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Isomorphism and Layout of Spiral Polygons

by

Glenn MacDonald
B.Sc.(Joint Honours, First Class)
Simon Fraser University 1990

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Computing Science

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APPROVAL

Name: Glenn MacDonald
Degree: Master of Science
Title of thesis: Isomorphism and Layout of Spiral Polygons

Examining Committee: Pavol Hell
Professor of Computing Science
Chair

Thomas Shermer
Senior Supervisor
Assistant Professor of Computing Science

Binay Bhattacharya
Professor of Computing Science

Arvind Gupta
Assistant Professor of Computing Science

Luis Goddyn
External Examiner
Assistant Professor of Mathematics

Date Approved: 13 April 1993
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Isomorphism and Layout of Spiral Polygons.

Author:

(signature)
Glenn MacDonald
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Abstract

Let $P$ be a polygon having $n$ vertices, exactly $r$ of which are reflex. Then $P$ is called spiral if its vertices may be labelled so that, counterclockwise around the boundary, $u_1, u_2, \ldots, u_r$ are all reflex and occur consecutively and $v_1, v_2, \ldots, v_{n-r}$ are all convex and also occur consecutively. If we consider $P$ to be the union of the actual polygon boundary and the interior region, then in $P$, a point $b$ is said to be visible to a point $a$ if the line segment $\overline{ab}$ does not intersect the exterior of $P$. Two polygons are considered to be isomorphic if there is a one-to-one mapping between their points that preserves visibility. This thesis establishes necessary and sufficient conditions for two spiral polygons to be isomorphic and gives an $O(n^2)$ detection algorithm. Also, it is shown that there exists an $O(r \log r)$ bit canonical representation for the visibility structure of a spiral polygon.

This thesis also investigates the layout of spiral polygons. If $P$ is a spiral having its vertices numbered as described above, then let $l_0$ and $l_{r+1}$ be the lines orthogonal to $\overrightarrow{u_0u_{r+1}}$ that pass through $u_0$ and $u_{r+1}$, respectively. Then $P$ is called a banana spiral if none of the three lines $\overrightarrow{u_0u_{r+1}}$, $l_0$ and $l_{r+1}$ intersect the interior of $P$. This thesis constructively shows that every canonical representation may be laid out as a banana spiral.

An orthogonal polygon is one whose edges are all either horizontal or vertical. This thesis establishes necessary and sufficient conditions for a canonical representation to be realized as an orthogonal spiral polygon and gives an inductive construction algorithm.
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First I want to thank Tom Shermer for his many ideas and invaluable assistance throughout my years as a graduate student. The second and third chapters of this thesis, "Isomorphism of Spiral Polygons" and "Canonical Representations", are joint results of Tom and myself for which he deserves the majority of the credit.

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Dedication

This thesis is dedicated to Tammy, Garvin and Shirley for their support and understanding (well, most of the time at least...) throughout my education.
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Chapter 1

Introduction

Computational geometry involves the study of the complexity of geometric problems and the analysis of algorithms on geometric objects. One subject that has received a great deal of attention is the area of visibility problems. We consider the concept of visibility within a simple polygon.

We define a polygon as an ordered sequence of at least three points \(v_1, v_2, \ldots, v_n\) in the plane, called vertices, and the \(n\) line segments \(\overline{v_1v_2}, \overline{v_2v_3}, \ldots, \overline{v_{n-1}v_n}, \overline{v_nv_1}\). A polygon is called simple if no two nonconsecutive edges intersect. A simple polygon is a Jordan curve and thus, divides the plane into three regions: the polygon itself, the interior region and the exterior region. Henceforth we will use the term polygon to mean a simple polygon and its interior. A vertex of \(P\) is called reflex if its interior angle exceeds \(180^\circ\), and convex otherwise.

Two points \(x\) and \(y\) in \(P\) are said to be visible if the line segment \(\overline{xy}\) is entirely contained in \(P\). We will address the problem of polygon isomorphism under this visibility relation; two polygons will be considered isomorphic if there exists a one to one mapping between their points that preserves visibility. First we will need to define our terminology and discuss some of the earlier work that has led to this thesis.

The visibility polygon of a point \(x \in P\), denoted \(VP(x)\), is the set of all points of \(P\) that are visible to \(x\), i.e. \(VP(x) = \{y \mid \overline{xy} \subset P\}\) (See Figure 1.1a). Note that \(VP(x)\) may fail to be a polygon (See Figure 1.1b). The kernel of a polygon \(P\), denoted \(K(P)\), is the (possibly empty) set of points that have all of \(P\) as their visibility polygon, i.e.
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Figure 1.1: a. The visibility polygon of $x$. b. The visibility polygon of $v$ is not a polygon.

$K(P) = \{x \in P \mid VP(x) = P\}$. If the kernel of $P$ is not empty, then $P$ is called starshaped.

We may generalize the notion of a kernel and consider sets of points of $P$ that have the same visibility polygon. A semikernel in $P$ is a maximal set of points that all have the same visibility polygon. Thus, if the kernel of $P$ is non-empty, it is a semikernel. When a semikernel consists of exactly one point, we call it trivial, and when it consists of an infinite number of points, we call it nontrivial. These are the only two possibilities.

A collection of subsets of $P$ is said to cover $P$ if their union is exactly $P$; $P$ is said to be covered by the collection. Most of the recent interest in visibility problems was sparked by the Art Gallery Problem: finding a minimum-cardinality set of points of $P$ whose visibility polygons cover $P$[16]. Chvátal[3] showed that the cardinality of such a set will never exceed $\lceil \frac{2n}{3} \rceil$, and this result has become known as the Art Gallery Theorem. The set of points of $P$ whose visibility polygons cover $P$ are called a guard set and they are said to guard $P$.

Many variations of the original art gallery problem have been studied, either changing the notion of the visibility, the properties of the guard or both. These variations were started by Toussaint in 1981 when he asked how the art gallery theorem would
change if the guards were now permitted to move along individual edges of the polygon and the portion of $P$ seen by a given guard would be the union of the visibility polygons of every point on the edge that the guard moved along. One important variation on the notion of visibility is $\textit{link-j visibility}$ ($L_j$-visibility) [14, 25, 24]. Two points in a polygon are called $L_j$ visible if there exists a path between them composed of $j$ or fewer line segments that lie entirely inside the polygon. Thus $L_1$ visibility is the usual notion of visibility. The most general variation on the type of guard is the use of $\textit{link-k convex}$ ($L_k$-convex) regions. A region is $L_k$-convex if every pair of points in the region is $L_k$-visible. The usual notion of a convex region is an $L_1$-convex region. Shermer[20] unified these variations, showing that when guarding an $n$ vertex polygon using $L_k$-convex regions as guards and $L_j$-visibility, that $\lceil n/(k + 2j + 1) \rceil$ guards are always sufficient.

We now need to define some graph theoretic terms and will do so following Chartrand and Lesniak[2]. Let $G$ be a connected graph; denote the vertex set of $G$ by $V(G)$ and denote the edge set of $G$ by $E(G)$. The complement graph of $G$, denoted $\overline{G}$ is the graph with the vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A \textit{path} in $G$ is an alternating sequence of vertices and edges that starts and ends with a vertex and such that no two vertices in the sequence are identical; the number of edges in a path is called its \textit{length}. A \textit{cycle} in $G$ is a path in which the first and last vertices are identical. A \textit{Hamiltonian cycle} is a cycle that includes every vertex of $G$ exactly once. The \textit{distance} between two vertices in $G$ is the length of the shortest path between the two vertices. The \textit{eccentricity} of a vertex $v$ is the maximum of the distances between $v$ and every other vertex of $G$. The \textit{radius} of $G$ is the minimum eccentricity taken over all the vertices, while the \textit{diameter} is the maximum eccentricity taken over all the vertices. The \textit{center} of $G$ is the set of vertices whose eccentricity equals the radius of $G$. A vertex $v$ is called a \textit{dominating} vertex if $v$ is adjacent to every other vertex of $G$. A set of vertices is called \textit{independent} if no two members of the set are adjacent to each other. The graph $G$ is called \textit{k-connected} if $k$ is the minimum number of vertices whose removal can result in a disconnected graph.

We need some additional graph theoretic concepts and will define these following
Golumbic[12] and Everett[7]. We call $G$ a complete graph if every pair of distinct vertices is adjacent. A subset of size $k$ of the vertex set $V(G)$ is called a $k$-clique if it induces a complete subgraph. A cycle $[v_0, v_1, \ldots, v_k, v_0]$ is chordless if $v_iv_j \notin E$ for $i$ and $j$ differing by more than one, except the edge $v_kv_0$. A graph $G$ is called an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals of a linearly ordered set (i.e. the real line) such that two vertices are connected by an edge if and only if their corresponding intervals have nonempty intersection. The clique number of $G$, denoted $\omega(G)$, is the size of the largest complete subgraph of $G$. A $c$-colouring of $G$ is a partition of the vertices into $c$ sets such that no two vertices in the same set are adjacent. The chromatic number of $G$, denoted $\chi(G)$, is the fewest number of colours needed to properly colour the vertices of $G$. The independence number of $G$, denoted $\alpha(G)$, is the size of the largest independent set of $G$. The clique cover number of $G$, denoted $k(G)$, is the fewest number of complete subgraphs such that the union of their vertex sets equals the vertex set of $G$. A perfect graph is a graph such that for every subset $A$ of $V(G)$, $\omega(G_A) = \chi(G_A)$ and $\alpha(G_A) = k(G_A)$, where $G_A$ denotes the subgraph induced by the vertices of $A$. The strong perfect graph conjecture states that a graph is perfect if and only if, for every $k \geq 2$, it contains no induced subgraph isomorphic to either the cycle having $2k + 1$ vertices or the complement of this cycle.

Assume that $G$ has a Hamiltonian cycle $H$ and define the chain from $v_a$ to $v_b$, denoted chain$(v_a, v_b)$, to be the ordered path in $H$ from $v_a$ to $v_b$, inclusive. A cycle in $G$ is ordered with respect to a Hamiltonian cycle $H$ if, for all vertices $v_i$ and $v_j$ in the cycle, $v_i$ precedes $v_j$ in the cycle implies that $v_i$ precedes $v_j$ in $H$. An invisible pair of vertices in $G$ is a pair of vertices that have no edge between them. Let $v_i, v_j$ and $v_k$ be vertices in $G$ such that $v_i \prec v_j \prec v_k$ with respect to the ordering of the vertices in $H$. The vertex $v_j$ is a blocking vertex with respect to $H$ for the invisible pair $v_i$ and $v_k$ if $i \neq k$ and no two vertices $v_s \in \text{chain}(v_i, v_{j-1})$ and $v_t \in \text{chain}(v_{j+1}, v_k)$ are adjacent in $G$. An invisible pair is minimal with respect to $v_j$ if $v_j$ is its only blocking vertex. Two invisible pairs $(v_i, v_k)$ and $(v_s, v_t)$ are separable with respect to $v_j$ if, when $H$ is traversed from $v_j$, either $v_i$ and $v_k$ are encountered before $v_s$ and $v_t$ or vice versa.

Avis and ElGindy introduced the combinatorial structure that they call a visibility
In 1983 [1], we will call this structure a *vertex visibility graph* and define it as follows: the vertex visibility graph of a simple polygon $P$, denoted $VVG(P)$, is the graph of the visibility relation on the vertices of $P$ (See Figure 1.2); i.e., $VVG(P) = (V, E)$ where

- $V = \{v \mid v$ is a vertex of $P\}$
- $E = \{(v, v') \mid vv' \subseteq P\}$

The vertex visibility graph has proven useful for solving some geometric problems, but these problems are usually restricted to questions about the vertices. For example, if a polygon is a fan (is starshaped and has a vertex in its kernel), then the vertex visibility graph will have a dominating vertex. In Figure 1.2, the vertex $h$ is in the kernel and is a dominating vertex of the vertex visibility graph.

There are three particularly interesting problems with vertex visibility graphs:

1. Given a polygon $P$, how do we compute $VVG(P)$?
2. Given a graph $G$, is it a vertex visibility graph of some polygon?
3. Given a vertex visibility graph $G$, determine a polygon $P$ such that $G = VVG(P)$?

Avis and ElGindy have developed an $O(n^2)$ algorithm for the first problem[1, 6]. Hershberger[13] improved on this, developing an $O(|E|)$ algorithm. Although $|E|$ can be as large as $\frac{n(n-1)}{2}$, (hence, in the worst case the two algorithms are the same), it can also be as small as $2n - 2$, thus representing an improvement in many cases. As Hershberger’s algorithm is linear in the size of the output, this problem may be considered satisfactorily solved. However, the second and third problems are both still open.

The second problem requires a characterization of vertex visibility graphs. Such a characterization should lead to an algorithm for recognizing if a graph is a vertex visibility graph of some polygon. Thus far, no characterization has been found, but Everett has shown that polynomial space is sufficient for this recognition problem[7]. Since a vertex visibility graph has an edge corresponding to each polygon edge, the graph must contain a Hamiltonian cycle (a cycle that goes through each vertex of the graph once and only once). Ghosh determined the following set of necessary conditions for a graph $G$ to be a vertex visibility graph and he conjectured that these conditions were also sufficient[11]:

1. $G$ has no ordered chordless cycle of length 4, where the ordering is with respect to the ordering of the vertices in the Hamiltonian cycle corresponding to the polygon boundary.

2. Every invisible pair in $G$ has a blocking vertex.

3. If two invisible pairs are separable with respect to a vertex $v_j$, then they are not both minimal with respect to $v_j$.

However, Everett proved that these conditions are not sufficient[7]. Coullard and Lubiw have established an additional necessary condition: each 3-connected component of $G$ must have a vertex ordering in which every vertex is adjacent to a previous
CHAPTER 1. INTRODUCTION

3-clique[4]. Vinay and Veni Madhavan[26] have also established some additional necessary conditions, but at present, it is still unknown whether all of these conditions are sufficient.

Another popular method of graph characterization is to provide a set of forbidden minors of a class of graphs. However, the vertex visibility graphs of convex polygons are cliques and thus the vertex visibility graphs of general polygons may contain large cliques corresponding to convex subpolygons. Thus vertex visibility graphs have no forbidden subgraphs, and most of the related work has been done on forbidden induced subgraphs instead. Unfortunately, Everett and Corneil have demonstrated an infinite family of minimal forbidden graphs; thus establishing the inability to characterize visibility graphs by these means[8, 9].

Since the characterization problem has defied a general solution, some work has been done restricting the problem to a specific class of polygons. Trivially, complete graphs are the vertex visibility graphs of convex polygons. Everett and Corneil have shown that the vertex visibility graphs of spiral polygons are interval graphs and if the strong perfect graph conjecture is true, then the vertex visibility graphs of 2-spiral polygons are perfect graphs[9]. They also provide a linear algorithm for recognizing whether a given graph is the visibility graph of a spiral polygon. Note, that a spiral polygon is one whose reflex vertices form a single chain along the boundary, while the reflex vertices of a 2-spiral form at most two separate chains (a more precise description is given later).

The third problem, constructing a polygon having a given vertex visibility graph, has also defied a satisfactory general solution, and some work has been done on restricted cases. Everett and Corneil determined a construction algorithm for the visibility graphs of spirals[9]. ElGindy established an $O(n \log n)$ construction algorithm for polygons having triangulation graphs as their vertex visibility graphs[5]. O'Rourke examined reconstructing the convex hull of a polygon from its internal and external vertex visibility graphs, determining an $O(n^2)$ algorithm for constructing the convex hull when the hull has at least four vertices, and for detecting the when the hull has three vertices[17]. He provides an example demonstrating that when the hull only has three vertices, it is possible to have two realizations of the same external visibility
Figure 1.3: Two polygons, one starshaped, one not, both having the shown vertex visibility graph.

Some of the difficulties solving the characterization and construction problems may be due to vertex visibility graphs not containing all of the visibility information of the polygon. For example, consider Figure 1.3, which shows a non-starshaped polygon, its vertex visibility graph, and a starshaped polygon having the same vertex visibility graph.

Shermer introduced a structure, called the point visibility graph of $P$, that includes all the visibility information of the polygon\[19, 20, 23\]. The point visibility graph of $P$, denoted $PVG(P)$, is the graph of the visibility relation on the points of $P$; i.e. $PVG(P) = (V, E)$ where

- $V(PVG(P)) = \{x \mid x \in P\}$
- $E(PVG(P)) = \{(x, y) \mid xy \subset P\}$

Notice that point visibility graphs are continuous graphs\[15, 27\]; there are $\aleph_1$ nodes in $PVG(P)$, each having degree $\aleph_1$ (for notational convenience, we will use $\aleph_1$ to denote the cardinality of the reals, but we neither assume nor use the continuum hypothesis).

