WREATH PRODUCTS AND VARIETIES OF INVERSE SEMIGROUPS

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WREATH PRODUCTS AND VARIETIES

OF INVERSE SEMIGROUPS

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ABSTRACT

Given two varieties $\mathcal{V}$ and $\mathcal{W}$ of inverse semigroups, define $\text{Wr}(\mathcal{V}, \mathcal{W})$ to be the variety of inverse semigroups generated by wreath products of semigroups in $\mathcal{V}$ with semigroups in $\mathcal{W}$. The principal result of this work is a description of the fully invariant congruence on the free inverse semigroup corresponding to $\text{Wr}(\mathcal{V}, \mathcal{W})$ in terms of the fully invariant congruences corresponding to $\mathcal{V}$ and $\mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are varieties of inverse semigroups. This description makes use of a graphical representation of inverse semigroups with presentations, due to Stephen, which is the inverse semigroup theoretic analogue to the Cayley graphs of group theory. We further show that $\text{Wr}$, considered as a binary operator on the lattice $\mathcal{L}(\mathcal{S})$ of varieties of inverse semigroups, is associative. Thus, the lattice of varieties of inverse semigroups is a semigroup $(\mathcal{L}(\mathcal{S}), \text{Wr})$ under the operation $\text{Wr}$.

Using these results we investigate properties possessed by varieties of the form $\text{Wr}(\mathcal{V}, \mathcal{W})$. We show that when $\mathcal{V}$ is a group variety, $\text{Wr}(\mathcal{V}, \mathcal{W})$ is the more familiar Mal'cev product variety $\mathcal{V} \circ \mathcal{W}$. The principal result also provides us with a solution to the word problem for the relatively free objects in $\text{Wr}(\mathcal{V}, \mathcal{W})$ given solutions to the word problem for the relatively free objects in the varieties $\mathcal{V}$ and $\mathcal{W}$. We show that when the varieties $\mathcal{V}$ and $\mathcal{W}$ have E-unitary covers over the group varieties $\mathcal{V}$ and $\mathcal{W}$, respectively, then $\text{Wr}(\mathcal{V}, \mathcal{W})$ has E-unitary covers over the group variety $\text{Wr}(\mathcal{V}, \mathcal{W})$. Further properties of varieties of this form are presented as well as a discussion of the basic properties of the semigroup $(\mathcal{L}(\mathcal{S}), \text{Wr})$. We conclude this work by showing that several special intervals in $\mathcal{L}(\mathcal{S})$ corresponding to v-classes and whose maximum member is of the form $\text{Wr}(\mathcal{V}, \mathcal{S}^1)$ are infinite, where $\mathcal{V}$ is a variety of abelian groups and $\mathcal{S}^1$ is the variety of inverse semigroups generated by the five element Brandt semigroup with an identity adjoined.

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CHAPTER ONE
Introduction

An inverse semigroup $S$ is a set with an associative binary operation, usually referred to as multiplication, and a unary operation of inversion which satisfies the property that every element of $S$ has a unique inverse in the sense of von Neumann. V.V. Wagner in 1952 was the first to study inverse semigroups, though he called them 'generalized groups' and defined them as regular semigroups in which the idempotents commute. In 1953 Liber proved that the two definitions are in fact equivalent. Preston later (and independently) rediscovered this class of semigroups and called them 'inverse semigroups', the name most widely used today.

Every inverse semigroup is isomorphic to a semigroup of partial one-to-one transformations on a nonempty set. This is the substance of the Wagner Representation Theorem which is the inverse semigroup theoric analogue to the Cayley Representation Theorem of group theory. Indeed, the Wagner representation of a group is the Cayley representation of that group. Thus, just as we often find it convenient to think of groups as permutations, we often think of inverse semigroups as semigroups of partial one-to-one transformations.

Given two inverse semigroups, a new inverse semigroup can be obtained by forming their wreath product. By the wreath product $S \text{wr} (T,I)$ of $S$ and $T$, where $T$ is a semigroup of partial one-to-one transformations on the set $I$, we mean the set of pairs $(\psi,\beta)$ where $\beta \in T$, $\psi$ is a mapping from $I$ into $S$ and the domains of $\psi$ and $\beta$ are equal, with products defined by

$$(\psi_1,\beta_1)(\psi_2,\beta_2) = (\psi_1 \beta_1 \psi_2, \beta_1 \beta_2) \quad ( (\psi_1,\beta_1),(\psi_2,\beta_2) \in S \text{wr} (T,I) )$$

where, for all $i$ in the domain of $\beta_1 \beta_2$, $i \psi_1 \beta_1 \psi_2 = (i\psi_1)(i\beta_1 \psi_2)$. Note that, given any two inverse semigroups $S$ and $T$, we can always form the wreath product of $S$ and $T$ by taking
the Wagner representation of $T$. Wreath products are of fundamental importance in the study of inverse semigroups and play a central role in our investigations.

Inverse semigroups are determined by associativity and the laws $xx^{-1}x = x$, $(x^{-1})^{-1} = x$ and $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Thus, the class of inverse semigroups (considered as algebras of type $(2,1)$) forms a variety and we may approach the study of inverse semigroups from the perspective of their lattice of varieties. This approach not only suggests a possible framework from which we may tackle classification problems, but it has proved itself essential in the study of the structure of inverse semigroups by the identities they satisfy.

The focus of our investigations is a binary operator $Wr$ on the lattice of varieties of inverse semigroups. Given two varieties $\mathcal{V}$ and $\mathcal{Y}$ of inverse semigroups, we define $Wr(\mathcal{V},\mathcal{Y})$ to be the variety generated by wreath products of members of $\mathcal{V}$ with members of $\mathcal{Y}$ which are represented as partial one-to-one transformations on some nonempty set. The principal result of this thesis is a characterization of the fully invariant congruence on the free inverse semigroup corresponding to $Wr(\mathcal{V},\mathcal{Y})$ in terms of the fully invariant congruences corresponding to $\mathcal{V}$ and $\mathcal{Y}$, where $\mathcal{V}$ and $\mathcal{Y}$ are varieties of inverse semigroups. Our motivation for studying this class of varieties is essentially twofold. First of all, every completely semisimple inverse semigroup is a subdirect product of inverse subsemigroups of wreath products of the form $G \wr \mathcal{I}(I)$, where $G$ is a group and $\mathcal{I}(I)$ is the inverse semigroup of all partial one-to-one transformations on the nonempty set $I$. Thus, every variety whose free objects are completely semisimple (and they are many) is generated by inverse subsemigroups of wreath products. Secondly, the relation $\nu$ defined on the lattice of varieties of inverse semigroups by $\mathcal{V} \vee \mathcal{Y}$ if and only if $\mathcal{V} \cap \mathcal{G} = \mathcal{Y} \cap \mathcal{G}$ and $\mathcal{V} \vee \mathcal{G} = \mathcal{Y} \vee \mathcal{G}$, where $\mathcal{G}$ is the variety of all groups, is a congruence $[K1]$ and, by a result due to Reilly [Re1], if $\mathcal{Y}$ is a combinatorial variety, the $\nu$-class of $\mathcal{Y} \vee \mathcal{V}$, for some variety of groups $\mathcal{V}$, is the interval $[\mathcal{V} \vee \mathcal{Y}, \mathcal{Y} \circ \mathcal{Y}]$, where $\mathcal{Y} \circ \mathcal{Y}$ is
the Mal'cev product of \( \mathcal{Z} \) and \( \mathcal{V} \). It turns out that whenever \( \mathcal{Z} \) is a variety of groups, 
\[ \text{Wr}(\mathcal{Z}, \mathcal{V}) = \mathcal{Z} \circ \mathcal{V}, \]
and so a description of the fully invariant congruence corresponding to 
\[ \text{Wr}(\mathcal{Z}, \mathcal{V}) \] is of some interest.

There is a connection between these two motivating factors and this connection forms the basis for our principal result, which is generalized beyond the specific classes of varieties mentioned above. The first factor is closely related to representations by right translations, which we must 'decode' in order to determine the laws of the varieties mentioned in the second factor. This 'decoding' is made possible by yet another representation of inverse semigroups, this time as directed inverse word graphs over some label set \( X \). This representation, due to Stephen [S], is called the Schützenberger representation and is the inverse semigroup analogue to the Cayley graphs of group theory.

Unlike the group case, in which there is one underlying graph representing a group with respect to some presentation, an inverse semigroup (with presentation) has one underlying graph for each \( \mathcal{D} \)-class. When considering whether the variety \( \text{Wr}(\mathcal{Z}, \mathcal{V}) \) satisfies the identity \( u = v \), where \( u \) and \( v \) are words over some alphabet \( X \), we look at, first of all, whether the variety \( \mathcal{V} \) satisfies \( u = v \) and if so, whether \( \mathcal{Z} \) satisfies an identity determined by the paths labelled by \( u \) and \( v \) in the Schützenberger representation of \( u \) and \( v \) relative to the presentation \( (X; p(\mathcal{V})) \), where \( p(\mathcal{V}) \) is the fully invariant congruence on the free inverse semigroup corresponding to \( \mathcal{V} \).

It turns out that \( \text{Wr} \) is an associative operator on the lattice of varieties of inverse semigroups which, when restricted to the lattice of varieties of groups, is the well-known product operator. While the lattice of varieties of groups under \( \text{Wr} \) is freely generated by its indecomposable members, the same cannot be said for the lattice of varieties of inverse semigroups. We can however, use our description of the fully invariant congruence corresponding to \( \text{Wr}(\mathcal{Z}, \mathcal{V}) \) to discover some interesting results concerning familiar classes.
of varieties, including Mal'cev products of the form \( \mathcal{Z} \circ \mathcal{Y} \) where \( \mathcal{Z} \) is a variety of groups and varieties whose free objects are E-unitary and their subvarieties.

The following is a brief outline of each chapter of this thesis.

Chapter 2 is devoted to preliminary material required in the sequel including fundamental results and definitions of inverse semigroup theory as well as the basics on the Wagner representation, the Translational Hull of an inverse semigroup, Varieties of inverse semigroups and Schützenberger graphs.

Since wreath products figure prominently in our investigations, Chapter 3 is concerned with the basic results we will require in subsequent chapters on this subject. The first section of this chapter deals with the definition of wreath product. Section 2 deals with showing that completely semisimple inverse semigroups are isomorphic to subdirect products of inverse subsemigroups of semigroups of the form \( G \ wr (T,I) \) where \( G \) is a group and \( T \) is an antigroup. The final section of chapter 3 presents some basic facts concerning wreath products of inverse semigroups.

In Chapter 4 we present our Main Theorem which characterizes the fully invariant congruence on the free inverse semigroup corresponding to \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \) in terms of the fully invariant congruences corresponding to \( \mathcal{Z} \) and \( \mathcal{Y} \), for varieties \( \mathcal{Z} \) and \( \mathcal{Y} \) of inverse semigroups. The first section introduces the notion of the doubly labelled Schützenberger graph and from it we define, for any given word \( w \) and variety \( \mathcal{Y} \), an associated word dependent upon \( \mathcal{Y} \) called the derived word of \( w \) with respect to \( \mathcal{Y} \). The derived word of \( w \) with respect to \( \mathcal{Y} \) is an 'encoding' of the path labelled by \( w \) in the Schützenberger graph of \( w \) with respect to \( \mathcal{Y} \). We use this encoding in our Main Theorem, which is the subject of Section 2. Section 3 is concerned with basic properties of varieties of the form \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \), including the result that, when \( \mathcal{Z} \) is a group variety, \( \text{Wr}(\mathcal{Z},\mathcal{Y}) = \mathcal{Z} \circ \mathcal{Y} \), the Mal'cev product of \( \mathcal{Z} \) and \( \mathcal{Y} \). The last section of this chapter deals with the associativity of the \( \text{Wr} \) operator.
Consequences of the Main Theorem are presented in Chapter 5. In section 1 we show that the \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \)-free semigroups have solvable word problem if both the \( \mathcal{Z} \)-free and the \( \mathcal{Y} \)-free semigroups have solvable word problem and also that \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \) is locally finite if and only if both \( \mathcal{Z} \) and \( \mathcal{Y} \) are locally finite. Section 2 contains results concerning E-unitary covers which utilize a graphical description, due to Meakin and Margolis, of varieties of the form \( \mathcal{Z}^{\text{max}} = [w = w^2 : w = w^2 \text{ is a law in } \mathcal{Z}] \), where \( \mathcal{Z} \) is a variety of groups. The third section is devoted to results concerning varieties of the form \( \text{Wr}(\mathcal{S},\mathcal{Y}) \). It turns out that \( \text{Wr}(\mathcal{S},\mathcal{Y}) \) is the largest variety satisfying those identities \( w = w^2 \) that are satisfied by \( \mathcal{Y} \). This chapter concludes with some basic results concerning the semigroup of varieties of inverse semigroups under the operation of \( \text{Wr} \).

In the final chapter we look at the intervals \([\mathcal{Z} \circ \mathcal{B}^1, \mathcal{Z} \circ \mathcal{B}^1]\) where \( \mathcal{Z} \) is a variety of abelian groups and \( \mathcal{B}^1 \) is the variety generated by a special six-element inverse semigroup (the five-element Brandt semigroup with an identity adjoined). For each of these intervals, we construct an infinite chain of varieties using only a minimal knowledge of the relatively free object on a countably infinite set in the variety \( \mathcal{B}^1 \).
CHAPTER TWO

Preliminaries

The fundamental definitions and results of inverse semigroup theory which are required in the sequel are presented in this chapter. The principle source used is Inverse Semigroups by Mario Petrich [P]. For the fundamentals of semigroup theory, the reader is referred to Clifford and Preston [CP]. The material on Schützenberger graphs comes from Stephen [S]. For basic universal algebraic results concerning varieties, we refer the reader to either Burris and Sankapanavar [BS] or Grätzer [G]. It is assumed that the reader is familiar with the notion of a lattice and the basic definitions and results concerning lattices. A standard text on this subject is Birkhoff's Lattice Theory [Bi1]. Most of the results of sections 2.3 through 2.7 can be found in [P]. We will cite the reference to [P] when the result is stated and provide the original source in the final paragraphs of these sections.

2.1 Semigroups

A semigroup is a pair $(S, \cdot)$ where $S$ is a set and $\cdot$ is an associative binary operation, usually referred to as multiplication. Unless there is the possibility of ambiguity, we denote the semigroup $(S, \cdot)$ by $S$ and denote products in $S$ by juxtaposition. A familiar example of a semigroup is the set of functions on a nonempty set $X$ under the operation of composition.

A semigroup may possess special elements which are distinguished by certain characteristics. Let $S$ be a semigroup.

An element $s \in S$ is an identity if $sx = xs = x$, for all $x \in S$. If $S$ possesses an identity then it is unique and is denoted by $1$ or $1_S$ if we wish to emphasize that it is the identity of $S$. A semigroup which has an identity is called a monoid. Given an arbitrary
A semigroup $S$, we define $S^1$ to be $S$ if $S$ is a monoid or $(S \cup \{1\}, \cdot )$ with $1 \cdot x = x \cdot 1 = x$, for all $x \in S$, if $S$ is not a monoid. It is easy to see that $S^1$ is a monoid.

An element $s \in S$ is a zero if $sx = xs = s$, for all $x \in S$. If $S$ possesses a zero then it is unique and is denoted by 0 or $0_S$ if we wish to emphasize that it is the zero of $S$. The semigroup $S^0$ is defined to be $S$, if $S$ possesses a zero, or $(S \cup \{0\}, \cdot )$ with $0 \cdot x = x \cdot 0 = 0$, for all $x \in S$, otherwise. If $T$ is a subset of $S$, but not a semigroup with 0, and $T$ satisfies the property that, for any $a,b,c \in T$, $ab$ and $(ab)c$ are elements of $T$ if and only if $bc$ and $a(bc)$ are elements of $T$, then we define $T^0$ to be the set $T \cup \{0\}$ with multiplication given by $t_1 \cdot t_2 = t_1 t_2$ if $t_1 t_2 \in T$, $t_1 \cdot t_2 = 0$, otherwise and $0 \cdot t_1 = t_1 \cdot 0 = 0$, for all $t_1, t_2 \in T$. One easily verifies that $T^0$ is a semigroup.

An element $e \in S$ is an idempotent if $e = e^2$. The set of idempotents of $S$ is denoted $E_S$. The relation $\leq$ on $E_S$ defined by $e \leq f$ if and only if $e = ef = fe$, for all $e,f \in E_S$, is a partial order and is called the natural partial order of $E_S$. If $S$ has no zero, an idempotent $e \in E_S$ is primitive if it is minimal in the natural partial order of $E_S$. If $S$ has a zero, $e \in E_S$ is primitive if it is minimal in $E_S \setminus \{0\}$.

### 2.2 Inverse semigroups

Let $S$ be a semigroup. An element $s \in S$ is regular if there exists an $x \in S$ such that $s = sx s$. The semigroup $S$ is said to be regular if all its elements are regular. The element $x$ is an inverse of $s$ if $s = sx s$ and $x = xs x$.

A regular semigroup whose idempotents commute is an inverse semigroup. An equivalent definition of inverse semigroup is a semigroup in which each element has a unique inverse [P;II.1.2]. The former definition is due to Wagner [Wa1] who was the first to study inverse semigroups, though he called them 'generalized groups'. The latter definition is due to Liber who, in [L], showed that the two definitions are equivalent. An
An inverse semigroup which is also a monoid is called an *inverse monoid*. For any element \( s \) in an inverse semigroup \( S \), we denote the unique inverse of \( s \) by \( s^{-1} \).

The set of partial one-to-one transformations on a nonempty set \( X \) under the operation of composition is an important example of inverse semigroups. This semigroup is called the *symmetric inverse semigroup on \( X \)* and is denoted by \( \mathcal{S}(X) \). It is easy to verify that if \( S \) is an inverse semigroup then both \( S^1 \) and \( S^0 \) are inverse semigroups. Moreover, if \( T \) is a subset of \( S \) such that \( t \in T \) implies that \( t^{-1}, tt^{-1} \in T \), and \( T \) satisfies the property mentioned in the definition of \( T^0 \), then \( T^0 \) is an inverse semigroup.

### 2.3 Fundamentals

Throughout this section \( S \) is an inverse semigroup.

Inverse semigroups are partially ordered algebras. Define the relation \( \leq \) on \( S \) by

\[
s \leq t \iff s = et \text{ for some } e \in E_S \quad (s,t \in S).
\]

It is a simple task to verify that \( \leq \) is a partial order on \( S \). The relation \( \leq \) is called the *natural partial order* on \( S \). The following are equivalent characterizations of \( \leq \) (See [P;II.1.6]):

\[
s \leq t \iff s = te \text{ for some } e \in E_S
\]

\[
\iff s = ss^{-1}t
\]

\[
\iff s = ts^{-1}s
\]

(\( s,t \in S \)).

Observe that the natural partial order on \( S \) restricted to \( E_S \) coincides with the natural partial order on \( E_S \) defined in the previous section.

Let \( S \) be an inverse semigroup. A subset \( T \) of \( S \) is an *inverse subsemigroup* of \( S \) if \( T \) is closed under the operations of \( S \); that is, for all \( t_1, t_2 \in T \), \( t_1 t_2 \in T \) and \( t_1^{-1} \in T \). It is not true in general that a subsemigroup of an inverse semigroup is an inverse semigroup. An example which illustrates this is \( T = \{ (1 \rightarrow 2), \emptyset \} \), where \( (1 \rightarrow 2) \) is the member of \( \mathcal{S}([1,2]) \) with domain \( \{1\} \) which maps 1 to 2. \( T \) is a subsemigroup, but not an inverse subsemigroup, of \( \mathcal{S}([1,2]) \). If \( S \) is a monoid and \( T \) is a subsemigroup of \( S \) such that
1 \in T, then T is an inverse submonoid of S. If K is a subset of S then the inverse subsemigroup of S generated by K is the intersection of all subsemigroups of S containing K. We say that the inverse subsemigroup T of S is full if $E \subseteq T$, and we say that T is closed if, for all $x \in T$, $y \in S$, $x \leq y$ implies that $y \in T$. The closure of T in S, denoted $\text{To}$, is the set $\{ s \in S : s \geq t \text{ for some } t \in T \}$. If $\text{To} = T$ then we say that T is closed.

A nonempty subset I of S is a right ideal if
\[ IS = \{ ts : s \in S, t \in I \} \subseteq I. \]
A nonempty subset I of S is a left ideal if
\[ SI = \{ st : s \in S, t \in I \} \subseteq I. \]
A subset I of S is a (two-sided) ideal of S if it is both a right ideal and a left ideal. Equivalently, I is an ideal of S if
\[ SIS = \{ s_1ts_2 : s_1,s_2 \in S, t \in I \}. \]
For any element $s \in S$, the principal right ideal generated by $s$ is the intersection of all right ideals containing $s$ and is denoted by $R(s)$. The principal left ideal generated by $s$ and the principal ideal generated by $s$ are defined similarly and are denoted by $L(s)$ and $J(s)$, respectively. It is not difficult to show that $R(s) = sS$, $L(s) = Ss$ and $J(s) = SsS$.

An inverse semigroup is simple if it has no proper ideals. If S has a zero, then S is 0-simple if $S^2 \neq 0$ and S has no proper nonzero ideals. A simple inverse semigroup possessing a primitive idempotent is called a completely simple inverse semigroup and likewise, a 0-simple inverse semigroup possessing a primitive idempotent is called a completely 0-simple inverse semigroup. The intersection, if nonempty, of all ideals of S is called the kernel of S. Note that the kernel of S, if it exists, is a simple semigroup.

The relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on S, called Green's relations, are of fundamental importance and are defined as follows. For all $s,t \in S$,
\[ s \mathcal{R} t \iff R(s) = R(t); \]
\[ s \mathcal{L} t \iff L(s) = L(t); \]
Clearly, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{H}$ are equivalence relations. Furthermore, it can be shown that $\mathcal{D}$ is an equivalence relation which can equivalently be defined by $s \mathcal{D} t$ if and only if there exists an $x \in S$ such that $s \mathcal{L} x$ and $x \mathcal{R} t$.

For any $x \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}\}$, define the $x$-class of $s \in S$ by $K_s = \{x \in S : s x \}$. For $x \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$ there is a partial order on the $x$-classes of $S$ given by $K_s \leq K_t$ if and only if $K(s) \subseteq K(t)$.

The following is a list of basic results concerning Green's relations in inverse semigroups.

**Lemma 2.3.1.** Let $S$ be an inverse semigroup.

a) Every $\mathcal{R}$-class and every $\mathcal{L}$-class of $S$ contains exactly one idempotent [P;II.1.2];

b) If $e$ is an idempotent of $S$ then $H_e$ is a maximal subgroup of $S$ and conversely, if $G$ is a maximal subgroup of $S$ then $G$ is an $\mathcal{H}$-class of $S$ [P;I.7.5,I.7.6];

c) For any $s, t \in S$,

\[
\begin{align*}
\mathcal{R} & \quad s \mathcal{R} t \iff ss^{-1} = tt^{-1}, \\
\mathcal{L} & \quad s \mathcal{L} t \iff s^{-1}s = t^{-1}t, \\
\mathcal{D} & \quad s \mathcal{D} t \iff \text{there exists an } x \in S \text{ such that } ss^{-1} = xx^{-1} \text{ and } x^{-1}x = t^{-1}t;
\end{align*}
\]

d) For any $e, f \in E_S$, $J(e) \subseteq J(f)$ if and only if $e = aa^{-1}$ and $a^{-1}a \leq f$, for some $a \in S$.

(both (c) and (d) are from [P;II.1.7])

In fact, property a) is an equivalent definition of an inverse semigroup.

A homomorphism from $S$ into a semigroup $T$ is a function $\phi$ such that, for all $s, t \in S$, $(s \phi)(t \phi) = st \phi$. Any homomorphic image of an inverse semigroup is necessarily
an inverse semigroup [P; II.1.10]. Furthermore, by the definition of inverse semigroup, homomorphisms preserve inverses. That is, if \( \phi \) is a homomorphism of \( S \) into \( T \) and \( s \in S \), then \( s^{-1}\phi = (s\phi)^{-1} \).

A congruence \( \rho \) on \( S \) is an equivalence relation satisfying the property that, for all \( s,t,x \in S \), \( s \rho t \) implies that \( xs \rho xt \) and \( sx \rho tx \). If \( \rho \) is a congruence on \( S \) then \( S / \rho \) is an inverse semigroup with multiplication given by \( (sp)(tp) = stp \). \( S / \rho \) is called the \textit{quotient semigroup} induced by \( \rho \). We denote by \( \omega \) and \( \varepsilon \) the universal relation on \( S \) and the identical relation on \( S \), respectively. The set of all congruences on an inverse semigroup \( S \) forms a complete lattice under inclusion with greatest element \( \omega \) and least element \( \varepsilon \).

There is a strong connection between congruences and homomorphisms. Given a homomorphism \( \phi : S \rightarrow T \), there is an associated congruence \( \phi^* \) on \( S \) defined by \( s \phi^* t \) if and only if \( s\phi = t\phi \), for all \( s,t \in S \). Conversely, given a congruence \( \rho \) on \( S \), there is an associated homomorphism \( \rho^# : S \rightarrow S / \rho \) given by \( sp^# = sp \), for all \( s \in S \).

Because congruences (and hence homomorphisms) play such an important role in our investigations, we present here some basic facts concerning congruences and list some special types.

Any congruence on an inverse semigroup \( S \) is uniquely determined by the union of its classes which contain idempotents and by its restriction to \( E_S \). Let \( \rho \) be a congruence on \( S \). Define the \textit{trace} and \textit{kernel} of \( \rho \) by

\[
\text{tr} \ \rho = \rho \cap (E_S \times E_S) \\
\ker \rho = \{ s \in S : s \rho e \text{ for some } e \in E_S \},
\]

respectively. \( \rho \) is the unique congruence on \( S \) with trace equal to \( \text{tr} \ \rho \) and kernel equal to \( \ker \rho \) [P;III.1.5]. If we think of \( \text{tr} \) as a mapping from the lattice of congruences on \( S \) into the lattice of congruences on \( E_S \), then \( \text{tr} \) is a complete lattice homomorphism [P;III.2.5].
Likewise, ker, considered as a mapping from the lattice of congruences on \( S \) into the lattice of kernels (of congruences) of \( S \), is a complete \( \cap \)-homomorphism [P;III.4.8].

For any congruences \( \rho \) and \( \tau \) on \( S \) such that \( \rho \subseteq \tau \), define the relation \( \tau / \rho \) on \( S / \rho \) by \((xp)(\tau / \rho)(yp)\) if and only if \( x \tau y \). Then \( \tau / \rho \) is a congruence on \( S / \rho \) and \((S / \rho) / (\tau / \rho) \equiv S / \tau \), [P;I.4.15].

Let \( I \) be an ideal of \( S \). Then the relation \( \rho_I \) on \( S \) defined by
\[
\text{for } s, t \in I \text{ or } s = t \\
\text{is a congruence and is called the Rees congruence on } S \text{ relative to } I. \text{ The quotient semigroup } S / \rho_I \text{ induced by } \rho_I \text{ is called the Rees quotient semigroup (See [P;I.5.3]).}
\]

A congruence \( \rho \) on \( S \) is idempotent separating if, for any \( e, f \in E_S \), \( e \rho f \) implies that \( e = f \). Thus, \( \rho \) is idempotent separating if and only if \( \text{tr} \rho = \varepsilon \), the identical relation. Equivalently, \( \rho \) is idempotent separating if and only if \( \rho \subseteq \mathcal{R} \) [P;III.3.2]. We denote by \( \mu_S \) the greatest idempotent separating congruence on \( S \). That \( \mu_S \) exists is guaranteed by the fact that it is characterized by being the greatest congruence on \( S \) contained in \( \mathcal{R} \). A further characterization is given by
\[
s \mu_S t \iff s^{-1}es = t^{-1}et \text{ for all } e \in E_S \\
\text{ (} s,t \in S \text{ ).}
\]

A congruence \( \rho \) on \( S \) is idempotent pure if \( E_S \) is the union of \( \rho \)-classes. That is, \( \rho \) is idempotent pure if for all \( s \in S, e \in E_S \), \( s \rho e \) implies that \( s \in E_S \). Thus \( \rho \) is idempotent pure if and only if \( \text{ker} \rho = E_S \). A useful characterization is \( \rho \) is idempotent pure if and only if \( \rho \cap \mathcal{R} = \varepsilon \), [P;III.4.2].

A congruence \( \rho \) on \( S \) is a group congruence if \( S / \rho \) is a group. The least group congruence on \( S \), denoted \( \sigma_S \), is given by
\[
s \sigma_S t \iff se = te \text{ for some } e \in E_S \\
\text{ (} s,t \in S \text{ ) [P;III.5.2].}
\]

The last concept which we introduce in this section is that of direct product. If \( \{ S_i \}_{i \in I} \) is a family of inverse semigroups, their direct product is the inverse semigroup
with underlying set the Cartesian product $\prod_{i \in I} S_i$ and coordinatewise multiplication. If, for all $i \in I$, $S_i = S$, then we write $S^I$, and call this direct product the **direct power** of $S$ by $I$.

An inverse semigroup $S$ is a **subdirect product** of an indexed family $\{S_i\}_{i \in I}$ of inverse semigroups if

i) $S$ is an inverse subsemigroup of $\prod_{i \in I} S_i$;

ii) $(S)\pi_i = S_i$ for each $i \in I$ where $\pi_i$ is the $i^{th}$ projection map.

An embedding $\alpha : S \to \prod_{i \in I} S_i$ is a **subdirect embedding** if $(S)\alpha$ is a subdirect product of the $S_i$.

Green's relations are named for J.A. Green who introduced them in 1951 [GrJ]. The natural partial order on inverse semigroups was introduced by Wagner in [Wa1]. He was also the first to show that a congruence on an inverse semigroup is completely determined by its classes containing idempotents [Wa2]. The kernel-trace approach to the study of congruences on an inverse semigroup is due to Scheiblich [Sc]. This approach differs from the traditional 'kernel normal system' approach which we do not use here. That $\ker$ is a homomorphism was proved by Reilly-Scheiblich [RS] and D.G. Green showed that $\ker$ is a $\rho$-homomorphism in [GrD]. Munn [Mu2] showed that idempotent separating congruences are contained in $\mathcal{H}$ and Howie [Ho] proved the existence of $\mu$, the greatest idempotent separating congruence. The characterizations of $\sigma$ are due to Munn [Mu1] and Wagner [Wa2].

2.4 Special Classes

There are several important classes of inverse semigroups which we find necessary to distinguish. The following is a list of those classes which figure prominently in our investigations.
2.4.1. Groups. It is immediate from the definition of an inverse semigroup that all groups are inverse semigroups. Furthermore, the class of completely simple inverse semigroups coincides with the class of groups. We denote the class of all groups by $\mathcal{G}$.

2.4.2. Semilattices and Clifford semigroups. A semilattice is an inverse semigroup in which every element is an idempotent. Such a semigroup is called a semilattice because under the natural partial order it forms a meet semilattice. Moreover, any meet semilattice $Y$ is a semilattice under the operation given by $e \cdot f = e \wedge f$, for all $e, f \in Y$. Note that, for any inverse semigroup $S$, $E_S$ is a semilattice.

An Clifford semigroup is an inverse semigroup which is a semilattice of groups. That is, the inverse semigroup $S$ is a Clifford semigroup if there is a congruence $\rho$ on $S$ such that $S/\rho$ is a semilattice and each of the $\rho$-classes is a group.

2.4.3. Brandt semigroups. A completely 0-simple inverse semigroup is a Brandt semigroup.

Let $G$ be a group and $I$ a nonempty set. Let $B(G,I) = I \times G \times I \cup \{0\}$, where $0 \notin I \times G \times I$, with multiplication $(i, g, j) \cdot (j, h, k) = (i, gh, k)$ and all other products equal to 0. It is a simple task to verify that with this multiplication $B(G,I)$ is an inverse semigroup. In fact, an inverse semigroup $S$ is a Brandt semigroup if and only if $S$ is isomorphic to $B(G,I)$ for some group $G$ and nonempty set $I$ [P;II.3.5]. The 'smallest' Brandt semigroup which is not a semilattice of groups is isomorphic to $B(G,I)$ for $G = \{1\}$ and $|I| = 2$, and is denoted by $B_2$. We sometimes refer to $B_2$ as the five-element Brandt semigroup.

An inverse semigroup which is a subdirect product of Brandt semigroups and/or groups is called a strict inverse semigroup. A property that characterizes strict inverse semigroups is $\mathcal{D}$-majorization: For any $e, f, g \in E_S$, $e \geq f, e \geq g, f \mathcal{D} g$ imply $f = g$.
Note that, in particular, if \( e \) and \( f \) are two comparable idempotents belonging to the same \( D \)-class then \( e = f \).

### 2.4.4. Completely semisimple inverse semigroups.

Let \( S \) be an inverse semigroup. For every \( a \in S \), define \( I(a) = J(a) \setminus J_a = \{ s \in J(a) : J(s) \neq J(a) \} \). Whenever \( I(a) \neq \emptyset \), \( I(a) \) is an ideal of \( S \). The Rees quotient semigroup \( J(a) / I(a) \), where \( J(a) / \emptyset = J(a) \), is called a principal factor of \( S \). A semigroup in which every principal factor is completely simple or completely 0-simple is a completely semisimple semigroup. Thus, an inverse semigroup is completely semisimple if and only if all of its principal factors are Brandt semigroups or groups. Indeed, at most one principal factor of a completely semisimple inverse semigroup can be a group and that is the kernel, if it exists. Note that in a completely semisimple inverse semigroup \( D = F \).

### 2.4.5. Combinatorial inverse semigroups.

An inverse semigroup is combinatorial if the Green's relation \( K \) is the identical relation. That is, an inverse semigroup is combinatorial if its maximal subgroups are trivial.

### 2.4.6. Cryptic inverse semigroups.

An inverse semigroup is cryptic if the Green's relation \( \beta \) is a congruence.

### 2.4.7. Antigroups.

An inverse semigroup \( S \) is an antigroup if \( \epsilon \) is the only congruence on \( S \) contained in \( K \). Equivalently, \( S \) is an antigroup if and only if \( \mu_S = \epsilon \). Note that all combinatorial inverse semigroups are antigroups and a cryptic inverse semigroup is an antigroup if and only if it is combinatorial. We denote the class of all antigroups by \( A \).

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2.4.8. **E-untary inverse semigroups.** An inverse semigroup $S$ is E-unitary if and only if, for all $a \in S$, $e \in E_S$, $a \geq e$ implies that $a \in E_S$. Equivalently, $S$ is E-unitary if and only if $\sigma_S$, the least group congruence on $S$, is idempotent pure.

Semilattices of groups were introduced by Clifford in [Cl]. In the same paper, Clifford also showed that Brandt semigroups are isomorphic to $B(G,I)$ for some group $G$ and some nonempty set $I$, though he was considering Brandt groupoids (first studied by H. Brandt in 1927) with a zero adjoined with all undefined products set to zero. Munn $[\text{Mu2}]$, was the first to recognize that Brandt semigroups (that is, Brandt groupoids with a zero adjoined and all undefined products set to zero) were precisely the inverse completely 0-simple semigroups. What we call antigroups was introduced by Wagner, though another terminology, 'fundamental inverse semigroup', was coined by Munn. E-unitary inverse semigroups were first studied by Saitô [Sa] who called them proper and later by, among others, McAlister [McA1], who called them reduced inverse semigroups. A great deal of research has concerned itself with E-unitary inverse semigroups; we mention only the work done by McAlister on the P-representation of E-unitary inverse semigroups $[\text{McA1,McA2}]$.

2.5 **The Wagner representation**

A fundamental result in group theory is the Cayley representation theorem which states that every group is isomorphic to a permutation group. The analogous result in the theory of inverse semigroups is the Wagner representation of an inverse semigroup by partial one-one transformations of a set. Every inverse semigroup is isomorphic to an inverse semigroup of one-one partial transformations on a nonempty set.

For any $\beta \in \mathcal{S}(X)$, the symmetric inverse semigroup on $X$, we denote by $d\beta$ and $r\beta$ the domain of $\beta$ and the range of $\beta$, respectively.
Theorem 2.5.1 [P;IV.1.6]. Let $S$ be an inverse semigroup. For each $s \in S$, let $\beta_s \in \mathcal{I}(S)$ be defined by
\[ x\beta_s = xs \quad [x \in d\beta_s = Ss^{-1}]. \]
Then the mapping
\[ \beta : S \to \mathcal{I}(S) \text{ defined by } s\beta = \beta_s \]
is an embedding of $S$ into $\mathcal{I}(S)$.

The Wagner representation of $S$ restricted to a given $\mathcal{R}$-class provides another representation of $S$, though in general it is not faithful (that is, not one-to-one). For a given $\mathcal{R}$-class $R$ of $S$, we call the following representation the **Wagner representation of $S$ restricted to $R$**.

Theorem 2.5.2. Let $S$ be an inverse semigroup and let $R$ be a fixed $\mathcal{R}$-class of $S$. For each $s \in S$, let $\alpha_s \in \mathcal{I}(R)$ be defined by
\[ x\alpha_s = xs \quad [x \in d\alpha_s = \{ y \in R : ys \in R \}]. \]
Then the mapping
\[ \alpha : S \to \mathcal{I}(R) \text{ defined by } s\alpha = \alpha_s \]
is a homomorphism.

**Proof:** Let $s \in S$ and suppose that for some $x, y \in d\alpha_s = \{ y \in R : ys \in R \}$, $x\alpha_s = y\alpha_s$. Then $x = xs$ and $y \alpha_s = y\alpha_s$. Therefore, $x = xx^{-1}x = (xs)(xs)^{-1}x = xss^{-1} = yss^{-1} = yss^{-1}y^{-1}y = y$. Therefore, $\alpha_s$ is indeed an element of $\mathcal{I}(R)$.

Let $s, t \in S$. In order to show that $\alpha$ is a homomorphism we must show that $\alpha_s\alpha_t = \alpha_{st}$. We first compare their domains.
\[
\begin{align*}
\text{d} \alpha_s &= \{ y \in R : ys \in R \} \\
\text{d} \alpha_t &= \{ y \in R : yt \in R \} \\
\text{d} \alpha_{st} &= \{ y \in R : yst \in R \}.
\end{align*}
\]

Therefore, \( \text{d} \alpha_s \alpha_t = \{ y \in R : ys \in R \text{ and } yst \in R \} \) and this is a subset of \( \text{d} \alpha_{st} \). On the other hand, if \( y \in \text{d} \alpha_{st} \), then \( y \) and \( yst \) are \( \mathcal{R} \)-related and so there is some \( z \in S \) such that \( ystz = y \). But then \( y \) and \( yst \) are \( \mathcal{R} \)-related and so \( y \in \text{d} \alpha_s \alpha_t \). Thus, \( \alpha_s \alpha_t \) and \( \alpha_{st} \) have identical domains. Since \( (xs)t = x(st) \) for all \( x \) in their common domain, \( \alpha_s \alpha_t = \alpha_{st} \). As a result, \( \alpha \) is a homomorphism.

The Wagner representation is due to Wagner [Wa1] and was discovered independently by Preston [Pr].

2.6 The translational hull of an inverse semigroup

Though it plays a minor role in our investigations, the translational hull of an inverse semigroup has strong connections with the Wagner representation and the Schützenberger representations (discussed below in §2.8), both of which figure prominently in the sequel.

Let \( S \) be an inverse semigroup. A transformation \( \rho \) on \( S \) is a right translation of \( S \) if, for all \( x,y \in S \), \( (xy)\rho = x(yp) \). Likewise, a transformation \( \lambda \) is a left translation if \( \lambda(xy) = (\lambda x)y \), for all \( x,y \in S \). If, in addition, the left translation \( \lambda \) and the right translation \( \rho \) satisfy \( x(\lambda y) = (xp)y \), for all \( x,y \in S \), then the two are linked and the pair \((\lambda, \rho)\) is a bitranslation. The set of all bitranslations on \( S \) under the operation of componentwise composition is an inverse semigroup and is called the translational hull of \( S \) [P;V.1.4]. We denote this semigroup by \( \Omega(S) \). We note that either of the projection maps on \( \Omega(S) \) is a monomorphism [P;V.1.2].
For any \( s \in S \), the functions \( \lambda_s \) and \( \rho_s \) defined by \( \lambda_s x = sx \) and \( x\rho_s = xs \), for all \( x \in S \), are left and right translations, respectively. In fact, \((\lambda_s, \rho_s)\) is a bitranslation and so is a member of \( \Omega(S) \). The mapping

\[
\pi : s \rightarrow (\lambda_s, \rho_s) \quad (s \in S),
\]

is a monomorphism of \( S \) into \( \Omega(S) \) and is called the **canonical homomorphism of \( S \) into \( \Omega(S) \)**.

It turns out that \( \Omega(S) \) is isomorphic to the idealizer of the Wagner representation of \( S \) in \( \mathcal{J}(S) \) [P;V.1.3]. That is, \( \Omega(S) \) is isomorphic to the largest inverse subsemigroup of \( \mathcal{J}(S) \) containing the Wagner representation of \( S \) as an ideal. Perhaps more to the point, \( \Omega(S) \) is isomorphic to the idealizer of the Wagner representation of \( S \) in the inverse semigroup of all one-to-one partial right translations on \( S \) (a partial one-to-one right translation on \( S \) is a right translation whose domain is a left ideal of \( S \); see [P;V.2]). It is this fact which makes plain the connection between the translational hull of \( S \) and both the Wagner representation and the Schützenberger representations of \( S \).

If \( S \) is an ideal of the inverse semigroup \( V \) then \( V \) is an **ideal extension** of \( S \) (by the Rees quotient semigroup \( V / S \)). The translational hull is particularly useful when considering ideal extensions of inverse semigroups \( S \) for which we know \( \Omega(S) \).

Let \( V \) be an ideal extension of \( S \). For each \( v \in V \), define

\[
\lambda^v_s = vs \quad \text{and} \quad sp^v = sv \quad (s \in S).
\]

Then the mapping

\[
\tau(V:S) : V \rightarrow \Omega(S)
\]

defined by

\[
\nu\tau(V:S) = (\lambda^v, \rho^v) \quad (v \in V)
\]

is a homomorphism of \( V \) into \( \Omega(S) \) which extends \( \pi \). Moreover, \( \tau(V:S) \) is the unique extension of \( \pi \) to a homomorphism of \( V \) into \( \Omega(S) \) [P;I.9.2]. \( \tau(V:S) \) is called the **canonical homomorphism of \( V \) into \( \Omega(S) \)**.
Theorem 2.6.1. Let $S$ be a completely semisimple inverse semigroup and let $D$ be a $D$-class of $S$ which is not the kernel of $S$. Let $I = \{ x \in S : J_x \not\geq D \}$. Then $S / I$ is an ideal extension of the Brandt semigroup $D^0$ and the image of $S / I$ in $\Omega(D^0)$ under the canonical homomorphism is isomorphic to the Wagner representation of $S$ restricted to any $\mathcal{R}$-class belonging to $D$.

Proof: First of all, identify $S / I$ with $(S \setminus I)^0$.

Let $R$ be an $\mathcal{R}$-class of $S$ contained in $D$. Let $\alpha$ be the Wagner representation of $S$ restricted to $R$ and denote $s\alpha$ by $s\alpha_s$, for all $s \in S$. Let $T$ be the projection of $\tau(S / I : D^0)$ onto its second coordinate. The elements of $T$ are right translations of the form $\rho^v$, for $v \in S / I$. We first prove the following statement:

Let $\phi : S \to S / I$ be the natural homomorphism of $S$ onto the Rees quotient semigroup $S / I$. Let $s,t \in S$. Then $\alpha_s = \alpha_t$ if and only if $\rho^{s\phi} = \rho^{t\phi}$.

First of all, observe that if $a$ and $as$ both belong to the same $D$-class in a completely semisimple inverse semigroup, then a $\mathcal{R}$ as. This is because $aa'^{-1} \geq ass'^{-1}$ and $D_a^0$ is a Brandt semigroup (and hence satisfies $D$-majorization) and so $aa^{-1} = ass^{-1}$. Secondly, observe that if $s \in S \setminus I$ then $\{ x \in R : xs \in R \} \neq \emptyset$. To see this, note that $D \subseteq J(s)$ and so, by Lemma 2.3.1 (d), there is an $a \in D$ such that $a^{-1}a \leq ss^{-1}$. Thus, $a^{-1}a = a^{-1}ass^{-1}$ and as $\neq 0$. But if $a$ and as both belong to the Brandt semigroup $D^0$, then a $\mathcal{R}$ as. Let $y \in R$ be such that $y \mathcal{L} a$. Then $y \mathcal{R} ys$ and, as a result, the set $\{ x \in R : xs \in R \}$ is nonempty.

If $\rho^{s\phi} = \rho^{t\phi}$ then for all $x \in D^0$, $xs\phi = xt\phi$. If $s \in I$, then $\{ x \in R : xs \in R \} \neq \emptyset$ and for all $x \in \{ x \in R : xs \in R \}$, $xs\phi \neq 0$. Thus, $s\phi = 0$ if and only if $t\phi = 0$. If $s \in I$ then we must have $xs = xt$ for all $x \in \{ x \in R : xs \in R \}$ since $\phi$ is one-to-one on $S \setminus I$. Likewise, we must have that $xs = xt$ for all $x \in \{ x \in R : xt \in R \}$. Therefore,
\{x \in R : xs \in R\} = \{x \in R : xt \in R\} \text{ and } \alpha_s = \alpha_t. \text{ If } s \in I \text{ then } s\phi = 0 \text{ and so, for all } x \in D^0, 0 = xs\phi = xt\phi. \text{ Consequently, } \\
\{x \in R : xs \in R\} = \{x \in R : xt \in R\} = \emptyset, \text{ and } \alpha_s = \alpha_t. \\
Conversely, \text{ if } \alpha_s = \alpha_t \text{ then for all } x \in \{x \in R : xs \in R\} = \{x \in R : xt \in R\}, xs = xt. \text{ If } s \in I \text{ then } \\
\{x \in R : xs \in R\} = \{x \in R : xt \in R\} = \emptyset \text{ and so } t \in I. \text{ Then } s\phi = t\phi = 0 \text{ and } p^{s\phi} = p^{t\phi}. \text{ So suppose that } s \notin I \text{ and hence that } t \notin I. \text{ Let } x \in D^0 \text{ and let } y \in R \text{ be such that } y \mathcal{L} x. \text{ Now } ys \in D \text{ if and only if } ys \in R \text{ and so } ys = yt. \text{ Since } y \mathcal{L} x, x = xy^{-1}y \text{ and so } xs = xy^{-1}ys = xy^{-1}yt = xt. \text{ Therefore, } p^{s\phi} = p^{t\phi} \text{ and the claim is proved.} \\

By what we have just done, it follows that the mapping 
\[ \Theta : \alpha_s \to p^{s\phi} \quad (s \in S) \]
is a well-defined bijection from the restricted Wagner representation onto the image of the projection onto the second coordinate of \(\Omega(S/I)\). Since \(\phi\) is a homomorphism, so is \(\Theta\). The projection map of \(\Omega(S/I)\) onto its second coordinate is an isomorphism and so the desired result is obtained. 

The connection between the Wagner representation restricted to an \(\mathcal{R}\)-class \(R\) and the translational hull of a semigroup related to the \(\mathcal{D}\)-class containing \(R\) was made in a more general setting by Petrich (See [Pe1] or [Pe2]). 

