CYCLE DECOMPOSITIONS OF COMPLETE GRAPHS

by

Susan Marshall
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Approval

Name: Susan Marshall
Degree: Master of Science (Mathematics)
Title of Thesis: Cycle Decompositions of Complete Graphs

Examining Committee:
Chairman: A. H. Lachlan

B. R. Alspach, Senior Supervisor

T. C. Brown

H. Gerber

K. Heinrich, External Examiner

Date Approved September 14, 1989
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CYCLE DECOMPOSITIONS OF COMPLETE GRAPHS

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Author: ________________________________

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SUSAN MARSHALL

(name)

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Abstract

The problem of finding a decomposition of the complete graph, directed or undirected, into cycles of a fixed length is one on which there has been much research, and for which there are still many cases left unsolved. We investigate in particular the decomposition of $K_{2n} - I$, the complete graph on $2n$ vertices with a one-factor removed, into cycles of fixed even length.

We begin with a brief exposition of known results in the area. We then construct a decomposition of the graph $K_{2n} - I$ into cycles of even length $2m$, for cases when $n$ is even and $3m/2 \leq n < 2m$. 
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Chapter 1

§1.1 Definitions and Notation.

1.1.1 $K_n$ and $K^*_n$ will be used to denote the complete undirected graph and the complete directed graph, respectively. $K_{n,s}$ and $K^*_{n,s}$ will be used to denote the complete undirected and directed bipartite graphs, respectively. The graph $K_{2n}-I$ is the complete undirected graph with a one-factor removed.

1.1.2 A path is a sequence $P = (x_1, ..., x_{n+1})$ of vertices together with the edges $x_ix_{i+1}, i = 1, ..., n$, where $x_i \neq x_j$ if $i \neq j$. The vertices $x_1$ and $x_{n+1}$ are the endvertices of $P$. The length of $P$, denoted $l(P)$, is the number of edges in $P$. A path of length $m$ is also called an $m$-path. A Hamilton path in a graph $G$ is a path which meets every vertex of $G$.

A cycle is a sequence $C = (x_1, ..., x_{n+1})$ of vertices together with the edges $x_ix_{i+1}, i = 1, ..., n$, where $x_1 = x_{n+1}$, and $x_i \neq x_j$ if $i \neq j$ and $\{x_i, x_j\} \neq \{x_1, x_{n+1}\}$. The length $l(C)$ of $C$ is the number of edges in $C$. An $m$-cycle is a cycle of length $m$, and a Hamilton cycle in a graph $G$ is one which meets every vertex of $G$.

A directed cycle or dicycle is a cycle in a directed graph, where $x_ix_{i+1}$ is the arc directed from $x_i$ to $x_{i+1}$ for $i = 1, ..., n$.

1.1.3 If $A = (x_1, ..., x_n)$ and $B = (x_n, ..., x_{n+k})$, then $A+B$ is the concatenation of $A$ and $B$, that is, $A+B = (x_1, ..., x_n, x_{n+1}, ..., x_{n+k})$. 

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1.1.4 A one-factor of a graph $G$ (also called a perfect matching) is a spanning subgraph of $G$ in which every vertex has degree 1. A 2-factor of $G$ is a spanning subgraph of $G$ in which every vertex has degree 2.

1.1.5 We write $G = H_1 \oplus H_2$ if $G$ is the edge-disjoint union of the subgraphs $H_1$ and $H_2$.

1.1.6 If $G$ is a graph and $n$ is a natural number, then $nG$ denotes the graph consisting of $n$ vertex-disjoint copies of $G$.

1.1.7 If $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, where $H_1 \equiv H_2 \equiv \ldots \equiv H_k \equiv H$, then we say that $G$ has an $H$-decomposition, or that $G$ may be decomposed into subgraphs isomorphic to $H$.

1.1.8 Let $G$ and $H$ be graphs. The wreath product of $G$ and $H$, denoted $G_{wr}H$, is formed by replacing each vertex of $G$ with a copy of $H$, and joining vertices in different copies of $H$ by an edge if and only if the corresponding vertices of $G$ are adjacent.

We state here for convenience some basic results which are used repeatedly throughout the thesis.

1.1.9 Lemma ([Lu]) The graph $K_{2n+1}$ can be decomposed into Hamilton cycles for every natural number $n$.

1.1.10 Corollary The graph $K_{2n}$ can be decomposed into Hamilton paths for every natural number $n$. 
1.1.11 Lemma ([Lu]) The graph $K_{2n} - I$ can be decomposed into Hamilton cycles for every natural number $n$.

1.1.12 Lemma The graph $K_{2n} - I$ is isomorphic to the graph $K_n \wr \overline{K}_2$.

**Proof** The graph $K_n \wr \overline{K}_2$ is formed by replacing each vertex $v$ of $K_n$ with a pair of independent vertices $v_1$ and $v_2$, and adding the edges $u_1v_1$, $u_1v_2$, $u_2v_1$ and $u_2v_2$ for every edge $uv$ of $K_n$. Thus each vertex $v_1$ of $K_n \wr \overline{K}_2$ is adjacent to every other vertex of $K_n \wr \overline{K}_2$ except its 'partner' $v_2$.

§ 1.2 Introduction

In this thesis we consider the decomposition of complete graphs (the complete directed graph, the complete undirected graph, and the graph $K_{2n} - I$) into cycles of fixed length. The problem has interested many authors and has proven to be quite challenging. Hamilton decompositions of the complete undirected graph and the graph $K_{2n} - I$ (stated here as Lemmas 1.1.9 and 1.1.11) appear in Lucas' Récréations Mathématiques, where they are attributed to Walecki. More recently, the question of finding a decomposition of the complete graph into 2-factors whose components are all 3-cycles appeared in [R1] in 1963, as a reformulation of Kirkman's Schoolgirl problem. (Ray-Chaudhuri and Wilson's solution to the Schoolgirl problem was later used by Bermond in finding a decomposition of the complete directed graph into 3-dicycles.) Since then the problem has been studied in general, and has been solved for various cases. In
Chapter 2 we give an outline of the known solutions, primarily for decompositions into even-length cycles.

In Chapter 3 we construct a decomposition of the graph $K_{2n} - I$ into cycles of length $2m$, where $n$ is even and $3m/2 \leq n < 2m$. The construction uses the following method of composition, due to Häggkvist: If a graph $G$ may be decomposed into paths and cycles of length $m$, then $GwrK_2$ may be decomposed into cycles of length $2m$. Now if $G$ is the complete graph $K_n$, then $GwrK_2 \cong K_{2n} - I$ (see Lemma 1.1.1). Häggkvist's result therefore allows us to use methods of decomposing the graph $K_n$, in solving the problem for the graph $K_{2n} - I$. 
Chapter 2

In this chapter we give a brief survey of known results in the area of decompositions of the complete graph (or the complete directed graph) into cycles (or dicycles) of fixed even length. We consider three different classes of decompositions: the decomposition of the complete directed graph into even-length dicycles (§2.1); the decomposition of the complete undirected graph into even-length cycles (§2.2); and the decomposition of the graph $K_{2n} - I$ into even-length cycles (§2.3).

§ 2.1 Directed cycles

We first look at the directed case - that is, decompositions of the complete directed graph $K_n^*$ into dicycles of fixed, even length $m$. The necessary conditions (NC1) for such a decomposition are:

1. $n(n-1) \equiv 0 \pmod{m}$ (the number of arcs of $K_n^*$ is a multiple of $m$); and
2. $n \geq m$.

It has been conjectured (see for example [B/F]) that these necessary conditions are also sufficient, for $m$ even or odd, except for the cases $(n,m) = (6,3), (4,4)$ and $(6,6)$ when the decomposition is known not to exist. The conjecture has been shown to be true if either $n \equiv 0 \pmod{m}$ or $(n-1) \equiv 0 \pmod{m}$, but apart from this relatively few cases have been solved.

We begin with some lemmas which are useful in recursive constructions. The first is from [S] and will be used repeatedly in later sections.
2.1.1 Lemma ([S])

(i) The complete directed bipartite graph $K_{n,s}^*$ can be decomposed into $2m$-dicycles if and only if $n, s \geq m$ and $ns \equiv 0 \pmod{m}$.

(ii) The complete (undirected) bipartite graph $K_{n,s}$ can be decomposed into $2m$-cycles if and only if $n, s \geq m, ns \equiv 0 \pmod{2m}$ and $n$ and $s$ are even.

We refer the reader to the proof given in [S], which is constructive and too long to include here.

The next two lemmas are straightforward.

2.1.2 Lemma If $K_n^*$, $K_s^*$, and $K_{n,s}^*$ can be decomposed into $m$-dicycles, then $K_{n+s}^*$ can be decomposed into $m$-dicycles.

Proof We write $K_{n+s}^* = K_n^* \cup K_{n,s}^* \cup K_s^*$. The result is obvious.  

2.1.3 Lemma ([B/F]) If $K_{n+1}^*$, $K_{s+1}^*$ and $K_{n,s}^*$ can be decomposed into $m$-dicycles, then $K_{n+s+1}^*$ can be decomposed into $m$-dicycles.

Proof Let $V(K_{n+s+1}^*) = X \cup Y \cup \{x_0\}$, where $|X| = n$ and $|Y| = s$. Then we may partition the arcs of $K_{n+s+1}^*$ into those of $K_{n+1}^*$ on vertex-set $X \cup \{x_0\}$, those of $K_{s+1}^*$ on vertex-set $Y \cup \{x_0\}$, and those of $K_{n,s}^*$ on vertex-set $X \cup Y$. Thus a decomposition of $K_{n+1}^*$, $K_{s+1}^*$ and $K_{n,s}^*$ into $m$-dicycles gives us a decomposition of $K_{n+s+1}^*$ into $m$-dicycles.

The following three lemmas give solutions for the cases $n = m$ and $n = m+1$. These will be used in the solutions for the cases with $n \equiv 0, 1 \pmod{m}$. 

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2.1.4 **Lemma** ([B/F]) For every integer \( m \), even or odd, \( K_{m+1}^* \) can be decomposed into \( m \)-dicycles.

2.1.5 **Lemma** ([T]) For \( 2m \geq 8 \), \( K_{2m}^* \) can be decomposed into \( 2m \)-dicycles.

2.1.6 **Lemma** ([B/F]) There is no decomposition of \( K_{2m}^* \) into \( 2m \)-dicycles for \( 2m = 4, 6 \).

We now give solutions to the problem for the cases when \( n \equiv 0, 1 \) (mod \( m \)). They are due to Bermond and Faber and appear in [B/F]. We first consider \( n \equiv 0 \) (mod \( m \)).

2.1.7 **Theorem** ([B/F]) If \( m \) is even and \( K_m^* \) can be decomposed into \( m \)-dicycles, then \( K_{qm}^* \) can be decomposed into \( m \)-dicycles for all \( q \geq 1 \).

**Proof** The proof is by induction on \( q \). When \( q = 1 \), the theorem is true by hypothesis. Let \( q \geq 1 \) and assume that \( K_{qm}^* \) can be decomposed into \( m \)-dicycles. Now \( K_{(q+1)m}^* = K_{qm}^* \cup K_{qm,m}^* \cup K_m^* \). By Lemma 2.1.1 \( K_{qm,m}^* \) is decomposable into \( m \)-dicycles, and by hypothesis \( K_m^* \) is also; therefore \( K_{(q+1)m}^* \) is decomposable into \( m \)-dicycles. The theorem follows by induction.

Combining Theorem 2.1.7 with Lemma 2.1.5 we have the following corollary.

