see the graphs $B_{3,2,0,0}$ and $B_{2,4,1,0}$. In Figure 2.8, we see the graphs $B_{3,2,0,1}$ and $B_{7,5,2,3}$.

![Diagram](image)

(a) $B_{3,2,0,0}$  
(b) $B_{2,4,1,0}$

Figure 2.7  $B_{3,2,0,0}$ and $B_{2,4,1,0}$
(dotted lines are the deleted edges)

**Lemma 2.4.8** [18] Let $G$ be an acyclic digraph. Let $r = \max \{d_1 + d_2 : \mu(x) = (d_1, d_2), \ x \in V(G) \}$, and define

$$h(x) = \begin{cases} 
01 & \text{if } \mu(x) = (d_1, d_2), \ d_1 = d_2 = 0 \\
\frac{d_1 d_2}{d_1 + d_2} & \text{if } \mu(x) = (d_1, d_2), \ 1 \leq d_1 + d_2 .
\end{cases}$$

Then $h$ is a homomorphism $G \rightarrow F_r$. 

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**Proof**  Take two vertices \( v_1 \) and \( v_2 \) of \( V(G) \). Let \( \mu(v_1) = (d_1, d_2) \), \( \mu(v_2) = (d_1', d_2') \). If \( v_1, v_2 \in E(G) \), then \( d_1 < d_1' \) and \( d_2 > d_2' \); i.e.,
\[
(d_1d_2, d_1'd_2') \in E_r.
\]
Hence \( h(x) \) is a homomorphism.  \( \Box \)

**Lemma 2.4.9** [18]  Let \( m, n, p, q \) be integers, \( 0 \leq p \leq n-2 \), \( 0 \leq q \leq m-2 \). Then
\[
\kappa(A_{m,n,p,q} \rightarrow B_{m,n,p,q})
\]

**Proof**  We include the proof for completeness (since it illuminates the idea behind these definitions). We shall first prove that
\[
\kappa(A_{m,n,p,q} \rightarrow) \subseteq \kappa(\rightarrow B_{m,n,p,q}) \quad (*)
\]
Assume \( G \in \kappa(A_{m,n,p,q} \rightarrow) \), i.e., there is no homomorphism from \( A_{m,n,p,q} \) to \( G \). Hence \( G \) contains no directed path \( \overrightarrow{P}_k \) with \( k \geq m+n-1 \) and no directed cycle. By Lemma 2.4.8, there exists a homomorphism \( h \) from \( G \) to \( F_{m-n-2} \). Suppose that there exists no homomorphism from \( G \) to \( B_{m,n,p,q} \). Then there exists an edge \( v_1, v_2 \in E(G) \) such that \( h(v_1) \in V_{L(q)} \), \( h(v_2) \in V_{U(p)} \). There exists a directed path \( \overrightarrow{P}_s \) (\( s \geq m \)) ending in \( v_2 \) and another directed path \( \overrightarrow{P}_s' \) (\( s' \geq p \)) starting in \( v_2 \). There exists also a directed path \( \overrightarrow{P}_t \) (\( t \geq q \)) ending in \( v_1 \) and another directed path \( \overrightarrow{P}_t' \) (\( t' \geq n \)) starting in \( v_1 \). This is in contradiction with the assumption \( G \in \kappa(A_{m,n,p,q} \rightarrow) \). Therefore we have \((*)\).

It remains to prove that
\[
\kappa(\rightarrow B_{m,n,p,q}) \subseteq \kappa(A_{m,n,p,q} \rightarrow) \quad (**)
\]
Let \( G \in \kappa(\rightarrow B_{m,n,p,q}) \). Suppose, on the contrary, that
\( G \notin \kappa(A_{m,n,p,q} \rightarrow) \). The composition of the two homomorphisms
\[
A_{m,n,p,q} \rightarrow G \rightarrow B_{m,n,p,q}
\]

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will give a homomorphism
\[ h: A_{m,n,p,q} \rightarrow B_{m,n,p,q}. \]

Since there is a directed path \( \vec{P}_m \) ending in the vertex \( m \) of \( A_{m,n,p,q} \) and also another directed path \( \vec{P}_p \) starting in \( m \), we have \( h(m) \in V_{U(p)} \).

There is a directed path \( \vec{P}_q \) ending in the vertex \( 0' \) of \( A_{m,n,p,q} \) and also another directed path \( \vec{P}_n \) starting in \( 0' \). This means that \( h(0') \in V_{L(q)} \). We have \((0', m) \in E(A_{m,n,p,q})\), but there are no edges between \( V_{L(q)} \) and \( V_{U(p)} \) in \( B_{m,n,p,q} \). This contradiction implies (**).

\[ \square \]

We give two examples of this construction in Figure 2.8.

(a) \( \xi( A_{7,5,2,3} \rightarrow ) = \xi( \rightarrow B_{7,5,2,3} ) \) (edges not shown)
Figure 2.8

Theorem 2.4.10 Let $m, n, p$ and $q$ be non-negative integers satisfying $0 \leq p \leq n-2$ and $0 \leq q \leq m-2$. Then $B_{m,n,p,q}$ is multiplicative.

Proof Apply Corollary 2.4.7 and Lemma 2.4.9.

We also have the following lemma.

Lemma 2.4.11 $\% (\rightarrow F_r) = \% (\vec{P}_{r+1} \rightarrow)$

Proof Suppose a graph $G \in \% (\vec{P}_{r+1} \rightarrow)$ (i.e., $\vec{P}_{r+1} \rightarrow G$), then $G \leftrightarrow F_r$ for otherwise $\vec{P}_{r+1} \rightarrow F_r$ which is impossible since the maximum length of a directed path in $F_r$ is $r$. So $G \in \% (\rightarrow F_r)$.

Suppose now that a graph $G \in \% (\vec{P}_{r+1} \rightarrow)$ (i.e., $P_{r+1} \nleftrightarrow G$ or equivalently $G$ does not contain a directed walk of length $r+1$). We map any vertex $v \in V(G)$, for which the maximum directed walk
ending at \( v \) has length \( k \) \((0 \leq k < r+1)\), to the vertex \( k(r-k) \in V(F_r) \). This mapping is a homomorphism \( G \rightarrow F_r \). Thus \( G \in \mathfrak{S}(F_r) \). \( \square \)

Therefore we have the following theorem from Corollary 2.4.7.

**Theorem 2.4.12** Let \( r \) be a positive integer. Then \( F_r \) is multiplicative.

\( \square \)

**Note.** It is easy to see that \( TT_n \leftrightarrow F_{n-1} \). We remark that the fact that each \( TT_n \) is multiplicative was already obtained in [11, 20].
Chapter 3  Oriented Paths

3.1 Introduction

This chapter will be devoted to the study of oriented paths with respect to multiplicativity, weak multiplicativity and very weak multiplicativity (as outlined in Table 3.1), and also to the closely related applications of these results to oriented cycles and digraphs in general. We shall need more definitions.

Let $\mathcal{P}$ be the set of all oriented paths. Let

$\mathcal{P}_1 = \{P_n : n = 1, 2, \ldots \}$,

$\mathcal{P}_2 = \{P \in \mathcal{P} : P \not\equiv P_k \text{ for some } k = 1, 2, \ldots, \text{ but } P \text{ itself is not a directed path}\}$,

$\mathcal{P}_3 = \{P \in \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2) : \text{P is a core}\}$, and

$\mathcal{P}_4 = \{P \in \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2) : \text{P is not a core}\}$.

<table>
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<th>$\vee \psi \mathcal{M}$</th>
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</thead>
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<td>yes(3.2.14)</td>
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<tr>
<td>$\mathcal{P}_2$</td>
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<td>no(3.2.19)</td>
<td>yes(3.2.14)</td>
</tr>
<tr>
<td>$\mathcal{P}_3$</td>
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<td>no(3.3.4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}_4$</td>
<td>no</td>
<td>no(3.3.4)</td>
<td>no(3.2.19)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1 Multiplicative properties of oriented paths
The total length of an oriented path (cycle) is its number of arcs. The net length (defined in Section 1.1) of an oriented path \( P \) is denoted by \( nl(P) \). An oriented path \( P \) of net length \( k \) is minimal if no subpath (i.e., subgraph which is a path) of \( P \) has net length strictly greater than \( k \). Note that there could be subpaths of net length \( k \).

The level of a vertex \( v \) in an oriented path \( P \) with respect to a chosen vertex \( x \) of \( P \), denoted by \( f_{P,x}(v) \), or simply \( f(v) \) if no confusion will result, is the difference between the number of edges directed forward and the number of edges directed backward on the subpath from the chosen vertex \( x \) to \( v \). We often write \( f(v) \) and describe \( x \) by saying \( f(x) = 0 \); this does not define \( x \) uniquely, but it does give the function \( f \) uniquely. Let \( x_0, x_1, ..., x_n \) be the vertices of an oriented path \( P \) in a fixed traversal order of \( P \). Then \( P[x_i, x_{i+1}, ..., x_j] \), or briefly \( P[x_i, x_j] \) is an interval of \( P \) consisting of the subgraph induced by the vertices \( x_i, x_{i+1}, ..., x_j \). If all edges in the interval \( P[x_i, x_j] \) are going in one direction (forward, backward), we call \( P[x_i, x_j] \) one-directional (forward, backward respectively) interval of \( P \). Furthermore if the one-directional interval \( P[x_i, x_j] \) cannot be extended to a larger one-directional interval of \( P \), then we call \( P[x_i, x_j] \) a maximal one-directional (forward, backward respectively) interval. Let \( P \) be an oriented path with vertices \( x_0, x_1, ..., x_n \). If we want to emphasize that \( P \) is given in a specific order of traversal from \( x_0 \) to \( x_n \), we write \( P = P^+ = P[x_0, x_1, ..., x_n] = P[x_0, x_n] \). We also write \( P^- = P[x_n, ..., x_0] = P[x_n, x_0] \) for the reversed order of traversal with respect to the order of traversal of \( P = P^+ \). \( P^- \) is the oriented path obtained by reversing the direction of each arc of \( P \).

Let \( P_1 \) and \( P_2 \) be two oriented paths with the specified orders of
traversal. Then the concatenation of $P_1$ and $P_2$, denoted by $P_1P_2$, is the oriented path obtained by identifying the last vertex of $P_1$ and the first vertex of $P_2$.

3.2 Multiplicativity of Oriented Paths (Part I)

Our main results here are that $P_1 \cup P_2 \subseteq \mathcal{M}$ (Theorem 3.2.13 and Corollary 3.2.14); and that $P_2 \cup P_4 \subseteq \mathcal{W}_n$ (with two exceptions) (Theorem 3.2.19).

Before proving the main theorems we need to analyse the structure of the products of two oriented paths. If one path is directed and the other is minimal, we give in Lemma 3.2.1 the details of each component (especially the component located on the "diagonal") of the product graph. Lemmas 3.2.2 - 3.2.5 are preliminary to Lemma 3.2.8. Lemmas 3.2.6-3.2.7 are preliminary to Lemma 3.2.9. In Lemma 3.2.8 we describe some properties of the levels of vertices in the product of two balanced graphs (and specialize to oriented paths in Lemma 3.2.9). In Lemma 3.2.10 we treat the product of a directed path and a general oriented path. The existence of a special minimal oriented path is discussed in Lemma 3.2.11, which deals with the product of minimal oriented paths, and in Lemma 3.2.12 which treats the product of general oriented paths.

**Lemma 3.2.1** Let $P_2[y_0, y_1, \ldots, y_k]$ be a directed path of length $k$, $P_1[x_0, x_1, \ldots, x_m]$ a minimal oriented path of net length $k$, and let $f$ be the level function in $P_1$ with $f(x_0) = 0$. Then each component of
\( P_1 \times P_2 \) is an oriented path. Furthermore, if the levels of \( x_0 \) and \( y_0 \) are minimum in \( P_1 \) (respectively \( P_2 \)), then any component \( C \) of \( P_1 \times P_2 \) satisfies one of the following five cases:

1. \( C = [(x_i, y_{f(x_i)}) : i = 0, 1, \ldots, m] \), is isomorphic to \( P_1 \), starts at \( (x_0, y_0) \) and ends at \( (x, y_k) \);
2. \( C \) starts at \( (x_i, y_0) \) and ends at \( (x_m, y_j) \), is isomorphic to \( P_1[x_i, x_m] \), and is minimal with net length \( j \);
3. \( C \) starts at \( (x_0, y_j) \) and ends at \( (x_i, y_k) \), is isomorphic to \( P_1[x_0, x_i] \), and is minimal with net length \( k-j \);
4. \( C \) starts at \( (x_i, y_0) \) and ends at \( (x_j, y_0) \), is isomorphic to \( P_1[x_i, x_j] \), and can be obtained by concatenating two minimal oriented paths with the same net lengths at their maximum level vertices; and
5. \( C \) starts at \( (x_i, y_k) \) and ends at \( (x_j, y_k) \), is isomorphic to \( P_1[x_i, x_j] \), and can be obtained by concatenating two minimal oriented paths with the same net lengths at their minimum level vertices.

**Proof** Since in \( P_2 \) each vertex has indegree and outdegree at most 1, in \( P_1 \times P_2 \) each vertex has total degree (indegree plus outdegree) at most 2. Therefore each component of \( P_1 \times P_2 \) is an oriented path.

The two endvertices of each component must be at the four edges of the rectangle \([x_0, x_m] \times [y_0, y_k]\), since any vertex \((x_i, y_j)\) with \( 0 < i < m, 0 < j < k \) has degree 2. Therefore the starting and ending vertices of any component are located at the four edges of the rectangle \([x_0, x_m] \times [y_0, y_k]\).
Consider, for example, the component starting at \((x_0, y_0)\) in (1). For each \(i (i = 0, 1, ..., m-1)\) \(\overrightarrow{x_i x_{i+1}}\) is an arc if and only if \(f(x_{i+1}) = f(x_i) + 1\); i.e., if and only if \(\overrightarrow{yf(x_i) yf(x_{i+1})}\) is an arc. Thus \(\overrightarrow{x_i x_{i+1}}\) is an arc if and only if \(\overrightarrow{(x_i, yf(x_i)) (x_{i+1}, yf(x_{i+1}))}\) is an arc. Similarly \(\overrightarrow{x_i x_{i+1}}\) is an arc if and only if \(f(x_{i+1}) = f(x_i) - 1\); i.e., if and only if \(\overrightarrow{yf(x_i) yf(x_{i+1})}\) is an arc. Thus \(\overrightarrow{x_i x_{i+1}}\) is an arc if and only if \((x_i, yf(x_i)) (x_{i+1}, yf(x_{i+1}))\) is an arc. Therefore

\[
C = \{(x_i, yf(x_i)) : i = 0, 1, ..., m\}
\]

is a component of \(P_1 \times P_2\) which is isomorphic to \(P_1\).

The proofs for (2)-(5) are similar to the proof of (1). Furthermore, if one endvertex is \((u, y_0)\), then the other endvertex must be either \((v, y_0)\) or \((x_m, w)\), with \(u, v\) in \(P_1\), \(w\) in \(P_2\). These are the cases given in (2) and (4). If one endvertex is \((u, y_k)\), then the other endvertex must be either \((v, y_k)\) or \((x_0, w)\); these are the cases given in (3) and (5).

\[
\square
\]

Note: Some components \(C\) may consist of a single vertex; these degenerate components may only occur in \([x_0, x_m] \times \{y_0, y_k\}\).

Let the diagonal component in (1) of the above lemma be denoted \(C^*\). From the proof above it is easy to see that

\[
f_{C^*}(x_0, y_0)(x_i, yf(x_i)) = f_{P_1}(x_0)(x_i) = f_{P_2}(y_0)(yf(x_i)).
\]

Choose the level functions so that \(f_{C^*}(x_0, y_0) = f_{P_1}(x_0) = f_{P_2}(y_0) = 0\). Then we may write \(f_{C^*}(x, y) = f_{P_1}(x) = f_{P_2}(y)\). Similar conclusions also hold for the other components. Note that the particular choice of level functions (i.e., choice of the starting vertex) is irrelevant if only
their differences are used. Thus if \((x_i, y_j)\) and \((x_s, y_t)\) are two vertices in a component \(C\) of the product graph \(P_1 \times P_2\), then
\[
f_C(x_s, y_t) - f_C(x_i, y_j) = f_{P_2}(y_t) - f_{P_2}(y_j) = f_{P_1}(x_s) - f_{P_1}(x_i).
\]

A more general statement is proved later (Lemma 3.2.8).

We introduce the following definitions and give some more general results.

A digraph \(G\) is said to be **connected** if the underlying graph is connected. A *connected digraph* \(G\) is said to be **balanced** if the net length of any oriented path connecting two vertices \(x\) and \(y\) only depends on \(x\) and \(y\). A *digraph* is said to be **balanced** if each component is balanced. Let \(G\) be any connected balanced digraph and let \(a \in V(G)\). For any \(x \in V(G)\) and any oriented subpath \(P\) of \(G\) joining \(a\) and \(x\), let the level function \(f_{G, P, a}(x)\) be the difference between the number of edges directed forward and the number of edges directed backward on \(P\) from \(a\) to \(x\). Since \(G\) is balanced, \(f_{G, P, a}(x)\) does not depend on \(P\), and we write \(f_{G, a}(x)\); the function \(f_{G, a}\) is called the *level function* of \(G\). Clearly we have the following result.

**Lemma 3.2.2**

(a) The digraph \(G\) is balanced if and only if any oriented cycle in \(G\) has net length 0.

(b) any oriented path is balanced.

(c) any oriented cycle with net length 0 is balanced. 

\[\square\]
It now follows that in the definition of a balanced digraph, oriented paths can be replaced by oriented walks (i.e., vertices and arcs may be repeated).

Now we can give more general results about the levels of vertices in the product of balanced graphs.

**Lemma 3.2.3** If $G$ is a balanced digraph, then any subgraph $G'$ of $G$ is a balanced graph. Furthermore, if $a$ and $x$ are two vertices in the same component of $G'$, then for the two level functions $f_{G',a}$ and $f_{G,a}$ in $G$ and $G'$ we have

$$f_{G',a}(x) = f_{G,a}(x).$$

An oriented walk $P[x_0, x_1, ..., x_n]$ in a digraph $G$, is a sequence of (not necessarily distinct) vertices $x_0, x_1, ..., x_n$ and arcs $x_0x_1, x_1x_2, ..., x_{n-1}x_n$, where $x_ix_{i+1}$ denotes either $\overrightarrow{x_ix_{i+1}}$ or $\overleftarrow{x_ix_{i+1}}$.

**Lemma 3.2.4** Let $G$ and $H$ be two balanced digraphs. Let $P[x_0, x_1, ..., x_n]$ be an oriented walk in $H$ and $Q[y_0, y_1, ..., y_m]$ an oriented walk in $G$. If $h: G \to H$ is a homomorphism with $f(y_0) = x_0$, and if $h(Q)$ is contained in $P$, then we have

$$f_{Q,y_0}(y_i) = f_{P,x_0}(f(y_i)) \text{ for } i = 0, 1, ..., m.$$

The proof is easily done by induction on the subscript of $y$.

**Lemma 3.2.5** Let $G$ and $H$ be two connected balanced digraphs. Let $h: G \to H$ be a homomorphism and let $a \in V(G)$. Then we have

$$f_{G,a}(x) = f_{H,h(a)}(h(x))$$

for any $x \in V(G)$. 

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Proof  It follows from Lemma 3.2.4 and Lemma 3.2.3.

Lemma 3.2.6  Let P be an oriented path and \( x_0 \in V(P) \). Let \( f \) be the level function for which \( f(x_0) = 0 \). Then \( nl(P[x_0, x]) = |f(x)| \) for any \( x \in V(P) \).

Proof  Apply induction on the number of arcs between \( x_0 \) and \( x \).

Corollary 3.2.7  Let Q and P be two oriented paths and let \( h: Q \rightarrow P \) be an homomorphism. If \( a \in V(Q) \), then for any \( x \in V(G) \)

\[
\begin{align*}
    f_{Q,a}(x) &= f_{P,f(a)}(f(x)); \\
    nl(Q[a, x]) &= nl(P[h(a), h(x)]).
\end{align*}
\]

Proof  Apply Lemma 3.2.5 and Lemma 3.2.6.

Lemma 3.2.8  Let G and H be two balanced graphs. Then \( G \times H \) is a balanced graph. Furthermore, if Q is a component of \( G \times H \) and \( u = (x_0, y_0), v = (x_n, y_n) \in Q \), then we have

\[
    f_{Q,u}(v) = f_{G,x_0}(x_n) = f_{H,y_0}(y_n).
\]

Proof  We only need to prove the lemma when both G and H are connected. Take any two vertices \( u = (x_0, y_0) \) and \( v = (x_n, y_n) \) in the component Q of \( G \times H \). For any two oriented paths P and P' of Q which join u and v, the images \( \pi_1(P), \pi_1(P') \) under the projection mapping \( \pi_1: G \times H \rightarrow G \) are balanced oriented walks in G joining \( x_0 \) and \( x_n \). By Lemma 3.2.5 we have

\[
    f_{P,u}(v) = f_{\pi_1(P),x_0}(x_n), \text{ and}
\]

\[
    f_{P',u}(v) = f_{\pi_1(P'),x_0}(x_n).
\]

But we have, by Lemma 3.2.3,
\[ f_{\pi_1}(P)_* x_0(x_n) = f_{\pi_1}(P')_* x_0(x_n) = f_G x_0(x_n), \]
since \( G \) is balanced. Therefore we have
\[ f_{P,u}(v) = f_{P',u}(v); \]
i.e., \( G \) is balanced. Furthermore we have
\[ f_{Q,u}(v) = f_{P,u}(v) = f_{\pi_1(p)} x_0(x_n) = f_G x_0(x_n) \]
and
\[ f_{Q,u}(v) = f_{H,y_0}(y_n). \]

**Corollary 3.2.9** Let \( P_1[x_0, x_1, \ldots, x_n] \) and \( P_2[y_0, y_1, \ldots, y_m] \) be two oriented paths. Let \( P \) be any oriented path in the product \( P_1 \times P_2 \) with endvertices \( (x_s, y_i) \) and \( (x_t, y_j) \). Then we have
\[ f_{P,(x_s,y_i)}(x, y) = f_{P_1,x_s}(x) = f_{P_2,y_i}(y), \quad \text{for any } (x, y) \in P, \]
and
\[ n(P) = n(P_1[x_s, x_t]) = n(P_2[y_i, y_j]). \]

**Proof** Apply Lemma 3.2.8 and Corollary 3.5.7.

**Lemma 3.2.10** Let \( P_2[y_0, y_1, \ldots, y_k] \) be a directed path of length \( k \) and \( P_1[x_0, x_1, \ldots, x_n] \) any oriented path. If \( P \) is an oriented path in the product \( P_1 \times P_2 \) with endvertices \( (x_i, y_0) \) and \( (x_j, y_k) \), then \( P \) is minimal and isomorphic to \( P_1[x_i, x_j] \). In general each component of \( P_1 \times P_2 \) is an oriented path isomorphic to a subpath of \( P_1 \).