Many of the visibility properties of a polygon’s points are easily extracted from the point visibility graph. Let $x$ be a point in a polygon $P$ and let $y \in V(PVG(P))$
be the graph node corresponding to $x$. Then, the neighbourhood of $y$, denoted $N(y)$, gives all the nodes that correspond to the points in the visibility polygon of $x$. Thus, the kernel of $P$ is exactly that set of nodes of $PVG(P)$ whose members each dominate $V(PVG(P))$ and semikernels of $P$ correspond to sets of nodes having identical neighbourhoods. All the nodes of $PVG(P)$ that are distance at most $k$ from $y$ correspond to the points that are $L_k$-visible to $x$. Because points in the same semikernel have identical neighbourhoods, they are indistinguishable in the point visibility graph.

When the context is clear, a node of $V(PVG(P))$ and the corresponding point in $P$ will be referred to by the same name. However, when the context is ambiguous, the term node shall always be used to refer to members of $V(PVG(P))$, while the terms point and vertex shall be reserved for referring to parts of the polygon. Similarly, the term edge will be reserved for polygon or cell decomposition edges (defined later), while the members of $E(PVG(P))$ will be called arcs.

Many visibility problems on $P$ may be expressed as graph theoretic problems on $PVG(P)$; we call such problems pure visibility problems. For example, the art gallery problem for $P$ is identical to the dominating node set problem for $PVG(P)$. Table 1.1 shows a list of pure visibility problems and their graph theoretic counterparts.
conditions occur? We define two polygons $P$ and $P'$ to be isomorphic if $PVG(P)$ and $PVG(P')$ are isomorphic. We will investigate this question, restricting our considerations to spiral polygons (defined below).

Suppose that $P$ has $n$ vertices, exactly $r$ of which are reflex and for convenience, assume that the vertices of $P$ are given in counterclockwise order. Then, $P$ is called spiral if in a counterclockwise traversal of the vertices of $P$ starting from some convex vertex, the $r$ reflex vertices occur consecutively. The reflex vertices of $P$ may be labelled so that in this traversal, the label $u_i$ corresponds to the $i$th reflex vertex encountered; i.e. the sequence $u_1, u_2, \ldots, u_r$ corresponds to the counterclockwise ordering of the reflex vertices. The convex vertices of $P$ may be labelled so that $v_i$ corresponds to the $i$th convex vertex counterclockwise of $u_r$; i.e. the sequence $v_1, v_2, \ldots, v_{n-r}$ corresponds to the counterclockwise ordering of the convex vertices with $v_1$ being the first convex vertex after the last reflex vertex $u_r$. For convenience, we let $u_0 = v_{n-r}$ and $u_{r+1} = v_1$. Then, the reflex chain is defined to be $u_0, u_1, \ldots, u_{r+1}$ and the convex chain is defined to be $v_1, v_2, \ldots, v_{n-r}$. Throughout the remainder of this thesis, unless explicitly stated otherwise, $P$ is a spiral polygon having $n$ vertices, $r$ of which are reflex. Notice that trivially, all convex polygons are spiral. Figure 1.4 shows three additional example spirals having the vertices $u_1, u_r, \bar{1}$ and $v_{n-r}$ labelled.

Another interesting class of polygons are the orthogonal polygons. An orthogonal
CHAPTER 1. INTRODUCTION

A polygon is a polygon with edges that alternate between horizontal and vertical. Orthogonal polygons arise in many areas of computing, such as VLSI chip design and windowing systems, and in the design of many machines, such as plotters. This is mainly because orthogonal polygons are easily represented and manipulated. We will consider the restricted case of orthogonal polygons that are also spiral.

Since we will be performing repeated arithmetic calculations with geometric objects, the number of bits required to represent the solutions using the RAM computational model can grow exponentially. We avoid this problem by using the real RAM as our computational model. In this model, every register of the machine may store a real number exactly and each arithmetic operation will take unit time. In some cases we will use the RAM model, but we will point out when we do so.

This thesis is organized as follows. In Chapter 2, we describe how to decompose a spiral polygon into cells and then we show that two spirals are isomorphic if and only if these cell decompositions are isomorphic. We also characterize the point visibility graphs of spirals as interval graphs. In Chapter 3, we develop a concise descriptor, which we call a canonical representation, for cell decompositions (and hence, spiral polygons) and then we consider algorithms for deciding if two spirals are isomorphic. In Chapter 4, we consider the layout of canonical representations, constraining our constructions to a subclass of spirals that we call bananas. In Chapter 5, we establish necessary and sufficient conditions on a canonical representation for it to be laid out as an orthogonal spiral, providing an algorithm to do so. Finally, in Chapter 6, we discuss open problems and give our concluding remarks.
Chapter 2

Isomorphism of Spiral Polygons

In this chapter, we determine necessary and sufficient conditions for two spiral polygons to be isomorphic. In the first section, we define a structure which we call the cell decomposition of $P$. In the second section, we establish that the point visibility graph of $P$ completely determines this cell decomposition. In the third section, we establish the converse; the cell decomposition completely determines the point visibility graph. In the fourth section, we use the cell decomposition to help us characterize the point visibility graphs of spiral polygons.

2.1 Introduction

Let $P$ be a spiral polygon having $n$ vertices, $r$ of which are reflex, and having its vertices labelled according to our previously described conventions for spiral polygons. Let $q$ be the first point of the convex chain of $P$ encountered when we extend $u_0u_1$ through $u_1$. The segment $u_1q$ divides $P$ into two subpolygons; we define the left end region of $P$ to be the subpolygon that contains $u_0$ (See Figure 2.1). Similarly, we define the right end region to be the subpolygon of $P$ that contains $u_{r+1}$ when $P$ is divided into two subpolygons by cutting it along the extension of $u_{r+1}u_r$ through $u_r$.

Let $x$ be a point in $P$ and let $u_i$ be the highest numbered reflex vertex seen by $x$. The forward extension from $x$ is the line segment from $x$ through $u_i$, extended until it intersects the convex chain at $x'$ (See Figure 2.2). If $x$ is in the right end region, then
no forward extension is defined. The point \( x' \) is called the forward extension point of \( x \). The segment \( x'_0u_i \) is called the former half of the extension, while the segment \( u_i x' \) is called the latter half. Similarly, the backward extension from \( x \) is the line segment from \( x \), extended through \( u_j \), the lowest numbered reflex vertex seen, until it intersects the convex chain at \( x'' \) and this is undefined when \( x \) is in the left end region. We analogously define the backward extension point \( (x'') \), the former half of the backward extension \( (x'_0u_j) \) and the latter half of the backward extension \( (u_jx'') \).

The first order cell decomposition of \( P \), denoted \( CD_1(P) \), is the diagram consisting of the boundary of \( P \) together with the forward and backward extensions of the vertices of the reflex chain (See Figure 2.3). Let \( B \) be the set of extension points of the extensions of \( CD_1(P) \). The second order cell decomposition of \( P \), denoted \( CD_2(P) \), is obtained by adding the forward and backward extensions of each member of \( B \) to \( CD_1(P) \) (See Figure 2.4). Notice that for each member of \( B \), either the forward or backward extension is already in \( CD_1(P) \); this extension is not added again to \( CD_2(P) \). The third and higher order cell decompositions are determined in a similar manner, by adding the forward and backward extensions of each extension.
point new to the previous cell decomposition. A cell decomposition is well defined for each possible order. However, it is possible that for a given order \( i \), \( CD_i(P) \) is identical to \( CD_k(P) \) for all \( k > i \). Such a cell decomposition is called a full cell decomposition (or just the cell decomposition of \( P \)) and is denoted \( CD(P) \) (See Figure 2.5). In fact, it is always the case that \( CD_r(P) = CD(P) \). Consider the chain of forward extensions that begins with the forward extension of \( u_0 \) and includes the extension of each successive extension point. First notice that the chain ends when the new extension point is in the right end region. Since each extension in this chain must be through a unique reflex vertex, the chain may contain at most \( r \) extensions. Since this is true for every reflex vertex, the \( r \)th order cell decomposition must contain all the forward extensions that are in the full cell decomposition. The case is symmetric with backward extensions.

Two cell decompositions, \( CD(P) \) and \( CD(P') \), are called isomorphic if their elements (cell, edges and vertices) can be placed in a one-to-one correspondence that preserves adjacency. Since convex vertices do not affect the visibility properties of other points, we consider the portion of the convex chain between two consecutive convex vertices to be a single cell decomposition edge, regardless of the number of convex vertices along this chain. Figure 2.6 shows two isomorphic cell decompositions.
Figure 2.3: A First Order Cell Decomposition

Figure 2.4: A Second Order Cell Decomposition
CHAPTER 2. ISOMORPHISM OF SPIRAL POLYGONS

Figure 2.5: A Full Cell Decomposition

Figure 2.6: Two Isomorphic Cell Decompositions
2.2 PVG Uniquely Determines CD

We will now show that the point visibility graph uniquely determines the cell decomposition. Since two points in the same semikernel are indistinguishable in the point visibility graph, nontrivial semikernels will require special treatment in our analysis; we will start by characterizing them.

**Lemma 2.1** Let \( P \) be a spiral polygon. Then the nontrivial semikernels of \( P \) are:

- the kernel of \( P \), if one exists and is not a single point,
- edges in the reflex chain, open at the reflex vertices
- line segments in the left end region of \( P \) between the boundary and the highest numbered reflex vertex seen, open at the reflex vertex if it is \( u_1 \) and closed at the boundary of the left end region otherwise, and
- line segments in the right end region of \( P \) between the boundary and the lowest numbered reflex vertex seen, open at the reflex vertex if it is \( u_r \) and closed at the boundary of the right end region otherwise.

**Proof** Since \( P \) is spiral, the visibility polygon of a point \( x \) in \( P \) is found by cutting \( P \) along the latter halves of the forward and backward extensions, if these extensions exist. Thus, two points \( x \) and \( y \) are in the same semikernel if their extensions have the same latter halves.

Consider when both \( x \) and \( y \) have neither a forward nor a backward extension. Then \( x \) and \( y \) are both simultaneously in the right end region and the left end region. Therefore, they are in the kernel of \( P \), which is non-trivial, and hence, are in the same nontrivial semikernel. If \( x \) and \( y \) both have a forward extension and neither has a backward extension, then they both are in the left end region of \( P \). To have the same latter half of their forward extensions, they must lie on the same line segment from the highest numbered reflex vertex seen. Similarly, if \( x \) and \( y \) both have a backward extension and neither has a forward extension, then they both are in the right end region of \( P \). To have the same latter half of their backward extensions, they must
lie on the same line segment from the lowest numbered reflex vertex seen. If \( x \) and \( y \) both have a forward and a backwards extension, then they both lie on the same line from the lowest numbered reflex vertex seen and from the highest numbered reflex vertex seen. This may only occur along a segment between two adjacent reflex vertices. □

A node \( x \) of a graph is called \textit{simplicial} if \( N(x) \) is a clique. In a point visibility graph, the simplicial nodes correspond to the polygon points that have convex visibility polygons; such points will be called \textit{simplicial points}.

Two points \( x \) and \( y \) in a polygon are said to be \textit{clearly visible} if the segment \( xy \) does not intersect the exterior of \( P \) or its boundary, except possibly at \( x \) and \( y \). The following lemma uses the concept of clear visibility and will enable us to determine where simplicial points may occur in a spiral polygon.

\textbf{Lemma 2.2} In any polygon, if a point \( x \) is simplicial, then no reflex vertices are clearly visible from \( x \).

\textit{Proof} Suppose that \( x \) has a convex visibility polygon and is clearly visible to a reflex vertex \( r \). Then, \( x \) sees two points in the neighbourhood of \( r \) that are not visible to each other, one clockwise about \( x \) from \( r \), and one counterclockwise. □

The converse of Lemma 2.2 may fail to be true in polygons having degeneracies, such as a reflex vertex that lies on the extension of an edge through a reflex endpoint (See Figure 2.7).

We now state the main result of this section.

\textbf{Theorem 2.3} Let \( G \) be the point visibility graph of some spiral polygon \( P \). Then the cell decomposition of \( P \) is uniquely determined by \( G \).

The remainder of this section contains the proof of this theorem. The proof is constructive and consists of four main parts:

1. Determining the points and edges in the reflex chain and finding a relative ordering of the edges along the chain
2. Defining forward and backward ordering relations on the nodes of $G$
3. Determining the forward and backward extensions of nodes of $G$
4. Determining the elements of the cell decomposition.

2.2.1 Extracting the Reflex Chain

In this section, we will determine sets of nodes of $G$ that correspond to the edges of the reflex chain and the reflex vertices of $P$. Then, we will order these sets of nodes to reflect the counterclockwise order in which they occur along the boundary of $P$.

First, observe that the visibility polygon of a point in the interior of a reflex edge must be convex because the only reflex vertices seen by such a point are exactly the endpoints of the edge on which it lies; by definition, such a point is simplicial. Also, notice that in a spiral polygon the only points that do not clearly see a reflex vertex are the points along the reflex chain, excluding the actual reflex vertices. Since these points are simplicial, and by Lemma 2.2, the simplicial nodes of $G$ correspond exactly to the points in the reflex chain, except the reflex vertices.

Let $S_v$ be the set of simplicial nodes in $V(G)$. If $S_v = V(G)$ then $P$ is a convex polygon and its cell decomposition contains a single region, all of $P$. Thus, we are
done. We henceforth assume that \( P \) is nonconvex.

Partition the set \( S_V \) into a minimum number of classes, with the members of each class all having identical neighbourhoods. By definition, each class is a semikernel of \( P \). By Lemmas 2.1 and 2.2, we know that we obtain one class for each edge in the reflex chain and these classes give all of the points in the edges of the reflex chain, except the actual reflex vertices. Furthermore, if we obtained \( r + 1 \) classes, then \( P \) contains \( r \) reflex vertices. Call these classes \( E_1, E_2, \ldots, E_{r+1} \).

We need to determine the reflex vertices and order the classes \( E_1, E_2, \ldots, E_{r+1} \) as they occur along the boundary of \( P \). We will do so by examining the neighbourhood of each \( E_i \); let \( R_i = N(E_i) \) (See Figure 2.8). Let \( Q \) be the set of nodes of \( G \) whose neighbourhoods are exactly the union of two of the \( R_i \)'s; i.e., \( Q = \{ v \in V(G) \mid N(v) = R_i \cup R_j \text{ where } i \neq j \text{ and } 0 \leq i, j \leq r \} \). Notice that, in general, the neighbourhood of a reflex vertex is exactly the union of the neighbourhoods of the two edges adjacent to it; thus, each reflex vertex is in \( Q \).

**Lemma 2.4** Let \( q \in Q \). If \( N(q) = R_i \cup R_j \), then the sets \( E_i \) and \( E_j \) correspond to reflex edges that are adjacent in \( P \).
PROOF Suppose that $E_i$ and $E_j$ correspond to edges that are not adjacent in $P$. Then there exists a set $E_k$ that corresponds to a reflex edge that lies between the two edges that $E_i$ and $E_j$ correspond to. To see the points of both $E_i$ and $E_j$, $q$ must lie in both $R_i$ and $R_j$; i.e., $q \in R_i \cap R_j$. Furthermore, every point in $R_i \cap R_j$ will see the points of $E_k$ (See Figure 2.9). However, $E_k$ is not in $R_i$ or $R_j$, and therefore $N(q)$ contains points other than $R_i \cup R_j$, a contradiction. 

Since each reflex vertex has a different visibility polygon, Lemma 2.4 implies that we may partition $Q$ into exactly $r$ semikernels, $Q_1, Q_2, \ldots, Q_r$. First consider when $r = 1$. The semikernel $Q_1 = Q$ is nontrivial and gives the kernel of $P$. Furthermore, the regions in the cell decomposition are given by $Q, R_1 \setminus Q, R_2 \setminus Q$ (See Figure 2.10) and we are finished for the case when $r = 1$.

Now consider the general case, $r \geq 2$. Lemma 2.1 implies that the partitioning of $Q$ into $Q_1, Q_2, \ldots, Q_r$ gives two nontrivial semikernels, corresponding to $u_1$ and $u_r$, 

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Figure 2.9: $q$ sees $E_k$
and \( r - 2 \) trivial semikernels, corresponding to \( u_2, u_3, \ldots, u_{r-1} \). We need to order the \( Q_i \)'s and \( E_i \)'s according to their order of appearance along the boundary of \( P \) and will do so by considering how the \( R_i \)'s define the \( Q_i \)'s.