2.1.8 **Corollary** For all even \( m \geq 8 \) and for all \( q \geq 1 \), \( K_{qm}^* \) can be decomposed into \( m \)-dicycles.

Secondly, we consider \( n \equiv 1 \) (mod \( m \)).
2.1.9 Theorem ([B/F]) If \( m \) is even, then \( K_{qm+1}^* \) can be decomposed into \( m \)-dicycles for all \( q \geq 1 \).

Proof Again the proof is by induction on \( q \). If \( q = 1 \), the theorem follows from Lemma 2.1.4. Let \( q \geq 1 \) and assume \( K_{qm+1}^* \) can be decomposed into \( m \)-dicycles. Then by the induction base and Lemma 2.1.3, \( K_{qm+m+1}^* = K_{(q+1)m+1}^* \) can be decomposed into \( m \)-dicycles. By induction the theorem holds for all \( q \geq 1 \). \( \Box \)

Thus if \( n \equiv 1 \pmod{m} \), or \( n \equiv 0 \pmod{m} \) and \( m > 6 \), where \( m \) is even, there is a decomposition of \( K_n^* \) into \( m \)-dicycles if and only if \( n \) and \( m \) satisfy the necessary conditions (NC1). Complete solutions for \( m \leq 6 \) are given below. For the cases when \( n \not\equiv 0,1 \pmod{m} \), the problem is not yet completely solved. The known results are the following.

2.1.10 Theorem (Hartnell and Milgram, [H/M]) If \( n = p^e \) for some prime \( p \) and integer \( e \geq 1 \), then \( K_n^* \) can be decomposed into \( m \)-dicycles if and only if \( n \) and \( m \) satisfy the necessary conditions (NC1) \( (m \) may be even or odd).

2.1.11 Theorem (Hartnell, [Hl]) If \( x \) and \( y \) are odd integers and \( y \geq 5 \), then \( K_{xy}^* \) can be decomposed into \( 2x \)-dicycles.

Finally, the problem has been completely solved for several small values of \( m \), that is, for even \( m \) satisfying \( 4 \leq m \leq 16 \). These results can all be found in [B/H/S].

2.1.12 Theorem ([B/H/S])

(i) \( K_n^* \) can be decomposed into 4-dicycles if and only if \( n \equiv 0,1 \pmod{4} \) and \( n > 4 \).
(ii) $K_n^*$ can be decomposed into 6-dicycles if and only if $n \equiv 0,1 \pmod{3}$ and $n > 6$.

(iii) $K_n^*$ can be decomposed into 8-dicycles if and only if $n \equiv 0,1 \pmod{8}$.

(iv) $K_n^*$ can be decomposed into 10-dicycles if and only if $n \equiv 0,1 \pmod{5}$.

(v) $K_n^*$ can be decomposed into 12-dicycles if and only if $n \equiv 0,1,4 \text{ or } 9 \pmod{12}$.

(vi) $K_n^*$ can be decomposed into 14-dicycles if and only if $n \equiv 0,1 \pmod{7}$.

(vii) $K_n^*$ can be decomposed into 16-dicycles if and only if $n \equiv 0,1 \pmod{16}$.

In [B/H/S] the authors show, using composition methods, that the problem of finding decompositions of $K_n^*$ into $m$-dicycles (for even $m$) can be reduced to checking a finite number of cases for each $m$. More specifically, they derive from the necessary conditions (NC1) the following necessary conditions, which we will call (NC1*):

1. $n \geq m,$

and 2. there exist positive integers $y$ and $z$ such that $yz = m$ and $n \equiv ly \pmod{yz}$, where $0 \leq l < z$ and $ly \equiv 1 \pmod{z}$ if $l > 0$.

They then show that, for a given $m$, there is an integer $n_0(m)$ such that if the conditions (NC1*) are sufficient for the existence of a decomposition of $K_n^*$ into $m$-dicycles when $n_0(m) \leq n < n_0(m) + m$, then these conditions are sufficient for the existence of such a decomposition for all $n$.

§2.2 Undirected cycles

We now consider decompositions of the complete undirected graph $K_n$ into $m$-cycles, for even $m$. Notice that if $C$ is an $m$-cycle in $K_n$, then every vertex of $K_n$ has degree either 2 or 0 in $C$. Thus if there is a decomposition of $K_n$ into $m$-cycles,
every vertex of $K_n$ must have even degree. Therefore, $n$ must be odd. The necessary conditions (NC2) are then:

1. $n(n-1) \equiv 0 \pmod{2m}$ (the number of edges of $K_n$ is a multiple of $m$);
2. $n$ is odd;
3. $n \geq m$.

The first two results in this area are from 1965 and 1966, respectively.

2.2.1 Theorem (Kotzig, [K]) If $n$ is odd, $(n-1)/2 \equiv 0 \pmod{2m}$, and $m \equiv 0 \pmod{4}$, then $K_n$ can be decomposed into $m$-cycles.

2.2.2 Theorem (Rosa, [R]) If $n$ is odd, $(n-1)/2 \equiv 0 \pmod{2m}$, and $m \equiv 2 \pmod{4}$, then $K_n$ can be decomposed into $m$-cycles.

Thus the problem has been solved for the case $n \equiv 1 \pmod{m}$. Notice that, since $m$ is even and $n$ must be odd, $n \equiv 0 \pmod{m}$ is impossible. For the cases when $n \equiv 1 \pmod{m}$, there are few known results. The following two theorems give solutions for the cases when $4 \leq m \leq 16$, and when $m = 2p^e$ for some prime $p$ and integer $e \geq 1$. In the first case the results appear in various papers but can all be found in [B/H/S].

2.2.3 Theorem ([B/H/S])

(i) $K_n$ can be decomposed into 4-cycles if and only if $n \equiv 1 \pmod{8}$.
(ii) $K_n$ can be decomposed into 6-cycles if and only if $n \equiv 1,9 \pmod{12}$.
(iii) $K_n$ can be decomposed into 8-cycles if and only if $n \equiv 1 \pmod{16}$.
(iv) $K_n$ can be decomposed into 10-cycles if and only if $n \equiv 1,5 \pmod{20}$.
(v) $K_n$ can be decomposed into 12-cycles if and only if $n \equiv 1,9 \pmod{24}$.
(vi) $K_n$ can be decomposed into 14-cycles if and only if $n \equiv 1, 21 \pmod{28}$.

(vii) $K_n$ can be decomposed into 16-cycles if and only if $n \equiv 1 \pmod{32}$.

2.2.4 Theorem (Alspach/Varma, [A/V]) If $m = 2p^e$ for some prime $p$ and integer $e \geq 1$, then $K_n$ can be decomposed into $m$-cycles if and only if $m$ and $n$ satisfy the necessary conditions (NC2).

§2.3 The graph $K_{2n} - I$

In this section we look at decompositions of the graph $K_{2n} - I$, the complete graph on $2n$ vertices with a one-factor removed, into cycles of fixed, even length $2m$. The necessary conditions (NC3) for such a decomposition are the following:

(i) $n(n-1) \equiv 0 \pmod{m}$ (the number of edges of $K_{2n} - I$ is a multiple of $2m$);

and (ii) $n \geq m$.

Some results for this problem are given in [H/K/R], although this paper is primarily concerned with finding those 2-factors which decompose $K_{2n} - I$. Their results are the following.

2.3.1 Proposition ([H/K/R]) For all $m \geq 2$, $K_{6m} - I$ can be decomposed into $2m$-cycles.

2.3.2 Proposition ([H/K/R]) For all $m \geq 2$, $K_{4m} - I$ can be decomposed into $2m$-cycles.

2.3.3 Proposition ([H/K/R]) If $n$ is even, the graph $K_{2n} - I$ can be decomposed into 4-cycles.
The remaining results in this section rely on the following lemma which provides us with a particular method for recursively constructing cycle decompositions. The construction in Chapter 3 will also use this method of composition.

2.3.4 Lemma (Häggkvist, [H]) Let $G$ be a path or a cycle with $m$ edges, and let $H$ be a 2-regular graph on $2m$ vertices with all components even. Then $G \cup K_2$ is the edge-disjoint union of $G'$ and $G''$, where $G' \cong G'' \cong H$.

Proof. Let $H$ consist of $k$ disjoint cycles with lengths $2m_1, 2m_2, \ldots, 2m_k$, where $\sum_{i=1}^{k} m_i = m$. Let $s_j = \sum_{i=1}^{j} m_i$ and $s_0 = 0$.

Let $G$ be the path or cycle $(u_0, u_1, \ldots, u_m)$, where $u_0 = u_m$ if $G$ is a cycle. Let $G_i$ be the segment $(u_{s_{i-1}}, u_{s_{i-1}+1}, \ldots, u_s)$ of $G$, of length $m_i$, $i = 1, 2, \ldots, k$.

Let $G \cup K_2$ be obtained from $G$ by replacing each vertex $u_i$, $0 \leq i \leq m$, by the independent vertices $x(u_i)$ and $y(u_i)$.

Let $G'_i$ have vertex set $V(G \cup K_2) \setminus \{y(u_{s_{i-1}}), x(u_s)\}$, and edges $(x(u_{s_{i-1}}), x(u_{s_{i-1}+1}))$, $(x(u_{s_{i-1}}), y(u_{s_{i-1}+1}))$, together with any pair of independent edges between $(x(u_j), y(u_j))$ and $(x(u_{j+1}), y(u_{j+1}))$ for $j = s_{i-1}, s_{i-1}+1, \ldots, s_i-2$, and finally the edges $(x(u_{s_{i-1}}), y(u_s))$, $(y(u_{s_{i-1}}), y(u_s))$ [see Fig. 2.1]

Clearly $G'_i$ is a cycle of length $2m_i$, whose edge-induced complement in $G \cup K_2$ is another $2m_i$-cycle, $G''_i$ [see Fig. 2.1].
Also, the graphs \( G_i, i = 1, 2, \ldots, k, \) are pairwise disjoint, and each of
\[
G' = \bigcup_{i=1}^{k} G_i \quad \text{and} \quad G'' = \bigcup_{i=1}^{k} G_i''
\]
is isomorphic to \( H. \) Moreover \( G \wr K_2 = G' \oplus G'', \)
and the lemma follows.

2.3.5 Corollary Let \( G \) be a path or a cycle with \( m \) edges. Then \( G \wr K_2 \) is the
edge-disjoint union of two \( 2m \)-cycles.

Proof Let \( k = 1 \) in Lemma 2.3.4.

One consequence of this is the following. Suppose that a graph \( G \) can be
decomposed into \( m \)-paths and \( m \)-cycles, say \( P_1, \ldots, P_k, C_1, \ldots, C_l. \) Then \( G \wr K_2 \) is
the edge-disjoint union of the graphs \( P_1 \wr K_2, \ldots, P_k \wr K_2, C_1 \wr K_2, \ldots, C_l \wr K_2. \)
Applying Lemma 2.3.4 to each \( P_i \) and \( C_j \) gives us a decomposition of \( G \wr K_2 \) into
2m-cycles. The following lemma allows us to use this method to decompose $K_{2n} - I$ into 2m-cycles when $n(n-1) \equiv 0 \pmod{2m}$.

2.3.6 Lemma (Tarsi, [T]) Necessary and sufficient conditions for a decomposition of $K_n$ into m-paths are:

(i) $n(n-1) \equiv 0 \pmod{2m}$;

and (ii) $n > m$.

2.3.7 Corollary If $m,n$ satisfy (NC3) and in addition $n(n-1) \equiv 0 \pmod{2m}$, then $K_{2n} - I$ can be decomposed into 2m-cycles.

Remark This result is also proven in [H], for odd $n$ and $m \neq n-1$, using the same method.