**Proof** By Corollary 3.2.9, \( f_{P,(x_i,y_0)}(x, y) = f_{P_1,x_i}(x) = f_{P_2,y_0}(y) \) for any \( (x, y) \in P \). Thus
\[ 0 = f_{P_2,y_0}(y_0) \leq f_{P,(x_i,y_0)}(x, y) = f_{P_1,x_i}(x) \leq f_{P_2,y_0}(y_k) = k, \]
for any \( (x, y) \in P \), i.e., \( P \) and \( P_1[x_i, y_j] \) are minimal. Then by Lemma 3.2.1 (1), \( P \) is isomorphic to \( P_1[x_i, x_j] \).
Clearly each component of $P_1 \times P_2$ is an oriented path. Any oriented path can be divided into several minimal oriented paths and this original oriented path can thus be constructed by concatenating these minimal oriented paths. Applying the above result to each minimal oriented path we have the conclusion that each component of $P_1 \times P_2$ is isomorphic to a subpath of $P$.

**Lemma 3.2.11** Let $P_1[x_0, x_1, ..., x_n]$ and $P_2[y_0, y_1, ..., y_m]$ be two minimal oriented paths with net length $k$. If $x_0$ ($y_0$) has the minimum level in $P_1$ (respectively $P_2$), then in $P_1 \times P_2$ there exists a minimal oriented path $Q$ of net length $k$ from $(x_0, y_0)$ to $(x_n, y_m)$, such that for any vertex $(x, y) \in Q$

$$f_{Q,(x_0,y_0)}(x, y) = f_{P_1,x_0}(x) = f_{P_2,y_0}(y).$$

**Proof** The proof can be done by induction on $k$. If $k = 1$, this is clear. Now suppose the lemma is true for net lengths 1, 2, ..., $k-1$; we shall prove it for $k$. This will also be proved by induction, on the total number $t$ of vertices of level $k$ in both $P_1$ and $P_2$ (here we take the level function on $P_1$ with respect to $x_0$ and the level function on $P_2$ with respect to $y_0$).

If $t = 2$, then there is no proper subpath of $P_1$ and $P_2$ of net length $k$. In this case both $P_1[x_0, x_1, ..., x_{n-1}]$ and $P_2[y_0, y_1, ..., y_{m-1}]$ are minimal oriented paths with net length $k-1$, and $x_{n-1}x_n$ and $y_{m-1}y_m$ are arcs. By the induction hypothesis, there exists a minimal oriented path $Q[(x_0, y_0), ..., (x_{n-1}, y_{m-1})]$ of net length $k-1$ such that for any $(x, y) \in Q$, $f_{Q,(x_0,y_0)}(x, y) = f_{P_1,x_0}(x) = f_{P_2,y_0}(y)$. Then $Q[(x_0, y_0), ..., (x_{n-1}, y_{m-1}), (x_n, y_m)]$ is the required oriented path in the product $P_1 \times P_2$. 

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Suppose now that we have \( t \) \((t > 2)\) vertices of level \( k \) in \( P_1[x_1, \ldots, x_n] \) and \( P_2[y_0, \ldots, y_m] \). Without loss of generality let \( x_{n_1} \) \((n_1 < n)\) be the last vertex of level \( k \) on \( P_1 \). Then we shall have \( t-1 \) vertices of level \( k \) in \( P_1[x_0, \ldots, x_{n_1}] \) and \( P_2[y_0, \ldots, y_m] \). By the induction hypothesis we shall have a minimal oriented path \( Q_1[(x_0, y_0), \ldots, (x_{n_1}, y_m)] \) with net length \( k \). Now suppose \( x_{n_2} \) is a vertex of the smallest level between \( x_{n_1} \) and \( x_n \), and \( f(x_{n_2}) = k_1 \). Then both \( P_1[x_{n_2}, x_{n_1}] \) and \( P_1[x_{n_2}, x_n] \) are minimal. Let \( y_j \) be the last vertex such that \( f(y_i) = k_1 \). Then \( P_2[y_j, y_m] \) is minimal. If \( k_1 > 0 \), then we apply the induction hypothesis on \( k \); if \( k_1 = 0 \), then we apply the induction hypothesis on the number of vertices of level \( k \). Thus there exists a minimal oriented path \( Q_2 \) from \((x_{n_2}, y_j)\) to \((x_{n_1}, y_m)\) (the traversal order is from \((x_{n_2}, y_j)\) to \((x_{n_1}, y_m)\)) and a minimal oriented path \( Q_3 \) from \((x_{n_2}, y_j)\) to \((x_n, y_m)\). Then \( Q_1Q_2(\cdot)Q_3 \) is the required minimal oriented path by Corollary 3.2.9.

**Corollary 3.2.12** Let \( P_1[x_0, x_1, \ldots, x_n] \) and \( P_2[y_0, y_1, \ldots, y_m] \) be two oriented paths. Let \( P_1[x_s, x_t] \subseteq P_1 \) and \( P_2[y_i, y_j] \subseteq P_2 \) be two intervals such that \( f_{P_1}(x_t)-f_{P_1}(x_s) = f_{P_2}(y_j)-f_{P_2}(y_i) \) and that both \( P_1[x_s, x_t] \) and \( P_2[y_i, y_j] \) are minimal. If \( x_s \) \((y_i)\) has the minimum level in \( P_1[x_s, x_t] \) \((\text{respectively } P_2[y_i, y_j])\), then there exists a minimal oriented path \( Q \) in \( P_1 \times P_2 \) starting at \((x_s, y_i)\), ending at \((x_t, y_j)\) and of net length \( f_{P_1}(x_t)-f_{P_1}(x_s) \); and for each vertex \((x, y) \in Q\), we have \( f_{Q}(x, y) = f_{P_1}(x)-f_{P_1}(x_s) \).

**Proof** Apply Lemma 3.2.11 to \( P_1[x_s, x_t] \times P_2[y_i, y_j] \).
Here we again note that the difference of levels on two vertices of a path does not depend on the starting vertex.

**Theorem 3.2.13** \(\mathcal{P}_1 \subseteq \mathcal{M}\).

**Proof** (Also see [11] and [20].) Apply Lemmas 3.2.11 and 1.3.9 (b).

**Corollary 3.2.14** \(\mathcal{P}_1 \subseteq \mathcal{W} \cap \mathcal{V} \mathcal{W} \mathcal{M}\) and \(\mathcal{P}_2 \subseteq \mathcal{M} \subseteq \mathcal{V} \mathcal{W} \mathcal{M}\).

Apply Claim 1.4.4, Claim 1.4.3, Lemma 1.3.2 and Theorem 3.2.13.

**Lemma 3.2.15** Let \(P[x_0, x_1, ..., x_n]\) be any oriented path, and \(P^*\) the subgraph of \(P \times P\) induced by \(V(P^*) = \{(x_i, x_i) i = 0, 1, ..., n\}\). Then the mapping \(\pi: V(P) \times V(P) \rightarrow V(P^*)\) defined by 

\[\pi(x_i, x_j) = (x_{\max(i,j)}, x_{\max(i,j)}) \text{ (or } \pi(x_i, x_j) = (x_{\min(i,j)}, x_{\min(i,j)})\)

is a retraction of \(P \times P\) to \(P^*\).

**Proof** Let \(G_1 (G_2)\) be the induced subgraph of \(P \times P\) determined by all vertices \((x_i, x_j)\) with \(i \leq j\) (respectively \(i \geq j\)). Then 

\[V(G_1) \cap V(G_2) = V(P^*).\]

For any vertices \(v_1 \in v(G_1 \setminus P^*)\) and \(v_2 \in (G_2 \setminus P^*)\), \(v_1\) and \(v_2\) cannot be adjacent, otherwise \(v_1\) will be \((x_i, x_{i+1})\) and \(v_2\) will be \((x_{i+1}, x_i)\) for some \(i\). If \(\overrightarrow{v_1v_2}\) (or \(\overrightarrow{v_2v_1}\)) is an arc in \(P \times P\), then \(\overrightarrow{x_i x_{i+1}}\) and \(\overrightarrow{x_{i+1} x_i}\) are both arcs in \(P\), a contradiction. Now suppose that \(v_1 = (x_i, x_j)\) and \(v_2 = (x_s, x_t)\) are adjacent in the product graph \(P \times P\). They should both be in \(G_1\) (or \(G_2\)). Without loss of generality,
let \( v_1 \) and \( v_2 \) be in \( G_1 \), then \( i \leq j \) and \( s \leq t \). If \( \overrightarrow{v_1v_2} \) is an arc in \( P \times P \), then \( \overrightarrow{x_jx_i} \) is an arc in \( P \) and so \( (x_j, x_j)(x_t, x_t) \) is an arc in \( P^* \) \((j = \max(i, j), t = \max(s, t))\); \( \overrightarrow{x_sx_s} \) is also an arc in \( P \) and so \( (x_i, x_i)(x_s, x_s) \) is an arc in \( P^* \). Furthermore, these two mappings keep vertices of \( P^* \) fixed. Hence they are retractions of \( P \times P \) to \( P^* \).

**Lemma 3.2.16** If the oriented path \( P \) is a core of total length greater than 1, then the first two arcs of \( P \) have the same direction and the last two arcs of \( P \) also have the same direction.

**Proof** Otherwise the first (last) vertex of \( P \) can be mapped to the third (third last) vertex of \( P \) by an obvious retraction.

**Lemma 3.2.17** If \( W \) is a minimal oriented path with \( \text{nl}(W) = 1 \), then any subpath of \( W \) is a retract of \( W \).

**Proof** The minimal oriented paths of net length one are very special, consisting of an alternating sequences of forward and backward arcs.

**Lemma 3.2.18** Assume that \( W \) is an oriented path with level function \( f \) and \( a \in V(W) \) has minimum level. Then any \( x \in V(W) \) with \( \text{nl}(W[a, x]) = 0 \) has indegree zero.

**Proof** If \( a \) has nonzero indegree, then there is an arc \( \overrightarrow{ya} \) in \( W \), and \( y \) will have level \( f(a) - 1 \), contradicting the minimality of the level of \( a \). For any \( x \in V(W) \) with \( \text{nl}(W[a, x]) = 0 \), we have \( f(x) = f(a) \) by Lemma 3.2.6 and so the same argument applies.
We now proceed to one of our main constructions, showing that, with a minor exception, only cores can be weakly multiplicative oriented paths.

**Theorem 3.2.19** Any oriented path in $\mathcal{W}$, except for $U_1$ and $U_2$ given in Figure 2.2, is a core.

**Proof** Let $W$ be a non-core oriented path other than $U_1$ and $U_2$. Let $P_1$ be a core of $W$, then we can write $W = P_3P_1P_2$ where $P_i (i = 1, 2, 3)$ are oriented paths, $P_1$ is a retract of $W$, and $P_3 \cup P_2 \neq \emptyset$. We proceed to prove that $W \notin \mathcal{W}$ according to the following two cases.

**Case 1.** The total length of $P_1$ is 1.

In this case $P_1$ must be the directed path $\overrightarrow{P_1}$, and $W$ must not contain $\overrightarrow{P_2}$. Since the non-core oriented path $W$ is neither $U_1$ nor $U_2$, $W$ must contain the oriented path $W_1$ (see Figure 3.1).

\[
W_1 = \begin{array}{c}
\bullet \\
w_1 & w_2 & w_3 & w_4 \\
\bullet 
\end{array}
\]

**Figure 3.1**

Let $W = [w_0, \ldots, w_1, w_2, w_3, w_4, \ldots, w_{n-1}, w_n]$ where $w_0$ may be equal to $w_1$ and $w_4$ may be equal to $w_n$. So the subpath induced by vertices $\{w_i \mid i = 1, 2, 3, 4\}$ is $W_1$, and arcs of $W$ are going alternately
forward and backward along the traversal order of $W$. If $w_n$ has indegree zero, then we construct graphs $G$ and $H$ as follows:

$$V(H) = V(W), E(H) = E(W) \cup \{\overrightarrow{w_4w_1}\};$$

$$V(G) = V(W) \cup \{f, g, h\}, E(G) = E(W) \cup \{\overrightarrow{fg}, \overrightarrow{gh}, \overrightarrow{w_n}\}. $$

If $w_n$ has outdegree zero, then we construct a graph $H$ as above, and a graph $G$ where

$$V(G) = V(W) \cup \{f, g, h\}, E(G) = E(W) = \{\overrightarrow{gf}, \overrightarrow{hg}, \overrightarrow{hw_n}\}. $$

It is easy to see that $W \subseteq G$, $W \subseteq H$, and $W \not\subseteq G$, $W \not\subseteq H$.

Now we shall prove that $W \triangleleft G \times H$. Note that $W$ is isomorphic to the subgraph $W^*$ induced by the vertices $V(W^*) = \{(w, w) \mid w \in W\}$ on the main diagonal of $W \times W \subseteq G \times H$.

Note that if $P_1$ and $P_2$ are any two oriented paths and $v \in V(P_1 \times P_2)$, then $v$ is incident with at most four arcs, and all the various possibilities are illustrated in Figure 3.2.

![Figure 3.2](image-url)
Now consider the product $G \times H$. Let $v \in V(G \times H)$. Assume $v = (x, y)$. Recall that $G$ and $H$ are constructed from $W$ which is an oriented path with arcs alternatively forward and backward. Then the degrees of $v$ are basically as illustrated in (1) and (2) of Figure 3.2 except when $x = w_0$, $g$, $f$, or $y = w_0$, $w_n$ in which case (3) occurs. The product $G \times H$ has two non-trivial components $A_1$ and $A_2$; the other components are isolated vertices (cf. Figure 3.3). The component $A_1$ is a connected graph with vertices contained in $V[(G-f) \times H]$. The component $A_2$ is a connected graph with vertices contained in $\{g, f\} \times V(H)$. Clearly $A_2 \to \overline{P}_1$ by a projection. Therefore it will suffice to prove that $W^*$ is the retract of $A_1$.

Let $W^{**}$ be the subgraph of $G \times H$ induced by $V(W^*) \cup \{(h, w_{n-1}), (g, w_n)\}$. Then $W^{**}$ is a minimal oriented path with net length 1, hence $W^*$ is a retract of $W^{**}$ by Lemma 3.2.17. Therefore it is sufficient to prove that $W^{**}$ is a retract of $A_1$.

Define a mapping $\varphi: A_1 \to W^{**}$ by

$$
\varphi(x, y) = \begin{cases} 
(x, x) & \text{if } x \neq g, \ x \neq h, \\
(h, w_{n-1}) & \text{if } x = h, \\
(g, w_n) & \text{if } x = g.
\end{cases}
$$

Since $\varphi = \varepsilon \pi$, where $\pi$ is the projection from $(G-f) \times H$ to $G-f$, and $\varepsilon$ is an isomorphism from $G-f$ onto $W^{**}$, $\varphi$ is a homomorphism from $A_1$ to $W^{**}$, and hence a retraction.

See Figure 3.3 for an illustration.
The vertex $w_n$ has indegree 0.

$w_n$ has outdegree zero.

Figure 3.3
Figure 3.3

Case 2  The total length of $P_1$ is greater than 1, i.e., $P_1 \neq \overrightarrow{P}_1$.

Without loss of generality, let $P_2 \neq \emptyset$. Let $P_1 = [x_0, ..., x_{n-2}, x_{n-1}=d, x_n=b]$ ($n \geq 2$), $P_2 = [x_n=b, x_{n+1}=e, ... ]$. The vertex $d=x_{n-1}$ must have indegree one and outdegree one by Lemma 3.2.16. Suppose that we have arcs $\overrightarrow{x_{n-2}x_{n-1}}$, $\overrightarrow{x_{n-1}x_n}$ and thus (as $P_1$ is the retract of $W$) $\overrightarrow{x_{n+1}x_n}$. The other case $\overrightarrow{x_{n-1}x_{n-2}}$, $\overrightarrow{x_{n}x_{n-1}}$ and $\overrightarrow{x_{n}x_{n+1}}$ is treated similarly. We choose a level function $\phi$ on $W$ such that the minimum level of a vertex $x \in V(W)$ is equal to zero. According to Lemma 3.2.18, any vertex $x \in V(W)$ with $\phi(x) = 0$ has indegree zero. We fix one such vertex $x = a$.

Now construct $H$ to be the graph such that $V(H) = V(W) \cup \{c\}$, $E(H) = E(W) \cup \{\overrightarrow{cd}, \overrightarrow{ce}\}$. Construct $G$ to be the graph such that $V(G) = V(W) \cup \{f, g, h\}$ and $E(G) = E(W) \cup \{\overrightarrow{ah}, \overrightarrow{fg}, \overrightarrow{gh}\}$. It is easy to see that $W \not\rightarrow G$ since we cannot find in $W$ a vertex of level as small as that of $f \in V(G)$. On the other hand there are homomorphisms $H \rightarrow P_1 \rightarrow W$. However for any homomorphism $f: H \rightarrow W$, the vertex $c$ is mapped to some $f(c) = x$ from which there is an arc to both $f(e)$ and $f(d)$. Thus $f(e) = f(d)$ and therefore $W \not\rightarrow H$.

We now prove that $W \not\rightarrow G \times H$.

The product $G \times H$ has several components. Each component can be homomorphically mapped to $H$ and hence to $P_1$. In fact $G \times H$ has one component $Q$ which contains the induced subgraph $W^*$ with $V(W^*) = \{(x, x) \mid x \in V(W)\}$ which is isomorphic to $W$. Hence we only need to prove that $W^*$ is a retract of $Q$.
As usual, the traversal order on \( W \) is from \( P_3 \) to \( P_1 \) to \( P_2 \). We define order "<" as the traversal order. In order to simplify the notation, (i.e., to avoid writing many subscripts), we denote by \( y+1 \) and \( y-1 \) the next vertex (in the traversal order) and the previous vertex, respectively, of \( y \in V(W) \).

**Claim 1** If \( Q_1 \) is the component of \( W\times H \) containing \( W^* \), and if \((a, y) \in V(Q_1)\), then \( y \) has indegree zero.

If \( y = c \), then this is obvious, since \( c \) has indegree zero. In the following we assume \( y \in V(W) \).

There exists an oriented path \( P \) in \( Q_1 \) from some \((x, x) \in V(W^*)\) to \((a, y)\) since the connected graph \( Q_1 \) contains both \( W^* \) and \((a, y)\). Thus

\[
 nl(P[(x, x),(a, y)]) = nl(W[x, a]) = nl(W[x, y])
\]

by Corollary 3.2.9. Hence

\[
 nl(W[a, y]) = nl(W[a, x]) - nl(W[y, x]) = 0.
\]

By Lemma 3.2.18, \( y \) has indegree zero.

**Claim 2** \( Q_1 = Q \cap (W\times H) \), i.e., \( Q \cap (W\times H) \) is connected.

It is obvious that \( Q_1 \subseteq Q \cap (W\times H) \). Now assume that \( Q \cap (W\times H) \) has a component \( Q_2 \) other than \( Q_1 \). Both \( Q_1 \) and \( Q_2 \) are connected subgraphs of \( W\times H \). There are no paths in \( W\times H \) that join \( Q_1 \) with \( Q_2 \), but there are paths in \( G\times H \) that join \( Q_1 \) with \( Q_2 \). Since \( a \) is the only vertex of \( W \) adjacent to \( \{f, g, h\} \), any oriented path in \( Q \) joining \( Q_1 \) and \( Q_2 \) must contain a vertex \((a, z)\) of \( Q_2 \) and a vertex \((a, y)\) of \( Q_1 \) \((y, z \in H)\). Denote the interval of this oriented path between \((a, y)\) and \((a, z)\) by \( P \). Then

\[
P' = P - \{(a, y), (a, z)\} \subseteq (G-W)\times H
\]
To reach a contradiction, we shall prove that $Q_1$ and $Q_2$ are actually connected by an oriented path of $W \times H$, according to the following three cases. This contradiction shows that $Q \cap (W \times H)$ is connected and thus equal to $Q_1$.

**Case 2.1 $y, z \in W$.**

Without loss of generality we assume that $y < z$. Because of the structure of $(G-\overline{W}) \times H$, $P'$ must be an oriented path generated by concatenating several oriented paths $U_1$ and $U_3$ of Figure 3.4 in some given order.

\[ U_1 = \begin{array}{c} \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \end{array} \quad U_3 = \begin{array}{c} \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \end{array} \]

\[ \begin{array}{c} u_2 \quad u_1 \quad u_3 \end{array} \]

**Figure 3.4**

We shall prove that only $P' = U_1 U_1 ... U_1$ is possible. Let $P' = P^1 P^2 ... P^k$ where each $P^i (1 \leq i \leq k)$ is either a copy of $U_1$ or a copy of $U_3$. We shall prove by induction on $i$ that each $P_i \neq U_3 (1 \leq i \leq k)$.

$P^1$ cannot be $U_3$. In fact if $P^1$ is $U_3$, then the subgraph of $P$ induced by $(a, y) \cup V(U_3)$ will either be (a) or (b) as follows:

(a) Vertex set $(a, y), (h, y+1), (g, y+2), (f, y+3), (g, y+4)$ and $(h, y+5)$; arc set $(a, y)(h, y+1), (g, y+2)(h, y+1), (f, y+3)(g, y+2), (f, y+3)(g, y+4)$ and $(g, y+4)(h, y+5)$.

(b) Vertex set $(a, y), (h, y-1), (g, y), (f, y+1), (g, y+2)$ and $(h, y+3)$; arc set $(a, y)(h, y-1), (g, y)(h, y-1), (f, y+1)(g, y), (f, y+1)(g, y+2)$ and $(g, y+2)(h, y+3)$.

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Figure 3.5

Figure 3.6
See Figure 3.5 (a) and (b).

In (a), we must have \((a, y+2) \in Q_1\) since there is an oriented path from \((a, y)\) to \((a, y+2)\) in \(W \times H\) and \((a, y) \in Q_1\). But \(y+2\) has indegree one in \(W\). In (b) \(y\) has indegree one in \(W\). Hence both (a) and (b) yield a contradiction to Claim 1.