Two of the \( R_i \)'s, say \( R_a \) and \( R_b \), are each only used once to define a \( Q_i \), all other \( R_i \)'s are used exactly twice. The two \( Q_i \)'s defined by \( R_a \) and \( R_b \) are the nontrivial semikernels and correspond to \( u_1 \) and \( u_r \), and hence \( E_a \) and \( E_b \) correspond to the edges of \( P \) at the ends of the reflex chain. Without loss of generality, assume that \( E_a \) corresponds to \( u_0 u_1 \) and \( E_b \) corresponds to \( u_r u_{r+1} \); this assures that the order we determine for the \( Q_i \)'s and \( E_i \)'s will be counterclockwise.

We will successively determine the order of the remaining \( Q_i \)'s and \( E_i \)'s, starting with \( E_a \) and stopping when we reach \( E_b \). Assume we have just determined the position of \( E_b \) in the reflex chain. Since each \( Q_i \) is defined by two \( R_i \)'s, there will exist a unique \( E_j \) such that \( E_j \) has not yet been ordered and \( R_k \) and \( R_j \) define some \( Q_i \), say \( Q_h \). Then \( E_j \) corresponds to the next edge in the reflex chain and \( Q_h \) corresponds to the next reflex vertex. Eventually, this new \( E_j \) will be \( E_b \) and we will be finished.

This gives a complete ordering of the \( Q_i \)'s and the \( E_i \)'s as they occur along the
reflex chain. By assuming that $E_j$ corresponded to $u_0u_1$, we have assumed that
this ordering counterclockwise. If $E_j$ corresponded to $u_ru_{r+1}$, then we would get a
construction that is symmetric under reflection to the one we obtain. However, by
our definition, cell decompositions that are symmetric under reflection are isomorphic
to each other, and hence, this case does not require separate analysis.

We now renumber the $Q_i$'s, $E_i$'s and $R_i$'s to reflect this ordering; i.e. $Q_1$ corre-
sponds to $u_1$, $E_1$ corresponds to $u_0u_1$, etc. We now know all of the points in the reflex
chain and have an ordering of the edges and the reflex vertices counterclockwise along
the boundary of $P$.

2.2.2 Defining Forward and Backward Orderings

We now define two orderings on the nodes of $G$ based on the portions of the polygon
the corresponding points see. The orderings will be based on the $R_i$'s that nodes
belong to, and when two nodes belong to the same $R_i$'s, on how much of $R_{i+1}$ or $R_{i-1}$
is seen.

First, we define the forward ordering and we use the notations $\leq$, $\leq$, $\geq$, $\geq$
to denote the usual relations in this ordering. We define all of the nodes in $R_r$ to be
equal and to be the maximum elements.

Now consider the nodes in $R_{r-1} \setminus R_r$; all of these nodes are less forward than the
nodes of $R_r$. Let $x, y \in R_{r-1} \setminus R_r$ and consider the neighbourhoods of $x$ and $y$ and
their intersections with $R_r$. We have three possible cases: if $N(x) \cap R_r \subset N(y) \cap R_r$
then $x \leq y$ (See Figure 2.11a); if $N(x) \cap R_r = N(y) \cap R_r$ then $x \geq y$ (See Figure 2.11b);
and if $N(x) \cap R_r \supset N(y) \cap R_r$ then $x \geq y$ (See Figure 2.11c).

The set $N(x) \cap R_r$ is found in $P$ by cutting $R_r$ along the latter half of the forward
extension of $x$, while $N(y) \cap R_r$ is found in $P$ by cutting $R_r$ along the latter half of the
forward extension of $y$. Since both $x$ and $y$ are in $R_{r-1} \setminus R_r$, their forward extensions
are both through $u_r$, and thus, they do not intersect in $R_r$. Therefore, it is impossible
for there to simultaneously be a node of $N(x)$ in $R_r \setminus N(y)$ and a node of $N(y)$ in
$R_r \setminus N(x)$. Hence, the above three cases are the only possibilities. Thus, we have
ordered all of the nodes in $R_r$ and $R_{r-1}$. 
We order the remaining nodes by considering each \( R_i \setminus R_{i+1} \) in order of decreasing index. Every node in \( R_i \setminus R_{i+1} \) is less than all the nodes of \( R_{i+1} \). Two nodes in \( R_i \setminus R_{i+1} \) are ordered as follows: if \( N(x) \cap R_{i+1} \subset N(y) \cap R_{i+1} \) then \( x < y \); if \( N(x) \cap R_{i+1} = N(y) \cap R_{i+1} \) then \( x = y \); and if \( N(x) \cap R_{i+1} \supset N(y) \cap R_{i+1} \) then \( x > y \). For the same reasons as above, these are the only three cases. This gives us an ordering defined on all of the nodes of \( G \).

Conceptually, one may view the forward ordering as being determined by the location of the forward extension points. Let \( x, y \in P \) and let \( f_x \) and \( f_y \) be their forward extension points. Consider what may occur if we move clockwise along the convex chain: if \( f_x \) is encountered before \( f_y \) then \( x < y \); if \( f_x \) is encountered after \( f_y \) then \( x > y \); and if \( f_x = f_y \) then \( x = y \).

The backward ordering is defined in an analogous manner and we will use the notations \( <, \leq, \geq, > \) to denote the relations. The nodes in \( R_i \) are all equal and are the maximum elements. The nodes in \( R_{i+1} \setminus R_i \) are all less than the nodes in \( R_i \) and two nodes in \( R_{i+1} \setminus R_i \) are ordered by comparing the intersections of their neighbourhoods with \( R_i \).
2.2.3 Finding Extensions

Now we show how to find the forward and backward extensions of nodes of $G$. Since the edges of a cell decomposition (excluding the actual polygon boundary) are all extensions of either vertices of the reflex chain or extension points, we will only describe these two cases. Also, by definition, extension points must lie on the convex chain of $P$. Thus, we only need to consider how to find extensions of an arbitrary reflex vertex and an arbitrary point on the convex chain. Furthermore, we only give the details for forward extensions; finding backward extensions is symmetric. We will use the notation $F(x)$ to denote the set of nodes of $G$ that correspond to the forward extension of $x$.

Let $x$ be an arbitrary point on the convex chain of $P$; we will determine $F(x)$. If $x \in R_r$, then no forward extension is defined for $x$ and thus $F(x) = \emptyset$. Otherwise, $F(x)$ must be a subset of $N(x)$. Let $Q_k$ be the highest numbered $Q_i$ that is in $N(x)$; i.e. $Q_k$ corresponds to the highest numbered reflex vertex seen by $x$. If $Q_k$ contains a single element, $u_k$, then $u_k$ is the highest numbered reflex vertex that $x$ is adjacent to. Otherwise, $Q_k$ is infinite and corresponds to the last reflex vertex; i.e. $u_k = u_r$. By definition, $F(x)$ corresponds to the extension of $x$ through $u_k$. The latter half of $F(x)$ is $u_k$, along with the nodes of $N(x)$ that are least in the backward ordering. The former half of $F(x)$ is all the nodes of $N(x)$ that are equal to $x$ in the forward ordering.

Now we will find $F(u_i)$, where $u_i$ is an element of $Q_i$, $1 \leq i \leq r - 1$ (no forward extension is defined for $u_r$ because it is in $R_r$). The nodes of $F(u_i)$ must all be in $R_i$; in fact, they are the nodes of $R_i$ that are least in the backward ordering. The latter half of $F(u_i)$ is $F(u_i) \setminus E_i$, while the former half is just $E_i$ (which, like above, is the nodes of $N(u_i)$ that are equal to $u_i$ in the forward ordering). The forward extension of $u_0$ is slightly different because the nodes of $R_0$ are all equivalent in the backward ordering. The former half of $F(u_0)$ consists of $E_0$. For the latter half, we select one node from each semikernel in $R_0 \cap R_1$.

We also need to determine the forward extension point of $y$ (where $y$ is either the $x$ or the $u_i$ considered above). In general, the forward extension point of $y$ is the
unique node of $F(y)$ that is greatest in the forward ordering. However, if $F(x) \cap R_r$ is not empty, then this unique node does not exist; in fact, $F(x) \cap R_r$ corresponds to a semikernel of $P$ and all of its members are greatest in the forward ordering. In this case the forward extension point of $x$ is not determinable. However, since $F(x)$ intersected $R_r$, no forward extension is defined for the extension point of $x$ and hence, we do not need to distinguish the extension point from the rest of the latter half of $F(x)$.

Determining the backward extensions is completely symmetric: the latter halves are the nodes least in the forward ordering; the former halves are the nodes that are equal in the backward ordering; and the extension points are the nodes greatest in the backward ordering.

2.2.4 Determining the Cell Decomposition, $CD(P)$

Now we wish to determine the actual cell decomposition of $P$. Thus far we have determined the reflex chain of $P$ and know how to find both forward and backward extensions. We need to determine all of the elements of $CD(P)$: the vertices, the edges and the cells.

There are three types of cell decomposition edges: the edges of the reflex chain, which we already know; forward and backward extensions, which we know how to determine; and portions of the convex chain, which we will determine after determining the vertices and cells. We do not need to determine the actual vertices or edges of the convex chain of $P$, just the nodes that compose them.

The vertices of $CD(P)$ are either reflex vertices (which we already know), extension points (which we know how to determine) or the intersection point of two extensions. If two extensions intersect, then the node corresponding to their intersection point is the node in the intersection of the two sets corresponding to the extensions; if the extensions do not intersect, then their corresponding sets do not intersect either.

The boundary of each cell of $CD(P)$ is defined by half-extensions and either a portion of the convex chain, a reflex edge or neither. If the cell is not in either end region, then for each case, the nodes $x$ in the cell are such that $x_1 < x < x_2$ and $x_3 < x < x_4$. 
where \( x_1, x_2, x_3 \) and \( x_4 \) are points on the four half-extensions defining the cell. Figure 2.12a shows the case when the cell boundary does not contact the polygon boundary; Figure 2.12b demonstrates when the cell is on the convex chain; and Figure 2.12c shows when a reflex edge is part of the cell boundary.

The cells that lie in the left end region are defined by two backward extensions. If \( x_1 \) and \( x_2 \) are points on these backward extensions, then the nodes of the cell are such that \( x_1 < x < x_2 \) and \( x \) is a maximum element in the backward ordering. Similarly, cells in the right end region are defined by two forward extensions; the nodes must be between these forward extensions and be maximum in the forward ordering.

For the case of the cell being on the convex chain, we need to distinguish the points of the convex chain from the rest of the cell points. Let \( x \) be a point in such a cell that is not in either end region and consider the neighbourhood of \( x \). Let \( f_x \) be a member of the subset of \( N(x) \) consisting of all its most forward points (\( f_x \) might be unique). If \( x \) is equal to the backward extension point of \( f_x \), then \( x \) is on the convex chain. Thus, the cell decomposition edge along the convex chain consists of all such \( x \) in the cell. If the cell is in either end region, then each point \( y \) along the convex chain is indistinguishable from the other points in the same semikernel as \( y \); we select

Figure 2.12: Half-extensions defining cell points not in either end region.
2.3 CD Uniquely Determines PVG

In this section, we show the converse of Theorem 2.3. We may then combine our two theorems to obtain necessary and sufficient conditions for two spiral polygons to be isomorphic.

**Theorem 2.5** Let $CD(P)$ be the cell decomposition of some spiral polygon $P$. Then the point visibility graph of $P$ is uniquely determined by $CD(P)$.

The main idea of this proof is to use the cell decomposition of $P$ to coordinatize all of the points in $P$. The two coordinates given to a point will be based on the point's position in the forward and backward orderings defined in the proof of Theorem 2.3. This gives a labelling of the nodes in $PVG(P)$ and the adjacency of two nodes may then be immediately determined from these labels.

2.3.1 Coordinatizing $P$

The first step in assigning coordinates to the points of $P$ is to assign a real number to each point of the convex chain of $P$. First consider the portion of the convex chain contained in the left end region of $P$. Along this chain, there will be a series of intersection points with edges of the cell decomposition. Suppose that there are $r'$ such points. If there are no degeneracies in the cell decomposition, then the chain of backward extensions from each reflex vertex will terminate at a unique point in the left end region and thus $r' = r$; otherwise $r' < r$. The convex chain in the left end region will be assigned real values from 0 to 1, beginning with 0 at the vertex $v_{n-r}$ and ending with 1 at the forward extension point of $u_0$. The assigned values increase monotonically and continuously from 0 to 1 as one proceeds around the boundary in a clockwise fashion. Also, the values are assigned so that the numbers
Figure 2.13: Assignment of Values to Convex Chain in Left End Region

$\frac{1}{r'+1}, \frac{2}{r'+1}, \ldots, \frac{r'}{r'+1}$ correspond to the clockwise ordering of the intersection points with the cell decomposition edges (See Figure 2.13).

The remainder of the convex chain is assigned values based upon the values assigned above to the left end region. Let $x$ be a point on the convex chain of $P$ and let $\Phi(x)$ denote the value associated with $x$. If $x$ is not in the right end region of $P$, then the forward extension point of $x$ is well defined and we will associate the value $\Phi(x) + 1$ with this extension point; i.e. if $y$ is the forward extension point of $x$, then $\Phi(y) = \Phi(x) + 1$. Using this definition and the values already assigned for the left end region, we may work our way clockwise around the convex chain, from when $y$ is the forward extension point of $u_0$, until $y$ is $u_{r+1}$, defining a value for each point of the convex chain when it is the current $y$ (See Figures 2.14 and 2.15). Note that by assigning the values $\frac{1}{r'+1}, \frac{2}{r'+1}, \ldots, \frac{r'}{r'+1}$ to the points of the convex chain in the left end region that were intersected by cell decomposition edges, we force each successive location at which a cell decomposition edge intersects the convex chain to have a value that is $\frac{1}{r'+1}$ greater than the previous such intersection. We use the term cd-value to denote the value of $\Phi(\cdot)$ at any point that is the intersection point of the convex chain with an extension of the cell decomposition. Thus, the cd-values
are completely determined by the cell decomposition.

We may now define coordinates for every point in \( P \). Let \( p \) be an arbitrary point in the interior of \( P \). Let \( p_b \) denote the backward extension point of \( p \) and let \( p_f \) denote the forward extension point. If \( p \) lies in the left end region, then let \( p_b = u_0 \) and if \( p \) lies in the right end region, then let \( p_f = u_{r+1} \). We define the coordinates of \( p \) to be the ordered pair \((b, f)\) where \( b = \Phi(p_b) \) and \( f = \Phi(p_f) \) and we use the notation \( p = (b, f) \) (See Figure 2.16). Two additional points of interest are shown in Figure 2.16: \( x \) such that \( \Phi(x) = \Phi(p_f) - 1 \) and \( y \) such that \( \Phi(y) = \Phi(p_b) + 1 \). Note that, by definition, all the points in a nontrivial semikernel have the same extension points and hence will have the same coordinates.

The following lemma describes how to determine if two points are visible by comparing their coordinates.

**Lemma 2.6** Two points \( p = (b, f) \) and \( p' = (b', f') \) are visible if and only if \( f' \geq b+1 \) and \( b' \leq f - 1 \).

**Proof** Let \( p_b \) and \( p_f \) denote the backward and forward extension points of \( p \), respectively. The visibility polygon of \( p \) is formed by cutting \( P \) along the latter halves.
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Figure 2.15: Coordinates of Cell Decomposition Edges

Figure 2.16: Determining the Coordinates of \( p = (b, f) = (\Phi(p_b), \Phi(p_f)) \)
of the two extensions of \( p \). If \( p' \) is more backward than \( p_b \), then the forward extension point of \( p' \) must be less forward than the forward extension point of \( p_b \); that is, \( f' < b + 1 \) (See Figure 2.16). If \( p' \) is more forward than \( p_f \), then the backward extension point of \( p' \) must be less backward than the backward extension point of \( p_f \); that is, \( b' > f - 1 \) (See Figure 2.16). Therefore, \( p \) only sees \( p' \) when neither of these inequalities holds; that is, when \( f' \geq b + 1 \) and \( b' \leq f - 1 \).

\[ \square \]

### 2.3.2 Determining the Point Visibility Graph, \( PVG(P) \)

We will now construct a graph \( G \) based on the cell decomposition of \( P \). Each element in the cell decomposition of \( P \) will yield a specific set of nodes of \( G \). This correspondence between cell decomposition elements and nodes of \( G \) will be entirely determined by the cd-values and the coordinates of the points composing the element. The adjacency of two nodes will also be based on the coordinates of the points. Then, since the union of all the cell decomposition elements is exactly \( P \), our constructed graph \( G \) will be identical to the point visibility graph of \( P \). We will consider each type of element (vertex, edge and cell) separately.