Now if $G$ has a Hamilton decomposition, then by applying Lemma 2.3.4 to each cycle of the decomposition we get the following result.

2.3.8 Proposition ([H]) If $G$ has a Hamilton decomposition, then $G \wr rK_2$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times.

In addition we have the following proposition, due to Laskar ([L]).

2.3.9 Proposition ([L]) If $G$ has a Hamilton decomposition, then the graph $G \wr rK_m$ also has a Hamilton decomposition.
2.3.10 Corollary ([A/H]) If $G$ has a Hamilton decomposition, then $G \wedge K_{2m}$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times.

We now show how this method of composition may be used to find a decomposition of $K_{2n} - I$ into $2m$-cycles for the cases when $n \equiv 0 \pmod{m}$. The following result was proven in [A/H] in 1985.

2.3.11 Theorem ([A/H]) For any $m \geq 2$ and any natural number $n$, the graph $K_{2nm} - I$ can be decomposed into $2m$-cycles.

Proof The proof is divided into two cases:

Case 1. Let $n$ be odd. Notice that $K_{2nm} - I \cong K_{nm} \wedge K_n$, and that $K_{nm} \wedge K_2 = H_1 \oplus H_2$, where $H_1 \cong nK_{m} \wedge K_2$ and $H_2 \cong K_{n} \wedge K_{2m}$. Since $n$ is odd, $K_n$ has a Hamilton decomposition. Therefore by Corollary 2.3.10, $K_{n} \wedge K_{2m}$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times. In particular, $K_{n} \wedge K_{2m}$ can be decomposed into $m(n-1)$ copies of $nC_{2m}$, since $n-1$ is even. Therefore $H_2$ can be decomposed into $2m$-cycles. Moreover, $K_{m} \wedge K_{2} = K_{2n} - I$ can be decomposed into $2m$-cycles by Lemma 1.1.12, so $H_1$ can also be decomposed into $2m$-cycles. Therefore, $K_{2nm} - I$ can be decomposed into $2m$-cycles.

Case 2. Let $n$ be even. In this case $K_{nm} \wedge K_2 = H_1 \oplus H_2$, where $H_1 \cong (n/2)K_{2m} \wedge K_2$ and $H_2 \cong K_{n/2} \wedge K_{4m}$. If $n/2$ is odd, then $K_{n/2} \wedge K_4$ can be decomposed into $m(n-2)$ copies of $nC_{2m}$ as above ($n-2$ is even in this case). If $n/2$ is even, then $K_{n/2}$ has a one-factorisation. Let $K_{n/2} = F_1 \oplus \ldots \oplus F_{(n-2)/2}$.
where each $F_i$ is a one-factor. For each $i$, $F_i \wr K^r_{4m} \equiv (n/4)K_{4m,4m}$. Now $K_{2m,2m} = C_4 \wr K_m$ has a Hamilton decomposition, since $C_4$ has. Therefore $K_{4m,4m} \equiv K_{2m,2m} \wr K_2$ can be decomposed into $2m$ 2-factors each isomorphic to $4C_{2m}$, and so $K_{n/2} \wr K_{4m}$ can be also. Thus $H_2$ can be decomposed into $2m$-cycles. Finally, applying Proposition 2.3.2 to each component of $(n/2)K_{2m} \wr K_2 \equiv (n/2)(K_{4m} - I)$ yields a decomposition of $H_1$ into $2m$-cycles. Therefore, $K_{2n^m} - I$ can be decomposed into $2m$-cycles.
Chapter 3

As stated in the introduction, the new results we have found concern the decomposition of the graph \( K_{2n} - I \) into edge-disjoint 2m-cycles. We discussed in the previous chapter how this problem has been solved in some cases; in summary, there is known to be a decomposition of \( K_{2n} - I \) into edge-disjoint 2m-cycles if

1. \( n \equiv 0 \pmod{m} \)
2. \( n(n-1) \equiv 0 \pmod{2m} \)

or

3. \( m = 2 \).

In this chapter we give partial results for the case when m does not divide n, and the quotient \( n(n-1)/m \) is odd. In Section 3.1 we give an outline of the methods we use in the constructions; Section 3.2 contains the constructions themselves.

§ 3.1 Outline of the construction

Our constructions are based on the method of Lemma 2.3.4. In the case where \( n(n-1)/m \) is even, we are able to decompose \( K_n \) into edge-disjoint m-paths, and using Lemma 2.3.4, this decomposition of \( K_n \) yields a decomposition of \( K_{2n} - I \equiv K_n \wr \overline{K}_2 \) into 2m-cycles. When \( n(n-1)/m \) is odd, however, there is no decomposition of \( K_n \) into m-paths, since \( |E(K_n)| \not\equiv 0 \pmod{m} \). Thus we need to modify the construction somewhat.
Suppose that \( n(n-1)/m \) is odd. Then in fact \( |E(K_n)| \equiv m/2 \pmod{m} \). Our method will be to find a subgraph \( D \) of \( K_n \) with the following properties:

1. \( |E(D)| \equiv m/2 \pmod{m} \);
2. \( K_n - D \) can be decomposed into \( m \)-paths and \( m \)-cycles;
3. \( D \uplus K_2 \) can be decomposed into \( 2m \)-cycles.

Notice that since \( |E(K_n)| \equiv m/2 \pmod{m} \), property (1) implies that \( |E(K_n - D)| \equiv 0 \pmod{m} \).

We will apply Lemma 2.3.4 to the decomposition of \( K_n - D \) into \( m \)-paths and \( m \)-cycles, which will give us a decomposition of \( (K_n - D) \uplus K_2 \) into \( 2m \)-cycles. Since \( D \uplus K_2 \) may be decomposed into \( 2m \)-cycles, we will then have a decomposition of \([ (K_n - D) \uplus K_2 ] = K_n \uplus K_2 = K_{2n} - I \) into \( 2m \)-cycles.

We begin with a lemma which reduces the problem to 'small' values of \( n \), that is, to values of \( n \) satisfying \( m < n < 2m \).

**3.1.1 Lemma** To find decompositions of \( K_{2n} - I \) into \( 2m \)-cycles for all \( m, n \) satisfying (NC3), it is sufficient to find decompositions of \( K_{2n} - I \) into \( 2m \)-cycles for all \( m, n \) satisfying both (NC3) and \( m < n < 2m \).

**Proof** Suppose we have decompositions of \( K_{2n} - I \) into \( 2m \)-cycles for all \( m, n \) satisfying both (NC3) and \( m < n < 2m \).

Let \( n \) and \( m \) satisfy (NC3) but not \( m < n < 2m \). By (NC3), \( n \geq m \), so either \( m = n \) or \( n \geq 2m \). By Lemma 1.1.11, \( K_{2n} - I \) can be decomposed into Hamilton cycles, so we assume \( n \geq 2m \). Thus we may write \( n = km + (m+r) \), where \( k \geq 1 \).
and \( 0 \leq r < m \). Therefore \( 2n = k(2m) + (2m + 2r) \), and \( K_{2n} - I \) is the edge-disjoint union of the following:

- \( k \) copies of \( K_{2m} - I \);
- one copy of \( K_{2m+2r} - I \);
- \( \binom{k}{2} \) copies of \( K_{2m,2m} \);

and \( k \) copies of \( K_{2m,2m+2r} \).

Again by Lemma 1.1.11, \( K_{2m} - I \) can be decomposed into Hamilton cycles. By Lemma 2.1.1, each of \( K_{2m,2m} \) and \( K_{2m,2m+2r} \) can be decomposed into \( 2m \)-cycles. If \( r = 0 \) this gives us the desired decomposition of \( K_{2n} - I \) into \( 2m \)-cycles. Suppose \( r > 0 \). By (NC3), \( |E(K_{2n} - I)| \equiv 0 \pmod{2m} \), and so \( |E(K_{2m+2r} - I)| \equiv 0 \pmod{2m} \).

Since in addition \( m < m + r < 2m \), by hypothesis we have a decomposition of \( K_{2m+2r} - I \) into \( 2m \)-cycles. Thus \( K_{2n} - I \) may be decomposed into \( 2m \)-cycles, and the lemma follows.

In the light of Lemma 3.1.1, for the remainder of the chapter we restrict ourselves to cases where \( m < n < 2m \). Our aim is to find a subgraph \( D \) of \( K_n \) with the three properties described above. We first define a subgraph \( D \) with properties (1) and (3); and for \( n \geq 3m/2 \), where \( n \) is even, we construct a decomposition of \( K_n - D \) into \( m \)-paths and \( m \)-cycles (this is property (2) required of \( D \)). Thus we have solved the problem for \( m \) and \( n \) satisfying (NC3) and for which \( 3m/2 \leq n < 2m \), with \( n \) even.

In the construction we first write \( K_n = K_m \cup K_{m,r} \cup K_r \), where \( r = n - m \).

Now as \( n(n-1)/m \) is odd, then necessarily \( m \) is even (since \( n \) is even). Thus we can decompose \( K_m \) into \( m/2-1 \) \( m \)-cycles and a one-factor \( F \). Notice that \( |E(F)| = \)
We will show that there is a cycle $C_1$ in the decomposition of $K_m$ such that $C_1 \cup F$ has the properties (1) and (3) required for the subgraph $D$. From this point we will proceed to find a decomposition of $K_n - (C_1 \cup F)$ into $m$-paths and $m$-cycles.

Now since $n$ and $m$ are even, $r$ is also even. If in addition $r \geq m/2$ (or equivalently $n \geq 3m/2$), then by Lemma 2.1.1 $K_{m,r}$ can be decomposed into $m$-cycles. However $n < 2m$ implies $r < m$, so there is no decomposition of $K_r$ into $m$-paths and $m$-cycles. Instead, we will decompose $K_r$ into $r$ paths of length at most $r-1$. Using the edges of two $m$-cycles from the decomposition of $K_{m,r}$ as a 'bridge' between $K_r$ and $K_m$, we will extend each of these paths to an $m$-path with segments of one or more of the $m$-cycles from the decomposition of $K_m$. This will yield $r$ $m$-paths which together cover all the edges of $K_r$, the edges of two $m$-cycles from $K_{m,r}$, and the edges of some (or all) of the $m$-cycles comprising $K_m - (C_1 \cup F)$. In addition the construction will be such that the remainder (if any) of $K_m$ will be a collection of $m$-cycles, as will be the remainder of $K_{m,r}$. Thus we will have a decomposition of $K_n - (C_1 \cup F)$ into $m$-paths and $m$-cycles.

We now give the constructions.

§ 3.2 The construction

Throughout this section we assume that:

(*) $n$ and $m$ satisfy (NC3), $n$ is even, $m < n < 2m$ and the quotient $n(n-1)/m$ is odd.

We begin with the construction of the subgraph $D$. We write $K_n = K_m \cup K_{m,r} \cup K_r$, where $V(K_m) = \{0, 1, ..., m-1\}$ and $V(K_r) = \{z_1, z_2, ..., z_r\}$. We decompose $K_m$
(remember that $m$ is even) into $m/2-1$ $m$-cycles $C_1, ..., C_{m/2-1}$ and a one-factor $F$, defined as follows. Let $\sigma = (0)(1 \ 2 \ ... \ m-1)$, and $C_1 = (0, 1, 2, m-1, 3, m-2, ..., m/2-1, m/2+1, 0)$. For $i = 2, ..., m/2-1$, let $C_i = \sigma^{i-1}(C_1)$. Finally, let $F = \{(m-1,1), (m-2,2), ..., (m/2+1,m/2-1), (0,m/2)\}$.