Ponizovskii [Po] first proved that the translational hull of an inverse semigroup is an inverse semigroup. The relationship between the translational hull of an inverse semigroup \(S\) and the semigroup of all one-to-one partial right translations on \(S\) was established by McAlister by way of Schein's work on permissable subsets.
2.7 Varieties

A nonempty class of algebras $\mathcal{C}$ of the same type is a variety if it is closed under subalgebras, homomorphic images and direct products. By a theorem due to Birkhoff, an equivalent definition of variety is an equationally defined class of algebras of the same type. That is, if $\mathcal{F}$ is a nonempty family of equations over a language $\mathcal{L}$, then the class $\mathcal{V}$ of all algebras of type $\mathcal{L}$ satisfying each identity in $\mathcal{F}$ is a variety.

If $\mathcal{U}$ is a variety contained in the variety $\mathcal{V}$ then $\mathcal{U}$ is a subvariety of $\mathcal{V}$. It is apparent from the definition of variety that the intersection of a nonempty family of varieties contained in the variety $\mathcal{V}$ is also a variety contained in $\mathcal{V}$. Consequently, the collection of subvarieties of a variety $\mathcal{V}$ forms a complete lattice under inclusion, which we denote by $\mathcal{L}(\mathcal{V})$.

Given a class $\mathcal{C}$ of algebras, each member of which belongs to the variety $\mathcal{V}$, the variety generated by $\mathcal{C}$ is the intersection of all varieties contained in $\mathcal{V}$ which contain $\mathcal{C}$. We write $\langle \mathcal{C} \rangle$ to denote this variety. If $\mathcal{C}$ consists of the single algebra $S$, we write $\langle S \rangle$ instead of $\langle \mathcal{C} \rangle$. If $\mathcal{U}$ is a subvariety of $\mathcal{V}$ defined by the equations $\Sigma$ then we write $\mathcal{U} = [\Sigma]$. If $\Sigma$ is a finite set of equations $\{u_1 = v_1, \ldots, u_n = v_n\}$ we will often write $\mathcal{U} = [u_1 = v_1, \ldots, u_n = v_n]$ instead of $[\Sigma]$. We sometimes refer to the equations $\Sigma$ which define the variety $\mathcal{V}$ as laws.

A refinement of our first definition of variety is the so-called HSP Theorem. If $\mathcal{C}$ is a class of algebras belonging to the variety $\mathcal{V}$, the variety $\langle \mathcal{C} \rangle$ consists of homomorphic images of subalgebras of direct products of algebras in $\mathcal{C}$.

If $\mathcal{V}$ is a variety and $X$ is a nonempty set then $\mathcal{V}$ possesses a free algebra $F\mathcal{V}(X)$ on $X$ which has the universal mapping property. In fact, up to isomorphism, this free algebra is the unique algebra in $\mathcal{V}$ with the universal mapping property freely generated by a set of generators of size $|X|$. Thus, $F\mathcal{V}(X)$ may be defined as the unique algebra $F$ in $\mathcal{V}$, up to isomorphism, which satisfies: Let $\iota: X \rightarrow F$ map $X$ injectively onto a set of
generators of \(F\). Then for any \(S \in \mathcal{Y}\) and any mapping \(\phi : X \to S\), there is a unique homomorphism \(\phi^* : F \to S\) which extends \(\phi\). That is, there is a unique homomorphism \(\phi^* : F \to S\) such that, for all \(x \in X\), \(x\phi = x\phi^*\).

The class of all semigroups forms a variety as does the class of all monoids (considered as algebras with a binary operation and a nullary operation (constant)). The free semigroup on the set \(X\) consists of all nonempty finite sequences of elements of \(X\), called words, over \(X\), called an alphabet, given the multiplication of concatenation (or juxtaposition). We denote the free semigroup on \(X\) by \(X^+\). The free monoid on \(X\), denoted \(X^*\), consists of all words over \(X\) including the empty word, which serves as the identity of \(X^*\).

An inverse semigroup \(S\) is subdirectly irreducible if for every subdirect embedding \(\alpha : S \to \prod_{i \in I} S_i\) there is an \(i \in I\) such that \(\alpha\pi_i\) is an isomorphism.

The following is an equivalent definition of subdirectly irreducible and can be found in any Universal Algebra text.

**Theorem 2.7.1** [BS;II.8.4]. An inverse semigroup \(S\) is subdirectly irreducible if and only if \(S\) is trivial or there is a minimum congruence in \(\mathcal{C}(S) \setminus \{\varepsilon\}\) where \(\mathcal{C}(S)\) is the lattice of congruences on \(S\) and \(\varepsilon\) is the equality relation.

The following useful theorem is due to Birkhoff.

**Theorem 2.7.2** [BS;II.9.7]. Every variety \(\mathcal{Y}\) of inverse semigroups is completely determined by its subdirectly irreducible members.

Inverse semigroups, considered as algebras with a binary operation and a unary operation, is determined by associativity and the equations \(x = xx^{-1}x\), \((x^{-1})^{-1} = x\) and
Let $X^{-1}$ denote a set disjoint from $X$ and in one-to-one correspondence with $X$ via $x \leftrightarrow x^{-1}$. This correspondence can be extended to a unary operation on $(X \cup X^{-1})^+$ by defining $(x^{-1})^{-1} = x$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $x \in X$, $a, b \in (X \cup X^{-1})^+$. Throughout $(X \cup X^{-1})^+$ will denote the free semigroup on $X \cup X^{-1}$ with involution $^{-1}$. The *Wagner congruence* is the least congruence $\rho$ on $(X \cup X^{-1})^+$ such that $(a, aa^{-1}a) \in \rho$ and $(aa^{-1}bb^{-1}, bb^{-1}aa^{-1}) \in \rho$, for all $a, b \in (X \cup X^{-1})^+$. If $\rho$ is the Wagner congruence, then $(X \cup X^{-1})^+/\rho$ is the free inverse semigroup on $X$ [P; VIII.1.1]. For any word $w$ over $X \cup X^{-1}$ we will write $w$ for $wp$ and refer to elements of the free inverse semigroup on $X$ as words over $X \cup X^{-1}$. For any word $w \in (X \cup X^{-1})^+$, we define the *content of* $w$ by $c(w) = \{ x \in X : x$ or $x^{-1}$ occurs in $w \}$.

A congruence $\rho$ on an inverse semigroup $S$ is *fully invariant* if it is invariant under all endomorphisms of $S$. That is, if $u \rho w$ and $\phi$ is an endomorphism of $S$, then $(u\phi) \rho (w\phi)$. The set of all fully invariant congruences on $S$, denoted $\mathcal{FI}(S)$, is a complete sublattice of the lattice of congruences on $S$. Let $X$ be a countably infinite set and consider the free inverse semigroup $F\mathcal{I}(X)$. For any variety $\mathcal{V}$ of inverse semigroups, the relation $\rho(\mathcal{V})$ defined on $F\mathcal{I}(X)$ by $u \rho(\mathcal{V}) w$ if and only if $u = w$ is a law in $\mathcal{V}$ is a fully invariant congruence on $F\mathcal{I}(X)$. Conversely, given a fully invariant congruence $\rho$ on $F\mathcal{I}(X)$, let $\mathcal{V}(\rho)$ be the variety of inverse semigroups determined by the set of identities $u = w$, where $u \rho w$. Then the mappings $\rho : \mathcal{V} \to \rho(\mathcal{V})$ and $\mathcal{V} : \rho \to \mathcal{V}(\rho)$ are mutually inverse order antiisomorphisms of $\mathcal{L}(\mathcal{F})$ and $\mathcal{FI}(F\mathcal{I}(X))$ [P; I.11.11]. We sometimes refer to $\rho(\mathcal{V})$ as the fully invariant congruence corresponding to $\mathcal{V}$. We will often find it necessary to consider fully invariant congruences on $F\mathcal{I}(Y)$, for some set $Y$, and $F\mathcal{I}(X)$ at the same time. Under these conditions, we will write $\rho_{Y}(\mathcal{V})$ to mean the fully invariant congruence on $F\mathcal{I}(Y)$ corresponding to $\mathcal{V}$, and simply $\rho(\mathcal{V})$ for the fully
invariant congruence on $\mathcal{F}(X)$ corresponding to $\mathcal{V}$. Throughout, $X$ is assumed to be a fixed countably infinite set, unless otherwise stated.

The variety $\mathcal{V}$ of inverse semigroups is said to be combinatorial if all the members are combinatorial. Equivalently, $\mathcal{V}$ is a combinatorial variety if and only if $\mathcal{V} \cap \mathcal{G} = \mathcal{I}$, the trivial variety (defined by the law $x = y$) if and only if $\mathcal{V} \subseteq \{x^n = x^{n+1}\}$, for some $n \in \omega$ [P; XII.1.10]. Likewise, the variety $\mathcal{V}$ is completely semisimple or cryptic if every member of $\mathcal{V}$ is completely semisimple or cryptic, respectively.

Let $S$ and $T$ be inverse semigroups and let $G$ be a group. $T$ is an $E$-unitary cover of $S$ over $G$ if $T$ is $E$-unitary, there exists an idempotent separating homomorphism of $T$ onto $S$ and $T/\sigma_T \cong G$. If $\mathcal{U}$ is a variety of groups then the inverse semigroup variety $\mathcal{V}$ has $E$-unitary covers over $\mathcal{U}$ if, for every $S \in \mathcal{V}$, there is a group $G \in \mathcal{U}$ for which there is an $E$-unitary cover of $S$ over $G$. A variety $\mathcal{V}$ of inverse semigroups has $E$-unitary covers if, for every $S \in \mathcal{V}$, there is an $E$-unitary cover of $S$ in $\mathcal{V}$.

**Theorem 2.7.3** [PR; 3.3, 5.4]. Let $\mathcal{V}$ be a variety of inverse semigroups. Then the following statements are equivalent:

i) $\mathcal{V}$ has $E$-unitary covers;

ii) the $\mathcal{V}$-free objects in $\mathcal{V}$ are $E$-unitary;

iii) the $\mathcal{V}$-free object on a countably infinite set is $E$-unitary;

iv) $\mathcal{V}$ has $E$-unitary covers over $\mathcal{V} \cap \mathcal{G}$.

**Theorem 2.7.4** [PR; 5.7]. Let $\mathcal{V}$ be a variety of inverse semigroups and $\mathcal{U}$ a variety of groups. Then $\mathcal{V}$ has $E$-unitary covers over $\mathcal{U}$ if and only if $\mathcal{V} \subseteq \{ u^2 = u : u^2 = u \text{ is a law in } \mathcal{U} \}$.
We will use the following notation. If $\mathcal{V}$ is a variety of inverse semigroups, we denote by $\mathcal{V}^{\text{max}}$ the variety of inverse semigroups [ $u^2 = u : u^2 = u$ is a law in $\mathcal{V}$ ] and by $\mathcal{V}_M^{\text{max}}$ the variety of inverse monoids [ $u^2 = u : u^2 = u$ is a law in $\mathcal{V}$ ].

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of inverse semigroups. The *Mal'cev product* of $\mathcal{V}$ and $\mathcal{W}$, denoted by $\mathcal{V} \circ \mathcal{W}$, is the collection of those inverse semigroups $S$ for which there exists a congruence $\rho$ on $S$ with the property that $e\rho \in \mathcal{V}$ for all $e \in E_S$ and $S/\rho \in \mathcal{W}$; we say that $\rho$ witnesses that $S \in \mathcal{V} \circ \mathcal{W}$.

In general, $\mathcal{V} \circ \mathcal{W}$ is not a variety. For example, if $\mathcal{W}$ is any nontrivial group variety and $\mathcal{V} = \mathcal{S}$, the variety of semilattices, then the five element Brandt semigroup $B_2$ is a member of $\langle \mathcal{V} \circ \mathcal{W} \rangle$ but $B_2$ is not a member of $\mathcal{V} \circ \mathcal{W}$. To see that $B_2 \notin \mathcal{V} \circ \mathcal{W}$, observe that any congruence $\rho$ for which $B_2/\rho$ is a group must be the universal relation and hence any idempotent $\rho$-class is just $B_2$ which is not a semilattice. On the other hand, since $B_2$ has an E-unitary cover over any nontrivial group variety ([PR] or [P;XII.9.8]), $B_2 \in \langle \mathcal{V} \circ \mathcal{W} \rangle$ ([PR] or [P;XII.9.11]).

However, when $\mathcal{V}$ is a variety of groups, $\mathcal{V} \circ \mathcal{W}$ is a variety [See [P; XII 8.3] or [Ba]]. Note that, if $\mathcal{V}$ and $\mathcal{W}$ are varieties such that $\mathcal{V} \subseteq \mathcal{W}$ then, for any variety $\mathcal{Y}$, $\mathcal{V} \circ \mathcal{Y} \subseteq \mathcal{V} \circ \mathcal{W}$ and $\mathcal{Y} \circ \mathcal{V} \subseteq \mathcal{W} \circ \mathcal{V}$.

**Lemma 2.7.5.** Let $\mathcal{V}$ be a variety of groups and let $\mathcal{W}$ be a variety of inverse semigroups. Then $S \in \mathcal{V} \circ \mathcal{W}$ implies that $S/\mu_S \in \mathcal{W}$. Moreover,

$$\text{tr } \rho(\mathcal{V}) = \text{tr } \rho(\mathcal{V} \circ \mathcal{W}).$$

**Proof:** If $\rho$ witnesses that $S \in \mathcal{V} \circ \mathcal{W}$, then $\rho$ is idempotent separating and so $\rho \subseteq \mu_S$.

Now, $S/\mu_S$ is isomorphic to $(S/\rho)/(\mu_S/\rho)$ and $S/\rho \in \mathcal{W}$ so we may conclude that $S/\mu_S \in \mathcal{W}$.

If $A$ is an antigroup belonging to $\mathcal{V} \circ \mathcal{W}$, then $A/\mu_S \equiv A \in \mathcal{Y}$. Thus,
(\mathcal{V} \circ \mathcal{V}) \cap \mathcal{A} \subseteq \mathcal{V} \cap \mathcal{A}. \text{ Since } \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{V}, \text{ we have } \mathcal{V} \cap \mathcal{A} \subseteq (\mathcal{V} \circ \mathcal{V}) \cap \mathcal{A}.

Therefore, \((\mathcal{V} \circ \mathcal{V}) \cap \mathcal{A} = \mathcal{V} \cap \mathcal{A}.\) It follows from [P; XII.2] that

\[ \mathcal{V} \lor \mathcal{G} = (\mathcal{V} \circ \mathcal{V}) \lor \mathcal{G}, \] and hence, \(\text{tr} \rho(\mathcal{V}) = \text{tr} \rho(\mathcal{V} \circ \mathcal{V}).\)

Mal'cev products play an important role in the study of varieties of inverse semigroups. For example, if \(\mathcal{V}\) is a group variety and \(\mathcal{V}\) is a combinatorial variety, then \(\mathcal{V} \circ \mathcal{V}\) is the maximum variety in the \(\lor\)-class of \(\mathcal{V} \lor \mathcal{V}\), where \(\lor\) is the congruence on \(\mathcal{L}(\mathcal{F})\) defined by \(\mathcal{V}_1 \lor \mathcal{V}_2\) if and only if \(\mathcal{V}_1 \cap \mathcal{G} = \mathcal{V}_2 \cap \mathcal{G}\) and \(\mathcal{V}_1 \lor \mathcal{G} = \mathcal{V}_2 \lor \mathcal{G}\), for all \(\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{L}(\mathcal{F})\). (See, for e.g., [P; XII.2, XII.3]). For strict inverse varieties it turns out that the \(\lor\)-classes are trivial (and in fact \(\mathcal{L}(\mathcal{F})\) is isomorphic to three copies of \(\mathcal{L}(\mathcal{G})\), the so-called 'first three layers' of \(\mathcal{L}(\mathcal{F})\) [P; XII.4.16], but this is by no means true throughout \(\mathcal{L}(\mathcal{F})\) as we shall see in Chapter Six. For further information on Mal'cev products we refer the reader to [P] or [R1].

Before we proceed, we provide a list of notation introduced in this section as well as the notation we will use for certain special varieties and classes of inverse semigroups.

Varieties and classes:

\[ \mathcal{I} \] — the variety of all inverse semigroups

\[ \mathcal{J} \] — \([x = y]\) the trivial variety

\[ \mathcal{G} \] — \([xx^{-1} = yy^{-1}]\) the variety of all groups

\[ \mathcal{S} \] — \([x = x^2]\) the variety of semilattices

\[ \mathcal{S}_{\mathcal{G}} \] — \([xx^{-1} = x^{-1}x, u_\alpha v_\alpha^{-1} = (u_\alpha v_\alpha^{-1})^2\] \(\alpha \in \mathcal{A}\) the variety of semilattices of groups in \(\mathcal{G}\) (Clifford semigroups over \(\mathcal{Z}\)) where

\[ \mathcal{G} = [u_\alpha = v_\alpha]_{\alpha \in \mathcal{A}} \]
\[ \mathcal{FG} \] — \[ xx^{-1} = x^{-1}x \] the variety of Clifford semigroups or the variety of semilattices of groups

\[ \mathcal{B} \] — \( \langle B_2 \rangle = [ xyx^{-1} = (xyx^{-1})^2 ] \) the variety generated by the five-element Brandt semigroup

\[ \mathcal{SG} \] — \( [ (xyx^{-1})(xyx^{-1})^{-1} = (xyx^{-1})^{-1}(xyx^{-1}) ] \) the variety of strict inverse semigroups

\[ \mathcal{B}^1 \] — \( \langle B_2^1 \rangle \) the variety generated by the five-element Brandt semigroup with an identity adjoined

\[ \mathcal{A}_n \] — the variety of abelian groups of exponent \( n \)

\[ \mathcal{AG} \] — the variety of abelian groups

\[ \mathcal{C}_n \] — \( [ x^n = x^{n+1} ] \) for every natural number \( n \)

\[ \mathcal{Z} \] — \( [ u^2 = u : u^2 = u \text{ is a law in } \mathcal{Z} ] \)

\( \mathcal{Z} \circ \mathcal{V} \) — the Mal'cev product of the varieties \( \mathcal{Z} \) and \( \mathcal{V} \) (not necessarily a variety)

\[ \mathcal{A} \] — the class of all antigroups (not a variety)

Further Notation

\( \mathcal{L}(\mathcal{V}) \) — the lattice of all subvarieties of \( \mathcal{V} \)

\( \langle \mathcal{E} \rangle \) — the variety of inverse semigroups generated by the nonempty class \( \mathcal{E} \) of inverse semigroups; when \( \mathcal{E} = \{ S \} \), we write \( \langle S \rangle \) instead of \( \langle \mathcal{E} \rangle \)

\( [ \Sigma ] \) — the variety of inverse semigroups satisfying \( u = w \) for all equations \( u = w \text{ in } \Sigma \)

\[ w \in E \] — the equation \( w = w^2 \)

\( \mathcal{F}\mathcal{V}(X) \) — the \( \mathcal{V} \)-free inverse semigroup on \( X \)
for a word \( w \) over \( X \cup X^{-1} \), the content of \( w \)

for a variety \( \mathcal{V} \) of inverse semigroups, the fully invariant
congruence on \( F\mathcal{J}(X) \) corresponding to \( \mathcal{V} \)

Many of the results we have mentioned here are of a fundamental nature and can be
found in virtually any text on Universal Algebra ([Gr] or [BS], for example); we do,
however, mention Birkhoff's important paper [Bi2] of 1935 in which he proved his
famous theorem that \( \mathcal{V} \) is a variety if and only if \( \mathcal{V} \) is an equational class. The Wagner
congruence is, of course, due to Wagner [Wa3]. Completely semisimple varieties were
studied by Reilly [Re2]. The congruence \( \nu \) was introduced by Kleiman who is responsible
for the result cited on the first three layers of \( \mathcal{L}(\mathcal{I}) \) [K1]. Reilly [Re2] also studied the
congruence \( \nu \) and showed that \( \mathcal{L}(\mathcal{I}) \) is not a modular lattice. For results concerning the
Mal'cev product of inverse semigroup varieties we refer the reader to Reilly [Re1], and for
results concerning E-unitary covers we refer the reader to [PR].

2.8 Presentations and Schützenberger graphs

A presentation of an inverse semigroup is a pair \( P = (X;R) \) where \( R \) is a binary
relation on \( F\mathcal{J}(X) \). If \( P = (X;R) \), the inverse semigroup presented by \( P \) is
\( F\mathcal{J}(X) / \theta \) where \( \theta \) is the congruence on \( F\mathcal{J}(X) \) generated by \( R \). Equivalently, we may
consider \( P = (X;R) \), where \( R \) is a binary relation on \( (X \cup X^{-1})^+ \). Then the inverse
semigroup presented by \( P \) is \( (X \cup X^{-1})^+ / \tau \), where \( \tau \) is the congruence on \( (X \cup X^{-1})^+ \)
generated by \( R \cup \rho \). We will consider only those presentations for which \( R \) (and hence
\( \theta \) ) is \( \rho(\mathcal{V}) \) for some variety \( \mathcal{V} \) of inverse semigroups.

The definitions and results of this section can be found in Stephen [S] to which we
refer the reader for additional information concerning Schützenberger graphs.
A labelled digraph $\Gamma$ over a nonempty set $X$ consists of a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, where $E(\Gamma) \subseteq V \times X \times V$. An edge $(v_1,x,v_2) \in E(\Gamma)$ is labelled by $x$ and directed from $v_1$ to $v_2$. We call $v_1$ the initial or start vertex and $v_2$ the terminal or end vertex of the edge $(v_1,x,v_2)$. A path $p$ is a sequence of edges such that the end vertex of an edge in the sequence is the start vertex of the next edge in the sequence.

$\Gamma$ is strongly connected if, given any two vertices $v_1, v_2 \in V(\Gamma)$, there is a path $p$ from $v_1$ to $v_2$. We will often call a path from $v_1$ to $v_2$ a $v_1$-$v_2$ walk. An inverse word graph $\Gamma$ over $X \cup X^{-1}$ is a strongly connected labelled digraph over $X \cup X^{-1}$ satisfying the condition: $(v_1,x,v_2) \in E(\Gamma)$ implies $(v_2,x^{-1},v_1) \in E(\Gamma)$, for all $x \in X \cup X^{-1}$. An inverse word graph $\Gamma$ is deterministic if all edges directed away from a vertex are labelled by different letters, and injective if all edges directed toward a vertex are labelled by different letters. Thus, a deterministic inverse word graph over $X \cup X^{-1}$ is necessarily injective.

If $\Gamma$ and $\Gamma'$ are inverse word graphs over $X \cup X^{-1}$, a $V$-homomorphism $\phi: \Gamma \rightarrow \Gamma'$ is a map on the vertices of $\Gamma$ which preserves incidence, orientation and labelling. More precisely, $\phi$ is a pair of functions $\phi_V: V(\Gamma) \rightarrow V(\Gamma')$ and $\phi_E: E(\Gamma) \rightarrow E(\Gamma')$ such that $(v_1,x,v_2) \phi_E = (v_1 \phi_V x, v_2 \phi_V x)$. $\phi$ is a $V$-monomorphism if it is one-one on the vertices of $\Gamma$; a $V$-epimorphism if it is surjective on both the set of edges and the set of vertices of $\Gamma$; a $V$-isomorphism if it is both a $V$-monomorphism and a $V$-epimorphism. An inverse birooted word graph is a triple $(s, \Gamma, e)$ where $\Gamma$ is an inverse word graph and $s$ and $e$ are distinguished vertices called, respectively, the start and end vertices.

Let $P = (X; R)$ be a fixed presentation of the inverse semigroup $S$ with $\tau$ the corresponding congruence on $F_\mathcal{S}(X)$. Let $w \in S$ and $R_w$ the $\mathcal{R}$-class of $w$ in $S$. The Schützenberger graph of $R_w$ with respect to $P$ is the labelled digraph $\Gamma(w)$, where
\( V(\Gamma(w)) = R_w \)
\( E(\Gamma(w)) = \{ (v_1,x,v_2) : v_1,v_2 \in R_w, x \in X \cup X^{-1} \text{ and } v_1(x \tau) = v_2 \} \).

Dually, we define the Schützenberger graph of \( L_w \) with respect to \( P \) to be the labelled digraph \( \Delta(w) \) with
\[ V(\Delta(w)) = L_w \]
\[ E(\Delta(w)) = \{ (v_1,x,v_2) : v_1,v_2 \in L_w, x \in X \cup X^{-1} \text{ and } (x \tau)v_1 = v_2 \} \).

**Lemma 2.8.1** [S; 3.1]. Let \( v \in S, \Gamma(v) \) be the Schützenberger graph of \( R_v \) with respect to \( P, v_1,v_2,v \in R_v, e = vv^{-1} \text{ and } w \in (X \cup X)^{+} \).

a) \( \Gamma(v) \) is a deterministic inverse word graph;

b) \( v_1(wt) = v_2 \) if and only if \( w \) labels a \( v_1-v_2 \) walk;

c) \( (wt) \geq v \) if and only if \( w \) labels an \( e-v \) walk;

The lemma above can be dualized for \( \Delta(v) \) for any \( \mathcal{L} \)-class \( L_v \) of \( S \). We remark that if \( S \) is a group, then for any \( w \in S, \Gamma(w) \) is the Cayley graph of \( S \) (See [S; 3.7]). For a discussion of Cayley graphs, we refer the reader to [W].

The following lemma characterizes Green's relations on \( S \) in terms of the Schützenberger graphs of \( S \).

**Lemma 2.8.2** [S; 3.4]. Let \( v_1, v_2 \in S \) and let \( e = v_1v_1^{-1} \text{ and } f = v_2v_2^{-1} \). Then

a) \( v_1 \mathcal{D} v_2 \) if and only if there exists a \( V \)-isomorphism \( \phi : \Gamma(v_1) \to \Gamma(v_2) \);

b) \( v_1 \mathcal{R} v_2 \) if and only if there exists a \( V \)-isomorphism \( \phi : \Gamma(v_1) \to \Gamma(v_2) \) such that \( e\phi = f \).

c) \( v_1 \mathcal{L} v_2 \) if and only if there exists a \( V \)-isomorphism \( \phi : \Gamma(v_1) \to \Gamma(v_2) \) such that \( v_1\phi = v_2 \).
d) \( v_1 \not\sim v_2 \) if and only if there exist \( V \)-isomorphisms \( \phi, \psi : \Gamma(v_1) \to \Gamma(v_2) \) such that 
\[ e\phi = f \text{ and } v_1\psi = v_2. \]

e) \( v_1 = v_2 \) if and only if there exists a \( V \)-isomorphism \( \phi : \Gamma(v_1) \to \Gamma(v_2) \) such that 
\[ e\phi = f \text{ and } v_1\phi = v_2. \]

For any \( v \in S \), the Schützenberger representation of \( v \) (with respect to \( P \)) is the birooted inverse word graph \((vv^{-1}, \Gamma(v), v)\). We will also use \( \Gamma(v) \) to denote the birooted graph and specify the roots whenever required. We are considering presentations in which the relation \( R \) is always a fully invariant congruence on \( F_{\mathcal{R}}(X) \) corresponding to some variety \( \mathcal{Y} \). Thus, for any word \( w \in (X \cup X^{-1})^+ \) and congruence \( \rho(\mathcal{Y}) \), we will write \( \Gamma(w) \) (or \( \Gamma_{\mathcal{Y}}(w) \) if we wish to emphasize the variety being considered) to denote \((ww^{-1}\rho(\mathcal{Y}), \Gamma(w\rho(\mathcal{Y})), w\rho(\mathcal{Y}))\) with respect to \( P = (X; \rho(\mathcal{Y})) \), and call \( \Gamma_{\mathcal{Y}}(w) \) the Schützenberger representation of \( w \) with respect to \( \mathcal{Y} \). We remark that the Schützenberger representation of the free inverse semigroup is the representation of \( F_{\mathcal{R}}(X) \) by birooted inverse word trees, which is due to Munn [Mu4] (See Stephen [S] for the connection between Schützenberger graphs of the free inverse semigroup and Munn trees). For further properties of Schützenberger graphs, we refer the reader to Stephen [S].

The following result will be used throughout, but is presented here so that we may look at what are probably the simplest examples of Schützenberger graphs relative to some variety.

**Proposition 2.8.3.** If \( w \in (X \cup X^{-1})^+ \), then \( \Gamma_{\mathcal{Y}}(w) \) is just a single vertex with \( 2|c(w)| \) loops. For each \( x \in c(w) \) there is precisely one loop labelled \( x \) and one loop labelled \( x^{-1} \).

**Proof:** For any \( u,v \in (X \cup X^{-1})^+ \), \( u \rho(\mathcal{Y}) v \) if and only if \( c(u) = c(v) \). Furthermore, \( u \rho(\mathcal{Y}) \not\sim u a \rho(\mathcal{Y}) \), for some \( a \in (X \cup X^{-1})^+ \), if and only if \( a \) or \( a^{-1} \) is an element of
c(u). From these two facts and the definition of Schützenberger graph, one easily obtains the desired result.

Examples. 1)  

![Figure 2.1. Schützenberger graphs in the free semilattice on three generators.](image)

The graphs above in Figure 2.1 form the collection of Schützenberger graphs (up to $V$-isomorphism) of the free semilattice on three generators (see Proposition 2.8.3). We follow the standard practice of providing edges labelled by $x \in X$ but not edges labelled by elements of $X^{-1}$ as these edges are implicitly determined by those edges labelled by elements of $X$. Also, we follow the convention of drawing a single edge with more than one label if there are several edges between two given vertices.

2) Figure 2.2 is the Schützenberger graph of the word $w = x_1x_2x_1^{-1}x_2^{-1}$ with respect to the variety $\mathcal{R}^1$. The proof of this can be found in Theorem 6.1.7.
This example will be used again in the sequel to illustrate concepts related to Schützenberger graphs.
CHAPTER THREE
Wreath Products

In this chapter we present the definition of the wreath product of two inverse semigroups. Our definition is a slightly more general definition than that of Houghton [H] and a generalization to arbitrary inverse semigroups of Petrich's definition of the (right) wreath product of a group and an inverse semigroup [P]. The restriction to groups of our definition is dual to the definition of standard (unrestricted) wreath product found in Neumann [N], as Neumann writes her operators on the left and we write our operators on the right. The only material of this chapter required for the sequel can be found in section 1. The material in section 2 serves as motivation for the work in subsequent chapters, particularly chapter 6. The third section contains some structural results concerning wreath products and, while these results are of independent interest, they are not required for the remaining chapters.

3.1 Definition of wreath product

Let S and T be inverse semigroups and suppose that T is an inverse subsemigroup of \( S(I) \), the symmetric inverse semigroup on I. Let \( I^S \) denote the set of functions (written on the right) from subsets of I into S. For any \( \psi \in I^S \), denote the domain of \( \psi \) by \( d\psi \).

Define a multiplication on \( I^S \) by

\[
(\psi \cdot \psi') = (i\psi) \cdot (i\psi') \quad [i \in \bigcap d\psi \cap d\psi'].
\]

For any \( \beta \in S(I) \) and \( \psi \in I^S \), we define a mapping \( \beta \psi \) by

\[
(\beta \psi) = (i\beta)\psi \quad [i \in d\beta, i\beta \in d\psi].
\]

The (right) wreath product of S and T is the set

\[
S \text{ wr } T = \{ (\psi,\beta) \in I^S \times T : d\psi = d\beta \}
\]
with multiplication given by
\[(\psi,\beta) \cdot (\psi',\beta') = (\psi \beta \psi',\beta \beta').\]
If T is an inverse subsemigroup of \(\mathcal{J}(I)\), we will sometimes write \((T,I)\) for T if we wish to emphasize the set I on which T acts. We will write \((T,T)\) to denote the Wagner representation of T by partial right translations.

Our definition of wreath product follows that of Houghton [H]. In [H] the wreath product \(W(S,T)\) of inverse semigroups S and T is, in our notation, \(S \wr (T,T)\) where T is given the Wagner representation by partial right translations. Our notation follows Petrich \([P;V.4]\).

**Proposition 3.1.1.** Let S and \((T,I)\) be inverse semigroups.

a) \(S \wr (T,I)\) is a semigroup;
b) \(S \wr (T,I)\) is regular;
c) If \((\psi,\beta) \in S \wr (T,I)\) then \((\psi,\beta)\) is an idempotent if and only if \(\beta\) is the identity map on \(d\beta\) and for all \(i \in d\beta, i\psi \in E_S\).
d) \(S \wr (T,I)\) is an inverse semigroup. If \((\psi,\beta) \in S \wr (T,I)\) then the inverse of \((\psi,\beta)\) is the pair \((\psi^{-1},\beta^{-1})\) where \(\beta^{-1}\) is the inverse of \(\beta\) in \((T,I)\) and for all \(i \in d\beta^{-1}, i\psi^{-1} = [i\beta^{-1}\psi]^{-1}\).

**Proof:** a) Let \((\psi,\beta), (\psi',\beta') \in S \wr (T,I)\).

Then
\[i \in d\psi \beta \psi' \iff i \in d\psi \text{ and } i \in d\beta \psi' \iff i \in d\psi = d\beta \text{ and } i\beta \in d\psi' = d\beta' \iff i \in d\beta \beta'.\]

Therefore, \((\psi,\beta)(\psi',\beta') = (\psi \beta \psi',\beta \beta') \in S \wr (T,I)\) and \(S \wr (T,I)\) is closed under the operation defined above.

Next, let \((\psi_1,\beta_1), (\psi_2,\beta_2)\) and \((\psi_3,\beta_3) \in S \wr (T,I)\).
Then
\[(\psi_1, \beta_1)(\psi_2, \beta_2)(\psi_3, \beta_3) = (\psi_1^{\beta_1} \psi_2, \beta_1 \beta_2)(\psi_3, \beta_3) = ((\psi_1^{\beta_1} \psi_2)\beta_1 \beta_2 \psi_3, (\beta_1 \beta_2) \beta_3)\]
and
\[(\psi_1, \beta_1)((\psi_2, \beta_2)(\psi_3, \beta_3)) = (\psi_1, \beta_1)(\psi_2^{\beta_2} \psi_3, \beta_2 \beta_3) = (\psi_1^{\beta_1}(\psi_2^{\beta_2} \psi_3), \beta_1(\beta_2 \beta_3)).\]

Since \((\beta_1 \beta_2) \beta_3 = \beta_1(\beta_2 \beta_3)\) and S wr (T,I) is closed under the operation, we need only check that the first components agree on \(d\beta_1 \beta_2 \beta_3\). Let \(i \in d\beta_1 \beta_2 \beta_3\). Then
\[i(\psi_1^{\beta_1} \psi_2)\beta_1 \beta_2 \psi_3 = (i\psi_1^{\beta_1} \psi_2)(i\beta_1 \beta_2 \psi_3) = [(i\psi_1)(i\beta_1 \psi_2)](i\beta_1 \beta_2 \psi_3) = (i\psi_1)(i\beta_1 \psi_2)(i\beta_1 \beta_2 \psi_3) \text{ (associativity of S)} = (i\psi_1)[i\beta_1(\psi_2^{\beta_2} \psi_3)] = i[\psi_1^{\beta_1}(\psi_2^{\beta_2} \psi_3)].\]

It follows that the operation is associative and so S wr (T,I) is a semigroup.

b) Let \((\psi, \beta) \in S \text{ wr } (T,I)\). Define \((\psi', \beta')\) by setting \(\beta' = \beta^{-1}\), \(d\psi' = d\beta'\) and \(j\psi' = [j\beta^{-1}]^{-1}\) for all \(j \in d\beta'\). It is immediate that \((\psi', \beta') \in S \text{ wr } (T,I)\). We have \((\psi, \beta)(\psi', \beta')(\psi, \beta) = (\psi^\beta \psi' \beta' \psi, \beta' \beta)\). Since \(\beta' = \beta^{-1}\), \(\beta' \beta = \beta\) and \(\beta' \beta'\) is the identity map on \(d\beta\). Therefore, for all \(i \in d\beta = d\beta \beta' \beta = d\psi \psi' \beta' \psi',\)
\[i \psi^\beta \psi' \beta' \psi = (i\psi)(i\beta \psi')(i\psi) = (i\psi)(i\beta \psi^{-1})(i\psi) = (i\psi)(i\psi)^{-1}(i\psi) = i\psi.\]

It now follows that \((\psi, \beta)(\psi', \beta')(\psi, \beta) = (\psi, \beta)\) and so S wr (T,I) is regular.

c) Let \((\psi, \beta) \in S \text{ wr } (T,I)\). Then \((\psi, \beta)\) is an idempotent in S wr (T,I) means that \((\psi^\beta \psi, \beta) = (\psi, \beta)\). But \(\beta \beta = \beta\) and \(\psi^\beta \psi = \psi\) if and only if \(\beta\) is the identity map on its domain and for all \(i \in d\beta = d\psi, i\psi \in E(S)\).

d) If \((\psi, \beta)\) and \((\psi', \beta')\) are idempotents in S wr (T,I), then
Therefore, the idempotents of \( S \wr_r (T,I) \) commute which, combined with the fact that \( S \wr_r (T,I) \) is regular, implies that \( S \wr_r (T,I) \) is an inverse semigroup.

If \( (\psi, \beta) \in S \wr_r (T,I) \) then define \( (\psi, \beta)^{-1} \) to be the pair \( (\psi^{-1}, \beta^{-1}) \) where \( \psi^{-1} \in S \) and \( \beta^{-1} \in T \) are defined by

\[
\beta^{-1} = \beta^{-1}_T = \{ i \beta : i \in d \beta \},
\]

\( \beta^{-1} \) is the inverse of \( \beta \) in \( T \) and

\[
\psi^{-1} = (i \beta^{-1} \psi)^{-1} \quad (i \in d \beta^{-1}).
\]

We have seen in the proof of the regularity of \( S \wr_r (T,I) \) that

\[
(\psi, \beta)(\psi, \beta)^{-1}(\psi, \beta) = (\psi, \beta). \quad \text{We also have that}
\]

\[
(\psi, \beta)^{-1}(\psi, \beta) = (\psi^{-1} \beta^{-1} \psi \beta^{-1} \beta^{-1} \psi^{-1}, \beta^{-1} \beta^{-1} \beta^{-1})
\]

\[
= (\psi^{-1} \beta^{-1} \psi \beta^{-1} \beta^{-1})^{-1}.
\]

For any \( i \in d \beta^{-1}, \)

\[
i \psi^{-1} \beta^{-1} \psi \psi^{-1} = (i \psi^{-1})(i \beta^{-1} \psi)(i \psi^{-1})
\]

\[
= (i \beta^{-1} \psi)^{-1}(i \beta^{-1} \psi)(i \beta^{-1} \psi)^{-1}
\]

\[
= (i \beta^{-1} \psi)^{-1}
\]

\[
= (i \psi^{-1}).
\]

Therefore, \( (\psi^{-1}, \beta^{-1}) \) is the inverse of \( (\psi, \beta) \) in \( S \wr_r (T,I) \). Note that we may equivalently define \( \psi^{-1} \) by

\[
\beta \psi^{-1} = (j \psi)^{-1} \quad (j \in d \beta).
\]
Remark. For any $(\psi, \beta)$ belonging to $S \wr (T,I)$, we have written $(\psi, \beta)^{-1}$ as $(\psi^{-1}, \beta^{-1})$ even though the definition of $\psi^{-1}$ depends on $\beta$. This is not to suggest that if $(\psi, \beta')$ is another member of $S \wr (T,I)$, then the first coordinate of $(\psi, \beta')^{-1}$ is the same as the first coordinate of $(\psi, \beta)^{-1}$. We use $\psi^{-1}$ to avoid notational difficulties and simply note that when $\psi^{-1}$ is used, the member of $(T,I)$ to which it is paired will be understood.

In [N], Neumann defines the (unrestricted) wreath product $A \Wr B$ of the groups $A$ and $B$ as follows.

$$A \Wr B = B \times A^B$$

with products defined by

$$(b, \phi)(c, \psi) = (bc, \phi^c \psi)$$

where, for all $y \in B$, $\phi^c(y) = \phi(yc^{-1})$.

Let $A^d$ be the group defined as $A$ with multiplication $*$ given by, for all $g,h \in A$,

$$g * h = h \cdot g$$

with this last product as in $A$. Then $A \Wr B$ is antiisomorphic to $A^d \wr (B,B)$ using the definition in section 3.1 with $B$ given its Wagner (Cayley) representation:

Define $\Theta : A \Wr B \to A^d \wr (B,B)$ by setting $(b, \phi)\Theta = (\phi^c, \alpha_{b^{-1}})$ where $\phi^c$ is defined by $y\phi^c = \phi(y)$ for all $y \in B$ and $\alpha_{b^{-1}}$ is the permutation corresponding to $b^{-1}$ in the Wagner representation of $B$. Then for all $(b, \phi),(c, \psi) \in A \Wr B$,

$$(c, \psi)\Theta(b, \phi)\Theta = (\psi^c, \alpha_{c^{-1}})(\phi^c, \alpha_{b^{-1}}) = (\psi^c \alpha_{c^{-1}} \phi^c, \alpha_{c^{-1}b^{-1}})$$

while

$$(bc, \phi^c \psi)\Theta = ((\phi^c \psi)^c, \alpha_{c^{-1}b^{-1}}).$$

For all $y \in B$,

$$y(\phi^c \psi)^c = \phi^c \psi(y) = \phi(yc^{-1}) \cdot \psi(y)$$

with this product in $A$ and

$$y \psi^c \alpha_{c^{-1}} \phi^c = (y \psi^c) \cdot (yc^{-1} \phi^c) = (\psi(y)) \cdot (\phi(yc^{-1})) = \phi(yc^{-1}) \cdot \psi(y).$$
Thus, \((c,\psi)\Theta(b,\varphi)\Theta = (bc,\varphi^c\psi)\Theta\). \(\Theta\) is easily seen to be a bijection and so \(\Theta\) is an antiisomorphism.

As a consequence of these remarks, the results concerning wreath products of groups and product varieties of groups found in [N] are valid in the context presented here.

We conclude this section with a remark concerning wreath products of semigroups. In the study of finite semigroups and automata theory wreath products play a significant role (see, for example, [E]). In general, however, the definition of wreath product for semigroups does not ensure that the wreath product of two inverse semigroups will be an inverse semigroup, as Houghton points out in [H]. In fact, the wealth of research on wreath products and pseudovarieties of semigroups did not serve as motivation for our investigations, though some of the ideas presented here have their analogues in finite semigroup theory.

3.2 Subdirectly irreducible inverse semigroups in completely semisimple varieties

The principal factors of a completely semisimple inverse semigroup \(S\) are Brandt semigroups and groups. In fact, at most one principal factor of \(S\) can be a group and this is the case only if \(S\) possesses a minimum ideal which is a group. If \(D\) is a \(\mathcal{J}\)-class of \(S\), but not the minimum \(\mathcal{J}\)-class of \(S\), then the Rees quotient semigroup corresponding to the ideal of \(S\) consisting of those elements \(x\) for which \(\mathcal{J}_x \not\subseteq D\) is an ideal extension of the Brandt semigroup \(D^0\). The canonical homomorphism of this ideal extension of \(D^0\) into the translational hull \(\Omega(D^0)\) of \(D^0\) is one-to-one on \(D\). Consequently, it can be shown that \(S\) can be subdirectly embedded into a product of inverse subsemigroups of \(\Omega(D^0_\alpha)\), where the \(D_\alpha\) are the non-minimum \(\mathcal{J}\)-classes of \(S\), and possibly a group. For any non-minimum \(\mathcal{J}\)-class \(D_\alpha\) of \(S\), the translational hull of \(D^0_\alpha\) is a wreath product of a group \(G\) and \(\mathcal{J}(1)\),
where \( G \) and \( I \) depend on \( \mathcal{D}_G \). Thus, wreath products play an important role in the study of completely semisimple inverse semigroups. In fact, as we will discover in subsequent chapters, wreath products of inverse semigroups in general prove to be useful tools in studying varieties of inverse semigroups.

The following two theorems make clear the connection between wreath products and completely semisimple inverse semigroups and are of fundamental importance.

**Theorem 3.2.1 [P;V.4.6].** For any Brandt semigroup \( S = B(G,I) \), we have
\[
\Omega(S) \equiv G \wr \mathcal{I}(I).
\]

In light of Theorem 3.2.1, wreath products of the form \( G \wr \mathcal{I}(I) \) are related to ideal extensions of Brandt semigroups. The following result is a general description of ideal extensions of Brandt semigroups which we will find useful. For a semigroup \( S \) with zero, we denote \( S \) with its zero removed by \( S^* \).

**Theorem 3.2.2 [P;V.4.7].** Let \( S = B(G,I) \) be a Brandt semigroup and \( Q \) be an inverse semigroup with zero disjoint from \( S \). Let \( \varphi : Q^* \to G \wr \mathcal{I}(I), \) denoted by \( \varphi : q \mapsto (\psi_q,\beta_q), \) be a partial homomorphism such that \( |d\beta_q| \leq 1 \) if \( qr = 0 \) in \( Q \). On \( V = S \cup Q^* \) define a multiplication \( * \) by:
\[
\begin{align*}
(i, g, j) \ast q &= (i, g(j\psi_q), j\beta_q) & \text{if } j \in d\beta_q, \\
q \ast (i, g, j) &= (i\beta_q^{-1}, (i\beta_q^{-1}\psi_q)g, j) & \text{if } i \in r\beta_q,
\end{align*}
\]
and if \( qr = 0 \) in \( Q \),
\[
q \ast r = (k\beta_q^{-1}, (k\beta_q^{-1}\psi_q)(k\psi_r), k\beta_r) & \quad \text{if } \{k\} = r\beta_q \cap d\beta_r,
\]
\[
a \ast b = ab & \quad \text{if } a, b \in S, \text{ or } a, b \in Q^* \text{ and } ab \neq 0,
\]

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and all other products equal to zero. Then V is an ideal extension of S by Q. Conversely, every ideal extension of S by Q can be so constructed.

The first result of this section states that every completely semisimple inverse semigroup is isomorphic to a subdirect product of ideal extensions of Brandt semigroups. This is nothing new. We refer the reader to [Pe1] and [Pe2]. We use the term kernel of S in this section to mean the intersection of all nonzero ideals of S. That is, the kernel of S is the minimum nonzero ideal of S, if it exists.

**Lemma 3.2.3.** Let S be a completely semisimple inverse semigroup. Then S is isomorphic to a subdirect product of ideal extensions of Brandt semigroups and possibly a group. Each of these ideal extensions of Brandt semigroups is an inverse subsemigroup of \( G \wr \mathcal{J}(I) \) where G and I are determined by kernel Brandt semigroup.

**Proof:** Let \( \{D_\alpha : \alpha \in A\} \) be the collection of \( \mathcal{D} \)-classes (or equivalently, \( \mathcal{J} \)-classes) of S. For each \( \alpha \in A \), let \( I_\alpha = \{x \in S : J_x \supseteq D_\alpha\} \). Then \( I_\alpha \) is an ideal of S and the Rees quotient \( S / I_\alpha \) is an ideal extension of \( D_0^\alpha \) or \( I_\alpha = \emptyset \). Observe that if \( I_\alpha = \emptyset \), then \( D_\alpha \) is the kernel of S and so must be a group. As S is completely semisimple, \( D_0^\alpha \) is a Brandt semigroup for each \( \alpha \) in A. Suppose that \( D_\alpha \equiv B(G_\alpha, K_\alpha) \). Let \( \tau_\alpha : S / I_\alpha \to G_\alpha \wr \mathcal{J}(K_\alpha) \) be the canonical homomorphism of \( S / I_\alpha \) into the translational hull of \( D_0^\alpha \). If S possesses a kernel group \( D_\alpha \), then \( \tau_\alpha \) is understood to be the canonical mapping of S into \( \Omega(D_\alpha) = D_\alpha \). Recall that for each \( \alpha \), \( \tau_\alpha \) is one-to-one on \( D_\alpha \).