We first consider a vertex \( v \) in the cell decomposition. Recall that the convex vertices of \( P \) are not considered to be vertices of the cell decomposition. Then \( v \) is one of the following:

1. a reflex vertex from \( u_2, u_3, \ldots, u_{r-1} \)
2. the intersection of two extensions (and perhaps an extension point)
3. on the backward extension of \( u_{r+1} \).
4. on the forward extension of \( u_0 \)

Notice that the third and fourth cases are subcases of the second, but due to the usual complications caused by the semikernels in the end regions, they warrant separate treatment. Also, note that both the forward and backward extensions of \( v \) must be edges of \( CD(P) \), and hence the coordinates of \( v \) are cd-values. In the first two cases,
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$v$ yields a single node of $G$ having the same coordinates as $v$. If $v$ is on the backward extension of $u_{r+1}$, then $v$ is part of a nontrivial semikernel in the right end region of $P$ that yields $\Re_1$ nodes in $G$, all having the same coordinates as $v$ (this includes the case $v = u_r$). If $v$ is on the forward extension of $u_0$, then again $v$ is part of a nontrivial semikernel, this time in the left end region of $P$, and yields $\Re_1$ nodes of $G$ all having the same coordinates as $v$ (this includes the case $v = u_1$). It is also important to notice that the vertices of $CD(P)$ along the convex chains in the end regions are members of the semikernels in the last two cases, and are not distinguishable from the rest of the semikernel.

Next, consider an edge $e$ of the cell decomposition and let $p$ be an arbitrary point on $e$. The edge $e$ is one of the following:

1. an edge of the reflex chain
2. a cell decomposition edge in the right end region
3. a cell decomposition edge in the left end region
4. a cell decomposition edge in neither end region
5. a portion of the convex chain of $P$.

When $e$ is an edge of the reflex chain, the forward and backward extensions of $p$ are in the cell decomposition, and thus the coordinates of $p$ are cd-values. The points of $e$ yield $\Re_1$ nodes of $G$ that all have the same coordinates as $p$.

If $e$ is in the right end region, then there are two cases, depending on where $e$ intersects the backward extension of $u_{r+1}$; call this point $p'$. If $p' \neq u_r$, then the nodes corresponding to the points of $e$ were all determined when we considered the vertex $p'$. Otherwise, $p' = u_r$ and the point $p$ has the coordinates $(x, \Phi(v_1))$, where $x$ is a cd-value; the edge $e$ yields $\Re_1$ nodes, each having the same coordinates as $p$. The case for the left end region is symmetric.

If $e$ is in neither end region, then the points of $e$ have either constant $b$ coordinate or constant $f$ coordinate, and this constant is a cd-value. Suppose that the $b$ coordinate is constant. Then, there exists cd-values $f_1$ and $f_2$ such that $f_1 < f_2$ and the endpoints
of $e$ are vertices having the coordinates $(b, f_1)$ and $(b, f_2)$. Again the edge $e$ yields $N_1$ nodes of $G$, one for each of the coordinates $\{(b, f) \mid f_1 < f < f_2\}$ where $b$ is the common coordinate value we assumed. The analysis is symmetric when the $f$ coordinate is constant.

If $e$ is a polygonal chain along the convex chain of $P$, then we have two possible cases, determined by whether or not $e$ is a portion of the boundary in one of the end regions. Note that outside of the two end regions, a point $p$ of the convex chain has coordinates $(b, b+2)$ for some $b$. Thus, if $e$ is outside of the end regions, the endpoints of $e$ have coordinates $(b_1, b_1+2)$ and $(b_2, b_2+2)$ where $b_1$ and $b_2$ are consecutive cd-values and $b_1 < b_2$. Again, $e$ yields $N_1$ nodes of $G$, one for each of the coordinates $\{(b, b+2) \mid b_1 < b < b_2\}$. Now, consider when $e$ is in the right end region. The endpoints of $e$ are vertices having coordinates $(b_1, \Phi(v_1))$ and $(b_2, \Phi(v_1))$, where $b_1$ and $b_2$ are consecutive cd-values and $b_1 < b_2$. Then, $e$ yields $N_1$ nodes of $G$, exactly one from each semikernel having constant $b$ value between $b_1$ and $b_2$ (remember, all the members of a semikernel in the right end region have the same coordinate, namely $(b, \Phi(v_1))$ for some fixed $b$). The case is symmetric when $e$ is in the left end region.

Finally, consider a cell $c$ of the cell decomposition. We have one of the following cases:

1. $c$ is in both end regions
2. $c$ is in the right or left end region
3. $c$ is in neither end region and has an edge along the convex chain
4. $c$ is in neither end region and does not have an edge along the convex chain.

If $c$ is in both end regions, then it is the kernel of $P$ and yields $N_1$ nodes of $G$, all having coordinates $(0, \Phi(v_1))$. If $c$ is in the right end region, then it is bounded by two edges having $b$ coordinates $b_1$ and $b_2$ (consecutive cd-values) with $b_1 < b_2$, and the points in $c$ have the coordinates $(b, \Phi(v_1))$ with $b_1 < b < b_2$. The cell is composed of a continuum of semikernels, each consisting of all the points having the coordinate $(b, \Phi(v_1))$ for some fixed $b$, $b_1 < b < b_2$. Thus, the cell yields $N_1$ nodes.
for each pair of coordinates in the set \( \{(b, \Phi(v_1)) \mid b_1 < b < b_2\} \). Similarly, a cell \( c \) in the left end region is bounded by two edges having constant (again consecutive) cd-value coordinates \( f_1 \) and \( f_2 \) with \( f_1 < f_2 \) and this cell yields \( n_1 \) nodes for each pair of coordinates in the set \( \{(0, f) \mid f_1 < f < f_2\} \).

If \( c \) is in neither end region and has an edge along the convex chain, then \( c \) is bounded by this edge and two extensions. One extension has constant \( b \) coordinate \( b_2 \) and the other has constant \( f \) coordinate \( b_1 + 2 \) where \( b_1 < b_2 \). The edge along the convex chain has endpoints that are vertices of \( CD(P) \), having coordinates \( (b_1, b_1 + 2) \) and \( (b_2, b_2 + 2) \) with \( b_1, b_2, b_1 + 2 \) and \( b_2 + 2 \) all cd-values. Thus, there will be a point \( p \) of \( c \) for each of the coordinates in the set \( \{(b, f) \mid b_1 < b < b_2 \text{ and } b_1 + 2 < f < b_2 + 2 \text{ and } b < f - 2\} \) and each point will yield a single node of \( G \) having the same coordinates as \( p \). The last constraint on the coordinates is due to the convex chain bounding the cell, instead of just the cell decomposition extensions, further restricting the possible coordinate values.

If \( c \) is in neither end region and does not have an edge along the convex chain, then \( c \) is bounded by two edges having constant \( b \) coordinate (cd-values \( b_1 \) and \( b_2 \), \( b_1 < b_2 \)) and two edges having constant \( f \) coordinate (cd-values \( f_1 \) and \( f_2 \), \( f_1 < f_2 \)).

If the cell has an edge along the reflex chain, then two of these edges are identical, that is, one edge has both constant \( b \) and \( f \). Again, \( c \) yields \( n_1 \) nodes of \( G \), one for each point having a coordinate in \( \{(b, f) \mid b_1 < b < b_2 \text{ and } f_1 < f < f_2\} \).

Now we have completely determined the vertex set of \( G \). We define the edge set of \( G \) as follows: two nodes \( v \) and \( v' \) having the coordinates \( (b, f) \) and \( (b', f') \), respectively, are adjacent if and only if \( f' \geq b + 1 \) and \( b' \leq f - 1 \).

A cell decomposition completely determines its elements (vertices, edges and cells) and cd-values. Also, the union of all the cell decomposition elements is exactly \( P \). The discussion of this section establishes that the cell decomposition elements and cd-values determine the nodes of \( G \) and a labelling of these nodes, using the coordinates we have defined. Furthermore, there is a one-to-one correspondence between nodes of \( G \) and points of \( P \); thus \( V(G) \) is equal to the vertex set of \( PVG(P) \). Two nodes of \( G \) are adjacent if and only if \( f' \geq b + 1 \) and \( b' \leq f - 1 \). Thus, Lemma 2.6 implies that \( G \) is a graph of the visibility relation and hence \( G \) is identical to \( PVG(P) \).
Therefore, $CD(P)$ completely determines $PVG(P)$. This completes the proof of Theorem 2.5.

We now combine Theorems 2.3 and 2.5 to get necessary and sufficient conditions for two spiral polygons to be isomorphic:

**Theorem 2.7** Two spiral polygons $P$ and $P'$ are isomorphic if and only if $CD(P)$ and $CD(P')$ are isomorphic.

Let $P$ and $P'$ be spiral polygons such that $P'$ is symmetric under reflection to $P$; by definition $P$ and $P'$ are isomorphic. Assume that we have coordinatized $P$ as in Section 2.3.1. When we were describing how to coordinatize the points of a spiral, we arbitrarily chose to start by assigning values to the convex chain in the left end region. Instead, we could have chosen to start by assigning values to the convex chain right end region and then defining the backward extension point of a point $x$ to have the value one greater than value of $x$. We choose to coordinatize $P'$ using this alternate scheme. Then, since $P$ and $P'$ are symmetric under reflection, the points of $P$ will have the same coordinates as the points of $P'$. Furthermore, the mapping from $P$ to $P'$ that maps a point of $P$ to the point having the same coordinates in $P'$ is an isomorphism.

### 2.4 Spiral PVGs are Interval Graphs

We call a continuous graph $G$ an *interval graph* if with each node of $G$ we may associate an interval $[x_1, x_2]$ of the real line, such that two nodes are adjacent if and only if their intervals intersect. We call a continuous graph *chordal* if it contains no chordless induced cycles of length four or greater. As with finite graphs, continuous interval graphs are a subset of continuous chordal graphs.

Let $G$ be the point visibility graph of some spiral polygon $P$ that has its nodes coordinatized as in Section 2.3.1. Let $v \in V(G)$ be the node associated with the point $p \in P$, where $p = (b, f)$. Then, with $v$, we will associate the interval $[x_1, x_2] = [b+1, f]$ (See Figure 2.17).
Lemma 2.8  Let $v$ and $v'$ be two nodes in $V(G)$. Let $[x_1, x_2]$ and $[x'_1, x'_2]$ be the intervals associated with $v$ and $v'$, respectively. Then $v$ and $v'$ are adjacent if and only if the intervals $[x_1, x_2]$ and $[x'_1, x'_2]$ intersect.

**Proof**  Let $p$ and $p'$ be the points corresponding to $v$ and $v'$, and let their coordinates be $(b, f)$ and $(b', f')$, respectively. Then, by definition, we know that $[x_1, x_2] = [b+1, f]$ and $[x'_1, x'_2] = [b'+1, f']$.

Suppose that $v$ and $v'$ are adjacent in $G$. Then, by definition, $p$ and $p'$ are visible. By Lemma 2.6, we know that $f' \geq b+1$ and $b' \leq f-1$. Then, substituting into these inequalities, we get $x'_2 \geq x_1$ and $x'_1 \leq x_2$. Hence, the intervals must intersect.

Conversely, suppose that the intervals $[x_1, x_2]$ and $[x'_1, x'_2]$ overlap. Then one of the following four cases must occur:

- $x_1 \leq x'_1 \leq x_2 \leq x'_2$
- $x'_1 \leq x_1 \leq x'_2 \leq x_2$
- $x_1 \leq x'_1 \leq x'_2 \leq x_2$
- $x'_1 \leq x_1 \leq x_2 \leq x'_2$
\begin{itemize}
  \item $x'_1 \leq x_1 \leq x_2 \leq x'_2$
\end{itemize}

In each case, $x'_2 \geq x_1$ and $x'_1 \leq x_2$. Thus, we know that $f' \geq b + 1$ and $b' \leq f - 1$. Therefore, by Lemma 2.6, $p$ and $p'$ see each other, and hence, $v$ and $v'$ are adjacent in $G$. \hfill \square

Then the definition of a continuous interval graph and Lemma 2.8 immediately give the following theorem:

**Theorem 2.9** Let $P$ be a spiral polygon. Then $PVG(P)$ is a (continuous) interval graph.

The vertex visibility graph of a polygon is the subgraph of the point visibility graph induced by the polygon vertices. Then, since any induced subgraph of an interval graph is also an interval graph, the vertex visibility graph of a spiral polygon $P$ must be an interval graph. Thus, we have provided an alternate proof to that of Everett for Theorem 3.2.8 of [7].

To demonstrate that spiral polygons are the only polygons that have point visibility graphs that are interval graphs, we show the following, stronger, theorem:

**Theorem 2.10** If $P$ is a polygon that is not spiral, then $PVG(P)$ is not chordal.

**PROOF** We define a reflex chain of $P$ to be a maximal sequence of reflex vertices that occur consecutively along the boundary of $P$. Notice that unlike the reflex chain of a spiral polygon, we do not include the bounding convex vertices.

Consider two reflex vertices, $u_1$ and $u_2$, on different reflex chains. Let $\mathcal{P}$ be the shortest path in $P$ between $u_1$ and $u_2$. Then, the interior vertices of $\mathcal{P}$ will be reflex vertices of $P$. Since $u_1$ and $u_2$ are on different reflex chains, there exists at least one edge of $\mathcal{P}$ between two reflex vertices of $P$ that are not on the same reflex chain; call these reflex vertices $u$ and $u'$. Since there is only a single edge of $\mathcal{P}$ between $u$ and $u'$, there is no vertex of $P$ between them, and hence, they are clearly visible.

Since $u$ and $u'$ are clearly visible, there exist four points $p_1$, $p_2$, $p_3$ and $p_4$ having the following properties:
1. \(p_1\) and \(p_2\) are in the neighbourhood of \(u\) but not visible to each other
2. \(p_3\) and \(p_4\) are in the neighbourhood of \(u'\) but not visible to each other
3. \(p_3\) is seen by \(p_1\) and \(p_2\)
4. \(p_4\) is seen by \(p_1\) and \(p_2\).

This implies that the nodes of \(PVG(P)\) corresponding to \(p_1, p_4, p_2\) and \(p_3\) form a chordless 4-cycle in \(PVG(P)\), in that order (See Figure 2.18). Hence, \(PVG(P)\) is not chordal.

Theorems 2.9 and 2.10 combine to give the following theorem:

**Theorem 2.11** Let \(P\) be an arbitrary polygon. Then, \(PVG(P)\) is chordal iff \(PVG(P)\) is an interval graph iff \(P\) is a spiral.
Chapter 3

Canonical Representations

In this chapter, we define a canonical representation for spiral polygons. First, we define the concept of an extension sequence, which is based on the cell decomposition. Next, we show that this extension sequence completely determines the cell decomposition. Then, we define the canonical representation, based on the extension sequence. Finally, we describe two algorithms for determining whether two spiral polygons are isomorphic, the asymptotically faster of which makes use of this canonical representation.

3.1 Introduction

Consider a reflex vertex \( u_i \) of \( P \). We define the forward extension chain of \( u_i \) to be the forward extensions in \( CD(P) \) that start with the forward extension of \( u_i \), then continue with successive forward extensions of the current extension point, until it lies in the right end region. That is, the forward extension chain of \( u_i \) consists of \( u_i \), its forward extension \( u_if_1 \), the forward extension \( f_1f_2 \) of \( f_1 \), the forward extension \( f_2f_3 \) of \( f_2 \), etc. until \( f_i \) lies in the right end region. The backward extension chain of \( u_i \) is defined similarly. Now consider the edge \( e_i = u_iu_{i+1} \). The extension chain of \( e_i \) is the union of the forward extension chain of \( u_i \) and the backward extension chain of \( u_{i+1} \) (See Figure 3.1). For convenience, the forward extension of \( u_i \) may be referred to as the forward extension of \( e_i \), and similarly, the backward extension of \( u_{i+1} \) may
be referred to as the backward extension of $e_i$. If two edges $e_i$ and $e_j$, $i < j$, have identical extension chains, then $CD(P)$ is said to contain a degeneracy. Notice that in such a case, it is impossible to have $j = i + 1$.

### 3.2 Defining the Canonical Representation

#### 3.2.1 Defining the Extension Sequence

First, we will define the concept of an *extension sequence*. To do this, we need to consider the extension chain of each edge in the reflex chain. Let $C_i$ be the extension chain of the edge $e_i = u_iu_{i+1}$. We assign the label $i$ to each intersection point of $C_i$ and the convex chain. By assigning labels like this for each extension chain, we obtain a labelling all along the convex chain, with a label at each cell decomposition vertex that is on the convex chain (See Figure 3.2). For consistency, we consider $e_0 = u_0u_1$ to have a degenerate backward extension and $e_r = u_{r-1}u_r$ to have a degenerate forward extension; this implies that we assign a 0 label to $u_0(= v_{n-r})$ and an $r$ label to $u_{r+1}(= v_1)$. If the extension chains of more than one edge are identical, then the label we assign is the set of indices of these edges,
Figure 3.2: A Complete labelling of the convex chain.