Thus $C_1$ is the $m$-cycle of Fig. 3.1, and for $i \geq 2$, $C_i$ is obtained from $C_1$ by a clockwise rotation of $i$ places, with the vertex 0 fixed.

![Diagram of graphs $C_1$ and $D = C_1 \cup F$](image)

**Fig. 3.1**

Letting $D = C_1 \cup F$ (see Fig. 3.1) we have

3.2.1 **Lemma** The graph $D_{MRK_2}$ is the edge-disjoint union of three $2m$-cycles.
Proof We show that \( D \) is the edge-disjoint union of three perfect matchings with the property that the union of any two perfect matchings is a Hamilton cycle. We then use the three resulting Hamilton cycles to construct three edge-disjoint 2m-cycles in \( D \wr K_2 \).

It is easy to see that the perfect matchings \( M_1, M_2 \) and \( M_3 \), where \( M_1 \) and \( M_2 \) consist of alternate edges of the \( m \)-cycle \( C_1 \) and \( M_3 = F \), partition the edges of \( D \). By construction, the union of \( M_1 \) and \( M_2 \) is the \( m \)-cycle \( C_1 \).

Let \( M_1 = \{ (0,1), (2,m-1), (3,m-2), \ldots, (m/2,m/2+1) \} \), and \( M_2 = \{ (1,2), (m-1,3), (m-2,4), \ldots, (m/2+2,m/2), (m/2+1,0) \} \). We have \( M_3 = F = \{ (m-1,1), (m-2,2), \ldots, (m/2+1,m/2-1), (m/2,0) \} \).

To show that \( M_1 \cup M_3 \) is an \( m \)-cycle, define a permutation \( \pi \) of \( V(K_m) \) by \( \pi(0) = 0, \pi(1) = 1, \) and \( \pi(i) = m-i+1, \) \( 2 \leq i \leq m-1. \) The effect of \( \pi \) is to interchange the endvertices of each edge of \( M_1 \), except for \( (0,1) \) which is unchanged. Thus \( \pi(M_1) = M_1. \)

Now \( M_3 = \{ (m/2,0), (m-1,1) \} \cup \{ (m-i,i) : 2 \leq i \leq m/2-1 \}. \)
Thus \( \pi(M_3) = \{ (m/2+1,0), (2,1) \} \cup \{ (i+1,m-i+1) : 2 \leq i \leq m/2-1 \} \)
\[ = \{ (m/2+1,0), (1,2) \} \cup \{ (j+2,m-j) : 1 \leq j \leq m/2-2 \} \]
\[ = M_2. \]

Similarly \( \pi(M_2) = M_3. \) So \( \pi : V(D) \rightarrow V(D) \) is an automorphism, and \( \pi(M_1 \cup M_3) = M_1 \cup M_2. \) Thus, \( M_1 \cup M_3 \) is also an \( m \)-cycle.
We now show that $M_2 \cup M_3$ is an $m$-cycle. We have

$$M_2 \cup M_3 = \{(1,2), (m/2+1,0)\} \cup \{(j+2,m-j) : 1 \leq j \leq m/2-2\} \cup$$

$$\{(m-1,1), (m/2,0)\} \cup \{(m-i,i) : 2 \leq i \leq m/2-1\}$$

$$= \{(1,2), (1,m-1)\} \cup \{(m-j,2+j), (1+j, m-j-1) : 1 \leq j \leq m/2-2\} \cup$$

$$\{(m/2,0), (m/2+1,0)\}.$$

Now $M_2 \cup M_3$ clearly contains the 2-path $(m-1,1,2)$. To this we add the pair of edges $(m-1,3)$ and $(2,m-2)$, which gives us a 4-path with endvertices $m-2$ and 3. We continue adding edges in pairs of the form $(m-j,2+j), (1+j,m-j-1)$, for $j = 1, 2, ..., m/2-2$, until a cycle is formed; after each addition we obtain a path or cycle of length two more than the previous path, and with endvertices $m-j-1$ and $j+2$ (where $2 \leq j \leq m/2-2$). A cycle will be formed only if $m-i-1 = i+2$ for some $i$; that is, if $m-3 = 2i$. But $m$ is even, so $m-3$ is odd, and this is impossible. Therefore after the addition of the last pair of edges $(m/2+2,m/2)$ and $(m/2-1,m/2+1)$ we have a path in $M_2 \cup M_3$ of length $m-2$ with endvertices $m/2+1$ and $m/2$. The addition of the two remaining edges $(m/2+1,0)$ and $(m/2,0)$ completes this path to an $m$-cycle; thus $M_2 \cup M_3$ is an $m$-cycle.

Let $M_1 \cup M_2 = C_{12}$ ( $C_{12} = C_1$ ), $M_2 \cup M_3 = C_{23}$ and $M_3 \cup M_1 = C_{31}$.

We now use these three $m$-cycles to partition the edges of $D_{wre}$ into three $2m$-cycles. We first observe that every edge of $D$ lies on exactly one of the perfect matchings $M_i$, and so on exactly two of the $m$-cycles $C_{ij}$. 

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Let $V(D_{wrK_2}) = \{x(i), y(i) : i \in V(D)\}$, so that

$$E(D_{wrK_2}) = \{(x(i), y(j)), (x(i), x(j)), (y(i), y(j)), (y(i), x(j)) : ij \in E(D)\}.$$ 

For each $ij \in E(D)$, let

$$p(ij) = \{(x(i), x(j)), (y(i), y(j))\} \text{ (the corresponding 'parallel' edges of } D_{wrK_2},$$

and $c(ij) = \{(x(i), y(j)), (y(i), x(j))\}$ (the corresponding 'crossing' edges of $D_{wrK_2}$).

We define $C_{12}'$ from $C_{12}$ as follows. We let $C_{12}'$ consist of the edges $(x(0), x(1))$, $(x(0), y(1))$, together with either pair of independent edges $p(ij)$ or $c(ij)$ for each subsequent edge $ij, j \neq 0$, of $C_{12}$, and finally the edges $(x(m/2+1), y(0))$, $(y(m/2+1), y(0))$ (see Fig. 3.2).

Fig. 3.2
We let $C_{23}'$ consist of the edges $(x(0), x(m/2+1)), (x(0), y(m/2+1))$, together with either pair of independent edges $p(ij)$ or $c(ij)$ for each subsequent edge $ij$, $j \neq 0$, of $C_{23} \setminus C_{12}$, and whichever pair of independent edges $p(ij)$, $c(ij)$ does not lie on $C_{12}'$ for each subsequent edge of $C_{23} \cap C_{12}$, and finally the two edges $(x(m/2), y(0)), (y(m/2), y(0))$.

Finally we let $C_{31}'$ consist of the edges $(x(0), x(m/2)), (x(0), y(m/2))$, together with whichever pair of independent edges $p(ij)$, $c(ij)$ lies on neither $C_{12}'$ nor $C_{23}'$, for each subsequent edge $ij$, $j \neq 0$, of $C_{31}$, and finally the edges $(x(1), y(0))$ and $(y(1), y(0))$.

It is clear that each of $C_{12}'$, $C_{23}'$ and $C_{31}'$ is a $2m$-cycle in $D \cup K_2$. In addition, if $C_{pq}'$ and $C_{kl}'$ are not edge-disjoint, then they share either a pair $p(ij)$ or $c(ij)$ for some $ij \in E(D)$, $i, j \neq 0$, or an edge incident with either $x(0)$ or $y(0)$. The first case cannot occur since in defining each $C_{pq}'$ we choose a pair $p(ij)$ or $c(ij)$ only if it does not lie on a previously defined cycle $C_{kl}'$. It is easy to check in the construction that the second case cannot occur. Thus the three $2m$-cycles $C_{12}'$, $C_{23}'$ and $C_{31}'$ partition the edges of $D \cup K_2$.

Therefore $D$ has property (3) described earlier, and clearly $|E(D)| = 3m/2$ so $D$ also has property (2). We now show that $D$ has property (1).
3.2.1 Lemma The graph $K_n - D$ can be decomposed into $m$-paths and $m$-cycles.

Proof We begin with an important observation about the cycles $C_1, ..., C_{m/2-1}$ of the decomposition of $K_n - D$. Let $W$ be the Eulerian walk of $K_m - D$ defined by

$$W = C_2 + C_3 + ... + C_{m/2-1}$$

(where the cycles $C_i$ are oriented so that $(0,i)$ is the first directed edge of $C_i$)

$$= (0, 2, 3, ..., m/2+1, m/2+2, 0, 3, 4, ..., m/2+2, m/2+3, 0, 4, 5, ..., 0, m/2-1, m/2, ..., m-2, m-1, 0).$$

3.2.2 Claim The shortest cycle in $W$ has length $m-2$.

Proof First, as we have already pointed out, each cycle $C_i$, $i \geq 3$, is simply a rotation of $C_2$. Also, any segment of $W$ which lies entirely within some $C_i$ must be either a path or an $m$-cycle. Thus it is sufficient to show that any cycle in $W$ which begins in $C_2$ and ends in $C_3$ has length at least $m-2$. To do this we find the length of the cycle which begins at the occurrence of a vertex $v$ in $C_2$ and ends at the occurrence of the same vertex $v$ in $C_3$, for each $v \neq 0$ of $K_m$ (clearly the cycle which begins and ends with 0 has length $m$).

Let $v$ be the $k^{th}$ vertex of $C_2$ (where we orient $C_i$ as in $W$ so that 0 is the first vertex, $i$ the second, and so on) (see Fig. 3.3).

Now $C_3$ is obtained from $C_2$ by a clockwise rotation through one place, with 0 fixed.
Thus (i) if $4 \leq v \leq m/2+1$, then $v$ is the $(k-2)^{\text{nd}}$ vertex of $C_3$;

(ii) if $m/2+3 \leq v \leq m-1$ or $v = 1$ or 2, then $v$ is the $(k+2)^{\text{nd}}$ vertex of $C_3$;

(iii) if $v = 3$, then $v$ is the third vertex of $C_2$ and the second of $C_3$;

and (iv) if $v = m/2+2$, then $v$ is the $m^{\text{th}}$ vertex of $C_2$ and the $(m-1)^{\text{st}}$ of $C_3$.

Therefore the closed walk of $C_2 + C_3$ which begins and ends at vertex $v \neq 0$ of $K_m$ has length $m-2$ (if $4 \leq v \leq m/2+1$), $m+2$ (if $m/2+3 \leq v \leq m-1$ or $v = 1$ or 2) or $m-1$ (if $v = 3$ or $v = m/2+2$). So the shortest cycle in $W$ has length $m-2$.

Thus $W$ is a trail in $K_m - D$ with the property that any segment of length at most $m-3$ is a path, and such that $E(K_m - D) = E(W)$. Notice also that if $S$ is an initial segment of $W$ whose length is a multiple of $m$, then $W \setminus S$ consists of a
collection of entire \( m \)-cycles \( C_j, \ldots, C_{m/2-1} \) for some \( j \). These properties of \( W \) are crucial for the constructions.

The remainder of the proof of Lemma 3.2.1 is divided into three cases. Case 1 contains the basic construction which is valid for all \( n \) and \( m \) satisfying (*) with \( r > m/2+1 \) and \( r > 8 \). Cases 2 and 3 contain modifications of this construction for the cases \( r = m/2+1 \) and \( r \leq 8 \), respectively. Observe that if \( r = m/2 \), then \( n(n-1) = (3m/2)(n-1) \), which is not divisible by \( m \) when \( n-1 \) is odd. Thus we may assume that \( r > m/2 \).