Let \( \phi_\alpha \) be the natural homomorphism of S onto \( S / I_\alpha \), for each \( \alpha \) in A. Define \( \Phi : S \to \prod_{\alpha \in A} \tau_\alpha(S / I_\alpha) \leq \prod_{\alpha \in A} (G_\alpha \wr \mathcal{J}(K_\alpha)) \) by \( (s\Phi)\pi_\alpha = s\phi_\alpha \tau_\alpha \), where if \( D_\alpha \) is the kernel of S, \( G_\alpha \wr \mathcal{J}(K_\alpha) \) is understood to be \( G_\alpha \). \( \Phi \) is clearly a homomorphism. Let \( x, y \in S \) and suppose that \( x\Phi = y\Phi \). If \( J_x \neq J_y \) then either \( J_x \supseteq J_y \) or \( J_y \supseteq J_x \). If \( J_x \supseteq J_y \) and \( y \in D_\alpha \) then \( x\phi_\alpha \tau_\alpha = y\phi_\alpha \tau_\alpha \) and so
xΦ ≠ yΦ. Likewise, if Jy ≠ Jx then xΦ ≠ yΦ. If Jx = Jy and x,y ∈ Dα, then
xφατα = yφατα implies that x = y as φατα is one-to-one on Dα. It follows that Φ is an
embedding which is rather obviously subdirect.

We will call an inverse subsemigroup S of G wr \( F(I) \) \( k \)-full if it contains all
\((\psi,\beta)\in G wr F(I)\) such that \(|\beta|\leq|\psi|\leq1\). That is, S is a k-full subsemigroup of
G wr \( F(I) \) if S contains the Brandt semigroup of which G wr \( F(I) \) is an extension.

Lemma 3.2.4. Let S be an ideal extension of B(G,I) and let φ be a congruence on G.
Define a binary relation \( \phi^* \) on S by

\[
x \phi^* y \iff \begin{array}{l}
i) \quad x,y \in B(G,I), \quad x = (i,g,j), y = (i,h,j) \text{ and } g \phi h, \text{ or} \\
ii) \quad x = y.
\end{array}
\]

Then \( \phi^* \) is a congruence on S. Moreover, if \( \theta^* \) is a congruence on S and \( \theta \) is its
restriction to some group \( \mathcal{H} \)-class of B(G,I), then i) \( \phi \subseteq \theta \) implies that \( \phi^* \subseteq \theta^* \); and
ii) \( \theta^* \subseteq \phi^* \) implies that \( \theta \subseteq \phi \).

Proof: It is easy to see that \( \phi^* \) is an equivalence relation. Suppose that \( x \phi^* y \) and let
z ∈ S. If \( x = y \) then zx = zy and xz = yz. If \( x = (i,g,j) \) and \( y = (i,h,j) \) with g \( \phi \) h then
a) if \( z = (i',g',j') \) then zx \( \phi^* \) zy and xz \( \phi^* \) yz because \( \phi \) is a congruence; b) using Theorem
3.2.2, \( xz = (i,g(j\psi_z),j\beta_z) \) and \( yz = (i,h(j\psi_z),j\beta_z) \) where \( (\psi_z,\beta_z)\in G wr F(I) \). Since \( \phi \)
is a congruence, xz \( \phi^* \) yz. Likewise, Theorem 3.2.2 also implies that \( xz \phi^* yz \). Thus,
\( \phi^* \) is a congruence.

Let \( \theta^* \) be a congruence on S and suppose that \( \theta \) is the restriction of \( \theta^* \) to the group
\( \mathcal{H} \)-class \( H = \{(i,g,i) : g \in G\} \). If \( x, y \in S \) and \( x \phi^* y \) then either \( x = y \), in which case \( x \theta^* y \), or \( x = (j,g',k), y = (j,h',k) \) and \( g^\prime \phi h' \). But if \( g^\prime \phi h' \) then \( (i,g',i) \theta (i,h',i) \) and so
for any \( j,k \in I \), \( (j,g',k) = (j,1_G,i)(i,g',i)(i,1_G,k) \theta^* (j,1_G,i)(i,h',i)(i,1_G,k) = (j,h',k) \).
Therefore, \( \phi^* \subseteq \theta^* \). Now suppose that \( \theta^* \subseteq \phi^* \). From the definition of \( \phi^* \) we have that
\( \phi^*|_H = \phi \). That is, \((i,g,i) \varphi (i,h,i)\) if and only if \(g \varphi h\). Thus, \(\theta^* \subseteq \varphi^*\) implies that \(\theta \subseteq \varphi\).

**Theorem 3.2.5.** Let \(S\) be a completely semisimple inverse semigroup. Then \(S\) is subdirectly irreducible if and only if \(S\) is a subdirectly irreducible group or \(S\) is a \(k\)-full inverse subsemigroup of \(G \wr \mathcal{F}(I)\) for some set \(I\) and some subdirectly irreducible group \(G\).

**Proof:** Let \(S\) be a subdirectly irreducible completely semisimple inverse semigroup. By Lemma 3.2.3 above, \(S\) is isomorphic to a \(k\)-full inverse subsemigroup of \(\Omega(D^0)\) for some \(D\)-class \(D\) of \(S\) or \(S\) is a group. If \(S\) is a group then it is a subdirectly irreducible group, so assume that \(S\) is isomorphic to a \(k\)-full inverse subsemigroup of \(\Omega(D^0)\) where \(D^0 \equiv B(G,I)\), since \(S\) is completely semisimple. By Theorem 3.2.1, we need only show that \(G\) is subdirectly irreducible. Let \(\theta\) be the minimum non-equality congruence on \(S\) (where we think of \(S\) as a \(k\)-full inverse subsemigroup of \(\Omega(B(G,I))\)). Then \(\theta\) is contained in the Rees congruence relative to \(D^0\). If \((x,y)\) generates \(\theta\) and \(x\) is not \(\mathcal{H}\)-related to \(y\) then it is not difficult to show that \(\theta\) must be the Rees congruence relative to \(D^0\).

[If \(x = (i_1,g_1,j_1)\) and \(y = (i_2,g_2,j_2)\) then for any \((i_3,g_3,j_3) \in B(G,I)\),

\[
(i_3,g_3,j_3) = (i_3,g_3g_1^{-1},i_1)(i_1,g_1,j_1)(j_1,1G,j_3)
\]

and

\[
(i_3,g_3g_1^{-1},i_1)(i_1,g_1,j_1)(j_1,1G,j_3) \theta (i_3,g_3g_1^{-1},i_1)(i_2,g_2,j_2)(j_1,1G,j_3) \neq 0 \text{ if and only if } i_1 = i_2 \text{ and } j_1 = j_2 \text{ if and only if } x \mathcal{H} y.
\]

Therefore, every \((i_3,g_3,j_3) \in B(G,I)\) is \(\theta\)-related to 0 and so \(\theta\) is the Rees congruence relative to \(D^0\).] By Lemma 3.2.4, \(G\) must be simple and hence subdirectly irreducible. So suppose that \((x,y)\) generates \(\theta\) and \(x \mathcal{H} y\). Let \(\varphi\) be any non-identity congruence on \(G\). Then \(\theta \subseteq \varphi^*\) and so by Lemma 3.2.4 (ii), the restriction of \(\theta\) to any group \(\mathcal{H}\)-class, \(\theta^* \subseteq \varphi\). Thus, \(G\) has a minimum non-identity congruence and so must be subdirectly irreducible.
Conversely, suppose that $S$ is a $k$-full inverse subsemigroup of $G \wr \mathcal{J}(I)$ where $G$ is subdirectly irreducible. We identify the minimum non-zero ideal of $S$ with $B(G,I)$. Let $\varphi$ be the minimum non-identity congruence on $G$. We claim that $\varphi^*$ is the minimum non-identity congruence on $S$. Let $\theta$ be the non-identity congruence on $S$ generated by the pair $(x,y)$. Since $S$ is $k$-full and $x \neq y$, there is a $z \in B(G,I)$ such that $z \leq x$, $z \neq y$ (or $z \leq y$, $z \neq x$). Then $z = zz^{-1}x \vartheta zz^{-1}y \neq z$ and $z$ and $zz^{-1}y$ are $\mathcal{D}$-related. If $z$ and $zz^{-1}y$ are not $\mathcal{H}$-related then it is not difficult to show that $\theta$ contains the Rees congruence relative to the ideal $B(G,I)$ which in turn contains $\varphi^*$. If $z \not\sim zz^{-1}y$ then suppose that $z = (i,g,j)$ and $zz^{-1}y = (i,h,j)$ where $g \neq h$. Then $(i,g,i) = (i,g,j)(j,1_G,i) \vartheta (i,h,j)(j,1_G,i) = (i,h,i)$ and so $\theta$ restricted to the group $\mathcal{H}$-class $H = \{(i,g,i) : g \in G\}$ is not the equality. Therefore, $\varphi$ is contained in $\theta$ restricted to $H$ and so by Lemma 3.2.4, $\varphi^* \subseteq \theta$. It now follows that $S$ is subdirectly irreducible.

The subdirectly irreducible completely semisimple inverse semigroups are not only inverse subsemigroups of wreath products of the form $G \wr \mathcal{J}(I)$ for some subdirectly irreducible group $G$, but in fact inverse subsemigroups of wreath products of the form $G \wr (T,I)$ where $G$ is a subdirectly irreducible group and $(T,I)$ is a $k$-full antigroup.

**Lemma 3.2.6.** Let $S$ be a $k$-full inverse subsemigroup of $G \wr \mathcal{J}(I)$, for some group $G$ and some nonempty set $I$, and let $\pi$ denote the natural homomorphism of $S$ into $\mathcal{J}(I)$ given by $(\psi,\beta)\pi = \beta$ for all $(\psi,\beta) \in S$. Then $S\pi$ is an antigroup.

**Proof:** Let $\mu$ denote the greatest idempotent separating congruence on $S\pi$ and suppose that $\beta_1 \mu \beta_2$ for some $\beta_1, \beta_2 \in S\pi$. Since $\mu \subseteq \mathcal{H}$, $\beta_1 \mathcal{H} \beta_2$ and, as a consequence $\beta_1 \beta_1^{-1} = \beta_2 \beta_2^{-1}$, whence $d\beta_1 = d\beta_2$.

Let $i \in d\beta_1 = d\beta_2$. Since $S$ is a $k$-full inverse subsemigroup of $G \wr \mathcal{J}(I)$, the element $\beta$ of $\mathcal{J}(I)$ defined by $d\beta = \{i\}$ and $i\beta = i$, is an idempotent of $S\pi$. By the
definition of $\mu$, $\beta_1^{-1}\beta_1 = \beta_2^{-1}\beta_2$. Now $i\beta_1 \in d\beta_1^{-1}\beta_1$ and so
$(i\beta_1)\beta_1^{-1}\beta_1 = (i\beta_1)\beta_2^{-1}\beta_2$. But $(i\beta_1)\beta_1^{-1}\beta_1 = i\beta_1 = i\beta_1$ and, in order for
$(i\beta_1)\beta_2^{-1}\beta_2$ to be defined, we must have that $(i\beta_1)\beta_2^{-1} = i$ and so
$(i\beta_1)\beta_2^{-1}\beta_2 = i\beta_2 = i\beta_2$. Thus, $i\beta_1 = i\beta_2$ and, since our choice of $i$ was arbitrary, it
follows that $\beta_1 = \beta_2$. Consequently, $S\pi$ is an antigroup.

**Theorem 3.2.7.** Let $\mathcal{V}$ be a completely semisimple variety of inverse semigroups.
Then $\mathcal{V}$ is generated by those members of $\mathcal{V}$ which are subdirectly irreducible groups and
inverse subsemigroups of wreath products of subdirectly irreducible groups and $k$-full
antigroups.

**Proof:** $\mathcal{V}$ is completely determined by its subdirectly irreducible members. By Theorem
3.2.5, these are subdirectly irreducible groups and $k$-full inverse subsemigroups of wreath
products of a subdirectly irreducible group and $\mathcal{S}(I)$ for some $I$. A $k$-full inverse
subsemigroup of a wreath product of a subdirectly irreducible group and $\mathcal{S}(I)$ is an inverse
subsemigroup of a wreath product of a subdirectly irreducible group and a $k$-full antigroup,
by Lemma 3.2.6.

### 3.3 Isomorphic wreath products and connections with varieties

This section contains some structural results concerning wreath products of inverse
semigroups and some connections with varieties.

**Lemma 3.3.1.** Let $T$ and $A$ be inverse semigroups. Then $T \wr (A,A)$ can be embedded
in $\mathcal{S}(T \times A)$.

**Proof:** Define $\Theta : T \wr (A,A) \rightarrow \mathcal{S}(T \times A)$ by $(\psi,\beta)\Theta = f_{(\psi,\beta)}$ where
$$
\operatorname{df}_{(\psi,\beta)} = \{ (t,a) : a \in d\beta \text{ and } t \in T(a\psi)^{-1} \}
$$
and

\[(t,a)f(\psi,\beta) = (t(\alpha\psi), a\beta).\]

We first show that \(f(\psi, \beta) \in \mathcal{S}(T \times A)\). Suppose that for some \((t_1, a_1), (t_2, a_2) \in df(\psi, \beta)\) we have that \((t_1, a_1)f(\psi, \beta) = (t_2, a_2)f(\psi, \beta)\). Then \((t_1(\alpha\psi), a_1\beta) = (t_2(\alpha\psi), a_2\beta)\) and so \(t_1(\alpha\psi) = t_2(\alpha\psi)\) and \(a_1\beta = a_2\beta\). Since \(\beta\) is one-to-one, it follows that \(a_1 = a_2 = a\), say. As a consequence, we have that both \(t_1\) and \(t_2\) belong to \(T(\alpha\psi)^{-1}\) and that \(t_1(\alpha\psi) = t_2(\alpha\psi)\). Therefore, \(t_1 = t_1(\alpha\psi)(\alpha\psi)^{-1} = t_2(\alpha\psi)(\alpha\psi)^{-1} = t_2\). Thus, \((t_1, a_1) = (t_2, a_2)\) and so \(f(\psi, \beta)\) is one-to-one and \(f(\psi, \beta) \in \mathcal{S}(T \times A)\).

Let \((\psi_1, \beta_1), (\psi_2, \beta_2) \in T \text{ wr } (A, A)\). Let \(f_1\) denote \((\psi_1, \beta_1)\Theta, f_2\) denote \((\psi_2, \beta_2)\Theta\) and \(f_3\) denote \((\psi_1^{\beta_1}\psi_2, \beta_1\beta_2)\Theta\). In order to show that \(\Theta\) is a homomorphism we must show that \(f_1f_2 = f_3\). Our first step is to show that \(df_1f_2 = df_3\). From the definition of \(\Theta\) we have that

\[df_1 = \{(t, a) : a \in d\beta_1 \text{ and } t \in T(\alpha\psi_1)^{-1}\},\]
\[df_2 = \{(t, a) : a \in d\beta_2 \text{ and } t \in T(\alpha\psi_2)^{-1}\},\]
\[df_3 = \{(t, a) : a \in d\beta_1\beta_2 \text{ and } t \in T(\alpha(\psi_1^{\beta_1}\psi_2))^{-1}\}.
\]

It follows that

\[df_1f_2 = \{(t, a) : a \in d\beta_1, t \in T(\alpha\psi_1)^{-1} \text{ and } (t(\alpha\psi_1), a\beta_1) \in df_2\}\]
\[= \{(t, a) : a \in d\beta_1, a\beta_1 \in d\beta_2, t \in T(\alpha\psi_1)^{-1} \text{ and } t(\alpha\psi_1) \in T(\alpha\beta_1\psi_2)^{-1}\}\]
\[= \{(t, a) : a \in d\beta_1\beta_2, t \in T(\alpha\psi_1)^{-1} \text{ and } t(\alpha\psi_1) \in T(\alpha\beta_1\psi_2)^{-1}\}.\]

If \((t, a) \in df_3\) then \(t \in T(\alpha(\psi_1^{\beta_1}\psi_2))^{-1} = T((\alpha\psi_1)(\alpha\beta_1\psi_2))^{-1} = T(\alpha\beta_1\psi_2)^{-1}(\alpha\psi_1)^{-1}\) and so \(t \in T(\alpha\psi_1)^{-1}\) and \(t(\alpha\psi_1) \in T(\alpha\beta_1\psi_2)^{-1}(\alpha\psi_1)^{-1}(\alpha\psi_1) = T(\alpha\beta_1\psi_2)^{-1}(\alpha\psi_1)^{-1}(\alpha\psi_1)(\alpha\beta_1\psi_2)(\alpha\beta_1\psi_2)^{-1} \subseteq T(\alpha\beta_1\psi_2)^{-1}\). Moreover, \(a \in d\beta_1\beta_2\) and so \((t, a) \in df_1f_2\). On the other hand, if \((t, a) \in df_1f_2\) then \(t \in T(\alpha\psi_1)^{-1}\) and \(t(\alpha\psi_1) \in T(\alpha\beta_1\psi_2)^{-1}\) and so

\[t = t(\alpha\psi_1)(\alpha\psi_1)^{-1} \in T(\alpha\beta_1\psi_2)^{-1}(\alpha\psi_1)^{-1} = T(\alpha(\psi_1^{\beta_1}\psi_2))^{-1}.\]
Also $a \in d\beta_1\beta_2$ and so $(t,a) \in df_3$. Therefore, $df_1f_2 = df_3$.

Let $(t,a) \in df_1f_2 = df_3$. Then
\[(t,a)f_1f_2 = (t(a\psi_1),a\beta_1)f_2 = (t(a\psi_1)(a\beta_1\psi_2), a\beta_1\beta_2) = (t(a(\psi_1\beta_1\psi_2)), a\beta_1\beta_2) = (t,a)f_3.
\]
It now follows that $\Theta$ is a homomorphism.

Finally, we show that $\Theta$ is one-to-one. Suppose that $(\psi_1,\beta_1)\Theta = (\psi_2,\beta_2)\Theta = f$. Then $df = \{(t,a): a \in d\beta_1$ and $t \in T(a\psi_1)^{-1}\} = \{(t,a): a \in d\beta_2$ and $t \in T(a\psi_2)^{-1}\}$ and
\[(t,a)f = (t(a\psi_1),a\beta_1) = (t(a\psi_2),a\beta_2).
\] Let $a \in d\beta_1 = d\psi_1$. Then $(a\psi_1)^{-1}, a \in df$ and so $a \in d\beta_1$ whence $d\beta_1 \subseteq d\beta_2$. Symmetrically, we obtain that $d\beta_2 \subseteq d\beta_1$ and so $d\beta_1 = d\beta_2$. For any $a \in d\beta_1 = d\beta_2$,
\[((a\psi_1)^{-1},a)f = ((a\psi_1)^{-1}(a\psi_1),a\beta_1) = ((a\psi_1)^{-1}(a\psi_2),a\beta_2).
\]
Thus, $a\beta_1 = a\beta_2$ and so $\beta_1 = \beta_2$. Furthermore,
\[(a\psi_1)^{-1}(a\psi_1) = (a\psi_2)^{-1}(a\psi_2) \text{ and we can likewise obtain}
\]
\[(a\psi_2)^{-1}(a\psi_2) = (a\psi_2)^{-1}(a\psi_1) \text{ by considering } ((a\psi_2)^{-1},a)f. \text{ We thus have that}
\]
\[a\psi_1 = (a\psi_1)(a\psi_1)^{-1}(a\psi_1) = (a\psi_1)(a\psi_1)^{-1}(a\psi_2) = (a\psi_1)(a\psi_1)^{-1}(a\psi_2)(a\psi_2)^{-1}(a\psi_2) = (a\psi_1)(a\psi_1)^{-1}(a\psi_2)(a\psi_2)^{-1}(a\psi_1) = (a\psi_2)(a\psi_2)^{-1}(a\psi_1)(a\psi_1)^{-1}(a\psi_1) = (a\psi_2)(a\psi_2)^{-1}(a\psi_1) = (a\psi_2)(a\psi_2)^{-1}(a\psi_2) = a\psi_2.
\]
Therefore, $\psi_1 = \psi_2$ and as a consequence, $\Theta$ is a monomorphism.

Thus, $\Theta$ is an embedding of $T \wr (A,A)$ into $F(T \times A)$.
We will call the representation of $T \mathrm{wr} (A,A)$ described in Lemma 3.3.1 the **cartesian representation of** $T \mathrm{wr} (A,A)$ and write $(T \mathrm{wr} A, T \times A)$ to denote this representation.

**Lemma 3.3.2.** Let $S, T$ and $A$ be inverse semigroups. Then

$$[S \mathrm{wr} (T,T)] \mathrm{wr} (A,A) \equiv S \mathrm{wr} (T \mathrm{wr} A, T \times A).$$

**Proof:** Let $(\Psi, B) \in [S \mathrm{wr} (T,T)] \mathrm{wr} (A,A)$. Set $a\Psi = (\psi_a, \beta_a)$, for all $a \in d\Psi = dB$.

Let $\Gamma \in \mathcal{F}(T \times A)$ be defined by setting $d\Gamma = \{(t,a) : a \in dB$ and $t \in d\beta_a \}$ and defining $(t,a)\Gamma = (t\beta_a, aB)$. Define $\Phi$, a partial map from $T \times A$ to $S$ by setting $d\Phi = d\Gamma$ and defining $(t,a)\Phi = t\psi_a$. Now $\Gamma$ corresponds to the pair $(\psi', B)$ in the cartesian representation of $T \mathrm{wr} A$, where for all $a \in dB$, $a\psi'$ is the element of $T$ which maps to $\beta_a$ in the Wagner representation of $T$. Thus, the pair $(\Phi, \Gamma) \in S \mathrm{wr} (T \mathrm{wr} A, T \times A)$.

Define $\Theta : [S \mathrm{wr} (T,T)] \mathrm{wr} (A,A) \to S \mathrm{wr} (T \mathrm{wr} A, T \times A)$ by mapping $(\Psi, B)$ to $(\Phi, \Gamma)$, as above.

We first show that $\Theta$ is a homomorphism.

Let $(\Psi_1, B_1), (\Psi_2, B_2) \in [S \mathrm{wr} (T,T)] \mathrm{wr} (A,A)$ and set $(\Psi_1, B_1)\Theta = (\Phi_1, \Gamma_1)$, $(\Psi_2, B_2)\Theta = (\Phi_2, \Gamma_2)$ and $(\Psi_1^{-1} B_2, B_1^{-1} B_2)\Theta = (\Phi_3, \Gamma_3)$. We must show that $(\Phi_1^{-1} \Phi_2, \Gamma_1 \Gamma_2) = (\Phi_3, \Gamma_3)$. For all $a \in dB_i$ set $a\Psi_i = (\psi_{a_i}, \beta_{a_i})$, $i = 1,2$, and set $a(\Psi_1 B_1 \Psi_2) = (\psi_a, \beta_a)$.

$$d\Gamma_1 = \{(t,a) : a \in dB_1$ and $t \in d\beta_{a_1} \}.$$

$$d\Gamma_2 = \{(t,a) : a \in dB_2$ and $t \in d\beta_{a_2} \}.$$

$$d\Gamma_3 = \{(t,a) : a \in dB_1 B_2$ and $t \in d\beta_a \}.$$

Now

$$d\Gamma_1 \Gamma_2 = \{(t,a) : a \in dB_1, aB_1 \in dB_2, t \in d\beta_{a_1}$ and $t\beta_{a_2} \in dB_{a_2} \},$$

where $c = aB_1$, while for all $a \in dB_1 B_2$.

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a(Ψ₁B₁Ψ₂) = (aΨ₁)(aB₁Ψ₂)
= (ψ₁⁺β₁⁺)(ψ₂⁺β₂⁺)
= (ψ₁⁺β₁⁺ψ₂⁺β₁⁺β₂⁺)

and so, as a consequence,

dΓ₃ = \{(t,a) : a ∈ dB₁B₂ and t ∈ dB₁B₂ \} = dΓ₁dΓ₂.

Also, for any (t,a) ∈ dΓ₁dΓ₂ = dΓ₃, (t,a)dΓ₃ = (tβ₁⁺β₂⁺, aB₁B₂), while
(t,a)Γ₁Γ₂ = (tβ₁⁺aB₁)Γ₂ = (tβ₁⁺β₂⁺, aB₁B₂) and so Γ₁Γ₂ = Γ₃. Moreover, for any
(t,a) ∈ dΓ₁Γ₂ = dΓ₃,

(t,a)Φ₁Γ₁Φ₂ = (t,a)Φ₁(t,a)Γ₁Φ₂
= (tψ₁⁺)(tβ₁⁺c)Φ₂
= (tψ₁⁺)(tβ₁⁺ψ₂⁺)

and

(t,a)Φ₃ = tψₐ
= t(ψ₁⁺β₁⁺ψ₂⁺)
= (tψ₁⁺)(tβ₁⁺ψ₂⁺).

Therefore, Φ₃ = Φ₁Γ₁Φ₂ which combined with Γ₃ = Γ₁Γ₂ implies that Θ is a
homomorphism.

Let (Ψ₁,B₁), (Ψ₂,B₂) ∈ [S wr (T,T)] wr (A,A) and suppose that
(Ψ₁,B₁)Θ = (Φ,Γ) = (Ψ₂,B₂)Θ. For all a ∈ dB₁ set aΨ₁ = (ψ₁⁺β₁⁺) and for all
a ∈ dB₂ set aΨ₂ = (ψ₂⁺β₂⁺). By the definition of Θ, we have

dΓ = \{(t,a) : a ∈ dB₁ and t ∈ dB₁ \}
= \{(t,a) : a ∈ dB₂ and t ∈ dB₂ \},

and for all (t,a) ∈ dΓ

(tβ₁⁺aB₁) = (tβ₂⁺aB₂),

and ₜψ₁⁺ = ₜψ₂⁺.
Since T is given the Wagner representation in $S \wr (T, T)$, for all $a \in d B_1$, $d \beta_1 \neq \emptyset$ and for all $a \in d B_2$, $d \beta_2 \neq \emptyset$. Thus, given $a \in d B_1$ there is a $t \in d \beta_1$ so that $(t,a) \in d \Gamma$ and so $aB_1 = aB_2$. Therefore, $d B_1 \subseteq d B_2$ and $B_1$ and $B_2$ agree on the domain of $B_1$. Symmetrically we obtain that $d B_2 \subseteq d B_1$ and $B_1$ and $B_2$ agree on the domain of $B_2$, and so, as a consequence, $B_1 = B_2$. Moreover, we have that $d \Psi_1 = d \Psi_2$, and so in order to show that $\Theta$ is a monomorphism, it remains to show that for all $a \in d B_1 = d B_2$, $(\Psi_1, \beta_1) = (\Psi_2, \beta_2)$. From the definition of $\Gamma$ we have that $t \in d \beta_1$ if and only if $(t,a) \in d \Gamma$ if and only if $t \in d \beta_2$, and so $d \beta_1 = d \beta_2$. Furthermore, for any $t \in d \beta_1 = d \beta_2$ by the definition of $\Theta$, $t \beta_1 = t \beta_2$ and so $\beta_1 = \beta_2$. Also, $d \beta_1 = d \beta_2$ implies that $d \Psi_1 = d \Psi_2$, and again by the definition of $\Theta$, $\Psi_1 = \Psi_2$. It follows that $a \Psi_1 = a \Psi_2$. Therefore, $(\Psi_1, B_1) = (\Psi_2, B_2)$ and $\Theta$ is a monomorphism.

Finally, we show that $\Theta$ is surjective. Let $(\Phi, f) \in S \wr (T \wr A, T \times A)$ and suppose that $f = f(\psi, \beta)$ for some $(\psi, \beta) \in T \wr A$. Consider the pair $(\Psi, \beta)$ where, for all $a \in d \beta$, $a \Psi = (\psi_a, \beta_a)$ and $d \beta_a = \{ t \in T : (t,a) \in df \}$ and $t \beta_a = t(a\psi)$, $t \psi_a = (t,a)\Phi$. Now $\beta_a$ is the representation of $(a\psi)$ in $(T, T)$ and so $(\psi_a, \beta_a) \in S \wr (T,T)$. Also, $\beta \in (A,A)$ and so $(\Psi, \beta) \in [S \wr (T,T)] \wr (A,A)$. We claim that $(\Psi, \beta)\Theta = (\Phi, f)$. Set $(\Psi, \beta)\Theta = (\Psi \Theta, \beta \Theta)$. Then

$$d \beta \Theta = \{ (t,a) : a \in d \beta \text{ and } t \in d \beta_a \}$$

$$= \{ (t,a) : a \in d \beta \text{ and } (t,a) \in df \}$$

$$= \{ (t,a) : (t,a) \in df \}$$

$$= df,$$

and, for all $(t,a) \in df = d \beta \Theta$,

$$(t,a)\beta \Theta = (t \beta_a, a \beta)$$

$$= (t(a\psi), a \beta)$$

$$= (t,a)f.$$
Also, for all \((t,a) \in d \Phi = df = d \beta \Theta = d \psi \Theta\), \((t,a) \psi \Theta = t \psi_a = (t,a) \Phi\). It follows that \((\psi, \beta) \Theta = (\Phi, f)\) and so \(\Theta\) is surjective. Therefore, \(\Theta\) is an isomorphism.

The following proposition is a collection of simple properties of wreath products which suggest a connection between \(A \wr B\) and the variety it generates.

**Proposition 3.3.3.** Let \(A\) and \(B\) be inverse semigroups and let \(\{A_i\}_{i \in I}\) be a collection of inverse semigroups.

a) If \(S\) is an inverse subsemigroup of \(A\) then \(S \wr B\) is an inverse subsemigroup of \(A \wr B\).

b) If \(\alpha : A \to S\) is an epimorphism then there exists an epimorphism \(\mu : A \wr B \to S \wr B\).

c) \(\prod_{i \in I} A_i \wr B\) can be embedded in \(\prod_{i \in I} (A_i \wr B)\).

**Proof:**

a) If \((\psi, \beta) \in S \wr B\) then \(d \beta = d \psi\) and for all \(i \in d \beta, i \psi \in S \subseteq A\). Therefore, \((\psi, \beta) \in A \wr B\). Since \(S \wr B\) is an inverse semigroup, it is an inverse subsemigroup of \(A \wr B\).

b) Define \(\mu : A \wr B \to S \wr B\) by \((\psi, \beta) \mu = (\psi^*, \beta)\) where \(\psi^*\) is defined by setting \(d \psi^* = d \beta\) and for all \(i \in d \psi^*\), defining \(i \psi^* = (i \psi) \alpha\). It is clear that \((\psi^*, \beta) \in S \wr B\). It follows from the definition of the multiplication in wreath products that \(\mu\) is a homomorphism provided that for any \((\psi_1, \beta_1), (\psi_2, \beta_2) \in A \wr B\), we have \(\psi_1^* \beta_1 \psi_2^* = (\psi_1 \beta_1 \psi_2)^*\). From the definition of the multiplication we have that \(d \psi_1^* \beta_1 \psi_2^* = d (\psi_1 \beta_1 \psi_2)^* = d \beta_1 \beta_2\). Let \(i \in d \beta_1 \beta_2\). Then

\[
i (\psi_1 \beta_1 \psi_2)^* = [i (\psi_1 \beta_1 \psi_2)] \alpha
= [(i \psi_1)(i \beta_1 \psi_2)] \alpha
= (i \psi_1) \alpha (i \beta_1 \psi_2) \alpha
\]

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Therefore, μ is a homomorphism.

Let (ψ, β) ∈ S wr B. Define (ψ', β) ∈ A wr B by, for all i ∈ dβ, iψ' ∈ (iψ)α⁻¹. Then (ψ', β)μ = ((ψ')*, β) and for all i ∈ dβ, i(ψ')* = (iψ')α = iψ. Thus, μ is an epimorphism.

c) Define Φ: (∐i∈IAi) wr B → ∐i∈IAi wr B by

(ψ, β)Φ = (ψi, β)i∈I

where if i ∈ dβ and iψ = (aj)j∈I, then iψj = aj.

Suppose that (ψ₁, β₁), (ψ₂, β₂) ∈ (∐i∈IAi) wr B. In order to show that Φ is a homomorphism, we must show that for all j ∈ I and for all i ∈ dβ₁β₂,

i((ψ₁)jβ₁(ψ₂)j) = i(ψ₁β₁ψ₂)j. Suppose that iψ₁ = (aj)j∈I and (iβ₁)ψ₂ = (bj)j∈I. Then i(ψ₁)j = aj and (iβ₁)(ψ₂)j = bj and so i((ψ₁)jβ₁(ψ₂)j) = i(ψ₁)j(iβ₁)(ψ₂)j = ajbj. On the other hand, i(ψ₁β₁ψ₂)j = jth coordinate of iψ₁β₁ψ₂ = jth coordinate of (iψ₁)(iβ₁ψ₂). But this is just the jth coordinate of (aj)j∈I · (bj)j∈I which is simply ajbj. Therefore, Φ is indeed a homomorphism.

Suppose now that (ψ₁, β₁)Φ = (ψ₂, β₂)Φ. From the definition of Φ we obtain that β₁ = β₂ and for all j ∈ I, (ψ₁)j = (ψ₂)j. For all i ∈ dβ₁ = dβ₂, i(ψ₁)j is the jth coordinate of iψ₁ and i(ψ₂)j is the jth coordinate of iψ₂. Therefore, iψ₁ and iψ₂ agree in each of their coordinates and so iψ₁ = iψ₂. This is true for all i ∈ dβ₁ = dβ₂ and so it follows that Φ is a monomorphism. Thus, (∐i∈IAi) wr B can be embedded in ∐i∈IAi wr B.

Corollary 3.3.4. Let $\mathcal{V}$ be a variety of inverse semigroups and suppose that A generates $\mathcal{V}$. Then for any $S ∈ \mathcal{V}$, $S$ wr B ∈ $\langle A$ wr B $\rangle$. 

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Proof: If \( S \in \mathcal{R} \) then \( S \) is a homomorphic image of an inverse subsemigroup \( T \) of a direct power \( A^I \) of \( A \), for some index set \( I \). By Proposition 3.3.8 (c), \( A^I \wr B \) can be embedded in \( (A \wr B)^I \) and as a consequence, \( A^I \wr B \in \langle A \wr B \rangle \). By Proposition 3.3.8 (a), \( T \wr B \) is an inverse subsemigroup of \( A^I \wr B \), since \( T \) is an inverse subsemigroup of \( A^I \). Thus, \( T \wr B \in \langle A \wr B \rangle \). \( S \) is a homomorphic image of \( T \) and so, by Proposition 3.3.8 (b), there is an epimorphism of \( T \wr B \) onto \( S \wr B \). Therefore, \( S \wr B \in \langle A \wr B \rangle \).
CHAPTER FOUR
The Principal Result

Given two varieties $\mathcal{Z}$ and $\mathcal{Y}$ of inverse semigroups, denote by $\text{Wr} (\mathcal{Z}, \mathcal{Y})$ the variety generated by wreath products of semigroups in $\mathcal{Z}$ with semigroups in $\mathcal{Y}$. The principal result of this chapter is a description of the fully invariant congruence on $F_3(X)$ corresponding to $\text{Wr} (\mathcal{Z}, \mathcal{Y})$ in terms of $\rho(\mathcal{Z})$ and $\rho(\mathcal{Y})$ for any pair of varieties $\mathcal{Z}$ and $\mathcal{Y}$ of inverse semigroups. Our description makes use of the Schützenberger graphs of the $\mathcal{Y}$-free inverse semigroup given by the presentation $P = (X; \rho(\mathcal{Y}))$. For any words $w$ and $v$ over $X$, $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ satisfies the equation $w = v$ if and only if $\mathcal{Y}$ satisfies $w = v$ and $\mathcal{Z}$ satisfies an equation dependent upon the paths in the Schützenberger representation of $w$ (and hence $v$) relative to $\mathcal{Y}$ labelled by $w$ and $v$. Given two varieties $\mathcal{Z}$ and $\mathcal{Y}$, we can thus describe a more 'complicated' variety both in terms of its generators and the equations it satisfies if we know the equations satisfied by $\mathcal{Z}$ and $\mathcal{Y}$.

The first section of this chapter deals with associating the path labelled by $w$ in the Schützenberger representation of the word $w$ relative to the variety $\mathcal{Y}$ with a word over some alphabet $Y$. This enables us to prove the main result of this chapter which is concerned with describing the fully invariant congruence corresponding to $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ in terms of the fully invariant congruences corresponding to $\mathcal{Z}$ and $\mathcal{Y}$. The third section concerns itself with basic properties of the $\text{Wr}$ operator, including the result that when $\mathcal{Z}$ is a group variety then $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ is the more familiar Mal'cev product variety $\mathcal{Z} \circ \mathcal{Y}$. Finally, it is shown in the fourth section that $\text{Wr}$ is an associative operator and so $\mathcal{Z}(\mathcal{F})$ is a semigroup under the operation of $\text{Wr}$. 

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4.1 Doubly Labelled Schützenberger Graphs

For any word $w$ over $X$ we require an 'encoding' of the path labelled by $w$ in the Schützenberger representation of $w$ with respect to $\mathcal{F}$ as a word over some alphabet $Y$. In order to do this we extend our definition of Schützenberger graph to what we call the \textit{doubly labelled Schützenberger graph}.

**Definition 4.1.1.** Let $\mathcal{F}$ be a variety of inverse semigroups and $\rho$ the fully invariant congruence on $\mathcal{F}(X)$ corresponding to $\mathcal{F}$. Let $w \in (X \cup X^{-1})^+$ and let $\Gamma_\mathcal{F}(w)$ be the Schützenberger graph of $w$ in the $\mathcal{F}$-free inverse semigroup on $X$. Let $Y$ be a countably infinite set and $Y^{-1}$ a set disjoint from $Y$ and in one-to-one correspondence with $Y$ via $y \leftrightarrow y^{-1}$. Assume that $X \cup X^{-1}$ and $Y \cup Y^{-1}$ are disjoint. From $\Gamma_\mathcal{F}(w)$ we obtain the \textit{doubly labelled Schützenberger graph} $\overline{\Gamma_\mathcal{F}(w)}$ of $w$ relative to $\mathcal{F}$, as follows:

$$\overline{\Gamma_\mathcal{F}(w)} = (\Gamma_\mathcal{F}(w), \lambda_w)$$

where

$$\lambda_w : E(\Gamma_\mathcal{F}(w)) \rightarrow Y \cup Y^{-1}$$

satisfies

(i) $(v_1, x, v_2) \in E(\Gamma_\mathcal{F}(w))$ and $x \in X$ implies that $\lambda_w(v_1, x, v_2) \in Y$;

(ii) $\lambda_w(v_2, x^{-1}, v_1) = [\lambda_w(v_1, x, v_2)]^{-1}$;

(iii) $\lambda_w(v_1, x, v_2) = \lambda_w(v_3, z, v_4)$ implies that $v_1 = v_3$, $v_2 = v_4$, and $x = z$.

We call $x$ the \textit{primary label} and $\lambda_w(v_1, x, v_2)$ the \textit{secondary label} of the edge $(v_1, x, v_2)$. Condition (iii) says that no two distinct edges have the same secondary label, condition (ii) says that inverse edges have inverse secondary labels and condition (i) is just convenient. Thus, the doubly labelled Schützenberger graph of $w$ is just $\Gamma_\mathcal{F}(w)$ with a secondary label attached to each edge such that inverse edges have inverse secondary labels and no two distinct edges have the same secondary label.
We define the derived word $d_{\mathcal{V}}(w)$ of $w$ relative to $\mathcal{V}$ as follows:

Let $v_1$ and $v_2$ be the start and end vertices respectively, of the Schützenberger graph $\Gamma_{\mathcal{V}}(w)$ of $w$ relative to $\mathcal{V}$. Then $w$ labels a $v_1$-$v_2$ walk in $\Gamma_{\mathcal{V}}(w)$ by primary labels, by Lemma 2.8.1 (b) and the definition of the Schützenberger representation of $w$ with respect to $\mathcal{V}$. Let $e_1, \ldots, e_n$ be the edge sequence corresponding to this walk. Define $d_{\mathcal{V}}(w) = \lambda_w(e_1)\lambda_w(e_2)\ldots\lambda_w(e_n) \in (Y \cup Y^{-1})^+$. That is, $d_{\mathcal{V}}(w)$ is just the word obtained by taking the secondary labels from each edge in our $v_1$-$v_2$ walk.

Note that if $w = a_1 \ldots a_k$, $a_i \in X \cup X^{-1}$ for $i = 1, \ldots, k$, and $d_{\mathcal{V}}(w) = b_1 \ldots b_m$, $b_i \in Y \cup Y^{-1}$, then $m = k$ and if $e$ is the edge corresponding to $a_i$ in the start-end path labelled by $w$ in $\Gamma_{\mathcal{V}}(w)$ then $b_i = \lambda_w(e)$ is the secondary label of $e$ in $\Gamma_{\mathcal{V}}(w)$. Note also that $w$ is an instance of its derived word $d_{\mathcal{V}}(w)$ relative to $\mathcal{V}$. That is, $w$ can be obtained from $d_{\mathcal{V}}(w)$ by a substitution of variables.

Example.

![Figure 4.1. The doubly labelled Schützenberger graph $\Gamma_{\mathcal{V}}(w)$.](image)

Figure 4.1 is the doubly labelled Schützenberger graph of the word $w = x_1x_2x_1^{-1}x_2^{-1}$ relative to the variety $\mathcal{V}^1$, the variety of inverse semigroups generated by the five-element
Brandt semigroup with an identity adjoined. Both the start vertex and the end vertex are $v_1$. Reading directly from the graph, we have that the derived word of $w$ with respect to $\mathcal{G}^1$ is $d_{\mathcal{G}^1}(w) = y_1y_2y_3^{-1}y_4^{-1}$.

**Proposition 4.1.2** Let $\mathcal{V}$ be a variety of inverse semigroups and let $w \in X \cup X^{-1}$. Suppose that $w \rho(\mathcal{V}) w^2$. Then $d_{\mathcal{V}}(w^2) = [d_{\mathcal{V}}(w)]^2$.

**Proof:** Let the two roots in the Schützenberger representation of $w$ with respect to $\mathcal{V}$ be $s$ and $e$. Since $w \rho(\mathcal{V}) w^2$ we have that $w \rho(\mathcal{V}) w w^{-1}$ and so, as a consequence, $s = e$. By Lemma 2.8.1 (c), $w$ and $w^2$ both label $s$-$s$ walks in $\Gamma_{\mathcal{V}}(w)$ by primary labels. By Lemma 2.8.1 (a), $\Gamma_{\mathcal{V}}(w)$ is deterministic which implies that the $s$-$s$ walk labelled by $w^2$ is just the $s$-$s$ walk labelled by $w$ taken twice. Thus, $d_{\mathcal{V}}(w^2) = [d_{\mathcal{V}}(w)]^2$, as required.

**Proposition 4.1.3.** Let $\mathcal{V}$ be a variety of inverse semigroups and let $v$ and $w$ be words over $X \cup X^{-1}$. Then $v \rho w$ if and only if $d_{\mathcal{V}}(v) \rho d_{\mathcal{V}}(w)$, where $\rho$ is the Wagner congruence.

**Proof:** The Wagner congruence $\rho$ is generated by the relation

$$\varphi = \{ (aa^{-1}a,a) : a \in (X \cup X^{-1})^+ \} \cup \{ (aa^{-1}bb^{-1},bb^{-1}aa^{-1}) : a,b \in (X \cup X^{-1})^+ \}$$

[P;VIII.1.1].

Now, $w \rho v$ if and only if $w = v$ or

$$w = x_1c_1y_1$$

$$x_1d_1y_1 = x_2c_2y_2$$

$$x_2d_2y_2 = x_3c_3y_3$$

.$$
\[ x_k d_k y_k = v, \]

for some words \( x_i, y_i, c_i, d_i \) such that \( c_i \varphi d_i \) or \( d_i \varphi c_i \), for \( i = 1, \ldots, k \).

If \( w = v \), then \( d_\mathcal{V}(w) = d_\mathcal{V}(v) \) and so \( d_\mathcal{V}(w) \vartriangleleft d_\mathcal{V}(v) \). Otherwise, we proceed by induction on \( k \).

If \( w = x_1 c_1 y_1, x_1 d_1 y_1 = v \) and \( c_1 \varphi d_1 \) then \( w \vartriangleleft v \) implies that \( w \vartriangleleft v \) and so both \( w \) and \( v \) label s-e walks in \( \Gamma_\mathcal{V}(w) \). Because \( \Gamma_\mathcal{V}(w) \) is deterministic, \( d_\mathcal{V}(w) = xcy \) and \( d_\mathcal{V}(v) = xdy \), where \( x, y, c, d \in (Y \cup Y^{-1})^+ \) and \( c, d \) depend upon the paths labelled by \( c_1 \) and \( d_1 \) in the Schützenberger graph of \( w \) with respect to \( \mathcal{V} \). If \( c_1 = a \) and \( d_1 = aa^{-1}a \) then \( d = cc^{-1}c \) since \( \Gamma_\mathcal{V}(x_1 c_1 y_1) \) is deterministic. That is, the path labelled by \( d_1 \) must be the path labelled by \( c_1 \) followed by the path labelled by \( c_1 \) in reverse followed by the path labelled by \( c_1 \). Likewise, if \( d_1 = a \) and \( c_1 = aa^{-1}a \) then \( c = dd^{-1}d \). Thus, in this case, \( d_\mathcal{V}(w) = xcy \vartriangleleft xdy = d_\mathcal{V}(v) \). If \( c_1 = aa^{-1}bb^{-1} \) and \( d_1 = bb^{-1}aa^{-1} \) then \( c = u_1 u_1^{-1}u_2 u_2^{-1} \) and \( d = u_2 u_2^{-1}u_1 u_1^{-1} \), again because \( \Gamma_\mathcal{V}(x_1 c_1 y_1) \) is deterministic and the paths labelled by \( aa^{-1} \) and \( bb^{-1} \) both start and end at the same vertex. Thus, \( d_\mathcal{V}(w) = xu_1 u_1^{-1}u_2 u_2^{-1}y \vartriangleleft xu_2 u_2^{-1}u_1 u_1^{-1}y = d_\mathcal{V}(v) \). In either case, we have that \( d_\mathcal{V}(w) \vartriangleleft d_\mathcal{V}(v) \).

If \( k > 1 \), then \( d_\mathcal{V}(w) \vartriangleleft d_\mathcal{V}(x_k c_k y_k) \) and \( d_\mathcal{V}(x_k c_k y_k) \vartriangleleft d_\mathcal{V}(v) \), by the induction hypothesis, and so \( d_\mathcal{V}(w) \vartriangleleft d_\mathcal{V}(v) \).

Conversely, \( w \) and \( v \) are instances of \( d_\mathcal{V}(w) \) and \( d_\mathcal{V}(v) \), respectively, whence \( d_\mathcal{V}(w) \vartriangleleft d_\mathcal{V}(v) \) implies that \( w \vartriangleleft v \).