Figure 3.3: A Cell Decomposition with degeneracies, \{0, 3\} and \{1, 4\}.
listed in ascending order (See Figure 3.3).

The list of these labels, taken as we move clockwise along the convex chain, starting from \( v_{n-r} \), is called the long extension sequence of \( P \). For Figure 3.2, we obtain \( 0 2 5 1 4 3 0 2 5 1 4 3 0 2 5 1 4 3 0 2 5 \), and for Figure 3.3, we obtain \( \{0 3\} \{1 4\} \{1 4\} 2 \{0 3\} \{0 3\} \{1 4\} \{1 4\} \). Notice that, as \( v_{n-r} \) is always labelled zero, the first label of a long extension sequence is always a 0 or a set containing 0. Similarly, the last label is always an \( r \) or a set containing \( r \). By taking the prefix of the long extension sequence, up to, but not including the second 0, we obtain what is called the extension sequence. For Figure 3.2, the extension sequence is \( 0 2 5 1 4 3 \), and for Figure 3.3, it is \( \{0 3\} \{1 4\} 2 \). Notice that this is exactly the labels that occur along the boundary of the left end region.

**Lemma 3.1** Using the RAM computational model, at most \( O(r \log r) \) bits are required to store the extension sequence of a spiral polygon having exactly \( r \) reflex vertices.

**PROOF** The extension sequence of \( P \) is essentially just an ordered list of the numbers \( 0, 1, \ldots, r \), with some of the numbers possibly being grouped into sets. Thus, we may store it as an ordered list of the \( r + 1 \) integers \( 0, 1, \ldots, r \), along with one bit for each number to indicate if it is in a set with the next element in the list. Therefore, it requires at most \( O(r \log r) \) bits to store the extension sequence. \( \square \)

Then, using the real RAM computational model, the space required to store the extension sequence of a spiral polygon having \( r \) reflex vertices will be \( O(r) \).

Each label in the extension sequence is considered to be an equivalence class of reflex edges of \( P \). Then, by definition, an equivalence class is a maximal set of reflex edges that all have identical extension chains. For brevity, when the class contains a single element, we choose to write it without the braces; i.e. we write 0 instead of \( \{0\} \).

**3.2.2 Determining \( CD(P) \) from the Extension Sequence**

In this section, we describe some of the properties of extension sequences, then show how to determine \( CD(P) \) from the extension sequence of \( P \).
Lemma 3.2 Let $E$ be a long extension sequence of a spiral polygon $P$ having $r$ reflex vertices. Then, in $E$, for every $i$, $0 \leq i \leq r$, between any two consecutive occurrences of $i$, every $j \neq i$, $0 \leq j \leq r$, must occur.

PROOF Suppose that there exists some $j \neq i$, such that $j$ always occurs either before the two $i$'s, after the two $i$'s or both before and after the two $i$'s. If $j$ occurs before the two $i$'s, then it is not in the right end region, and the last $j$ before the two $i$'s must have a forward extension (labelled $j$). Thus, this case can not occur. The case of $j$ occurring after the two $i$'s is symmetric. If $j$ occurs before and after the two $i$'s, then the last $j$ before the two $i$'s and the first $j$ after the two $i$'s are not visible, and thus, can not be the endpoints of an edge of the extension chain of $e_j$ (See Figure 3.4).

Figure 3.4: The two $j$'s are not visible.

Notice that Lemma 3.2 implies that the order in which the numbers occur can not vary, and hence, the long extension sequence must consist of a repetition of the extension sequence, with the last repetition possibly only being partial. It follows that we may exactly determine the long extension sequence from the extension sequence. The forward extension point of $u_0$ is the second 0 label in the long extension sequence. The forward extension point of $u_1$ is the first 1 label after the forward extension
CHAPTER 3. CANONICAL REPRESENTATIONS

point of $u_0$. Similarly, the forward extension point of $u_2$ is the first 2 label after the forward extension point of $u_1$. In general, the forward extension point of $u_{i+1}$ is the first $i + 1$ label after the forward extension point of $u_i$. Therefore, the long extension sequence is exactly the extension sequence repeated until the numbers 0 0 1 2 ... $r - 1$ $r$ have appeared in that order. The 0 label occurs twice because the first occurrence corresponds to the degenerate backward extension of $u_0$; after this backward extension, the first occurrence of each label $i$ in increasing order corresponds to the forward extension point of $u_i$. The long extension sequence stops when the $r$ label corresponding to the (degenerate) forward extension of $u_r$ is encountered. Note that the backward extension points may now be determined by working backwards through the long extension sequence in a similar manner as above for the forward extensions.

Each point $p$ of the convex chain outside of the left end region lies (non-strictly) between the forward extensions of two consecutive reflex vertices, $u_i$ and $u_{i+1}$. Then, $u_i$ is the lowest-numbered reflex vertex seen by $p$ and, by definition, the backward extension of $p$ will be through $u_i$. Similarly, $p$ lies (non-strictly) between the backward extensions of $u_j$ and $u_{j+1}$, $u_{j+1}$ is the highest numbered reflex vertex seen and the forward extension of $p$ is through $u_{j+1}$.

To derive the cell decomposition of $P$ from the long extension sequence, first note that the largest number in the sequence is the number of reflex vertices. Each number in the long extension sequence corresponds to a point $p$ that is the intersection point of the convex chain with an edge of $CD(P)$. Also, since we can determine which reflex vertex each extension passes through, we can determine all of the edges defining the cell decomposition, and where they intersect. This also gives us the cells that are present. Therefore, via the long extension sequence, the extension sequence determines the cell decomposition of $P$.

3.2.3 The Canonical Representation

Recall that cell decompositions that are planar reflections of each other are isomorphic under our definitions. Thus, it is possible that two different extension sequences
describe isomorphic cell decompositions. When a spiral polygon is reflected, each $e_i$ of the reflex chain becomes $e_{r-i}$ and their order is reversed. To obtain the \textit{reflection} of the extension sequence of $P$, the extension sequence of the reflected polygon, from that of $P$, we must first reverse the sequence, including the order of elements in any sets of the sequence. Then, we must cyclically permute the sequence so that it starts with $r$. Finally, we replace each number $i$ with $r - i$. Notice that the first two steps of this transformation produces the equivalent results as listing the labels of $P$, starting from $u_{r+1}$, and moving counterclockwise along the convex chain of $P$ in the right end region (remember that the extension sequence is effectively just the labels as they occur clockwise in the left end region). The reflections of the extension sequences for Figures 3.2 and 3.3 are $0 \ 3 \ 5 \ 2 \ 1 \ 4$ and $(0 \ 3) \ (1 \ 4) \ 2$, respectively, with the latter being the same under reflection.

We now define the \textit{canonical representation} of $P$, denoted $CR(P)$, as whichever is lexicographically smaller, the extension sequence of $P$ or the reflection of the extension sequence. In the lexicographic ordering, we consider sets to occur after single numbers. The canonical representations for our examples are $0 \ 2 \ 5 \ 1 \ 4 \ 3$ and $(0 \ 3) \ (1 \ 4) \ 2$. By the above discussion, and Theorem 2.7, we have:

\textbf{Theorem 3.3} Two spiral polygons $P$ and $P'$ are isomorphic if and only if their canonical representations are identical.

From Lemma 3.1 and the above definition for canonical representations, we immediately get:

\textbf{Theorem 3.4} Using the RAM computational model, at most $O(r \log r)$ bits are required to store the canonical representation of a spiral polygon having exactly $r$ reflex vertices.

\section{3.3 Algorithms for Isomorphism}

In this section, we investigate the computational complexity of determining if two spirals $P$ and $P'$ are isomorphic. First, we will consider an algorithm based on comparing
the cell decompositions of the two polygons. A lower bound for such an algorithm is determined by the size of the cell decomposition. Then, we will consider an algorithm based on the canonical representations of the polygons. We will show that the complexity of this algorithm is less than the size of the cell decomposition.

3.3.1 Comparing Cell Decompositions

To determine if $P$ and $P'$ are isomorphic, Theorem 2.7 implies that it is sufficient to compute $CD(P)$ and $CD(P')$ and then check these for isomorphism. Thus, a bound on the size of the cell decomposition will provide us with a lower bound on the time required to compute and compare them.

Lemma 3.5 Let $P$ be a spiral polygon having $n$ vertices. Then, the size of the cell decomposition of $P$ may be $\Theta(n^3)$, but no larger.

Proof First, we will show that the number of vertices in $CD(P)$ may be of size $\Theta(n^3)$. We only consider vertices (and not edges or cells) because the decomposition is planar and the number of vertices is a linear function of the number of cells and the number of edges. We define a corridor of a spiral polygon to be the region visible to an open edge of the reflex chain; i.e. a corridor corresponds to a region $R_i$ in the proof of Theorem 2.3 (See Figure 2.8).

Consider an orthogonal spiral $P_o$ that has very narrow corridors (See Figure 3.5). If the corridors are sufficiently narrow, each edge extension chain will be incident on every reflex vertex and the cell decomposition will contain no degeneracies. Consider the corridor that is visible to the edge $u_iu_{i+1}$ (See Figure 3.6). In this corridor, each extension chain consists of two segments, from $u_i$ to the convex chain, then from the convex chain to $u_{i+1}$. Furthermore, since there are $\Theta(n)$ extension chains and each pair of extension chains intersect in this corridor, there will be $\Theta(n^2)$ intersection points (cell decomposition vertices) in this corridor. Note also, that by choosing the corridor to be sufficiently narrow, we can assure that none of these vertices will simultaneously lie in any other corridor. Then, since there are $\Theta(n)$ corridors, each having $\Theta(n^2)$ vertices in it, there will be at least $\Theta(n^3)$ vertices in $CD(P_o)$. The two
CHAPTER 3. CANONICAL REPRESENTATIONS

Figure 3.5: An orthogonal spiral with narrow corridors.

Figure 3.6: $O(n^2)$ vertices due to intersection of extension chains in this corridor.
end regions only have $\Theta(n)$ vertices in them, but this does not affect the asymptotic bound.

We now show that $CD(P)$ can get no larger than $O(n^3)$, by again considering the corridor of the edge $\overline{u_iu_{i+1}}$. In this corridor, each extension chain consists of two segments, one that crosses the latter half of the forward extension of $u_i$ then intersects the convex chain at a point $p$, and one that crosses the latter half of the backward extension of $u_{i+1}$ and also intersects the convex chain at $p$. Thus, in this corridor, each extension chain creates at most three cell decomposition vertices due to intersections with the convex chain. Also, as every extension chain is a two segment chain in this corridor, each pair of chains intersect in at most four places. Therefore, since there are $O(n)$ corridors, each containing at most $O(n^2)$ vertices, the cell decompositions must contain at most $O(n^3)$ vertices.

Lemma 3.5 implies that an algorithm to compute the elements of a cell decomposition must require at least $O(n^3)$ time. Trivially, an algorithm which determines the isomorphism of $P$ and $P'$ by generating the cell decompositions and checking them for isomorphism must also require at least $O(n^3)$ time. In the next section, we show that we can achieve a faster algorithm by comparing the canonical representations the spirals.

### 3.3.2 Comparing Canonical Representations

We use Theorem 3.3 to obtain an algorithm for spiral isomorphism detection that is asymptotically faster than the brute force method of computing and comparing cell decompositions. It is not known if this algorithm is optimal.

**Theorem 3.6** Let $P$ be a spiral polygon having $n$ vertices, $r$ of which are reflex. Then $CR(P)$ may be computed in $O(nr)$ time using the real RAM computational model.

**Proof** To determine $CR(P)$, we first need to compute the backward extension chain for each edge of the reflex chain, except $\overline{u_0u_1}$. Each of these $r$ chains may be computed in $O(n)$ time by simultaneously traversing the convex chain in counterclockwise order to determine the next extension point and traversing the reflex
chain in clockwise order to determine the next reflex vertex extended through. Thus $O(nr)$ time is required to determine the chain for every edge of the reflex chain. The last extension of each of these chains intersects the convex chain of $P$ in the left end region. Note that the extension sequence of $P$ corresponds to the clockwise ordering of these extension points along the boundary. Thus, by sorting these points into the clockwise order along the convex chain, we get the extension sequence of $P$ in $O(r \log r)$ time. However, we must also determine the reflection extension sequence of $P$. This may be done using string manipulations, as described earlier, and requires $O(r)$ time. Finally, we choose the lexicographically smaller of the extension sequence and its reflection, to be $CR(P)$. This requires $O(r)$ time. This entire process requires $O(nr) + O(r \log r) + O(r) + O(r) \in O(nr)$ time.

**Corollary 3.7** Let $P_1$ and $P_2$ be spiral polygons having $n_1$ and $n_2$ vertices, of which $r_1$ and $r_2$ are reflex, respectively. Then, we can determine if $P_1$ and $P_2$ are isomorphic in $O(n_1 r_1 + n_2 r_2)$ time using the real RAM computational model.

**Proof** If $r_1 \neq r_2$ then $P_1$ and $P_2$ can not be isomorphic. The value of $r_i$ may be determined in $O(n_i)$ time for each polygon and thus this comparison requires $O(n_1 + n_2)$ time. If $r_1 = r_2$, then by Theorem 3.3, to determine if $P_1$ and $P_2$ are isomorphic, we only need to compare $CR(P_1)$ and $CR(P_2)$. We can determine $CR(P_1)$ in $O(n_1 r_1)$ time, and $CR(P_2)$ in $O(n_2 r_2)$ time. If $CR(P_1)$ and $CR(P_2)$ are identical, then $P_1$ and $P_2$ are isomorphic, otherwise, they are not. Then, since comparing the two canonical representations only requires time proportional to their length, this entire process requires $O(n_1 r_1 + n_2 r_2)$ time. \qed
Chapter 4

Layout of Banana Spirals

4.1 Introduction

Let $P$ be a spiral polygon having its vertices numbered according to our conventions for spiral polygons. Let $l_0$ and $l_{r+1}$ be the lines orthogonal to $\overrightarrow{u_0u_{r+1}}$ that pass through $u_0$ and $u_{r+1}$, respectively. Then, $P$ is called a banana spiral if none of $l_0$, $l_{r+1}$ and $\overrightarrow{u_0u_{r+1}}$ intersect the interior of $P$ (See Figure 4.1). Notice that this definition allows the edges $\overrightarrow{v_{n-r}v_{n-r-1}}$ and $\overrightarrow{v_1v_2}$ to lie along $l_0$ and $l_{r+1}$, respectively (remember that $u_0 = v_{n-r}$ and $u_{r+1} = v_1$).

In this chapter, we will construct a banana spiral for an arbitrary extension sequence and discuss the implications of such a construction. In the next section, we consider where the extension chains may intersect in $P$. Then, from this, we determine that the convex chains in the end regions of a spiral may undergo a constrained modification without changing the extension sequence of the spiral. In the third section, we will use this fact to help us inductively construct a banana spiral having a given extension sequence. Finally, in the fourth section, we consider the vertex visibility graphs of banana spirals.
4.2 Modifying Spiral Polygons

Let $P$ be a spiral polygon having $r$ reflex vertices and let $p_{r+1}$ be the backward extension point of $u_{r+1}$. We define the right end chain of $P$ to be the portion of the boundary of $P$ from $p_{r+1}$ clockwise to $u_r$; i.e. the portion of the convex chain in the right end region together with $\overline{u_{r+1}u_r}$ (See Figure 4.2). Similarly, we define the left end chain of $P$ be the portion of the boundary of $P$ from the forward extension point of $u_0$ counterclockwise to $u_1$. In this section we will show that the right end chain may undergo a constrained modification without changing the extension sequence of $P$. We omit demonstrating this for the left end chain as the proof is symmetric. First we need to show that no pair of the forward extensions of $CD(P)$ intersect each other in the right end region.

Let $\mathcal{E}_r$ be the extension sequence of $P$ and let $k$ be the number of equivalence classes in $\mathcal{E}_r$ (remember that an equivalence class is a maximal set of reflex edges that have identical extension chains). Since $\overline{u_r p_{r+1}}$ bounds the right end region, $k-1$ forward extensions in the cell decomposition of $P$ intersect $\overline{u_r p_{r+1}}$, one for every equivalence class except the class that $r+1$ belongs to. Let $\mathcal{L} = \{L_1, L_2, \ldots, L_{k-1}\}$ be this set of forward extensions. The line $\overline{u_{r+1}u_r}$ divides the plane into two open half-planes, one of which intersects the right end region; call this half-plane $P$.

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**Figure 4.1:** A banana spiral.
Lemma 4.1 Let \( L_i, L_j \in \mathcal{L} \). Then \( L_i \) and \( L_j \) do not intersect in \( \mathcal{P} \).