**Case 1** Let \( r > m/2+1 \) and \( r > 8 \).

Recall that since \( n \) and \( m \) are even, then \( r \) is also even. In addition, since both \( n(n-1)/m \) and \( n-1 \) are odd, then for any integer \( e \), \( 2^e \) divides \( m \) if and only if \( 2^e \) divides \( n \). In particular, \( m = n \pmod{4} \), and so \( r = 0 \pmod{4} \).

Now since \( r \) is even, we may decompose \( K_r \) into \( r/2-1 \) \( r \)-cycles and a one-factor. If we add one edge of the one-factor to each \( r \)-cycle, we obtain a decomposition of \( K_r \) into \( r/2-1 \) subgraphs \( G_i \) and a single edge \( zz' \), where each \( G_i \) is an \( r \)-cycle with a chord. These subgraphs have the following useful property.

**3.2.3 Claim** Given \( 2 \leq y_i \leq r-1 \), there is a vertex \( x_i \) of \( G_i \) such that \( G_i \) may be divided into two paths \( P_i \) and \( P'_i \) of lengths \( y_i \) and \( r+1-y_i \) respectively, with a common endvertex \( x_i \). Moreover, we may choose the vertices \( x_i \) in such a way that distinct values of \( y_i \) will determine distinct vertices \( x_i \).
Proof Each $G_i$ is an $r$-cycle with a chord. Let $p$ and $q$ be the endvertices of the chord. Thus $p$ and $q$ have degree 3 in $G_i$ while all other vertices of $G_i$ have degree 2.

Now $p$ and $q$ divide the $r$-cycle of $G_i$ into two segments $S_1$ and $S_2$, where we assume $k(S_1) \leq k(S_2)$ (Fig. 3.4).

![Diagram](image)

Fig. 3.4

If $y_i \leq k(S_1)$, we let $P_i$ be the path which consists of the edge $qp$ and the first (beginning with $p$) $y_i-1$ edges of $S_1$. We let $x_i$ be the terminal vertex of $P_i$. Since $2 \leq y_i \leq k(S_1)$, $x_i$ lies on $S_1$, $x_i \neq p,q$, and for distinct values of $y_i$ with $2 \leq y_i \leq k(S_1)$, the corresponding vertices $x_i$ are distinct. Since $x_i \neq p$, the remainder $P_i'$ of $G_i$, which consists of a segment of $S_1$ and all of $S_2$, is also a path (see Fig. 3.5)

If $y_i > k(S_1)$, we let $P_i$ be the path of length $y_i$ which begins at $p$, follows $S_1$ to $q$, and continues along $S_2$; and we let $x_i$ be the terminal vertex of $P_i$. In this case,
since \( l(S_1) < y_i \leq r-1 \), \( x_i \) lies on \( S_2 \) and \( x_i \neq p, q \). Thus the remainder \( P'_i \) of \( G_i \), which consists of a segment of \( S_2 \) and the edge \( pq \), is a path. For distinct values of \( y_i \) with \( l(S_1) < y_i \leq r-1 \), the corresponding vertices \( x_i \) are distinct (see Fig. 3.5).

Notice that each vertex of degree 2 in \( G_i \) will be the vertex \( x_i \) corresponding to exactly one value of \( y_i \), \( 2 \leq y_i \leq r-1 \), and that no value of \( y_i \) gives \( x_i = p \) or \( q \). Notice also that we cannot divide \( G_i \) into paths \( P_i \) and \( P'_i \) for which \( l(P_i) < 2 \) or \( l(P'_i) < 2 \). Finally, \( 2 \leq y_i \leq r-1 \) implies \( 2 \leq r+1-y_i \leq r-1 \), so that for each \( i \), \( 2 \leq l(P_i), l(P'_i) \leq r-1 \).

In the construction, we divide each subgraph \( G_i \) into two paths, \( P_i \) and \( P'_i \), of lengths \( y_i \) and \( r+1-y_i \), respectively, as in the claim. We use the edges of one \( m \)-cycle \( C \) from \( K_{m,r} \), together with an initial segment of \( W \), to complete each of the paths \( P_i, P'_i \), \( 1 \leq i \leq r/2-1 \), to an \( m \)-path in \( K_n - D \).
To do this, we choose \( r-2 \) edge-disjoint paths \( Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_{r/2-1}, Q'_{r/2-1} \), each a segment of \( W \), so that \( Q_i \) and \( Q'_i \) will complete \( P_i \) and \( P'_i \), respectively, to a path of length \( m \), using one or more edges of \( C \) as a bridge between \( K_r \) and \( K_m \). We will use all the edges of \( C \) to do this, so that the remainder of \( K_{m,r} \) will consist of entire \( m \)-cycles. The union of the paths \( Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_{r/2-1}, Q'_{r/2-1} \) will be an initial segment of \( W \).

We will then use a second \( m \)-cycle \( C' \) from \( K_{m,r} \), together with the single edge \( zz' \) from \( K_r \) and the first \( m-1 \) edges of \( W \setminus \bigcup \{Q_i \cup Q'_i : 1 \leq i \leq r/2-1\} \), to construct two more \( m \)-paths in \( K_n - D \).

Thus we will use the edges of \( K_r \), some or possibly all of those of \( K_m - D \), and those of two \( m \)-cycles from \( K_{m,r} \), to construct \( r \) \( m \)-paths in \( K_n - D \). We must check that there are enough edges in \( E(K_r) \cup E(K_m - D) \) to construct these paths. We need

\[
|E(K_r)| + |E(K_m - D)| + 2m \geq rm.
\]

Now \( |E(K_r)| = r(r-1)/2 \), and \( |E(K_m - D)| = m(m/2 - 2) = m^2/2 - 2m \).

So we need

\[
r(r-1)/2 + m^2/2 - 2m + 2m \geq rm,
\]

or equivalently,

\[
m^2 - 2rm + r^2 - r \geq 0.
\] (1)

We are assuming that \( m > r \), and that both \( m \) and \( r \) are even. So we may let \( r = m - 2k \), for some positive integer \( k \). Then (1) becomes

\[
m^2 - 2(m - 2k)m + (m - 2k)(m - 2k - 1) \geq 0,
\]
or \( 4k^2 + 2k \geq m \).
Now by (NC3), $m|(n(n-1))$. Therefore $m|(2m-2k)(2m-2k-1)$.

Equivalently, $m|(4k^2 + 2k)$. But this implies that $4k^2 + 2k \geq m$. Therefore we have

$$|E(K_r)| + |E(K_{m-D})| + 2m \geq rm,$$

as required. So there are enough edges in $K_r \cup (K_{m-D})$ for the construction of our $r \cdot m$-paths.

To construct these $m$-paths we will need to use two particular $m$-cycles $C$ and $C'$ from $K_{m,r}$ (for example, if $v$ is the endvertex of $Q_i$ and $x$ the endvertex of $P_i$, then in order to use edges of $C$ to cross from $K_r$ to $K_m$, and so join $P_i$ to $Q_i$, we require that $x$ and $v$ lie on $C$).

By Lemma 2.1.1 we know that there is a decomposition of $K_{m,r}$ into $m$-cycles. For our construction we want the decomposition to contain two particular cycles $C$ and $C'$. So we choose two cycles from the given decomposition and relabel their vertices with those of the required cycles $C$ and $C'$. This of course induces a relabelling of the entire decomposition of $K_{m,r}$. The cycles $C$ and $C'$ are related to some extent, so we must choose the original cycles from the given decomposition of $K_{m,r}$ carefully.

In particular, we want to use $z z'$, $C'$, and the first $m-1$ edges of $W \setminus \cup \{Q_i \cup Q'_i : 1 \leq i \leq r/2-1\}$ to construct two $m$-paths. We do not know at this point the length of the initial segment $\cup \{Q_i \cup Q'_i : 1 \leq i \leq r/2-1\}$ of $W$. However the $r \cdot m$-paths which we are constructing in $K_n-D$ together cover $rm$ edges of $K_n-D$. Since $E(K_n-D) = E(K_{m-D}) \cup E(K_{m,r}) \cup E(K_r)$, and we know that $m$ divides each of $|E(K_n-D)|$, $|E(K_{m-D})|$ and $|E(K_{m,r})|$, then $m$ also divides $|E(K_r)|$. In addition, $m$ divides $|E(C \cup C')|$. Therefore, since we are using $E(K_r)$, $E(C \cup C')$ and a segment of $W$ to construct these $r \cdot m$-paths, the total number of
edges we use from $W$ is also a multiple of $m$. Since $W$ begins at 0 in the cycle $C_2$, the segment of $W$ which we use (including the first $m-1$ edges of $W \setminus (Q_i \cup Q'_i : 1 \leq i \leq \lfloor r/2 \rfloor - 1)$) consists of a collection $C_2, C_3, ..., C_j$ of $m$-cycles from $K_m$, where $j \leq \lfloor m/2 \rfloor - 1$. Therefore the $m-1$ edges of $W \setminus (Q_i \cup Q'_i : 1 \leq i \leq \lfloor r/2 \rfloor - 1)$ which we use will be the last $m-1$ edges of the cycle $C_j$. Since $C_j = (0, j, j+1, j-1, ..., j+m/2, 0)$, these $m-1$ edges will be the segment $S = (j, j+1, j-1, ..., j+m/2, 0)$ of $C_j$.

We use $S$, $C'$ and $(z, z')$ to construct two $m$-paths as follows. We label one edge of $C'$ $(j+1, z)$. The two paths are

$R_{r/2} = [S \setminus (j, j+1)] + (j+1, z) + (z, z')$,

and $R'_{r/2} = (j, j+1) + [C' \setminus (j+1, z)]$ (Fig. 3.6).

![Fig. 3.6](image)

In order for $R_{r/2}$ to be a path, we must ensure that $j$ does not lie on $C'$. This is the only restriction on the labelling of the $m$ remaining vertices of $C'$. However, requiring that $(j+1, z) \in E(C')$ and $j \notin V(C')$ will put some restriction on how we may label the other $m$-cycle, $C$. For this reason we choose $C$ and $C'$ as follows.
3.2.4 Claim In any decomposition of $K_{m,r}$ into $m$-cycles we can find two vertices $a$ and $b$ of $K_m$, and a vertex $w$ of $K_r$, such that the cycle containing the edge $aw$ does not contain $b$, and the cycle containing the edge $bw$ does not contain $a$.

Proof There are $\binom{m}{2}$ possible pairs $\{a,b\} \subseteq V(K_m)$. For each cycle $C^*$ of $K_{m,r}$, $V(C^*)$ contains $\left\lfloor \frac{m}{2} \right\rfloor$ pairs $\{a,b\}$. There are $r$ cycles in the decomposition, and so $r \cdot \frac{m}{2}$ pairs in total. But $r \cdot \frac{m}{2} = \frac{r}{4}(m/2)(m/2-1)$

$$= \frac{r}{4}(m/2-1)$$

$$< \frac{r}{4}(m/2-1/2)$$

$$= \frac{r}{4}(m(m-1))/2$$

$$= \frac{r}{4} \left\lfloor \frac{m}{2} \right\rfloor$$

Therefore at least one pair $\{a,b\} \subseteq V(K_m)$ occurs on fewer than $r/4$ cycles. Let $\{a,b\}$ be such a pair. Each cycle $C^*$ containing both $a$ and $b$ contains four edges of the form $aw_0$ or $bw_0$, for $w_0 \in V(K_r)$. Thus the set $C$ of all cycles containing both $a$ and $b$ covers at most $4(r/4 - 1) = r - 4$ edges of the form $aw_0$ or $bw_0$, for $w_0 \in V(K_r)$. So there is a vertex $w_0$ of $K_r$ for which no cycle of $C$ contains either $aw_0$ or $bw_0$ (in fact there are at least four such vertices). The three vertices $a$, $b$, and $w_0$ satisfy the claim.