In the following lemma and throughout this thesis we use the following shorthand notation. For any words \( v \) and \( w \) over some alphabet \( Z \) and any variety \( \mathcal{V} \) of inverse semigroups, we write \( w \leq_\mathcal{V} v \) to mean \( w \vartriangleleft v \) in the natural partial order on the \( \mathcal{V}\)-free object over the set \( Z \).
Lemma 4.1.4. Let \( w = a_1 \ldots a_k \) and \( v = d_1 \ldots d_m \) with \( w \rho(\mathcal{Y}) v \). Set \( d_\mathcal{Y}(w) = b_1 \ldots b_k \) and \( d_\mathcal{Y}(v) = c_1 \ldots c_m \), where we construct both \( d_\mathcal{Y}(w) \) and \( d_\mathcal{Y}(v) \) from the same doubly labelled Schützenberger graph \( \Gamma_\mathcal{Y}(w) \). Then

\[
\begin{align*}
\text{a)} & \quad b_i = c_j \quad \iff \quad w \leq_Y a_1 \ldots a_{i-1} d_j \ldots d_m \quad \text{and} \quad a_i = d_j \\
\text{b)} & \quad b_i = c_{j-1} \quad \iff \quad w \leq_Y a_1 \ldots a_d d_j \ldots d_m \quad \text{and} \quad a_i = d_{j-1}
\end{align*}
\]

Proof: The proofs of a) and b) are similar. We provide a proof of a). Let \( s \) and \( e \) be the start and end vertices, respectively, corresponding to \( w \) and \( v \) in \( \Gamma_\mathcal{Y}(w) \) (and so also in \( \Gamma_\mathcal{Y}(v) \)). If \( b_i = c_j \) then \( a_i \) and \( d_j \) are primary labels for the same edge in \( \Gamma_\mathcal{Y}(w) \) and so \( a_i = d_j \). Moreover, \( a_1 \ldots a_{i-1} d_j \ldots d_m \) must label an \( s \)-\( e \) walk by primary labels in \( \Gamma_\mathcal{Y}(w) \) and so, by Lemma 2.8.1 (c), \( a_1 \ldots a_{i-1} d_j \ldots d_m \geq_Y w \). Conversely, if \( a_i = d_j \) and \( a_1 \ldots a_{i-1} d_j \ldots d_m \geq_Y w \) then, by Lemma 2.8.1 (c), \( a_1 \ldots a_{i-1} d_j \ldots d_m \) must label an \( s \)-\( e \) walk by primary labels \( \Gamma_\mathcal{Y}(w) \). Since both \( w \) and \( v \) label \( s \)-\( e \) walks by primary labels in \( \Gamma_\mathcal{Y}(w) \) and since \( \Gamma_\mathcal{Y}(w) \) is deterministic by Lemma 2.8.1 (a), we must have that \( a_i \) and \( d_j \) are primary labels for the same edge. It follows that \( b_i = c_j \).

Remark. If we take \( v = w \) in Lemma 4.1.4 we obtain

\[
\begin{align*}
\text{a)} & \quad b_i = b_j \quad \iff \quad w \leq_Y a_1 \ldots a_{i-1} a_j \ldots a_k \quad \text{and} \quad a_i = a_j \\
\text{b)} & \quad b_i = b_{j-1} \quad \iff \quad w \leq_Y a_1 \ldots a_{j-1} a_j \ldots a_k, \quad \text{and} \quad a_i = a_{j-1}
\end{align*}
\]

4.2 The Main Theorem

Definition 4.2.1. For every pair \( \mathcal{Z} \) and \( \mathcal{Y} \) of varieties of inverse semigroups, let

\[
\text{Wr}(\mathcal{Z}, \mathcal{Y}) = \langle S \text{ wr } (T,I) : S \in \mathcal{Z} \text{ and } T \in \mathcal{Y} \rangle.
\]
Varieties of the form \( \text{Wr}(\mathcal{V}, \mathcal{W}) \) will be the focus of our investigations throughout this chapter and chapter five. Our first task is to describe the fully invariant congruence on the free inverse semigroup corresponding to \( \text{Wr}(\mathcal{V}, \mathcal{W}) \) in terms of the fully invariant congruences corresponding to the varieties \( \mathcal{V} \) and \( \mathcal{W} \). Observe that, for any varieties \( \mathcal{V} \) and \( \mathcal{W} \) of inverse semigroups, \( \mathcal{V}, \mathcal{W} \subseteq \text{Wr}(\mathcal{V}, \mathcal{W}) \). This fact will be used throughout this text without explicit reference.

**Definition 4.2.2.** Let \( \mathcal{V} \) and \( \mathcal{W} \) be varieties of inverse semigroups. Define a relation \( \Phi(\mathcal{V}, \mathcal{W}) \) on \( F_\mathcal{I}(X) \) as follows:

\[
u \Phi(\mathcal{V}, \mathcal{W}) w \Leftrightarrow u \rho(\mathcal{W}) w \text{ and } d_\mathcal{W}(u) \rho_\mathcal{V}(\mathcal{Z}) d_\mathcal{W}(w),
\]

where \( d_\mathcal{W}(u) \) and \( d_\mathcal{W}(w) \) are both obtained from the same doubly labelled Schützenberger graph \( \Gamma_\mathcal{W}(w) \).

Observe that \( \Phi(\mathcal{V}, \mathcal{W}) \) is an equivalence relation. We will see in Theorem 4.2.3 that it is not only an equivalence relation, but a fully invariant congruence on \( F_\mathcal{I}(X) \). Note that, if we think of \( \Phi(\mathcal{V}, \mathcal{W}) \) as a relation on \( (X \cup X^{-1})^+ \), then by Proposition 4.1.3, the Wagner congruence \( \rho \subseteq \Phi(\mathcal{V}, \mathcal{W}) \) and so, as a relation on \( F_\mathcal{I}(X) \), \( \Phi(\mathcal{V}, \mathcal{W}) \) is well-defined.

**Example.** If we let \( \mathcal{W} \) be the variety \( \mathcal{S} \) of semilattices and \( \mathcal{V} \) be any variety of inverse semigroups then \( d_\mathcal{W}(u) \rho_\mathcal{V}(\mathcal{Z}) d_\mathcal{W}(w) \) if and only if \( u \rho(\mathcal{W}) w \) because \( d_\mathcal{S}(u) \) is just a relabelling of \( u \), for any word \( u \) over \( X \) (see Proposition 2.8.3 and the example which
accompanies it). Thus, \( u \Phi(\mathcal{Y},\mathcal{F}) w \) if and only if \( u \rho(\mathcal{F}) w \) and \( u \rho(\mathcal{Y}) w \). That is, \( \Phi \) is just \( \rho(\mathcal{F} \vee \mathcal{Y}) \) in this case.

**Example.** If \( \mathcal{Y} \) is any variety of inverse semigroups then \( u \Phi(\mathcal{F},\mathcal{Y}) w \) if and only if \( u \rho(\mathcal{F}) w \) and \( d_{\mathcal{F}}(u) \rho d_{\mathcal{F}}(w) \), where \( \rho \) is the Wagner congruence. By Proposition 4.1.3, \( d_{\mathcal{F}}(u) \rho d_{\mathcal{F}}(w) \) if and only if \( u \rho w \). Thus, \( u \Phi(\mathcal{F},\mathcal{Y}) w \) if and only if \( u \rho(\mathcal{F} \vee \mathcal{Y}) w \). That is, \( \Phi(\mathcal{F},\mathcal{Y}) \) is just \( \rho \).

The following is the principal result of this work. It connects the variety \( \text{Wr}(\mathcal{Y},\mathcal{F}) \) to the relation \( \Phi(\mathcal{Y},\mathcal{F}) \).

**Theorem 4.2.3.** Let \( \text{Wr}(\mathcal{Y},\mathcal{F}) = \{ T \text{ wr } (F,I) : T \in \mathcal{Y}, F \in \mathcal{F} \} \). Then

\[
\Phi(\mathcal{Y},\mathcal{F}) = \rho(\text{Wr}(\mathcal{Y},\mathcal{F})).
\]

**Proof:** For ease of notation, set \( \Phi(\mathcal{Y},\mathcal{F}) = \Phi \) and \( \rho(\text{Wr}(\mathcal{Y},\mathcal{F})) = \rho \).

We first show that \( \Phi \subseteq \rho \). Suppose that \( u \Phi w \). We will show that \( u = w \) is a law in \( \text{Wr}(\mathcal{Y},\mathcal{F}) \). It is sufficient to show that every \( S = T \text{ wr } (F,I) \) which is in the generating set of \( \text{Wr}(\mathcal{Y},\mathcal{F}) \) satisfies \( u = w \).

Let

\[
u = u(x_1, \ldots, x_n) = a_1 \ldots a_k \quad \text{and} \quad w = w(x_1, \ldots, x_n) = d_1 \ldots d_m
\]

where

\[c(u) \cup c(w) = \{x_1, \ldots, x_n\} \quad \text{and} \quad a_i, d_j \in X \cup X^{-1} \text{ for } i = 1, \ldots, k \text{ and } j = 1, \ldots, m.\]

Let \( S = T \text{ wr } (F,I) \) where \( T \in \mathcal{Y} \) and \( F \in \mathcal{F} \). Let \( (\psi_1, \beta_1), \ldots, (\psi_n, \beta_n) \in S \) and suppose that

\[u[(\psi_1, \beta_1), \ldots, (\psi_n, \beta_n)] = (\psi, \beta),\]
Let \( \pi : S \rightarrow F \) be given by \( \pi : (\varphi, \alpha) \rightarrow \alpha \), for all \((\varphi, \alpha) \in S\). Then \( \pi \) is an epimorphism of \( S \) onto \( F \) which we shall call the natural homomorphism of \( S \) onto \( F \). Since \( F \in \mathcal{V} \), it follows from the hypothesis that \( F = S\pi \) satisfies \( u = w \).

Therefore,

\[
\begin{align*}
\beta &= u((\psi_1, \beta_1), \ldots, (\psi_n, \beta_n))\pi \\
&= u((\psi_1, \beta_1))\pi, \ldots, (\psi_n, \beta_n))\pi \\
&= w((\psi_1, \beta_1))\pi, \ldots, (\psi_n, \beta_n))\pi \\
&= [w((\psi_1, \beta_1), \ldots, (\psi_n, \beta_n))\pi \\
&= \beta'.
\end{align*}
\]

Thus, \( \beta = \beta' \) and \( d\psi = d\beta = d\beta' = d\psi' \), so that all we need to show is that \( i\psi = i\psi' \) for all \( i \in d\psi \).

We will write

\[
\psi_{a_i} = \begin{cases} 
\psi_j & \text{if } a_i = x_j \\
\psi_{j^{-1}} & \text{if } a_i = x_j^{-1}
\end{cases}
\]

\[
\beta_{a_i} = \begin{cases} 
\beta_j & \text{if } a_i = x_j \\
\beta_{j^{-1}} & \text{if } a_i = x_j^{-1}
\end{cases}
\]

We will also write \( \beta_{a_1 \ldots a_i} \) for \( \beta_{a_1} \beta_{a_2} \ldots \beta_{a_i} \). Observe that with this notation \( \psi_{a_i^{-1}} = \psi_{a_i}^{-1} \) and \( \beta_{a_i^{-1}} = \beta_{a_i}^{-1} \).

Let \( d_{\mathcal{V}}(u) = b_1 \ldots b_k \) and \( d_{\mathcal{V}}(w) = c_1 \ldots c_m \). Let \( i \in d\psi = d\beta \). We first prove the following statements.

1) \( b_p = b_j \quad \Rightarrow \quad (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p} = (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_j} \);
2) \( b_p = b_j^{-1} \quad \Rightarrow \quad (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p} = [(i\beta_{a_1 \ldots a_{p-1}})\psi_{a_j}]^{-1} \);
3) \( b_p = c_j \quad \Rightarrow \quad (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p} = (i\beta_{d_1 \ldots d_{p-1}})\psi_{d_j} \);
4) $b_p = c_j^{-1} \Rightarrow (i\beta_{a_1...a_{p-1}})\psi_{a_p} = [(i\beta_{d_1...d_{k-1}})\psi_{d_j}]^{-1};$

5) $c_p = c_j \Rightarrow (i\beta_{d_1...d_{p-1}})\psi_{d_j} = (i\beta_{d_1...d_{j-1}})\psi_{d_j};$

6) $c_p = c_j^{-1} \Rightarrow (i\beta_{d_1...d_{p-1}})\psi_{d_j} = [(i\beta_{d_1...d_{k-1}})\psi_{d_j}]^{-1}.$

1) Suppose that $b_p = b_j$ for $p < j \leq k$. Then $a_p = a_j$ and $u' = a_1...a_{p-1}a_j...a_k \geq u$, by Lemma 4.1.4. Again $S\pi \in \mathcal{Y}$ and so $\beta'' = \beta_{a_1...a_{p-1}a_j...a_k} \geq \beta = \beta_{a_1...a_k}$. This means that $d\beta \subseteq d\beta''$ and $\beta$ and $\beta''$ agree when both are defined, and so $i\beta = i\beta''$. But then, $(i\beta_{a_1...a_{j-1}})\beta_{a_j...a_k} = i\beta = i\beta'' = (i\beta_{a_1...a_{p-1}})\beta_{a_j...a_k}$. Since $\beta_{a_j...a_k}$ is one-to-one, we have $i\beta_{a_1...a_{j-1}} = i\beta_{a_1...a_{p-1}}$. This, combined with $a_p = a_j$, gives $(i\beta_{a_1...a_{j-1}})\psi_{a_j} = (i\beta_{a_1...a_{p-1}})\psi_{a_p}.$

2) Suppose that $b_p = b_j^{-1}$ with $p < j \leq k$. Then $a_p = a_j^{-1}$ and $u' = a_1...a_{p-1}a_j...a_k \geq u$, by Lemma 4.1.4. Again $S\pi \in \mathcal{Y}$ and so $\beta'' = \beta_{a_1...a_{p-1}a_j...a_k} \geq \beta = \beta_{a_1...a_k}$. This means that $d\beta \subseteq d\beta''$ and $\beta$ and $\beta''$ agree when both are defined. As in 1) we obtain $i\beta_{a_1...a_p} = i\beta_{a_1...a_{j-1}}$. Then

\[
[i(\beta_{a_1...a_{j-1}})\psi_{a_j}]^{-1} = [(i\beta_{a_1...a_p})\psi_{a_{p-1}}]^{-1} = [(i\beta_{a_1...a_{p-1}})\psi_{a_{p-1}}]^{-1} = [((i\beta_{a_1...a_{p-1}})\psi_{a_{p-1}}]^{-1} (\text{definition of } \psi_{a_{p-1}}) = (i\beta_{a_1...a_{p-1}})\psi_{a_p}.
\]

3) Suppose that $b_p = c_j$. By Lemma 4.1.4, $a_p = d_j$ and $u' = a_1...a_{p-1}d_j...d_m \geq u$. Since $S\pi \in \mathcal{Y}$, we have that $\beta'' = \beta_{a_1...a_{p-1}d_j...d_m} \geq \beta = \beta_{a_1...a_k}$. This means that $d\beta \subseteq d\beta''$ and both $\beta$ and $\beta''$ agree on $d\beta$. In particular, $i\beta'' = i\beta = i\beta'$. But then, $(i\beta_{a_1...a_{j-1}})\beta_{d_j...d_m} = i\beta' = (i\beta_{d_1...d_{j-1}})\beta_{d_j...d_m}$. Since $\beta_{d_j...d_m}$ is one-to-one, $i\beta_{a_1...a_{p-1}} = i\beta_{d_1...d_{j-1}}$. Combining this with the fact that $a_p = d_j$ gives

$(i\beta_{a_1...a_{p-1}})\psi_{a_p} = (i\beta_{d_1...d_{j-1}})\psi_{d_j}.$
4) Suppose that \( b_p = c_{j-1} \). By Lemma 4.1.4, \( a_p = d_{j-1} \) and \( u^{*} = a_1...a_pd_j...d_m \geq u \).

Again \( S \pi \in \mathcal{Y} \) implies that \( \beta^{\prime\prime} = \beta a_1...a_pd_j...d_m \geq \beta a_1...a_k = \beta \). This means that \( d\beta \preceq d\beta^{\prime\prime} \) and both \( \beta \) and \( \beta^{\prime\prime} \) agree on \( d\beta \). In particular, \( i\beta^{\prime\prime} = i\beta = i\beta^{\prime} \). But then, \( (i\beta a_1...a_p)\beta d_j...d_m = i\beta^{\prime\prime} = i\beta^{\prime} = (i\beta d_1...d_{j-1})\beta d_j...d_m \). Since \( \beta d_j...d_m \) is one-to-one, \( i\beta a_1...a_p = i\beta d_1...d_{j-1} \).

\[
[i\beta d_1...d_{j-1})\psi d_j]^{-1} = [(i\beta a_1...a_p)\psi a_p^{-1}]^{-1} \\
= [(i\beta a_1...a_p)\psi a_p^{-1}]^{-1} \\
= [(i\beta a_1...a_p)\psi a_p^{-1}]^{-1} \text{ (definition of } \psi a_p^{-1}) \\
= (i\beta a_1...a_{p-1})\psi a_p.
\]

5) and 6) The proofs use Lemma 4.1.4 and are similar to the proofs of 1) and 2).

Multiplying \( u[(\psi_1,\beta_1),...(\psi_n,\beta_n)] \) from left to right we obtain

\[ i\psi = (i\psi a_1)(i\beta a_1,\psi a_2)(i\beta a_1a_2,\psi a_3)...(i\beta a_1...a_{k-1},\psi a_k). \]

Likewise, we obtain

\[ i\psi^{\prime} = (i\psi d_1)(i\beta d_1,\psi d_2)(i\beta d_1d_2,\psi d_3)...(i\beta d_1...d_{m-1},\psi d_m). \]

By 1)–6) above, the expressions on the right-hand side are instances of \( d_{\mathcal{Y}}(u) \) and \( d_{\mathcal{Y}}(w) \) by the same substitution of variables. Since \( T \in \mathcal{Y} \), \( T \) satisfies \( d_{\mathcal{Y}}(u) = d_{\mathcal{Y}}(w) \) and so, as a consequence, \( i\psi = i\psi^{\prime} \). It now follows that \( (\psi,\beta) = (\psi^{\prime},\beta^{\prime}) \) and hence that \( T wr (F,I) \) satisfies \( u = w \). Therefore, the generating semigroups of \( Wr(\mathcal{Z},\mathcal{Y}) \) satisfy \( u = w \) and so \( Wr(\mathcal{Z},\mathcal{Y}) \) also satisfies \( u = w \), whence \( \Phi \subseteq \rho \).
Before we prove that \( p \subseteq \Phi \), we require a construction and a preliminary lemma.

**Construction 4.2.4.** Let \( w, u \in (X \cup X)^{+} \) be such that \( w \rho(\mathcal{F}) u \) and let \( \Gamma_{\mathcal{F}}(w) \) be their doubly labelled Schützenberger graph relative to \( \mathcal{F} \). Let \( s \) and \( e \) be the start and end vertices, respectively, corresponding to \( w \) (and \( u \)) in \( \Gamma_{\mathcal{F}}(w) \) and let \( V \) denote the set of vertices of \( \Gamma_{\mathcal{F}}(w) \). Suppose that \( c(w) \cup c(u) = \{x_1, \ldots, x_m\} \) and \( c(d_{\mathcal{F}}(w)) \cup c(d_{\mathcal{F}}(u)) = \{y_1, \ldots, y_n\} \), where \( x_1, \ldots, x_m \in X \) and \( y_1, \ldots, y_n \in Y \). Here \( X \) is the set of primary labels and \( Y \) is the set of secondary labels in \( \Gamma_{\mathcal{F}}(w) \). Let \( T \) be any inverse semigroup and let \( t_1, \ldots, t_n \in T \). We use \( \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_n\} \) and \( t_1, \ldots, t_n \) to construct an inverse semigroup \( S \) as follows.

For \( i = 1, \ldots, m \) let \( s_i = (\psi_i, \beta_i) \) where \( \psi_i \in T^V, \beta_i \in \mathcal{S}(V) \) are defined by:

\[
\begin{align*}
d\beta_i &= d\psi_i = \{v \in V : (v, x_i, v') \in E(\Gamma_{\mathcal{F}}(w)) \text{ for some } v' \in V\} \\
v\beta_i &= v' \quad \text{where } (v, x_i, v') \in E(\Gamma_{\mathcal{F}}(w)), \\
v\psi_i &= \begin{cases} t_k & \text{if } \lambda_w(v, x_i, v') = y_k, \\
t & \text{if } \lambda_w(v, x_i, v') \notin \{y_1, \ldots, y_n\}. \end{cases}
\end{align*}
\]

Here, \( t \) is some fixed element of \( T \).

Then \( s_i \in T \wr \mathcal{S}(V) \), for \( i = 1, \ldots, m \). Let \( S \) be the inverse subsemigroup of \( T \wr \mathcal{S}(V) \) generated by \( \{s_1, \ldots, s_m\} \). Note that \( S \) depends on \( T, t_1, \ldots, t_n, \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_n\} \) and \( \Gamma_{\mathcal{F}}(w) \).
Observe that if \( u \) is a word in \( \{ x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1} \}^+ \) and \( (\psi, \beta) = s \) is the element of \( S \) obtained from \( u \) by substituting \( s_j \) for \( x_j \), \( j = 1, \ldots, m \), then for all \( v \in \text{d} \beta \), \( u \) labels a \( v \)-\( v' \) walk by primary labels in \( \Gamma_{\Psi}(w) \) if and only if \( v\beta = v' \).

**Lemma 4.2.5.** Let \( \mathcal{V} \) be a variety of inverse semigroups and suppose that \( T \) is an inverse semigroup. Let \( u, w \in (X \cup X)^+ \) be such that \( u \rho(\mathcal{V}) w \) and set \( F = F_{\mathcal{V}}(X) \). Let \( S \) be as constructed in 4.2.4 using any \( t_1, \ldots, t_n \in T \) and \( \Gamma_{\mathcal{V}}(w) \). Let \( (F,F) \) be the Wagner representation of \( F \) by partial right translations. Then \( S \in \langle T \wr (F,F) \rangle \). If \( T \) is a member of the variety \( \mathcal{Z} \) then \( S \in \text{Wr}(\mathcal{Z}, \mathcal{V}) \).

**Proof:** Let \( R_w \) be the \( \mathcal{A} \)-class of \( w\rho(\mathcal{V}) \) in \( F_{\mathcal{V}}(X) \).

Define

\[
\theta : T \wr (F,F) \to T \wr \mathcal{A}(R_w) \text{ by}
\]

\[
\theta : (\psi, \beta) \to (\psi \theta, \beta \theta) \text{ where}
\]

\[
d \psi \theta = d \beta \theta = \{ u \in d \beta : u \in R_w, u \beta \in R_w \} \text{ and for all } u \in d \beta \theta, u \beta \theta = u \beta, u \psi \theta = u \psi.
\]

We first show that \( \theta \) is a homomorphism. Observe that \( \theta \) as defined maps \( T \wr (F,F) \) into \( T \wr \mathcal{A}(R_w) \). Let \( (\psi_1, \beta_1), (\psi_2, \beta_2) \in T \wr (F,F) \). Now \( F \) is given the Wagner representation by partial right translations of itself, so there exist \( v_1, v_2 \in F \) such that \( d \beta_1 = Fv_1^{-1}, d \beta_2 = Fv_2^{-1} \) and for all \( v \) in the domain of \( \beta_1 \), \( v \beta_1 = vv_1 \) and for all \( v \) in the domain of \( \beta_2 \), \( v \beta_2 = vv_2 \).

Since \( (\psi_1, \beta_1) \theta (\psi_2, \beta_2) \theta = (\psi_1 \theta \beta_1 \theta, \beta_1 \theta \beta_2 \theta) \) and \( (\psi_1, \beta_1)(\psi_2, \beta_2) = ((\psi_1 \beta_1 \psi_2), \beta_1 \beta_2 \theta) \), we must show that

\[
\beta_1 \beta_2 \theta = \beta_1 \beta_2 \theta \text{ and } \psi_1 \theta \beta_1 \theta = (\psi_1 \beta_1 \psi_2) \theta.
\]

The domain of \( \beta_1 \theta \beta_2 \theta \) is the set \( \{ u : u \in R_w, u \beta_1 \in R_w \text{ and } u \beta_1 \beta_2 \in R_w \} \), while the domain of \( \beta_1 \beta_2 \theta \) is the set \( \{ u : u \in R_w \text{ and } u \beta_1 \beta_2 \in R_w \} \). But if \( u \in R_w \) and \( u \beta_1 \beta_2 \in R_w \) then \( u \in R_w \) and \( uv_1 v_2 \in R_w \) and so \( uv_1 \in R_w \). As a consequence,

\[
d \beta_1 \beta_2 \theta = d \beta_1 \theta \beta_2 \theta \text{ and so, for all } v \in d \beta_1 \beta_2 \theta, v \beta_1 \beta_2 \theta = v \beta_1 \beta_2 = v \beta_1 \theta \beta_2 \theta.
\]
Since $d\beta_1\beta_2\theta = d\beta_1\theta\beta_2\theta$ we have $d\psi_1\theta \beta_1\psi_2\theta = d(\psi_1\beta_1\psi_2)\theta$. For any $v \in d\beta_1\beta_2\theta$, $v(\psi_1\beta_1\psi_2)\theta = v(\psi_1\beta_1\psi_2) = (v\psi_1)(v\beta_1\psi_2)$, while $v\psi_1\theta \beta_1\psi_2\theta = (v\psi_1\theta)(v\beta_1\theta\psi_2\theta) = (v\psi_1)(v\beta_1\psi_2)$ since $v \in d\beta_1\theta\beta_2\theta$ implies that $v\beta_1\theta = v\beta_1$ and $(v\beta_1)\psi_2\theta = v\beta_1\psi_2$. Therefore, $\psi_1\theta \beta_1\psi_2\theta = (\psi_1\beta_1\psi_2)\theta$. It now follows that $\theta$ is a homomorphism.

We now claim that $S$ is an inverse subsemigroup of the image of $\theta$ in $T \wr \mathcal{I}(R_w)$. It is enough to show that each generator of $S$ is in the image of $\theta$. Let $s_i = (\psi_i, \beta_i)$ be a generator of $S$. Then,

$$d\psi_i = d\beta_i = \{ v \in R_w : (v, x_i, v') \in E(\Gamma(\mathcal{I}(w))) \text{ for some } v' \in R_w \}$$

$$= \{ v \in R_w : v x_i \rho(\mathcal{I}) \in R_w \}$$

$$\subseteq F x_i^{-1} \rho(\mathcal{I}),$$

where the last containment follows from the more general fact that if $a$ and $ax$ are $\mathcal{R}$-related elements of the same inverse semigroup, then $a = axx^{-1}$.

We choose $(\psi, \beta) \in T \wr (F, F)$ as follows. Let $\beta$ be the representation in the Wagner representation of $F$ of $x_i \rho(\mathcal{I})$ so that $d\beta = F x_i^{-1} \rho(\mathcal{I})$. Let $\psi$ be any mapping from $d\beta$ into $T$ such that, for any $v \in R_w \cap d\beta$ such that $v\beta \in R_w$, $v\psi = v\psi_i$. Such a $\psi$ exists since $d\psi_i \subseteq d\beta$. We clearly have that $(\psi, \beta) \in T \wr (F, F)$.

Consider now $(\psi, \beta)\theta = (\psi\theta, \beta\theta)$, where

$$d\psi\theta = d\beta\theta = \{ v \in d\beta : v \in R_w \text{ and } v\beta \in R_w \}$$

$$= \{ v \in d\beta : v \in R_w \text{ and } v x_i \rho(\mathcal{I}) \in R_w \}$$

$$= \{ v \in F x_i^{-1} \rho(\mathcal{I}) : v \in R_w \text{ and } v x_i \rho(\mathcal{I}) \in R_w \}$$

$$= \{ v \in R_w : v = v x_i x_i^{-1} \rho(\mathcal{I}) \text{ and } v x_i \rho(\mathcal{I}) \in R_w \}$$

$$= \{ v \in R_w : v x_i \rho(\mathcal{I}) \in R_w \}$$

$$= d\beta_i = d\psi_i.$$

Moreover, for any $v \in d\beta\theta$, $v\beta\theta = v\beta = v x_i \rho(\mathcal{I}) = v\beta_i$ and $v\psi\theta = v\psi = v\psi_i$ by our choice of $(\psi, \beta)$.

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Thus, \( \langle \psi, \beta \rangle \Theta = (\psi_i, \beta_i) \) and so \( S \) is an inverse subsemigroup of the image of \( \Theta \) in \( T \wr \mathcal{F}(R_w) \). It follows that \( S \in \langle T \wr (F,F) \rangle \). If \( T \) is a member of the variety \( \mathcal{V} \) then \( T \wr (F,F) \in \mathsf{Wr}(\mathcal{V}, \mathcal{V}) \) and so we also have that \( S \in \mathsf{Wr}(\mathcal{V}, \mathcal{V}) \).

We now show that \( \rho \subseteq \Phi \). Suppose that \( u = w \) is a law in \( \mathsf{Wr}(\mathcal{V}, \mathcal{V}) \). Since \( \mathcal{V} \subseteq \mathsf{Wr}(\mathcal{V}, \mathcal{V}) \), \( \mathcal{V} \) satisfies \( u = w \). Therefore, to prove the theorem, we need only show that \( d_{\mathcal{V}}(u) \rho_{\mathcal{V}}(\mathcal{V}) \) \( d_{\mathcal{V}}(w) \).

Suppose that \( c(u) \cup c(w) = \{x_1, \ldots, x_m\} \) and that \( c(d_{\mathcal{V}}(u)) \cup c(d_{\mathcal{V}}(w)) = \{y_1, \ldots, y_n\} \). Let \( T \) be any generator of \( \mathcal{V} \) (for e.g., we may take \( T = F\mathcal{V}(X) \)). It is sufficient to show that \( T \) satisfies \( d_{\mathcal{V}}(u) = d_{\mathcal{V}}(w) \).

Let \( t_1, \ldots, t_n \in T \), and let \( S \) be the inverse semigroup which is constructed as in 4.2.4 using \( t_1, \ldots, t_n \) and \( \Gamma_{\mathcal{V}}(w) \). By Lemma 4.2.5, \( S \in \mathsf{Wr}(\mathcal{V}, \mathcal{V}) \) and so \( S \) satisfies \( u = w \). Therefore, with \( s_i = (\psi_i, \beta_i) \), \( u(s_1, \ldots, s_m) = w(s_1, \ldots, s_m) = (\psi, \beta) \), say. Let \( v \) be the start vertex of \( u \) and \( w \) in \( \Gamma_{\mathcal{V}}(w) \) (\( u \) and \( w \) have the same start vertex in \( \Gamma_{\mathcal{V}}(w) \) since \( u \rho(\mathcal{V}) w \)). Set \( u = d_1 \ldots d_k \), where \( d_i \in X \cup X^{-1} \).

As before, we write
\[
\psi_{d_i} = \begin{cases} 
\psi_j & \text{if } d_i = x_j \\
\psi_{j^{-1}} & \text{if } d_i = x_j^{-1}
\end{cases}
\]
\[
\beta_{d_i} = \begin{cases} 
\beta_j & \text{if } d_i = x_j \\
\beta_{j^{-1}} & \text{if } d_i = x_j^{-1}
\end{cases}
\]

Again, we write \( \beta_{d_1} \ldots d_i \) for \( \beta_{d_1} \beta_{d_2} \ldots \beta_{d_i} \).

Now \( u \) labels a \( v \)-\( v \beta \) walk in \( \Gamma_{\mathcal{V}}(w) \) (see the observation made after the construction) and the edge sequence corresponding to this walk is \( (v, d_1, v \beta_{d_1}), \ldots, (v \beta_{d_1} d_2, v \beta_{d_1} \beta_{d_2} d_3), \ldots, (v \beta_{d_1} \ldots d_{k-1}, v \beta_{d_1} \ldots d_k) \). By definition, for any
\( i < k, \quad v\beta_{d_1 \ldots d_{i-1}}w_{d_i} = t_q \) if and only if the edge between \( v\beta_{d_1 \ldots d_{i-1}} \) and \( v\beta_{d_1 \ldots d_i} \) with primary label \( d_i \) has secondary label \( y_q \). Thus,

\[
\nu \psi = (\nu \psi_{d_i})(v\beta_{d_1}w_{d_2})(v\beta_{d_1}d_2w_{d_3}) \ldots (v\beta_{d_1 \ldots d_{i-1}}w_{d_i})
\]

\[= d_\psi(u)[t_1, \ldots, t_n].\]

Similarly, we obtain \( \nu \psi = d_\psi(w)[t_1, \ldots, t_n] \) and so we conclude that

\[d_\psi(u)[t_1, \ldots, t_n] = d_\psi(w)[t_1, \ldots, t_n].\]

Since the \( t_i \) were chosen arbitrarily, we have \( T \) (and hence \( Z \)) satisfies \( d_\psi(u) = d_\psi(w) \). Therefore, \( d_\psi(u) \rho Y(Z) d_\psi(w) \) and \( u \Phi w \). •

**Theorem 4.2.6.** Let \( Z \) and \( \Psi \) be varieties of inverse semigroups. Let \( (F\Psi(X),F\Psi(X)) \) be the Wagner representation of \( F\Psi(X) \) by partial right translations. If \( T \) generates \( Z \) then \( T \) \( {\text{wr}} \) \( (F\Psi(X),F\Psi(X)) \) generates \( Wr(Z,\Psi) \). In particular,

\[F\Psi(X) \ {\text{wr}} \ (F\Psi(X),F\Psi(X)) \text{ generates } Wr(Z,\Psi).\]

**Proof:** Clearly \( \langle T \ {\text{wr}} \ (F\Psi(X),F\Psi(X)) \rangle \subseteq Wr(Z,\Psi) \). Thus, to prove the corollary we need only show that if \( T \) \( {\text{wr}} \) \( (F\Psi(X),F\Psi(X)) \) satisfies the equation \( u = w \) then \( \Psi \) satisfies \( u = w \) and \( Z \) satisfies \( d_\psi(u) = d_\psi(w) \).

Since \( F\Psi(X) \in \langle T \ {\text{wr}} \ (F\Psi(X),F\Psi(X)) \rangle \) we have that

\[\Psi \subseteq \langle T \ {\text{wr}} \ (F\Psi(X),F\Psi(X)) \rangle \text{ and so } \Psi \text{ satisfies } u = w.\]

Since \( T \) generates \( Z \), it is sufficient to show that \( T \) satisfies \( d_\psi(u) = d_\psi(w) \). We may now use the proof of \( \rho \subseteq \Phi \) in Theorem 4.2.3 to demonstrate this, noting that any semigroup \( S \), as constructed in 4.2.4, which is used in this proof is a member of the variety \( \langle T \ {\text{wr}} \ (F\Psi(X),F\Psi(X)) \rangle \) by Lemma 4.2.5. •
**Remark.** We cannot in general replace $\mathcal{F}(X)$ in Theorem 4.2.6 by an arbitrary inverse semigroup which generates $\mathcal{V}$. An example well known to group theorists illustrates this (cf [N;22.23]):

Let $\mathcal{A}_2$ be the variety of abelian groups of exponent 2 and let $C_2$ be the cyclic group of order 2. $F_{\mathcal{A}_2}(X) \wr C_2$ is nilpotent of class 2 and so satisfies the identity $[[x,y],z] = [[x,y],z]^2$. On the other hand $Wr(\mathcal{A}_2,\mathcal{A}_2)$ does not satisfy this identity. One can demonstrate this directly by showing that $F_{\mathcal{A}_2} \wr C_2^2$ does not satisfy $[[x,y],z] = [[x,y],z]^2$ or, by appealing to Theorem 4.2.3, one can simply note that $\mathcal{A}_2$ does not satisfy the identity $y_1y_2y_3^{-1}y_4^{-1}y_5y_6y_7y_8^{-1}y_9^{-1}y_5^{-1} \in E$; that is, $d_{\mathcal{A}_2}([[x,y],z])$ is not a law in $\mathcal{A}_2$.

**Example.** The following diagram (Figure 4.2) is the Schützenberger graph of $w = x_1x_2x_1^{-1}x_2^{-1}$ relative to the variety $\mathcal{B}^1$, where $v_1 = s = e$, the start and end vertices corresponding to $w$. Here $d_{\mathcal{B}}(w) = y_1y_2y_3^{-1}y_4^{-1}$.

From this we can conclude that, for any nontrivial group variety $\mathcal{V}$, $\text{Wr}(\mathcal{V},\mathcal{B}^1) = \mathcal{V} \circ \mathcal{B}^1$ does not satisfy the equation

$$x_1x_2x_1^{-1}x_2^{-1} = (x_1x_2x_1^{-1}x_2^{-1})^2,$$

since no group variety other than the trivial variety satisfies the equation $y_1y_2y_3^{-1}y_4^{-1} = (y_1y_2y_3^{-1}y_4^{-1})^2$. Moreover, $\text{Wr}(\mathcal{V},\mathcal{B})$ does not satisfy $x_1x_2x_1^{-1}x_2^{-1} \in E$ whenever $\mathcal{V} \supseteq \mathcal{B}^1$ and $\mathcal{V} \in \{\mathcal{F},\mathcal{P}\}$. This is a consequence of the fact that only $\mathcal{F}$ and $\mathcal{P}$ satisfy the equation $y_1y_2y_3^{-1}y_4^{-1} \in E$ and Proposition 4.3.1, the first result of the next section.
4.3 Basic Properties of $\text{Wr}(\_,\_)$

This section is devoted to several consequences of the main theorem discussed in the previous section. We first present some properties of $\text{Wr}(\mathcal{Z},\mathcal{V})$ and then show that when $\mathcal{Z}$ is a group variety, $\text{Wr}(\mathcal{Z},\mathcal{V})$ is the more familiar variety $\mathcal{Z} \circ \mathcal{V}$, the Mal'cev product of $\mathcal{Z}$ and $\mathcal{V}$.

Proposition 4.3.1. Let $\mathcal{Z}, \mathcal{V}, \mathcal{W}$ and $\mathcal{X}$ be varieties of inverse semigroups. If $\mathcal{Z} \subseteq \mathcal{W}$ and $\mathcal{V} \subseteq \mathcal{X}$, then $\text{Wr}(\mathcal{Z},\mathcal{V}) \subseteq \text{Wr}(\mathcal{W},\mathcal{X})$.

Proof: This is immediate from the definition of $\text{Wr}(\_,\_)$. 

Proposition 4.3.2. Let $\mathcal{Z}$ and $\mathcal{V}$ be varieties of inverse semigroups. Then $\{ S \text{ wr } (T,I) : S \in \mathcal{Z}, T \in \mathcal{V} \} \subseteq \mathcal{Z} \circ \mathcal{V}$ and hence, $\text{Wr}(\mathcal{Z},\mathcal{V}) \subseteq \langle \mathcal{Z} \circ \mathcal{V} \rangle$.

Proof: Let $S$ and $(T,I)$ be inverse semigroups with $S \in \mathcal{Z}$ and $(T,I) \in \mathcal{V}$. Let $\pi$ be the natural map of $S \text{ wr } (T,I)$ onto $T$ and let $\rho$ be the congruence induced by $\pi$. Let $e \in E_S$. 

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Then \( e = (\psi, \beta) \) where for all \( i \in d\psi = d\beta \), \( i\beta = i \) and \( i\psi \in E_S \). Therefore, \( ep = \{ (\psi', \beta) : d\psi' = d\beta \} \). Since \( \beta \) is the identity map on its domain we have, for all \( \psi', \psi'' \) with \( d\psi' = d\psi'' = d\beta \), that \((\psi', \beta)(\psi'', \beta) = (\psi \psi'', \beta)\). Therefore, the map \( \phi : ep \to Sd\beta \) defined by \((\psi', \beta)\phi = \psi'\) is a homomorphism. It is clearly one-to-one and so the fact that \( Sd\beta \in \mathcal{Z} \) implies that \( ep \in \mathcal{Z} \). Since \( S \wr (T, I) / \rho \equiv T \in \mathcal{Y} \), we have that \( S \wr (T, I) \in \mathcal{Z} \circ \mathcal{Y} \). Since \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) is generated by \( \{ S \wr (T, I) : S \in \mathcal{Z}, T \in \mathcal{Y} \} \), it follows that \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \subseteq \langle \mathcal{Z} \circ \mathcal{Y} \rangle \). 

When \( \mathcal{Z} \) is a group variety it turns out that \( \mathcal{Z} \circ \mathcal{Y} \) is a variety and \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) = \mathcal{Z} \circ \mathcal{Y} \). In order to show this, we require a special case of a result due to Houghton.

**Lemma 4.3.3 [H].** Let \( S \) be an inverse semigroup and let \( \rho \) be an idempotent separating homomorphism of \( S \) onto \( T \). Then there is a monomorphism of \( S \) into \( (\ker \rho) \wr (T, T) \) where \( T \) is given the Wagner representation by partial right translations.

**Theorem 4.3.4.** Let \( \mathcal{Z} \) be a variety of groups and \( \mathcal{Y} \) a variety of inverse semigroups. Then \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) = \mathcal{Z} \circ \mathcal{Y} \).

**Proof:** Note that, in the setting of the theorem, \( \mathcal{Z} \circ \mathcal{Y} \) is a variety.

Let \( S \in \mathcal{Z} \circ \mathcal{Y} \) and let \( \rho \) be the congruence which witnesses this. Then \( \rho \) is idempotent separating and so, by Lemma 4.3.3, \( S \) can be embedded in \( (\ker \rho) \wr S/\rho \), where \( S/\rho \) is given its Wagner representation. Since \( \ker \rho \) is a semilattice of groups belonging to \( \mathcal{Z} \), \((\ker \rho) \wr S/\rho \in \text{Wr}(\mathcal{Z}, \mathcal{Y})\) and hence \( S \in \text{Wr}(\mathcal{Z}, \mathcal{Y}) \). Therefore, \( \mathcal{Z} \circ \mathcal{Y} \subseteq \text{Wr}(\mathcal{Z}, \mathcal{Y}) \).

Now, \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \subseteq \mathcal{Z} \circ \mathcal{Y} \), by Proposition 4.3.2 and \( \mathcal{Z} \circ \mathcal{Y} \subseteq \text{Wr}(\mathcal{Z}, \mathcal{Y}) \). By Theorem 4.2.3, \( \ker \rho(\text{Wr}(\mathcal{Z}, \mathcal{Y})) = \ker \rho(\text{Wr}(\mathcal{Z}, \mathcal{Y})) \) since \( d_{\mathcal{Y}}(w) \rho(\mathcal{Z}) d_{\mathcal{Y}}(w^2) \) if
and only if \( d_{\mathcal{V}}(w) \rho (\mathcal{Z}) d_{\mathcal{V}}(w^2) \). Therefore, \( \ker \rho (\mathcal{Z} \circ \mathcal{V}) = \ker \rho (\text{Wr}(\mathcal{Z}, \mathcal{V})) \).

But, \( \mathcal{V} \subseteq \text{Wr}(\mathcal{Z}, \mathcal{V}) \subseteq \mathcal{Z} \circ \mathcal{V} \) and \( \text{tr} \rho (\mathcal{V}) = \text{tr} \rho (\mathcal{Z} \circ \mathcal{V}) \) by Lemma 2.7.5, so that \( \text{tr} \rho (\text{Wr}(\mathcal{Z}, \mathcal{V})) = \text{tr} \rho (\mathcal{Z} \circ \mathcal{V}) \). It now follows that \( \rho (\text{Wr}(\mathcal{Z}, \mathcal{V})) = \rho (\mathcal{Z} \circ \mathcal{V}) \) and hence that \( \text{Wr}(\mathcal{Z}, \mathcal{V}) = \mathcal{Z} \circ \mathcal{V} \).

It is not true in general that \( \text{Wr}(\mathcal{Z}, \mathcal{V}) = \langle \mathcal{Z} \circ \mathcal{V} \rangle \) for varieties \( \mathcal{Z} \) and \( \mathcal{V} \) of inverse semigroups, as the following example illustrates.

**Example.** Let \( \mathcal{Z} = \mathcal{B} \), the variety of inverse semigroups generated by the five element Brandt semigroup \( B_2 \) and let \( \mathcal{V} = \mathcal{S} \), the variety of semilattices. \( \mathcal{B} \) is defined by the identity \( xyx^{-1} = (xyx^{-1})^2 \) (See [P;XII.4.8],[K1] or [R2]). Let \( w = xyx^{-1} \). Now \( w \rho (\mathcal{S}) w^2 \) and \( d_{\mathcal{B}}(w) \) is just a relabelling of \( w \) since the Schützenberger graph of \( w \) relative to \( \mathcal{S} \) has no two edges with the same (primary) label. (In fact, \( \Gamma_{\mathcal{B}}(w) \) is just a single vertex with four loops labelled by \( x,y,x^{-1},y^{-1} \) — see Proposition 2.8.3.) Therefore, \( d_{\mathcal{B}}(w) \rho (\mathcal{B}) d_{\mathcal{B}}(w^2) \). By Theorem 4.2.3, \( \text{Wr}(\mathcal{B}, \mathcal{S}) \) satisfies the equation \( w = w^2 \), and so \( \text{Wr}(\mathcal{B}, \mathcal{S}) \subseteq \mathcal{B} \). Clearly \( \mathcal{B} \subseteq \text{Wr}(\mathcal{B}, \mathcal{S}) \), and so \( \mathcal{B} = \text{Wr}(\mathcal{B}, \mathcal{S}) \). But \( \mathcal{B}_1 \), the variety of inverse semigroups generated by the five element Brandt semigroup with an identity adjoined, denoted by \( B_2^1 \) , is contained in \( \langle \mathcal{B} \circ \mathcal{S} \rangle \) since \( B_2^1 \) is a semilattice of \( B_2 \) and \( \{1\} \). Since \( \mathcal{B} \neq \mathcal{B}_1 \), we have \( \text{Wr}(\mathcal{B}, \mathcal{S}) \neq \langle \mathcal{B} \circ \mathcal{S} \rangle \).

**Proposition 4.3.5.** Let \( \mathcal{Z}, \mathcal{V}, \mathcal{W} \) be varieties of inverse semigroups. Then

\[
\text{Wr}(\mathcal{Z} \vee \mathcal{V}, \mathcal{W}) = \text{Wr}(\mathcal{Z}, \mathcal{W}) \vee \text{Wr}(\mathcal{V}, \mathcal{W}).
\]

**Proof:** Set \( \rho = \rho (\text{Wr}(\mathcal{Z} \vee \mathcal{V}, \mathcal{W})) \). Then, for any \( u,v \in (X \cup X^{-1})^+ \) we have

\[
\begin{align*}
&u \rho v \iff u \rho (\mathcal{W}) v \quad \text{and} \quad d_{\mathcal{W}}(u) \rho (\mathcal{Z} \vee \mathcal{V}) d_{\mathcal{W}}(v) \quad \text{(Theorem 4.2.3)} \\
&\iff u \rho (\mathcal{W}) v \quad \text{and} \quad d_{\mathcal{W}}(u) \rho (\mathcal{Z}) \cap \rho (\mathcal{V}) d_{\mathcal{W}}(v)
\end{align*}
\]

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We do not know whether $\text{Wr}(\mathcal{U}, \mathcal{W}) \land \text{Wr}(\mathcal{X}, \mathcal{W}) = \text{Wr}(\mathcal{U} \land \mathcal{X}, \mathcal{W})$ for arbitrary varieties $\mathcal{U}, \mathcal{X}$ and $\mathcal{W}$, but the equality does hold when $\mathcal{U}$ and $\mathcal{X}$ are varieties of groups and $\mathcal{W}$ is a combinatorial variety. The proof of this requires further results which are presented in the following chapter and so we leave this proposition until section 5.3. Another case in which this equality holds is when $\mathcal{U}, \mathcal{X}$ and $\mathcal{W}$ are varieties of groups [N;21.23].