Now suppose that each of \(P_1, P_2, ..., P_{i-1}\) \((i < k)\) is a copy of \(U_1\) but \(P_i\) is a copy of \(U_3\). Then the subgraph \(P - \{P_{i+1} \ldots P_k\}\) of \(P\) will have vertices (see Figure 3.6)

\[
(a, y), (h, y+1), (g, y+2), (h, y+3), ..., (h, y+2i-3), (g, y+2i-2)
\]

\[
(h, y+2i-1), (g, y+2i), (f, y+2i+1), (g, y+2i+2), (h, y+2i+3)
\]

and arcs

\[
(a, y)(h, y+1), \quad (g, y+2j)(h, y+2j+1)
\]

\[
(g, y+2i)(h, y+2i-1), \quad (f, y+2i+1)(g, y+2i)
\]

It can then be concluded that there is an oriented path from 
\((a, y)\) to \((a, y+2i)\) in \(W \times H\) with vertices \((a, y+2j)\) \((j = 0, 1, ..., i)\) and 
\((a+1, y+2j-1)\) \((j = 1, 2, ..., i)\) with arcs alternatively forward and backward. Moreover, \(y+2i\) has indegree one in \(W\). This contradicts Claim 1.

If \(P' = U_1 U_1 ... U_1\), then by the same argument as above \((a, y)\) and \((a, z)\) will be joined by an oriented path in \(W \times H\) with alternate forward and backward arcs.

**Case 2.2.** \(y \in W, z = c\).

**Case 2.3.** \(z \in W, y = c\).

The proofs in Cases 2.2 and 2.3 are very similar to the proof in Case 2.1.
Now we shall find a retraction from $Q_1 = Q \cap (W \times H)$ to $W^*$. Then we prove that this retraction can be extended to $Q$.

Define $\rho: Q_1 \to W^*$ by $\rho(x, y) = (x, x)$. Then $\rho$ keeps vertices of $W^*$ fixed. Moreover if $(x_1, y_1)(x_2, y_2)$ is an arc of $Q \cap (W \times H)$, then $x_1x_2$ and $y_1y_2$ are arcs of $W$. Hence $(x_1, x_1)(x_2, x_2)$ is an arc of $W^*$. Therefore $\rho$ is a retraction of $Q_1$ to $W^*$.

Now we extend $\rho$ to $Q$. If there is an arc between the vertices of $Q_1 = Q \cap (W \times H)$ and the vertices of $Q \cap ((G-W) \times H)$, it should be either (1) $(a, y)(h, y+1)$ or (2) $(a, c)(h, e)$, $(a, c)(h, d)$. The component of $Q \cap ((G-W) \times H)$ which is connected to $Q_1$ in $G \times H$ can only be

$$P^* = U_1U_1 \ldots U_1$$

for some $i$ (see Figure 3.8). For each copy $U_1$, (see Figures 3.4 and 3.8) $u_1$ is some $(g, y)$, $u_2$ and $u_3$ are some $(h, y+1)$ or $(h, y-1)$. The proof is the same as the proof of that $P' = U_1U_1 \ldots U_1$ in Claim 2. Thus we can define

$$\rho(u_1) = (a, a), \quad \rho(u_2) = (a+1, a+1), \quad \rho(u_3) = (a+1, a+1) \quad \text{(or} \quad (a-1, a-1)),$$

for all copies $U_1$ of all $P^*$'s, to extend $\rho$ to $Q$ while preserving the homomorphism.

**Illustration** For the graph $W$ in Figure 3.7, see the main component of $G \times H$ in Figure 3.8.

![Figure 3.7](image-url)
Figure 3.8
### 3.3 Multiplicativity of Oriented Paths (Part II)

Here we prove that all paths in the classes $\mathcal{P}_3$ and $\mathcal{P}_4$, (i.e., all those oriented paths which are not homomorphically equivalent to a directed path), are non-multiplicative. Our main construction is the so-called basic path.

A basic forward $b$-path with parameters $(k_1, 2m_1-1; k_2, 2m_2-1; \ldots; k_{n-1}, 2m_{n-1}-1; k_n)$ $(n \geq 2$ and all $k_i > b)$ is an oriented path such that:

(i) All maximal one-directional intervals have length at least $b$;
(ii) All maximal backward intervals have length exactly $b$;
(iii) The first and last maximal one-directional intervals have length strictly greater than $b$ (and so, in particular, they are forward intervals);
(iv) The lengths of $n$ of the maximal forward intervals exceed $b$; these lengths are $k_1, k_2, \ldots, k_n$ in that order (including the first, $k_1$, and the last, $k_n$); and
(v) There are $2m_i-1$ maximal one-directional intervals of length $b$ between the maximal forward interval of length $k_i$ and of length $k_{i+1}$ (of which $m_i$ are backward and $m_i-1$ forward intervals).

In other words, such a basic b-path consists of a $\overrightarrow{P}_{k_1}$ followed by $\overrightarrow{P}_b, \overrightarrow{P}_b, \ldots, \overrightarrow{P}_b$ ($2m_1-1$ repetitions), $\overrightarrow{P}_{k_2}$, then $\overrightarrow{P}_b, \overrightarrow{P}_b, \ldots, \overrightarrow{P}_b$ ($2m_2-1$ repetitions), etc., and ending with $\overrightarrow{P}_{k_n}$ (cf. Figure 3.9). When we write $S = \cdots \overrightarrow{P}_i \cdots \overrightarrow{P}_j \cdots$, we always assume that each $\overrightarrow{P}_i$ (and $\overrightarrow{P}_j$) is a maximal one-directional interval. We also use the terms
(\(k_1, k_2, \ldots, k_m\)) basic forward b-path, if the exact numbers of the repetitions of \(\overrightarrow{P}_{b}, \overleftarrow{P}_{b}\) need not be specified (i.e., conditions (i) - (iv)); or just basic forward b-path if the parameters are not needed (i.e., conditions (i) - (iii)). Basic backward b-paths are defined similarly. A basic b-path is a basic forward or backward b-path.

A \((b+1, 2m-1)^n\) basic b-path is a basic b-path with parameters \((k_1, 2m_1-1; \ldots; k_{n-1}, 2m_{n-1}-1; k_n)\) where \(k_1 = \ldots = k_n = b+1\) and \(m_1 = \ldots = m_{n-1} = m\).

(a) \((3, 4, 3, 5)\) basic (forward) 2-path or
\((3, 3; 4, 5; 3, 1; 5)\) basic (forward) 2-path

(b) \((3, 2, 4, 4)\) basic (backward) 1-path or
\((3, 1; 2, 3; 4, 5; 4)\) basic (backward) 1-path

Figure 3.10  Basic b-path
A degenerate \((b+1, 2m-1)^n\) basic b-path is defined exactly the same way except that \(k_1 = b\) or \(k_n = b\) (or both). (Thus a degenerate basic b-path violates condition (iii).)

We give examples for these definitions in Figures 3.10 and 3.11.

![Diagram](image)

(2,3)\(^4\) basic (forward)1-path

(2,3)\(^4\) degenerate basic (forward)1-path

Figure 3.11 (degenerate) basic b-path

**Lemma 3.3.1** Let \(P_2[y_0, y_1, \ldots, y_k]\) be a directed path of length \(k\), \(P_1[x_0, x_1, \ldots, x_t]\) a \((b+1, 2m-1)^n\) basic forward b-path \((k > 1)\). Then any component \(C\) of \(P_1 \times P_2\) is one of the following five cases:

1. \(C\) is an isolated vertex located in \(\{(x, y_0): x \in P_1\}\) or \(\{(x, y_k): x \in P_1\}\);

2. \(C\) is a directed path \(\overrightarrow{P}_{b'}\) \((1 \leq b' \leq b)\) either starting at some \((x_0, y)\) \((y \in P_2)\) and ending at some \((x, y_k)\) \((v \in P_1)\); or starting at some \((x, y_0)\) \((x \in P_1)\) and ending at some \((x_t, y)\) \((y \in P_2)\);

3. \(C\) is \(\overrightarrow{P}_{b'}\overrightarrow{P}_{b'}\) \((1 \leq b' \leq b)\) starting at some \((x, y_0)\) and ending at some \((x', y_0)\) \((x, x' \in P_1)\);
(4) $C$ is $\overrightarrow{P}_b$, $\overrightarrow{P}_b$, $(1 \leq b' \leq b)$ starting at some $(x, y_k)$ and ending at some $(x', y_k)$ $(x, x' \in P_1)$; or

(5) $C$ is a $(b+1, 2m-1)^n$ basic or degenerate basic b-path for some $n_1$ $(2 \leq n_1 \leq n)$ starting at some $(x, y_0)$ $(x \in P_1)$ and ending at some $(x', y_k)$ $(x' \in P_1)$; or starting at some $(x_0, y)$ $(y \in P_2)$ and ending at some $(x, y_k)$ $(x \in P_1)$; or starting at some $(x, y_0)$ $(x \in P_1)$ and ending at some $(x_t, y)$ $(y \in P_2)$.

Proof The proof can easily be obtained by Lemmas 3.2.1 and 3.2.10.

Lemma 3.3.2 Let $G$ be a $(b+1, 2m-1)^n$ basic b-path and $W$ any oriented path. Then there exists an homomorphism from $G$ onto $W$ if and only if $W$ is a directed path of net length $nl(G)$ or a basic b-path with parameters $(k_1, 2m_1-1; \ldots; k_t-1, 2m_t-1-1; k_t)$ where $t \leq n$, all $m_i \leq m$, and $nl(W) = nl(G)$.

Proof If $h: G \rightarrow W$ is an onto homomorphism, then $nl(G) = nl(W)$ by Corollary 3.2.9. Since $h(\overrightarrow{P}_i) = \overrightarrow{P}_i$ because $W$ is an oriented path, any maximal one-directional interval of $W$ must have length at least $b$. Similarly $h(\overrightarrow{P}_i \overrightarrow{P}_j) = \overrightarrow{P}_i \overrightarrow{P}_j$ or $\overrightarrow{P}_{\max(i,j)}$, as the underlying graph of $W$ has no vertices of degree 3. Therefore,

$$h(\overrightarrow{P}_{b+1} \overrightarrow{P}_b \overrightarrow{P}_b \ldots \overrightarrow{P}_b \overrightarrow{P}_{b+1})$$

is either $\overrightarrow{P}_{b+2}$ or some $\overrightarrow{P}_{b+1} \overrightarrow{P}_b \overrightarrow{P}_b \ldots \overrightarrow{P}_b \overrightarrow{P}_{b+1}$ with the number of repetitions of $\overrightarrow{P}_b \overrightarrow{P}_b$ in the image at most equal to that in the original. These observations amount to the proof that if $n = 2$ then $W = \overrightarrow{P}_{b+2}$ or is a basic b-path with parameters $(b+1, 2m_1-1; b+1)$ where
Let \( m_1 \leq m \). The general case easily follows by induction on \( n \).

Conversely, if \( W \) is a directed path of net length \( nl(G) \), there is an obvious homomorphism of \( G \) onto \( W \). (Choose a level function on each of \( G \) and \( W \) which assigns 0 to the first vertex and maps each vertex of \( G \) to the unique vertex of \( W \) with the same level.) Let
\[
W = \overrightarrow{P}_{k_1} H_1 \overrightarrow{P}_{k_2} H_2 \ldots \overrightarrow{P}_{k_t}
\]
where \( t \leq n \) and
\[
H_i = \overrightarrow{P}_b \overrightarrow{P}_b \ldots \overrightarrow{P}_b \quad \text{(repeated } 2m_i - 1 \text{ times).}
\]
A homomorphism of \( G \) onto \( W \) is easily accomplished by writing \( G \) as
\[
G = G_1 H G_2 H \ldots G_{t-1} H G_t
\]
where \( G_i \) is a subpath of \( G \) which is a \((b+1, 2m-1)^{k_i-1}\) basic \( b \)-path, and
\[
H = \overrightarrow{P}_b \overrightarrow{P}_b \ldots \overrightarrow{P}_b \quad \text{(repeated } 2m-1 \text{ times).}
\]
Then clearly \( G_i \) is mapped onto \( \overrightarrow{P}_{k_i} \) and \( H \) is mapped onto \( H_i \) (\( i = 1, 2, \ldots, t \)), with endvertices mapped to endvertices. \( \square \)

**Lemma 3.3.3** Let \( G \) be a \((b+1, 2m-1)^n\) basic or degenerate basic \( b \)-path and \( W \) a directed path or a \((k_1, 2m_1-1, \ldots; k_t-1, 2m_t-1; k_t)\) basic \( b \)-path with \( m \geq m_i \) for \( i = 1, \ldots, t-1 \). Assume that \( nl(G) \leq nl(W) \). Then

1. \( G \rightarrow W \),
2. we can map the vertex of \( G \) with smallest (or greatest) level to the vertex of \( W \) with smallest (or greatest) level in the homomorphic mapping of \( G \) to \( W \), and
(3) if \( \text{nl}(G) = \text{nl}(W) \), then the two endvertices of \( G \) will be mapped to the two endvertices of \( W \) in the homomorphic mapping of \( G \) to \( W \).

**Proof**  Similar to the proof of sufficiency in Lemma 3.3.2. \( \square \)

**Theorem 3.3.4**  \( P_3 \cup P_4 \subseteq \overline{\mathcal{P}}. \)

**Proof**  Let \( W \in P_3. \) (If \( W \in P_4 \), then by Corollary 1.4.3 we only need to consider the core of \( W \).)

Let \( b_0 \) be the minimum length of any maximal one-directional intervals in \( W \). A \( b_0 \)-run is a subpath of \( W \) consisting of alternating consecutive \( \overrightarrow{p_{b_0}} \) and \( \overleftarrow{p_{b_0}} \) (i.e., \( \overrightarrow{p_{b_0}} \overleftarrow{p_{b_0}} \ldots \), or \( \overrightarrow{p_{b_0}} \overleftarrow{p_{b_0}} \ldots \)). A maximal \( b_0 \)-run is a \( b_0 \)-run not included in a larger \( b_0 \)-run. Since \( W \) is a core, any maximal \( b_0 \)-run must be preceded and followed by a one-directional interval of length greater than \( b_0 \). We shall prove the theorem by constructing two oriented paths \( G \) and \( H \) with \( G \leftarrow W, \ H \leftrightarrow W \) and \( G \times H \rightarrow W \); we shall do this according to the following two cases:

**Case 1**  The digraph \( W \) has a maximal \( b_0 \)-run with an odd number of maximal one-directional intervals, which occurs at neither the beginning nor the end of \( W \).

In this case we must have a subpath \( W_1 \) of \( W \) which is a \((b_0+1, 2m-1; b_0+1)\) basic \( b_0 \)-path for some \( m \). We may assume that \( W_1 \) is a basic forward \( b_0 \)-path.
Write $W = U y_1 W_1 y_2 V$ where we denote by $y_1$ the common vertex of $U$ and $W_1$ and $y_2$ the common vertex of $W_1$ and $V$. Recall that $\mathcal{P}$ is the set of all oriented paths. Let

\begin{align*}
\mathcal{A}_1 &= \{S \in \mathcal{P} : S \text{ is a subpath of } W\}, \\
\mathcal{A}_2 &= \{S \in \mathcal{A}_1 : S \text{ is a maximal one-directional interval in } W\}, \\
\mathcal{A}_3 &= \{S \in \mathcal{A}_1 : S \text{ is a basic } b_0\text{-path}\}.
\end{align*}

For an illustration, assume $W$ is the oriented path in Figure 3.11. Then the basic 2-paths in Figure 3.12 are the members of $\mathcal{A}_3$.

![Figure 3.11](image_url)

![Figure 3.12](image_url)
Figure 3.13 One component of $G \times H$
Let 
\[ b = \max \{ \text{nl}(s) : s \in \mathcal{A}_2 \cup \mathcal{A}_3 \} \]

Now we construct \( G \) and \( H \) as follows:

Let \( G \) be a \((b_0 + 1, 2m-1)^{b+1-b_0}\) basic forward \( b_0 \)-path and let \( H \) be \( Uz_1P_{b_0+2}z_2V \).

For the example \( W \) given in Figure 3.11, see Figure 3.13 for the graphs \( W_1, G \) and \( H \).

It now follows that \( G \rightarrow W \) by Lemma 3.3.2 since \( \text{nl}(G) = b+1 \).

Furthermore \( H \rightarrow W \). Otherwise \( H \rightarrow W \) by a homomorphism which is not onto, because \( W \) has more vertices than \( H \). Since \( W \rightarrow H \), we have \( W \rightarrow H \rightarrow W \) by a homomorphism of \( W \) to a proper subgraph, contradicting the fact that \( W \) is a core.

**Claim** \( G \times H \rightarrow W \).

Let
\[
Q_1 = G \times U,
Q_2 = G \times P_{b_0+2},
Q_3 = G \times V,
\]
and let
\[
X_1 = \{(x, z_1) : x \in G\}, \text{ and}
X_2 = \{(x, z_2) : x \in G\}.
\]

Then the vertex set \( X_1 \) is contained in both \( Q_1 \) and \( Q_2 \) and the vertex set \( X_2 \) is contained in both \( Q_2 \) and \( Q_3 \).
The projection \( \pi \) defined by \( \pi(x, y) = y \) is a homomorphism of \( G \times H \) onto \( H \) which maps \( Q_1 \) to \( U \), \( Q_2 \) to \( \overrightarrow{P_{b_0+2}} \), and \( Q_3 \) to \( V \), with the set \( X_i \) mapped to \( z_i \) \((i = 1, 2)\). Lemma 3.3.1 classifies the possible components of \( Q_2 \). We now show that \( \pi \) can be modified to be a homomorphism \( G \times H \rightarrow W \) by showing that each component \( C \) of \( Q_2 \) may be homomorphically mapped to \( W_1 \) so that the images of \( X_1 \cap C \) and \( X_2 \cap C \) are \( y_1 \) and \( y_2 \) respectively. This is clear for the components of the type (1-4) in the statement of Lemma 3.3.1, and follow from Lemma 3.3.3 for the component of type (5).

See Figure 3.13 for illustration.

**Case 2** All maximal \( b_0 \)-runs have an even number of maximal one-directional intervals. Let \( I \) be one such run.

Write \( W = A_1 x I y A_2 \). We can perform an I-deleting operation on the interval \([x, y]\) of \( W \) to obtain \( A = A_1 x A_2 \) \( (i.e., \) cancel \( I \) and identify \( x \) and \( y \)). Conversely, we can perform an I-squeezing operation at \( x \) of \( A \) to return to \( W \). If we don't wish to specify \( I \), we call these two operations a \( b_0 \)-even-deleting operation and a \( b_0 \)-even-squeezing operation. Let \( W^1 \) be the oriented path obtained from \( W \) after performing \( b_0 \)-even-deleting operations on all maximal \( b_0 \)-runs.

Let \( b_1 \) be the minimum length of all maximal one-directional intervals in \( W^1 \). Clearly \( b_1 > b_0 \). Let \( I \) be one of the deleted intervals of \( W \) during the \( b_0 \)-even-deleting operation. Then \( I \) has the following three properties:

(i) \( nl(I) = 0; \)
(ii) the level of any vertex of I is between the levels of the two endvertices of either one of the neighboring maximal one-directional intervals in W (for any fixed level function on W); and

(iii) any subpath of I has net length smaller than $b_1$.

Now it is easy to see that $W \rightarrow W^1$ by a homomorphism which maps the deleted interval to either one of its neighboring maximal one-directional intervals.

If $W^1$ is still in Case 2, then we again do all the $b_1$-even-deleting operations on $W^1$. Continuing this process, let the oriented path $W^k$ be obtained from $W^{k-1}$ by doing all the $b_{k-1}$-even-deleting operations, provided $W^{k-1}$ is in Case 2, and $b_{k-1}$ is the minimum length of all maximal one-directional intervals of $W^{k-1}$. Furthermore, let $I^k$ be one of the intervals which was removed from W in the process of constructing $W^k$ and I the corresponding interval of $W^{k-1}$ (each maximal one-directional interval of I has length $b_{k-1}$) and let $b_k$ the minimum length of all maximal one-directional intervals of $W^k$. One can prove by induction on $k$ that $I^k$ has the following three properties (see Figure 3.14 for an illustration):

(iv) $nl(I^k) = 0$;

(v) the level of any vertex of $I^k$ is between the levels of two endvertices of either one of the neighboring maximal one-directional intervals in W (for any fixed level function on W); and

(vi) any subpath of $I^k$ has net length smaller than $b_k$. 

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Let $W_k$ be the first graph obtained in this process in which there is a maximal $b_k$-run $I$ with an odd number of maximal one-directional intervals of length $b_k$ where $b_k$ is the minimum length of all the maximal one-directional intervals of $W_k$. (We shall call $I$ a maximal odd $b_k$-run.)

Let $I_k$ be any one of the intervals which were removed from $W$ in order to construct $W_k$. Then $I_k$ satisfies (iv)-(vi).

Now it is easy to see that $W \rightarrow W_k$ by a homomorphism which maps each deleted interval to either one of its two neighboring maximal one-directional intervals.
It is also easy to see that the maximal odd $b_k$-run I cannot occur at the beginning or at the end of $W^k$. Otherwise $W$ will not be a core, since by (v) we can homomorphically map the corresponding $I^k$ (see Figure 3.15) to the neighboring maximal one-directional interval in $W$.

Hence Case 1 applies to $W^k$, and let

$$W^k = U^k W^k_1 V^k$$

where $W^k_1$ is a $(b_k+1, 2m-1; b_k +1)$ basic $b_k$-path. Correspondingly we have $\delta_k_1, \delta_k_2, \delta_k_3$ and $b^*$ as in Case1.

$$\delta_k_1 = \{S \mid S \text{ is a subpath of } W^k\},$$
$$\delta_k_2 = \{S \in \delta_k_1 \mid S \text{ is a maximal one-directional interval of } W^k\},$$
$$\delta_k_3 = \{S \in \delta_k_1 \mid S \text{ is a basic } b_k\text{-path }\}, \text{ and }$$
$$b^* = \max\{nl(S) \mid S \in \delta_k_2 \cup \delta_k_3\}.$$ 

We now construct $G^k$ and $H^k$ as follows:

$G^k$ is a $(b_k+1, 2m-1)^{b^*+1-b_k}$ basic $b_k$-path; and

$H^k = U^k \overrightarrow{P_{b_k+2}} V^k.$

We have $G^k \rightarrow W^k, G^k \times H^k \rightarrow W^k$ as proved in Case1. If we write

$$W^k_1 = \overrightarrow{P_{b_k+1}} a_1 \overrightarrow{P_{b_k}} a_2 \ldots \overrightarrow{P_{b_k}} a_{2m} \overrightarrow{P_{b_k+1}}$$

$$= \overrightarrow{P_{b_k+1}} W^k_2 \overrightarrow{P_{b_k+1}},$$

where $a_i \ (i=1, 2, \ldots, 2m)$ denotes the common endvertex of the two neighboring maximal one-directional intervals, see Figure 3.16, then $G^k$ can be written as

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Now we can construct $G$ and $H$ as follows.