Therefore we may choose $a$, $b$, and $w_0$ as in the claim, relabel $a$ with $j+1$, $b$ with $j$, and $w_0$ with $z$, and let $C'$ be the cycle containing $(j+1, z)$ but not $j$, and $C$ be the cycle containing $(j, z)$ but not $j+1$.

We now proceed to label $C$. 

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We arrange the paths $Q'_i, Q_i, 1 \leq i \leq r/2 - 1$, along $W$ so that

$$W = Q'_1 + Q_2 + Q'_3 + Q_4 + Q'_5 + \ldots + Q_i + Q_{i-1} + \ldots + Q_{r/2-2} + Q'_{r/2-3} + Q_{r/2-1} + Q'_{r/2-2} + Q'_{r/2-1} + Q_1 + S + R,$$

where $R$ is the remaining segment (if any) of $W$ which will not lie on any of the $r$ $m$-paths.

Recall that $W$ begins with the cycle $C_2$, so that $Q'_1 = (0, 2, 3, 1, \ldots)$, and that $S$ consists of the last $m-1$ edges of $C_j$, so that $S = (j, j+1, j-1, \ldots, j+m/2, 0)$. Thus $R$ consists of the $(m/2 - 1 - j)$ $m$-cycles $C_{j+1}, C_{j+2}, \ldots, C_{m/2-1}$.

Let $v_1$ be the terminal vertex of $Q'_1$ (and so also the initial vertex of $Q_2$). For $2 \leq i \leq r/2 - 2$, let $v_i$ be the terminal vertex of $Q_{i+1}$ (and the initial vertex of $Q'_i$). Let $v_{r/2-1}$ be the terminal vertex of $Q'_{r/2-1}$ (and the initial vertex of $Q_1$).

We will label $r-2$ of the vertices of $C$ with $v_1, \ldots, v_{r/2-1}$ and $x_1, \ldots, x_{r/2-1}$ so that the paths $P_i, Q_i$ and $P'_i, Q'_i$ match up as in Fig. 3.7. Thus we need the vertices $v_1, \ldots, v_{r/2-1}$ to be distinct, and the corresponding vertices $x_1, \ldots, x_{r/2-1}$ to be distinct. Now from Claim 3.2.3, we may choose $l(P_i), l(P'_i) \in \{2, 3, \ldots, r-1\}$, where we require of course that $l(P_i) + l(P'_i) = r+1$, and distinct values of $l(P_i)$ will give us distinct vertices $x_i$ to label on $C$. Similarly the vertices $v_1, \ldots, v_{r/2-1}$ will be determined by our choices of $l(Q_1), l(Q'_1), \ldots, l(Q_{r/2-1}), l(Q'_{r/2-1})$. For $i \neq 2$, we will use precisely one edge of $C$ to join $P_i$ and $P'_i$ to $Q_i$ and $Q'_i$, respectively. Thus for $i \neq 2$

$$l(Q_i) = m - 1 - l(P_i),$$

and

$$l(Q'_i) = m - 1 - l(P'_i),$$

so

$$m - r \leq l(Q_i), l(Q'_i) \leq m - 3.$$
Therefore by Claim 3.2.2, for any choice of \( l(P_i) \) from \( \{2, 3, \ldots, r-1\} \), \( Q_i \) and \( Q'_i \) will be segments in \( W \) and hence paths in \( K_m \). Our procedure is to choose the lengths \( l(P_i) \), \( 1 \leq i \leq r/2-1 \), so that the resulting vertices \( v_1, \ldots, v_{r/2-1} \) and \( x_1, \ldots, x_{r/2-1} \) are distinct. In addition, recall that for the construction of the two \( m \)-paths \( R_{r/2} \) and \( R'_{r/2} \) we require \( (j,z) \in E(C) \) and \( j+1 \in V(C) \), as in Claim 3.2.4.

First let us look at what happens when \( i = 2 \).

When \( i = 2 \) we have a segment \( T \) of \( C \), of length \( m - r + 3 \), joining \( x_2 \) to \( v_2 \) (and consequently joining \( P'_2 \) to \( Q'_2 \) ). In order for \( P'_2 + T + Q'_2 \) to be a path we must ensure that the internal vertices of \( T \) lie on neither \( P'_2 \) nor \( Q'_2 \). So having labelled \( x_1, \ldots, x_{r/2-1} \) and \( v_1, \ldots, v_{r/2-1} \), we will need to be able to label the remaining \( m - r + 2 \) vertices of \( C \) (which are exactly the internal vertices of \( T \)) with vertices which do not lie on \( P'_2 \) or \( Q'_2 \). To do this we will need at least \( (m - r + 2)/2 \) vertices of \( K_m \) which do not belong to \( \{v_1, \ldots, v_{r/2-1}\} \cup V(Q'_2) \), and at least \( (m - r + 2)/2 \) vertices of \( K_r \) which do not belong to \( \{x_1, \ldots, x_{r/2-1}\} \cup V(P'_2) \). Therefore we need

\[
m - |\{v_1, \ldots, v_{r/2-1}\} \cup V(Q'_2)| \geq (m - r + 2)/2, \quad \text{and} \\
r - |\{x_1, \ldots, x_{r/2-1}\} \cup V(P'_2)| \geq (m - r + 2)/2.
\]

Since \( v_2 \in V(Q'_2) \) and \( x_2 \in V(P'_2) \) and \( r > 4 \), these conditions reduce to

\[
l(Q'_2) \leq m/2 \quad \text{and} \quad l(P'_2) \leq r - m/2.
\]

We cannot satisfy these unless \( r - m/2 \geq 2 \), since from Claim 3.2.3 we need \( l(P'_2) \geq 2 \). It is for this reason that we made this assumption at the outset.

Since \( r \geq m/2+2 \), we may set \( l(P'_2) = r - m/2 \) and consequently \( l(Q'_2) = m/2 - 3 \) (clearly \( m/2 - 3 \leq m - 3 \), so \( Q'_2 \) will indeed be a path). This will give us enough
freedom in the labelling of the last \( m - r + 2 \) vertices of \( C \) to ensure that 

\[ P_2' + T + Q_2' \] will be a path.

We now begin the labelling of \( C \). We first choose \( v_1 \). Notice that we have arranged the paths \( Q_i, Q_i' \), \( 1 \leq i \leq r/2 - 1 \), along \( W \) in such a way that the length of the segment between \( v_1 \) and \( v_{r/2-1} \) is precisely

\[ |l(Q_2) + l(Q_2')| + |l(Q_3) + l(Q_3')| + \ldots + |l(Q_{r/2-1}) + l(Q'_{r/2-1})|. \]

Since \( l(Q_i) + l(Q'_i) = \begin{cases} 2m - r - 3, & i \neq 2 \\ m - 5, & i = 2, \end{cases} \)

the length of this segment depends only upon \( m \) and \( r \). Therefore the choice of \( v_1 \) will uniquely determine \( v_{r/2-1} \). In addition, the length of the segment of \( C_{j-1} \cup C_j \) between \( v_{r/2-1} \) and \( j \) (travelling in the direction specified for \( W \)) is precisely \( l(Q_1) \).

Therefore \( v_{r/2-1} \) will in turn determine \( l(Q_1) \).

Since we must ensure that \( (j, z) \) is an edge of \( C \), we will show how to choose \( l(Q_1') \) so that \( v_1 = j \), and \( v_{r/2-1} \neq j, j+1 \) (this is because \( j+1 \) may not lie on \( C \), and \( v_1 \) must be different from \( v_{r/2-1} \)). We will then choose \( G_1 \) so that \( x_1 = z \).

Now since \( C_j \) is the last cycle from \( W \) used in constructing these \( m \)-paths, we use in total \( j-1 \) cycles \( C_i \) from \( W \) (recall that \( C_1 \) is contained in \( D \)). We construct \( r \) \( m \)-paths in total, and so

\[ |E(K_r)| + |E(C \cup C')| + (j-1)m = rm. \]

Thus \( r(r-1)/2 + 2m + (j-1)m = rm \),

so \( j = r - 1 - r(r - 1)/2m \). \hspace{1cm} (2)

We want to set \( v_1 = j \), so that \( j \in V(C) \). Since \( v_1 \) determines \( v_{r/2-1} \), we must make sure that setting \( v_1 = j \) does not force \( v_{r/2-1} = j \) or \( j+1 \). Consider \( C_2 \), the
first cycle from $W$ we use in constructing the $m$-paths. Clearly, $Q_1'$ will be a segment of $C_2$, beginning $(0, 2, 3, 1, \ldots)$ (see Fig. 3.8)

The (forward) path from 0 to $j$ in $C_2$ has length $2(j - 2)$, provided that $2 < j < m/2 + 2$. Now $j$ is the index of one of the cycles in the decomposition of $K_m$, and so $j \leq m/2 - 1 < m/2 + 2$. Secondly, by (2) $j > 2$ if and only if $r - 1 - r(r - 1)/2m > 2$, or equivalently, $m > r(r - 1)/(2r - 6)$. Now since we are assuming $r > 8$, we have $r - 1 < 2r - 6$ and so $(r - 1)/(2r - 6) < 1$. Therefore $r[(r - 1)/(2r - 6)] < r < m$, and so $j > 2$ as required. So setting $l(Q_1') = 2(j - 2)$ we will have $v_1 = j$.

![Fig. 3.8](image)

First, we must check that this will not force $v_{r/2-1} = j, j+1$. Now the segment $(C_1 + \ldots + C_j) \setminus S$ of $W$ ends at the vertex $j$ (more precisely, at the edge $(0, j)$) of $C_j$. Therefore, if $l(Q_1) \leq m - 3$ we will have $v_{r/2-1} \neq j$ (by Claim 3.2.2). In addition, $j+1$ is the second vertex of $S$ (recall that $S$ is the segment...
(j, j+1, j-1, ..., 0) of Cj). So again by Claim 3.2.2, if \( l(Q_1) \leq m - 4 \) we will have \( v_{r/2-1} \neq j+1 \).

We also need \( m - r \leq l(Q'_1) \leq m - 3 \). Combining this with the above requirement that \( l(Q_1) \leq m - 4 \) and the fact that \( l(Q_1) + l(Q'_1) = 2m - r - 3 \), we need \( m - r < l(Q'_1) \leq m - 3 \). First, since \( j \leq m/2 - 1 \), then \( l(Q'_1) = 2(j - 2) \leq m - 6 < m - 3 \). Second, we need \( m - r < 2(j - 2) = 2j - 4 \), or equivalently, \( j > (m - r + 4)/2 \). So by (2) we need \( r - 1 - r(r - 1)/2m > (m - r + 4)/2 \), or equivalently,

\[-m^2 + (3r - 6)m - r(r - 1) > 0.\]

Let \( F(m) = -m^2 + (3r - 6)m - r(r - 1) \). The roots of \( F \) are

\[ r_1 = (3r - 6)/2 - \sqrt{5r^2/4 - 8r + 9}, \]

and

\[ r_2 = (3r - 6)/2 + \sqrt{5r^2/4 - 8r + 9}, \]

and \( F(m) > 0 \) whenever \( r_1 < m < r_2 \). We are assuming that \( r \geq m/2 + 1 \) and \( r \leq m - 2 \). Therefore \( r + 2 \leq m \leq 2r - 2 \). It is straightforward to check that, if \( r > 8 \), then \( r_1 < r + 2 < 2r - 2 < r_2 \). Therefore, since \( r > 8 \), we have \( F(m) > 0 \) whenever \( r + 2 \leq m \leq 2r - 2 \). This gives \( 2(j - 2) > m - r \) as required, and so we may set \( l(Q'_1) = 2(j - 2) \). We have \( m - r \leq l(Q'_1) \leq m - 3 \), and \( v_{r/2-1} \neq j, j+1 \).