**Remark.** Let $\mathcal{U}, \mathcal{X}$ and $\mathcal{Y}$ be varieties of inverse semigroups. It is not true in general that $\text{Wr}(\mathcal{U}, \mathcal{X} \land \mathcal{Y}) = \text{Wr}(\mathcal{U}, \mathcal{X}) \land \text{Wr}(\mathcal{Y}, \mathcal{X})$, nor is it true in general that $\text{Wr}(\mathcal{U}, \mathcal{X} \lor \mathcal{Y}) = \text{Wr}(\mathcal{U}, \mathcal{X}) \lor \text{Wr}(\mathcal{Y}, \mathcal{X})$.

Consider $\text{Wr}(\mathcal{S}, \mathcal{G} \land \mathcal{B})$, where $\mathcal{S}$ is the variety of semilattices, $\mathcal{G}$ is the variety of groups and $\mathcal{B}$ is the variety of inverse semigroups generated by the five element Brandt semigroup.

$$\text{Wr}(\mathcal{S}, \mathcal{G} \land \mathcal{B}) = \text{Wr}(\mathcal{S}, \mathcal{G}) = \mathcal{S},$$

while

$$\text{Wr}(\mathcal{S}, \mathcal{G}) \land \text{Wr}(\mathcal{S}, \mathcal{B}) = \mathcal{G}^{\text{max}} \land \mathcal{B}^{\text{max}} = \mathcal{S} \land \mathcal{B} = \mathcal{B} \neq \mathcal{S}.,$$

(We will prove in Theorem 5.3.3 that, for any variety of inverse semigroups $\mathcal{X}$, $\text{Wr}(\mathcal{S}, \mathcal{X}) = \mathcal{X}^{\text{max}}$.)

Now consider $\text{Wr}(\mathcal{S}, \mathcal{A}_2 \lor \mathcal{A}_3)$ where $\mathcal{A}_2$ and $\mathcal{A}_3$ are, respectively, the variety of abelian groups of exponent 2 and the variety of abelian groups of exponent 3. It is clear that $\text{Wr}(\mathcal{S}, \mathcal{A}_2) \lor \text{Wr}(\mathcal{S}, \mathcal{A}_3) \subseteq \text{Wr}(\mathcal{S}, \mathcal{A}_2 \lor \mathcal{A}_3) = \text{Wr}(\mathcal{S}, \mathcal{A}_6)$, where $\mathcal{A}_6$ is the
variety of abelian groups of exponent 6. The containment is proper however, as both
\( \text{Wr}(\mathcal{A}, \mathcal{A}_2) \) and \( \text{Wr}(\mathcal{A}, \mathcal{A}_3) \) satisfy the equation \( x^3 = x^9 \), while \( \text{Wr}(\mathcal{A}, \mathcal{A}_6) \) does not satisfy \( x^3 = x^9 \). This can be seen by considering the Cayley graphs of \( \mathbb{Z}_2, \mathbb{Z}_3 \) and \( \mathbb{Z}_6 \) and using Theorem 4.2.3.

**Lemma 4.3.6.** Let \( T \) be a full, closed inverse subsemigroup of \( S \). Then \( \sigma_S \) saturates \( T \) and \( T/\sigma_T \) is isomorphic to the subgroup of \( S/\sigma_S \) consisting of those \( \sigma_S \)-classes contained in \( T \).

**Proof:** Let \( t_1, t_2 \in T \). By the definition of \( \sigma_S \), \( t_1 \sigma_S t_2 \) if and only if for some idempotent \( e \) in \( S \), \( t_1 e = t_2 e \). Since \( T \) is full, it follows from the definition of \( \sigma_T \) that \( t_1 \sigma_S t_2 \) if and only if \( t_1 \sigma_T t_2 \). Thus, \( \sigma_S \) restricted to \( T \) is \( \sigma_T \). If \( t \sigma_S s \) for some \( t \in T \) and \( s \in S \) then for some idempotent \( e \) in \( S \), we have \( te = se \leq s \). Since \( T \) is full, \( te \in T \) and so, since \( T \) is closed, \( s \in T \). Thus, \( \sigma_S \) saturates \( T \) and these \( \sigma_S \)-classes form a group isomorphic to \( T/\sigma_T \).

**Theorem 4.3.7.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be varieties of inverse semigroups. Then
\[
\text{Wr}(\mathcal{U}, \mathcal{V}) \cap \mathcal{G} = \text{Wr}(\mathcal{U} \cap \mathcal{G}, \mathcal{V} \cap \mathcal{G}) = (\mathcal{U} \cap \mathcal{G}) \circ (\mathcal{V} \cap \mathcal{G}) = (\mathcal{U} \circ \mathcal{V}) \cap \mathcal{G}.
\]

**Proof:** Let \( S \in \mathcal{U} \circ \mathcal{V} \) and let \( \rho \) be the congruence on \( S \) which witnesses this. We consider \( S/\sigma_S \), the maximal group homomorphic image of \( S \). Now \( \sigma_S \leq \rho \circ \sigma_S \) and so \( \rho \circ \sigma_S / \sigma_S \) is a congruence on \( S/\sigma_S \). Moreover, \( (S/\sigma_S)/(\rho \circ \sigma_S / \sigma_S) \equiv S / \rho \circ \sigma_S \) and \( S / \rho \circ \sigma_S \equiv (S/\rho) / (\rho \circ \sigma_S / \rho) \in \mathcal{V} \), since \( S/\rho \in \mathcal{V} \). Therefore, \( (S/\sigma_S)/(\rho \circ \sigma_S / \sigma_S) \in \mathcal{V} \cap \mathcal{G} \). The single idempotent \( (\rho \circ \sigma_S / \sigma_S) \)-class is just \( \ker (\rho \circ \sigma_S / \sigma_S) = \{ x \sigma_S : x \in (\ker \rho) \omega \} \) since \( e \sigma_S \rho \circ \sigma_S \) if and only if \( e (\rho \circ \sigma_S) x \); that is, \( x \sigma_S \in \ker (\rho \circ \sigma_S / \sigma_S) \) if and only if \( x \in \ker (\rho \circ \sigma_S) = (\ker \rho) \omega \) \([P;III.5.5]\). Now \( (\ker \rho) \omega \) is a closed, full inverse
subsemigroup of $S$ and so it follows from Lemma 4.3.6 that
\[
\ker (\rho \vee (\sigma S / \sigma S)) \equiv (\ker \rho) \omega / \sigma(\ker \rho) \omega.
\]

We claim that $(\ker \rho) \omega / \sigma(\ker \rho) \omega \equiv \ker \rho / \sigma(\ker \rho)$. To see this observe that if $s \in (\ker \rho) \omega$ then there is a $t \in \ker \rho$ such that $t \leq s$. But this means that $t = se$ for some idempotent $e$ in $(\ker \rho) \omega$ and hence in $\ker \rho$. Thus, $te = see = se$ and so $t \sigma(\ker \rho) \omega s$. Moreover, for any $t_1, t_2 \in \ker \rho$, $t_1 \sigma(\ker \rho) \omega t_2$ if and only if $t_1 \sigma(\ker \rho) t_2$ since $\ker \rho$ is full.

It follows from these remarks that
\[
(\ker \rho) \omega / \sigma(\ker \rho) \omega \equiv \ker \rho / \sigma(\ker \rho) \in (\mathcal{U} \circ \mathcal{I}) \cap \mathcal{F} = \mathcal{U} \cap \mathcal{F}.
\]

It now follows that $\ker (\rho \vee (\sigma S / \sigma S)) \in \mathcal{U} \cap \mathcal{F}$. Thus, the congruence $\rho \vee (\sigma S / \sigma S)$ on $S/\sigma S$ witnesses that $S/\sigma S \in (\mathcal{U} \cap \mathcal{F}) \circ (\mathcal{V} \cap \mathcal{G})$.

Let $G \in \text{Wr}(\mathcal{U}, \mathcal{V}) \cap \mathcal{G}$. Then $G \leq (\mathcal{U} \circ \mathcal{V}) \cap \mathcal{G}$. Since $\mathcal{U} \circ \mathcal{V}$ is closed under the formation of direct products and subsemigroups [P; XII.8.2], $G = S/\rho$ for some $S \in \mathcal{U} \circ \mathcal{V}$. We have just shown that $S/\sigma S \in (\mathcal{U} \cap \mathcal{F}) \circ (\mathcal{V} \cap \mathcal{G})$ so we must have that $G \in (\mathcal{U} \cap \mathcal{F}) \circ (\mathcal{V} \cap \mathcal{G})$ since $G$ is a homomorphic image of $S/\sigma S$. Therefore,
\[
\text{Wr}(\mathcal{U}, \mathcal{V}) \cap \mathcal{G} \subseteq (\mathcal{U} \circ \mathcal{V}) \cap \mathcal{G} \subseteq (\mathcal{U} \cap \mathcal{G}) \circ (\mathcal{V} \cap \mathcal{G}) = \text{Wr}(\mathcal{U} \cap \mathcal{G}, \mathcal{V} \cap \mathcal{G})
\]
by Theorem 4.3.4. It follows immediately from Proposition 4.3.1 that
\[
\text{Wr}(\mathcal{U} \cap \mathcal{G}, \mathcal{V} \cap \mathcal{G}) \subseteq \text{Wr}(\mathcal{U}, \mathcal{V}) \cap \mathcal{G}
\]
and so
\[
\text{Wr}(\mathcal{U}, \mathcal{V}) \cap \mathcal{G} = \text{Wr}(\mathcal{U} \cap \mathcal{G}, \mathcal{V} \cap \mathcal{G}) = (\mathcal{U} \cap \mathcal{G}) \circ (\mathcal{V} \cap \mathcal{G}) = (\mathcal{U} \circ \mathcal{V}) \cap \mathcal{G}.
\]

### 4.4 The Associativity of Wr

The binary operator $\text{Wr}$ on the lattice of varieties of inverse semigroups is, in fact, an associative operator and so $(\mathcal{L}(\mathcal{F}), \text{Wr})$ is a semigroup. The proof of this makes use of Theorem 4.2.2 — the description of the fully invariant congruence on the free inverse semigroup corresponding to $\text{Wr}(\mathcal{U}, \mathcal{V})$, for any pair of varieties $\mathcal{U}$ and $\mathcal{V}$ of inverse
semigroups, and Lemma 4.1.4 — the description of the derived word relative to the variety \( \mathcal{Y} \).

We say that the two equations \( u_1 = u_2 \) and \( v_1 = v_2 \) are equivalent if each is a consequence of the other. Another way of saying this is \( u_1 = u_2 \) and \( v_1 = v_2 \) are equivalent if and only if, for any variety \( \mathcal{Z} \) of inverse semigroups, \( \mathcal{Z} \) satisfies the equation \( u_1 = u_2 \) if and only if \( \mathcal{Z} \) satisfies the equation \( v_1 = v_2 \).

**Lemma 4.4.1.** Let \( \mathcal{Z} \) and \( \mathcal{Y} \) be varieties of inverse semigroups and let \( v, w \in (X \cup X^{-1})^+ \) be such that \( w \rho(\mathcal{Z})v \) and \( d_{\mathcal{Z}}(w) \rho(\mathcal{Y}) d_{\mathcal{Z}}(v) \) (or, equivalently, \( w \rho(Wr(\mathcal{Y}, \mathcal{Z})v) \)). Set

\[
w = a_1 \ldots a_n, \quad v = b_1 \ldots b_m,
\]

where \( a_i, b_j \in X \cup X^{-1}, i = 1, \ldots, n, j = 1, \ldots, m.\)

\[
d_{\mathcal{Z}}(w) = c_1 \ldots c_n, \quad d_{\mathcal{Y}}(v) = d_1 \ldots d_m,
\]

where both words are constructed from the same doubly labelled Schützenberger graph \( \overline{\Gamma_{\mathcal{Z}}(w)} \).

\[
d_{\mathcal{Y}}(d_{\mathcal{Z}}(w)) = e_1 \ldots e_n, \quad d_{\mathcal{Y}}(d_{\mathcal{Y}}(v)) = f_1 \ldots f_m,
\]

where both words are constructed from the same doubly labelled Schützenberger graph \( \overline{\Gamma_{\mathcal{Y}}(d_{\mathcal{Z}}(w))} \).

\[
d_{Wr(\mathcal{Y}, \mathcal{Z})}(w) = g_1 \ldots g_n, \quad d_{Wr(\mathcal{Y}, \mathcal{Z})}(v) = h_1 \ldots h_m,
\]

where both words are constructed from the same doubly labelled Schützenberger graph \( \overline{\Gamma_{Wr(\mathcal{Y}, \mathcal{Z})}(w)} \).

Then the equations \( d_{\mathcal{Y}}(d_{\mathcal{Z}}(w)) = d_{\mathcal{Y}}(d_{\mathcal{Y}}(v)) \) and \( d_{Wr(\mathcal{Y}, \mathcal{Z})}(w) = d_{Wr(\mathcal{Y}, \mathcal{Z})}(v) \) are equivalent.

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Proof: We prove the stronger statement that each of the two equations is a one-to-one relabelling of the other. That is, we prove the following statements.

For all $i,j,$

1) $g_i = h_j \iff e_i = f_j$;

2) $g_i = h_i^{-1} \iff e_i = f_i^{-1}$;

3) $g_i = g_j \iff e_i = e_j$;

4) $g_i = g_j^{-1} \iff e_i = e_j^{-1}$;

5) $h_i = h_j \iff f_i = f_j$;

6) $h_i = h_j^{-1} \iff f_i = f_j^{-1}$.

1) First of all, observe that

$$g_i = h_j \implies w \leq \text{Wr}(\forall, \forall) a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

( Lemma 4.1.4 since $w \rho(\text{Wr}(\forall, \forall)) v$ )

$$\implies w \rho(\text{Wr}(\forall, \forall)) w w^{-1} a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

$$\implies w \rho(\forall) w w^{-1} a_1 \ldots a_{i-1} b_j \ldots b_m,$$

$$d_g(w) \rho(\forall) d_g(w w^{-1} a_1 \ldots a_{i-1} b_j \ldots b_m) \text{ and } a_i = b_j$$

( Theorem 4.2.2 )

Next, observe that

$$e_i = f_j \implies d_g(w) \leq \forall c_1 \ldots c_{i-1} d_j \ldots d_m \text{ and } c_i = d_j$$

( Lemma 4.1.4 since $d_g(w) \rho(\forall) d_g(v)$ )

$$\implies (c_1 \ldots c_n) \rho(\forall) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m \text{ and } c_i = d_j$$

$$\implies (c_1 \ldots c_n) \rho(\forall) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m$$

$$w \leq \forall a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

( Lemma 4.1.4 since $w \rho(\forall) v$ ).

$$\implies (c_1 \ldots c_n) \rho(\forall) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m$$

$$w \rho(\forall) w w^{-1} a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j.$$
By the hypothesis \( \rho(2) v \) and so if \( \rho(2) \) \( \text{ww}^{-1}a_1...a_i-1b_j...b_m \) we must have that \( d_2(\text{ww}^{-1}a_1...a_i-1b_j...b_m) = (c_1...c_n)(c_1...c_n)^{-1}c_1...c_i-1d_j...d_m \). This is because each of \( w, v \) and \( \text{ww}^{-1}a_1...a_i-1b_j...b_m \) label s-e paths in \( \Gamma_2(w) \) by primary labels. It is clear that \( \text{ww}^{-1}a_1...a_i-1 \) corresponds to the path labelled \( (c_1...c_n)(c_1...c_n)^{-1}c_1...c_i-1 \) by secondary labels, since \( \Gamma_2(w) \) is deterministic. While there may be many paths in \( \Gamma_2(w) \) labelled \( b_j...b_m \), the part of \( \text{ww}^{-1}a_1...a_i-1b_j...b_m \) labelled \( b_j...b_m \) ends at vertex e. Since the s-e path labelled \( v \) ends at e, both the \( b_j...b_m \) of \( v \) and the \( b_j...b_m \) of \( \text{ww}^{-1}a_1...a_i-1b_j...b_m \) must correspond to the same edges, since \( \Gamma_2(w) \) is deterministic. It follows that \( d_2(\text{ww}^{-1}a_1...a_i-1b_j...b_m) = (c_1...c_n)(c_1...c_n)^{-1}c_1...c_i-1d_j...d_m \). From these remarks we have that

\[
\text{w } \rho(2) \text{ww}^{-1}a_1...a_i-1b_j...b_m, \; d_2(\text{w}) \rho(\gamma) \; d_2(\text{ww}^{-1}a_1...a_i-1b_j...b_m) \text{ and } a_i = b_j
\]

\[
(c_1...c_n) \rho(\gamma) (c_1...c_n)(c_1...c_n)^{-1}c_1...c_i-1d_j...d_m, \; \text{w } \rho(2) \text{ww}^{-1}a_1...a_i-1b_j...b_m \text{ and } a_i = b_j.
\]

From this it follows that \( g_i = h_j \) if and only if \( e_i = f_j \).

2) We proceed in a similar manner:

\[
g_i = h_j^{-1} \iff \text{w} \leq_{\text{Wr}(\gamma, 2)} a_1...a_i b_j...b_m \text{ and } a_i = b_j^{-1}
\]

( Lemma 4.1.4 since \( \text{w} \rho(\text{Wr}(\gamma, 2) v) \) )

\[
\iff \text{w} \rho(\text{Wr}(\gamma, 2)) \text{ww}^{-1}a_1...a_i b_j...b_m \text{ and } a_i = b_j^{-1}
\]

\[
\iff \text{w} \rho(2) \text{ww}^{-1}a_1...a_i b_j...b_m, \; d_2(\text{w}) \rho(\gamma) \; d_2(\text{ww}^{-1}a_1...a_i b_j...b_m) \text{ and } a_i = b_j^{-1}
\]

( Theorem 4.2.2 )

Also,

\[
e_i = f_j^{-1} \iff \text{d}_2(\text{w}) \leq_{\gamma} c_1...c_i d_j...d_m \text{ and } c_i = d_j^{-1}
\]

( Lemma 4.1.4 since \( \text{d}_2(\text{w}) \rho(\gamma) \text{d}_2(\gamma) \) )

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\[ \Leftrightarrow \quad (c_1 \ldots c_n) \rho(\mathcal{Y}) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1}c_1 \ldots c_i d_j \ldots d_m \quad \text{and} \quad c_i = d_{j-1} \]
\[ \Leftrightarrow \quad (c_1 \ldots c_n) \rho(\mathcal{Y}) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1}c_1 \ldots c_i d_j \ldots d_m \]
\[ w \leq_\mathcal{Y} a_1 \ldots a_i b_j \ldots b_m \quad \text{and} \quad a_i = b_{j-1} \]

( Lemma 4.1.4 since \( w \rho(\mathcal{Z}) v \).)

\[ \Leftrightarrow \quad (c_1 \ldots c_n) \rho(\mathcal{Y}) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1}c_1 \ldots c_i d_j \ldots d_m \]
\[ w \rho(\mathcal{Z}) w w^{-1} a_1 \ldots a_i b_j \ldots b_m \quad \text{and} \quad a_i = b_{j-1}. \]

We have by the hypothesis \( w \rho(\mathcal{Z}) v \) and so if \( w \rho(\mathcal{Z}) w w^{-1} a_1 \ldots a_i b_j \ldots b_m \) we must have, as in 1), that \( d_\mathcal{Z}(w w^{-1} a_1 \ldots a_i b_j \ldots b_m) = (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1}c_1 \ldots c_i d_j \ldots d_m \). It follows that

\[ w \rho(\mathcal{Z}) w w^{-1} a_1 \ldots a_i b_j \ldots b_m, \quad d_\mathcal{Z}(w) \rho(\mathcal{Y}) d_\mathcal{Z}(w w^{-1} a_1 \ldots a_i b_j \ldots b_m) \quad \text{and} \quad a_i = b_{j-1} \]

\[ \Leftrightarrow \]
\[ (c_1 \ldots c_n) \rho(\mathcal{Y}) (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1}c_1 \ldots c_i d_j \ldots d_m, \quad w \rho(\mathcal{Z}) w w^{-1} a_1 \ldots a_i b_j \ldots b_m \]
\[ \quad \text{and} \quad a_i = b_{j-1}. \]

Consequently, \( g_i = h_{j-1} \) if and only if \( e_i = f_{j-1} \).

The proofs of 3), 4), 5) and 6) are similar, noting the remark immediately following Lemma 4.1.4.

**Theorem 4.4.2.** The operator \( Wr \) is associative.

**Proof:** Let \( \mathcal{Y}, \mathcal{Y}' \) and \( \mathcal{Z} \) be varieties of inverse semigroups. For any \( w, v \in F\mathcal{S}(X) \),

\[ w \rho(Wr(\mathcal{Y}, Wr(\mathcal{Y}, \mathcal{Z}))) v \Leftrightarrow w \rho(Wr(\mathcal{Y}, \mathcal{Z})) v \quad \text{and} \]
\[ d_{\mathcal{Y}, \mathcal{Z}}(w) \rho(\mathcal{Y}) d_{\mathcal{Y}, \mathcal{Z}}(v) \]

( Theorem 4.2.3 )

\[ \Leftrightarrow w \rho(\mathcal{Z}) v, \quad d_{\mathcal{Z}}(w) \rho(\mathcal{Y}) d_{\mathcal{Z}}(v) \quad \text{and} \]
\[ d_{\mathcal{Y}, \mathcal{Z}}(w) \rho(\mathcal{Y}) d_{\mathcal{Y}, \mathcal{Z}}(v) \]

( Theorem 4.2.3 )
Theorem 4.4.3. \( Y(3) \) is a monoid with zero under the operation \( Wr \).

Proof: That \( (Y(3), Wr) \) is a semigroup is a consequence of Theorem 4.4.2. Since 
\[
wr(\emptyset, Y) = Wr(\emptyset, Y) = \emptyset,
\]
for any variety \( Y \), the variety \( \mathcal{F} \) is an identity for \( (\mathcal{L}(\mathcal{F}), Wr) \) and so \( (\mathcal{L}(\mathcal{F}), Wr) \) is a monoid. For any variety \( Y \) of inverse semigroups, 
\[
Wr(\emptyset, Y) = Wr(\emptyset, \mathcal{F}) = \emptyset \quad \text{and so} \quad \mathcal{F} \quad \text{is a zero of the monoid} \quad (\mathcal{L}(\mathcal{F}), Wr).
\]

There are several natural questions which arise as a result of Theorem 4.4.3. For example: Which varieties are idempotents? Which varieties, if any, can be expressed as a product of two non-trivial varieties? Which familiar classes of varieties of inverse semigroups form a subsemigroup of \( (\mathcal{L}(\mathcal{F}), Wr) \)? What is the structure of the semigroup \( (\mathcal{L}(\mathcal{F}), Wr) \)? Is it free? Many of these problems do not have obvious solutions. In Chapter 5, once we have equipped ourselves with some facts, we discuss some of these questions.

We conclude this section with some results concerning generators of varieties of the form \( Wr(\emptyset, Y) \).

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Theorem 4.4.4. \( F(\text{Wr}(\mathcal{Z},\mathcal{Y}))(X) \) can be embedded in \( F(\mathcal{Z}) \) wr \( (F(\mathcal{Y}), F(\mathcal{Y})) \).

Proof: Set \( \rho = \rho(\text{Wr}(\mathcal{Z},\mathcal{Y})) \). Let \( X = \{ x_i : i \in \omega \} \) and \( Y = \cup \{ x_{i_n} : u \in F(\mathcal{Y})x_{i_n}^{-1}\rho(\mathcal{Y}) \} \), where the union is over all \( i \in \omega \).

Define

\[ \Theta : F(\text{Wr}(\mathcal{Z},\mathcal{Y}))(X) \rightarrow F(\mathcal{Z})(Y) \text{ wr } (F(\mathcal{Y})(X), F(\mathcal{Y})(X)) \]

as follows:

For each \( i \in \omega \), set

\[ (x_i \rho) \Theta = (\psi_i, \beta_i) \]

Here \( \beta_i \) corresponds to \( x_i \rho(\mathcal{Y}) \); that is,

\[ d\beta_i = F(\mathcal{Y})(x_i^{-1}\rho(\mathcal{Y})), \]
\[ u\beta_i = ux_i \rho(\mathcal{Y}) \quad (u \in d\beta_i), \]
\[ u\psi_i = x_i \rho(\mathcal{Y}) \quad (u \in d\beta_i). \]

It is immediate that \( \Theta \) maps \( \{ x_i \rho : i \in \omega \} \) into \( F(\mathcal{Z})(Y) \text{ wr } (F(\mathcal{Y})(X), F(\mathcal{Y})(X)) \). Since \( F(\mathcal{Z})(Y) \text{ wr } (F(\mathcal{Y})(X), F(\mathcal{Y})(X)) \) is a member of \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \) and \( F(\text{Wr}(\mathcal{Z},\mathcal{Y}))(X) \) is \( \text{Wr}(\mathcal{Z},\mathcal{Y}) \)-free, we let \( \Theta \) be the unique extension of \( \Theta \) thus far defined, to \( F(\text{Wr}(\mathcal{Z},\mathcal{Y})). \)

Let \( w = a_1 \ldots a_n \) and \( v = b_1 \ldots b_m \) where \( a_i, b_j \in X \cup X^{-1}, i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Suppose that \( w \rho \Theta = v \rho \Theta \). As before we write

\[ \psi_{d_i} = \begin{cases} \psi_j & \text{if } d_i = x_j \\ \psi_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases} \]

\[ \beta_{d_i} = \begin{cases} \beta_j & \text{if } d_i = x_j \\ \beta_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases} \]

Again, we write \( \beta_{d_1} \ldots d_i \) for \( \beta_{d_1} \beta_{d_2} \ldots \beta_{d_i} \).

Now

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\[ \wp \Theta = (\psi_{a_1}, \beta_{a_1}) \ldots (\psi_{a_n}, \beta_{a_n}) \]
\[ = (\psi_{a_1}^{\beta_{a_1}}, \psi_{a_2}^{\beta_{a_2}}, \ldots, \psi_{a_n}^{\beta_{a_n}}, \beta_{a_1}, \ldots, \beta_{a_n}) \]

and

\[ \nu \Theta = (\psi_{b_1}, \beta_{b_1}) \ldots (\psi_{b_m}, \beta_{b_m}) \]
\[ = (\psi_{b_1}^{\beta_{b_1}}, \psi_{b_2}^{\beta_{b_2}}, \ldots, \psi_{b_m}^{\beta_{b_m}}, \beta_{b_1}, \ldots, \beta_{b_m}). \]

First of all, \( \beta_{a_1}, \ldots, \beta_{a_n} = \beta_{b_1}, \ldots, \beta_{b_m} \) and so \( \wp(\gamma) \nu \). Secondly, observe that

\[ \wp \gamma^{-1} \rho(\gamma) = \nu \gamma^{-1} \rho(\gamma) \in d(\beta_{a_1}, \ldots, \beta_{a_n} = d(\beta_{b_1}, \ldots, \beta_{b_m}). \]

Thus,

\[ [\wp \gamma^{-1} \rho(\gamma)] \psi_{a_1}^{\beta_{a_1}}, \psi_{a_2}^{\beta_{a_2}}, \ldots, \psi_{a_n}^{\beta_{a_n}} \]
\[ = \gamma_1 \ldots \gamma_n \rho(\gamma) \text{, where } \gamma_i = \wp \gamma^{-1} \rho(\gamma) \beta_{a_{\gamma_i}} \psi_{a_{\gamma_i}} \in Y \cup Y^{-1} \]
\[ = [\wp \gamma^{-1} \rho(\gamma)] \psi_{b_1}^{\beta_{b_1}}, \psi_{b_2}^{\beta_{b_2}}, \ldots, \psi_{b_m}^{\beta_{b_m}} \]
\[ = \gamma_1 \ldots \gamma_m \rho(\gamma) \text{, where } \gamma_i = \wp \gamma^{-1} \rho(\gamma) \beta_{b_{\gamma_i}} \psi_{b_{\gamma_i}} \in Y \cup Y^{-1}. \]

Now observe that

\[ y_i = z_j \iff \wp \gamma^{-1} \rho(\gamma) \beta_{a_1}, \ldots, \beta_{a_{\gamma_i}} \psi_{a_{\gamma_i}} = \wp \gamma^{-1} \rho(\gamma) \beta_{b_1}, \ldots, \beta_{b_{\gamma_i}} \psi_{b_{\gamma_i}} \]
\[ \iff \wp \gamma^{-1} a_1, \ldots, a_{i-1} \rho(\gamma) \psi_{a_i} = \wp \gamma^{-1} b_1, \ldots, b_{i-1} \rho(\gamma) \psi_{b_i} \]
\[ \iff a_i \equiv b_j \text{ and } \wp \gamma^{-1} a_1, \ldots, a_{i-1} \rho(\gamma) \wp \gamma^{-1} b_1, \ldots, b_{i-1} \rho(\gamma) \psi_{b_{\gamma_i}} \]
\[ \iff a_i \equiv b_j \text{ and } \wp \gamma^{-1} a_1, \ldots, a_{i-1} b_j, \ldots, b_m \rho(\gamma) \wp \gamma^{-1} v(\gamma) \rho(\gamma) w \]
\[ \iff a_i \equiv b_j \text{ and } a_1, a_{i-1} b_j, \ldots, b_m \geq v(\gamma) w \]

and

\[ y_i = z_j^{-1} \iff \wp \gamma^{-1} \rho(\gamma) \beta_{a_1}, \ldots, \beta_{a_{\gamma_i}} \psi_{a_{\gamma_i}} = [\wp \gamma^{-1} \rho(\gamma)] \beta_{b_1}, \ldots, \beta_{b_{\gamma_i}} \psi_{b_{\gamma_i}} \]
\[ \iff \wp \gamma^{-1} a_1, \ldots, a_{i-1} \rho(\gamma) \psi_{a_i} = \wp \gamma^{-1} b_1, \ldots, b_{i-1} \rho(\gamma) \beta_{b_{\gamma_i}} \psi_{b_{\gamma_i}} \]
\[ \iff \wp \gamma^{-1} a_1, \ldots, a_{i-1} \rho(\gamma) \psi_{a_i} = \wp \gamma^{-1} b_1, \ldots, b_{i-1} \rho(\gamma) \beta_{b_{\gamma_i}} \psi_{b_{\gamma_i}} \]
\[ \iff a_i \equiv b_{j^{-1}} \text{ and } \wp \gamma^{-1} a_1, \ldots, a_{i-1} \rho(\gamma) \wp \gamma^{-1} b_1, \ldots, b_{j^{-1}} \rho(\gamma) \psi_{b_{\gamma_i}} \]
\[ \iff a_i \equiv b_{j^{-1}} \text{ and } \wp \gamma^{-1} a_1, \ldots, a_{i-1} b_{j^{-1}}, \ldots, b_m \rho(\gamma) \wp \gamma^{-1} v(\gamma) \rho(\gamma) w \]
\[ \iff a_i \equiv b_{j^{-1}} \text{ and } a_1, a_{i-1} b_{j^{-1}}, \ldots, b_m \geq v(\gamma) w \]

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We similarly obtain
\[
\begin{align*}
y_i = y_j & \iff a_i = a_j \text{ and } a_1 \cdots a_i a_j \cdots a_n \geq \gamma w \\
y_i = y_j^{-1} & \iff a_i = a_j^{-1} \text{ and } a_1 \cdots a_i a_j+1 \cdots a_n \geq \gamma w \\
z_i = z_j & \iff b_i = b_j \text{ and } b_1 \cdots b_i b_j \cdots b_m \geq \gamma w \\
z_i = z_j^{-1} & \iff b_i = b_j^{-1} \text{ and } b_1 \cdots b_i b_j+1 \cdots b_m \geq \gamma w
\end{align*}
\]

Since \( y_1 \cdots y_n \rho (\mathcal{Z}) z_1 \cdots z_m \) we have by the above and Lemma 4.1.4 that \( d\gamma(w) \rho (\mathcal{Z}) d\gamma(v) \). It now follows from Theorem 4.2.2 that \( w \rho v \) and hence that \( \Theta \) is a monomorphism. Thus, \( \Theta \) is an embedding of \( \mathcal{F}(\mathrm{Wr}(\mathcal{Z},\mathcal{Y}))(X) \) into \( \mathcal{F}(\mathcal{Z}) \wr \mathcal{F}(\mathcal{Y}) \).

**Corollary 4.4.5.** For any pair of varieties \( \mathcal{Z} \) and \( \mathcal{Y} \) of inverse semigroups, the variety \( \mathrm{Wr}(\mathcal{Z},\mathcal{Y}) \) is generated by \( \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \).

**Proof:** By the definition of \( \mathrm{Wr} \), \( \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \in \mathrm{Wr}(\mathcal{Z},\mathcal{Y}) \) and so \( \langle \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \rangle \subseteq \mathrm{Wr}(\mathcal{Z},\mathcal{Y}) \). On the other hand, by Theorem 4.4.4, \( \mathcal{F}(\mathrm{Wr}(\mathcal{Z},\mathcal{Y}))(X) \in \langle \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \rangle \), and so \( \mathrm{Wr}(\mathcal{Z},\mathcal{Y}) \subseteq \langle \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \rangle \). Thus, \( \langle \mathcal{F}(\mathcal{Z}) \wr (\mathcal{Y}, \mathcal{Y}) \rangle = \mathrm{Wr}(\mathcal{Z},\mathcal{Y}) \).

**Lemma 4.4.6.** Let \( A \) and \( S \) be inverse semigroups and suppose that \( T \) is an inverse subsemigroup of \( S \). Then

1) If \( T' \) is isomorphic to \( T \) then \( A \wr (T',T') \) is isomorphic to \( A \wr (T,T) \);

2) \( A \wr (T,T) \in \langle A \wr (S,S) \rangle \).

**Proof:** 1) Let \( \Phi \) be the isomorphism from \( T \) to \( T' \). Let \((\psi,\beta) \in A \wr (T,T) \) with \( d\beta = Tt^{-1} \) (and so \( \beta \) corresponds to \( t \in T \)). Define \((\psi',\beta') \in A \wr (T',T') \) by setting \( d\beta' = T'(t^{-1})\Phi \) and defining \( u\beta' = (u\Phi^{-1}\beta)\Phi = u(t\Phi) \) (and so \( \beta' \) corresponds to \( t\Phi \)) and \( u\psi' = u\Phi^{-1}\psi \) (for all \( u \in d\beta' \)). The map which sends \((\psi,\beta)\) to \((\psi',\beta')\) is the desired
isomorphism: Let \((y, \beta, (\varphi, \alpha)) \in A \text{ wr } (T, T)\). To see that this map is a homomorphism, it is enough to show that \(\beta' \alpha' = (\beta \alpha)'\) and that \(\psi' \beta' \varphi' = (\psi \beta \varphi)'\). It is clear that \(\beta' \alpha' = (\beta \alpha)\)'\), so let \(u \in d \beta' \alpha' = d(\beta \alpha)'\). Then \(u(\psi \beta \varphi)' = (u \Phi^{-1}) \psi \beta \varphi = (u \Phi^{-1} \psi)(u \Phi^{-1} \beta \varphi)\), while \(u(\psi' \beta' \varphi') = (u \psi')(u \beta' \varphi') = (u \Phi^{-1} \psi)(u \Phi^{-1} \beta \Phi^{-1} \varphi) = (u \Phi^{-1} \psi)(u \Phi^{-1} \beta \varphi)\). Thus, the map is a homomorphism. Since \(\Phi\) is an isomorphism, it is not difficult to verify that this map is indeed a bijection and hence, an isomorphism.

2) If \((y, \beta) \in A \text{ wr } (T, T)\) then \(\beta\) corresponds to some \(t \in T \subseteq S\) and \(d \beta = T^{-1}\). Let \(\beta'\) be the element of \((S, S)\) corresponding to \(t\). Then \(d \beta' = S \alpha' \subseteq T^{-1}\) and it follows that there exists a \(\psi'\) such that \((\psi', \beta') \in A \text{ wr } (S, S)\) and \(\psi'\) restricted to \(T^{-1}\) is \(\psi\). Given any identity satisfied by \(A \text{ wr } (S, S)\) to see that \(A \text{ wr } (T, T)\) satisfies this identity, observe that for any substitution of variables from \(A \text{ wr } (T, T)\), say \((\psi_1, \beta_1), \ldots, (\psi_n, \beta_n)\), the identity holds by substituting \((\psi_1', \beta_1'), \ldots, (\psi_n', \beta_n')\) and so it must hold when substituting \((\psi_1, \beta_1), \ldots, (\psi_n, \beta_n)\).

\[\text{Theorem 4.4.7.}\] Let \(\mathcal{Z}, \mathcal{V}\) and \(\mathcal{W}\) be varieties of inverse semigroups. Then

\[
\langle F_\mathcal{Z}(X) \text{ wr } F(W_{\mathcal{V}}(\mathcal{W}, \mathcal{W})) \rangle = \langle F_\mathcal{Z}(X) \text{ wr } (F_\mathcal{V}(X) \text{ wr } F_\mathcal{W}(X), F_\mathcal{V}(X) \text{ wr } F_\mathcal{W}(X)) \rangle
\]

\[
= \langle F_\mathcal{Z}(X) \text{ wr } (F_\mathcal{V}(X) \text{ wr } F_\mathcal{W}(X), F_\mathcal{V}(X) \times F_\mathcal{W}(X)) \rangle
\]

\[
= \langle [F_\mathcal{Z}(X) \text{ wr } (F_\mathcal{V}(X), F_\mathcal{W}(X))] \text{ wr } (F_\mathcal{V}(X), F_\mathcal{W}(X)) \rangle
\]

\[
= \langle F(W_\mathcal{Z}(\mathcal{V}), \mathcal{W})(X) \text{ wr } (F_\mathcal{V}(X), F_\mathcal{W}(X)) \rangle
\]

\[\text{Proof:}\]

\[W_\mathcal{Z}(\mathcal{V}, W_{\mathcal{W}}) = \langle F_\mathcal{Z}(X) \text{ wr } F(W_{\mathcal{V}}(\mathcal{W}, \mathcal{W})) \rangle\]

\[
\subseteq \langle F_\mathcal{Z}(X) \text{ wr } (F_\mathcal{V}(X) \text{ wr } F_\mathcal{W}(X), F_\mathcal{V}(X) \times F_\mathcal{W}(X)) \rangle
\]

\[
= \langle F(W_{\mathcal{Z}}(\mathcal{V}) \mathcal{W})(X) \text{ wr } (F_\mathcal{V}(X), F_\mathcal{W}(X)) \rangle
\]  

( Theorem 4.4.4 and Lemma 4.4.6 )

\[
\subseteq W_{\mathcal{Z}}(\mathcal{V}, W_{\mathcal{W}}) \quad \text{(by the definition of Wr)}
\]

and,

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\[
\text{Wr}(\text{Wr}(\mathcal{Z},\mathcal{Y}),\mathcal{W}) = \langle [F\mathcal{Y}(X) \text{ wr } (F\mathcal{Y}(X),F\mathcal{Y}(X))] \text{ wr } (F\mathcal{W}(X),F\mathcal{W}(X)) \rangle \\
= \langle F\mathcal{Y}(X) \text{ wr } (F\mathcal{Y}(X) \text{ wr } F\mathcal{W}(X), F\mathcal{Y}(X) \times F\mathcal{W}(X)) \rangle
\]
( Theorem 3.3.3 )

\subseteq \text{Wr}(\mathcal{Z},\text{Wr}(\mathcal{Y},\mathcal{W})).

Therefore, by the associativity of \text{Wr}, all of these varieties are the same.

\text{•}
CHAPTER FIVE
Consequences

Armed with the main result of the previous chapter, we set about proving various properties of \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) for a given pair of varieties \( \mathcal{Z} \) and \( \mathcal{Y} \). We first discuss general properties preserved under the \( \text{Wr} \) operation. Included among these are that the \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \)-free semigroups have solvable word problem (or, \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) has decidable equational theory) whenever the \( \mathcal{Y} \)-free semigroups and the \( \mathcal{Z} \)-free semigroups have solvable word problem. Also, if \( \mathcal{Z} \) and \( \mathcal{Y} \) are locally finite then so is \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \). In the second section we discuss properties preserved under \( \text{Wr} \) which are more inverse semigroup related. Included here are results concerning E-unitary covers. The penultimate section is devoted to showing that \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) is in fact the largest variety of inverse semigroups satisfying the same idempotent laws as \( \mathcal{Y} \). In the final section we look at some basic properties of the semigroup \( (\mathcal{Z}(\mathcal{Z}), \text{Wr}) \).

5.1 Further properties of \( \text{Wr} \)

**Theorem 5.1.1.** Let \( \mathcal{Z} \) and \( \mathcal{Y} \) be varieties of inverse semigroups. If \( \text{FZ}(X) \) and \( \text{FY}(X) \) have solvable word problems then so does \( \text{FWr}(\mathcal{Z}, \mathcal{Y})(X) \).

**Proof:** By Theorem 4.2.3, we need only show that we can determine whether or not \( \tau(w) \rho(\mathcal{Z}) \tau(u) \) whenever \( w \rho(\mathcal{Y}) u \).

Suppose that \( w = a_1 \ldots a_m \) and \( u = d_1 \ldots d_k \) where \( c(w) \cup c(u) = \{x_1, \ldots, x_n\} \) and \( a_i, d_j \in X \cup X^{-1} \). We construct words \( v_1 = b_1 \ldots b_m \) and \( v_2 = c_1 \ldots c_k \) over \( X \cup X^{-1} \) satisfying: for \( i < j \leq m \),

\[
\begin{align*}
\text{if } b_i &= b_j \quad \Leftrightarrow \quad a_i &= a_j \text{ and } a_1 \ldots a_{i-1}a_j \ldots a_m \geq_{\mathcal{Y}} a_1 \ldots a_m, \\
\text{if } b_i &= b_j^{-1} \quad \Leftrightarrow \quad a_i &= a_j^{-1} \text{ and } a_1 \ldots a_j a_j \ldots a_m \geq_{\mathcal{Y}} a_1 \ldots a_m,
\end{align*}
\]

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for $i < j \leq k$,

\begin{align*}
  c_i &= c_j & \Leftrightarrow & & d_i = d_j \text{ and } d_1 \ldots d_{i-1} d_j \ldots d_k \succeq_Y d_1 \ldots d_k, \\
  c_i &= c_{j-1} & \Leftrightarrow & & d_i = d_{j-1} \text{ and } d_1 \ldots d_{i-1} d_j \ldots d_k \succeq_Y d_1 \ldots d_k,
\end{align*}

and for any $i, j$,

\begin{align*}
  b_i &= c_j & \Leftrightarrow & & a_i = d_j \text{ and } a_1 \ldots a_{i-1} d_j \ldots d_k \succeq_Y a_1 \ldots a_k, \\
  b_i &= c_{j-1} & \Leftrightarrow & & a_i = d_{j-1} \text{ and } a_1 \ldots a_{i-1} d_j \ldots d_k \succeq_Y a_1 \ldots a_k.
\end{align*}

It is clear from Lemma 4.1.4 that $v_1$ and $v_2$ are one-to-one relabellings of $d_Y(w)$ and $d_Y(u)$ via the correspondence

\begin{align*}
  \lambda_w(e_{a_i}) & \leftrightarrow b_i \\
  \lambda_w(e_{d_j}) & \leftrightarrow c_j
\end{align*}

where $e_{a_i}$ ($e_{d_j}$) is the edge in $\Gamma_Y(w)$ corresponding to $a_i$ ($d_j$) in the path in $\Gamma_Y(w)$ labelled by $w$ ($u$). It follows that $v_1 \rho(\mathcal{Z}) v_2$ if and only if $d_Y(w) \rho_Y(\mathcal{Z}) d_Y(u)$. Since $F\mathcal{Z}(X)$ has solvable word problem, we can determine whether or not $v_1 \rho(\mathcal{Z}) v_2$ and hence, whether or not $d_Y(w) \rho_Y(\mathcal{Z}) d_Y(u)$. Therefore, if $F\mathcal{Z}(X)$ and $F\mathcal{Z}(X)$ have solvable word problems then so does $F\mathcal{Z}(\mathcal{Z},\mathcal{Y})(X)$.

We have used the fact that, if $\mathcal{Y}$ has solvable word problem, then the natural partial order $\leq_Y$ is solvable, since $w \leq_Y u$ if and only if $w \rho(\mathcal{Y}) w w^{-1} u$.

Corollary 5.1.2. If $\mathcal{Z}$ is a group variety and $\mathcal{Y}$ is any variety of inverse semigroups, then $F\mathcal{Z} \circ \mathcal{Y}(X)$ has solvable word problem if both $F\mathcal{Z}(X)$ and $F\mathcal{Y}(X)$ have solvable word problems.

Proof: This follows immediately from 4.3.4 and 5.1.1.

A variety $\mathcal{Y}$ is said to be \textit{locally finite} if every finitely generated member of $\mathcal{Y}$ is finite. Equivalently, $\mathcal{Y}$ is locally finite if and only if every $\mathcal{Y}$-free inverse semigroup on a finite set of generators is finite.
Theorem 5.1.3. $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ is locally finite if and only if both $\mathcal{Z}$ and $\mathcal{Y}$ are locally finite.

Proof: Since both $\mathcal{Z}$ and $\mathcal{Y}$ are contained in $\text{Wr}(\mathcal{Z}, \mathcal{Y})$, if $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ is locally finite then so are $\mathcal{Z}$ and $\mathcal{Y}$.

Suppose that $\mathcal{Z}$ and $\mathcal{Y}$ are locally finite but $\text{Wr}(\mathcal{Z}, \mathcal{Y})$ is not. Then for some $n \in \omega$, the $\text{Wr}(\mathcal{Z}, \mathcal{Y})$–free inverse semigroup on $n$ generators is not finite. Let $X_n$ be a subset of $X$ of cardinality $n$. It follows that there exists an infinite set of words $\{ w_i : i \in \omega \}$ over $X_n \cup X_n^{-1}$ such that, for all $i, j \in \omega$, $w_i$ is not $\rho(\text{Wr}(\mathcal{Z}, \mathcal{Y}))$–equivalent to $w_j$. Since $\mathcal{Y}$ is locally finite, we have as a consequence of Theorem 4.2.3 that there exists an infinite subset $\{ w_{ij} : j \in \omega \}$ of $\{ w_i : i \in \omega \}$ such that, for all $j, k \in \omega$, $w_{ij} \rho(\mathcal{Y}) w_{ik}$ but $d(w_{ij})$ is not $\rho(\mathcal{Z})$–equivalent to $d(w_{ik})$.

Let $V$ be the set of vertices and $E$ the set of edges incident to any of the paths from $s$ to $e$ in $\Gamma_{\mathcal{Y}}(w_{ij})$ labelled by the $w_{ik}$, $k \in \omega$, where $s$ and $e$ are the start and end vertices, respectively, of $\Gamma_{\mathcal{Y}}(w_{ij})$.

Let $v \in V$. Then for some $w_{ik}$, $v$ is incident to the path in $\Gamma_{\mathcal{Y}}(w_{ij})$ labelled by $w_{ik}$. Thus, there is an initial segment $w'$ of $w_{ik}$ such that $w'$ labels an $s$–$v$ walk in $\Gamma_{\mathcal{Y}}(w_{ij})$. It follows from the definition of Schützenberger graphs that $w_{ik} w_{ik}^{-1} w' \rho(\mathcal{Y}) v$. Since $c(w_{ik} w_{ik}^{-1} w') \subseteq X_n$ and since $F_{\mathcal{Y}}(X_n)$ is finite by our hypothesis, there can only be finitely many members of $V$.

Let $(v_1, z, v_2) \in E$. Then $v_1, v_2 \in V$ and $z \in \{ x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1} \}$. Since $V$ is finite, we have $|E| \leq |V|^2 \cdot 2n$ and so $E$ is finite.