Suppose that the interval $I$ of $W$ was removed in the process of constructing $W^k$. If the vertex of $W^k$ where this occurred is in $U^k$ or $V^k$, then we insert a copy of $I$ in the corresponding vertex of $H^k$. If the vertex was some $a_i$ ($i = 1, ..., 2m$), then we insert a copy of $I$ at the corresponding vertex $a_i$ of each copy of $W^{k2}$ in $G^k$. Note that there are $b^* - b_k$ copies of $W^{k2}$ in $G^k$.

This way we obtain graphs $G$ and $H$, and $\overrightarrow{P}_{b_k+2}$ is a subpath of $H$.

We write

$$H = UP_{b_k+2} V, \quad W = UW_1 V$$

where $U$ ($V$) corresponds to $U^k$ ($V^k$).

Claim 1 $H \not	o W$. 
Assume on the contrary that \( H \to W \). Since we have \( W^k \to H^k \), because \( U^k \to U^k \), \( V^k \to V^k \) and \( W^{k+1} \to \overrightarrow{P}_{b_{k+2}} \), and we have inserted a copy of \( I \) at the corresponding vertices of \( W^k \) and \( H^k \), we have \( W \to H \). The mapping from \( H \) to \( W \) is not onto since \( H \) has fewer vertices than \( W \). Thus the composition of \( W \to H \to W \) is a homomorphism of \( W \) onto a proper subgraph, contradicting the fact that \( W \) is a core.

**Claim 2** \( G \not\to W \).

Assume on the contrary that \( f: G \to W \) is a homomorphism. If we write

\[
G^k = \overrightarrow{P}_{s_1} \overrightarrow{P}_{s_2} \ldots, \quad W^k = \overrightarrow{P}_{t_1} \overrightarrow{P}_{t_2} \ldots,
\]

then

\[
G = \overrightarrow{P}_{s_1} I_1 \overrightarrow{P}_{s_2} I_2 \ldots, \quad W = \overrightarrow{P}_{t_1} J_1 \overrightarrow{P}_{t_2} J_2 \ldots,
\]

where the intervals \( I_i, J_i \ (i = 1, 2, \ldots) \) satisfy (iv-vi) and contain only one-directional paths of lengths smaller than all \( s_i \) \( (t_i) \). The homomorphism \( f \) must therefore map each \( P_{s_i} \) to some \( P_{t_j} \).

Moreover, the images under \( f \) of the endpoints of any \( I_i \) are the endpoints of some \( J_m \). This assures that \( f \) can be used to define (in the natural way) a homomorphism \( f^k: G^k \to W^k \), contrary to our construction.

**Claim 3** \( G \times H \to W \).

The product \( G \times H \) has three parts:

\[
Q_1 = G \times U;
\]
\[ Q_2 = G \times \overrightarrow{P}_{b_k+2} \text{; and} \]
\[ Q_3 = G \times V. \]

Let
\[ H = U_{z_1} \overrightarrow{P}_{b_k+2} z_2 V, \quad W = U_{y_1} W_1 y_2 V, \]
\[ X_1 = \{(x, z_1) : x \in G\}, \text{ and} \]
\[ X_2 = \{(x, z_2) : x \in G\}. \]

The projection \( \pi \) defined by \( \pi(x, y) = y \) is a homomorphism of \( G \times H \) onto \( H \) which maps \( Q_1 \) to \( U \), \( Q_2 \) to \( \overrightarrow{P}_{b_k+2} \), and \( Q_3 \) to \( V \), with the set \( X_i \) mapped to \( z_i \) \((i = 1, 2)\). Like Lemma 3.3.1, we have the following lemma (Lemma 3.3.5) to classify the possible components of \( Q_2 \).

Then as in Case 1, we can modify \( \pi \) to a homomorphism \( G \times H \rightarrow W \) by showing that each component \( C \) of \( Q_2 \) may be homomorphically mapped to \( W_1 \) so that the image of \( X_1 \cap C \) and \( X_2 \cap C \) are \( y_1 \) and \( y_2 \) respectively.

See Figure 3.17 and Figure 3.18 for illustrations.

**Lemma 3.3.5** Let \( P_2[y_0, y_1, \ldots, y_{b_k+2}] \) be a directed path of length \( b_k + 2 \), and \( P_1[x_0, \ldots, x_t] \) be the graph \( G \) constructed in Case 2 of Theorem 3.3.4. Then any component \( C \) of \( P_1 \times P_2 \) is one of the following five cases:

1. \( C \) is an isolated vertex located in \( X_1 \) or \( X_2 \);
2. \( C \) is a directed path \( \overrightarrow{P}_a \) \((1 \leq a \leq b_k)\) starting at \((x_0, y_i)\) \((i > 1)\) and ending at \((x, y_{b_k+2})\) \((x \in P_1)\), or starting at \((x, y_0)\) \((x \in P_1)\) and ending at \((x_t, y_j)\) \((j \leq b_k)\);
3. \( C \) is \( \overrightarrow{P}_{b_k+1} I_1 \overrightarrow{P}_{b_k} I_2 \overrightarrow{P}_{b_k} \ldots \) starting at \((x_0, y_1)\) and ending at
(4) $C$ is $\overrightarrow{P_{b_k}} I_1 \overrightarrow{P_{b_k}} I_2 \ldots \overrightarrow{P_{b_k+1}}$ starting at $(x, y_0)$ $(x \in P_1)$ and ending at $(x, y_{b_k+1})$; or

(5) $C$ is $\overrightarrow{P_{s_1}} I_1 \overrightarrow{P_{s_2}} I_2 \ldots \overrightarrow{P_{s_m}}$ where $\overrightarrow{P_{s_1}} \overrightarrow{P_{s_2}} \ldots \overrightarrow{P_{s_m}}$ is a basic or degenerate basic $b_k$-path with net length $b_k+2$ and $s_i = b_k$ or $b_k+1$ for $i = 1, 2, \ldots, m$. $I_1, I_2, \ldots$ are oriented path satisfying (iv)-(vi), starting at $(x, y_0)$ and ending at $(x', y_k)$ $(x, x' \in P_1)$.

**Proof**  
Apply Lemma 3.2.1 and consider the structure of $P_1$ and $P_2$.  
\[\Box\]
Figure 3.17  From $W$ to $W^k$, $G^k$, $H^k$ and $G^k \times H^k$
3.4 Digraphs

In this section we apply our methods to show that a large class of digraphs are not multiplicatve.

Let $G$ be a digraph. The directed walk $v_0v_1\ldots v_n$ in $G$ has a sequence of vertices $v_0, v_1, \ldots, v_n$ and arcs $\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{n-1}v_n}$. The vertices and arcs do not have to be distinct in the directed walk. Similarly we have the definition of oriented walk which is given before Lemma 3.2.4. Similar to the oriented path we can define the interval, the one-directional (forward, or backward with respect to the specified order of traversal) interval, the maximal one-directional (forward, or backward) interval etc., in the oriented walk, with vertices and arcs within one interval or among several intervals not necessarily being distinct. We can also define in $G$ a basic (forward, backward) $b$-walk with parameters $(k_1, 2m_1-1; \ldots; k_{n-1}, 2m_{n-1}-1; k_n)$, a $(b+1, 2m-1)^n$ basic $b$-walk etc., in analogy with definitions in 3.3, except that vertices and arcs can now be repeated. Net length can similarly be defined for walks. Thus in Figure 3.19, $x_0x_1x_2x_0x_1x_2x_3x_4x_9$ is a directed walk of net length 8, $x_0x_1x_2x_3x_4x_5x_2x_6x_7x_8x_4x_9$ is a $(4,1; 4)$ basic 3-walk of net length 5. There are also other basic $b$-walks as well.

![Figure 3.19 Directed walk and basic b-walk](image-url)
Any \((k_1, 2m_1-1; k_2)\) basic b-walk can also be written as a \((k_1, 2m-1; k_2)\) basic b-walk for any \(m \geq m_1\). Any directed path of length \(k_1 + k_2 - b\) can also be written as a \((k_1, 2m_1-1; k_2)\) basic b-walk for \(m = 1, 2, 3, \ldots\). For the ease of consideration, when we write the parameters \((k_1, 2m_1-1; k_2, 2m_2-1; \ldots; k_{n-1}, 2m_{n-1}; k_n)\) for a basic b-walk, we usually take each \(m_i (i = 1, 2, \ldots, n-1)\) as small as possible. By this convention a directed walk is not usually written as some basic b-walk.

In a directed cycle, there exist directed walks with any net length. We also note that a directed cycle contains an infinite directed walk. Similar remarks apply to oriented cycles. Consider the oriented cycles in Figure 3.20. In (a) there exist a \((3, 1; 3, 3; 3)\) basic 2-walk with net length 5, a \((3, 1; 3, 3; 3, 1; 3, 3; 3)\) basic 2-walk with net length 7, etc., and a basic 2-walk with infinite net length, called an infinite basic 2-walk. In (b) and (c), there exist \((3, 1)^n\) basic 2-walks for all \(n = 2, 3, \ldots\), and a basic 2-walk with infinite net length, which will be described as a \((3, 1)^\infty\) basic 2-walk. There also exists a \((2, 1)^\infty\) basic 1-walk in (c).

![Figure 3.20 Infinite basic b-walks](image)
A basic b-circuit in a graph G is a closed walk:
\[ a \vec{P}_k \vec{P}_b \vec{P}_b \ldots \vec{P}_b a, \text{ or } \]
\[ a \vec{P}_k \vec{P}_b \vec{P}_b \ldots \vec{P}_b W \vec{P}_b \vec{P}_b \ldots \vec{P}_b a \]
where W is a basic b-walk, and \( k > b \). (Note that the entire b-circuit is a walk, i.e., any of its vertices and arcs may be repeated.) A simple basic b-circuit is a basic b-circuit in which all vertices and arcs are distinct, except that the first vertex is the same as the last vertex.

The following lemma is true since we only consider finite graphs.

**Lemma 3.4.1** The following statements are equivalent in any digraph G:

1. G has a basic b-circuit;
2. G has a basic b-walk with infinite net length; and
3. the net lengths of all basic b-walks in G are unbounded.

**Proof** We only need to consider the case when G is acyclic.

1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are clear. If C is a basic b-circuit, then \( \text{nl}(C) > 0 \), and so the basic b-walk CCC ... has infinite net length. If P is a basic b-walk of infinite net length, then we can take a finite subwalk of P with arbitrarily large net length.

3) \( \Rightarrow \) (1) Let a be the maximum length of a directed walk in G. The value of a is finite since G is acyclic and finite. Since (3) holds, there exists a basic b-walk P in G with \( \text{nl}(P) \geq (|V(G)|+1) \cdot a \). Let \( v_0, v_1, \ldots, v_q \) be the first vertices of all maximal forward intervals of length greater than b, in the order in which they appear in P. Note
that the net length of each subwalk \([v_i, v_{i+1}]\) is less than \(a\). Therefore \(q > |V(G)|\) and hence some \(v_i = v_j\). Then the subwalk \([v_i, v_j]\) in \(P\) is a basic \(b\)-circuit.

In analogy with Lemma 3.3.3, we have the following lemma.

**Lemma 3.4.2** Let \(G\) be a \((b+1, 2m-1)^n\) basic (or degenerate basic) \(b\)-path and \(P\) any directed walk or any \((k_1, 2m_1-1; \ldots; k_t-1, 2m_{t-1}-1; k_t)\) basic \(b\)-walk with \(m \geq m_i (i = 1, 2, \ldots, t-1)\) in a given graph \(W\). Let \(nl(G) \leq nl(P)\). Then

1. \(G \rightarrow P\),
2. we can map the vertex of \(G\) with smallest (or greatest) level to the vertex of \(P\) with smallest (or greatest) level under the homomorphic mapping of \(G\) to \(P\),
3. if \(nl(G) = nl(P)\), then the two endvertices of \(G\) will be mapped to the two endvertices of \(P\) under the homomorphic mapping of \(G\) to \(P\).

Since the homomorphic image of a directed path must be a directed walk with the same net length, we have the following lemma.

**Lemma 3.4.3** Let \(P = [x_0, x_1, \ldots, x_n]\) be a basic \(b\)-path and \(f\) a homomorphism of \(P\) to some digraph \(G\). Then

\[
f(P) = [f(x_0), f(x_1), \ldots, f(x_n)]
\]

is a directed walk or a basic \(b\)-walk with net length \(nl(P)\) in \(G\).
Theorem 3.4.4 Let $W$ be a digraph and $b_1$ a positive integer. If the maximum net length of all basic $b_1$-walks in $W$ is finite and greater than the length of any directed walk in $W$, then $W \in \overline{\mathcal{M}}$.

Proof Take a basic $b_1$-walk $P$ of maximum net length $b$ with parameters $(k_1, 2m_1-1; \ldots; k, 2m_n-1; k_n)$

Let

$m = \max \{ m_i : i = 1, 2, \ldots, n-1 \}$.

Let

$H = \overline{P}_b$, and

$G$ be a $(b_1+1, 2m-1)^{b+1-b_1}$ basic $b_1$-path.

Clearly $H \not\rightarrow W$, since $b$ is greater than the length of any directed walk in $W$. Furthermore $G \not\rightarrow W$, otherwise the homomorphic image of $G$ is a basic $b_1$-walk with net length $b+1$ contradicting the maximality of $P$. By Lemma 3.3.1, each component of $G \times H$ is an oriented path in one of the five cases of Lemma 3.3.1 and can be homomorphically mapped to $P$ by Lemma 3.4.2. Thus $G \times H \rightarrow W$. $\square$

In Theorem 3.4.4, the condition that the maximum net length $b$ is greater than the length of any directed walk in $W$ is equivalent to the conditions that $W$ has no directed cycle and $b$ is greater than the length of any directed path in $W$.

In the following theorem, we allow neither infinite directed walks (i.e., directed cycles), nor infinite basic $b_1$-walks (i.e., basic $b_1$-circuits). But we allow directed paths of very large length.
Theorem 3.4.5  Let \( b_1 \) be a positive integer, and \( W \) an acyclic core digraph without basic \( b_1 \)-circuits. If \( W \) contains a basic \( b_1 \)-path \( P \) which is adjacent to the other vertices of \( W \) by its two endvertices only, then \( W \in \mathcal{M} \).

Proof  Suppose that \( P \) is a basic \( b_1 \)-path with net length \( b_2 \) and parameters \((k_1, 2m_1-1; \ldots; k_{n-1}, 2m_{n-1}-1; k_n)\). Take \( m = \max\{m_i : i = 1, 2, \ldots, n-1\} \).

By Lemma 3.4.1, the net lengths of all basic \( b_1 \)-walks and the lengths of all directed paths in \( W \) are bounded. Let \( b \) be an integer greater than both the net length of any basic \( b_1 \)-walk of \( W \) and the length of any directed path of \( W \).

Construct \( H \) from \( W \) by replacing \( P \) with a directed path \( P_{b_2} \), and construct \( G \) to be a \((b_1+1, 2m-1)^{b-b_1}\) basic \( b_1 \)-path.

We now have \( G \rightarrow W \), otherwise the homomorphic image of \( G \) is a directed path or a basic \( b_1 \)-walk of net length \( b \) by Lemma 3.4.3, a contradiction. We also have \( H \rightarrow W \). Suppose on the contrary that \( H \rightarrow W \). Then the composition of \( W \rightarrow H \) and \( H \rightarrow W \) will give a homomorphic mapping from \( W \) to a proper subgraph since \( H \) has fewer vertices than \( W \). This contradicts the fact that \( W \) is a core.

Claim  \( G \times H \rightarrow W \).

Let the two endvertices of \( P_{b_2} \) in \( H \) be \( y_1 \) and \( y_2 \). And use the same notation \( y_1, y_2 \) to denote the two endvertices of \( P \) in \( W \). Use \( H \setminus P_{b_2} \) to denote the subgraph of \( H \) by deleting \( P_{b_2} \) but keeping \( y_1 \) and \( y_2 \). Any component \( Q \) of \( G \times H \) has two parts:

\[ Q_1 = Q \cap (P_{b_2} \times G) \]
\[ Q_2 = Q \cap (\overrightarrow{(H \setminus \mathcal{P}_{b_2}) \times G}) \]

If \( Q_1 = \emptyset \), then \( Q_2 \mapsto H \setminus \mathcal{P}_{b_2} \subseteq W \) by the projection map. If \( Q_2 = \emptyset \), then by Lemmas 3.3.1 and 3.3.3, each component of \( Q_1 \to P \subseteq W \).

Now suppose that both \( Q_1 \) and \( Q_2 \) are not empty. Let
\[
X_1 = \{ x \in G : (x, y_1) \in Q \} \quad \text{and} \quad X_2 = \{ x \in G : (x, y_2) \in Q \}.
\]

Then \( (X_1, y_1) \cup (X_2, y_2) = Q_1 \cap Q_2 \). The projection \( \pi \) defined by \( \pi(x, y) = y \) is a homomorphism of \( G \times H \) onto \( H \) which maps \( Q_1 \) to \( \overrightarrow{P}_{b_2} \) and \( Q_2 \) to \( H \setminus \mathcal{P}_{b_2} \), with the set \( (X_1, y_1) \) mapped to \( y_1 \) and \( (X_2, y_2) \) mapped to \( y_2 \). If \( Q_1 \) has several components, then by Lemmas 3.3.1 and 3.3.3, each component can be homomorphically mapped to \( P \), with the vertices in \( (X_1, y_1) \) mapped to \( y_1 \) and vertices in \( (X_2, y_2) \) mapped to \( y_2 \). Thus we can modify \( \pi \) to a homomorphism: \( Q \to W \).

Therefore \( G \times H \to W \).

**Corollary 3.4.6** Let \( W \) be a core oriented cycle and \( b_1 \) a positive integer. If \( W \) is not a simple basic \( b_1 \)-circuit and \( W \) contains a basic \( b_1 \)-path, then \( W \in \overline{\mathcal{M}} \).

**Proof** Apply Theorem 3.4.5.

We say that almost all oriented cycles have property \( P \), if
\[
\lim_{n \to \infty} \frac{p_n}{v_n} = 1
\]
where \( p_n \) is the number of orientations of \( C_n \) which have property \( P \) and \( v_n \) is the total number of orientations of \( C_n \).

**Lemma 3.4.7** Almost all oriented cycles contain \( \overrightarrow{P}_2 \overrightarrow{P}_1 \overrightarrow{P}_2 \).

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Proof Consider all orientations of some $C_n$, and consider $q = \left\lfloor \frac{n}{6} \right\rfloor$ edge-disjoint subintervals $I_1, I_2, \ldots, I_q$ of $C_n$, each being an undirected path of length five. The proportion of orientations of $C_n$ which have $I_j$ oriented to be $\overrightarrow{P_2} \overrightarrow{P_1} \overrightarrow{P_2}$ is

$$\frac{2^{n-5}}{2^n} = \frac{1}{32}.$$ 

Therefore the proportion of all orientations of $C_n$ which have none of the $I_j$ oriented to be $\overrightarrow{P_2} \overrightarrow{P_1} \overrightarrow{P_2}$ is $(\frac{31}{32})^q \to 0$ as $n \to \infty$. Hence the proportion of all orientations of $C_n$ which contain $\overrightarrow{P_2} \overrightarrow{P_1} \overrightarrow{P_2}$ somewhere tends to 1, as $n \to \infty$. \hfill $\square$

**Theorem 3.4.8** Almost all oriented cycles are non-multiplicative.

**Proof** Apply Corollary 3.4.6 and Lemma 3.4.7, and observe that the proportion of orientations of $C_n$ which are simple basic 1-circuits goes to 0. (By an argument analogous to the one above, almost all orientations contain $\overrightarrow{P_2} \overrightarrow{P_1} \overrightarrow{P_2}$ and hence cannot be simple basic 1-circuits.) \hfill $\square$

### 3.5 Some Other Oriented Paths Which Are Not Weakly Multiplicative

**Theorem 3.5.1** Let $W \in \mathcal{F}_3$. Assume that $W = UW_1X$ where $W_1$ is a $(k_1, 2m_1-1; \ldots; k_{n-1}; 2m_{n-1}-1; k_n)$ basic forward (backward) $b_1$-path with net length $b$, and both $U$ and $X$ may be empty. Let $G_1$ be the $(b_1+1, 2m-1)^{b_1+1-b_1}$ basic backward (forward) $b_1$-path where
m = \max\{m_i : i = 1, 2, ..., n-1\}. If either \( W \notin WX(-)P_b \) (respectively \( WX(-)P_b \)) and \( W \notin WX(-)G_1 \), or \( W \notin P_b U(-)W \) (respectively \( P_b U(-)W \)) and \( W \notin G_1 U(-)W \), then \( W \in \overline{W \cap m} \).

**Proof** Without loss of generality we only need to prove the case when \( W_1 \) is forward and \( W \notin WX(-)P_b \), \( W \notin WX(-)G_1 \). And for convenience we set \( H = WX(-)P_b \), \( G = WX(-)G_1 \), and denote the last vertex of \( WX(-) \) by \( a \).

If we can prove \( W \prec G \times H \), then we shall have \( W \in \overline{W \cap m} \).

Let \( W^* \) be the subgraph induced by the vertices \( \{(x, x) : x \in W\} \); \( X(-)^* \) the subgraph induced by the vertices \( \{(x, x) : x \in X(-)\} \); \( G_1^* \) the component of \( G_1 \times P_b \) containing the vertex \((a, a)\). Then \( G_1^* \) is a degenerate \((b_1+1, 2m-1)^{b_1+1-b_1}\) basic backward \( b_1 \)-path and \( \mathit{nl}(G_1^*) = b \) by Lemma 3.3.1. Thus \( W^* \) is a retract of \( W^*X(-)^*G_1^* \) with \( X(-)^* \) mapped to \( X^* \) and \( G_1^* \) mapped to \( W_1^* \) by Lemma 3.3.3. See Figure 3.21.

Let \( Q^* \) be the component of \( G \times H \) containing \( W^*X(-)^*G_1^* \). Then \( Q^* \cap (G_1 \times P_1) = G_1^* \). For any \((x, y) \in Q^*\), the mapping

\[
\varphi(x, y) = \begin{cases} 
(x, y) & \text{if } (x, y) \in G_1^*, \\
(x, x) & \text{if } (x, y) \in WX(-) \times H, \ x \leq y, \ \text{and} \\
(y, y) & \text{if } (x, y) \in G \times WX(-), \ x > y
\end{cases}
\]

is obviously a retraction of \( Q^* \) onto \( W^*X(-)^*G_1^* \). Since \( W^* \) is a retract of \( W^*X(-)^*G_1^* \), then \( W^* \) is a retract of \( Q^* \). Therefore we have \( W \prec Q^* \).