Finally we want to set \( x_1 = z \). We have a decomposition of \( K_r \) into \( r \)-cycles and a one-factor, and we want to add one edge of the one-factor to each \( r \)-cycle to obtain a decomposition of \( K_r \) into the \( r/2-1 \) subgraphs \( G_i \) and a single edge \( zz' \). First, choose any \( r \)-cycle and any edge of the one-factor, and let \( G_1 \) be the union of the chosen \( r \)-cycle with the chosen edge. Since we have set \( l(Q'_1) = 2(j - 2) \), we know \( l(P_1) \) and \( l(P'_1) \). Moreover, \( m - r < l(Q'_1) < m - 3 \) and

\[ 40 \]
l(Q_1) + l(Q'_1) = 2m - r - 3 \implies 2 < l(P_1), l(P'_1) < r - 1. As in Claim 3.2.3, the value of $l(P_1)$ will uniquely determine the vertex $x_1$. Also by Claim 3.2.3, the vertex $x_1$ has degree 2 in $G_1$, that is, $x_1$ is not an endvertex of the chord of $G_1$. So the edge $e$ of the one-factor $F$ which contains $x_1$ is not the chord of $G_1$. Thus we may set $z = x_1$, and let $z'$ be the other endvertex of $e$. In other words, we choose $e$ to be the single edge of $K_r$ not contained in any $G_i$.

We may now construct the subgraphs $G_2, \ldots, G_{r/2-1}$, adding each remaining edge of the one-factor to one of the $r$-cycles. All we require is that each $G_i$ be an $r$-cycle with a chord, where the chord is not $e$ (and of course that $E(G_1), \ldots, E(G_{r/2-1})$ and $e$ partition $E(K_r)$).

We have now labelled the vertices $v_1, v_{r/2-1}$ and $x_1$ on $C$. For the labelling of $v_2$ and $x_2$, recall that we have set $l(P'_2) = r - m/2$ and $l(Q'_2) = m/2 - 3$, which gives $l(P_2) = m/2 + 1$ and $l(Q_2) = m/2 - 2$. Now $l(P_2)$ will determine the vertex $x_2$ of $G_2$. Suppose $x_2 = x_1 = z$. Since $l(P_2) + l(P'_2) = r + 1$ is odd, $l(P_2) \neq l(P'_2)$. So we may interchange $P_2$ and $P'_2$ so that $l(P'_2) = m/2 + 1$ and $l(P_2) = r - m/2$. This will give us a different vertex $x_2$ so that $x_2 \neq x_1$. The two $m$-paths we construct will now be $P'_2 + x_2v_1 + Q_2$ and $P_2 + T + Q'_2$. Thus we may assume $x_2 \neq x_1$.

Notice that the vertex $v_2$ will depend on $l(Q_3)$ (see Fig. 3.7). Since $m - r \leq l(Q_3) \leq m - 3$, there are $r - 2$ choices for $l(Q_3)$. Each of these $r - 2$ values of $l(Q_3)$ determines a vertex $v_2$, and these vertices are consecutive vertices of $W$. Since $r - 2 < m - 2$, then by Claim 3.2.2 they are all different. Thus we have $r - 2$ different choices for $v_2$. Now we must choose $l(Q_3)$ so that $v_2 \neq v_1, v_{r/2-1}, j+1$. In addition, each choice of $l(Q_3)$ will determine a
corresponding vertex $x_3$ of $G_3$. We must choose $l(Q_3)$ so that $x_3 \neq x_2, x_1$.

Therefore in total we might have to exclude five of the possible values of $l(Q_3)$. But we are assuming $r > 8$, so $r - 2 > 6$, and so we may certainly choose $l(Q_3)$ so that both $v_2 \neq v_1, v_{r/2-1}, j+1$ and $x_3 \neq x_2, x_1$.

Notice that for $2 \leq i \leq r/2 - 2$, once we have chosen $v_1, \ldots, v_{i-1}, v_{r/2-1}$, the vertices $x_1, \ldots, x_i$ are all fixed, and the choice of $v_i$ (equivalently the choice of $l(Q_{i+1})$) will determine $x_{i+1}$.

Assume that we have chosen distinct vertices $v_1, \ldots, v_{i-1}, v_{r/2-1}$ and that the resulting vertices $x_1, \ldots, x_i$ are all different, where $3 \leq i \leq r/2 - 2$. We now choose $v_i$. The value of $l(Q_{i-1})$ is fixed. Thus the vertex $v_i$ depends on $l(Q_{i+1})$.

Again $m - r \leq l(Q_{i+1}) \leq m - 3$, giving us $r - 2$ choices of $l(Q_{i+1})$, and consequently $r - 2$ distinct choices of $v_i$. Of these, we will have to exclude at most $i + 1$ to ensure that $v_i \notin \{v_1, \ldots, v_{i-1}, v_{r/2-1}, j+1\}$, and at most $i$ others might result in $x_{i+1} \in \{x_1, \ldots, x_i\}$. We therefore have at least $r - 2 - (i + 1) - i = r - 2i - 3$ valid choices for $v_i$. But $i \leq r/2 - 2$, so $r - 2i - 3 \geq 1$. Thus there is at least one valid choice of $v_i$ and hence $x_{i+1}$.

Once $v_{r/2-2}$ has been chosen, we have labelled $v_1, \ldots, v_{r/2-1}$ and $x_1, \ldots, x_{r/2-1}$. We must now label the remaining $m - r + 2$ vertices of $C$ (recall that these are the internal vertices of the segment $T$ of $C$).

We have $(m - r + 2)/2$ vertices of $C$ to label in $K_r$. None of these may be labelled with vertices from $\{x_1, \ldots, x_{r/2-1}\} \cup V(P_2')$. Since $x_2$ lies on $P_2'$, this leaves us with at least $r - (r/2 - 1 + r - m/2) = (m - r + 2)/2$ available labels, which is just enough. Similarly we have $(m - r + 2)/2$ vertices of $C$ to label in $K_m$. 

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In this case we may not use vertices from \( \{v_1, \ldots, v_{r/2-1}, j+1\} \cup V(Q'_2) \). We have \( v_2 \in V(Q'_2) \), and so there are at least
\[
m - (r/2 - 1 + 1 + m/2 - 3) = (m - r + 6)/2 > (m - r + 2)/2 \quad \text{available labels.}
\]

We have now labelled the cycle \( C \). Since we also labelled \( z \) and \( j+1 \) on \( C' \), we have in total labelled \( m/2 + 1 \) vertices of \( K_m \) and \( m/2 \) vertices of \( K_r \) (in the decomposition of \( K_{m,r} \)). We may label the remaining vertices of \( K_{m,r} \) arbitrarily.

The construction is now complete. We have constructed the following \( r \) \( m \)-paths in \( K_n - D \):
\[
R_1 = P_1 + (x_1, v_{r/2-1}) + Q_1,
R_2 = P_2 + (x_2, v_1) + Q_2,
\vdots
R_{r/2-1} = P_{r/2-1} + (x_{r/2-1}, v_{r/2-2}) + Q_{r/2-1},
R_1' = P_1' + (x_1, v_1) + Q_1',
R_2' = P_2' + T + Q_2',
R_3' = P_3' + (x_3, v_3) + Q_3',
\vdots
R_{r/2-1}' = P_{r/2-1}' + (x_{r/2-1}, v_{r/2-2}) + Q_{r/2-1}',
R_{r/2} = (S \setminus (j, j+1)) + (j+1, z) + (z, z'),
\]
and \( R_{r/2}' = (j, j+1) + (C' \setminus (j+1, z)) \).

These paths cover all the edges of \( K_r \), \( 2m \) edges of \( K_{m,r} \), and an initial segment (or possibly all) of \( W \). Now the remaining edges of \( K_{m,r} \) are partitioned into \( m \)-cycles. In addition, the 'unused' portion \( R \) of \( W \) consists of the \((m/2 - 1 - j) \) \( m \)-cycles \( C_{j+1}, \ldots, C_{m/2-1} \). Thus we have a decomposition of \( K_n - D \) into \( m \)-paths and \( m \)-cycles. \( \blacksquare \)
Case 2 Let $n$ be even, and let $r = m/2 + 1$, where $r > 8$. As in Case 1, $m$ and $r$ are even, and $r \equiv 0 \pmod{4}$.

The problem with the previous construction, when $r = m/2 + 1$, was that we might not be able to label the internal vertices of $T$ so that $R'_2 = P'_2 + T + Q'_2$ would be a path. In this case we set $l(P'_2) = r - m/2 + 1 = 2$ and we choose $G_2$ so that $x_1$ is one endvertex of its chord (so of course we can no longer set $x_1 = z$).

By Claim 3.2.3, the vertex $x_2$ will be neither endvertex of this chord (in particular, we will have $x_2 \neq x_1$). Moreover, each endvertex of the chord lies on both $P_2$ and $P'_2$, so this will force $x_1 \in V(P'_2)$. This will leave us, when we come to label the internal vertices of $T$, with

$$r - |\{x_1, \ldots, x_{r/2-1}\} \cup V(P'_2)| \geq r - [(r/2 - 1) + (r - m/2 + 1) - 1] = (m - r + 2)/2$$

available vertices to use as labels.

In order to let $x_1$ be an endvertex of the chord in $G_2$, we must first ensure that $x_1 \neq z$, since the edge $(z, z')$ of the one-factor lies on no $G_i$. In addition, $x_1$ must not be an endvertex of the chord in $G_1$. (But this is guaranteed by Claim 3.2.3.)

Since we now require $x_1 \neq z$ (whereas before we set $x_1 = z$), we cannot label the edge $v_1x_1$ of $C$ as $jz$. However, since $r = m/2 + 1$, and since each vertex of $K_r$ lies on $m/2$ cycles in the decomposition, then each vertex of $K_r$ in fact lies on $(r-1)$ of the $r$ $m$-cycles in the decomposition. Now if we choose the vertices $a$, $b$ and $z$ of $K_n$ as in Claim 3.2.4, we have at most $(r/4 - 1)$ cycles containing both $a$ and $b$. Since each vertex of $K_m$ lies in total on $r/2$ of the cycles in the $m$-cycle decomposition of $K_{m,r}$, there are then at least $(r/4 + 1)$ cycles containing $b$ and
not containing $a$. Since at most one of these cycles does not contain $z$, we have, in addition to the cycle containing the edge $bz$ but not $a$, at least $(r/4 + 1) - 2 \geq 1$ cycles containing both $b$ and $z$ (but not $a$) on which $b$ and $z$ are non-adjacent. Consequently it does not matter whether or not we label $j$ and $z$ as adjacent on $C$.

The modified construction is as follows. As before we let $l(Q_1) = 2(j - 2)$, so that $v_1 = j$ while $v_{r/2-1} \neq j, j + 1$. This is still valid since we are again assuming that $r > 8$. For $G_1$ we choose any $r$-cycle and any edge of the one-factor from the decomposition of $K_r$, and we determine the resulting vertex $x_1$.