Since $E$ is finite, it follows from the definition of the derived word that there exists an $m \in \omega$ such that $c( d(w_{ij}) ) \subseteq \{ y_1, \ldots, y_m \} = Y_m$ for all $j \in \omega$. Therefore, $\{ d(w_{ij}) \rho(\mathcal{Z}) : j \in \omega \}$ is contained in the $\mathcal{Z}$–free inverse semigroup on $Y_m$. Since $\mathcal{Z}$ is locally finite, the $\mathcal{Z}$–free inverse semigroup on $Y_m$ is finite and as a consequence,
\{ d(w_{ij}) \rho(\mathcal{Z}) : j \in \omega \} must be finite, contradicting the assertion above that, for all
j,k \in \omega, \ d( w_{ij} ) \ is not \ \rho(\mathcal{Z})-equivalent \ to \ d( w_{ik} ).

Therefore, if \mathcal{Z} and \mathcal{V} are locally finite then \text{Wr}(\mathcal{Z},\mathcal{V}) is locally finite.

In [K1], Kleiman proved that \mathcal{L}(\mathcal{P}) \ \text{the lattice of varieties of strict inverse}
semigroups, is isomorphic to three copies of \mathcal{L}(\mathcal{G}) \ \text{the lattice of group varieties and that}
every strict inverse variety \mathcal{V} \ is equal to (\mathcal{V} \cap \mathcal{G}) \lor (\mathcal{V} \cap \mathcal{B}). \ As a consequence, the \text{-}
class of a strict inverse variety is trivial. Thus, for any group variety \mathcal{U}, \mathcal{U} \circ \mathcal{I} = \mathcal{U} \lor \mathcal{I} = \text{Wr}(\mathcal{U},\mathcal{I}) \text{ and } \mathcal{U} \circ \mathcal{B} = \mathcal{U} \lor \mathcal{B} = \text{Wr}(\mathcal{U},\mathcal{B}). \ This \ is \ also \ an \ immediate \ consequence \ of
\text{Theorem 4.2.3}, \text{Proposition 2.8.3} \text{ and the following result.}

**Theorem 5.1.4 [Re3].** The collection of Schützenberger graphs corresponding to the
variety \mathcal{B} \ \text{is the collection of all finite birooted inverse word graphs in which each label}
(from X \cup X^{-1}) \ occurs at most once.

Reilly in fact showed that the \mathcal{B}-free semigroup on countably infinite X can be
represented faithfully by birooted labelled digraphs which, as it turns out, are the
Schützenberger graphs of \text{F}_\mathcal{B}(X). \ Stephen showed directly that the Schützenberger
graphs of \text{F}_\mathcal{B}(X) \ are the ones mentioned above. \ We remark that, in the following
theorem, we do not need to know what the Schützenberger representation relative to \mathcal{B} \ of a
given word is, but simply that in its Schützenberger representation relative to \mathcal{B} each label
occurs at most once.

**Theorem 5.1.5.** If \mathcal{V} \in \{ \mathcal{I}, \mathcal{P}, \mathcal{B} \} then \text{Wr}(\mathcal{U},\mathcal{V}) = \mathcal{U} \lor \mathcal{V} \ for any variety \mathcal{U} \ of
inverse semigroups.
Proof: It follows from the definition of the derived word, Theorem 5.1.4 and Proposition 2.8.3 that for $\mathcal{Y} \in \{ \mathcal{X}, \mathcal{Y}, \mathcal{Z} \}$, $d_{\mathcal{Y}}(w)$ is a relabelling of $w$ in $Y \cup Y^{-1}$ for any word $w \in (X \cup X^{-1})^+$. It follows from Theorem 4.2.3 that $\text{Wr}(\mathcal{Y}, \mathcal{Y}') = \mathcal{Y} \vee \mathcal{Y}'$.

5.2 E-unitary covers

The results of this section are concerned with conditions under which varieties of the form $\text{Wr}(\mathcal{Y}, \mathcal{W})$ have E-unitary covers and E-unitary covers over some group variety. If we are to use Theorem 4.2.3 in this effort, we require some information about the Schützenberger graphs of $F_{\mathcal{W}}^{\text{max}}(X)$, for $\mathcal{W}$ a variety of groups. This information is provided by a graphical representation of $F_{\mathcal{W}}^{\text{max}}(X)$ due to Meakin and Margolis [MM] which we present forthwith.

Let $\mathcal{W}$ be a variety of groups. Then $P = (X; \rho(\mathcal{W}))$ is a presentation of $F_{\mathcal{W}}(X)$ for which $\{ x\rho(\mathcal{W}) : x \in X \}$ freely generates $F_{\mathcal{W}}(X)$. Let $\Gamma(X; \rho(\mathcal{W}))$ denote the Cayley graph of $F_{\mathcal{W}}(X)$ relative to $P$.

Let $M(X; \rho(\mathcal{W})) = \{ (\Gamma, g) : g \in F_{\mathcal{W}}(X) \}$ and $\Gamma$ is a finite connected subgraph of $\Gamma(X; \rho(\mathcal{W}))$ containing $1$ and $g$ as vertices, where $1$ is the identity of $F_{\mathcal{W}}(X)$. For each finite subgraph $\Gamma'$ of $\Gamma(X; \rho(\mathcal{W}))$ and each $g \in F_{\mathcal{W}}(X)$, let $g\Gamma'$ be the subgraph of $\Gamma(X; \rho(\mathcal{W}))$ obtained by acting on $\Gamma'$ on the left. The set of vertices of $g\Gamma'$ is $\{ gh : h \text{ is a vertex of } \Gamma' \}$ and the edges of $g\Gamma'$ are of the form $(gh, x, ghx)$ whenever $(h, x, hx)$ is an edge in $\Gamma'$. Observe that $g\Gamma'$ is $V$-isomorphic to $\Gamma'$.

On $M(X; \rho(\mathcal{W}))$ define a multiplication by setting

$$(\Gamma, g)(\Gamma', g') = (\Gamma \cup g\Gamma', gg')$$
where $\Gamma \cup g\Gamma^*$ is the graph whose vertices and edges are the union of the vertices and edges of $\Gamma$ and $g\Gamma^*$.

**Theorem 5.2.1 [MM]**. $M(X; \rho(\mathcal{Z}))$ is an $E$-unitary inverse monoid generated by the graphs $(\Gamma_x, xp(\mathcal{Z}))$ for $x \in X$, where $\Gamma_x$ is the graph with vertex set $\{1, xp(\mathcal{Z})\}$ and (directed) edge set $\{(1, x, xp(\mathcal{Z})), (xp(\mathcal{Z}), x^{-1}, 1)\}$. Furthermore, $M(X; \rho(\mathcal{Z}))$ is (isomorphic to) the relatively free $X$-generated inverse monoid in the variety $\mathcal{Z}_M^{\text{max}}$.

$M(X; \rho(\mathcal{Z}))$ satisfies the following properties:

i) $(\Gamma, g) \not\sim (\Gamma', g')$ if and only if $\Gamma = \Gamma'$;

ii) $(\Gamma, g) \not\simeq (\Gamma', g')$ if and only if $g^{-1}\Gamma = (g')^{-1}\Gamma'$;

iii) $(\Gamma, g) \not\simeq (\Gamma', g')$ if and only if $\Gamma$ is $\mathcal{V}$-isomorphic to $\Gamma'$;

iv) $(\Gamma, g)$ is an idempotent if and only if $g = 1$.

We are interested in varieties of inverse semigroups (as opposed to monoids) and so to make use of Theorem 5.2.1, we require the following result.

**Lemma 5.2.2**. Let $\mathcal{Z}$ be a variety of groups. Then $F\mathcal{Z}^{\text{max}}(X)$ is isomorphic (as inverse semigroups) to $F\mathcal{Z}_M^{\text{max}}(X) \setminus \{1\}$, where 1 is the identity of $F\mathcal{Z}_M^{\text{max}}(X)$.

**Proof**: We first must show that $S = F\mathcal{Z}_M^{\text{max}}(X) \setminus \{1\}$ is an inverse semigroup. If $S$ is not an inverse semigroup then there is a $w \in (X \cup X^{-1})^+$ such that $\mathcal{Z}_M^{\text{max}}$ satisfies the equation $w = 1$. Let $a \in X \cup X^{-1}$ be the initial letter in $w$; that is, $w = aw'$ for some $w' \in (X \cup X^{-1})^*$. Then, since $aa^{-1}$ is a left identity for $w$, $\mathcal{Z}_M^{\text{max}}$ must satisfy the equation $aa^{-1} = 1$. But then for any $v \in \mathcal{Z}_M^{\text{max}}$ we must have $vv^{-1} = 1$ and $v^{-1}v = 1$ (by substituting $v$ for $a$ in the first case and $v^{-1}$ for $a$ in the latter case). Thus, $F\mathcal{Z}_M^{\text{max}}(X)$ has a single $\mathcal{H}$-class and so must be a group. Since this is not the case, $F\mathcal{Z}_M^{\text{max}}(X) \setminus \{1\}$ must be an inverse semigroup.
If \( T \in \mathcal{Z}^{\text{max}} \), then the monoid \( T^1 \in \mathcal{Z}^\text{max}_M \). This follows from the observation that the set of identities \( \{ u = u^2 : u = u^2 \text{ is a law in } \mathcal{Z} \} \) is closed under deletion. That is, if \( A \subseteq c(u) \) and \( u_A \) is the word obtained from \( u \) by deleting all occurrences of \( x \) and \( x^{-1} \) in \( u \) for all \( x \in A \), then \( u_A = u_A^2 \) is a law in \( \mathcal{Z} \).

Finally, we show that \( S \) is free in \( \mathcal{Z}^{\text{max}} \). Let \( T \) be an inverse semigroup in \( \mathcal{Z}^{\text{max}} \) and let \( f : X \to T \) be a function. Then \( T^1 \in \mathcal{Z}^\text{max}_M \) and \( f \) can be extended uniquely to a homomorphism \( \psi \) of \( F\mathcal{Z}^\text{max}_M(X) \) into \( T^1 \). Since \( S \) and \( T \) are inverse semigroups, \( \psi \) maps \( S \) into \( T \) and hence maps only the identity of \( F\mathcal{Z}^\text{max}_M(X) \) onto the identity of \( T \). Thus, \( \chi = \psi |_S \) is a homomorphism of \( S \) into \( T \) which extends \( f \). If \( \theta \) is another homomorphism of \( S \) into \( T \) extending \( f \), then the uniqueness of \( \psi \) implies that \( \theta = \chi \). Therefore, \( \chi \) is the unique homomorphism of \( S \) into \( T \) extending \( f \). It follows that \( S \cong F\mathcal{Z}^{\text{max}}(X) \).

**Remark.** The argument above can be extended in the obvious way to obtain the following result. If \( \mathcal{Y}_M \) is a variety of inverse monoids which is not a variety of groups, then

1) for all \( w \in (X \cup X^{-1})^+ \), \( \mathcal{Y}_M \) does not satisfy the equation \( w = 1 \);

2) if \( \Sigma \) is a basis of identities for \( \mathcal{Y}_M \) such that no equation in \( \Sigma \) contains an occurrence of \( 1 \), then \( F\mathcal{Y}_M(X) \setminus \{1\} \) is isomorphic to the relatively free object on \( X \) in the variety of inverse semigroups defined by \( \Sigma \).

It is not difficult to verify that the identity of \( M( X ; \rho(\mathcal{Z}) ) \) is \( (\Gamma_1,1) \), where \( \Gamma_1 \) is the graph consisting of the single vertex \( \{1\} \) and no edges. Thus, \( M( X ; \rho(\mathcal{Z}) ) \setminus \{ (\Gamma_1,1) \} \) is isomorphic to \( F\mathcal{Z}^{\text{max}}(X) \) via the map \( \chi \) which takes \( (\Gamma_x, x\rho(\mathcal{Z}) ) \) to \( x\rho(\mathcal{Z}^{\text{max}}) \) for all \( x \in X \).

**Lemma 5.2.3.** Let \( w \in (X \cup X^{-1})^+ \) and suppose that
(\Gamma, g) \in M(\mathcal{X}; \rho(\mathcal{Z}')) \setminus \{(\Gamma_1, 1)\} is such that (\Gamma, g)\chi = w\rho(\mathcal{Z}^{\text{max}}). Then the Schützenberger graph \Gamma_{\mathcal{Z}^{\text{max}}}(w) of w relative to \mathcal{Z}^{\text{max}} is \text{V}-isomorphic to \Gamma.

\textbf{Proof:} By Theorem 5.2.1, (\Gamma, g) \mathcal{R} (\Gamma', g') if and only if \Gamma = \Gamma' and so there is a one-to-one correspondence between the vertices of \Gamma and the members of the \mathcal{R}-class of (\Gamma, g). Let the vertex set of \Gamma, V(\Gamma) = \{g_1, \ldots, g_n\}. Then the function \phi_V : V(\Gamma) \to V(\Gamma_{\mathcal{Z}^{\text{max}}}(w)) defined by \phi_i(\Gamma, g_i) = (\Gamma, g_i)\chi, i = 1, \ldots, n, is a one-to-one map of the vertices of \Gamma onto the vertices of \Gamma_{\mathcal{Z}^{\text{max}}}(w). We then define \phi_E : E(\Gamma) \to E(\Gamma_{\mathcal{Z}^{\text{max}}}(w)) by \phi_E = ((\Gamma, g_i)\chi, x, (\Gamma, g_j)\chi), for all edges \phi_i(x, g_j) in \Gamma. Now,

(\Gamma, g_i)(\Gamma, x) = (\Gamma \cup g_i\Gamma x, g_i \rho(\mathcal{Z}')) = (\Gamma, g_j)

Thus, \phi_E maps edges of \Gamma to edges of \Gamma_{\mathcal{Z}^{\text{max}}}(w). Clearly \phi_E is one-to-one. To see that \phi_E is onto, let (v_1, x, v_2) be an edge in \Gamma_{\mathcal{Z}^{\text{max}}}(w). Then v_1 = (\Gamma, g_i)\chi and v_2 = (\Gamma, g_j)\chi, say. By the definition of Schützenberger graph, (\Gamma, g_i)\chi \rho(\mathcal{Z}^{\text{max}}) = (\Gamma, g_j)\chi which implies that (\Gamma, g_i)(\Gamma x, x) = (\Gamma, g_j) since \chi is an isomorphism and maps (\Gamma x, x) onto \rho(\mathcal{Z}^{\text{max}}). But then, (\Gamma \cup g_i\Gamma x, g_i x) = (\Gamma, g_j) and so \Gamma \cup g_i\Gamma x = \Gamma and g_i \rho(\mathcal{Z}') = g_j, whence (g_i, x, g_j) is an edge in \Gamma. Since (g_i, x, g_j) \phi_E = ((\Gamma, g_i)\chi, x, (\Gamma, g_j)\chi) = (v_1, x, v_2), we have that \phi_E is surjective. Therefore, \phi = (\phi_V, \phi_E) is a \text{V}-isomorphism of \Gamma onto \Gamma_{\mathcal{Z}^{\text{max}}}(w). Finally, g\phi_E = w\rho(\mathcal{Z}^{\text{max}}) and l\phi_E = (\Gamma, 1)\chi = ww^{-1}\rho(\mathcal{Z}^{\text{max}}), since \chi is an isomorphism and so must map the idempotent of the \mathcal{R}-class of (\Gamma, g) onto the idempotent of the \mathcal{R}-class of (\Gamma, g)\chi = w\rho(\mathcal{Z}^{\text{max}}). Thus, \phi is a \text{V}-isomorphism which maps roots to roots, as required.
If \((\Gamma, g)\) and \((\Gamma', g')\) are members of \(M(X; \rho(Z'))\) then both \(\Gamma\) and \(\Gamma'\) are \(V\)-embeddable in \(\Gamma \cup g\Gamma'\), since \(g\Gamma'\) is \(V\)-isomorphic to \(\Gamma'\). Lemma 5.2.3 then says that if \(u\) and \(w\) are words in \(X\), then \(\Gamma_{\text{max}}(u)\) and \(\Gamma_{\text{max}}(w)\) are both \(V\)-embeddable in \(\Gamma_{\text{max}}(uw)\). Indeed, we have the following:

**Lemma 5.2.4.** Let \(\mathcal{V}\) be a variety of inverse semigroups which has \(E\)-unitary covers over its group part and let \(\mathcal{Z}\) be a variety of groups. Let \(u, w \in (X \cup X^{-1})^+\). If

\[
d_{\text{max}}(uw) \rho(\mathcal{V}) d_{\text{max}}(uw)^2 \quad \text{and} \quad d_{\text{max}}(u) \rho(\mathcal{V}) d_{\text{max}}(u)^2
\]

then

\[
d_{\text{max}}(w) \rho(\mathcal{V}) d_{\text{max}}(w)^2.
\]

**Proof:** Let \(\chi\) be the isomorphism which maps \(M(X; \rho(Z')) \setminus \{(\Gamma_1, 1)\}\) onto \(\Gamma_{\text{max}}(X)\). Let \(u_{\mathcal{V}}(\Gamma_{\text{max}}) = (\Gamma, g)\chi\) and \(w_{\mathcal{V}}(\Gamma_{\text{max}}) = (\Gamma', g')\chi\). Then \(u_{\mathcal{V}}(\Gamma_{\text{max}}) = (\Gamma \cup g\Gamma', gg')\chi\). Now both \(\Gamma\) and \(g\Gamma'\) are \(V\)-embeddable in \(\Gamma \cup g\Gamma'\) and so by Lemma 5.2.3, \(\Gamma_{\text{max}}(u)\) and \(\Gamma_{\text{max}}(w)\) are \(V\)-embeddable in \(\Gamma_{\text{max}}(uw)\). Let \(Y_1, Y_2\) and \(Y_3\) be the secondary label sets of \(\overline{\Gamma_{\text{max}}(u)}\), \(\overline{\Gamma_{\text{max}}(w)}\) and \(\overline{\Gamma_{\text{max}}(uw)}\), respectively. We may assume that \(Y_1\) and \(Y_2\) are disjoint.

Define \(f: Y_1 \cup Y_2 \to Y_3\) as follows:

If \(y \in Y_1\) then \(y\) labels an edge \(e_{y_1}\) in \(\Gamma_{\text{max}}(u)\). Since \(\Gamma_{\text{max}}(u)\) is \(V\)-isomorphic to \(\Gamma\), \(e_{y_1}\) corresponds to an edge \(e_{y_2}\) in \(\Gamma\) which in turn corresponds to an edge \(e_{y_3}\) in \(\Gamma \cup g\Gamma'\) via the obvious embedding. Since \(\Gamma \cup g\Gamma'\) is \(V\)-isomorphic to \(\Gamma_{\text{max}}(uw)\), \(e_{y_3}\) corresponds to an edge \(e_{y_4}\) in \(\Gamma_{\text{max}}(uw)\). Define \(yf\) to be the secondary label \(\lambda_{uw}(e_{y_4})\) in \(\Gamma_{\text{max}}(uw)\).

If \(y \in Y_2\) then \(y\) labels an edge \(e_{y_1}\) in \(\Gamma_{\text{max}}(w)\). Since \(\Gamma_{\text{max}}(w)\) is \(V\)-isomorphic to \(\Gamma'\), \(e_{y_1}\) correspond to an edge \(e_{y_2}\) in \(\Gamma'\) which in turn corresponds to an edge \(e_{y_3}\) in \(g\Gamma'\) via the obvious \(V\)-isomorphism. Now, \(e_{y_3}\) corresponds to an edge \(e_{y_4}\) in
$\Gamma \cup g\Gamma'$ via the obvious embedding and since $\Gamma \cup g\Gamma'$ is $V$–isomorphic to $\Gamma_{\text{max}}(uw)$, $e_y$ corresponds to an edge $e_y$ in $\Gamma_{\text{max}}(uw)$. Define $y_f$ to be the secondary label $\lambda_{uw}(e_y)$ in $\Gamma_{\text{max}}(uw)$.

It follows that $f$ is one-to-one on $Y_1$ and one-to-one on $Y_2$ (but not necessarily one-to-one on $Y_1 \cup Y_2$). Furthermore, $f$ maps $Y_1 \cup Y_2$ onto $Y_3$ which is a consequence of Lemma 5.2.3 and the fact that the edge set of $\Gamma \cup g\Gamma'$ is the union of the edge sets of $\Gamma$ and $g\Gamma'$. Also, $f$ extends uniquely to a homomorphism (also denoted $f$) which maps $(Y_1 \cup Y_2)^+$ onto $Y_3^+$. It follows from our definition of $f$ that $d_{\text{max}}(uw) = (d_{\text{max}}(u)f)(d_{\text{max}}(w)f)$. By the hypothesis, $d_{\text{max}}(uw) \rho(\mathcal{Y}) d_{\text{max}}(uw)^2$ and $d_{\text{max}}(u) \rho(\mathcal{Y}) d_{\text{max}}(u)^2$ and so, $d_{\text{max}}(u)f \rho(\mathcal{Y}) (d_{\text{max}}(u)f)^2$, since $f$ is one-to-one on $Y_1$. Thus, $d_{\text{max}}(uw) \leq d_{\text{max}}(w)f$. By Theorem 2.7.3, $F\mathcal{Y}(Y_3)$ is $E$-unitary and so, as a consequence, $d_{\text{max}}(w)f \rho(\mathcal{Y}) (d_{\text{max}}(w)f)^2$. But $f$ is one-to-one on $Y_2$, so $d_{\text{max}}(w) \rho(\mathcal{Y}) (d_{\text{max}}(w))^2$.

Corollary 5.2.5. Let $\mathcal{Y}$ be a variety of inverse semigroups which has $E$-unitary covers (over $\mathcal{Y} \cap \mathcal{G}$) and let $\mathcal{Z}$ be a variety of groups. Then $FWr(\mathcal{Y},Z_{\text{max}})(X)$ is $E$-unitary.

Proof: Set $\rho(\mathcal{W}(\mathcal{Y},Z_{\text{max}})) = \rho$ and write $d(\_)$ for $d_{\text{max}}(\_)$.

Let $e,w \in (X \cup X^{-1})^+$ be such that $e \rho ew$ where $e \rho e^2$. By Theorem 4.2.3 we have $e \rho(Z_{\text{max}}) ew$, $e \rho(Z_{\text{max}}) e^2$, $d(e) \rho(\mathcal{Y}) d(ew)$ and $d(e) \rho(\mathcal{Y}) d(e^2) = d(e)^2$ with this last equality holding by Proposition 4.1.2 since $e \rho(Z_{\text{max}}) e^2$. Now $FZ_{\text{max}}(X)$ is $E$-unitary by Theorems 2.7.3 and 2.7.4, so $w \rho(Z_{\text{max}}) w^2$. But $d(e) \rho(\mathcal{Y}) d(e)^2$ and $d(ew) \rho(\mathcal{Y}) d(e)$ which implies $d(ew) \rho(\mathcal{Y}) d(ew)^2$. Thus, by Lemma 5.2.4, $d(w) \rho(\mathcal{Y}) d(w)^2 = d(w^2)$ where again the last equality holds by Proposition 4.1.2. Theorem 4.2.3 now gives $w \rho w^2$. Therefore, $FWr(\mathcal{Y},Z_{\text{max}})(X)$ is $E$-unitary.
Corollary 5.2.6. Let \( \mathcal{V} \) be a variety of inverse semigroups which has E-unitary covers and let \( \mathcal{Z} \) be a variety of groups. Then \( \text{Wr}(\mathcal{V}, \mathcal{Z}^{\max}) \) has E-unitary covers (over \( \text{Wr}(\mathcal{V}, \mathcal{Z}^{\max}) \cap \mathcal{G} \)).

Proof: By Corollary 5.2.5 and Theorem 2.7.3.

Theorem 5.2.7. Let \( \mathcal{Z} \) and \( \mathcal{V} \) be varieties of groups and let \( \mathcal{A} \) and \( \mathcal{W} \) be varieties of inverse semigroups such that \( \mathcal{A} \) has E-unitary covers over \( \mathcal{Z} \) and \( \mathcal{W} \) has E-unitary covers over \( \mathcal{V} \). Then \( \text{Wr}(\mathcal{A}, \mathcal{W}) \) has E-unitary covers over \( \text{Wr}(\mathcal{Z}, \mathcal{V}) = \mathcal{Z} \circ \mathcal{V} \).

Proof: We know that \( \text{Wr}(\mathcal{A}, \mathcal{W}) \subseteq \text{Wr}(\mathcal{Z}^{\max}, \mathcal{V}^{\max}) \) by the hypothesis and Proposition 4.3.1. The theorem will follow from Corollary 5.2.6 and Theorem 2.7.4 if we can show that \( \text{Wr}(\mathcal{Z}^{\max}, \mathcal{V}^{\max}) \cap \mathcal{G} = \text{Wr}(\mathcal{Z}, \mathcal{V}) \). This follows immediately from Theorem 4.3.7, however, we include the following argument as it deals with this specific case and provides us with a better 'intuitive feel' for why this result should be true.

Set \( U = \{ w \in (X \cup X^{-1})^\dagger : w \text{ is a law in } \mathcal{Z} \} \) and \( V = \{ w \in (X \cup X^{-1})^\dagger : w \text{ is a law in } \mathcal{V} \} \).

Let \( U(V) = \{ u(v_1, \ldots, v_n) : u = u(x_1, \ldots, x_n) \in U \text{ and } v_1, \ldots, v_n \in V \} \).

Our first claim is that \( \text{Wr}(\mathcal{Z}^{\max}, \mathcal{V}^{\max}) \subseteq [ w = w^2 : w \in U(V) ] \). It is sufficient to show that \( S \text{ wr } (T, I) \) satisfies \( w = w^2 \) for all \( S \in \mathcal{Z}^{\max}, (T, I) \in \mathcal{V}^{\max} \) and \( w \in U(V) \).

Let \( w \in U(V) \), say \( w = u(v_1, \ldots, v_n) \), where \( v_i = v_i[x_1, \ldots, x_m(i)] \), for \( i = 1, \ldots, n \), and \( u = u[x_1, \ldots, x_n] \). Suppose that for an arbitrary substitution of variables, \( v_i \) takes the value \( (\psi_i, \beta_i) \) in \( S \text{ wr } (T, I) \), for \( i = 1, \ldots, n \). Since \( T \in \mathcal{V}^{\max} \) and \( \mathcal{V}^{\max} \) satisfies the identities \( v_i = v_i^2 \), for \( i = 1, \ldots, n \), each \( \beta_i \) is an idempotent in \( (T, I) \). That is, each \( \beta_i \) is the identity map on its domain. We wish to show that \( u[(\psi_1, \beta_1), \ldots, (\psi_n, \beta_n)] \) is an idempotent in \( S \text{ wr } (T, I) \). Let \( u = a_1 \ldots a_k \). Using the same notation as before, we wish to show that

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$(\psi_{a_1}\beta_{a_1}\psi_{a_2}...\beta_{a_k}\psi_{a_k})$ is an idempotent in $S \wr (T,I)$. Since each of the $\beta_i$s is an idempotent, we have that $\beta_{a_1}...\beta_{a_k}$ is an idempotent. Moreover, for all $i \in d \beta_{a_1}...\beta_{a_k}$,

$$i(\psi_{a_1}\beta_{a_1}\psi_{a_2}...\beta_{a_k}\psi_{a_k}) = (i\psi_{a_1})(i\beta_{a_1}\psi_{a_2})...(i\beta_{a_k}\psi_{a_k}) = (i\psi_{a_1})(i\psi_{a_2})...(i\psi_{a_k})$$

since each $\beta_i$ is the identity map on its domain. Since $S \in Z_{\text{max}}$, $S$ satisfies the equation $u = u^2$. Thus, $(i\psi_{a_1})(i\psi_{a_2})...(i\psi_{a_k})$ and hence, $i(\psi_{a_1}\beta_{a_1}\psi_{a_2}...\beta_{a_k}\psi_{a_k})$ is an idempotent of $S$. It follows from Proposition 3.1.1 (c) that $u[\!(\psi_1,\beta_1),...,(\psi_n,\beta_n)\!]$ is an idempotent of $S \wr (T,I)$. Therefore, $S \wr (T,I)$ satisfies the equation $w = w^2$. From this we obtain that $\text{Wr}(Z_{\text{max}},Z_{\text{max}}) \subseteq [ w = w^2 : w \in U(V) ]$.

Our second claim is that $[ w = w^2 : w \in U(V) ] \cap \mathcal{G} = \mathcal{Z} \circ \mathcal{Y}$. Observe that $U \rho(\mathcal{G}) = \{ u \rho(\mathcal{G}) : u \in U \}$ and $V \rho(\mathcal{G}) = \{ v \rho(\mathcal{G}) : v \in V \}$ are the fully invariant subgroups of $FG(X)$ corresponding to $\mathcal{Z}$ and $\mathcal{Y}$, respectively. It follows from Neumann [N;21.12] that $\{ w = w^2 : w \in U(V) \} \cup \{ xx^{-1} = yy^{-1} \}$ forms a basis of identities for $\mathcal{Z} \circ \mathcal{Y}$.

We may now conclude that

$$\text{Wr}(Z_{\text{max}},Z_{\text{max}}) \cap \mathcal{G} \subseteq [ w = w^2 : w \in U(V) ] \cap \mathcal{G} = \mathcal{Z} \circ \mathcal{Y}.$$ 

By Theorem 4.3.4 and Proposition 4.3.1,

$$\mathcal{Z} \circ \mathcal{Y} = \text{Wr}(\mathcal{Z},\mathcal{Y}) \subseteq \text{Wr}(Z_{\text{max}},Z_{\text{max}}) \cap \mathcal{G}.$$ 

Thus, $\text{Wr}(Z_{\text{max}},Z_{\text{max}}) \cap \mathcal{G} = \text{Wr}(\mathcal{Z},\mathcal{Y})$ and so Theorem 5.2.7 is proved.

**Corollary 5.2.8.** Let $\mathcal{Y}$ be a variety of inverse semigroups.

1) If $\mathcal{Y}$ has $E$-unitary covers then, for any group variety $\mathcal{Z}$,

$$\mathcal{Z} \circ \mathcal{Y} = \mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G}) \cup \mathcal{Y}.$$ 

2) If $\mathcal{Y}$ has $E$-unitary covers over the group variety $\mathcal{Z}$ then, for any group variety $\mathcal{W}$,

$$(\mathcal{W} \circ \mathcal{Z}) \cup \mathcal{Y} = (\mathcal{W} \circ \mathcal{Z}) \cup (\mathcal{W} \circ \mathcal{Y}).$$ 

**Proof:** 1) By Theorem 5.2.7, $\mathcal{Z} \circ \mathcal{Y}$ has $E$-unitary covers over $\mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G})$. Therefore,

$$\mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G}) \subseteq \mathcal{Z} \circ \mathcal{Y} \subseteq [ \mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G}) ]_{\text{max}}.$$ 

But $\mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G}) \subseteq \mathcal{Z} \circ (\mathcal{Y} \cap \mathcal{G}) \cup \mathcal{Y}$. 

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and $\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{V} \subseteq [\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G})]^{\max}$. Thus, $\ker \rho (\mathcal{V} \circ \mathcal{V}) = \ker \rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V})$. On the other hand, $tr(\mathcal{V}) = tr(\mathcal{V} \circ \mathcal{V})$ by Lemma 2.7.5 and $tr \rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}) = tr [\rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G})) \cap \rho (\mathcal{V})] = tr \rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G})) \cap tr \rho (\mathcal{V})$ and $tr \rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G})) \cap tr \rho (\mathcal{V}) = tr \rho (\mathcal{V})$. Thus, $tr(\mathcal{V} \circ \mathcal{V}) = tr \rho (\mathcal{V} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V})$. We thus obtain $\mathcal{V} \circ \mathcal{V} = \mathcal{V} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}$.

2) By Theorem 5.2.7, $\mathcal{W} \circ \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{V})^{\max}$ and by the hypothesis $\mathcal{V} \subseteq \mathcal{V}^{\max}$ and hence, $\mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{V})^{\max}$. Thus, $\mathcal{W} \circ \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{V}) \vee \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{V})^{\max}$ and $\mathcal{W} \circ \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{V}) \vee (\mathcal{W} \circ \mathcal{V}) \subseteq (\mathcal{W} \circ \mathcal{V})^{\max}$. Therefore, $\ker \rho ((\mathcal{W} \circ \mathcal{V}) \vee \mathcal{V}) = \ker \rho ((\mathcal{W} \circ \mathcal{V}) \vee (\mathcal{W} \circ \mathcal{V}))$. On the other hand, $tr \rho [(\mathcal{W} \circ \mathcal{V}) \vee \mathcal{V}] = tr [\rho (\mathcal{W} \circ \mathcal{V}) \cap \rho (\mathcal{V})] = tr \rho (\mathcal{W} \circ \mathcal{V}) \cap tr \rho (\mathcal{V}) = tr \rho (\mathcal{V})$ and $tr \rho [(\mathcal{W} \circ \mathcal{V}) \vee (\mathcal{W} \circ \mathcal{V})] = tr [\rho (\mathcal{W} \circ \mathcal{V}) \cap \rho (\mathcal{W} \circ \mathcal{V})] = tr \rho (\mathcal{W} \circ \mathcal{V}) \cap tr \rho (\mathcal{W} \circ \mathcal{V}) = tr \rho (\mathcal{W} \circ \mathcal{V}) \cap tr \rho (\mathcal{V})$ (by Lemma 2.7.5) $= tr \rho (\mathcal{V})$. It now follows that $(\mathcal{W} \circ \mathcal{V}) \vee \mathcal{V} = (\mathcal{W} \circ \mathcal{V}) \vee (\mathcal{W} \circ \mathcal{V})$.

5.3 $\text{Wr}(\mathcal{I}, \mathcal{V})$

The principal result of this section is the following. For any variety $\mathcal{V}$ of inverse semigroups, $\text{Wr}(\mathcal{I}, \mathcal{V})$ is the largest variety of inverse semigroups which satisfies the equations $w = w^2$ whenever $\mathcal{V}$ satisfies $w = w^2$. Throughout this section we will use the following convention. If $w \in (X \cup X^{-1})^+$ and $\mathcal{V}$ is a variety of inverse semigroups, we will write $w_\mathcal{V}$ to denote $wp(\mathcal{V})$.

**Theorem 5.3.1.** Let $\mathcal{U} \subseteq \mathcal{V}$ be varieties of inverse semigroups and let $\rho$ be the congruence on $F\mathcal{V}(X)$ such that $F\mathcal{V}(X) / \rho \equiv F\mathcal{U}(X)$. Then $\rho$ is idempotent pure if and only if for every $w \in (X \cup X^{-1})^+$, $\Gamma_\mathcal{V}(w)$ is $V$-embeddable in $\Gamma_\mathcal{U}(w)$. 100
Proof: Suppose that \( p \) is idempotent pure and let \( w \in (X \cup X^{-1})^+ \). Define a map \( \phi \) on \( \mathcal{R}_{w,}\), the set of vertices of \( \Gamma_{\mathcal{V}}(w) \), by setting \( v\phi = vp \). Green's relation \( \mathcal{R} \) is preserved under homomorphism so \( \phi \) maps \( \mathcal{R}_{w,} \) into \( \mathcal{R}_{w,p} \), which is the vertex set of \( \Gamma_{\mathcal{V}}(w) \) since, for any \( v \in (X \cup X^{-1})^+ \), \( v\mathcal{R}p = v_{2}. \) If \( (v_{1},x,v_{2}) \) is an edge in \( \Gamma_{\mathcal{V}}(w) \) then \( v_{1}x\mathcal{V} = v_{2} \) and so \( (v_{1}p)(x\mathcal{R}p) = (v_{2}p) \). But this means that \( (v_{1}p)x_{2} = v_{2}\phi \) from which it follows that \( (v_{1}p,x,v_{2}) \) is an edge in \( \Gamma_{\mathcal{V}}(w) \). Therefore, \( \phi \) is a \( \mathcal{V} \)-homomorphism.

Suppose that \( v_{1}\phi = v_{2}\phi \) for some \( v_{1},v_{2} \in \mathcal{R}_{w,} \). Then \( v_{1}p v_{2} \) and so \( v_{1} = v_{2} \) since \( p \cap \mathcal{R} = \varepsilon \) whenever \( p \) is idempotent pure. Thus, \( \phi \) is a \( \mathcal{V} \)-embedding of \( \Gamma_{\mathcal{V}}(w) \) into \( \Gamma_{\mathcal{V}}(w) \).

Conversely, suppose that \( \Gamma_{\mathcal{V}}(w) \) is \( \mathcal{V} \)-embeddable in \( \Gamma_{\mathcal{V}}(w) \) for every \( w \in (X \cup X^{-1})^+ \). Let \( e,a \in (X \cup X^{-1})^+ \) be such that \( e\mathcal{V} = e_{2} \) and \( e\mathcal{R} a_{\mathcal{V}} \). Then \( a_{2} = aa^{-1}_{\mathcal{V}} \). If \( \phi \) is the \( \mathcal{V} \)-embedding of \( \Gamma_{\mathcal{V}}(a) \) into \( \Gamma_{\mathcal{V}}(a) \) then \( a\mathcal{V} \phi = a_{2} \) and \( aa^{-1}_{\mathcal{V}} \phi = aa^{-1}_{\mathcal{V}} \), since \( \phi \) maps roots to roots. Since \( \phi \) is one-to-one on the vertices of \( \Gamma_{\mathcal{V}}(a) \) and \( a_{2} = aa^{-1}_{\mathcal{V}} \), we must have that \( a\mathcal{V} = aa^{-1}_{\mathcal{V}} \) and so \( p \) is idempotent pure.

Lemma 5.3.2. Let \( \mathcal{Y} \subseteq \mathcal{V} \) be varieties of inverse semigroups and suppose that \( \mathcal{Y}^{\max} = \mathcal{V}^{\max} \). If \( p \) is the congruence on \( F_{\mathcal{V}}(X) \) such that \( F_{\mathcal{V}}(X)/p \equiv F_{\mathcal{Y}}(X) \) then \( p \) is idempotent pure.

Proof: Let \( w,a \in (X \cup X^{-1})^+ \) be such that \( w\mathcal{V} = w_{2} \) and \( w\mathcal{R} a_{\mathcal{V}} \).

Then \( a\mathcal{V} = a_{2} \); that is, \( a_{2} = a_{2} \). But then \( a_{2}^{\max} = a_{2}^{\max} \) and so, since \( \mathcal{Y} \subseteq \mathcal{Y}^{\max} = \mathcal{Y}^{\max} \), we have that \( a_{\mathcal{V}} = a_{\mathcal{V}} \) and as a consequence, \( p \) is idempotent pure.

Theorem 5.3.3. Let \( \mathcal{V} \) be a variety of inverse semigroups. Then \( \text{Wr}(\mathcal{S},\mathcal{V}) = \mathcal{V}^{\max} \).

Proof: First of all, observe that \( \text{Wr}(\mathcal{S},\mathcal{V}) \subseteq \mathcal{V}^{\max} \) because, for any \( w \in (X \cup X^{-1})^+ \), \( w\mathcal{R}(\text{Wr}(\mathcal{S},\mathcal{V}))w_{2} \) if and only if \( w\mathcal{R}(\mathcal{V})w_{2} \) and \( d_{\mathcal{V}}(w)\mathcal{R}(\mathcal{S})d_{\mathcal{V}}(w_{2}) = d_{\mathcal{V}}(w_{2}) \), by
Theorem 4.2.3 with the last equality by Proposition 4.1.2, and so \( w \rho(\text{Wr}(\mathcal{S}, \mathcal{Y})) \) \( w^2 \) if and only if \( w \rho(\mathcal{Y}) \) \( w^2 \).

Let \( \rho_1, \rho_2 \) be the congruences on \( \mathcal{Y}^{\text{max}}(X) \) such that
\[
\mathcal{Y}^{\text{max}}(X) / \rho_1 \cong \text{FWr}(\mathcal{S}, \mathcal{Y})(X) \\
\mathcal{Y}^{\text{max}}(X) / \rho_2 \cong \mathcal{Y}(X)
\]
and let \( \rho_3 \) be the congruence on \( \text{FWr}(\mathcal{S}, \mathcal{Y})(X) \) such that
\[
\text{FWr}(\mathcal{S}, \mathcal{Y})(X) / \rho_3 \cong \mathcal{Y}(X).
\]

From the preceding lemma we obtain that \( \rho_1, \rho_2 \) and \( \rho_3 \) are idempotent pure and so, by the theorem above, for all \( w \in (X \cup X^{-1})^+ \), \( \Gamma \mathcal{Y}^{\text{max}}(w) \) is \( V \)-embeddable in \( \Gamma \text{Wr}(\mathcal{S}, \mathcal{Y})(w) \) which in turn is \( V \)-embeddable in \( \Gamma \mathcal{Y}(w) \). Let \( w, u \in (X \cup X^{-1})^+ \) be such that \( \mathcal{Y}^{\text{max}} = \mathcal{Y}^{\text{max}}_2 \), \( u \mathcal{Y}^{\text{max}} = u \mathcal{Y}^{\text{max}}_2 \) and \( w \mathcal{Y}^{\text{max}} = w \mathcal{Y}^{\text{max}}_1 \). By Theorem 4.2.3, \( w \mathcal{Y} = u \mathcal{Y} \) and \( c(d\mathcal{Y}(w)) = c(d\mathcal{Y}(u)) \). It follows that \( \Gamma \mathcal{Y}(w) = \Gamma \mathcal{Y}(u) \) and both \( u \) and \( w \) label \( w w^{-1} \mathcal{Y} - w \mathcal{Y} \) paths in \( \Gamma \mathcal{Y}(w) \). Moreover, the \( w w^{-1} \mathcal{Y} - w \mathcal{Y} \) path labelled by \( u \) in \( \Gamma \mathcal{Y}(w) \) uses only the edges in the \( w w^{-1} \mathcal{Y} - w \mathcal{Y} \) path in \( \Gamma \mathcal{Y}(w) \) labelled by \( w \). Thus, \( u \) labels a \( w w^{-1} \mathcal{Y} - w \mathcal{Y} \) path in the subgraph of \( \Gamma \mathcal{Y}(w) \) consisting of the \( w w^{-1} \mathcal{Y} - w \mathcal{Y} \) path labelled by \( w \). Since \( \Gamma \mathcal{Y}^{\text{max}}(w) \) is \( V \)-embeddable in \( \Gamma \mathcal{Y}(w) \), this subgraph is \( V \)-embeddable in \( \Gamma \mathcal{Y}^{\text{max}}(w) \) and so \( u \) labels a \( w w^{-1} \mathcal{Y}^{\text{max}} - w \mathcal{Y}^{\text{max}} \) path in \( \Gamma \mathcal{Y}^{\text{max}}(w) \). By Lemma 2.8.1 (c), we have that \( \mathcal{Y}^{\text{max}} \geq \mathcal{Y}^{\text{max}} \). In a similar fashion we may demonstrate that \( \mathcal{Y}^{\text{max}} \geq \mathcal{Y}^{\text{max}} \) and so obtain that \( \mathcal{Y}^{\text{max}} = \mathcal{Y}^{\text{max}} \). As a consequence, we have that \( \rho_1 \) is an idempotent separating congruence. But the only idempotent pure and idempotent separating congruence on any inverse semigroup is the identical relation \( \varepsilon \). Thus, \( \rho_1 = \varepsilon \) and \( \mathcal{Y}^{\text{max}}(X) \cong \text{FWr}(\mathcal{S}, \mathcal{Y})(X) \). Therefore, \( \text{Wr}(\mathcal{S}, \mathcal{Y}) = \langle \text{FWr}(\mathcal{S}, \mathcal{Y})(X) \rangle = \langle \mathcal{Y}^{\text{max}}(X) \rangle = \mathcal{Y}^{\text{max}} \).

We now present some immediate consequences of the preceding Theorem in light of some of the principle results obtained thus far.


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Corollary 5.3.4. Let $\mathcal{Z}$ and $\mathcal{Y}$ be varieties of inverse semigroups.

a) $[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y})$;

b) If $\mathcal{Z} = \mathcal{Z}^{\text{max}}$ then $[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y})$;

c) $\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}^{\text{max}}) = \mathrm{Wr}(\mathcal{Z} \vee \mathcal{S}, \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y}) \vee \mathcal{Y}^{\text{max}}$;

d) If $\mathcal{Z}$ is not a variety of groups then $\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}^{\text{max}}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y})$;

e) $[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}^{\text{max}})$;

Proof: a) $[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{S}, \mathrm{Wr}(\mathcal{Z}, \mathcal{Y}))$ by Theorem 5.3.3. Since Wr is associative, we have $\mathrm{Wr}(\mathcal{S}, \mathrm{Wr}(\mathcal{Z}, \mathcal{Y})) = \mathrm{Wr}(\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}), \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y})$, again by Theorem 5.3.3.

b) If $\mathcal{Z} = \mathcal{Z}^{\text{max}}$ then $[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y})$, by part a) and $\mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y})$ by our assumption.

c) $\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}^{\text{max}}) = \mathrm{Wr}(\mathcal{Z}, \mathrm{Wr}(\mathcal{S}, \mathcal{Y}))$ by Theorem 5.3.3. By the associativity of Wr we have that $\mathrm{Wr}(\mathcal{Z}, \mathrm{Wr}(\mathcal{S}, \mathcal{Y})) = \mathrm{Wr}(\mathrm{Wr}(\mathcal{Z}, \mathcal{S}), \mathcal{Y})$ and $\mathrm{Wr}(\mathrm{Wr}(\mathcal{Z}, \mathcal{S}), \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z} \vee \mathcal{S}, \mathcal{Y})$ by Theorem 5.1.5. By Proposition 4.3.5, $\mathrm{Wr}(\mathcal{Z} \vee \mathcal{S}, \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y}) \vee \mathrm{Wr}(\mathcal{S}, \mathcal{Y}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y}) \vee \mathcal{Y}^{\text{max}}$.

d) If $\mathcal{Z}$ is not a variety of groups then $\mathcal{Z} \vee \mathcal{S} = \mathcal{Z}$. By part c) above,

$\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}^{\text{max}}) = \mathrm{Wr}(\mathcal{Z}, \mathcal{Y})$.

e) For any variety $\mathcal{Z}$ of inverse semigroups, $\mathcal{Z}^{\text{max}}$ is not a variety of groups. By part d) above, $\mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}^{\text{max}}) = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y})$ and so, by part a),

$[\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})]^{\text{max}} = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}^{\text{max}})$.

If we let $\mathcal{Z}$ and $\mathcal{Y}$ be varieties of groups in Corollary 5.3.4 (e), then we have that $(\mathcal{Z} \circ \mathcal{Y})^{\text{max}} = \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}^{\text{max}})$. Thus, if the variety $\mathcal{Z}$ has E-unitary covers over $\mathcal{Z}$ and the variety $\mathcal{Y}$ has E-unitary covers over $\mathcal{Y}$ then $\mathcal{Z} \subseteq \mathcal{Z}^{\text{max}}$ and $\mathcal{Y} \subseteq \mathcal{Y}^{\text{max}}$ and so $\mathrm{Wr}(\mathcal{Z}, \mathcal{Y}) \subseteq \mathrm{Wr}(\mathcal{Z}^{\text{max}}, \mathcal{Y}^{\text{max}}) = (\mathcal{Z} \circ \mathcal{Y})^{\text{max}}$. Consequently, $\mathrm{Wr}(\mathcal{Z}, \mathcal{Y})$
has E-unitary covers over \( \mathcal{Z} \circ \mathcal{Y} = \text{Wr}(\mathcal{Z}, \mathcal{Y}) \). As a result, Theorem 5.2.7 is just a special case of the results of this section.