Next we shall show that any other component of the product \( G \times H \) can be homomorphically mapped to \( W^* \).
Let $x_0$ be the first vertex of $W$, $x_s$ be the second to last vertex of $G_1$, $x_{s+1}$ be the last vertex of $G_1$, and $y_t$ be the last vertex of $\overrightarrow{P_b}$. Then $W \times (-) \times G_1^*$ has starting vertex $(x_0, y_0)$ and ending vertex $(x_s, y_t)$ (since $G_1^*$ has starting vertex $(a, a)$ and ending vertex $(x_s, y_t)$ by Lemma 3.2.1).

Let $A_1 = \{(x, y) \in WX(-) \times H : y > x\}$ and $B_1 = \{(x, y) \in G \times WX(-) : y < x\}$ where $<$ is defined by traversal order on $WX(-)$, $G$ and $H$, which is defined from $W$ to $X(-)$. Let

$$A_2 = \{(x, y) \in G_1 \times \overrightarrow{P_b} : (x, y) \text{ is joined to } (x, y_t) \text{ by an oriented path}, x \in V(G_1) \setminus \{x_s, x_{s+1}\}\} \text{ and}$$

$$B_2 = \{(x, y) \in G_1 \times \overrightarrow{P_b} : (x, y) \text{ is joined to } (x, a) \text{ by an oriented path}, x \in V(G_1) \setminus \{a\}\}.$$

Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. See Figure 3.21.

Consider that $G \times H$ is a planar graph, and that $W \times (-) \times G_1^*$ has the starting vertex $(x_0, y_0)$ and ending vertex $(x_s, y_t)$. Therefore any component in $A \setminus Q^*$ cannot be joined to any component in $B \setminus Q^*$ by an oriented path.

Let $Q_1$ be any component of $G \times H$ in $A \setminus Q^*$. Define the mapping

$$\varphi_1(x, y) = (x, x)$$

for $(x, y) \in Q_1 \cap A_1$. Then $\varphi_1$ is a homomorphism, and $\varphi(a, y) = (a, a)$. Furthermore, we have $Q_1 \cap A_2 \rightarrow G_1^*$ with $(a, y)$ mapped to $(a, a)$, by Lemmas 3.3.1 and 3.3.3. Thus $Q_1 \rightarrow W \times (-) \times G_1^* \rightarrow W^*$.

Let $Q_2$ be any component of $G \times H$ in $B \setminus Q^*$. Define the mapping
\( \varphi_2(x, y) = (y, y) \)

for \((x, y) \in Q_2 \cap B_1\). Then \( \varphi_2 \) is a homomorphism and \( \varphi(x, a) = (a, a) \).

Moreover, we have \( Q_2 \cap B_2 \rightarrow G_1^* \) with \((a, y)\) mapped to \((a, a)\) by Lemmas 3.3.1 and 3.3.3. Thus \( Q_2 \rightarrow W^*X(-)^*G_1^* \rightarrow W^* \).

Therefore \( W \triangleleft G \times H \).

\[ \square \]

**Figure 3.21**

**Corollary 3.5.2** Let \( W = UW_1X \in \mathcal{F}_3 \), where \( W_1 \) is a \((k_1, 2m_1-1; \ldots; k_n-1, 2m_n-1; k_n) \) basic forward \( b_1 \)-path which is maximal for some fixed \( b_1 \), and \( nl(W_1) = b \). Let \( G_1 \) be a \((b_1+1, 2m-1)^{b+1-b_1} \) basic forward \( b_1 \)-path. If either \( \overrightarrow{P_b}X \rightarrow X \) and \( G_1X \rightarrow X \), or \( U\overrightarrow{P_b} \rightarrow U \) and \( UG_1 \rightarrow U \), then \( W \in \overline{\mathcal{F}_m} \).

**Proof** \( \overrightarrow{P_b}X \rightarrow X \) implies \( W \triangleleft UW_1XX^{(-)}\overrightarrow{P_b} \) and \( G_1X \rightarrow X \) implies \( W \triangleleft UW_1XX^{(-)}G_1^{(-)} \). Here we use the fact that \( W_1 \) is maximal, that the homomorphistic image of a directed path in an oriented path
must be directed, and that the homomorphic image of a basic $b_1$-path in an oriented path must be either directed or a basic $b_1$-path.

We make a similar argument for $U\vec{P}_b \mapsto U$ and $U\vec{G}_1 \mapsto U_1$. □

**Corollary 3.5.3** Let $W \in \mathcal{P}_3$ and let $b_1$ be a positive integer. If $b$ is the maximum net length of all basic $b_1$-paths in $W$, and if $W = UW_1X$ where $W_1$ is a basic forward $b_1$-path of net length $b$, then $W \in \overline{W \mathcal{M}}$ provided that $\vec{P}_b X \mapsto X$ or $U\vec{P}_b \mapsto U$.

**Proof** We can construct $G_1$ as in Corollary 3.5.2. We have $G_1X \mapsto X$ and $U\vec{G}_1 \mapsto U$ by the maximality of $b$. □

**Corollary 3.5.4** Let $W \in \mathcal{P}_3$ and let $b_1$ be a positive integer. Let $b$ be the maximum net length of all basic $b_1$-paths in $W$. If $b$ is greater than the length of any directed path in $W$, then $W \in \overline{W \mathcal{M}}$. □

**Corollary 3.5.5** Let $W = UW_1X \in \mathcal{P}_3$, where $W_1$ is a basic $b_1$-path. If either $X = \emptyset$ or $U = \emptyset$, then $W \in \overline{W \mathcal{M}}$. □

A path $P$ is called a **forward minimal oriented path** if $P$ is a minimal oriented path, the level of the first vertex is 0 and the levels of the other vertices are non-negative. Let $P$ be a forward minimal oriented path with net length $b_1$. If each $\vec{P}_{b_1}$ is replaced by $P$ and each $\vec{P}_{b_1}$ is replaced by $P(-)$ in a $(k_1, 2m_1-1; ...; k_{n-1}, 2m_{n-1}-1; k_n)$ basic (forward, backward) $b_1$-path, then we obtain a $(k_1, 2m_1-1; ...; k_{n-1}, 2m_{n-1}-1; k_n)$ basic (forward, backward) $P$-path. We can replace any basic $b_1$-path by the corresponding basic $P$-path in Theorems 3.3.2-3.3.3, Theorem 3.5.1 and Corollaries 3.5.2-3.5.5, and
also in Lemma 3.3.1 with corresponding changes. Those theorems
and corollaries still hold.

Although the criteria given above which imply that an oriented
path of $P_3$ belongs to $\mathcal{W}$ may seem somewhat unnatural, they are
quite strong and useful in practice. By applying these results, many
small oriented paths of $P_3$ are shown to be non-weakly-
multiplicative. The smallest graph for which the membership in
$\mathcal{W}$ is unknown is given in Figure 3.22 (b).

The graph in (a) is not weakly multiplicative and is smaller than
the graph in (b). But the cited results do not imply this. We construct
$G$ and $H$ as in Figure 3.23. Then this graph is not a retract of $G$ or $H$,
but is a retract of $G \times H$ by an ad hoc mapping. The same method
doesn't work for (b).

We close this chapter by raising the following natural conjecture:

The only weakly multiplicative oriented paths are directed
paths, or in other words

$$P_3 \subseteq \mathcal{W}$$
Figure 3.22

(a)  

(b)  

Figure 3.23

H  

G
Chapter 4  Oriented Cycles

4.1 Introduction

As outlined in Chapter 1, this chapter is devoted to the study of oriented cycles with respect to multiplicativity. We shall give a purely combinatorial proof of the multiplicativity of directed cycles of prime power order. We shall also give a new construction that produces a class of oriented cycles which are non-multiplicative, other than those given in Chapter 3. However, the complete classification of all oriented cycles with respect to multiplicativity, weak multiplicativity and very weak multiplicativity remains an open problem.

In addition to the definitions and notation given in 1.1, 2.1 and 3.1 we shall need the following definitions and notation.

In any oriented cycle the vertices with indegree zero and those with outdegree zero appear in pairs. We denote by \( \mathcal{S}(p, k) \) the set of oriented cycles of net length \( p \) with exactly \( k \) vertices of outdegree zero (and therefore exactly \( k \) vertices of indegree zero). The set of all oriented cycles of net length \( p \) will be denoted by \( \mathcal{S}(p) \).

Let \( W \) be a fixed digraph, and \( G \) an arbitrary digraph with vertices \( v_1, v_2, ..., v_n \) (in some fixed order). Let \( W(G) \) be the map graph defined in 1.1. The display of \( f \in V(W(G)) \) has boxes 1, 2, ..., \( n \) (with box \( i \) corresponding to vertex \( v_i \) of \( G \)) and the \( i \)-th box contains \( f_i = f(v_i) \in V(W) \), provided \( f \) maps vertex \( v_i \) of \( G \) to vertex \( f_i \) of \( W \).
(i = 1, 2, ..., n). We put the n boxes in a column, in the order 1, 2, ..., n from top to bottom. Suppose \( f^1 f^2 \) is an arc of \( W(G) \). We display the arc by putting the displays of \( f^1 \) and \( f^2 \) side by side, and putting an arrow from the i-th box of \( f^1 \) to the j-th box of \( f^2 \) whenever \( \overrightarrow{v_i v_j} \in E(G) \) for \( i, j = 1, 2, ..., n \). It follows from the definition of \( W(G) \) that in the display of \( f^1 f^2 \in E(W(G)) \) the contents \( f^1_i \) of the i-th box of \( f^1 \), and \( f^2_j \) of the j-th box of \( f^2 \) must satisfy \( f^1_i f^2_j \in E(W) \) whenever there is an arrow from \( v_i \) to \( v_j \). If each vertex and each arc of a subgraph \( S \) of \( W(G) \) is replaced by the display of the corresponding vertex and arc, then we have the display of \( S \) (c.f., Figures 4.1, 4.2 and 4.3). The main idea of our proof of the multiplicativity of directed cycles of prime power order is implicit in Figures 4.2 and 4.3.

We have the following lemma:

**Lemma 4.1.1** [11] If \( \overrightarrow{C_n} \) is multiplicative then \( n \) is a prime power.

**Proof** If \( n = km \) with \( k \) and \( m \) relatively prime, then \( \overrightarrow{C_n} \) is isomorphic to \( \overrightarrow{C_k \times C_m} \). On the other hand \( C_k \rightarrow \overrightarrow{C_n}, C_m \rightarrow \overrightarrow{C_n} \).

Thus if \( n \) is not a prime power, \( \overrightarrow{C_n} \) cannot be multiplicative.

In Section 4.2 we will prove the converse of the above lemma.
Figure 4.1 Display of an arc of $W(G)$

Figure 4.2 Display of a directed cycle of $\overrightarrow{C}_4(G)$
4.2 A Combinatorial Proof of Multiplicativity

The main theorem of this chapter is that $\mathcal{C}_n$ is multiplicative if and only if $n$ is a prime power. The difficult part is to show that $\mathcal{C}_n$ is multiplicative if $n$ is a prime power (cf. Lemma 4.1.1). A proof of this fact based on the Lefschetz duality theorem of homology theory has been given in [11]. Here we give the first purely combinational proof.
Lemma 4.2.1  Let $p$, $q$ and $n$ be positive integers. Then

\[ C_q \subseteq C_n(C_p) \]

if and only if the following system of equations in $\mathbb{Z}_n$ has a solution $f_0, f_1, ..., f_{p-1}$:

\[
\begin{align*}
f_q &= f_0 + q, \\
f_{1+q} &= f_1 + q, \\
& \quad \vdots \\
f_{p-1+q} &= f_{p-1} + q,
\end{align*}
\]

where the subscripts are reduced modulo $p$.

Proof  It is easy to see that each vertex of $C_n(C_p)$ has exactly one entering arc and exactly one leaving arc; hence each component of $C_n(C_p)$ is a directed cycle. Consider one component $C$ of $C_n(C_p)$ with vertices $f^0, f^1, ..., f^j, ..., f^p$, and arcs $f^i f^{i+1}$ ($j = 0, 1, ...$).

Let $V(C_p) = \{v^0, v^1, ..., v^p\}$, $V(C_n) = \{0, 1, ..., n-1\}$, and assume that $C_p$ has arcs $\overrightarrow{v^i v^{i+1}}$ ($i = 0, 1, ..., p-1$ and subscripts calculated modulo $p$), and $C_n$ has arcs $\overrightarrow{i, i+1}$ ($i = 0, 1, ..., n-1$ and addition is modulo $n$).

Suppose $f^0 \in C_n(C_p)$ has $f^0(v^i) = f_i$ ($i = 0, 1, ..., p-1$). Then $f^i(v^i+j) = f_i + j$ for all $i = 0, 1, ..., p-1, j = 0, 1, ...$; here the subscripts of $v$ are reduced modulo $p$ and the addition on $f_i$ is performed modulo $n$.

Therefore $C$ is a directed cycle $C_q$ if and only if $f^q$ is the same as $f^0$.

This happens just if the numbers $f_i$ satisfy (*) (for $i = 0, 1, ..., p-1$).

Conversely, if (*) has a solution $f_0, ..., f_{p-1}$, then letting $f^i(v^i+j) = f_i + j$ ($i = 0, 1, ..., p-1$ and $j = 0, 1, ..., q$) we obtain a directed cycle $C_q$ as the subgraph of $C_n(C_p)$. This completes the proof.
Instead of writing $f^0(v_i) = f_i$, we shall often write that the display has $f_i$ in the $i$-th box of the column $f^0$. See Figure 4.4 for the display of one component of $\overrightarrow{C}_n(\overrightarrow{C}_p)$.

Figure 4.4 Display of one component of $\overrightarrow{C}_n(\overrightarrow{C}_p)$

Lemma 4.2.2 Let $n = m^\alpha$, where $m$ is a prime. Let $p, q \not\equiv 0 \pmod{n}$, and let $p = ad, q = bd$ where $d = \gcd(p, q)$. Then

(a) \{q, q + d, ..., q + (a-1)d\} \pmod{p} = \{0, d, ..., (a-1)d\},

(b) $\text{lcm}(p, q) \not\equiv 0 \pmod{n}$, and

(c) the system of equations (*) has no solution $f_0, f_1, ..., f_{p-1}$ in $\mathbb{Z}_n$.

Proof (a) Suppose $q = bd \equiv jd \pmod{p}$ where $j \in \{0, 1, ..., a-1\}$. Then

\[ d + q = (b+1)d \equiv (j+1)d \pmod{p}, \]
\[ 2d + q = (b+2)d \equiv (j+2)d \pmod{p}, \]

...\[
(a-1)d + q = (a-1+b)d \equiv (a-1+j)d \pmod{p}.\]
Obviously \((jd, (j+1)d, \ldots, (a-1+j)d)(\mod p) = \{0, d, \ldots, (a-1)d\}\).

(b) Let \(p = cm^\beta, q = em^\gamma\). Then \(0 \leq \beta < \alpha,\ 0 \leq \gamma < \alpha,\ c \geq 1,\ e \geq 1,\) and \(m\) does not divide \(c\) or \(e\). As \(m\) is prime, it does not divide \(\text{lcm}(c, e)\). Letting \(\delta = \min\{\beta, \gamma\}\), we have \(\text{lcm}(p, q) = m^\delta \text{lcm}(c, e) \neq 0\ (\mod n)\).

(c) Otherwise, a solution to the system (*) would imply
\[
\begin{align*}
f_q &\equiv f_0 + q, \\
f_{d+q} &\equiv f_d + q, \\
&\vdots \\
f_{(a-1)d+q} &\equiv f_{(a-1)d} + q,
\end{align*}
\]
where the congruences are in \(\mathbb{Z}_n\) and the subscripts of \(f\) are reduced modulo \(p\). Adding these congruences we obtain
\[
f_q + f_{d+q} + \cdots + f_{(a-1)d+q} \equiv f_0 + f_d + \cdots + f_{(a-1)d} + aq \ (\mod n).
\]
By (a),
\[
0 \equiv aq \equiv \text{lcm}(p, q) \ (\mod n),
\]
which contradicts (b). Therefore (*) has no solution. \(\Box\)

**Lemma 4.2.3** If \(u\) is a prime power and \(p, q \neq 0\ (\mod n)\), then \(\overrightarrow{C}_q \not\in \overrightarrow{C}_n(\overrightarrow{C}_p)\).

**Proof** The proof follows from Lemma 4.2.1 and Lemma 4.2.2 (c).

**Lemma 4.2.4** The following statements are equivalent:
(a) \(\overrightarrow{C}_n\) is multiplicative.
(b) For any graph \(G, G \leftrightarrow \overrightarrow{C}_n\) implies \(\overrightarrow{C}_n(G) \rightarrow \overrightarrow{C}_n\).

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(c) For any graph $G$, if $C' \subseteq G$ for some $C' \in \mathcal{C}(p)$, $p \neq 0 \pmod{n}$, then $C'' \not\subseteq \overrightarrow{C_n}(G)$ for any oriented cycle $C'' \in \mathcal{C}(q)$ with $q \neq 0 \pmod{n}$.

(d) $C'' \not\subseteq \overrightarrow{C_n}(C')$ for any $C' \in \mathcal{C}(P)$ and any $C'' \in \mathcal{C}(q)$ where $p \neq 0 \pmod{n}$ and $q \neq 0 \pmod{n}$.

Proof The equivalence of (a) and (b) follows from Lemma 1.3.5 (f). The equivalence of (b) and (c) follows from Lemma 1.3.9 (d).

(d) $\Rightarrow$ (c) Let $C' \subseteq G$ for some $C' \in \mathcal{C}(p)$, $p \neq 0 \pmod{n}$. If $f \in \overrightarrow{C_n}(G)$, then $f|_{C'} \in \overrightarrow{C_n}(C')$. Moreover if $f^1$ and $f^2$ are adjacent in $\overrightarrow{C_n}(G)$, then $f^1|_{C'}$ and $f^2|_{C'}$ are adjacent in $\overrightarrow{C_n}(C')$. Now if the conclusion of (c) is not true, (i.e., if $C'' \not\subseteq \overrightarrow{C_n}(G)$ for some $C'' \in \mathcal{C}(q)$, $q \neq 0 \pmod{n}$), then $C'' \not\subseteq \overrightarrow{C_n}(C')$ for some $C' \in \mathcal{C}(p)$, $C'' \in \mathcal{C}(q)$, $p \neq 0 \pmod{n}$, $q \neq 0 \pmod{n}$, which is a contradiction to (d).

(b) $\Rightarrow$ (d) If $C'' \subseteq \overrightarrow{C_n}(C')$ for $C' \in \mathcal{C}(p)$, $C'' \in \mathcal{C}(q)$ with $p$, $q \neq 0 \pmod{n}$, then $G = C'$ violates (b), in view of Lemma 1.3.9 (d).

Lemma 4.2.5 Let $k$ be a nonnegative integer, $n$ and $q$ positive integers. If $\overrightarrow{C_q} \subseteq \overrightarrow{C_n}(G)$ for some $G \in \mathcal{C}(p, k+1)$, then $\overrightarrow{C_q} \subseteq \overrightarrow{C_n}(H)$ for some $H \in \mathcal{C}(p, k)$.

Proof Let $G \in \mathcal{C}(p, k+1)$. We may assume that there exists an interval of $G$, $[v_{i-a}, ..., v_i, ..., v_{i+a}]$, such that in the subinterval $[v_{i-a}, ..., v_i]$ the arcs are directed forward, in the subinterval $[v_i, ..., v_{i+a}]$ the arcs are directed backward, and $\overrightarrow{v_i+a v_i+a+1}$. Thus if we delete the interval $[v_{i-a}, v_{i+a}]$ and identify $v_{i-a}$ with $v_{i+a}$, then we shall have one fewer vertex of outdegree zero and one fewer vertex of indegree zero, and we obtain a graph $H \in \mathcal{C}(p, k)$.  

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Let \( \overrightarrow{C}_q \subseteq \overrightarrow{C}_n(G) \) and consider an \( f \in \overrightarrow{C}_n(G) \) which belongs to \( \overrightarrow{C}_q \). We shall construct a corresponding vertex \( f' \in \overrightarrow{C}_n(H) \). We first notice that, since \( f \in \overrightarrow{C}_q \), there are in \( \overrightarrow{C}_q(G) \) directed paths of arbitrarily large length which begin (respectively end) in \( f \). Because of the way \( \overrightarrow{C}_n(G) \) is defined, this guarantees that \( f(v_{i-j}) = f(v_{i+j}) \) for all \( i = 1, 2, \ldots, a \). Therefore we may define \( f'(v) = f(v) \) for \( v \in \{v_{i-a+1}, \ldots, v_{i}, \ldots, v_{i+a-1}\} \). By a similar argument, if \( f_1f_2 \) is an arc of \( \overrightarrow{C}_q \) in \( \overrightarrow{C}_n(G) \), we have \( f_1'f_2' \) an arc in \( \overrightarrow{C}_n(H) \). Therefore \( \overrightarrow{C}_n(H) \) also contains a copy of \( \overrightarrow{C}_q \). (Cf. Figure 4.2, \( G \in \mathcal{C}(3, 2), H \in \mathcal{C}(3, 1) \) and \( i = 3, a = 1 \) or \( i = 6, a = 1 \).)

Lemma 4.2.6 Let \( p, q \) and \( m \) be non-negative integers, and \( n \) a positive integer. If \( F \subseteq \overrightarrow{C}_n(H) \) for some \( F \in \mathcal{C}(q, m+1) \) and \( H \in \mathcal{C}(p) \), then there exists a \( G \in \mathcal{C}(q, m) \) such that \( G \subseteq \overrightarrow{C}_n(H) \).

Proof The idea of the proof is as follows.

Suppose that some \( F \in \mathcal{C}(q, m+1) \) and \( H \in \mathcal{C}(p) \) have \( F \subseteq \overrightarrow{C}_n(H) \). Note that \( H \) is not a directed cycle, as \( \overrightarrow{C}_n(\mathcal{C}_p) \) contains only directed cycles. Suppose that the minimum length of all maximal one directional intervals in \( F \) is \( d \) and suppose, without loss of generality, that it occurs in the backward direction. Then \( F \) contains an interval \([f^0, f^3d]\) such that \([f^0, f^d]\) is a forward interval, \([f^d, f^{2d}]\) a (maximal one-directional) backward interval, and \([f^{2d}, f^3d]\) another forward interval (cf. Figure 4.5).

\[ \begin{array}{ccccccccccc}
\vdots & \rightarrow & \rightarrow & \rightarrow & f_0 & f_1 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & f_d & f_{d+1} & \rightarrow & \rightarrow & \rightarrow & f_{2d} & \rightarrow & \rightarrow & \rightarrow & f_{3d} & \rightarrow \\
\end{array} \]

Figure 4.5 The interval \([f^0, f^3d]\)
The interval \([f^0, f^{3d}]\) has net length \(d\) and contains precisely one vertex, \(f^d\), of outdegree zero (and one vertex, \(f^{2d}\), of indegree zero).