For $G_2$ we choose any remaining $r$-cycle together with the edge of the one-factor containing $x_1$. We set $l(P_2) = r - m/2 + 1 = 2$, which determines the vertex $x_2$. For $z$, we then choose any vertex of $K_r$ different from $x_1$ and $x_2$ and the endvertices of the chords in $G_1$ and $G_2$ (recall that one of these is $x_1$). Notice that this will guarantee $z \in V(P_{2})$, since, letting $(x^*, x_1)$ be the chord in $G_2$, we have $V(P_{2}) = \{x_2, x_1, x^*\}$ [see Fig. 3.9]. Thus there are $r - 5$ choices for $z$, and since $r > 8$ we have $r - 5 > 3$. Our choice of $z$ determines the edge $(z, z')$ of the one-factor which will lie on no subgraph $G_i$.

We now proceed to construct $G_3, ..., G_{r/2-1}$ arbitrarily (of course not using $(z, z')$ on any $G_i$) and to label the vertices $\{v_2, ..., v_{r/2-1}\}$ and $\{x_3, ..., x_{r/2-1}\}$ exactly as before.

It now remains to label the internal vertices of $T$. For those which lie in $K_m$, since $l(Q_2') = m - l(T) - l(P_2') = m/2 - 4$, we have at least

$$m - l(\{v_1, ..., v_{r/2-1}\} \cup \{j+1\} \cup V(Q_2')) \geq m - (r/2 + m/2 - 4)$$

$$= (m - r + 8)/2$$

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available vertices to use as labels. For those which lie in $K_r$, as in the previous argument we have at least $(m - r + 2)/2$ available vertices to use as labels. If $z \notin \{x_1, ..., x_{r/2-1}\}$ then we label one vertex of $T$ with $z$, and the others arbitrarily (notice that $(m - r + 2)/2 > 0$, so there is a vertex of $T$ which we may label $z$, and recall that $z \notin V(P'_2)$). If $z \in \{x_1, ..., x_{r/2-1}\}$, then we may label all $(m - r + 2)/2$ of these vertices arbitrarily, as before.

Finally, we construct the paths $R_{r/2}$ and $R'_{r/2}$ exactly as before, using the cycle $C'$ containing the edge $(j+1, z)$ (and not containing $j$), the edge $(z, z')$, and the last $m - 1$ edges of $C_j$.

This gives us, as before, a decomposition of $K_n - D$ into the $r$ $m$-paths $R_i, R'_i$, $1 \leq i \leq r/2$, the remaining $(r - 2)$ $m$-cycles of $K_{m,r}$ and, if $j < m/2 - 1$, the $(m/2 - 1 - j)$ cycles $C_{j+1}, ..., C_{m/2-1}$ from $K_m$.

Case 3 Let $n$ be even and let $r \leq 8$. We have either $r = 4$ or $r = 8$, since $r \equiv 0 \pmod{4}$.
(i) Let \( r = 4 \). Then \( n = m + 4 \); so (NC3) imply \( m \mid (m + 4)(m + 3) \), or equivalently, \( m \mid (m^2 + 7m + 12) \). Since in addition we assume \( n(n-1)/m \) is odd, we have \( m \mid 12 \), where \( 12/m \) is even and \( m \) is even. This implies \( m = 6 \), so \( n = 10 \).

We give a decomposition of \( K_{10} - D \) into 6-paths and 6-cycles. We let

\[ K_{10} = K_6 \cup K_{6,4} \cup K_4 \]  
where \( V(K_6) = \{u_0, u_1, ..., u_5\} \) and

\[ V(K_4) = \{w_0, w_1, w_2, w_3\}. \]

We have

\[ D = C_1 \cup F = (u_0, u_1, u_2, u_5, u_3, u_4, u_0) \cup \{(u_5, u_1), (u_4, u_2), (u_3, u_0)\} \]  
and \[ W = C_2 = (u_0, u_2, u_3, u_1, u_4, u_5, u_0). \]

We decompose \( K_{6,4} \) into the following 6-cycles:

\[ A = (u_0, w_0, u_2, w_2, u_1, w_1, u_0), \]

\[ B = (u_1, w_0, u_4, w_1, u_2, w_3, u_1), \]

\[ C = (u_3, w_0, u_5, w_2, u_4, w_3, u_3), \]

and \( E = (u_0, w_2, u_3, w_1, u_5, w_3, u_0). \)

We decompose \( K_4 \) into the 3-paths \( (w_3, w_0, w_2, w_1) \) and \( (w_0, w_1, w_3, w_2) \).

We use these 3-paths together with the 6-cycles \( A \), \( B \), and \( C_2 \) to form the following 6-paths [see Fig. 3.10]:

\[ R_1 = (w_3, w_0, u_0, w_1, u_1, w_2, u_2), \]

\[ R_2 = (w_1, w_2, w_0, u_2, u_3, u_1, u_4), \]

\[ R_3 = (w_2, w_3, u_1, w_0, u_4, w_1, u_2), \]

and \( R_4 = (w_0, w_1, w_3, u_2, u_0, u_5, u_4). \)

We have decomposed \( K_{10} - D \) into the 6-paths \( R_1 \), \( R_2 \), \( R_3 \), and \( R_4 \), and the 6-cycles \( C \) and \( E \).
(ii) Let $r = 8$. Then $n = m + 8$, so (NC3) imply that $m(m + 8)(m + 7)$, or equivalently, $m(m^2 + 15m + 56)$. Since we assume $n(n - 1)/m$ is odd, we have $m|56$, where $56/m$ is even and $m$ is even. Finally, $r < m$ implies $m > 8$, and $r \geq m/2 + 1$ implies $m \leq 14$. Thus the only case to consider is $m = 14$ (and so $n = 22$).
We give a decomposition of $K_{22} - D$ into 14-paths and 14-cycles. We let

$$K_{22} = K_{14} \cup K_{14,8} \cup K_8,$$

where $V(K_{14}) = \{u_0, \ldots, u_{13}\}$ and $V(K_8) = \{w_0, \ldots, w_7\}$. We have $K_{14} = F \cup \bigcup\{C_i : 1 \leq i \leq 6\}$, so

$$D = C_1 \cup F$$

$$= (u_0, u_1, u_2, u_{13}, \ldots, u_7, u_8, u_0) \cup \{u_{13}u_1, u_{12}u_2, \ldots, u_6u_6, u_0u_7\},$$

and $W = C_2 + C_3 + \ldots + C_6$.

We decompose $K_8$ into the following four 7-paths:

$$G_1 = (w_0, w_1, w_7, w_2, w_6, w_3, w_5, w_4),$$

$$G_2 = (w_1, w_2, w_0, w_3, w_7, w_4, w_6, w_5),$$

$$G_3 = (w_2, w_3, w_1, w_4, w_0, w_5, w_7, w_6),$$

and $G_4 = (w_3, w_4, w_2, w_5, w_1, w_6, w_0, w_7)$.

By Lemma 2.1.1, there is a decomposition of $K_{14,8}$ into 14-cycles. Let $C$ be any cycle from this decomposition. As in the construction in Case 1, we will relabel the vertices of $C$ (and so the entire decomposition of $K_{14,8}$) so that we can use the edges of $C$ as a bridge between $K_{14}$ and $K_8$. We will divide each of the 7-paths $G_i$ into two paths $P_i$ and $P'_i$. If $i \neq 2$, we will set $l(P_i) = 3$ and $l(P'_i) = 4$, and we will extend each of $P_i$ and $P'_i$ to a 14-path in $K_{22} - D$ using one edge of $C$ and a segment of $W$. We will set $l(P_2) = 6$ and $l(P'_2) = 1$. To extend $P_2$ to a 14-path, we use one edge of $C$ and a segment of $W$ (of length 7). To extend $P'_2$ to a 14-path, we use the remaining 7 edges of $C$ (which we again call $T$) and a segment of $W$ (of length 6). As before, we let $Q_i$ and $Q'_i$ be the segments of $W$ which we use to extend $P_i$ and $P'_i$, respectively, and we arrange these segments along $W$ so that $W = Q'_1 + Q_2 + Q_3 + Q'_4 + Q_4 + Q'_3 + Q_3 + Q'_1$. We define $v_1, \ldots, v_4$ and $x_1, \ldots, x_4$ as before.
Here, we want to construct eight 14-paths from the edges of $K_8$, $K_{14} - D$, and one 14-cycle. We have $|E(K_{14} - D)| + 14 + |E(K_8)| = 14(5) + 14 + 28 = 14(8)$.

So our construction will use all the edges of $W$.

We relabel $C$ so that

$v_1, v_2, v_3, v_4 = u_{11}, u_9, u_0, u_8,$

$x_1, x_2, x_3, x_4 = w_2, w_6, w_4, w_5,$

and $T = (w_6, u_1, w_0, u_2, w_1, u_7, w_3, u_9)$.

The remaining vertices of $K_{14,8}$ may be relabelled arbitrarily.

Fig. 3.11
The eight 14-paths we construct are the following (see Fig. 3.11):

\[ R_1 = P_1 + x_1v_4 + Q_1 \]
\[ = (w_0, w_1, w_7, w_2, u_8, u_4, u_9, u_3, u_{10}, u_2, u_{11}, u_1, u_{12}, u_{13}, u_0), \]

\[ R'_1 = P'_1 + x_1v_1 + Q'_1 \]
\[ = (w_4, w_5, w_3, w_6, w_2, u_{11}, u_6, u_{12}, u_5, u_{13}, u_4, u_1, u_3, u_2, u_0), \]

\[ R_2 = P_2 + x_2v_1 + Q_2 \]
\[ = (w_1, w_2, w_0, w_3, w_7, w_4, w_6, u_{11}, u_7, u_{10}, u_8, u_9, u_0, u_3, u_4), \]

\[ R'_2 = P'_2 + T + Q'_2 \]
\[ = (w_5, w_6, u_1, w_0, u_2, w_1, u_7, w_3, u_9, u_{10}, u_0, u_4, u_5, u_3, u_6), \]

\[ R_3 = P_3 + x_3v_2 + Q_3 \]
\[ = (w_2, w_3, w_1, w_4, u_9, u_{11}, u_8, u_{12}, u_7, u_{13}, u_6, u_1, u_5, u_2, u_4), \]

\[ R'_3 = P'_3 + x_3v_3 + Q'_3 \]
\[ = (w_6, w_7, w_5, w_0, w_4, u_0, u_5, u_6, u_4, u_7, u_3, u_8, u_2, u_9, u_1), \]

\[ R_4 = P_4 + x_4v_3 + Q_4 \]
\[ = (w_3, w_4, w_2, w_5, u_0, u_{11}, u_{10}, u_{12}, u_9, u_{13}, u_8, u_1, u_7, u_2, u_6) \]

and \[ R'_4 = P'_4 + x_4v_4 + Q'_4 \]
\[ = (w_7, w_0, w_6, w_1, w_5, u_8, u_5, u_7, u_6, u_0, u_{12}, u_{11}, u_{13}, u_{10}, u_1). \]

These eight 14-paths together cover all the edges of \( K_{14-D} \), all the edges of \( K_{8} \), and the edges of one 14-cycle from the decomposition of \( K_{14,8} \). Since the remainder of \( K_{14,8} \) consists of 14-cycles (now relabelled), we have a decomposition of \( K_{22-D} \) into 14-paths and 14-cycles. By Corollary 2.3.5 and Lemma 3.2.1, this yields a decomposition of \( K_{44-I} \) into 28-cycles.

This completes the proof of Lemma 3.2.2.
Thus we have a decomposition of $K_{2n} - I$ into $2m$-cycles if $n$ and $m$ satisfy the conditions (*) on page 20. The remaining cases are those for which the quotient $n(n-1)/m$ is odd, $m < n < 2m$, and either $n$ is odd or $n < 3m/2$. We hope that a construction similar to the above will give results in some or all of these cases, particularly the case when $n$ is odd and at least $3m/2$. 
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