Corollary 5.3.5. Let \( \mathcal{W} \) and \( \mathcal{X} \) be non-trivial varieties of inverse semigroups and suppose that \( \mathcal{W} \) is a strict inverse semigroup with group part \( \mathcal{Z} \) and combinatorial part \( \mathcal{Y} \).

a) If \( \mathcal{X} \) is not a group variety then \( \text{Wr}(\mathcal{X}, \mathcal{W}) = \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) unless \( \mathcal{Z} = \mathcal{I} \) in which case \( \text{Wr}(\mathcal{X}, \mathcal{W}) = \mathcal{Z} \cup \mathcal{Y} \). If \( \mathcal{X} \) is a group variety then

\[
\text{Wr}(\mathcal{X}, \mathcal{W}) = \text{Wr}(\mathcal{X}, \mathcal{Z}) \cup \mathcal{Y}.
\]

b) If \( \mathcal{W} \) is not a group variety then \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \text{Wr}(\mathcal{V}, \mathcal{Z}) \cup \text{Wr}(\mathcal{V}, \mathcal{Y}) \). If \( \mathcal{W} \) is a group variety then \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \text{Wr}(\mathcal{V}, \mathcal{X}) \).

c) If \( \mathcal{X} \) is a strict inverse variety with group part \( \mathcal{Z}^* \) and combinatorial part \( \mathcal{Y}^* \) then

i) \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \text{Wr}(\mathcal{Z}, \mathcal{Y}^*) \cup \mathcal{Y}^* \) if \( \mathcal{W} \) is a group variety;

ii) \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \text{Wr}(\mathcal{W}, \mathcal{Y}^*) \) if \( \mathcal{W} \) is not a group variety and \( \mathcal{Z}^* \neq \mathcal{I} \);

iii) \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \mathcal{W} \cup \mathcal{X} \) if \( \mathcal{W} \) is not a group variety and \( \mathcal{Z}^* = \mathcal{I} \).

Proof: It follows from [K1] (See [P; XII.2 and XII.3]) that if \( \mathcal{W} \) is a strict inverse variety with group part \( \mathcal{Z} \) and combinatorial part \( \mathcal{Y} \) then \( \mathcal{W} = \mathcal{Z} \cup \mathcal{Y} \).

a) \( \text{Wr}(\mathcal{X}, \mathcal{W}) = \text{Wr}(\mathcal{X}, \mathcal{Z} \cup \mathcal{Y}) = \text{Wr}(\mathcal{X}, \text{Wr}(\mathcal{Z}, \mathcal{Y})) \) by Theorem 5.1.5 and

\[ \text{Wr}(\mathcal{X}, \text{Wr}(\mathcal{Z}, \mathcal{Y})) = \text{Wr}(\text{Wr}(\mathcal{X}, \mathcal{Z}), \mathcal{Y}) \] by the associativity of \( \text{Wr} \). Also from Theorem 5.1.5 we have that \( \text{Wr}(\text{Wr}(\mathcal{X}, \mathcal{Z}), \mathcal{Y}) = \text{Wr}(\mathcal{X}, \mathcal{Z}) \cup \mathcal{Y} \). If \( \mathcal{X} \) is not a group variety then \( \mathcal{I} \subseteq \mathcal{X} \) and so \( \mathcal{Y} \subseteq \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) whenever \( \mathcal{Z} \) is not trivial since \( \mathcal{Y} \) has E-unitary covers over every nontrivial group variety and so is contained in \( \mathcal{Z}_{\text{max}} \) by Theorem 2.7.4. But \( \text{Wr}(\mathcal{I}, \mathcal{Z}) \subseteq \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) and as a consequence,

\[ \text{Wr}(\mathcal{X}, \mathcal{Z}) \cup \mathcal{Y} = \text{Wr}(\mathcal{X}, \mathcal{Y}). \] If \( \mathcal{Z} \) is trivial then \( \text{Wr}(\mathcal{X}, \mathcal{Z}) \cup \mathcal{Y} = \mathcal{X} \cup \mathcal{Y} \).

b) \( \text{Wr}(\mathcal{W}, \mathcal{X}) = \text{Wr}(\mathcal{V} \cup \mathcal{Y}, \mathcal{X}) = \text{Wr}(\mathcal{V}, \mathcal{X}) \cup \text{Wr}(\mathcal{Y}, \mathcal{X}) \) by Theorem 5.1.5 and Proposition 4.3.5. If \( \mathcal{W} \) is a group variety then \( \mathcal{V} = \mathcal{I} \) and so \( \text{Wr}(\mathcal{V}, \mathcal{X}) = \mathcal{X} \). Therefore, \( \text{Wr}(\mathcal{V}, \mathcal{X}) \cup \text{Wr}(\mathcal{Y}, \mathcal{X}) = \text{Wr}(\mathcal{V}, \mathcal{X}) \cup \mathcal{X} = \text{Wr}(\mathcal{V}, \mathcal{X}). \)
c) If \( \mathcal{Z} = \mathcal{U}^* \lor \mathcal{Y}^* \) and \( \mathcal{W} \) is a group variety then by part b),
\[
\text{Wr}(\mathcal{W}, \mathcal{Z}) = \text{Wr}(\mathcal{U}, \mathcal{U}^* \lor \mathcal{Y}^*) = \text{Wr}(\mathcal{U}, \text{Wr}(\mathcal{U}^*, \mathcal{Y}^*)) \text{ by Theorem 5.1.5 and by the associativity of Wr, } \text{Wr}(\mathcal{U}, \text{Wr}(\mathcal{U}^*, \mathcal{Y}^*)) = \text{Wr}(\text{Wr}(\mathcal{U}, \mathcal{U}^*), \mathcal{Y}^*). \text{ But }
\]
\[
\text{Wr}(\text{Wr}(\mathcal{U}, \mathcal{U}^*), \mathcal{Y}^*) = \text{Wr}(\mathcal{U}, \mathcal{U}^*) \lor \mathcal{Y}^*. \text{ On the other hand, if } \mathcal{Z} = \mathcal{U}^* \lor \mathcal{Y}^* \text{ and } \mathcal{W} \text{ is not a group variety then by part b),}
\]
\[
\text{Wr}(\mathcal{W}, \mathcal{Z}) = \text{Wr}(\mathcal{U}, \mathcal{U}^* \lor \mathcal{Y}^*) \lor \text{Wr}(\mathcal{W}, \mathcal{U}^* \lor \mathcal{Y}^*). \text{ Using Theorem 5.1.5 and the associativity of Wr, we obtain }
\]
\[
\text{Wr}(\mathcal{W}, \mathcal{Z}) = \text{Wr}(\mathcal{U}, \mathcal{U}^*) \lor \mathcal{Y}^* \lor \text{Wr}(\mathcal{W}, \mathcal{U}^*) \lor \mathcal{Y}^*.
\]
But if \( \mathcal{W} \) is not a group variety then \( \mathcal{Y} \neq \mathcal{T} \) and so, as in part a), if \( \mathcal{U}^* \) is not trivial, we have that \( \mathcal{Y}^* \subseteq \text{Wr}(\mathcal{W}, \mathcal{U}^*) \) and so
\[
\text{Wr}(\mathcal{W}, \mathcal{Z}) = \mathcal{U} \lor \mathcal{Y}^* \lor \mathcal{Y} = \mathcal{W} \lor \mathcal{Z}.
\]

**Proposition 5.3.6.** Let \( \mathcal{U}, \mathcal{Y} \) and \( \mathcal{W} \) be varieties of inverse semigroups. If \( \mathcal{W} \) is not a variety of groups then \( \mathcal{U}^\text{max} = \mathcal{Y}^\text{max} \) implies that \( \text{Wr}(\mathcal{W}, \mathcal{U}) = \text{Wr}(\mathcal{W}, \mathcal{Y}) \).

**Proof:** If \( \mathcal{U}^\text{max} = \mathcal{Y}^\text{max} \) then \( \text{Wr}(\mathcal{W}, \mathcal{U}^\text{max}) = \text{Wr}(\mathcal{W}, \mathcal{Y}^\text{max}) \) and so, by Theorem 5.3.3 and the associativity of Wr, \( \text{Wr}(\mathcal{W}, \mathcal{U}) = \text{Wr}(\mathcal{W}, \mathcal{Y}) \). By Theorem 5.1.5,
\[
\text{Wr}(\mathcal{W}, \mathcal{U}) = \mathcal{W} \lor \mathcal{U} = \mathcal{W} \text{ since } \mathcal{W} \text{ is not a variety of groups. Therefore,}
\]
\[
\text{Wr}(\mathcal{W}, \mathcal{U}) = \text{Wr}(\mathcal{W}, \mathcal{Y}).
\]

**Proposition 5.3.7.** Let \( \mathcal{U}, \mathcal{Y} \) and \( \mathcal{W} \) be varieties of inverse semigroups. If \( \mathcal{U}^\text{max} = \mathcal{Y}^\text{max} \) then \( \text{Wr}(\mathcal{U}, \mathcal{W})^\text{max} = \text{Wr}(\mathcal{Y}, \mathcal{W})^\text{max} \), but the converse is not true.

**Proof:** By Corollary 5.3.4, \( \text{Wr}(\mathcal{U}, \mathcal{W})^\text{max} = \text{Wr}(\mathcal{U}^\text{max}, \mathcal{W}) \) and
\[
\text{Wr}(\mathcal{U}^\text{max}, \mathcal{W}) = \text{Wr}(\mathcal{Y}^\text{max}, \mathcal{W}) \text{ by the hypothesis. Again by Corollary 5.3.4,}
\]
\[
\text{Wr}(\mathcal{Y}^\text{max}, \mathcal{W}) = \text{Wr}(\mathcal{Y}, \mathcal{W})^\text{max}, \text{ and so Wr}(\mathcal{U}, \mathcal{W})^\text{max} = \text{Wr}(\mathcal{Y}, \mathcal{W})^\text{max}. \text{ As for the converse, consider the wreath-closed variety } \mathcal{C}_2 = [x^2 = x^3]. \text{ Now } \mathcal{B} \text{ and } \mathcal{B}^1 \text{ are both contained in } \mathcal{C}_2, \text{ so Wr}(\mathcal{B}, \mathcal{C}_2) = \mathcal{C}_2 = \text{Wr}(\mathcal{B}^1, \mathcal{C}_2) \text{ but } \mathcal{B}^\text{max} = \mathcal{B} \neq (\mathcal{B}^1)^\text{max}.\]
Theorem 5.3.3 deals with varieties which satisfy the same 'kernel identities'; that is, identities of the form \( w = w^2 \). The following results deal with 'trace identities' and are the companion results to Theorem 5.3.3.

**Theorem 5.3.8.** Let \( Z, \mathcal{V} \) and \( \mathcal{W} \) be varieties of inverse semigroups. If \( \text{tr} \rho(\mathcal{V}) = \text{tr} \rho(\mathcal{Z}) \) then \( \text{tr} \rho(\text{Wr}(Z, \mathcal{W})) = \text{tr} \rho(\text{Wr}(\mathcal{V}, \mathcal{W})) \).

**Proof:** Let \( v \) and \( w \) be idempotents in \( F\mathcal{X}(X) \) and suppose that \( v \rho(\text{Wr}(\mathcal{V}, \mathcal{W})) w \). Then, by Theorem 4.2.3, \( v \rho(\mathcal{V}) w \) and \( d_{\mathcal{V}}(v) \rho(\mathcal{Z}) d_{\mathcal{W}}(w) \). By Lemma 4.1.3, both \( d_{\mathcal{V}}(v) \) and \( d_{\mathcal{W}}(w) \) are idempotents of \( F\mathcal{Y}(Y) \). Consequently, \( v \rho(\mathcal{W}) w \) and \( d_{\mathcal{W}}(v) \rho(\mathcal{V}) d_{\mathcal{W}}(w) \), and so \( v \rho(\text{Wr}(\mathcal{V}, \mathcal{W})) w \). Similarly, \( v \rho(\text{Wr}(\mathcal{V}, \mathcal{W})) w \) implies that \( v \rho(\text{Wr}(\mathcal{V}, \mathcal{W})) w \), and the result follows.

**Corollary 5.3.9.** For any varieties \( Z \) and \( \mathcal{V} \) of inverse semigroups,

\[
\text{tr} \rho(\text{Wr}(Z, \mathcal{V})) = \text{tr} \rho(\text{Wr}(Z \vee G, \mathcal{V})),
\]

and

\[
\text{Wr}(Z, \mathcal{V}) \vee G = \text{Wr}(Z \vee G, \mathcal{V}).
\]

**Proof:** By Theorem 5.3.8, since \( \text{tr} \rho(Z) = \text{tr} \rho(Z \vee G) \) for any variety \( Z \) of inverse semigroups [P;XII.2.2]. Also by [P;XII.2.2],

\[
\text{Wr}(Z, \mathcal{V}) \vee G = \text{Wr}(Z \vee G, \mathcal{V}) \vee G = \text{Wr}(Z \vee G, \mathcal{V}).
\]

It is just a conjecture that \( \text{Wr}(Z_1 \land Z_2, \mathcal{V}) = \text{Wr}(Z_1, \mathcal{V}) \land \text{Wr}(Z_2, \mathcal{V}) \), for varieties \( Z_1, Z_2 \) and \( \mathcal{V} \), but we do have the following special case, as promised at the end of section 4.3.
Proposition 5.3.10. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be varieties of groups and let $\mathcal{V}$ be a combinatorial variety of inverse semigroups. Then

$$\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V}) = \text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V}).$$

Consequently, the mapping

$$\chi_\mathcal{V}: \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{F}) \text{ defined by } \mathcal{Z} \to \text{Wr}(\mathcal{Z}, \mathcal{V}) \quad (\mathcal{Z} \in \mathcal{L}(\mathcal{G}))$$

is a lattice homomorphism. Moreover, $\chi_\mathcal{V}$ is one-to-one and so is an embedding of $\mathcal{L}(\mathcal{G})$ into $\mathcal{L}(\mathcal{F})$.

**Proof:** First of all, $\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V}) \subseteq \text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V})$, by Proposition 4.3.1. Now,

$$\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V}) \land \mathcal{G} = (\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \mathcal{G}) \land (\text{Wr}(\mathcal{Z}_2, \mathcal{V}) \land \mathcal{G}),$$

and by Theorem 4.3.7, this expression is $\mathcal{Z}_1 \land \mathcal{Z}_2$. Therefore, both $\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V})$ and $\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V})$ have the same group parts. By Corollary 5.3.9,

$$\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V}) \land \mathcal{G} = \text{Wr}(\mathcal{G}, \mathcal{V})$$

and

$$(\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V})) \land \mathcal{G} = (\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \mathcal{G}) \land (\text{Wr}(\mathcal{Z}_2, \mathcal{V}) \land \mathcal{G})$$

$$= \text{Wr}(\mathcal{G}, \mathcal{V}).$$

It follows that both $\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V})$ and $\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V})$ belong to the same $\mathcal{V}$-class and so $\text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V}) \subseteq \text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V})$, since $\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V})$ is the maximum member of its $\mathcal{V}$-class. Therefore,

$$\text{Wr}(\mathcal{Z}_1 \land \mathcal{Z}_2, \mathcal{V}) = \text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \text{Wr}(\mathcal{Z}_2, \mathcal{V}).$$

By Proposition 4.3.5 and what we have just done, the map $\chi_\mathcal{V}$ is a homomorphism. To see that it is one-to-one, observe that $\text{Wr}(\mathcal{Z}_1, \mathcal{V}) = \text{Wr}(\mathcal{Z}_2, \mathcal{V})$ implies that $\mathcal{Z}_1 = \text{Wr}(\mathcal{Z}_1, \mathcal{V}) \land \mathcal{G} = \text{Wr}(\mathcal{Z}_2, \mathcal{V}) \land \mathcal{G} = \mathcal{Z}_2$, by Theorem 4.3.7, since $\mathcal{V}$ is combinatorial. 

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5.4 Some facts about the semigroup \((\mathcal{L}(\mathcal{F}), \text{Wr})\)

Before we address some of the questions concerning the monoid \((\mathcal{L}(\mathcal{F}), \text{Wr})\) alluded to at the end of the previous chapter, we introduce some terminology and notation.

Following the standard nomenclature of group theory, we call a variety of inverse semigroups indecomposable if it cannot be written as the product of two non-trivial factors. An obvious example is the variety \(\mathcal{A}_p\), the variety of abelian groups of exponent \(p\), for some prime \(p\). If \(\mathcal{A}_p = \text{Wr}(\mathcal{Z}, \mathcal{W})\) then both \(\mathcal{Z}\) and \(\mathcal{W}\) are subvarieties of \(\mathcal{A}_p\), and hence each must be either \(\mathcal{A}_p\) or \(\mathcal{F}\). Since \(\text{Wr}(\mathcal{A}_p, \mathcal{A}_p) \neq \mathcal{A}_p\), it follows that \(\mathcal{A}_p\) is indecomposable. A less obvious class of indecomposable varieties is the class of nilpotent varieties of groups [N;24.34].

We define a variety \(\mathcal{V}\) of inverse semigroups to be wreath-closed if for every pair of varieties \(\mathcal{Z}, \mathcal{W} \subseteq \mathcal{V}\), \(\text{Wr}(\mathcal{Z}, \mathcal{W}) \subseteq \mathcal{V}\). The most obvious example of a wreath-closed variety is \(\mathcal{F}\), the variety of all inverse semigroups.

**Proposition 5.4.1.** Let \(\mathcal{V}\) be a variety of inverse semigroups. Then \(\mathcal{V}\) is an idempotent in \((\mathcal{L}(\mathcal{F}), \text{Wr})\) if and only if \(\mathcal{V}\) is wreath-closed.

**Proof:** If \(\mathcal{V}\) is an idempotent then \(\text{Wr}(\mathcal{V}, \mathcal{V}) = \mathcal{V}\). If \(\mathcal{Z}, \mathcal{W} \subseteq \mathcal{V}\) then \(\text{Wr}(\mathcal{Z}, \mathcal{W}) \subseteq \text{Wr}(\mathcal{V}, \mathcal{V}) = \mathcal{V}\) and so \(\mathcal{V}\) is wreath-closed. On the other hand, if \(\mathcal{V}\) is wreath-closed then, in particular, \(\text{Wr}(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}\). Since \(\mathcal{V} \subseteq \text{Wr}(\mathcal{V}, \mathcal{V})\), we have that \(\text{Wr}(\mathcal{V}, \mathcal{V}) = \mathcal{V}\); and \(\mathcal{V}\) is an idempotent.

Exactly which varieties are wreath-closed is not obvious, though we can narrow down the class of candidates significantly. In the process we discover a familiar class of varieties which forms a subsemigroup of \((\mathcal{L}(\mathcal{F}), \text{Wr})\).
Proposition 5.4.2. If \( \mathcal{Z} \) and \( \mathcal{Y} \) are combinatorial varieties of inverse semigroups, then 
\( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) is combinatorial. The only non-combinatorial varieties which are wreath-closed 
are \( \mathcal{J} \) and \( \mathcal{I} \). Included among the combinatorial wreath-closed varieties are \( \mathcal{I}, \mathcal{B} \) and \( \mathcal{C}_n \) 
for all \( n \in \omega \) (we remind the reader that \( \mathcal{C}_0 = \mathcal{I} \) and \( \mathcal{C}_1 = \mathcal{I} \)).

Proof: Let \( n \in \omega \). It is not too difficult to see that the Schützenberger graph of \( x^n \) 
relative to the variety \( \mathcal{C}_n \) is just a single vertex with loops labelled \( x \) and \( x^{-1} \). It follows that 
\( d_{\mathcal{C}_n}(x^n) = y^n \) and \( d_{\mathcal{C}_n}(x^{n+1}) = y^{n+1} \) for some \( y \in \mathcal{Y} \cup \mathcal{Y}^{-1} \). It then follows from Theorem 
4.2.3 that \( \text{Wr}(\mathcal{C}_n, \mathcal{C}_n) \) satisfies the equation \( x^n = x^{n+1} \) and hence that 
\( \text{Wr}(\mathcal{C}_n, \mathcal{C}_n) \subseteq \mathcal{C}_n \). As a result, not only have we shown that \( \mathcal{C}_n \) is wreath closed for all 
\( n \in \omega \), but, since every combinatorial variety is contained in some \( \mathcal{C}_m \) for some 
\( m \in \omega \), we have that if \( \mathcal{Z} \) and \( \mathcal{Y} \) are combinatorial varieties then so is \( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \).

Since \( \text{Wr}(\mathcal{B}, \mathcal{B}) = \mathcal{B} \lor \mathcal{B} = \mathcal{B} \), \( \mathcal{B} \) is a wreath-closed variety. Since 
\( \text{Wr}(\mathcal{Z}, \mathcal{Y}) \) is a group variety if and only if \( \mathcal{Z} \) and \( \mathcal{Y} \) are both group varieties, \( \mathcal{J} \) is a wreath-
closed variety. Clearly both \( \mathcal{I} \) and \( \mathcal{J} \) are wreath-closed varieties. Suppose that \( \mathcal{Y} \) is 
some arbitrary wreath-closed variety. If \( \mathcal{Y} \) is a group variety then \( \mathcal{Y} \) is wreath-closed if 
and only if \( \mathcal{Y} = \mathcal{J} \) or \( \mathcal{I} \) [N;23.32]. Let \( \mathcal{Z} = \mathcal{Y} \cap \mathcal{J} \) be the group part of \( \mathcal{Y} \). Since 
\( \text{Wr}(\mathcal{Z}, \mathcal{Z}) \subseteq \text{Wr}(\mathcal{Y}, \mathcal{Y}) \cap \mathcal{J} = \mathcal{Y} \cap \mathcal{J} = \mathcal{Z} \), we must have that \( \mathcal{Z} \) is a wreath-closed variety 
and so must be either \( \mathcal{J} \) or \( \mathcal{I} \). Since \( \text{Wr}(\mathcal{J}, \mathcal{J}) = \mathcal{J}^{\text{max}} = \mathcal{J} \), the only wreath-closed 
varieties containing \( \mathcal{J} \) are \( \mathcal{I} \) and \( \mathcal{J} \) itself. It now follows that all wreath-closed varieties 
which do not belong to \( \{ \mathcal{J}, \mathcal{I} \} \) are combinatorial varieties.

Corollary 5.4.3. The class of combinatorial varieties of inverse semigroups forms a 
subsemigroup of \( (\mathcal{L}(\mathcal{J}), \text{Wr}) \).

Proof: By Proposition 5.4.2, the class of combinatorial varieties forms a subsemigroup 
of \( (\mathcal{L}(\mathcal{J}), \text{Wr}) \).
The definition of wreath-closed suggests the following connection between wreath-closed varieties and certain subsemigroups of \((L(P), \text{Wr})\).

**Proposition 5.4.4.** If \(\mathcal{V}\) is a wreath-closed variety then the interval \([\mathcal{I}, \mathcal{V}]\) is a subsemigroup of \((L(P), \text{Wr})\). Moreover, \(\mathcal{V}\) is a zero of this subsemigroup.

**Proof:** By the definition of wreath-closed, \([\mathcal{I}, \mathcal{V}]\) is closed under the operation \(\text{Wr}\). That \(\mathcal{V}\) is a zero for \([\mathcal{I}, \mathcal{V}]\) follows from the fact that, for any \(\mathcal{U} \in [\mathcal{I}, \mathcal{V}]\), \(\mathcal{V} \subseteq \text{Wr}(\mathcal{V}, \mathcal{U}) \subseteq \mathcal{V}\) and \(\mathcal{V} \subseteq \text{Wr}(\mathcal{U}, \mathcal{V}) \subseteq \mathcal{V}\). 

**Corollary 5.4.5.** The lattice of varieties of groups forms a subsemigroup of \((L(P), \text{Wr})\) with identity \(\mathcal{I}\) and zero \(\mathcal{J}\). For each \(n \in \omega\), the interval \([\mathcal{I}, \mathcal{C}_n]\) is a subsemigroup of \((L(P), \text{Wr})\) with identity \(\mathcal{I}\) and zero \(\mathcal{C}_n\). Also \([\mathcal{I}, \mathcal{I}, \mathcal{B}]\) is a subsemigroup of \((L(P), \text{Wr})\) which is also a three-element chain (semilattice).

**Proof:** This follows from Propositions 5.4.2 and 5.4.4. That \([\mathcal{I}, \mathcal{I}, \mathcal{B}]\) is a semilattice follows from Proposition 5.4.1 and the fact that each variety is wreath-closed. From Proposition 5.4.4 we obtain that \(\mathcal{B}\) is a zero for \(\mathcal{I}\) which in turn is a zero for \(\mathcal{I}\) and hence, \([\mathcal{I}, \mathcal{I}, \mathcal{B}]\) is a chain.

Not all subsemigroups of \((L(P), \text{Wr})\) have a direct connection with wreath closed varieties, as the following illustrates.

**Theorem 5.4.6.** Let \(\mathcal{F}_1\) be a subsemigroup of \((L(P), \text{Wr})\) and let \(\mathcal{F}_2\) be the family of varieties of inverse semigroups which have E-unitary covers over some variety in \(\mathcal{F}_1\). Then \(\mathcal{F}_2\) is a subsemigroup of \((L(P), \text{Wr})\).

**Proof:** Let \(\mathcal{U}, \mathcal{V} \in \mathcal{F}_2\) and suppose that \(\mathcal{U}\) has E-unitary covers over \(\mathcal{W} \in \mathcal{F}_1\) and \(\mathcal{V}\) has E-unitary covers over \(\mathcal{X} \in \mathcal{F}_1\). By Theorem 5.2.7, \(\text{Wr}(\mathcal{U}, \mathcal{V})\) has E-unitary covers...
over \(\text{Wr}(\mathcal{W}, \mathcal{H})\). Since \(\mathcal{F}_1\) is a subsemigroup of \(\mathcal{L}(\mathcal{G})\), \(\text{Wr}(\mathcal{W}, \mathcal{H}) \in \mathcal{F}_1\) and as a consequence, \(\text{Wr}(\mathcal{U}, \mathcal{Y}) \in \mathcal{F}_2\).

**Proposition 5.4.7.** Let \(\mathcal{V}\) be a variety of inverse semigroups. The interval \([\mathcal{V}, \mathcal{F}]\) is a subsemigroup of \((\mathcal{L}(\mathcal{F}), \text{Wr})\). If \(\mathcal{V} = \mathcal{B}\) or \(\mathcal{F}\) then \(\mathcal{V}\) is a right identity of the semigroup \([\mathcal{V}, \mathcal{F}]\). Consequently, the only indecomposable varieties in \(\mathcal{L}(\mathcal{F})\) are the indecomposable group varieties.

**Proof:** If \(\mathcal{U}\) and \(\mathcal{W}\) are varieties in the interval \([\mathcal{V}, \mathcal{F}]\) then \(\mathcal{V} \subseteq \mathcal{U} \subseteq \text{Wr}(\mathcal{U}, \mathcal{W})\) and so \([\mathcal{V}, \mathcal{F}]\) is closed under the operation \(\text{Wr}\). By Theorem 5.1.5, if \(\mathcal{V}\) is either \(\mathcal{B}\) or \(\mathcal{F}\) then, for any \(\mathcal{U} \in [\mathcal{V}, \mathcal{F}], \text{Wr}(\mathcal{U}, \mathcal{V}) = \mathcal{U} \vee \mathcal{V} = \mathcal{U}\). As a result, any variety \(\mathcal{V}\) which contains \(\mathcal{F}\) cannot be indecomposable since \(\text{Wr}(\mathcal{V}, \mathcal{F}) = \mathcal{V}\). That is, the only indecomposable varieties are the indecomposable group varieties.

Some familiar classes of varieties of inverse semigroups do not form a subsemigroup of \((\mathcal{L}(\mathcal{F}), \text{Wr})\).

**Proposition 5.4.8.** \(\text{Wr}(\mathcal{U}, \mathcal{V})\) need not be completely semisimple if both \(\mathcal{U}\) and \(\mathcal{V}\) are completely semisimple. \(\text{Wr}(\mathcal{U}, \mathcal{V})\) need not be cryptic if both \(\mathcal{U}\) and \(\mathcal{V}\) are cryptic.

**Proof:** Consider \(\text{Wr}(\mathcal{F}, \mathcal{G})\). Both of \(\mathcal{F}\) and \(\mathcal{G}\) are completely semisimple cryptic varieties but \(\text{Wr}(\mathcal{F}, \mathcal{G}) = \mathcal{F}\) which is neither completely semisimple nor cryptic.

As far as Green's relations are concerned, we have the following. By a \(\mathcal{F}\)-trivial semigroup we mean a semigroup \(S\) in which \(s \mathcal{F} t\) implies that \(s = t\), for all \(s,t \in S\).
Theorem 5.4.9. \((\mathcal{L}(\mathcal{F}), \text{Wr})\) is a \(\mathcal{F}\)-trivial semigroup.

Proof: If the variety \(\mathcal{Y}\) is in the principal ideal generated by the variety \(\mathcal{V}\) then \(\mathcal{V} \subseteq \mathcal{Y}\) by the definition of the operator \(\text{Wr}\). Thus, if \(\mathcal{Y}\) and \(\mathcal{V}\) are \(\mathcal{F}\)-related in \((\mathcal{L}(\mathcal{F}), \text{Wr})\) then \(\mathcal{Y} = \mathcal{V}\). Therefore, \((\mathcal{L}(\mathcal{F}), \text{Wr})\) is \(\mathcal{F}\)-trivial.

A well-known result from the study of varieties of groups is that the semigroup of group varieties other than \(\mathcal{G}\) is freely generated by the indecomposable varieties [N; 23.4]. That is, every variety of groups can be uniquely factored as a product of non-trivial indecomposable varieties. This is not true for \(\mathcal{L}(\mathcal{F})\), nor is it true for any of the intervals \([\mathcal{F}, \mathcal{G}_n], n \in \omega\).

Proposition 5.4.10. \((\mathcal{L}(\mathcal{F}), \text{Wr})\) is not freely generated by its indecomposable members. None of the subsemigroups \([\mathcal{F}, \mathcal{G}_n], n \geq 2\), is freely generated by its indecomposable members.

Proof: Consider the variety \(\mathcal{B}^1\). \(\mathcal{F} \subseteq \mathcal{B}^1 \neq (\mathcal{B}^1)_{\text{max}}\) and so, by Proposition 5.3.6, \(\text{Wr}(\mathcal{B}^1, \mathcal{B}^1) = \text{Wr}(\mathcal{B}^1, (\mathcal{B}^1)_{\text{max}})\) and so none of the semigroups mentioned in the statement of the theorem possess the property of unique factorization. As a result, none of the semigroups mentioned in the theorem are freely generated by their indecomposable members.

Theorem 5.4.11. \((\mathcal{L}(\mathcal{G}), \text{Wr})\) is a homomorphic image (as well as a subsemigroup) of \((\mathcal{L}(\mathcal{F}), \text{Wr})\).

Proof: Define the mapping \(\Theta : \mathcal{L}(\mathcal{F}) \rightarrow \mathcal{L}(\mathcal{G})\) by \(\mathcal{V} \Theta = \mathcal{V} \cap \mathcal{G}\). Since \(\text{Wr}(\mathcal{Y}, \mathcal{V}) \cap \mathcal{G} = \text{Wr}(\mathcal{Y} \cap \mathcal{G}, \mathcal{V} \cap \mathcal{G})\), for all \(\mathcal{Y}, \mathcal{V} \in \mathcal{L}(\mathcal{F})\), it follows that \(\Theta\) is a homomorphism. Since \(\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{F})\), \(\Theta\) is surjective.
Since \( \mathcal{L}(\mathcal{G}) \) is freely generated by its indecomposable members, the free semigroup on the indecomposable varieties of groups is a homomorphic image of \((\mathcal{L}(\mathcal{P}), \text{Wr})\).

While the relation \( \nu \) on the lattice of varieties is a congruence, the relation \( \nu \) on the semigroup \((\mathcal{L}(\mathcal{P}), \text{Wr})\) is only a right congruence.

**Theorem 5.4.12.** The relation \( \nu \) on \( \mathcal{L}(\mathcal{P}) \) is a right (semigroup) congruence but not a (semigroup) congruence.

**Proof:** Let \( \mathcal{U} \) and \( \mathcal{W} \) be varieties of inverse semigroups and suppose that \( \mathcal{U} \nu \mathcal{V} \).

Then, for any variety \( \mathcal{W} \),
\[
\text{Wr}(\mathcal{U}, \mathcal{W}) \cap \mathcal{G} = \text{Wr}(\mathcal{U} \cap \mathcal{G}, \mathcal{W} \cap \mathcal{G}) \quad \text{(Theorem 4.3.7)}
\]
\[
= \text{Wr}(\mathcal{V} \cap \mathcal{G}, \mathcal{W} \cap \mathcal{G}) \quad \text{(since } \mathcal{U} \nu \mathcal{V} \text{)}
\]
\[
= \text{Wr}(\mathcal{V}, \mathcal{W}) \cap \mathcal{G} \quad \text{(Theorem 4.3.7)}
\]

and
\[
\text{Wr}(\mathcal{U}, \mathcal{W}) \nu \mathcal{G} = \text{Wr}(\mathcal{U} \nu \mathcal{G}, \mathcal{W}) \quad \text{(Corollary 5.3.9)}
\]
\[
= \text{Wr}(\mathcal{V} \nu \mathcal{G}, \mathcal{W}) \quad \text{(since } \mathcal{U} \nu \mathcal{V} \text{)}
\]
\[
= \text{Wr}(\mathcal{V}, \mathcal{W}) \nu \mathcal{G} \quad \text{(Corollary 5.3.9).}
\]

Therefore, \( \text{Wr}(\mathcal{U}, \mathcal{W}) \nu \text{Wr}(\mathcal{V}, \mathcal{W}) \) and so \( \nu \) is a right (semigroup) congruence.

To see that \( \nu \) is not a semigroup congruence, consider the following varieties. Since \( \mathcal{B}^1 \) is combinatorial, the \( v \)-class of \( \mathcal{A}_2 \vee \mathcal{B}^1 \), where \( \mathcal{A}_2 \) is the variety of abelian groups of exponent two, is the interval \( [ \mathcal{A}_2 \vee \mathcal{B}^1, \mathcal{A}_2 \circ \mathcal{B}^1 ] \). Thus, \( \mathcal{A}_2 \vee \mathcal{B}^1 \) and \( \mathcal{A}_2 \circ \mathcal{B}^1 \) are \( v \)-related. We claim that \( \text{Wr}(\mathcal{P}, \mathcal{A}_2 \vee \mathcal{B}^1 \) is not \( v \)-related to the variety \( \text{Wr}(\mathcal{P}, \mathcal{A}_2 \circ \mathcal{B}^1 \). First of all, by Theorem 5.3.3,
\[
\text{Wr}(\mathcal{P}, \mathcal{A}_2 \vee \mathcal{B}^1) = (\mathcal{A}_2 \vee \mathcal{B}^1)^{\text{max}} = \mathcal{A}_2^{\text{max}}
\]

and, by Theorem 5.3.3, the associativity of \( \text{Wr} \) and Theorem 4.3.4,
\[ \text{Wr}(\mathcal{S}, \mathcal{A}_2 \circ \mathcal{B}^1) = \text{Wr}(\mathcal{S}, \text{Wr}(\mathcal{A}_2, \mathcal{B}^1)) = \text{Wr}(\text{Wr}(\mathcal{S}, \mathcal{A}_2), \mathcal{B}^1) = \text{Wr}(\mathcal{A}_2^{\max}, \mathcal{B}^1). \]

Let \( w \) be the word \( x_1x_2x_1^{-1}x_2^{-1} \). Now \( \mathcal{A}_2^{\max} \) satisfies the identity \( ww^{-1} = w^{-1}w \). This can easily be seen by considering the Cayley graph of the \( \mathcal{A}_2 \)-free group and using Theorem 4.2.3. \( \mathcal{B}^1 \) also satisfies this identity as it is contained in \( \mathcal{A}_2^{\max} \). The Schützenberger graph of \( ww^{-1} \) (and \( w^{-1}w \)) is the one given in Figures 2.2, 4.1 and 4.2. From the Schützenberger graph we read \( d_\mathcal{S}(ww^{-1}) = y_1y_2y_3^{-1}y_4^{-1}y_4y_3^{-1}y_2^{-1}y_1^{-1} \) and \( d_\mathcal{S}(w^{-1}w) = y_4y_3y_2^{-1}y_1^{-1}y_1y_2y_3^{-1}y_4^{-1} \). While it is true that \( \mathcal{A}_2 \) satisfies the identity \( d_\mathcal{S}(ww^{-1}) = d_\mathcal{S}(w^{-1}w) \), \( \mathcal{A}_2^{\max} \) does not. This is because in the Cayley graph of the \( \mathcal{A}_2 \)-free group on \( \{y_1, y_2, y_3, y_4\} \) (which is a 4-cube), the paths corresponding to \( d_\mathcal{S}(ww^{-1}) \) and \( d_\mathcal{S}(w^{-1}w) \) do not use precisely the same set of edges. It follows that \( \text{Wr}(\mathcal{A}_2^{\max}, \mathcal{B}^1) \) does not satisfy the identity \( ww^{-1} = w^{-1}w \). Therefore, the fully invariant congruences corresponding to \( \text{Wr}(\mathcal{S}, \mathcal{A}_2 \vee \mathcal{B}^1) \) and \( \text{Wr}(\mathcal{S}, \mathcal{A}_2 \circ \mathcal{B}^1) \) do not have the same trace and, as a consequence, these two varieties cannot be \( \vee \)-related. It follows that \( \vee \) is not a semigroup congruence on \( (\mathcal{S}(\mathcal{S}), \text{Wr}) \). 

\[ \bullet \]

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CHAPTER SIX
An Infinite Chain of Varieties

As was pointed out in the previous chapter, Kleiman [K1] showed that \( \mathcal{L}(\mathcal{P}) \) is isomorphic to three copies of \( \mathcal{L}(\mathcal{G}) \) and that each of the intervals \([\mathcal{I}, \mathcal{I} \lor \mathcal{G}] \) and \([\mathcal{G}, \mathcal{G} \lor \mathcal{G}] \) is isomorphic to \( \mathcal{L}(\mathcal{G}) \) (and so, as a consequence, \( \mathcal{L}(\mathcal{P}) \) is a modular lattice). \( \mathcal{L}(\mathcal{P}) \) is sometimes referred to colloquially as the first three layers of the lattice \( \mathcal{L}(\mathcal{P}) \). The 'fourth' layer, \([\mathcal{A}, \mathcal{A} \lor \mathcal{G}] \), is not nearly as nice. While it is a modular lattice (the collection of congruences on an inverse semigroup which have the same trace forms a complete modular sublattice of the lattice of congruences on that semigroup), the v-classes of its members are not all trivial and, as a result, \( \mathcal{L}(\mathcal{A} \lor \mathcal{G}) \) is not modular, and hence \( \mathcal{L}(\mathcal{P}) \) is not modular ([Re2] provides one example). In this chapter we show that the v-class of \( \mathcal{A} \lor \mathcal{G} \), for any abelian group variety \( \mathcal{A} \), contains an infinite chain of varieties and so is far from being trivial. The technique used is interesting in that we are only required to know the Schützenberger graphs of a given collection of words with respect to \( \mathcal{A} \) (and not the entire \( \mathcal{A} \)-free object on countably infinite \( X \)) in order to construct inverse semigroups which are then shown to generate distinct varieties. We remark that the variety \( \mathcal{B} \) has proved to be rather enigmatic. Even though it is generated by a small (6-element) inverse semigroup and \( \mathcal{L}(\mathcal{B}) \) is just a 4-element chain, its members are not easily characterized and, as Kleiman proved in [K2], it is not defined by a finite set of identities.
6.1 The variety $\mathcal{D}^1$

In this section we construct inverse semigroups which belong to the variety $\mathcal{D}^1$ which, in subsequent sections, will be used to construct inverse semigroups in $\text{Wr}(\mathcal{V},\mathcal{D}^1)$, where $\mathcal{V}$ is a variety of abelian groups of exponent $n$, for some $n \in \omega$. These semigroups will be used to define an infinite collection of varieties in the interval $[\mathcal{V} \vee \mathcal{D}^1, \text{Wr}(\mathcal{V},\mathcal{D}^1)]$. Throughout the remainder of this chapter $\rho$ will denote the fully invariant congruence on $F_\mathcal{F}(X)$ corresponding to $\mathcal{D}^1$.

Before we proceed, we require some notation. For any word $w \in X \cup X^{-1}$, denote by $w_A$ the word obtained from $w$ by deleting all occurrences of variables not in $A$. For example, $(x_1x_2x_1^{-1}x_3x_2x_1)(x_1)$ is the word $x_1x_2x_1^{-1}x_1$.

**Lemma 6.1.1.** Let $w$ and $v$ be words over $X \cup X^{-1}$. Then $w \rho v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho(\mathcal{D}) v_A$.

**Proof:** $w \rho v$ if and only if $B_2^1$ satisfies the equation $w = v$. Since $B_2^1$ possesses an identity, $B_2^1$ satisfies the equation $w = v$ if and only if $B_2$ satisfies $w_A = v_A$ for all $A \subseteq c(w_A) = c(v_A)$. This is equivalent to $c(w) = c(v)$ and for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho(\mathcal{D}) v_A$.

**Corollary 6.1.2.** Let $w$ and $v$ be words over $X \cup X^{-1}$. Then $w \rho v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho v_A$.

**Proof:** If $w \rho v$ then by Lemma 6.1.1, $c(w) = c(v)$ and for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho(\mathcal{D}) v_A$. But then for any $A \subseteq c(w) = c(v)$, for all $B \subseteq A, B \neq \emptyset$, $w_B \rho(\mathcal{D}) v_B$ and so by Lemma 6.1.1, $w_A \rho v_A$. On the other hand, if $c(w) = c(v)$ and for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho v_A$, then for all $A \subseteq c(w), A \neq \emptyset$, $w_A \rho(\mathcal{D}) v_A$. As a consequence of Lemma 6.1.1, $w \rho v$. 

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Lemma 6.1.3. If $S \in \mathcal{D}^1$ then $S^1 \in \mathcal{D}^1$.

Proof: Suppose that $\mathcal{D}^1$ satisfies the equation $w = v$, where $c(w) = c(v) = \{x_1, \ldots, x_n\}$. Let $s_1, \ldots, s_n$ be arbitrarily chosen elements of $S^1$ with repetitions allowed. Suppose that $s_{i_1}, \ldots, s_{i_k}$ are each the identity of $S^1$. Then $S^1$ satisfies $w[s_1, \ldots, s_n] = v[s_1, \ldots, s_n]$ if $S$ satisfies $w_A(s_1, \ldots, s_n) = v_A(s_1, \ldots, s_n)$ where $A = \{x_1, \ldots, x_n\}\{x_{j_1}, \ldots, x_{j_k}\}$. Since $S \in \mathcal{D}^1$, $S$ does satisfy $w_A(s_1, \ldots, s_n) = v_A(s_1, \ldots, s_n)$ by Corollary 6.1.2 and so, as a result, $w[s_1, \ldots, s_n] = v[s_1, \ldots, s_n]$ is true in $S^1$. Since the $s_i$ were chosen arbitrarily, $S^1$ satisfies the equation $w = v$. Therefore, $S^1 \in \mathcal{D}^1$.

We require some further notation for this section. Let $w \in (X \cup X^{-1})^+$. We write $w \equiv v$ to mean $w$ and $v$ are identical words, letter for letter, over a common alphabet (in this case $X \cup X^{-1}$). We say the word $v$ is a cyclic shift of $w$ if $w \equiv u_1u_2$ and $v \equiv u_2u_1$ for words $u_1, u_2$ over the alphabet of $w$. For each $n \in \omega$, we denote by $\tau_n$ the equation $x_1x_2x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1} \in \mathcal{E}$. Observe that if $w$ is the word $x_1x_2x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ then any cyclic shift of $w$ can be written $y_1y_2\ldots y ny_{n+1}^{-1}y_{n+1}^{-1}\ldots y_n^{-1}$.

The remainder of 6.1 is devoted to a construction of a family of inverse semigroups $\{S(\tau_n) : n \in \omega\}$ each of which belongs to the variety $\mathcal{D}^1$. For each $n \in \omega$, $S(\tau_n)$ is obtained from the $\mathcal{D}^1$-free inverse semigroup by first identifying the ideal consisting of those elements whose $\mathcal{R}$-class does not lie above the $\mathcal{R}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}(\rho$ (which results in an ideal extension of the $\mathcal{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}(\rho$, a Brandt semigroup) and then mapping this semigroup into the translational hull of the principal factor corresponding to the $\mathcal{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}(\rho$. In order to do this we require some knowledge of the $\mathcal{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}(\rho$. 

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Lemma 6.1.4. Let $w = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ and suppose that $v = y_1y_2\ldots yNy_1^{-1}y_2^{-1}\ldots y_n^{-1}$ is a cyclic shift of $w$. Let $a \in X \cup X^{-1}$.

a) $vp$ is an idempotent;

b) $(vap) \not\subseteq (vp)$ if and only if $a = y_1$ or $a = y_n$.

Proof: a) $A^1$ has E-unitary covers over the variety $A_2$ of abelian groups of exponent two and so is contained in $A_2^{\text{max}}$. Since $A_2$ satisfies the equation $v = v^2$, $A_2^{\text{max}}$ and hence $A^1$ satisfies $v = v^2$. Thus, $vp$ is an idempotent.

b) Since $vp$ is an idempotent, if $a = y_1$ or $a = y_n$ then $(vap) \not\subseteq (vp)$. On the other hand, suppose that $(vap) \subseteq (vp)$. Then $vaa^{-1}v^{-1} \not\subseteq v$ and so $c(va) = c(v)$. Thus, $a \in c(v)$. But $(vap) \subseteq (vp)$ also implies that $vaa^{-1}v^{-1} \not\subseteq v$. If $a = y_i^{-1}$ for some $i$, then $(vaa^{-1})_{(y_i, y_i, y_n)} = y_iy_i^{-1}y_i^{-1}y_i \not\subseteq y_i^2$, while $v_{(y_i)} = y_iy_i^{-1} \not\subseteq y_i^2$ and so, by Lemma 6.1.2, $vaa^{-1} \not\subseteq v$. Therefore, $a = y_i$ for some $i$. If $1 < i < n$ then $(vaa^{-1})_{(y_1, y_i, y_n)} = y_1y_iy_i^{-1}y_i^{-1}y_n^{-1}y_i^{-1}y_i^{-1}$ and $v_{(y_1, y_i, y_n)} = y_1y_iy_n^{-1}y_i^{-1}y_n^{-1}$. If $a$ is any non-idempotent element of $B_2$, then substituting $a$ for $y_1$ and $y_n$ and substituting $a^{-1}$ for $y_i$, yields that $(vaa^{-1})_{(y_1, y_i, y_n)} \not\subseteq v_{(y_1, y_i, y_n)}$. As a consequence, $y_i$ must be either $y_1$ or $y_n$.

Lemma 6.1.5. Let $w = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ and suppose that $u$ is an initial segment of $w$ with $w \equiv uu'$. Let $a \in X \cup X^{-1}$. Then $wup \subseteq wuap$ if and only if $a$ is the initial letter of $u$ or $a^{-1}$ is the terminal letter of $u$, unless $u$ is the empty word, in which case $a^{-1}$ is the terminal letter of $u$.