We shall identify in \(\overrightarrow{C_n}(H)\) a forward directed path \(g^0, g^1, ..., g^d\) such that \(g^0 = f^0\) and \(g^d = f^{3d}\). Then we may define \(G \in \mathscr{C}(q, m)\) as the graph obtained from \(F\) by replacing the interval \([f^0, f^{3d}]\) by the forward directed path \(g^0, g^1, ..., g^d\).

Consider any two consecutive maximal one-directional intervals \([v_{i-a}, v_i]\) and \([v_i, v_{i+b}]\) of \(H\), such that \([v_{i-a}, v_i]\) is a forward directed path and \([v_i, v_{i+b}]\) a backward directed path. (Recall that \(H\) is not a directed cycle.)

In Figure 4.6 we illustrate the display of \([f^0, f^{3d}]\) with each \(f^k\) \((0 \leq k \leq 3d)\) shown only on its restriction to \([v_{i-a}, v_{i+b}]\). Each \(f^k\) corresponds to one column. There are \(3d+1\) columns and \(a+b+1\) boxes in each column. If \(f^k(v_j) = f_j^k\), then we put \(f_j^k\) in the box \(j\) of column \(f^k\). Note that the vertices of \(\overrightarrow{C_n}\) are denoted by \(0, 1, ..., n-1\), and the arcs are \(t(t+1)\) for \(t = 0, 1, ..., n-1\). If there is an edge from \(v_j\) to \(v_{j+1}\) in \(H\) and an edge from \(f^k\) to \(f^{k+1}\) in \(F\), then we put an arrow from the box \(j\) of \(f^k\) to the box \(j+1\) of \(f^{k+1}\), and we must have \(f^{k+1}(v_{j+1}) = f^k(v_j) + 1\). Therefore the \((3d+1)(a+b+1)\) values of the \((3d+1)\) \(f^k\)'s on the \(a+b+1\) vertices are not all independent. We shall analyse their dependence and then do a replacement operation on the values of some boxes, i.e., redefine the values of some \(f^k\) \((0 \leq k \leq 3d)\), so that the resulting \(f^k\) \((0 \leq k \leq 3d)\) satisfy the following conditions:

(1) \(f^k\) \((0 \leq k \leq 3d)\) are still vertices of \(\overrightarrow{C_n}(H)\);
(2) \([f^0, ..., f^d]\) is a forward directed path, \([f^d, ..., f^{2d}]\) is a backward directed path and \([f^{2d}, ..., f^{3d}]\) is a forward directed path;

(3) \(f^k(v_j) = f^{2d+k}(v_j)\) for \(0 \leq k \leq d\); and

(4) \(f^k(v_j) = f^k(v_j)\) for \(k = 0\) and \(k = 3d\).

Then we may define

\[ g^k = f^k \text{ for } k = 1, ..., d \] (note that \(f^d = f^{3d}\)), and

\[ G = F \setminus [f^0, f^1, ..., f^{3d}] \cup [g^0, g^1, ..., g^d]. \]

We have therefore \(G \subseteq \vec{C}_n(H)\) and \(G \in \mathfrak{S}(q, m)\).

The replacement operation will be described separately on each interval \([v_{i-a}, v_{i+b}]\) consisting of a maximal forward interval followed by a maximal backward interval. We shall only consider three cases: both \(a\) and \(b\) are greater than \(d\) (Case 1), both \(a\) and \(b\) are smaller than \(d\) (Case 2), and \(a < d < b\) (Case 3). We shall analyse Case 1 in detail, and only illustrate Cases 2 and 3 in Figures 4.7 and 4.8, since the procedures are very similar. Other cases like \(a = d, b > d; a > d, b = d; a = d, b < d; a < d, b = d; a = b = d\) are similar too.

**Case 1** As illustrated in Figure 4.6, we partition the display of \(f^0, f^1, ..., f^{3d}\) on the vertices \(v_{i-a}, ..., v_i, ..., v_{i+b}\) into several regions:

\[ A = \{f^k(v_j) : 0 \leq k \leq d, \ i - a + k \leq j \leq i - d + k\} \cup \]
\[ \{f^k(v_j) : d < 2d, \ i - a + 2d - k \leq j \leq i + d - k\} \cup \]
\[ \{f^k(v_j) : 2d \leq k \leq 3d, \ i - a + k - 2d \leq j \leq i + k - 3d\} \cup \]
\[ f^k(v_j) : 0 \leq k \leq k, \ i + d - k \leq j \leq i + d - k\} \cup \]
\[ \{f^k(v_j) : d < k < 2d, \ i + d - k \leq j \leq i + b + k - 2d\} \cup \]
\[ \{f^k(v_j) : 2d \leq k \leq 3d, \ i + 3d - k \leq j \leq i + b + 2d - k\} \]

\[ A^* = \{f^0(v_j) : i - a \leq j \leq i - d, \ i + d \leq j \leq i + b\} \]
B_1 = \{f^k(v_j) : 1 \leq k \leq d, \ i - a \leq j \leq i - a + k - 1\}
B_1^* = \{f^k(v_{j-a}) : 1 \leq k \leq d\}
B_1' = \{f^k(v_j) : d + 1 \leq k \leq 2d - 1, \ i - a \leq j \leq i - a + 2d - k - 1\}
B_3 = \{f^k(v_j) : 2d + 1 \leq k \leq 3d, \ i - a \leq j \leq i - a + k - (2d + 1)\}
B_3^* = \{f^k(v_{i-a}) : 2d + 1 \leq k \leq 3d\}
B_2 = \{f^k(v_j) : 1 \leq k \leq d, \ i + b - k + 1 \leq j \leq i + b\}
B_2^* = \{f^k(v_{i+b}) : 1 \leq k \leq d\}
B_2' = \{f^k(v_j) : d + 1 \leq k \leq 2d - 1, \ i + b - 2d + k + 1 \leq j \leq i + b\}
B_4 = \{f^k(v_j) : 2d + 1 \leq k \leq 3d, \ i + b + 2d + 1 - k \leq j \leq i + b\}
B_4^* = \{f^k(v_{i+b}) : 2d + 1 \leq k \leq 3d\}
D^* = \{f^0(v_j) : i - d + 1 \leq j \leq i + d - 1\}
D = \{f^k(v_j) : 1 \leq k \leq d - 1, \ i - d + k + 1 \leq j \leq i + d - k - 1\}
E^{**} = \{f^k(v_j) : d + 1 \leq j \leq 2d\}
E_1 = \{f^k(v_j) : d + 1 \leq k \leq 2d - 1, \ i -(k - d - 1) \leq j \leq i + k - d - 1\}
E_2 = \{f^k(v_j) : 2d + 1 \leq k \leq 3d - 1, \ i - 3d + k + 1 \leq j \leq i + 3d - k - 1\}

Recall that [v_{i-a}, ..., v_i] is a forward interval and [v_i, ..., v_{i+b}] a backward interval in the graph H, and that [f^0, ..., f^d] is a forward interval in F. If we write f^0(v_j) = f^0_j \ (i-d+1 \leq j \leq i+d-1) for the values of the boxes in the region D^*, then we have

\[ f^k(v_j) = f^0_{j-k} + k \quad (0 \leq k \leq d-1, \ i-d+k+1 \leq j \leq i+d-k-1) \]

(Note also that \( f^0_{i-d+j} = f^0_{i+d-j} \) for \( j = 1, 2, ..., d-1 \). Thus the values of the boxes in region D are determined by the values in the boxes of region D^*.

Similarly the values of the boxes in region A are determined by the values of the boxes in region A^*; the values in region B_i are
determined by the values in region $B_i^*$ ($i = 1, 2, 3, 4$); and the values in region $B_i'$ are determined by the values in region $B_i^*$ ($i = 1, 2$).

Furthermore the values in regions $E^{**}$, $E_1$, $E^*$, $E_2$ are independent of any other values in $A$, $D$, $B_i$, etc. For example $f^{d+1}(v_i)$ is independent of the values of $f^d$ on $v_j$ ($i-a \leq j \leq j+b$). Hence we have the freedom to choose $f^{d+1}(v_i)$. Once $f^{d+1}(v_i)$ is fixed, then $f^{d+2}(v_{i-1}) = f^{d+2}(v_{i+1}) = f^{d+1}(v_i)+1$, $\ldots$, $f^{2d}(v_{i-d+1}) = f^{2d}(v_{i+d-1}) = f^{d+1}(v_i)+d-1$, $\ldots$.

Similar arguments apply to all $f^{d+k}(v_i)$ ($1 \leq k \leq d$) which belong to the region $E^{**}$. Once values in $E^{**}$ are defined, values in $E_1$, $E^*$ and $E_2$ are determined. Thus we may choose the values in $E^{**}$ so that $E^* = D^*$ (i.e., the corresponding values in the box of $E^*$ and $D^*$ are equal). For example we may choose $f^{d+k}(v_i) = f^0(v_{i-d+k}) - d + k$ for $1 \leq k \leq d$.

We may also change the values in $B_1^*$ and $B_2^*$ so that $B_1^* = B_3^*$, $B_2^* = B_4^*$, and change the values in $B_1$, $B_1'$, $B_2$ and $B_2'$ as well. Correspondingly we shall have $B_1 = B_3$, $B_2 = B_4$.

After all these changes (replacement of the values of boxes in regions $E^*$, $E_2$ by the values of boxes in regions $D^*$, $D$ respectively, and replacement of the values of boxes in regions $B_1^*$, $B_1$, $B_2^*$ and $B_2$ by the values of boxes in regions $B_3^*$, $B_3$, $B_4^*$ and $B_4$ respectively) have been done, the resulting $f^k$ ($0 \leq k \leq 3d$) will satisfy (1) - (4). □

For the display in Figure 4.3, see Figure 4.9 showing the replacement operation.

**Theorem 4.2.7** $\mathbb{C}_n$ is multiplicative if and only if $n$ is a prime power.
Proof  By Lemma 4.1.1, it will suffice to prove that $C_n$ is multiplicative when $n$ is a prime power. By Lemma 4.2.1 we only need to prove that $C'' \not< C_n(C')$ for any $C' \in \mathfrak{S}(p)$, $C'' \in \mathfrak{S}(q)$, $p \not\equiv 0 \pmod{n}$, $q \not\equiv 0 \pmod{n}$. This can be done by double induction on the number of vertices of outdegree zero in $C'$, and on the number of vertices of outdegree zero in $C''$.

As the starting step of the induction, suppose that neither $C'$ nor $C''$ contain a vertex of outdegree zero, i.e., that $C' = \overrightarrow{C}_p$ and $C'' = \overrightarrow{C}_q$. By Lemma 4.2.3 we have $\overrightarrow{C}_q \not< \overrightarrow{C}_n(\overrightarrow{C}_p)$.

Now we assume, as an induction hypothesis that $\overrightarrow{C}_q \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p, k)$, $(k \geq 0)$. We see by Lemma 4.2.5 that $\overrightarrow{C}_q \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p, k+1)$. Therefore $\overrightarrow{C}_q \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p)$.

Next we apply induction on the number of vertices of outdegree zero in $C''$. When this number is 0, $C'' = \overrightarrow{C}_q$. We have already proved that $\overrightarrow{C}_q \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p)$. We assume, as the induction hypothesis that $C'' \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p)$, $C'' \in \mathfrak{S}(q, m)$ $(m \geq 0)$. Then $C'' \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p)$, $C'' \in \mathfrak{S}(q, m+1)$ by Lemma 4.2.6. Therefore $C'' \not< \overrightarrow{C}_n(\overrightarrow{C}')$ for any $C' \in \mathfrak{S}(p)$, $C'' \in \mathfrak{S}(q)$.

Corollary 4.2.8  $\overrightarrow{C}_n \in \mathfrak{A} \mathfrak{N} \cap \mathfrak{U} \mathfrak{W} \mathfrak{M}$ provided that $n$ is a prime power.

Proof   Apply Theorem 4.2.7, Corollary 2.2.4 and Claim 1.4.4 to obtain the proof.
Figure 4.6 Replacement operation, both $a > d$ and $b > d$
Figure 4.7  Replacement operation, both $a < d$ and $b < d$

Figure 4.8  Replacement operation, $a < d < b$
Figure 4.9 Replacement operation
4.3 Another Class of Non-multiplicative Oriented Cycles.

We have seen in Theorem 3.4.8, that almost all oriented cycles are non-multiplicative. In the last section we have shown that all prime power directed cycles are multiplicative.

Let \( k \) and \( m \) be positive integers. Then \( C_{k,m} \) is the oriented cycle obtained by identifying the endvertex of \( P_k \) with outdegree (indegree) zero with the endvertex of \( P_m \) with outdegree (indegree) zero. Obviously any oriented cycle with one vertex of outdegree zero is some \( C_{k,m} \). Clearly \( C_{k,k} \) is multiplicative. (In fact any oriented cycle \( C \) obtained by identifying the endvertex of \( P_k \) of outdegree (indegree) zero with the endvertex of maximum (minimum) level of a minimal oriented path of net length \( k \) is multiplicative, as it is homomorphically equivalent with \( P_k \).) We do not know if \( C_{k+1,k} \) is or is not multiplicative. But we can prove the following theorem.

**Theorem 4.3.1** If \( |k-m| \geq 2 \) then \( C_{k,m} \) is non-multiplicative.

**Proof** Without loss of generality, let \( k-m \geq 2 \). Let \( c_0 \) (respectively \( c_i \)) be the vertex of \( C_{k,m} \) with outdegree two (respectively indegree two).

Construct \( G \) and \( H \) as follows:

\[
G = P_k P_{m+1} P_{m+2} ;
\]
\[
H = P_k P_m P_m P_m P_k .
\]

Let \( g_0 \) (respectively \( g_i \)) be the vertex of \( G \) with outdegree (respectively indegree) two.

**Claim** \( G \rightarrow C_{k,m} \), \( H \rightarrow C_{k,m} \), and \( G \times H \rightarrow C_{k,m} \).
Suppose $G \to C_{k,m}$. Then $g_i$ must be mapped to $c_i$ and hence $g_0$ must be mapped to some vertex of the $\vec{P}_k$ in $C_{k,m}$, which is strictly between $c_0$ and $c_i$. Then the path $\vec{P}_{m+2}$ of $G$ cannot be mapped homomorphically to $C_{k,m}$.

Suppose $H \to C_{k,m}$. Then the first $\vec{P}_k$ must be mapped to the unique $\vec{P}_k$ of $C_{k,m}$, and the next $\vec{P}_m$ must be mapped to the $\vec{P}_m$ of $C_{k,m}$ - else the next $\vec{P}_{m+1}$ couldn't be homomorphically mapped to $C_{k,m}$ (cf. above). Then the last vertex of the path $\vec{P}_{m+1}$ in $H$ must be mapped to some inner vertex of the $\vec{P}_k$ in $C_{k,m}$, and so must the last vertex of the following $\vec{P}_m$ in $H$. Therefore the last $\vec{P}_k$ in $H$ cannot be properly mapped.

Now we show that $G \times H \to C_{k,m}$. It is easy to verify that any proper subpath of $H$ admits a homomorphism to $C_{k,m}$, and that $\nu_l(G) = k+1$, $\nu_l(H) > k+1$, and $\nu_l(G') \leq k+1$ for any subpath $G'$ of $G$. Let $\pi_1$ and $\pi_2$ be the projections $G \times H \to G$ and $G \times H \to H$ respectively. For any component $A$ of $G \times H$, $A$ is balanced, and $\pi_1(A)$ is a subpath of $G$. Hence $\nu_l(\pi_2(A)) = \nu_l(\pi_1(A))$ (by Corollary 3.2.9) and is at most $k+1$. Therefore $\pi_2(A)$ is a proper subpath of $H$ and so admits a homomorphism to $C_{k,m}$. By composition, $A \to C_{k,m}$. Therefore $G \times H \to C_{k,m}$. \[\square\]

See Figure 4.10 and Figure 4.11 for illustrations.
Figure 4:10
\[ G \to C_{6,1}, \quad H \to C_{6,1}, \quad G \times H \to C_{6,1} \]

Figure 4.11
Chapter 5 The Chromatic Difference Sequence of The Cartesian Product of Graphs

5.1 Introduction

The concepts investigated in this chapter arose from a different point of view. Nevertheless they also concern a graph product and graph homomorphisms (in particular, colorings). Recall the definition of the chromatic difference sequence (cds) of an undirected graph:

$$\text{cds}(G) = (a(1), a(2), \ldots),$$

where

$$\sum_{j=1}^{t} a(j) = \alpha(t)$$

is the maximum number of vertices in an induced t-colorable subgraph of G.

If G has chromatic number n, then a(t) = 0 for t > n, and we usually omit the zero terms. Also recall that \(\text{ncds}(G) = \text{cds}(G)/|V(G)|\).

For example, the cds of the 5-cycle \(C_5\) is (2, 2, 1), and the cds of the Petersen graphs is (4, 3, 3). See Figures 5.1 and 5.2.

Figure 5.1 \(\text{cds}(C_5) = (2, 2, 1)\)
Greene and Kleitman were the first who studied the cds of some special graph, called the comparability graph \([9, 10]\). But they didn't use the name cds. The maximum \(k\)-colorable subgraph in a comparability graph \(C(P)\) of a poset \(P\) is a Sperner \(k\)-family of \(P\), i.e., a subset of \(P\) of maximum size which contains no chains of length \(k+1\).

Albertson and Berman have proposed the concept of cds and stated a conjecture for an \(n\)-term sequence to be the cds of some \(n\)-colorable graph \([1]\), and proven the conjecture for \(n\) up to 4.

Albertsen and Collins have given a "no-homomorphism lemma" \([2]\). The no-homomorphism lemma states: "If there exists a homomorphism from \(G\) to \(H\) and if \(H\) is vertex transitive, then \(ncds(G)\) dominates \(ncds(H)\)." Let \(P^1\) denote the Petersen graph. Recall that \(P^n\) is the Cartesian product of \(P^1\) with \(P^{n-1}\). They also proved that

\[
ncds(P^r) = \frac{1}{3} \left( 1 + \frac{2}{10^r}, 1 - \frac{1}{10^r}, 1 - \frac{1}{10^r} \right).
\]

Then they applied the no-homomorphism lemma to obtain the result that for any \(r > s\) there does not exist a homomorphism from \(P^r\) to \(P^s\).
These results motivated our research in Section 5.3. As an important application of the no-homomorphism lemma we obtain the ncds of powers of circulant graphs. We also generalize the result about the cds of the powers of the Petersen graph.

A sequence of integers with a finite number of non-zero terms is said to be flat if it is non-increasing and the largest and smallest nonzero terms differ by at most one. We denote by $F(s,n)$ the flat sequence with $n$ non-zero terms whose terms sum to $s$. The sequence $F(ns, n) = (s, s, ..., s)$ is called a non-drop flat ($ND$-flat) sequence. The sequence $F(ns+1, n) = (s+1, s, ..., s)$ is called a first-drop flat ($FD$-flat) sequence.

The sequence $\bar{r} = (r_1, r_2, ..., r_n)$ with $r_1 \geq r_2 \geq ... \geq r_n$ is called a coloring sequence of the graph $G$ with $\chi(G) = n$ if there exists an $n$-coloring of $V(G)$ with color-classes $V_1, V_2, ..., V_n$ such that $r_i = |V_i|$, $i = 1, 2, ..., n$. Any non-increasing sequence is a coloring sequence of some complete $n$-partite graph. Each graph has at least one coloring sequence. For example, both $(3, 3, 1)$ and $(3, 2, 2)$ are coloring sequences of $C_7$. The set of all coloring sequences of a graph $G$ is denoted by $cs(G)$.

The sequence $cds(G)$ is said to be achievable if it is also a coloring sequence of $G$. Every achievable cds sequence is non-increasing. Every non-increasing sequence is the achievable cds sequence of some complete $n$-partite graph.

For some graphs $G$, the sequence $cds(G)$ is not achievable. For instance this is the case for the graph in Figure 5.3.
Obviously

$$\tilde{r} \geq \ast F[\sum_{i=1}^{n} r_i, n]$$

if $\tilde{r} = (r_1, r_2, ..., r_n)$ is non-increasing. Thus

$$\tilde{r} \geq \ast F[\sum_{i=1}^{n} r_i, n]$$

if $\tilde{r} = (r_1, r_2, ..., r_n)$ is a coloring sequence of a graph $G$.

Let $G$ and $H$ be two graphs with $|V(G)| = r$, $|V(H)| = s$ and $\chi(G) = \chi(H) = n$. We define

$$F(G \square H) = F(rs, n).$$

A latin square is an $n \times n$ array with entries from $\{1, 2, ..., n\}$, such that each $t \in \{1, 2, ..., n\}$ appears exactly once in each row and each column.

Let $U_1, U_2, ..., U_n$ be the color-classes of an $n$-coloring of $V(G)$, and let $V_1, V_2, ..., V_n$ be the color-classes of an $n$-coloring of $V(H)$. Let $A = (a_{ij})$ be an $n \times n$ latin square. Let

$$W_t = \bigcup_{a_{ij} = t} U_i \times V_j \quad (t = 1, 2, ..., n).$$

It follows from the definitions that $W_1, W_2, ..., W_n$ is the color-classes of an $n$-coloring of $G \square H$. We say that $W_1, W_2, ..., W_n$ is the color-classes of the $n$-coloring of $G \square H$ induced by the two given $n$-
colorings of $G$ and $H$ and the latin square $A$. Obviously different $n$-colorings of $G$ and $H$ and different Latin Squares will induce different $n$-colorings of $G \square H$. We denote by $\text{npcs}(G \square H)$ (*natural product coloring sequences*), the set of all coloring sequences of $G \square H$ induced by any two $n$-colorings of $G$ and $H$ and any $n \times n$ latin square.

Obviously we have

$$\text{cds}(G \square H) \geq^* \text{npcs}(G \square H) \geq^* F(G \square H)$$

(where $\text{cds}(G \square H) \geq^* \text{npcs}(G \square H)$ means that $\text{cds}(G \square H)$ dominates any sequence in the set $\text{npcs}(G \square H)$, similarly $\text{npcs}(G \square H) \geq^* F(G \square H)$).

5.2 Products of Bipartite Graphs

The following facts are easy to see. The coloring sequence of any connected bipartite graph is unique. There are only two latin Squares of order 2, namely

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$ 

The Cartesian product of two bipartite graphs is bipartite. Any 2-coloring of $G \square H$ induces a 2-coloring of $G$ ($H$) which is unique if $G$ (respectively $H$) is connected. Moreover there is only one element in $\text{npcs}(G \square H)$. Therefore we have the following theorem.