PROOF First, partition \( \mathcal{L} \) into classes according to the highest numbered reflex vertex that the extension intersected and call these classes \( U_1, U_{t+1}, \ldots, U_r \), where the index corresponds to that of the reflex vertex. By definition, we know that all the members of class \( U_i \) intersect at \( u_i \). Furthermore, no \( u_i, i = t, t + 1, \ldots, r - 1 \), may lie in \( \mathcal{P} \) because there exists a forward extension through each such \( u_i \) that intersects \( \overline{u_r p_{r+1}} \). Trivially, \( u_r \) is not in \( \mathcal{P} \), because \( \mathcal{P} \) is open and \( u_r \) lies on the line that defines it. Thus, for any given class \( U_i \), no two members of the class intersect in \( \mathcal{P} \).

Consider two classes, \( U_i \) and \( U_j \), where \( i < j \). Let \( I \) be the region of \( \mathcal{P} \) between the latter halves of the forward extensions of \( u_{i-1} \) and \( u_i \). Let \( J \) be the region of \( \mathcal{P} \) between the latter halves of the backward extensions of \( u_j \) and \( u_{j+1} \) (See Figure 4.3). By definition, each member of \( U_i \) is a forward extension through \( u_i \) and each member of \( U_j \) is a forward extension through \( u_j \). Thus, the latter half of each extension in \( U_i \) lies in \( I \) and the former half of each of extension in \( U_j \) lies in \( J \). Thus, the intersection
of some $L_a \in U_i$ with some $L_b \in U_j$ must lie in $I \cap J$.

Now we show that $I \cap J$ does not intersect $\mathcal{P}$ by considering the two cases, $j = r$ and $j \neq r$. If $j = r$, then the backward extension of $u_{j+1} = u_{r+1}$ is part of the line defining $\mathcal{P}$. The vertex $u_{r-1}$ is on the backward extension of $u_r$ and thus is in $J$. However, $u_{r-1}$ cannot be in the right end region of $\mathcal{P}$ and hence is not in $\mathcal{P}$. Thus all of $J$ lies outside of $\mathcal{P}$ and trivially $I \cap J$ does not intersect $\mathcal{P}$. If $l < j < r$ then $u_j$ cannot lie in the right end region of $\mathcal{P}$. Since the forward extension of $u_j$ must intersect the right end region (and hence the defining line of $\mathcal{P}$), the backward extensions of both $u_j$ and $u_{j+1}$ cannot. Thus $J$ (and trivially $I \cap J$) does not intersect $\mathcal{P}$. Therefore no $L_a \in U_i$ and $L_b \in U_j$ intersect in $\mathcal{P}$.

Let $J_1$ and $J_2$ be two simple curves from $p_{r+1}$ to $u_r$ that lie entirely in $\mathcal{P} \cup \overline{u_{r+1}u_r}$, intersect each $L_i \in \mathcal{L}$ exactly once, at some point other than $u_r$, and intersect each other only at $p_{r+1}$ and some connected non-trivial portion of $\overline{u_{r+1}u_r}$, including $u_r$. Let $p_1$ be the first point after $p_{r+1}$ that $J_1$ and $J_2$ have in common. Due to the restrictions on the simple curves, $p_1$ must lie on $\overline{u_{r+1}u_r}$. Let $a_i$ be the point at which $L_i$ intersects $J_1$ and let $b_i$ be the point at which $L_i$ intersects $J_2$. For convenience, we assume that $J_1$ is the curve that lies nearer to $\overline{u_{r+1}u_r}$ (See Figure 4.4).
Lemma 4.2 Let $L_i, L_j \in L$. When we traverse $J_1$ starting from $p_{r+1}$, if we encounter $a_i$ before $a_j$, then when we traverse $J_2$ starting from $p_{r+1}$, we encounter $b_i$ before $b_j$.

**Proof** Suppose that $a_i$ is encountered before $a_j$ but $b_i$ is encountered after $b_j$. Let $J_0$ be the Jordan curve obtained by taking the portions of $J_1$ and $J_2$ between $p_{r+1}$ and $p_1$ together. Observe that the open segments $a_i b_i$ and $a_j b_j$ both lie entirely in the interior of $J_0$.

Consider the Jordan curve formed by $b_j a_j$, $J_1$ between $a_j$ and $p_{r+1}$, and $J_2$ between $p_{r+1}$ and $b_j$; call this curve $J'$. The point $a_i$ is on the boundary of $J'$ but not on the boundary of $J_0 \setminus J'$ and thus, all the points that are both in the neighbourhood of $a_i$ and in the interior of $J_0$ are also in the interior of $J'$. Therefore, there exists a point $a'_i$ on the open segment $a_i b_i$ in the interior of $J'$.

By definition, $b_i$ is on the boundary of $J_0$ and not on the boundary of $J'$. Thus, $b_i$ is in the exterior region of $J'$ and by the Jordan curve theorem, the segment $a'_i b_i$ intersects the boundary of $J'$. Furthermore, since $J_1$ and $J_2$ only intersect each member of $L$ once, $a'_i b_i$ must intersect the boundary of $J'$ somewhere along $a_j b_j$. However, by Lemma 4.1, this is impossible. \qed
Let $P'$ be the object resulting from replacing the right end chain of $P$ with some new polygonal chain $c'$. We say that $c'$ is a replacement chain if:

- $P'$ is a spiral polygon having $r$ reflex vertices
- the last segment of $c'$ is a non-trivial portion of $\overline{u_{r+1}u_r}$

**Theorem 4.3** Let $P$ be a spiral polygon. Then the right end chain of $P$ may be replaced by any replacement chain to obtain a new spiral having the same extension sequence as $P$.

**Proof** Let $c_{re}$ be the right end chain of $P$. Let $c'$ be a replacement chain and let $P'$ be the object we obtain by replacing $c_{re}$ with $c'$. By definition, we know that $P'$ is a spiral polygon having exactly $r$ reflex vertices. Notice that any replacement chain lies entirely in $P$ and intersects each line in $L$ exactly once. If $c'$ and $c_{re}$ only intersect at $p_{r+1}$ and a portion of $\overline{u_{r+1}u_r}$, then they satisfy the conditions on the curves $J_1$ and $J_2$ in Lemma 4.2. This implies that the intersection points of $L$ with $c'$ occur in the same order as they do with $c_{re}$. Thus, the extension sequences of $P$ and $P'$ are the same.

If $c_{re}$ and $c'$ do intersect somewhere other than $p_{r+1}$ or along $\overline{u_{r+1}u_r}$, we consider a third replacement chain, $c_3$, which does not intersect either $c_{re}$ or $c'$, except at $p_{r+1}$ and along $\overline{u_{r+1}u_r}$. We know that such a $c_3$ exists because both $c'$ and $c_{re}$ lie entirely in $P$, which is an open half-plane. So a replacement chain $c_3$ that lies closer to $\overline{u_{r+1}p_{r+1}}$ than either $c_{re}$ or $c'$ will exist. Then, $c_{re}$ and $c_3$ satisfy the conditions on the curves in Lemma 4.2, as do $c_3$ and $c'$. Thus, we apply Lemma 4.2 twice, first to $c_{re}$ and $c_3$ then to $c_3$ and $c'$. This will again give us the fact that $P$ and $P'$ have the same extension sequence.

Similarly, we define a replacement chain for the left end region as any polygonal chain that ends with a non-trivial portion of $\overline{u_0u_1}$ and results in a spiral polygon having $r$ reflex vertices when it is substituted for the left end chain. The following corollary follows from a symmetrical proof to that for the right end region.
Corollary 4.4 Let \( P \) be a spiral polygon. Then the left end chain of \( P \) may be replaced by any other replacement chain to obtain a new spiral having the same extension sequence as \( P \).

### 4.3 Construction of Banana Spirals

Let \( \mathcal{L} \) be a partitioning of the numbers \( 0,1,\ldots,r \) into \( k \) sets. Let \( \mathcal{E} \) be an ordered list of the members of \( \mathcal{L} \) that starts with the set containing 0. If no set in \( \mathcal{E} \) contains both \( i \) and \( i+1 \), \( i = 0,1,\ldots,r-1 \), then we call \( \mathcal{E} \) a valid extension sequence. Observe that the extension sequence of a spiral polygon \( P \) must be valid as \( i \) and \( i+1 \) can never be in the same equivalence class. In this section, we will show that every valid extension sequence may be realized as a banana spiral.

Let \( \mathcal{E}_r = E_1E_2\ldots E_k \) be a valid extension sequence for a spiral polygon having \( r \) reflex vertices and \( k \) equivalence classes. Let \( E_j \) be the equivalence class to which \( r \) belongs and let \( E'_j = E_j \setminus \{r\} \). If \( E'_j \) is empty, then let \( \mathcal{E}_{r-1} = E_1E_2\ldots E_{j-1}E_{j+1}\ldots E_k \) otherwise, let \( \mathcal{E}_{r-1} = E_1E_2\ldots E_{j-1}E'_jE_{j+1}\ldots E_k \). We say that \( \mathcal{E}_{r-1} \) is obtained from \( \mathcal{E}_r \) by removal of \( r \) (See Table 4.1).

**Lemma 4.5** Let \( \mathcal{E}_r \) be a valid extension sequence. Then \( \mathcal{E}_{r-1} \), obtained by removal of \( r \) from \( \mathcal{E}_r \), is also valid.

**Proof** Since \( \mathcal{E}_r \) is valid, it contains no illegal degeneracies. That is, for every

| 0 2 5 1 4 3 | \{0 5\} | 4 \{1 3\} | 2 \{0 2\} | 4 \{1 3 5\} |
| 0 2 1 4 3 | 0 4 \{1 3\} | 2 \{0 2\} | 4 \{1 3\} |
| 0 2 1 3 | 0 \{1 3\} | 2 \{0 2\} | \{1 3\} |
| 0 2 1 | 0 1 2 \{0 2\} | 1 |
| 0 1 | 0 1 | 0 1 |

Table 4.1: Removal of \( r \) from an extension sequence.
i = 0, 1, \ldots, r - 1, there is no \( E_g, g = 1, 2, \ldots, k \) such that \( i \in E_g \) and \( i + 1 \in E_g \). Trivially, this is also true for \( i = 0, 1, \ldots, r - 2 \). The modification to \( \mathcal{E}_r \) performed to get \( \mathcal{E}_{r-1} \) was to remove the equivalence class containing \( r \), if \( r \) was its only member, otherwise just remove \( r \) from the class; no other \( i = 0, 1, \ldots, r - 2 \) changes its class during this process. Thus, no illegal degeneracies are created and \( \mathcal{E}_{r-1} \) is valid. □

**Theorem 4.6** Let \( \mathcal{E}_r \) be a valid extension sequence for a spiral polygon having \( r \) reflex vertices and let \( \alpha \) be an angle, \( 0 < \alpha \leq \frac{180^\circ}{r+2} \). Then, there exists a banana spiral polygon having \( \mathcal{E}_r \) as its extension sequence and such that the exterior angle between consecutive edges of the reflex chain is at least \( 180^\circ - \alpha \).

**PROOF** Let \( \mathcal{E}_r, \mathcal{E}_{r-1}, \mathcal{E}_{r-2}, \ldots, \mathcal{E}_1 \) be the list of extension sequences where each \( \mathcal{E}_{i-1} \) is obtained from \( \mathcal{E}_i \) by removal of \( i \). We will inductively construct an \( \mathcal{E}_r \) spiral polygon that satisfies the stated angle constraint along its reflex chain, by inductively constructing a sequence of such polygons corresponding to this list of extension sequences. We arbitrarily choose to make the edges of the reflex chain \( d \) units long.

First, observe that \( \mathcal{E}_1 \) must be 01. An appropriate \( \mathcal{E}_1 \) spiral has the vertices (in counterclockwise order): \( u_0 = (0, 0), \ u_1 = (d \sin \alpha, d \cos \alpha), \ v_1 = (2d \sin \alpha, 0), \ v_2 = (2d \sin \alpha, 1), \ v_3 = (d \sin \alpha, d \cos \alpha + 1), \ v_4 = (0, 1) \) (See Figure 4.5).

In the remainder of this proof, when we say we **move** \( \overline{v_1v_2} \) to include a point \( x \) lying in the right end region, we perform the following operations: First, we replace
v_2$ with $v'_2$ where $v'_2 \in \overline{v_1v_2}$ and $\overrightarrow{v'_2x}$ is parallel to $\overrightarrow{v_2v_3}$. Then we define $v'_3$ to be the intersection point of the right end chain with the extension of $\overrightarrow{v'_2x}$ through $x$ (See Figure 4.6). Finally, we will replace the right end chain with the chain consisting of the convex chain between the backward extension point of $u_{r+1}$ and $v'_3$, the edge $\overrightarrow{v'_2v'_3}$ and the segment $\overrightarrow{v'_2v_1}$.

For induction, assume that for some $i$, $1 \leq i < r$, we have constructed an appropriate $\mathcal{E}_i$ spiral, say $P_i$. To construct an appropriate $\mathcal{E}_{i+1}$ spiral from $P_i$, first we will modify the right end chain of $P_i$ (if it is necessary), and then we will add a parallelogram to $P_i$ that introduces a new reflex vertex at $v_1$. This will result in an $\mathcal{E}_{i+1}$ spiral that may require modifications to both its left and right end chains to make it a banana (i.e. it may protrude beyond the two lines orthogonal to $\overrightarrow{u_0u_{r+1}}$ that pass through $u_0$ and $u_{r+1}$).

Let $p_0$ be the point such that the distance from $v_1$ to $p_0$ is $d$, $\angle u_1v_1p_0 = 180^\circ - \alpha$ and $\angle u_1v_1p_0$ forms a right turn (See Figure 4.7). Let $p'$ be the point at which the extension of $\overrightarrow{p_0v_1}$ through $v_1$ intersects the right end chain of $P_i$. We divide the equivalence classes of $\mathcal{E}_i$ into three classes determined by where the last forward extension of the class intersects the right end chain: $\Sigma_a$ contains the classes that occur clockwise of (after) $p'$; $\Sigma_b$ contains the classes that occur counterclockwise of (before) $p'$; $\Sigma_c$ contains the classes that occur at the same position as (coincident with) $p'$. We consider the
equivalence class of $i$ to be in both $\Sigma_a$, due to the degenerate forward extension of $e_i$, and in $\Sigma_b$, because the backward extension of $e_i$ bounds the right end region. Also, notice that $\Sigma_c$ contains either zero or one element.

To construct an $\mathcal{E}_{i+1}$ spiral, we will add to $P_i$ the parallelogram that has $v_1p_0$ and $v_1v_2$ as two of its edges. Thus, the location of $p'$ will determine the equivalence class that $i+1$ will belong to in the new polygon. There are two ways in which we will modify the position of $p'$:

- by moving $v_2v_3$ towards $u_1v_1$
- by rotating $v_1p_0$ counterclockwise about $v_1$

Theorem 4.3 assures that moving $v_2v_3$ is legal. We can not rotate $v_1p_0$ clockwise as this would make $\angle u_1v_1p_0$ less than $180^\circ - \alpha$.

Let $E_{i+1}$ be the class of $\mathcal{E}_{i+1}$ to which $i+1$ belongs. If $E_{i+1}$ contains more than just $i+1$, then $E_{i+1} \setminus \{i+1\}$ must be a class of $\mathcal{E}_i$ and we need $p'$ to intersect the right end chain at the same point as the last forward extension of this class. Also, $E_{i+1} \setminus \{i+1\}$ must be a member of one of $\Sigma_a$, $\Sigma_b$ and $\Sigma_c$. If $E_{i+1}$ contains only $i+1$, then in $\mathcal{E}_{i+1}$, the class $E_{i+1}$ must occur between two classes of $\mathcal{E}_i$ and we need $p'$ to lie between the last forward extension points of these classes.

First, we describe the two possible cases for which our current construction is sufficient: when $\Sigma_c$ is not empty and $i+1$ joins the equivalence class in $\Sigma_c$ and when
Σ_c is empty and E_{i+1} occurs after all the members of Σ_b and before all the members of Σ_a. Both of these case are satisfied by our current location of p' and hence, no modifications are required before we add the parallelogram to P_i.

To reduce the number of remaining cases, we henceforth consider the sole member of Σ_c (if it exists) to have been added to each of Σ_a and Σ_b. Then, there are four remaining cases:

1. i + 1 joins a member of Σ_a
2. i + 1 is in its own equivalence class and is in between two members of Σ_a
3. i + 1 joins a member of Σ_b
4. i + 1 is in its own equivalence class and is in between two members of Σ_b.