Proof: First suppose that $wup \subseteq wuap$. $wup = uu'up \subseteq uu'up$ since $u' \subseteq u$ is a cyclic shift of $w$ and any cyclic shift of $w$ is an idempotent modulo $\rho$. Therefore, $wup \subseteq wuap$ if and only if $u' \subseteq u'up \subseteq u'up$. (This follows from the more general result that $t \subseteq s$ implies that $t \subseteq ta$ if and only if $s \subseteq sa$.) Since $u' \subseteq u$ is a cyclic shift of $w$, we have by Lemma 6.1.4 that $a$ is either the initial letter of $u$ or $a^{-1}$ is the terminal letter of $u$. For the converse, first note that if $a$ is the initial letter of $u$ then $ua$ is an initial segment of
w and so, since wp is an idempotent, wup \( \not\equiv \) wuap. If a\(^{-1}\) is the terminal letter of u then letting u \( \equiv u^*a^{-1}\) we obtain that wua \( \equiv wu^*a^{-1}a \equiv u^*a^{-1}u^*a^{-1}a\). Since a\(^{-1}\)u\(^*\) is a cyclic shift of w, a\(^{-1}\)u\(^*\)p is an idempotent by Lemma 6.1.4 (a) and as a result, wua \( \equiv wu^*a^{-1}a \equiv u^*a^{-1}u^*a^{-1}a \rho u^*a^{-1}aa^{-1}u^*u^* \rho u^*a^{-1}u^*u^* \equiv uu^*u^* \equiv wu^*\). It is now immediate that wup \( \not\equiv \) wuap. Note that if u is the empty word then the statement becomes wp \( \not\equiv \) wap if and only if a is the initial letter of w or a\(^{-1}\) is the terminal letter of w (which is the terminal letter of u\(^*\), in this case), by Lemma 6.1.4.

**Lemma 6.1.6.** Let \( w = x_1x_2...x_nx_1^{-1}x_2^{-1}...x_n^{-1} \). For any word v over \( X \cup X^{-1} \), wp \( \not\equiv \) vp if and only if v \( \rho \) wu for some initial segment u of w.

**Proof:** Suppose that wp \( \not\equiv \) vp, say wa\(_1\)...ak \( \rho \) v, where a\(_1\),...,ak \( \in X \cup X^{-1} \). We prove by induction on k that wa\(_1\)...ak \( \rho \) v implies that wa\(_1\)...ak \( \rho \) wu for some initial segment u of w. If k = 1 then wa\(_1\) \( \rho \) wp implies by Lemma 6.1.4 that a\(_1\) = x\(_1\) or x\(_n\). If a = x\(_1\) then a\(_1\) is an initial segment of w already. If a\(_1\) = x\(_n\) then wa\(_1\) \( \rho \) xx\(_n\).

Now \( wwx_n \equiv x_1...x_nx_1^{-1}...x_n^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_n^{-1}]x_n^{-1}x_n \rho \)

\( x_1...x_nx_1^{-1}...x_n^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_n^{-1}] \) since \( [x_n^{-1}x_1...x_nx_1^{-1}...x_n^{-1}] \) is a cyclic shift of w and so \( [x_n^{-1}x_1...x_nx_1^{-1}...x_n^{-1}] \) \( \rho \) is an idempotent.

But \( x_1...x_nx_1^{-1}...x_n^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_n^{-1}] \equiv wx_1...x_nx_1^{-1}...x_n^{-1} \) and so as a consequence, v \( \rho \) wx\(_1\)...x\(_n\)x\(_1\)^{-1}...x\(_n\)^{-1}. Now suppose that k > 1. wa\(_1\)...ak \( \rho \) wp implies that wp \( \not\equiv \) wa\(_1\)...ak\(_{-1}\) \( \rho \) and so, by the induction hypothesis, wa\(_1\)...ak\(_{-1}\) \( \rho \) wu for some initial segment u of w \( \equiv uu^*\). By Lemma 6.1.5, wup \( \not\equiv \) wuak \( \rho \) implies that ak is the initial letter of u\(^*\) or ak\(_{-1}\) is the terminal letter of u. If a is the initial letter of u\(^*\) then v \( \rho \) wa\(_1\)...ak \( \rho \) wuak and uak is an initial segment of w. If ak\(_{-1}\) is the terminal letter of u then setting u \( \equiv b_1...b_m \) we obtain that v \( \rho \) wa\(_1\)...ak \( \rho \) wuak and wuak \( \equiv wb_1...b_mb_m^{-1} \equiv b_1...b_{m-1}[b_mb^*...b_{m-1}]b_mb_m^{-1} \rho b_1...b_{m-1}[b_mb^*...b_{m-1}] \) since \( [b_mb^*...b_{m-1}] \) is a cyclic shift of w and so must be an idempotent modulo \( \rho \). But
b_1 \ldots b_{m-1} [b_m u \cdot b_1 \ldots b_{m-1}] \equiv w b_1 \ldots b_{m-1} and so \nu \rho \omega b_1 \ldots b_{m-1} and b_1 \ldots b_{m-1} is an initial segment of w. Since wp is an idempotent, the converse is immediate.

Schützenberger graphs provide a concise, visual representation of a \( S \)-class. Because of this, in the following theorem we describe the \( S \)-classes of the words \( \{x_1 x_2 \ldots x_n x_1^{-1} x_2^{-1} \ldots x_n^{-1} : n \in \omega, n > 1 \} \) relative to the variety \( S^1 \) in this way.

**Theorem 6.1.7.** Let \( w = x_1 x_2 \ldots x_n x_1^{-1} x_2^{-1} \ldots x_n^{-1} \). The following graph is \( V \)-isomorphic to the Schützenberger graph of \( w \) relative to \( S^1 \), where \( v_1 \) is both the start and end vertex.

![Figure 6.1. The Schützenberger graph of \( w = x_1 x_2 \ldots x_n x_1^{-1} x_2^{-1} \ldots x_n^{-1} \) with respect to \( S^1 \).](image)

**Proof:** By Lemma 6.1.6 there are at most 2n vertices in the Schützenberger graph \( \Gamma \) of \( w \) relative to \( S^1 \) as there are 2n initial segments of \( w \) not identical to \( w \). It is a simple exercise to verify, using Lemma 6.1.1, that if \( u \) and \( u' \) are two proper initial segments of \( w \) (that is, \( u \) nor \( u' \) is identical to \( w \)) then \( w u \rho w u' \) implies that \( u \equiv u' \). By Lemma 6.1.5, \((w_1 u_1 \rho, x, w_2 u_2 \rho)\) is an edge of \( \Gamma \) if and only if \( x^{-1} \) is the terminal letter of \( u_1 \) or \( x \) is the initial letter of \( u_1 \), where \( u_1 u_1 \equiv w. \) If \( x \) is the initial letter of \( u_1 \), then \( w u_2 \) and \( w u_1 x \) are \( \rho \)-equivalent with both \( u_1 x \) and \( u_2 \) initial segments of \( w \). Thus, \( u_1 x \equiv u_2 \). If \( x^{-1} \) is the
terminal letter of \( u_1 \) then \( u_1 \equiv u_1^* x^{-1} \) and \( wu_1^* x^{-1} x \rho wu_2 \). Since \( wu_1^* \rho \not\equiv wu_1^* x^{-1} \rho \), we have that \( wu_1^* \rho wu_1^* x^{-1} x \rho wu_2 \). Since both \( u_1^* \) and \( u_2 \) are initial segments of \( w \), \( wu_1^* \equiv wu_2 \) and so \( wu_2 x^{-1} \equiv wu_1 \). Finally, if \( u_1 \) is the empty word and \( x^{-1} \) is the terminal letter of \( w \) then \( x^{-1} \) is the terminal letter of \( ww \equiv ww^* x^{-1} \rho w \) and hence, \( ww^* x^{-1} x \rho wu_2 \). But, \( ww^* x^{-1} x \rho ww^* \) and both \( w^* \) and \( u_2 \) are initial segments of \( w \), so \( wu_2 \equiv ww^* \), whence \( wu_2 x^{-1} \equiv ww \).

It follows from these remarks that \( \Gamma \) is \( V \)-isomorphic to the graph described above via the map which sends \( wu \rho \) to \( v_{u|u+1} \), for all proper initial segments \( u \) of \( w \).

**Definition 6.1.8.** Let \( F \) be the \( \mathcal{R} \)-free inverse semigroup on \( X = \{ x_i : i \in \omega \} \). Let \( w_n \) be the word \( x_1 \ldots x_n x_1^{-1} \ldots x_n^{-1} \) for each \( n \in \omega \). Denote the ideal \( \{ v \in F : J_v \not\subseteq J_{w_n \rho} \} \) of \( F \) by \( I(\tau_n) \) and let \( J(\tau_n) = F / I(\tau_n) \). Now \( J(\tau_n) \) is an ideal extension of \( J_{w_n \rho}^0 \) which is isomorphic to \( B(\{1\},2n) \). Let \( S(\tau_n) \) be the image of \( J(\tau_n) \) under the canonical homomorphism into the translational hull \( \Omega(J_{w_n \rho}^0) \) of \( J_{w_n \rho}^0 \).

**Lemma 6.1.9.** \( S(\tau_n) \in \mathcal{B}^1 \) and \( S(\tau_n)^1 \in \mathcal{B}^1 \), for all \( n \in \omega \), \( n \geq 2 \).

**Proof:** \( S(\tau_n) \) is a homomorphic image of the \( \mathcal{R} \)-free inverse semigroup on \( X \) and so is an element of \( \mathcal{B}^1 \). \( S(\tau_n)^1 \in \mathcal{B}^1 \) by Lemma 6.1.3.

In the following section we will use the \( S(\tau_n) \) to construct a family of inverse semigroups which belong to \( Wr(\mathcal{R}_m, \mathcal{B}^1) \) but not to \( \mathcal{R}_m \cap \mathcal{B}^1 \), for \( m \in \omega \). Before we do so, we describe the \( S(\tau_n) \).

**Lemma 6.1.10.** \( S(\tau_n) \) is isomorphic to the Wagner representation of the \( \mathcal{B}^1 \)-free inverse semigroup on \( X \) restricted to \( R_{w_n \rho} \).
Proof: By Theorem 2.6.1, since the $\mathcal{F}^1$-free inverse semigroup is completely semisimple.

An added advantage to using the Schützenberger graph description in Theorem 6.1.7 is that we can read directly from the graph the image of any word of $J(\tau_n)$ under the canonical homomorphism into $\Omega(J_{w_0}) \cong \mathcal{L}(R_{w_0})$. $S(\tau_n)$ is generated by the image of the $x_i$ under the canonical homomorphism and, for each $i = 1, \ldots, n$, the domain of the image of $x_i$ is the set of vertices $v$ for which there is an edge labelled by $x_i$ starting at $v$ and $v$ is mapped to the terminal vertex of that edge. It is straightforward to verify that $S(\tau_n)$ is (isomorphic to) the inverse subsemigroup of $\mathcal{L}(R_{w_0})$ generated by $\{\alpha_i : i = 1, \ldots, n\}$ where for each $i$,

$$d\alpha_i = \{w_n x_1 \ldots x_{i-1} \rho, w_n x_1 \ldots x_n x_1^{-1} \ldots x_{i-1} \rho\}$$

and

$$w_n x_1 \ldots x_{i-1} \rho \alpha_i = w_n x_1 \ldots x_i \rho$$

$$w_n x_1 \ldots x_n x_1^{-1} \ldots x_{i-1} \rho \alpha_i = w_n x_1 \ldots x_n x_1^{-1} \ldots x_{i-1} x_i \rho$$

$$w_n x_1 \ldots x_n x_1^{-1} \ldots x_{i-1}^{-1} \rho$$

6.2 Inverse semigroups in $\text{Wr}(\mathcal{A}_m, \mathcal{F}^1)$

The semigroups constructed in section 6.1 can be used to construct semigroups in $\text{Wr}(\mathcal{A}_m, \mathcal{F}^1)$ for $m \in \omega$. By Lemma 6.1.10, $S(\tau_n)$ can be represented as an inverse subsemigroup of $\mathcal{L}(R_{w_n})$ for all $n \in \omega$. Thus, for any group $G$ belonging to $\mathcal{A}_m$, $m \in \omega$, $G \text{ wr } (S(\tau_n), R_{w_n}) \in \text{Wr}(\mathcal{A}_m, \mathcal{F}^1)$. The semigroups we construct in this section are inverse subsemigroups of semigroups of this form and so belong to $\text{Wr}(\mathcal{A}_m, \mathcal{F}^1)$.

For each $n \in \omega$, $n \geq 2$, let $C_n$ denote the cyclic group of order $n$. 

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Definition 6.2.1. Let \( m,n \in \omega, m,n \geq 2 \). Let \( 1 \) denote the identity of \( \mathbb{C}_m \) and let \( g \) be a generator of \( \mathbb{C}_m \). Let \( A_{m,n} \subseteq \mathbb{C}_m \wr (\mathbb{S}(\tau_n), R_{w_n}) \) be defined as follows:

Let \( \{ \alpha_i : i = 1, \ldots, n \} \) be the generators of \( \mathbb{S}(\tau_n) \) as described at the end of the previous section. For \( i = 1, \ldots, n-1 \), define the map \( \phi_i \) from \( R_{w_n} \) into \( \mathbb{C}_m \) by setting

\[
d\phi_i = d\alpha_i = \{ w_nx_1 \ldots x_{i-1}\rho, w_nx_1 \ldots x_{n-1}\rho \} \quad \text{and defining} \quad (w_nx_1 \ldots x_{i-1}\rho)(\phi_i) = 1, \\
(w_nx_1 \ldots x_{n-1}\rho)(\phi_i) = 1.
\]

Define the map \( \phi_n \) from \( R_{w_n} \) into \( \mathbb{C}_m \) by setting

\[
d\phi_n = d\alpha_n = \{ w_nx_1 \ldots x_{n-1}\rho, w_n\rho \} \quad \text{and defining} \quad (w_nx_1 \ldots x_{n-1}\rho)(\phi_n) = 1, (w_n\rho)(\phi_n) = g.
\]

Then \( (\phi_i, \alpha_i) \in \mathbb{C}_m \wr (\mathbb{S}(\tau_n), R_{w_n}) \) for \( i = 1, \ldots, n \).

Let \( A_{m,n} = \{ (\psi, \beta) \in \mathbb{C}_m \wr (\mathbb{S}(\tau_n), R_{w_n}) : |d\psi| = |d\beta| \leq 1 \} \)

\[
\cup \{ (\phi_i, \alpha_i) : i = 1, \ldots, n \}.
\]

Define \( T_{m,n} \) to be the inverse subsemigroup of \( \mathbb{C}_m \wr (\mathbb{S}(\tau_n), R_{w_n}) \) generated by \( A_{m,n} \).

Observe that \( T_{m,n} \) is an ideal extension of a Brandt semigroup over the group \( \mathbb{C}_m \). It is not difficult to see that \( T_{m,n} \) is in fact the following:

\[
\{ (\psi, \beta) \in \mathbb{C}_m \wr (\mathbb{S}(\tau_n), R_{w_n}) : |d\psi| = |d\beta| \leq 1 \} \cup
\]

\[
\{ (\phi_i, \alpha_i), (\phi_i, \alpha_i)^{-1}, (\phi_i, \alpha_i)(\phi_i, \alpha_i)^{-1}, (\phi_i, \alpha_i)^{-1}(\phi_i, \alpha_i) : i = 1, \ldots, n \}.
\]

Lemma 6.2.2. For each \( m,n \in \omega, m,n \geq 2 \),

a) \( T_{m,n} \in \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \) but \( T_{m,n} \notin \mathcal{B}^1 \);

b) \( T_{m,n}^1 \in \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \) but \( T_{m,n}^1 \notin \mathcal{B}^1 \);

c) \( \mathcal{A}_m \vee \mathcal{B}^1 \subseteq \langle T_{m,n} \rangle \subseteq \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \);

d) \( \mathcal{A}_m \vee \mathcal{B}^1 \subseteq \langle T_{m,n}^1 \rangle \subseteq \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \).

Proof: \( T_{m,n}^1 \) is an inverse subsemigroup of \( \mathbb{C}_m \wr (\mathbb{S}(\tau_n)^1, R_{w_n}) \) and \( \mathbb{S}(\tau_n)^1 \in \mathcal{B}^1 \) by Lemma 6.1.9. Thus, \( T_{m,n}^1 \in \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \) by the definition of the Wr operator. As a consequence, \( T_{m,n} \in \text{Wr}(\mathcal{A}_m, \mathcal{B}^1) \) since \( T_{m,n} \) is an inverse subsemigroup of \( T_{m,n}^1 \). On the other hand, \( T_{m,n} \) is an ideal extension of a Brandt semigroup over \( \mathbb{C}_m \) and so contains a subgroup isomorphic to \( \mathbb{C}_m \). Thus, \( T_{m,n} \notin \mathcal{B}^1 \) since \( \mathcal{B}^1 \) is a combinatorial variety. Since
$T_{m,n}$ is an inverse subsemigroup of $T_{m,n}^1$, we also have that $T_{m,n}^1 \not\subseteq \mathcal{S}^1$. This proves both a) and b).

Both $T_{m,n}^1$ and $T_{m,n}$ contain subgroups isomorphic to $C_m$ and so $\mathcal{A}_m \subseteq \langle T_{m,n}^1 \rangle$ and $\mathcal{A}_m \subseteq \langle T_{m,n} \rangle$ since $\mathcal{A}_m$ is generated by $C_m$. The natural homomorphism onto the second coordinate maps $T_{m,n}$ onto an inverse semigroup isomorphic to $S(\tau_n) \in \mathcal{S}^1$, and maps $T_{m,n}^1$ onto an inverse semigroup isomorphic to $S(\tau_n)^1 \in \mathcal{S}^1$. Since both $S(\tau_n)$ and $S(\tau_n)^1$ contain copies of $B_2^1$, it follows that $\mathcal{S}^1 \subseteq \langle T_{m,n}^1 \rangle$ and $\mathcal{S}^1 \subseteq \langle T_{m,n} \rangle$. Consequently, we have that $\mathcal{A}_m \vee \mathcal{S} \subseteq \langle T_{m,n} \rangle$ and $\mathcal{A}_m \vee \mathcal{S} \subseteq \langle T_{m,n}^1 \rangle$. It is immediate from parts a) and b) that $\langle T_{m,n} \rangle \subseteq \text{Wr}(\mathcal{A}_m, \mathcal{S}^1)$ and $\langle T_{m,n}^1 \rangle \subseteq \text{Wr}(\mathcal{A}_m, \mathcal{S}^1)$. This completes the proofs of c) and d).

\begin{itemize}
  \item \textbf{Lemma 6.2.3.} Let $m,n \in \omega$, $m,n \geq 2$. Neither $T_{m,n}$ nor $T_{m,n}^1$ satisfies the equation $\tau_n$.

  \textbf{Proof:} Substitute $(\phi_i, \alpha_i)$ for $x_i$, $i = 1, \ldots, n$.

  In the following lemma we use the term \textit{kernel} to mean the minimum nonzero ideal of an inverse semigroup, if it exists.

  \begin{itemize}
    \item \textbf{Lemma 6.2.4.} Let $m,n \in \omega$, $m,n \geq 2$. $T_{m,n}$ satisfies the equation $\tau_k$ for $k < n$.

    \textbf{Proof:} Towards a contradiction, suppose that $T_{m,n}$ does not satisfy $\tau_k$ for some $k < n$. Assume that $k$ is the least such integer and let $(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k) \in T_{m,n}$ be such that $x_1 \cdots x_k x_1^{-1} \cdots x_k^{-1} [(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)] = (\psi, \beta)$ is not an idempotent in $T_{m,n}$.

    We first make a few observations.

    i) $|d\beta| = 1$: If $|d\beta| = 0$ then we immediately have that $(\psi, \beta)$ is an idempotent. If $|d\beta| = 2$ then the $(\psi_i, \beta_i)$ all belong to the same $\mathcal{D}$-class, namely, the $\mathcal{D}$-class $D$ of $(\psi, \beta)$. [This is because $T_{m,n}$ is completely semisimple and so $\mathcal{D} = \mathcal{J}$. Thus, the $\mathcal{D}$-class of
$(\psi, \beta)$ is contained in the $\mathcal{D}$-class of $(\psi_i, \beta_i)$ for all $i$. But if $|d\beta| = 2$, then the $\mathcal{D}$-class of $(\psi, \beta)$ is a maximal $\mathcal{D}$-class in $T_{m,n}$ and so $(\psi, \beta)$ is $\mathcal{D}$-related to $(\psi_i, \beta_i)$ for all $i$. But $D^0$ is a Brandt semigroup and as such satisfies $\tau_k$. Since $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)] = (\psi, \beta)$ in $D^0$ and $(\psi, \beta) \neq 0$, we conclude that, in this case, $(\psi, \beta)$ is an idempotent. The only remaining possibility is that $|d\beta| = 1$.

ii) If $d\beta = \{v\}$ then $v\beta = v$. We know that $\beta$ is an idempotent of $(S(\tau_n), R_{\tau_n})$ since the natural homomorphism of $T_{m,n}$ onto its second coordinate has image $S(\tau_n)$ which, by Lemma 6.1.9, is a member of $\mathcal{B}_1$ and $\mathcal{B}_1$ satisfies the equation $\tau_k$. Thus, $v\beta = v$.

iii) If $(\psi, \beta)$ is not an idempotent then for any cyclic shift $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}$ of $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}$ we have that $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)]$ is not an idempotent. To see this note that if $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}$ is a cyclic shift of $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}$ then $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)] = (\psi', \beta')$ can be expressed as $(\phi_1, \gamma_1)(\phi_2, \gamma_2)$ where $(\psi, \beta) = (\phi_2, \gamma_2)(\phi_1, \gamma_1)$. If $\{v\} = d\beta$ then $\nu \gamma_2 \gamma_1 = \nu \gamma_2$ because $\nu \gamma_2 \gamma_1 = v\beta = v$. Then $v\gamma_2 \psi' = (v \gamma_2 \phi_1)(v \gamma_2 \gamma_1 \phi_2) = (v \gamma_2 \phi_1)(v \phi_2) = (v \phi_2)(v \gamma_2 \phi_1)$ since $C_m$ is abelian. But $(v \phi_2)(v \gamma_2 \phi_1) = v\psi$ which is not an idempotent and so, as a result, $(\psi', \beta')$ is not an idempotent.

iv) For some $i \in \{1, \ldots, k\}$, $(\psi_i, \beta_i) = (\phi_n, \alpha_n)$ or $(\phi_n, \alpha_n)^{-1}$. By ii), if $d\beta = \{v\}$ then $v\beta = v$. Therefore, if $(\psi, \beta)$ is not an idempotent then $v\psi$ is not the identity of $C_m$. The only elements of $T_{m,n}$ which can contribute non-identity elements to $v\psi$ are those $(\psi, \beta)$ for which $|d\beta| = 1$, $(\phi_n, \alpha_n)$ and $(\phi_n^{-1}, \alpha_n^{-1})$. Now $v\psi = (v \psi_1)(v \beta_1 \psi_2) \ldots (v \beta_1 \ldots \beta_k \psi_k)(v \beta_1 \ldots \beta_k \psi_1^{-1})(v \beta_1 \ldots \beta_k \beta_1^{-1} \psi_2^{-1}) \ldots (v \beta_1 \ldots \beta_k \beta_1^{-1} \ldots \beta_k^{-1} \psi_k^{-1})$. If $(\psi_i, \beta_i)$ is such that $|d\beta_i| = 1$, then in this factorization of $v\psi$, $\psi_i$ contributes $v \beta_1 \ldots \beta_i \psi_i = g$, say, and $v \beta_1 \ldots \beta_k \beta_1^{-1} \ldots \beta_i^{-1} \psi_i^{-1} = g^{-1}$, since $g^{-1}$ is the only element of $r \psi_i^{-1}$. Thus, the contributions to this factorization of $v\psi$ by $\psi_i$ cancel and so, if $(\psi, \beta)$ is not an idempotent, one of the $(\psi_i, \beta_i)$ must be $(\phi_n, \alpha_n)$ or $(\phi_n, \alpha_n)^{-1}$.
v) None of the $(\psi_i, \beta_i)$ is an idempotent. This follows from the general observation that if $e = e^2$ and $aebec$ is not an idempotent then $aebec = aea^{-1}(abc)c^{-1}ec$ and so $abc$ cannot be an idempotent. Thus, $(\psi_i, \beta_i)$ an idempotent contradicts the minimality of $k$.

As a consequence of the aforementioned observations, the following assumptions concerning the $(\psi_i, \beta_i)$ can be made. First of all, by iii) and iv) we may assume that $(\psi_1, \beta_1) = (\phi_n, \alpha_n)$. Secondly, assume that the $k$-tuple $((\psi_1, \beta_1), \ldots, (\psi_k, \beta_k))$ contains a maximal number of elements from the kernel of $T_{m, n}$ among the collection of $k$-tuples from $T_{m, n}$ whose first element is $(\phi_n, \alpha_n)$ and which witness that $T_{m, n}$ does not satisfy $\tau_k$.

There are two stages to the remainder of the proof. The first stage is showing that exactly one of the $(\psi_i, \beta_i)$ is a member of the kernel of $T_{m, n}$. We do this in four parts.

1) For any $i \in \{1, \ldots, k\}$, both $(\psi_i, \beta_i)$ and $(\psi_{i+1}, \beta_{i+1})$ do not belong to the kernel of $T_{m, n}$.

Suppose that both $(\psi_i, \beta_i)$ and $(\psi_{i+1}, \beta_{i+1})$ belong to the kernel of $T_{m, n}$. If $d\beta_i = \{v_i\}$ and $d\beta_{i+1} = \{v_{i+1}\}$ then $v_i\beta_i = v_{i+1}$ since $\beta_i\beta_{i+1} \neq 0$ and $v_i+1\beta_{i+1} = v_i$ since $\beta_{i+1}^{-1}\beta_i^{-1} \neq 0$. It follows that

$$v_i\beta_i\beta_{i+1} = v_i \quad \text{and} \quad v_{i+1}\beta_{i+1}\beta_i = v_{i+1}$$

and

$$(v_{i+1}\psi_i^{-1})(v_{i+1}\beta_i^{-1}\psi_{i+1}^{-1}) = (v_i\beta_i\psi_i^{-1})(v_i\psi_{i+1}^{-1})$$

$$= (v_i\psi_i)^{-1}(v_i\beta_i^{-1}\psi_{i+1})^{-1}$$

$$= (v_i\psi_i)^{-1}(v_{i+1}\psi_{i+1})^{-1}$$

$$= (v_{i+1}\psi_{i+1})^{-1}(v_i\psi_i)^{-1} \quad \text{(since $C_m$ is abelian)}$$

$$= [(v_i\psi_i)(v_{i+1}\psi_{i+1})]^{-1}$$

As a consequence of this we have that

$$x_1 \ldots x_{i-1}x_{i+2} \ldots x_k x_1^{-1} \ldots x_{i-1}^{-1}x_{i+2}^{-1} \ldots x_k^{-1}[(\psi_1, \beta_1), \ldots, (\psi_{i-1}, \beta_{i-1}), (\psi_{i+2}, \beta_{i+2}), \ldots, (\psi_k, \beta_k)]$$

is equal to $(\psi, \beta)$, which is not an idempotent by assumption. Thus, $T_{m, n}$ does not satisfy the equation $\tau_{k-2}$, contrary to our choice of $k$. Note that under these conditions, $k \geq 3$, by
observation iv). In the case $k = 3$, the conclusion is that $T_{m,n}$ does not satisfy $\tau_1$ which is absurd since all inverse semigroups satisfy the equation $xx^{-1} \in E$.

2) If $(\psi_i, \beta_i)$ is an element of the kernel then

i) if $d\beta_i = \{wx_1...x_j\rho\}$, then $wx_1...x_j\rho\beta_i = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$;

ii) if $d\beta_i = \{wx_1...x_nx_1^{-1}...x_j^{-1}\rho\}$, then $wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i = wx_1...x_j\rho$;

i) We have assumed that $(\psi_1, \beta_1) = (\phi_n, \beta_n)$ and so $i \neq 1$. Let $d\beta_{i-1} = \{v_1, v_2\}$ (By (1) $ld\beta_{i-1} = 2)$, and suppose that $v_1\beta_{i-1} = u_1$ and $v_2\beta_{i-1} = u_2$. Now, $\beta_{i-1}\beta_i \neq 0$ so one of $u_1$ and $u_2$ must be $wx_1...x_j\rho$, say $u_1 = wx_1...x_j\rho$. Also, $\beta_{i-1}^{-1}\beta_i^{-1} \neq 0$ so one of $v_1$ and $v_2$ must be $wx_1...x_j\rho\beta_i$. If $v_1 = wx_1...x_j\rho\beta_i$ then $(\psi_{i-1}, \beta_{i-1})$ can be replaced by $(\psi_i, \beta_i)$ where $d\beta = \{v_1\}$ and $v_1\beta = u_1$ and $v_1\psi = v_1\psi_{i-1}$. This new substitution witnesses that $T_{m,n}$ does not satisfy $\tau_k$ which contradicts 1), above (that is, this new substitution yields $T_{m,n}$ does not satisfy $\tau_{k,2}$ following the argument in (1), above). Thus, $v_2 = wx_1...x_j\rho\beta_i$. By observation (v), $\beta_{i-1}$ is $\alpha_p$ or $\alpha_p^{-1}$ for some $p \in \{1,...,n\}$. If $\beta_{i-1} = \alpha_p$ then $v_1\beta_{i-1} = wx_1...x_j\rho$ implies that $v_1x_\rho = wx_1...x_j\rho$ and hence that either $p = j$ or $j = n$, $p = 1$ and $v_1 \rho = wx_1...x_{j-1}$ or $v_1 \rho = wx_1...x_{n}x_1^{-1}$. Thus, $wx_1...x_j\rho\beta_i = v_2 = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$, by the definition of $\alpha_p$ or $wx_1...x_n\rho\beta_i = v_2 = w\rho$, which is what we want to prove.

If $\beta_{i-1} = \alpha_p^{-1}$ then $v_1\beta_{i-1} = wx_1...x_j\rho$ implies that $v_1x_\rho = wx_1...x_j\rho$ and hence that $v_1 \rho = wx_1...x_p$ and $p = j + 1$. Note that in this case $j \neq n$ since if $u$ is an initial segment of $w$, then $wx_\rho^{-1} \rho = wx_1...x_n$ is impossible by Lemma 6.1.5. Therefore, $wx_1...x_j\rho\beta_i = v_2 = wx_1...x_nx_1^{-1}...x_{p-1}^{-1}\rho = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$, by the definition of $\alpha_p^{-1}$.

ii) As in (i) we can assume that $d\beta_{i-1} = \{v_1, wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i\}$ and that $v_1\beta_{i-1} = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$. Again, by observation (v), we may assume that $\beta_{i-1} = \alpha_p$ or $\alpha_p^{-1}$. If $\beta_{i-1} = \alpha_p$ then $v_1x_\rho = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$ and hence $p = j + 1$ and $v_1 \rho = wx_1...x_nx_1^{-1}...x_{j+1}^{-1}$. Note that if $j = n$, $wx_1...x_nx_1^{-1}...x_j^{-1}\rho w$ and so for any
initial segment \( u \) of \( w \), \( wux_p\rho w \) is impossible, by Lemma 6.1.5. Therefore, by the definition of \( \alpha_p \), \( wx_1...x_nx_1^{-1}...x_{j-1}^1\rho\beta_i = wx_1...x_j\rho \).

If \( \beta_{i-1} = \alpha_p^{-1} \) then \( v_1x_p^{-1}\rho = wx_1...x_nx_1^{-1}...x_{j-1}^{-1}\rho \) and so \( p = j \) and \( v_1\rho wx_1...x_nx_1^{-1}...x_{j-1}^{-1} \) or \( j = n, p = 1, v_1\rho wx_1 \). By the definition of \( \alpha_p^{-1} \), \( wx_1...x_nx_1^{-1}...x_{j-1}^{-1}\rho\beta_i = wx_1...x_j\rho \) and if \( j = n, p = 1, wp\beta_i = v_2 = wx_1...x_n\rho \).

3) At most one of the \((\psi_i, \beta_i)\) belongs to the kernel of \( T_{m,n} \). Suppose that \((\psi_j, \beta_j)\) and \((\psi_{j+p}, \beta_{j+p})\) are two members of the kernel of \( T_{m,n} \) and they are the first two such elements appearing in the sequence \((\psi_1, \beta_1),...,(\psi_k, \beta_k)\). Let \( d\beta_j = \{v_1\}, d\beta_{j+p} = \{u_1\}, v_1\beta_j = v_2 \) and \( v_1\psi_j = g_1 \), and \( u_1\beta_{j+p} = u_2 \) and \( u_1\psi_{j+p} = g_2 \). The claim is that if \((\psi, \beta)\) is not an idempotent then neither is the following:

\[
x_1...x_{j-1}x_{j+1}^{-1}...x_{j+p-1}x_{j+p+1}...x_kx_1^{-1}...x_{j-1}^{-1}x_{j+1}...x_{j+p-1}x_{j+p+1}^{-1}...x_k^{-1}
\]

when \((\psi_i, \beta_i)\) is substituted for \( x_i \) for all \( x_i \) appearing in the expression. If the claim is correct then \( T_{m,n} \) does not satisfy \( \tau_{k+2} \), contrary to our assumptions. Since \((\psi_j, \beta_j)\) and \((\psi_{j+p}, \beta_{j+p})\) do not contribute to \( v\psi \) (where \( v = d\beta \)) it is sufficient to show that the above expression in the second coordinate is identical to \( \beta \). Now, with \( d\beta = \{v\} \)

\[
v\beta_1...\beta_{j-1} = v_1; \\
v_1 = dx_{j+1}^{-1}...x_{j+p-1}^{-1}[(\psi_{j+1}, \beta_{j+1}),...,(\psi_{j+p-1}, \beta_{j+p-1})]\text{ and} \\
v_1\beta_{j+1}^{-1}...\beta_{j+p-1}^{-1} = u_2; \\
u_2 = dx_{j+p+1}...x_kx_1^{-1}...x_{j-1}^{-1}[(\psi_{j+p+1}, \beta_{j+p+1}),...,(\psi_k, \beta_k), (\psi_1, \beta_1),...,(\psi_{j-1}, \beta_{j-1})] \\
u_2\beta_{j+p+1}...\beta_k\beta_1^{-1}...\beta_{j-1}^{-1} = v_2; \\
v_2 = dx_{j+1}...x_{j+p-1}[(\psi_{j+1}, \beta_{j+1}),...,(\psi_{j+p-1}, \beta_{j+p-1})]\text{ and} \\
v_2\beta_{j+1}...\beta_{j+p-1} = u_1; \\
u_1 = dx_{j+p+1}^{-1}...x_k^{-1}[(\psi_{j+p+1}, \beta_{j+p+1}),...,(\psi_k, \beta_k)]\text{ and} \\
u_1\beta_{j+p+1}^{-1}...\beta_k^{-1} = v\beta = v.
\]

It now follows that at most one of the \((\psi_i, \beta_i)\) belongs to the kernel of \( T_{m,n} \).
4) Exactly one of the \((\psi_i, \beta_i)\) is a member of the kernel of \(T_{m,n}\). First of all, observe that if none of the \((\psi_i, \beta_i)\) belong to the kernel then each \((\psi_i, \beta_i)\) is \((\phi_p, \alpha_p)\) or \((\phi_p, \alpha_p)^{-1}\) for some \(p\). By the definition of the \(\alpha_p\), if \(\psi_1...\beta_k \in d\beta_1^{-1}\) then \(\psi_1...\beta_k\beta_1^{-1} = v\). This is because if \(v = wu\rho\) for some initial segment \(u\) of \(w\) then \(\psi_1...\beta_k = wu'\rho\) for some initial segment \(u'\) of \(w\) and the difference between the lengths of \(u\) and \(u'\) is not greater than \(k\) and hence strictly less than \(n\). It follows that \(\psi_1...\beta_k\) must be \(\psi_{\beta_1}\). By the same reasoning we can conclude that, for all \(1 \leq i \leq k\), \(\psi_1...\beta_k\beta_1^{-1}...\beta_i^{-1} = \psi_1...\beta_i^{-1}\). Since \(d\beta = \{v\}\), we can replace each \((\psi_i, \beta_i)\) with an element of the kernel and conclude that if \((\psi, \beta)\) is not an idempotent then neither is the result of this new substitution. But this cannot be since the kernel of \(T_{m,n}\) is a Brandt semigroup over an abelian group and so satisfies the equation \(\tau_k\). Therefore, exactly one of the \((\psi_i, \beta_i)\) belongs to the kernel of \(T_{m,n}\). This completes the first stage of the proof.

Let \((\psi_j, \beta_j)\) be the only member of \(\{(\psi_1, \beta_1), ..., (\psi_k, \beta_k)\}\) which belongs to the kernel of \(T_{m,n}\). Let \(d\beta_j = \{v\}\), \(\psi_1\beta_j = \psi_2\) and \(\psi_1\psi_j = g_1\). We consider the following two cases: i) \(\psi_1 \rho w x_1...x_p\); and ii) \(\psi_1 \rho w x_1...x_n x_1^{-1}...x_p^{-1}\).

i) If \(\psi_1 \rho w x_1...x_p\) then \(\psi_2 = w x_1...x_n x_1^{-1}...x_p^{-1}\rho\) by the first stage, part 2). Since \((\psi_1, \beta_1) = (\phi_n, \alpha_n)\) and \(k < n\), by the constraints on the \((\psi_i, \beta_i)\) discussed thus far, for some \(1 < q < j\), \((\psi_q, \beta_q) = (\phi_n, \alpha_n)^{-1}\). Assume \(q\) is the least such integer. Because \(k < n\) and each of the \((\psi_i, \beta_i)\) is either \((\phi_h, \alpha_h)\) or \((\phi_h, \alpha_h)^{-1}\), for some \(h\), for \(1 < i \leq q\), as a consequence of the definitions of the \((\phi_h, \alpha_h)\), we have that \(\psi_1...\beta_q = v\) and 

\[ (\psi_1...\beta_k)\beta_1^{-1}...\beta_q^{-1} = \psi_1...\beta_k \]

and

\[ [(\psi_1...\beta_k)\psi_1^{-1}][(\psi_1...\beta_k)\beta_1^{-1}\psi_2^{-1}]...[(\psi_1...\beta_k)\beta_1^{-1}...\beta_q^{-1}\psi_q^{-1}] = 1. \]
As a result, $x_{q+1}x_kx_{q+1}^{-1}...x_k^{-1}[(\psi_{q+1},\beta_{q+1})...,(\psi_k,\beta_k)]$ is not an idempotent if $(\psi,\beta)$ is not an idempotent, contrary to our choice of $k$.

ii) If $v_1 \rho wx_1...x_nx_1^{-1}...x_p^{-1}$ then $v_2 \rho wx_1...x_p$. Using a similar argument to that used in (i) above, we can assume that $(\psi_1,\beta_1)$ is the only $(\psi_i,\beta_i)$ equal to $(\phi_n,\alpha_n)$ for $i < j$. Moreover, the same argument can be used to show that at most one of the $(\psi_i,\beta_i)$ is equal to $(\phi_n,\alpha_n)$ for $j < i \leq k$. In this case, by the constraints on the $(\psi_i,\beta_i)$ and the definitions of the $(\phi_i,\alpha_i)$ and their inverses, $(\psi_k,\beta_k)$ is equal to $(\phi_n,\alpha_n)$. Thus, the only $(\psi_i,\beta_i)$ equal to $(\phi_n,\alpha_n)$ are $(\psi_1,\beta_1)$ and $(\psi_k,\beta_k)$. But for any inverse semigroup, $axaa^{-1}ya^{-1}$ is not an idempotent implies that $xy$ is not an idempotent. It would then follow that $T_{m,n}$ does not satisfy the equation $\tau_{k-2}$, a contradiction.

The proof is complete if we can show that, for $n > 2$, $T_{m,n}$ satisfies $\tau_2$. This is not difficult to verify directly: Suppose that $(\psi,\beta) \in T_{m,n}$ is such that $(\phi_n,\alpha_n)(\psi,\beta)(\phi_n,\alpha_n)^{-1}(\psi,\beta)^{-1}$ is not an idempotent. Since $\mathcal{G}$ does satisfy $\tau_2$, we have that $\alpha_n \beta \alpha_n^{-1} \beta^{-1}$ is an idempotent. Thus, for all $v \in d \alpha_n \beta \alpha_n^{-1} \beta^{-1} \subseteq d \alpha_n$, $v\alpha_n \beta \alpha_n^{-1} \beta^{-1} = v$. Therefore, both $v$ and $v\alpha_n$ (which are not equal) are in the domain of $\beta$. For either $v$ in the domain of $\alpha_n$, there is no pair $(\psi,\beta)$ in $T_{m,n}$ such that $d\beta = \{v, v\alpha_n\}$. It follows that $T_{m,n}$ must satisfy $\tau_2$.

**Lemma 6.2.5.** Let $m,n \in \omega$, $m,n \geq 2$. $T_{m,n}^1$ satisfies the equation $\tau_k$ for $k < n$, but $T_{m,n}^1$ does not satisfy the equation $\tau_k$ for $k \geq n$.

**Proof:** This is an immediate consequence of Lemma 6.2.4.

**Remark.** The only property of the varieties $\mathcal{A}_m$ that we used in the construction of the $T_{m,n}$'s was that they each satisfied the equations $\tau_n$, $n \in \omega$. This is also true of the variety $\mathcal{G}$, the variety of abelian groups. Thus, in a similar way, we can construct a family of inverse semigroups $\{T_n^1\}$ such that, for each $n$, $T_n^1$ satisfies the equations $\tau_k$, for $k < n$, 130
but $T_n^1$ does not satisfy the equations $\tau_k$, for $k \geq n$. Moreover, for each $n \in \omega$, $\mathcal{E} \cap \mathcal{B} \subseteq \langle T_n^1 \rangle \subseteq \mathcal{E} \cup \mathcal{B}$.

6.3 A class of varieties in the interval $[\mathcal{E}_m, \mathcal{B}_1]$

The inverse semigroups defined in the previous section can be used to define an infinite collection of varieties in the interval $[\mathcal{E}_m, \mathcal{B}_1]$. Once it is established that the interval $[\mathcal{E}_m, \mathcal{B}_1]$ is infinite, it can then be shown that other intervals which coincide with $v$-classes are infinite.

Notation 6.3.1. Let $m \in \omega$. For each $n \in \omega$, define the variety $\mathcal{Y}_{m,n}$ to be the variety of inverse semigroups generated by $\{ T_{m,k}^1 : k \geq n \}$.

Proposition 6.3.2. Let $m,n \in \omega$, with $m,n > 1$.

a) $\mathcal{Y}_{m,n}$ satisfies $\tau_j$ for $j < n$;

b) $\mathcal{Y}_{m,n}$ does not satisfy $\tau_j$ for $j \geq n$;

c) $\mathcal{Y}_{m,n} \supseteq \mathcal{Y}_{m,n+1}$.

Proof: a) By Lemma 6.2.5, $T_{m,k}^1$ satisfies $\tau_j$ for $j < k$. Therefore, each generator of $\mathcal{Y}_{m,n}$ satisfies $\tau_j$ for $j < n$, and hence $\mathcal{Y}_{m,n}$ satisfies $\tau_j$ for $j < n$.

b) By Lemma 6.2.3, $T_{m,n}^1$ does not satisfy $\tau_n$. Since $T_{m,n}^1$ is a generator of $\mathcal{Y}_{m,n}$, the equation $\tau_n$ is not satisfied by $\mathcal{Y}_{m,n}$.

c) $\{ T_{m,k}^1 : k \geq n \} \supseteq \{ T_{m,k}^1 : k \geq n + 1 \}$ and so $\mathcal{Y}_{m,n} = \langle T_{m,k}^1 : k \geq n \rangle \supseteq \langle T_{m,k}^1 : k \geq n + 1 \rangle = \mathcal{Y}_{m,n+1}$.

As a consequence of Proposition 6.3.2, the collection of varieties of inverse semigroups $\{ \mathcal{Y}_{m,n} : n > 1 \}$ forms an infinite chain in the lattice of varieties of inverse semigroups. Furthermore, by Lemma 6.2.2, $\mathcal{E} \cap \mathcal{B}_1 \subseteq \mathcal{Y}_{m,n} \subseteq \text{Wr}(\mathcal{E}_m, \mathcal{B}_1)$. Since
\[ \text{Wr}(A_m, B^1) = A_m \circ B^1, \] by Theorem 4.3.4, and the \( v \)-class of \( A_m \circ B^1 \) is the interval \([A_m \circ B^1, A_m \circ D^1]\), we have the following result.

**Theorem 6.3.3.** The \( v \)-class of the variety \( A_m \circ B^1 \) possesses an infinite descending chain of varieties.

Using Theorem 6.3.3, we can show that other intervals in \( L(F) \) are infinite.

**Lemma 6.3.4.** Let \( Y \in [A_m \circ B^1, A_m \circ D^1] \), where \( A_m \) is the variety of abelian groups of exponent \( m \), and let \( Y \in [A_m \circ B^1, A_m^{\max}] \). Then
\[ \ker \rho(Y \circ Y) = \ker \rho(Y) \quad \text{and} \quad \text{tr} \rho(Y \circ Y) = \text{tr} \rho(Y). \]

**Proof:** \( A_m \subseteq Y \) and so \( A_m^{\max} \subseteq Y^{\max} \). Therefore,
\[ Y \subseteq Y \circ Y \subseteq A_m^{\max} \circ Y \subseteq Y^{\max} \circ Y = Y^{\max}. \]

Since \( \ker \rho(Y) = \ker \rho(Y^{\max}) \), it follows that \( \ker \rho(Y \circ Y) = \ker \rho(Y) \).

Also,
\[ Y \subseteq Y \circ Y \subseteq Y \circ Y \circ G = Y \circ (A_m \circ B^1) \circ G = Y \circ G. \]

Since \( \text{tr} \rho(Y) = \text{tr} \rho(Y \circ G) \), we have that \( \text{tr} \rho(Y \circ Y) = \text{tr} \rho(Y) \).

**Theorem 6.3.5.** Let \( Y \in [A_m \circ B^1, A_m^{\max}] \). Then the interval \([Y, (A_m \circ B^1) \circ Y] \) contains an infinite descending chain.

**Proof:** The function \( \theta : [A_m \circ B^1, A_m \circ D^1] \rightarrow [Y, (A_m \circ B^1) \circ Y] \) defined by \( \forall Y = Y \circ Y \) is one-to-one on \([A_m \circ B^1, A_m \circ D^1]\) by Lemma 6.3.4 and the fact that all varieties \( Y \) in this interval are such that \( \text{tr} \rho(Y) = \text{tr} \rho(A_m \circ B^1) \). Clearly \( \theta \) is order-preserving, and the result follows from Theorem 6.3.3.
Corollary 6.3.6. Let $\mathcal{V}$ be a combinatorial variety contained in $\mathcal{A}_m^{\text{max}}$ and containing $\mathcal{B}^1$. Then the $\vee$-class of $\mathcal{V} \vee \mathcal{A}_m$, that is, $[\mathcal{V} \vee \mathcal{A}_m, \mathcal{A}_m \circ \mathcal{V}]$, contains an infinite descending chain.

Proof: By Theorem 6.3.5 since $\mathcal{V} \vee \mathcal{A}_m \subseteq [\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m^{\text{max}}]$. 

Remark. The results of this section are true for the variety $\mathcal{A}\mathcal{F}$ as well. That is, defining the variety $\mathcal{V}_n$ to be the variety of inverse semigroups generated by $\{ T_n^1 : k \geq n \}$, the analogous results to Proposition 6.3.2 hold and replacing $\mathcal{A}_m$ by $\mathcal{A}\mathcal{F}$ in the remaining results of this section yields valid statements.
REFERENCES


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