**Theorem 5.2.1** Let $G$ and $H$ be connected bipartite graphs with achievable cds given by $\text{cds}(G) = (r_1, r_2)$ and $\text{cds}(H) = (s_1, s_2)$. Then $\text{npcs}(G \square H) = \{\text{cds}(G \square H)\} = \{(r_1s_1 + r_2s_2, r_1s_2 + r_2s_1)\}$.
Let \( U_1 \) and \( U_2 \) be two color-classes of \( V(G) \) with \(|U_1| = r_1, |U_2| = r_2\). Let \( V_1 \) and \( V_2 \) be two color-classes of \( V(H) \) with \(|V_1| = s_1, |V_2| = s_2\). Then \( V(G \square H) \) has bipartitions \( U_1 \times V_1 \cup U_2 \times V_2 \) and \( U_1 \times V_2 \cup U_2 \times V_2 \) which have \( r_1 s_1 + r_2 s_2 \) and \( r_1 s_2 + r_2 s_1 \) vertices respectively. Furthermore, \( r_1 s_1 + r_2 s_2 - r_1 s_2 - r_2 s_1 = (r_1 - r_2)(s_1 - s_2) \geq 0. \)

Now suppose \( M \) is a maximum independent set of \( V(G \square H) \). We will consider the following two cases.

**Case 1** \( s_1 = s_2 \)

For any \( u \in V(G) \), \( M \) induces an independent set \( \{u\} \times H_u \) in the subgraph \( \{u\} \times H \) of \( G \square H \) which is a copy of \( H \). Hence \( |\{u\} \times H_u| \leq s_1 = s_2 \). Furthermore, \( |M| \leq |G| s_1 = (r_1 + r_2)s_1 = r_1 s_1 + r_2 s_2. \)

**Case 2** \( s_1 > s_2 \)

For any \( u \in V(G) \), \( M \) induces an independent set \( \{u\} \times H_u \) in the subgraph \( \{u\} \times H \) of \( G \square H \). Any two maximum size independent sets of \( H \) must have common vertices, otherwise the number of vertices of \( H \) will be at least greater than \( 2s_1 > s_1 + s_2 \), a contradiction. If \( \{u_i\} \times H_i \) is a maximum independent set in \( \{u_i\} \times H \) for \( i = 1, 2 \), then \( u_1 \) and \( u_2 \) are not adjacent in \( G \). Hence there can be no more than \( r_1 \) induced independent sets in \( H \) with size \( s_1 \). Each of the remaining \( r_2 \) independent sets can have size no more than \( s_2 \). So the independence number of \( G \square H \) is at most \( r_1 s_1 + r_2 s_2 \). In fact it is exactly \( r_1 s_1 + r_2 s_2 \) by our earlier argument. \( \square \)

Theorem 5.2.1 doesn't hold if the sequences \( \text{cds}(G) \) and \( \text{cds}(H) \) are not achievable. For example if the graph \( G \) is given by Figure 5.4,
Theorem 5.2.2 If both \( \text{cds}(G) = (r_1, r_2) \) and \( \text{cds}(H) = (s_1, s_2) \) are achievable, then \( \text{cds}(G \Box H) = (r_1s_1 + r_2s_2, r_1s_2 + r_2s_1) \in \text{npcs}(G \Box H) \).

Proof This follows from Theorem 5.2.1 if both \( G \) and \( H \) are connected. Now if \( G \) has \( n_1 \) bipartite components \( G_1, ..., G_{n_1} \), and \( H \) has \( n_2 \) bipartite components \( H_1, H_2, ..., H_{n_2} \), then \( G \Box H \) is a graph with \( n_1 \times n_2 \) components \( G_i \Box H_j \) (\( i = 1, ..., n_1, j = 1, 2, ..., n_2 \)) each of which is bipartite. Any 2-coloring of \( G \Box H \) induces a 2-coloring of \( G_i \Box H_j \) for each \( i = 1, ..., n_1; j = 1, ..., n_2 \). Any 2-coloring of \( G \Box H \) can be constructed by a suitable combination of some 2-colorings of \( G_i \Box H_j \) (\( i = 1, ..., n_1, j = 1, ..., n_2 \)). Therefore we only need to prove that if

\[
   r = r_1 + r_2 = r_3 + r_4, \quad s = s_1 + s_2 = s_3 + s_4, \quad r_1 \geq r_2, \quad r_3 \geq r_4, \quad s_1 \geq s_2, \quad s_3 \geq s_4, \quad r_1 > r_3, \quad \text{and} \quad s_1 > s_3,
\]

then \( s_1r_1 + s_2r_2 > s_3r_3 + s_4r_4 \).

Let \( r_1 = r_3 + u, \quad u > 0, \quad s_1 = s_3 + v, \quad v > 0 \), then \( r_2 = r_4 - u, \quad s_2 = s_4 - v \) and we have

\[
   s_1r_1 + s_2r_2 = (s_3 + v)(r_3 + u) + (s_4 - v)(r_4 - u)
   = s_3r_3 + vr_3 + us_3 + uv + s_4r_4 - vr_4 - us_4 + uv
   = s_3r_3 + s_4r_4 + [u(s_3 - s_4) + v(r_3 - r_4) + 2uv]
\]

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Corollary 5.2.3 Let $\text{cds}(G) = (r_1, r_2)$ be achievable. Then

$$\text{cds}(G^n) = \left( \sum_{i=0 \mod 2}^{n} \binom{n}{i} r_1^{n-i} r_2^i, \sum_{i=1 \mod 2}^{n} \binom{n}{i} r_1^{n-i} r_2^i \right).$$

Proof By Theorem 5.2.2, we have $\text{cds}(G^2) = (r_1^2 + r_2^2, 2r_1r_2)$. Suppose the result is true for a certain value $n$. Applying Theorem 5.2.2 again

$$\text{cds}(G^{n+1}) = \text{cds}(G^n \square G)$$

$$= \left( \sum_{i=0 \mod 2}^{n} \binom{n}{i} r_1^{n+1-i} r_2^i + \sum_{i=1 \mod 2}^{n} \binom{n}{i} r_1^{n+1-i} r_2^i, \sum_{i=0 \mod 2}^{n} \binom{n}{i} r_1^{n-i} r_2^{i+1} + \sum_{i=1 \mod 2}^{n} \binom{n}{i} r_1^{n-i} r_2^{i+1} \right).$$

Note that

$$\sum_{i=0 \mod 2}^{n} \binom{n}{i} r_1^{n+1-i} r_2^i + \sum_{i=1 \mod 2}^{n} \binom{n}{i} r_1^{n-i} r_2^{i+1}$$

$$= \binom{n}{0} r_1^{n+1} + \binom{n}{2} r_1^{n-2+1} r_2^2 + \binom{n}{4} r_1^{n-4+1} r_2^4 + \ldots + \binom{n}{2k} r_1^{n-2k+1} r_2^{2k} + \ldots$$

$$+ \binom{n}{1} r_1^{n-1} r_2 + \binom{n}{3} r_1^{n-3} r_2^4 + \ldots + \binom{n}{2k-1} r_1^{n-2k+1} r_2^{2k} + \ldots$$

$$= \binom{n+1}{0} r_1^{n+1} + \binom{n+1}{2} r_1^{n+1-2} r_2^2 + \binom{n+1}{4} r_1^{n+1-4} r_2^4 + \ldots + \binom{n+1}{2k} r_1^{n+1-2k} r_2^{2k} + \ldots$$

$$= \sum_{i=0 \mod 2}^{n+1} \binom{n+1}{i} r_1^{n+1-i} r_2^i.$$

Applying the same argument to the second term, we have

$$\text{cds}(G^{n+1}) = \left( \sum_{i=0 \mod 2}^{n+1} \binom{n+1}{i} r_1^{n+1-i} r_2^i, \sum_{i=1 \mod 2}^{n+1} \binom{n+1}{i} r_1^{n+1-i} r_2^i \right). \qed
Recall that \( \lceil n \rceil \) denotes the least integer greater than \( n \), \( \lfloor n \rfloor \) denotes the greatest integer smaller than \( n \).

**Corollary 5.2.4** Let \( P_{m-1} \) be a path of length \( m-1 \). Then

\[
\text{cgs}(P_{m-1}) = (\lfloor \frac{m-1}{2} \rfloor, \lfloor \frac{m}{2} \rfloor) = (\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + \frac{m-1}{2} \leq 1, 1)
\]

\[
= (\lfloor \frac{m}{2} \rfloor + \frac{m-1}{2}, \lfloor \frac{m}{2} \rfloor + \frac{m-1}{2} \).
\]

**Corollary 5.2.5** Let \( C_{2m} \) be a cycle of length \( 2m \). Then

\[
\text{cgs}(C_{2m}^n) = 2^{n-1}m^{n}(1,1) = (2^{n-1}m^{n}, 2^{n-1}m^{n}).
\]

5.3 Products of Graphs with ND Flat and FD Flat Chromatic Difference Sequence.

**Theorem 5.3.1** If \( \text{cgs}(G) = F(r, n) \), then \( \text{cgs}(G) \) is achievable.

**Proof** There exists an \( n \)-coloring of \( V(G) \) with color-classes \( V_1, V_2, ..., V_n \) since \( G \) is \( n \)-colorable. For any \( i \) \((i = 1, 2, ..., n)\) \( |V_i| \leq f + 1 \), where \( f = \lfloor \frac{r}{n} \rfloor \), otherwise the size of the maximum independent set will be greater than \( f + 1 \), a contradiction. On the other hand \( |V_i| \geq f \) for all \( i \), otherwise \( \bigcup_{j \neq i} V_j > r-f \) contradicting the fact that \( \alpha(n-1) = r-f \).

Therefore \( f \leq |V_i| \leq f + 1 \) for any \( i, i = 1, 2, ..., n. \) Since \( \sum_{i=1}^{n} |V_i| = r, \) we have \( (|V_1|, ..., |V_n|) = F(r, n). \)

It is not true that \( \text{cgs}(G) \) is achievable wherever \( F(r, n) \) is a coloring sequence of \( G \). For example, let \( A \) be the graph in Figure 5.3, Then
\[(2, 2, 2) = F(6, 3) \neq \text{cds}(A) = (3, 1, 2)\]
even though \((2, 2, 2)\) is a coloring sequence of \(A\).

**Theorem 5.3.2** Let \(\text{cds}(H) = F(ns+1, n)\) (an FD flat sequence), let \(\text{cds}(G) = (r_1, r_2, \ldots, r_n)\) be achievable, and let \(r = \sum_{i=1}^{n} r_i\). Then

\[
\text{cds}(G \Box H) = (r_1, \ldots, r_n) + rsF(n, n) \in \text{npcs}(G \Box H)
\]

**Proof** By Theorem 5.3.1, \(\text{cds}(H)\) is achievable, hence there exists an \(n\)-coloring of \(V(H)\) with color-classes \(V_1, \ldots, V_n\) such that

\[
(|V_1|, \ldots, |V_n|) = (s+1, s, \ldots, s).
\]

There also exists an \(n\)-coloring of \(V(G)\) with color-classes \(U_1, \ldots, U_n\) such that \((|U_1|, \ldots, |U_n|) = (r_1, \ldots, r_n)\).

Let \(A = (a_{ij})\) be an \(n \times n\) latin square given by \(a_{ij} \equiv i+j-1 \pmod{n}\). Then \(G \Box H\) has an \(n\)-coloring given by the color-classes \(W_1, \ldots, W_n\) where

\[
W_t = \bigcup_{a_{ij} = t} U_i \times V_j \quad (t=1, 2, \ldots, n).
\]

Obviously \(|W_m| = r_m(s+1) + (r-r_m)s = r_m + rs \quad (m = 1, 2, \ldots, n)\).

Now we shall consider the maximum independent sets in \(G \Box H\). Any maximum independent set in \(G \Box H\) induces \(|V(G)|\) independent sets in \(H\). Any two maximum independent sets of \(H\) must have common vertices, otherwise the number of vertices of a maximum 2-colorable subgraph of \(H\) will be \(2s+2\), a contradiction. If \(\{u_i\} \times H_i\) is a maximum independent set in \(\{u_i\} \times H\) for \(i = 1, 2\), then \(u_1\) and \(u_2\) are not adjacent in \(G\). Hence there can be no more than \(r_1\) induced independent set in \(H\) with size \(s+1\). Each of the remaining \(r-r_1\) independent sets can have size no more than \(s\). So the independence number of \(G \Box H\) is at most \(r_1(s+1) + (r-r_1)s\) and hence
\[ \alpha(1) = r_1(s+1) + (r-r_1)s = rs + r_1. \]

For \( m \) (\( m = 2, 3, \ldots, n \)) any given maximum \( m \)-colorable subgraph \( M_m \) in \( G \square H \) induces \( |V(G)| \) \( m \)-colorable subgraphs in \( H \). Denote by \( \{u_i\} \times H_i \) the \( m \)-colorable subgraph in \( \{u_i\} \times H \) induced by \( M_m \). Denote by \( \{\{u_i\} \times H_{i_k}\}_{k=1}^m \), the corresponding \( m \)-coloring of \( \{u_i\} \times H_i \).

If for some \( u_i \in V(G) \), \( \{u_i\} \times H_i \) happens to be a maximum \( m \)-colorable subgraph in \( \{u_i\} \times H \), then \( |H_i| = ms+1 \), and one color must occur on \( s+1 \) vertices, while each of the other colors must occur on \( s \) vertices. Otherwise the maximum independent set of \( H \) will have more than \( s+1 \) vertices, a contradiction.

Without loss of generality, we suppose that \( |H_i| = s+1 \) for each \( i \).

If \( \{u_i\} \times H_i \) and \( \{u_j\} \times H_j \) are two induced \( m \)-colorable subgraphs in \( \{u_i\} \times H \) and \( \{u_j\} \times H \) respectively, which are both maximum, then \( H_{i_i} \) and \( H_{j_j} \) must have common vertex. Otherwise we will get a 2-colorable subgraph with \( 2s+2 \) vertices for \( H \), a contradiction. If \( u_i \) and \( u_j \) are not adjacent, \( \{u_i\} \times H_{i_i} \) and \( \{u_j\} \times H_{j_j} \) can have the same color in \( M_m \). If \( u_i \) and \( u_j \) are adjacent, \( \{u_i\} \times H_{i_i} \) and \( \{u_j\} \times H_{j_j} \) must have different colors in \( M_m \). So no more than \( r_1 + \ldots + r_m \) subgraphs \( \{u\} \times H \) can have maximum \( m \)-colorable induced subgraphs of size \( ms+1 \).

The remaining \( r_{m+1} + \ldots + r_n \) induced \( m \)-colorable subgraphs can have size no more than \( ms \). So the number of vertices of the maximum \( m \)-colorable subgraph of \( G \square H \) is at most

\[ (r_1+ \ldots + r_m)(ms+1) + (r_{m+1}+ \ldots + r_n)ms = mrs + r_1 + \ldots + r_m. \]

By virtue of the argument at the beginning of this proof we obtain

\[ \alpha(m) = mrs + r_1 + \ldots + r_m. \]

Hence \( a(m) = \alpha(m) - \alpha(m-1) = rs + r_m \) for \( m = 1, 2, \ldots, n \). \( \square \)
Corollary 5.3.3  If \( \text{cds}(G) = F(nr+1, n) \) and \( \text{cds}(H) = F(ns+1, n) \), then \( \text{cds}(G \Box H) = F(n^2rs + nr + ns + 1, n) \).

Let \( P \) be any graph. Recall that \( P^m \) is recursively defined by

\[
P^m = P^{m-1} \Box P, \quad P^1 = P.
\]

Corollary 5.3.4  Let \( \text{cds}(P) = F(nr+1, n) \). Then

1. \( \text{cds}(P^m) = F(nr_m+1, n) \), where \( r_m = \sum_{i=1}^{m-1} \binom{m}{i} n^{m-(i+1)} r^{m-i} \);

2. \( \text{ncds}(P^m) = (N_m(1), \ldots, N_m(n)) \), where \( N_m(1) = \frac{1}{n} + \frac{n-1}{n(nr+1)^m} \);

\[
N_m(2) = \ldots = N_m(n) = \frac{1}{n} - \frac{1}{n(nr+1)^m} \; ; \text{and}
\]

3. \( \lim_{m \to \infty} \text{ncds}(P^m) = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \).

Proof  (1) We give a proof by induction. By definition

\[
P^2 = P \Box P,
\]

\[r_2 + 1 = (r+1)(r+1) + (n-1)rr = nr^2 + 2r + 1, \quad \text{cds}(P^2) = (r_2+1, r_2, \ldots, r_2).
\]

Now assume that

\[
r_{m-1} = \sum_{i=0}^{m-2} \binom{m-1}{i} n^{m-1-(i+1)} r^{m-1-i} \quad \text{and}
\]

\[
\text{cds}(P^{m-1}) = (1, 0, \ldots, 0) + r_{m-1}(1, \ldots, 1).
\]

Then

\[
r_m = nr_{m-1}r + r + r_{m-1}
\]

\[
= n \left[ \sum_{i=0}^{m-2} \binom{m-1}{i} n^{m-1-(i+1)} r^{m-1-i} \right] + r + \sum_{i=0}^{m-2} \binom{m-1}{i} n^{m-1-(i+1)} r^{m-1-i}
\]

\[
= n^{m-1}r + \binom{m-1}{1} n^{m-2} r^{m-1} + \binom{m-1}{2} n^{m-3} r^{m-2} + \ldots + \binom{m-1}{m-3} n^2 r^3 + \binom{m-1}{m-2} nr^2
\]

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and \( \text{cds}(P^m) = (1, 0, \ldots, 0) + r_m(1, \ldots, 1). \)

\[
\begin{align*}
N_m(1) &= \frac{a(1)}{|V(P^m)|} = \frac{r_m + 1}{(n+1)m} = \frac{(nr+1)m+n-1}{n(n+1)m} = \frac{1}{n} + \frac{n-1}{n(n+1)m} \\
N_m(2) &= \ldots = N_m(n) = \frac{n r_m}{n(n+1)m} = \frac{(nr+1)m-1}{n(n+1)m} = \frac{1}{n} - \frac{1}{n(n+1)m} 
\end{align*}
\]

\textbf{Theorem 5.3.5} Assume that \( \text{cds}(H) \) is the ND flat sequence \( sF(n, n) \). Then for any graph \( G \) with achievable \( \text{cds}(G) \), we have an ND flat \( \text{cds}(G \Box H) \) so that \( \text{cds}(G \Box H) = r sF(n, n) \in \text{npcs}(G \Box H) \), where \( r = |V(G)| \).

\textbf{Proof} The proof is easy to see by a similar (but much simpler) argument to that in Theorem 3.2. The \( m \)-colorable subgraphs \( (m = 1, \ldots, n-1) \) of the \( \{u\} \times H \) induced by the maximum \( m \)-colorable subgraphs of \( G \Box H \) always have size at most \( m s \).

What will happen to \( \text{cds}(G \Box H) \) if \( \text{cds}(H) \) is neither ND nor FD flat and if \( \text{cds}(G) \) is still achievable? Here we give two examples.

\textbf{Example 5.3.6}

\( \text{cds}(C_5) = (2, 2, 1) = \text{cs}(C_5), \quad \text{cds}(C_5 \Box C_5) = (10, 10, 5) \)

\( \text{npcs}(C_5 \Box C_5) = \{(9, 8, 8)\}, \quad \text{cds}(C_5 \Box C_5) \geq \ast \text{npcs}(C_5 \Box C_5) \)

\textbf{Example 5.3.7} Let \( C_5^* \) be the graph given by Figure 5.5.
Then
\[ \text{cds}(C_5^*) = (3, 2, 1), \quad \text{npcs}(C_5 \square C_5^*) = [(10, 10, 10), (11, 10, 9)] \]
\[ \text{cds}(C_5 \square C_5^*) = (12, 12, 6), \quad \text{cds}(C_5 \square C_5^*) \geq* \text{npcs}^*(C_5 \square C_5^*). \]

Figure 5.5

Since \( cs(C_5) = (2, 2, 1) \), \( cs(C_5^*) = \{(3, 2, 1), (2, 2, 2)\} \), we obtain \( \text{npcs}(C_5 \square C_5^*) \) from Table 5.1 and Table 5.2, and \( \text{cds}(C_5 \square C_5^*) \) from Figure 5.6.

\[
\begin{array}{ccc}
2 & 2 & 2 \\
2 & 4 & 4 & 4 \\
2 & 4 & 4 & 4 \\
1 & 2 & 2 & 2 \\
(10, 10, 10)
\end{array}
\quad
\begin{array}{ccc}
3 & 2 & 1 \\
2 & 6 & 4 & 2 \\
2 & 6 & 4 & 2 \\
1 & 3 & 2 & 1 \\
(11, 10, 9)
\end{array}
\]

Table 5.1 Table 5.2 Figure 5.6

5.4 The Non-increasing Theorem

**Theorem 5.4.1** \( \text{ncds}(G \square H) \leq* \min(\text{ncds}(G), \text{ncds}(H)). \)

**Proof** It suffices to prove that \( \text{ncds}(G \square H) \leq* \text{ncds}(H). \)

Let \( |V(G)| = r \) and \( |V(H)| = s \). Let \( M_k \) be a maximum \( k \)-colorable subgraph of \( G \square H \), and let \( s_k \) be the number of vertices of a maximum
k-colorable subgraph of $H$. $M_k$ induces a k-colorable subgraph 
\{u\}×H_u in \{u\}×H for any $u \in V(G)$. Hence $|V(\{u\}×H_u)| \leq s_k$ and we have

$$|M_k| = \sum_{u \in G} |V(\{u\}×H_u)| \leq \sum_{u \in G} s_k = rs_k.$$ 

Therefore

$$\frac{|M_k|}{|V(G □ H)|} = \frac{|M_k|}{rs} \leq \frac{rs_k}{rs} = \frac{s_k}{s} \text{ for } k = 1, 2, ..., n-1,$$

and

$$\frac{|M_n|}{|V(G □ H)|} = 1 = \frac{s_n}{s}. \quad \Box$$

**Corollary 5.4.2** \text{ncds}(G^{k+1}) ≤* \text{ncds}(G^k) for $k = 1, 2, ...$. \quad \Box

**Corollary 5.4.3** \text{lim}_{k \to \infty} \text{ncds}(G^k) \text{ exists.} \quad \Box

**Proof** Let \text{ncds}(G) = (a_{11}, a_{12}, ..., a_{1n}) and \text{ncds}(G^k) = (a_{k1}, a_{k2}, ..., a_{kn}). Then $a_{11} \geq a_{21} \geq ... \geq a_{k1} \geq ... \geq \frac{1}{n}$, and hence \text{lim}_{k \to \infty} a_{k1} exists.