For the first two cases, we will define a point x and then move \( \overline{v_2v_3} \) toward \( \overline{u_iu_1} \) to include x. First consider when \( i + 1 \) joins an equivalence class in Σ_a, say E_a. Let \( l_a \) be the last forward extension of the E_a and let \( p_a \) be the intersection point of \( l_a \) and \( \overline{v_1p'} \). Then, we move \( \overline{v_2v_3} \) to include \( p_a \) (See Figure 4.8a). Now consider when \( i + 1 \) is in its own equivalence class and is in between two members of Σ_a, say E_a and E_b. Let \( l_a \) and \( l_b \) be the last forward extensions of E_a and E_b, respectively, and define \( p_a \) and \( p_b \) to be the intersection points of \( l_a \) and \( l_b \) with \( \overline{v_1p'} \), respectively. Then, we move \( \overline{v_2v_3} \) to include the midpoint of \( \overline{p_ap_b} \) (See Figure 4.8b). If one of E_a and E_b
contains $i$, say $E_b$, then $l_b$ corresponds to the degenerate forward extension that we defined. In this case, we simply define $p_b$ to be equal to $v_1$, then move $\overline{v_2v_3}$ as above.

For the last two cases, we will define a point $x$ and then rotate $\overline{p_0v_1}$ counterclockwise about $v_1$ until it is collinear with $\overline{xx}$.

First consider when $i + 1$ joins an equivalence class of $\Sigma_b$, say $E_b$. Let $l_b$ be the last forward extension of $E_b$ and let $p_b$ be the point at which $l_b$ intersects the right end chain. Then, we rotate $\overline{p_0v_1}$ until it is collinear with $\overline{p_0v_1}$ (See Figure 4.9a). Now consider when $i + 1$ is in its own equivalence class and is inbetween two members of $\Sigma_b$, say $E_a$ and $E_b$. Let $l_a$ and $l_b$ be the last forward extensions of $E_a$ and $E_b$, respectively, and let $p_a$ and $p_b$ to be their intersection points with the right end chain. Then, we rotate $\overline{p_0v_1}$ until it is collinear with $\overline{xx}$, where $x$ is the midpoint of $\overline{p_ap_b}$ (See Figure 4.9b). If one of $E_a$ and $E_b$ contains $i$, say $E_a$, then we define $l_a$ to be the backward extension of $u_{i+1}$ and then proceed as above.

Let $p_1$ be the point such that the line segments $\overline{v_1p_0}$, $\overline{p_0p_1}$, $\overline{p_1v_2}$ and $\overline{v_2v_1}$ form the boundary of a parallelogram (See Figure 4.10). Let $P_{i+1}$ be the union of $P_i$ and this parallelogram. By our construction, $P_{i+1}$ is a spiral polygon having the extension sequence $\mathcal{E}_{i+1}$. However, $P_{i+1}$ might not be a banana because its end regions may protrude past the vertical lines orthogonal to $\overline{u_1p_0}$.

Suppose that the right end region of $P_{i+1}$ protrudes past the line $l_{i+2}$, where $l_{i+2}$
is the line through $u_{i+2}$ orthogonal to $\overline{u_0u_{i+2}}$ (remember that $P_{i+1}$ has $i+1$ reflex vertices). Let $p$ be the other intersection point of $l_{i+2}$ and the right end chain and let $p_{i+2}$ be the backward extension point of $u_{i+2}$. We know that $p$ must occur clockwise of $p_{i+2}$ along the convex chain (i.e. in the right end region) because of the restriction on the angle $\alpha$, $0 < \alpha \leq \frac{180^\circ}{i+2}$. Let $c$ be the convex chain consisting of the right end chain between $p_{i+2}$ and $p$ and the segment $\overline{pu_{i+2}}$ (See Figure 4.11). Then Theorem 4.3 assures that when we replace the right end chain of $P_{i+1}$ with $c$, the resulting polygon also has $E_{i+1}$ as its extension sequence. Corollary 4.4 assures that if the left end region protrudes past $l_0$, then we may modify the left end chain similarly.

We have now constructed a banana spiral having $E_{i+1}$ as its extension sequence. □

The above theorem shows that we may construct a banana spiral from any extension sequence. Then, if $E$ is the extension sequence of a spiral polygon $P$, we may construct a banana spiral $P'$ having $E$ as its extension sequence. Since $P$ and $P'$ have the same extension sequence, they have the same canonical representation and by Theorem 3.3, $P$ and $P'$ are isomorphic. This proves the following theorem (See Figure 4.12):

**Theorem 4.7** Let $P$ be a spiral polygon. There exists a banana spiral $P'$ such that $P$ and $P'$ are isomorphic.
Figure 4.11: The right end region protrudes past $l_{i+2}$.

Figure 4.12: A spiral and a banana isomorphic to it.
Theorem 4.8 Let $\mathcal{E}_r$ be a valid extension sequence. We can construct a banana spiral having $\mathcal{E}_r$ as its extension sequence in $O(r^2)$ time using the real RAM computational model.

**PROOF** Theorem 4.6 assures that a banana spiral having $\mathcal{E}_r$ as its extension sequence exists. We will consider the complexity of a construction algorithm based on the construction provided in the proof of Theorem 4.6. Assume that we have already constructed $P_i$ and we are constructing $P_{i+1}$; we will determine the amount of work required to do this. We store $\mathcal{E}_r$ in a list structure, as is described in Lemma 3.1, and we also use a list to store the set of last forward extensions of the extension chains.

First we need to determine the position of the equivalence class of $i+1$ in the extension sequence of $P_{i+1}$. We determine this by traversing $\mathcal{E}_r$ and determining either the equivalence class of $P_i$ that $i+1$ joins or the two classes that $i+1$ lies between. This can be done in a single traversal of the list and hence will require $O(r)$ time. We will only consider the case of $i+1$ lying between two classes; the case of $i+1$ joining an equivalence class is almost identical. Let $E_b$ and $E_a$, respectively, be the equivalence classes before and after the class of $i+1$.

Next we spend $O(1)$ time positioning the potential new reflex edge $v_1 \overrightarrow{p_0}$. Then, we have to decide if $i+1$ will be in the correct equivalence class in our current construction or if we have to move $\overrightarrow{v_2v_3}$ or rotate $\overrightarrow{v_1p_0}$ about $v_1$. Since we are storing the set of last forward extensions in a list, it will take $O(i)$ time to determine the location of the extension points of $E_b$ and $E_a$. Once the last forward extensions and extension points of $E_b$ and $E_a$ are obtained, only a constant amount of time is required to determine if the current construction is satisfactory or if a move or rotate needs to be performed.

Next consider performing a move of $\overrightarrow{v_2v_3}$; this requires determining a point $x$ that we want $\overrightarrow{v_2v_3}$ to include, determining the points $v'_2$ and $v'_3$ and updating the convex chain. Since we already know the two classes (and their forward extensions) that we want $i+1$ to be between, determining $x$ and $v'_2$ only require constant time. However, if we start from $v_1$ and proceed counterclockwise around the convex chain, we may traverse $O(i)$ edges before determining the edge that $v'_3$ lies on, thus requiring $O(i)$ time. Then, to update the convex chain, we remove the edges after $v'_2$ that we just
traversed while seeking \( v'_3 \).

Similarly, if we needed to rotate \( \overline{v_1p_0} \), then it only requires constant time to determine the point \( x \) that we want the extension of \( \overline{v_1p_0} \) through \( v_1 \) to include. This point \( x \) will be the midpoint of the extension points of \( E_b \) and \( E_a \). However, after we have rotated the segment, we need to determine where the extension of \( \overline{v_1p_0} \) through \( v_1 \) intersects the convex chain, and there may be \( O(i) \) edges between the two convex edges that the extension points of \( E_b \) and \( E_a \) are on.

In the next step, we only require constant time to add the parallelogram that causes \( v_1 \) to become a reflex vertex. Then we may need to remove portions of the end chains that protrude past the bounding lines in the definition of a banana spiral. Again, we may have to traverse \( O(i) \) edges to determine where the bounding lines through \( u_0 \) and \( u_{i+1} \) intersect the convex chain.

Finally, we need to extend all of the last forward extensions of \( P_i \) that may be extended through \( u_{i+1} \). Again, there may be \( O(i) \) such extensions. This is the dominant step of the algorithm. An amortized analysis could show that some of the other steps may require less time than that stated. However, it is possible that for every \( i \), every last forward extension in \( P_i \) will have to be extended again in \( P_{i+1} \). For example, consider constructing the banana spiral isomorphic to narrow orthogonal spiral in Figure 3.5.

Thus, since we require \( O(i) \) time for each \( i = 1, 2, \ldots, r \), we will require \( O(r^2) \) total time.

We may place an additional constraint on banana polygons and further extend the previous two results. A \textit{parallel banana spiral} is a banana spiral in which for all \( i, i = 0, 1, \ldots, r \), the reflex edge \( \overline{u_iu_{i+1}} \) is parallel to the convex edge \( \overline{v_{(n-r-1)}(n-r-1)(i+1)} \) (See Figure 4.13). Conceptually, if we start from second last edge of the convex chain \( \overline{v_{n-r-1}v_{n-r-2}} \) and traverse the chain clockwise, the \( i \)th edge encountered is parallel to the \( i \)th edge of the reflex chain when it is traversed clockwise starting from \( \overline{u_0u_1} \). If a parallel banana has \( r \) reflex vertices, then there are \( r + 2 \) vertices on the reflex chain and there must be \( r + 4 \) vertices on the convex chain; i.e. each edge in the reflex chain must have a parallel counterpart in the convex chain and there are two
additional convex edges that connect the convex chain to the reflex chain, namely $v_1v_2$ and $v_{n-r}v_{n-r-1}$. Therefore, the total number of vertices in a parallel banana having $r$ reflex vertices must be $n = 2r + 4$. Thus, we may restate the definition of a parallel banana spiral to require that $u_iu_{i+1}$ is parallel to $v_{(r+3)-(i+1)}$.

**Theorem 4.9** Let $E_r$ be a valid extension sequence for a spiral polygon having $r$ reflex vertices. Let $\alpha$ be an angle, $0 < \alpha \leq \frac{180^\circ}{r+2}$. Then a parallel banana spiral polygon exists such that the exterior angle between consecutive edges of the reflex chain is at least $180^\circ - \alpha$.

**Proof** The proof of Theorem 4.6 provides an inductive construction of a banana spiral satisfying the given angle constraint and having the extension sequence $E_r$. We will describe what changes are necessary to ensure that our construction is a parallel banana spiral. First notice that the $E_1$ banana spiral chosen for the induction is a parallel banana (See Figure 4.5). Also notice that when we add a polygon to $P_i$ to create $P_{i+1}$, we choose to add a parallelogram. Thus, in $P_{i+1}$, the reflex edge $u_{i+1}u_{i+2}$ and the convex edge $v_2v_3$ (which is $v_{(r+3)-(r+1)}$ since $r = i+1$ in $P_{i+1}$) are parallel. Therefore, we start our induction with a parallel banana and each new subpolygon we add satisfies this parallel constraint. However, during two steps of our construction we may modify the convex chain and possibly remove some of its edges.
Consider first when we move $v_2v_3$ in $P_i$ to include a point $x$. If the point $v'_3$ is not on $v_3v_4$, then we will have decreased the number of edges in the convex chain when we replaced the right end chain with the chain consisting of the convex chain between the backward extension point of $u_{i+1}$ and $v'_3$, the edge $v'_2v'_3$ and the segment $v'_2v'_1$ (See Figure 4.6b); call this modified polygon $P'_i$. More specifically, let $v_iv_{i+1}$ be the edge $v'_3$ is on in $P_i$; if $v'_3$ coincides with a vertex of $P_i$, then we choose the edge after this vertex to be the edge $v_iv_{i+1}$. Then we have removed the edges $v_3v_4, v_4v_5, \ldots, v_{i-1}v_i$ from $P_i$ to obtain $P'_i$; call this chain of edges $C$ and assume that $v_3v_4$ is the start of $C$. If $v'_3$ coincides with $v_i$, then we also add $v_iv_{i+1}$ to $C$.

Recall that the purpose of moving $v_2v_3$ to include $x$ was to control where the extension of $v_0v_1$ through $v_1$ intersected the right end chain. This fact and Theorem 4.3 assure that we may modify the portion of the right end chain between the backward extension point of $u_{i+1}$ and $x$ without changing the extension sequence of $P'_i$ or effecting the equivalence class of $i+1$ in our constructed $P_{i+1}$. Thus, we replace the right end chain between $x$ and the backward extension point of $u_{i+1}$ with a chain having the same number of edges as $C$ and such that the $i$th edge of the new chain (starting from $x$) is parallel to the $i$th edge of $C$.

Now consider the last step in the construction, fixing any protrusions of an end chain beyond one of the bounding lines $l_0$ and $l_{i+2}$ (See Figure 4.11). Again consider the right end region, with the case for the left end region being symmetric. Let $p$ be the point, distinct from $u_{i+2}$, at which $l_{i+2}$ intersects the right end chain and let $p_{i+2}$ be the backward extension point of $u_{i+2}$. Let $v_iv_{i+1}$ be the edge that $p_{i+2}$ is on; if $p_{i+2}$ coincides with a convex vertex, then we choose the edge after this vertex to be $v_iv_{i+1}$. We replace the chain $v_2v_3, v_3v_4, \ldots, v_{i+1}v_{i+2}$ with a chain $v'_2v'_3, v'_3v'_4, \ldots, v'_sv'_{s+1}$ such that $v'_sv'_{s+1}$ is parallel to $v_iv_{i+1}$ and its endpoints are the midpoint of the segment $u_{i+2}p$ and the point $p_{i+2}$. Again, Theorem 4.3 guarantees that the polygon resulting from this modification will have the same extension sequence.

Thus, the $P_{i+1}$ constructed using this modified construction algorithm will be a parallel banana spiral. \qed
4.4 VVGs and Banana Spirals

Theorem 4.10 Let $G$ be the vertex visibility graph of some spiral polygon. Then, there exists a banana spiral $P$ such that $VVG(P)$ is isomorphic to $G$.

**Proof** An algorithm for constructing a spiral polygon having a given vertex visibility graph was developed by Everett [7]. For input, this algorithm requires the graph $G$ and the two subchains of the Hamiltonian cycle that correspond to the vertices of the reflex chain and the convex chain. Let $P_G$ be the polygon constructed by this algorithm when it is given $G$ as input and let $E_G$ be its canonical representation. Also, assume that $G$ (and hence $P_G$) has $n$ vertices, $r$ of which are reflex, and that we have coordinatized the points of $P_G$ using the method described earlier. Let $v_i$, $i = 2, 3, \ldots, n - r - 1$ be the vertices of $P_G$ not on the reflex chain and let $(b_i, f_i)$ be the coordinates of $v_i$.

By Theorem 4.6, we may construct a banana spiral $P_B$ having $E_G$ as its canonical representation. Then, by Theorem 3.3, $P_G$ and $P_B$ are isomorphic. If $VVG(P_B)$ is isomorphic to $G$, then we are finished; in general, this will not be the case. Since the actual convex vertices are not important in the cell decomposition, it is possible that the number of convex vertices in $P_B$ varies greatly from the number in $P_G$. Thus, we must modify the convex chain of $P_B$ so that it has the same number of vertices as the convex chain of $P_G$ and so that the vertices of the two chains have the same visibility properties.

Let $p_i$, $i = 2, 3, \ldots, n - r - 1$ be the points of $P_B$ having the coordinates $(b_i, f_i)$; i.e. $p_i$ corresponds to $v_i$ in $P_G$. Since $P_G$ and $P_B$ are isomorphic, each $p_i$ must be a boundary point of $P_B$ and $p_i$ must have the same visibility properties as $v_i$ does in $P_G$. More specifically, $p_i$ must see both $p_{i-1}$ and $p_{i+1}$ and by definition, $\overline{p_ip_{i-1}}$ and $\overline{p_ip_{i+1}}$ are both subsets of $P_B$. We will construct a polygon $P'$ having the same reflex chain as $P_B$ and having the points $p_2, p_3, \ldots, p_{n-r-1}$ as its convex vertices $v_2, v_3, \ldots, v_{n-r-1}$.

Suppose that $p_i$ and $p_{i+1}$ are not on the same edge of $P_B$. The convex polygon whose boundary consists of $\overline{p_ip_{i+1}}$ and the portion of the convex chain between $p_i$ and $p_{i+1}$ can not have any reflex vertices of $P_B$ in its interior. Thus, replacing the convex chain between $p_i$ and $p_{i+1}$ with $\overline{p_ip_{i+1}}$ does not effect the visibilities of any of
CHAPTER 4. LAYOUT OF BANANA SPIRALS

the other \( p_j \)'s or any vertex on the reflex chain. Let \( P' \) be the polygon resulting from performing this replacement for all such \( p_i \) and \( p_{i+1} \).