Inductively, suppose that \text{lim}_{k \to \infty} a_{ki} exists for $i = 1, 2, ..., t-1$. Since

$$\sum_{i=1}^{t} a_{1i} \geq \sum_{i=1}^{t} a_{2i} \geq ... \geq \sum_{i=1}^{t} a_{ki} \geq ... \geq \frac{1}{n},$$

\text{lim}_{k \to \infty} \sum_{i=1}^{t} a_{ki} exists, and hence also \text{lim}_{k \to \infty} a_{kt} exists. Therefore

\text{lim}_{k \to \infty} \text{ncds}(G^k) \text{ exists.} \quad \Box

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5.5 Powers of Circulant Graphs

Let \( G \) be a graph. We denote by \( \text{Aut}(G) \) the automorphism group of \( G \). The graph \( G \) will be called vertex transitive if for any two vertices \( u \) and \( v \) there exists \( \tau \in \text{Aut}(G) \) such that \( \tau(u) = v \).

**Lemma 5.5.1** If both \( G \) and \( H \) are vertex transitive, then \( G \square H \) is vertex transitive.

**Proof** For any \( \tau \in \text{Aut}(H) \), define \( \tau^* \) by \( \tau^*(x, y) = (x, \tau(y)) \), then \( \tau^* \in \text{Aut}(G \square H) \). In fact, if \( \{(x_1, y_1), (x_2, y_2)\} \in E(G \square H) \), then either \( x_1 = x_2 \), \( \{y_1, y_2\} \in E(H) \) which imply \( \{\tau(y_1), \tau(y_2)\} \in E(H) \) (i.e., \( \{\tau^*(x_1, y_1), \tau^*(x_2, y_2)\} \in E(G \square H) \), or \( \{x_1, x_2\} \in E(G) \), \( y_1 = y_2 \) which imply \( \tau(y_1) = \tau(y_2) \) (i.e., \( \{\tau^*(x_1, y_1), \tau^*(x_2, y_2)\} \in E(G \square H) \)). It is easy to check that \( \tau^* \) is one-to-one and onto since \( \tau \) is one-to-one and onto. Finally, \( \tau^{*-1} \) is a homomorphism by a similar argument as that for \( \tau^* \).

For any \( \sigma \in \text{Aut}(G) \), define \( \sigma^* \) by \( \sigma^*(x, y) = (\sigma(x), y) \). Then \( \sigma^* \in \text{Aut}(G \square H) \), the proof is the same as above.

Now for any two vertices \((x_1, y_1)\) and \((x_2, y_2)\) in \( V(G \square H) \), there exist \( \tau \in \text{Aut}(H) \) such that \( \tau(y_1) = y_2 \), and \( \sigma \in \text{Aut}(G) \) such that \( \sigma(x_1) = x_2 \). Define \( \sigma^* \) and \( \tau^* \) as above. Then \( \sigma^*\tau^*(x_1, y_1) = (x_2, y_2) \) and \( \sigma^*\tau^* \in \text{Aut}(G \square H) \). Therefore \( G \square H \) is vertex transitive.

**Lemma 5.5.2** If \( G \) is a circulant with \( p \) vertices and symbol \( N \), then there exists a homomorphism \( G^{k+1} \rightarrow G^k \).

**Proof** For any vertex \( \vec{x} = (x_1, x_2, ..., x_{k+1}) \) of \( G^{k+1} \), define \( \tau(\vec{x}) = (x_1, x_2, ..., x_{k-1}, x_k - x_{k+1}) \in G^k \). The arithmetic here is done modulo \( p \).
Now suppose \( \vec{x} = (x_1, ..., x_{k+1}) \) and \( \vec{y} = (y_1, ..., y_{k+1}) \) are two adjacent vertices in \( G^{k+1} \). Then there exists \( j \in \{1, 2, ..., k+1\} \), such that \( x_j - y_j \in \mathbb{N} \) and such that for all \( i = 1, 2, ..., k+1 \), \( i \neq j \), \( x_i = y_i \). It is easy to check that \( \{\tau(\vec{x}), \tau(\vec{y})\} \in E(G^k) \). (Distinguish the cases \( j \in \{1, ..., k-1\} \) and \( j \in \{k, k+1\} \).)

**Lemma 5.5.3** (The No-homomorphism Lemma) If there exists a homomorphism from \( G \) to \( H \) and \( H \) is vertex transitive, then \( \text{ncds}(G) \geq \text{ncds}(H) \).

**Proof** See [2].

**Theorem 5.5.4** Let \( G \) be a circulant. Then \( \text{ncds}(G^k) = \text{ncds}(G) \) for all positive integers \( k \).

**Proof** For any \( k \), \( \text{ncds}(G^k) \geq \text{ncds}(G^{k+1}) \), by the Non-increasing Theorem, and \( \text{ncds}(G^k) \leq \text{ncds}(G^{k+1}) \) by the No-homomorphism Lemma. (Each circulant is vertex-transitive, and so Lemma 5.5.1 assures that Lemma 5.5.3 applies.) Therefore \( \text{ncds}(G^k) = \text{ncds}(G^{k+1}) \).

**Corollary 5.5.5** For any vertex-transitive graph \( G \) with a prime number of vertices, \( \lim_{k \to \infty} \text{ncds}(G^k) = \text{ncds}(G) \).

**Proof** The graph \( G \) is circulant under the given conditions, see [22].

**Corollary 5.5.6** Let \( G \) be a circulant graph with \( p \) vertices and the symbol \( \mathbb{N} = \{1, 2, ..., n, p-n, ..., p-2, p-1\} \). Then \( \text{cds}(G) \) is achievable and
(1) if \( p = 2n \) or \( 2n+1 \), then \( \text{ncds}(G^k) = (\frac{1}{p}, \ldots, \frac{1}{p}) \) for \( k = 1, 2, \ldots \);

(2) if \( p = (n+1)r \) for some \( r \), then \( \text{ncds}(G^k) = (\frac{1}{p}, \ldots, \frac{1}{p}) \) for \( k = 1, 2, \ldots \);

(3) if \( p = (n+1)r + \alpha \) for some \( \alpha = mr \) (\( 0 < \alpha < n+1 \)), then

\[
\text{ncds}(G^k) = (\frac{r}{p}, \ldots, \frac{r}{p}) \text{ for } k = 1, 2, \ldots \text{ and }
\]

\[
\frac{r}{n+m+1}
\]

(4) if \( p = (n+1)r + \alpha \) for some \( \alpha = mr+\beta \) (\( 0 < \alpha < n+1 \), \( 0 < \beta < r \)), then \( \text{ncds}(G^k) = (\frac{r}{p}, \ldots, \frac{r}{p}, \frac{\beta}{p}) \) for \( k = 1, 2, \ldots \).

**Proof**

(1) is immediate since \( G \) is a complete graph.

(2) is immediate since \( G \) has \( n+1 \) pairwise disjoint maximum independent sets of \( r \) vertices and \( |V(G)| = (n+1)r \).

(3) The independence number of \( G \) is \( r \) from \( p = (n+1)r + \alpha \), \( (0 < \alpha < n+1) \). Furthermore since \( p = (n+m+1)r \), \( G \) has \( n+m+1 \) independent sets each of which has size \( r \).

(4) The independence number of \( G \) is \( r \) since \( p = (n+1)r + \alpha \), \( (0 < \alpha < n+1) \). Furthermore since \( p = (n+m+1)r + \beta \), \( 0 < \beta < r \), \( V(G) \) admits the following partition, \( V_1, V_2, \ldots, V_{n+2+m} \):

\[
V_1 = \{0, n+2+m, 2(n+2+m), \ldots, (\beta-1)(n+2+m)\}
\]

\[
V_2 = \{v+1 | v \in V_1 \} \cup \{\beta(n+2+m), \beta(n+2+m) + (n+m+1), \ldots, \\
\beta(n+2+m)+(r-\beta-1)(n+m+1)\}
\]

\[
V_3 = \{v+1 | v \in V_2\}
\]

....
\[ V_{n+2+m} = \{ v+n+m \mid v \in V_2 \}. \]

Here each \( V_i \) (\( i = 1, 2, ..., n+2+m \)) is independent, and \(|V_1| = \beta, |V_2| = ... = |V_{n+m+2}| = r\). Hence \( \text{cds}(G) \) is achieved by
\[
(|V_2|, ..., |V_{n+m+2}|, |V_1|). 
\]

We don't know whether the sequence \( \text{cds}(G) \) (respectively \( \text{cds}(G^k) \)) is achievable for an arbitrary circulant graph \( G \).

**Problem 1** If \( G \) is a circulant, prove that each \( \text{cds}(G^k) \) is achievable for \( k = 1, 2, ... \).

Although we cannot solve this problem for a general circulant graph, we can prove the following result, which implies that in order to solve Problem 1 we only have to verify that each \( \text{cds}(G) \) is achievable.

**Theorem 5.5.7** Let \( G \) be a circulant graph. If \( \text{cds}(G) \) is achievable, then \( \text{cds}(G^k) \) is achievable for all \( k \).

**Proof** We proceed by induction on \( k \). Let \( G \) have vertices 0, 1, ..., \( p-1 \), and symbol \( N \), and let \( \text{cds}(G) = (r_1, r_2, ..., r_n) \). Since \( \text{cds}(G) \) is assumed achievable there exist \( n \) subsets \( I_0^1, ..., I_{n-1}^1 \) of \( V(G) \) satisfying the following three conditions:

1. \( |I_i^1| = r_i \) for \( i = 0, 1, ..., n-1 \);
2. \( I_0^1, ..., I_{n-1}^1 \) are pairwise disjoint;
3. each \( I_0^1 \cup ... \cup I_{t-1}^1 \) (\( t = 1, 2, ..., n \)) induces a maximum \( t \)-colorable subgraph in \( G \), and the partition \( I_0^1, ..., I_{t-1}^1 \) achieves the \( \text{cds} \) of this subgraph.
Obviously, for any fixed j, \( I_0^1 + j, I_1^1 + j, ..., I_{n-1}^1 + j \) are pairwise disjoint and each is an independent set since G is a circulant. Hence they still satisfy the above three conditions.

Thus the sequence \( \text{cds}(G) \) can be achieved by p partitions:
\[
(I_0^1 + j), (I_1^1 + j), ..., (I_{n-1}^1 + j) \quad \text{for } j = 0, 1, ..., p-1.
\]

Now as the induction hypothesis suppose
\[
\text{cds}(G^{k-1}) = p^{k-2}(r_0, r_1, ..., r_{n-1})
\]
is achieved by p partitions:
\[
I_0^{k-1} + je_1^{k-1}, I_1^{k-1} + je_1^{k-1}, ..., I_{n-1}^{k-1} + je_1^{k-1} \quad \text{for } j = 0, 1, ..., p-1.
\]
(Here \( e_1^{k-1} = (1, 0, ..., 0) \) has \( k-1 \) coordinates, and \( I_0^{k-1} + je_1^{k-1} = \{(x_1, x_2, ..., x_{k-1}) + (j, 0, ..., 0) : (x_1, x_2, ..., x_{k-1}) \in I_0^{k-1}\} \).

That is, for each \( j = 0, 1, 2, ..., p-1 \), we have
\[
(4) \quad |I_i^{k-1} + je_1^{k-1}| = p^{k-2}r_i \quad \text{for } i = 0, 1, ..., n-1;
\]
\[
(5) \quad I_0^{k-1} + je_1^{k-1}, I_1^{k-1} + je_1^{k-1}, ..., I_{n-1}^{k-1} + je_1^{k-1} \text{ are pairwise disjoint;}
\]
\[
(6) \quad (I_0^{k-1} + je_1^{k-1}) \cup (I_1^{k-1} + je_1^{k-1}) \cup ... \cup (I_{t-1}^{k-1} + je_1^{k-1}) \text{ is a maximum } t\text{-colorable subgraph in } G^{k-1} \text{ and}
\]
\[
I_0^{k-1} + je_1^{k-1}, I_1^{k-1} + je_1^{k-1}, ..., I_{t-1}^{k-1} + je_1^{k-1} \text{ achieves the cd's of this subgraph for } t = 1, 2, ..., n.
\]

Now we first construct the following n subsets of vertices of \( G^k \):
\[
I_i^k = (0, I_i^{k-1}) \cup (1, I_i^{k-1} + 1e_1^{k-1}) \cup ... \cup (p-1, I_i^{k-1} + (p-1)e_1^{k-1})
\]
for \( i = 0, 1, ..., n-1 \).
(Here \( (0, I_i^{k-1}) = \{(0, x_1, x_2, ..., x_{k-1}) : (x_1, x_2, ..., x_{k-1}) \in I_i^{k-1}\} \).

Then we construct, for each \( j = 0, 1, ..., p-1 \), n subsets of \( V(G) \) as follows:
\[ I_0^k + j e_1^k, \ I_1^k + j e_1^k, \ldots, I_{n-1}^k + j e_1^k. \]

We need to check that these subsets of vertices satisfy conditions (4) - (6) with \( k-1 \) replaced by \( k \) for each \( j = 0, 1, \ldots, p-1 \).

The condition (4) is obvious by the construction and direct calculation.

Suppose \( (I_a^k + j e_1^k) \cap (I_b^k + j e_1^k) \neq \emptyset \) for \( 0 \leq a, b \leq n, a \neq b \), and the first coordinate of the common element is \( c+j \). Then

\[
(c+j, I_a^{k-1} + j e_1^{k-1}) \cap (c+j, I_b^{k-1} + j e_1^{k-1}) \neq \emptyset,
\]

\[
(I_a^{k-1} + j e_1^{k-1}) \cap (I_b^{k-1} + j e_1^{k-1}) \neq \emptyset,
\]

\[
I_a^{k-1} \cap I_b^{k-1} \neq \emptyset,
\]

which contradicts the induction hypothesis. Hence (5) is true.

Every maximum \( t \)-colorable subgraph of \( G^k \) induces a \( t \)-colorable subgraph in \( \{i\} \times G^{k-1} \) for \( t = 1, 2, \ldots, n \). The size of a maximum \( t \)-colorable subgraph in \( G^{k-1} \) is \( p^{k-2}(r_0 + \ldots + r_{t-1}) \). So the size of maximum \( t \)-colorable subgraph of \( G^k \) is at most

\[
p^{k-2}(r_0 + \ldots + r_{t-1}) = P^{k-1}(r_0 + \ldots + r_{t-1})
\]

Now

\[
| (I_0^k + j e_1^k) \cup \cdots \cup (I_{t-1}^k + j e_1^k) |
\]

\[
= p^{k-1}r_0 + \ldots + p^{k-1}r_{t-1} = p^{k-1}(r_0 + \ldots + r_{t-1}).
\]

In order that (6) is true we only need to prove that

\[
(I_0^k + j e_1^k) \cup \cdots \cup (I_{t-1}^k + j e_1^k)
\]

is a \( t \)-colorable subgraph in \( G^k \).
If we can prove that for \( j, 0 \leq j \leq p-1 \), each of the following

\[
I^k_0 + je_1^k, \ I^k_1 + je_1^k, \ldots, I^k_{i-1} + je_1^k, \ldots, I^k_{n-1} + je_1^k
\]

is an independent set in \( G^k \), then the above statement will surely be true for any \( t, 1 \leq t \leq n \).

Take any two vertices from \( I^k_{i} + je_1^k \) \((i = 0, 1, \ldots, n-1, j = 0,1, \ldots, p-1)\), say \( x \) and \( y \). If \( x \) and \( y \) belong to the same set \((a+j, I^k_{i-1} + ae_1^{k-1})\) then they are not adjacent since any two vertices in \( I^k_{i-1} \) are not adjacent. If \( x \in (a+j, I^k_{i-1} + ae_1^{k-1}) \) and \( y \in (b+j, I^k_{i-1} + be_1^{k-1}) \) with \( a \neq b \), \( (a+j,b+j) \)
in \( E(G) \), then \( \{a, b\} \in E(G) \) and \( b-a \in \mathbb{N} \). Suppose \( \{x, y\} \in E(G^k) \). Then

\[
(I^k_{i-1} + ae_1^{k-1}) \cap (I^k_{i-1} + be_1^{k-1}) \neq \emptyset
\]

and hence there is a

\[
(x_1, \ldots, x_{k-1}) \in I^k_{i-1} \quad \text{and} \quad (y_1, \ldots, y_{k-1}) \in I^k_{i-1}
\]
such that \( x_1 + a = y_1 + b \)

and \( x_i = y_i \) \((i = 2, \ldots, k-1)\). So \( x_1 - y_1 = b - a \in \mathbb{N} \), \( x_i = y_i \) \((i = 2, \ldots, k-1)\), which contradicts the fact that \( I^k_{i-1} \) is an independent set. If

\[
x \in (a+j, I^k_{i-1} + ae_1^{k-1}) \quad \text{and} \quad y \in (b+j, I^k_{i-1} + be_1) \quad \text{with} \quad a \neq b, \ (a+j,b+j) \in E(G),
\]

then \( \{a,b\} \notin E(G) \) and \( b-a \notin \mathbb{N} \). Thus there is no edge between \( x \) and \( y \).

\[ \square \]

5.6 Miscellaneous

(5.6.1) Powers of the Generalized Butterfly

The generalized butterfly \( B_n \) is the graph which formed from two disjoint copies of the complete graph \( K_n \) by identifying a vertex of one \( K_n \) with a vertex of the other.
Obviously \(\text{cds}(B_n) = (2, 2, \ldots, 2, 1) = (1, \ldots, 1, 0) + (1, \ldots, 1)\). We have the following theorem.

**Theorem 5.6.1** Let \(B_n\) be the generalized butterfly. Then

1. \(\text{cds}(B_{2k}^n) = (1, 0, \ldots, 0) + \frac{(2n-1)^{2k-1}}{n} (1, \ldots, 1)\);
2. \(\text{cds}(B_{2k+1}^n) = (1, \ldots, 1, 0) + \frac{(2n-1)^{2k+1-(n-1)}}{n} (1, \ldots, 1)\);
3. \(\lim_{k \to \infty} n\text{cds}(B_n^k) = (-\frac{1}{n}, \ldots, -\frac{1}{n})\).

**Proof** It is routine work to check that

\(\text{cds}(B^2) = (1, 0, \ldots, 0) + \frac{(2n-1)^{2-1}}{n} (1, \ldots, 1)\).

Since \(\text{cds}(B^2)\) is an FD flat sequence, we obtain \(\text{cds}(B_{2k}^n)\) by Corollary 5.3.4 and \(\text{cds}(B_{2k+1}^n)\) by Theorem 5.3.2.

Let \(G\) be a circulant graph with seven vertices and symbol \(\{1, 2, 5, 6\}\). Then \(\text{cds}(G) = (2, 2, 2, 1)\) which is the same as \(\text{cds}(B_4)\). But the ncds sequences of \(G\) and \(B_4\) have different limit behaviour.

We see that

\[\lim_{k \to \infty} n\text{cds}(G^k) = (\frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{1}{7})\] but

\[\lim_{k \to \infty} n\text{cds}(B_4^k) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\).

5.6.2 Powers of the Product of the Butterfly \(B_3\) and Cycle \(C_5\)

**Theorem 5.6.2**

1. \(\text{cds}(B_3^{2k} \Box C_5^{2k}) = \text{cds}((B_3 \Box C_5)^{2k})\)
\[
\begin{align*}
&= 5^{2k-1}(2,2,1) + \frac{5^{2k}-1}{3} 5^{2k} (1,1,1); \\
\end{align*}
\]

(2) \( \lim_{k \to \infty} \text{ncds} \left( B^{k+k_1} \square C_5^{k+k_2} \right) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) for fixed \( k_1, k_2. \)

**Proof** \( \text{cds}(C_5^k) = 5^{k-1}(2, 2, 1) \) by Corollary 5.5.6,

\[ \text{cds}(B^{2k}_3) = (1, 0, 0) + \frac{5^{2k}-1}{3} (1, 1, 1) \] by Theorem 5.6.1, and

\[ \text{cds}(C_5^{2k}) = 5^{2k-1}(2, 2, 1). \]

So \( \text{cds}(B^{2k}_3 \square C_5^{2k}) = 5^{2k-1}(2, 2, 1) + \frac{5^{2k}-1}{3} 5^{2k} (1, 1, 1) \) by Theorem 5.3.2,

\[ \text{ncds}(B^{2k}_3 \square C_5^{2k}) = \frac{1}{54k} \left( \frac{5^{2k+5} \cdot 5^{2k}}{15}, \frac{5^{2k+5} \cdot 5^{4k}}{15}, \frac{5 \cdot 5^{4k-2} \cdot 5^{2k}}{15} \right), \]

\[ \lim_{k \to \infty} \text{ncds}((B_3 \square C_5)^{2k}) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \text{ and} \]

\[ \lim_{k \to \infty} \text{ncds}(B^{k+k_1} \square C_5^{k+k_2}) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \text{ for any fixed } k_1, k_2. \]

(The last result is from Non-increasing Theorem)

**Note** \( \text{cds}(B_3 \square C_5) \) is not even achievable, and

\[ \text{cds}(B_3 \square C_5) = (9, 9, 7). \] See Figure 5.7.

In the left hand side of Figure 5.7, the maximum 2-colorable subgraph has 18 vertices, but the three parts of 3-coloring of that graph can only have 9, 8 and 8 vertices, see the graph in the right hand side.
In Figure 5.8, \( \text{cds}(A) = (3, 1, 2), \text{ncds}(A) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) \),

\( \text{cds}(A^2) = (12, 12, 12), \text{ncds}(A^2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Hence

\( \text{ncds}(A^k) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) for \( k \geq 2 \)

by the Non-increasing Theorem. In fact, \( \text{cds}(A^2) \) is a special example of the following.

**Theorem 5.6.3** If \( |V(G)| = n \), \( k \) divides \( n \), the maximum size of an independent set of \( G \) is \( \frac{n}{k} \), and there exists a \( k \)-coloring of \( V(G) \):

\( V_1, \ldots, V_k \) such that \( (|V_1|, \ldots, |V_k|) = (\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}) \), then

\( \text{cds}(G) = (\frac{n}{k}, \ldots, \frac{n}{k}) \).
**Proof**  The proof follows by observing that \( V_1 \cup \ldots \cup V_t \) is a maximum \( t \)-colorable subgraph of \( G \) for \( t = 1, 2, \ldots, k \).

For all the graphs we studied we found that \( \text{ncds}(G^k) \) behaves well as \( k \) goes to infinity. It is either stable or its limit is a balanced sequence. Therefore we propose the following problem.

**Problem 2**  Prove that \( \lim_{k \to \infty} \text{ncds}(G^k) \) is equal to either \( \text{ncds}(G) \) or \( \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \).
Bibliography