If \( p_i \) and \( p_{i+1} \) are on the same edge \( e \), but neither \( p_{i-1} \) nor \( p_{i+2} \) are, then \( P' \) lies along the convex chain and we do not need to modify this portion of the convex chain. However, if \( p_{i-1} \) was also on \( e \), then the segments \( p_{i-1}p_i \) and \( p_ip_{i+1} \) would be collinear and thus \( p_i \) would be a degenerate vertex in \( P' \). Similarly, \( p_{i+1} \) would be a degenerate vertex if \( p_{i-2} \) was on \( e \).

For the general case, suppose that the vertices \( p_j, p_{j+1}, \ldots, p_k \) are all on the edge \( c = p_jp_k \) and \( j + 1 < k \); i.e. there are at least two vertices in the interior of \( e \) (the case of only one vertex being in the interior of \( e \) is a simple restriction of this case).

We will perturb the vertices \( p_{j+1}, p_{j+2}, \ldots, p_{k-1} \) so that none of them is degenerate. Consider some vertex \( p_i, j + 1 \leq i \leq k - 1 \). Let \( p_f \) be the first convex vertex clockwise of \( p_i \) that \( p_i \) does not see and let \( u_f \) be the highest numbered vertex of the reflex chain that is not seen by \( p_i \); if \( p_i \) is in the right end region, then \( p_f \) and \( u_f \) are not defined.

Similarly, let \( p_b \) be the first convex vertex counterclockwise of \( p_i \) that \( p_i \) does not see and let \( u_b \) be the lowest numbered vertex of the reflex chain that is not seen by \( p_i \); if \( p_i \) is in the left end region, then \( p_b \) and \( u_b \) are not defined.

First, suppose that \( p_f, p_b, u_f \) and \( u_b \) are all defined. We wish to perturb \( p_i \) so that it is no longer degenerate and such that \( p_f, p_b, u_f \) and \( u_b \) are all still not seen by \( p_i \). Let \( x_f \) be whichever of \( p_f \) and \( u_f \) has the lowest \( b \)-coordinate and let \( x_b \) be whichever of \( p_b \) and \( u_b \) has the highest \( f \)-coordinate. Since \( p_i \) sees neither \( x_f \) nor \( x_b \), the backward extension of \( x_f \) and the forward extension of \( x_b \) must intersect outside of \( P' \) and it is possible to perturb \( p_i \) outwards slightly without it becoming able to see \( x_f \) or \( x_b \); by perturbed outwards we mean perturbed so that its \( b \)-coordinate is decreased and its \( f \)-coordinate is increased. However, if we perturbed just \( p_i \), then we would introduce a reflex vertex adjacent to \( p_i \) and our polygon would no longer be spiral. Thus, we must simultaneously perturb every \( p_i, i = j + 1, j + 2, \ldots, k - 1 \), so that we have a single convex chain between \( p_j \) and \( p_k \). Assuming that the vertices \( p_f, p_b, u_f \) and \( u_b \) are all defined for each \( p_i, i = j + 1, j + 2, \ldots, k - 1 \), this simultaneous perturbation is possible because we can perturb each \( p_i \) outwards a small amount. All of the \( p_i \)'s on \( e \) were initially visible to each other and thus we need this to also be the case after the
perturbation. Trivially, the polygon $p_j, p_{j+1}, \ldots, p_k$ is convex and all of its vertices are visible to each. Furthermore, Lemma 2.6 indicates that since the $b$-coordinate is decreased and the $f$-coordinate is increased for each $p_i$, $i = j + 1, j + 2, \ldots, k - 1$, all of the unperturbed vertices that $p_i$ originally saw will still be seen. Thus, the vertices of $P'$ that $p_i$ sees are not changed.

Now suppose that $p_j$ and $u_j$ are not defined; i.e. $p_i$ is in the right end region. Let $p'_b$ be the first convex vertex clockwise of $p_b$ and let $u'_b$ be the lowest numbered reflex vertex that $p_i$ sees. By definition, $p_i$ sees $p'_b$ but not $p_b$. Thus we perturb $p_i$, keeping it in the right end region, to the right of the backward extension of $p_b$ and to the left of the backward extensions of $p'_b$ and $u'_b$; we consider a point to be to the left (right) of an extension, if it is to the left (right) of the extension when the extension is traversed from its starting point to its extension point. The case is symmetric when $p_i$ is in the left end region.

Thus, for any given vertex $p_i$, we may perturb it slightly without changing the vertices that it sees. Therefore, we may drop the assumption that we made when we simultaneously perturbed all the $p_i$'s in the interior of $e$; i.e. that $p_j, p_b, u_j$ and $u_b$ are all defined for each $p_i$, $i = j + 1, j + 2, \ldots, k - 1$. Hence, we may replace the convex chain of $P_B$ with the $p_i$'s and then perturb any $p_i$'s that may be degenerate; let $P'$ be the polygon resulting from these modifications.

Then $P'$ has $n$ vertices, exactly $r$ of which are reflex. Furthermore, each $p_i$ has the same visibility properties as $v_i$ does in $P_G$. Hence the reflex chains of $P'$ and $P_G$ must also have the same visibility properties and thus, the mapping of $u_i$ of $P'$ to $u_i$ of $P_G$, $i = 0, 1, \ldots, r + 1$, and $p_i$ of $P'$ to $v_i$ of $P_G$, $i = 2, 3, \ldots, n - r - 1$ is an isomorphism.

Notice that although the polygon $P'$ in the above theorem has its vertex visibility graph isomorphic to $G$, it is possible that $P'$ is not isomorphic to $P_G$. Remember that we obtained $P'$ by first replacing the portion of the convex chain between $p_i$ and $p_{i+1}$ with the segment $\overline{p_i p_{i+1}}$, and then possibly perturbing some of the $p_i$'s. Whenever $p_i$ and $p_{i+1}$ were not on the same edge of $P_G$, it was possible that we were changing the extension sequence of the polygon when we forced $\overline{p_i p_{i+1}}$ to become part of the
convex chain. This is because the convex polygon whose boundary was the portion of the convex chain between $p_i$ and $p_{i+1}$ and the segment $p_i p_{i+1}$ may have contained a vertex $v$ of the cell decomposition in its interior. Removing this convex polygon removed $v$ and altered the cell decomposition.
Chapter 5

Extension Sequences of Orthogonal Spirals

In this chapter, we will describe a set of necessary and sufficient conditions on an extension sequence $\mathcal{E}$ for there to exist an orthogonal spiral having $\mathcal{E}$ as its extension sequence. The sufficiency proof given describes an inductive method for constructing such an orthogonal spiral.

5.1 Introduction

Let $\mathcal{E}$ be an extension sequence of a spiral polygon having $k$ equivalence classes. We may let $\mathcal{E} = E_1 E_2 \ldots E_k$ where each $E_i$ represents an equivalence class. We say that $i$ occurs strictly between $h$ and $j$ if one of the following is true:

1. $h \in E_s$ and $j \in E_t$ with $s < t - 1$ and $i$ belongs to one of $E_{s+1}, E_{s+2}, \ldots, E_{t-1}$

2. $h \in E_s$ and $j \in E_t$ with $s > t$ and $i$ belongs to one of $E_{s+1}, E_{s+2}, \ldots, E_k$ or $E_1, E_2, \ldots, E_{t-1}$.

Note that this is not a commutative property; i.e. if $i$ occurs strictly between $h$ and $j$, then $i$ does not occur strictly between $j$ and $h$. Conceptually, we are considering $\mathcal{E}$ to be written clockwise around a circle. If $i$ occurs strictly between $h$ and $j$, then
as we move clockwise, starting from the class containing $h$, we will encounter $i$ before we encounter $j$, regardless of where the class containing $0$ lies. The first case in the definition handles when the equivalence class of $h$ occurs before that of $j$ in the extension sequence; the second case handles when it occurs after the class of $j$.

Let $E$ be an extension sequence for a polygon having $r \geq 2$ vertices. We say that $E$ is orthogonally valid if $i$ occurs strictly between $i - 2$ and $i - 1$ for every $i = 2, 3, \ldots, r$. For example, $0 5 4 3 2 1$ and $0 2 \{4 1\} 3 5$ are orthogonally valid extensions sequences, but $0 2 3 5 1 4$ is not because 3 does not occur strictly between 1 and 2. For convenience, we also define the extension sequence $E = 01$ to be orthogonally valid.

An extension sequence $E$ is called orthogonally realizable if there exists an orthogonal spiral polygon $P$ having $E$ as its extension sequence. Such a polygon $P$ is said to be an orthogonal realization of $E$.

**Lemma 5.1** Let $E$ be an extension sequence for a spiral polygon having one reflex vertex. Then $E$ has an orthogonal realization.

**Proof** Since $E$ is an extension sequence for a spiral polygon having one reflex vertex, $E$ must be 01. An orthogonal spiral having this extension sequence has the vertices: $(0, 0), (1, 0), (1, 1), (-1, 1), (-1, -1), (0, -1)$ (See Figure 5.1).

**Theorem 5.2** Let $E$ be an extension sequence of a spiral polygon having $r$ reflex vertices. Then $E$ is orthogonally realizable if and only if $E$ is orthogonally valid.
Lemma 5.1 establishes the proof for the case of \( r = 1 \). The remainder of this chapter is a proof of Theorem 5.2 for \( r \geq 2 \). First, we establish the necessity condition: the extension sequence of an arbitrary orthogonal spiral must be orthogonally valid. Then, to establish sufficiency, we will inductively construct an orthogonal spiral having a given orthogonally valid extension sequence.

5.2 Necessity

Suppose that \( \mathcal{E} \) has an orthogonal realization, say \( P_0 \). Let the edges of the reflex chain of \( P_0 \) be numbered from 0 to \( r \), in counterclockwise order. In the same manner as when we were defining the long extension sequence, we label the cell decomposition vertices along the convex chain using the edge numbers.

Consider some reflex edge \( e_i, 2 \leq i \leq r \). Since \( P_0 \) is orthogonal, \( e_i \) is orthogonal to \( e_{i-1} \) and parallel to \( e_{i-2} \). Thus, the forward extension of \( e_{i-2} \) and the backward extension of \( e_i \) do not intersect each other and also intersect the same edge, say \( f_j \), of the convex chain of \( P_0 \) (See Figure 5.2). Also, the backward extension of \( e_i \) must occur more clockwise along the convex chain than the forward extension of \( e_{i-2} \). Furthermore, neither extension of \( e_{i-1} \) can intersect \( f_j \), as \( e_{i-1} \) must be parallel to \( f_j \). Therefore, neither extension of \( e_{i-1} \) may occur between the labels \( i - 2 \) and \( i \) on \( f_j \) and clockwise along the convex chain of \( P_0 \), \( i \) occurs after \( i - 2 \) and before \( i - 1 \).

Consider the first occurrence of the label 0, on the convex chain, at or counterclockwise of the backward extension point of \( e_i \); let \( x \) be the extension point that was given this 0 label. As the backward extension point of \( e_i \) may be extended through \( u_{i-1} \), Lemma 3.2 guarantees that such a 0 label exists. From the point \( x \), a forward extension will exist (through either \( u_{i-1} \) or \( u_i \)), implying that a more clockwise 0 label exists. Thus, the labels along the convex chain must occur exactly as \( \mathcal{E} \) as we move clockwise starting from \( x \).

There are three possible locations for \( x \) (See Figure 5.2):

1. at or counterclockwise of the backward extension point of \( e_i \) and clockwise of the forward extension point of \( e_{i-2} \)
2. at or counterclockwise of the forward extension point of $e_{i-2}$ and clockwise of the backward extension point of $e_{i-1}$.

3. at or counterclockwise of the backward extension point of $e_{i-1}$

In the first and third cases, $i$ occurs strictly between $i - 2$ and $i - 1$ with the second condition for occurring strictly between being satisfied. The second case satisfies the first condition for occurring strictly between. Thus, in $E$, for every $i = 2, 3, \ldots, r$, $i$ occurs strictly between $i - 2$ and $i - 1$.

5.3 Sufficiency

Let $E_r$ be an orthogonally valid extension sequence for a spiral having $r$ reflex vertices. Let $E_{r-1}$ be the extension sequence obtained from $E_r$ by removal of $r$. We claim that $E_{r-1}$ is orthogonally valid. Since $E_r$ is orthogonally valid, by definition, for every $i = 2, 3, \ldots, r$, $i$ occurs strictly between $i - 2$ and $i - 1$. The term $r$ plays no role in whether or not any $i = 2, 3, \ldots, r - 1$ occurs strictly between $i - 2$ and $i - 1$ and
thus, removal of $r$ from $E_r$ can not cause this condition to longer be satisfied by some $i$. Therefore, $E_{r-1}$ must be orthogonally valid.

To establish sufficiency, we will show how to construct an orthogonal spiral having $E_r$ as its extension sequence. Let $E_r, E_{r-1}, E_{r-2}, \ldots, E_2$ be the sequence of extension sequences where $E_{i-1}$ is obtained from $E_i$ by removal of $i$. We will inductively construct a polygon $P_r$ having $E_r$ as its extension sequence by constructing a sequence of polygons, $P_2, P_3, \ldots, P_r$ where $P_1$ has $E_1$ as its extension sequence. We will label the vertices of each $P_i$ according to our conventions for spirals; i.e. the reflex vertices will be $u_1, u_2, \ldots, u_i$ and the convex vertices will be $v_1, v_2, \ldots, v_{i+4}$ (every orthogonal spiral having $r$ reflex vertices must have $r + 4$ convex vertices). The spirals will be constructed so that they spiral outwards; i.e. when we add a rectangle to $P_i$ to form $P_{i+1}$, the convex hull of $P_{i+1}$ will differ from that of $P_i$. This is achieved by always positioning $v_2$ so that $v_2v_3$ intersects $\overrightarrow{v_5v_6}$ (See Figure 5.3). This ensures that the convex hull of $P_i$ is exactly the polygon having the vertices $v_1, v_2, v_3, v_4, v_5$ and that every $P_i$ may be rotated so that $v_2$ is the bottommost rightmost vertex.

First we will directly construct $P_2$. Then, in the general case, $P_{i+1}$ will be constructed from $P_i$ by modifying the right of end region of $P_i$, as is permitted by Theorem 4.3, and then adding a rectangle to the modified $P_i$, to introduce a new reflex vertex.
Chapter 5. Extension Sequences of Orthogonal Spirals

The manner in which we perform these steps ensures that the resulting orthogonal spiral has $E_{i+1}$ as its extension sequence, thus making it a satisfactory $P_{i+1}$.

If $r = 2$, then the only extension sequence that satisfies the orthogonal validity constraint is $E = 021$. In this case, let $P_2$ have the vertices (in counterclockwise order): $u_1 = (0,1)$, $u_2 = (1,1)$, $v_1 = (1,-2)$, $v_2 = (2,-2)$, $v_3 = (2,2)$, $v_4 = (-1,2)$, $v_5 = (-1,0)$, $v_6 = (0,0)$ (See Figure 5.4). Notice that the edge $v_2v_3$ intersects $v_5v_6$.

Suppose that for some $k$, $2 \leq k < r$, a polygon $P_k$ exists which is an orthogonal realization of $E_k$ and has $v_2v_3$ intersected by $v_5v_6$. We will construct $P_{k+1}$, an orthogonal realization of $E_{k+1}$, from $P_k$.

By our construction, we may assume wlog that $v_2$ is the bottommost rightmost vertex and $v_5$ is the bottommost leftmost vertex. By Theorem 4.3, we may slide $v_1v_2$ in the negative $y$ direction any amount, extending $u_kv_1$ and $v_2v_3$ simultaneously, and still have an orthogonal spiral with $E_k$ as its extension sequence. In general, when we say that we slide some edge $e$, we implicitly mean that we also extend the two edges adjacent to $e$ so that we still have a proper orthogonal polygon. Notice that, by induction, $v_1v_2$ is always at least two units below $v_5$. We slide $v_1v_2$ down until
it is also at least two units below any forward extension of the cell decomposition (excluding those in the equivalence class containing \( k \)). That is, let \( q \) be a member of the equivalence class immediately before the equivalence class of \( k \) in \( E_k \). Then, if we move counterclockwise along the convex chain of \( P_k \), starting from \( v_1 \), the last forward extension in the extension chain of \( e_q \) will be the first cell decomposition edge encountered. We slide \( v_1v_2 \) down until it is two units below the intersection point of this last forward extension and \( \overline{v_2v_3} \) (See Figure 5.5). Let this modified polygon be called \( P'_k \).

Let \( u_{k+1} \) be the point of \( \overline{u_kv_1} \) exactly one unit above \( v_1 \). Because we are constructing an orthogonal realization of \( E_{k+1} \), we eventually need to add a reflex vertex; we will do so at the point \( u_{k+1} \). Let \( l_{k+1} \) be the horizontal line that passes through \( u_{k+1} \). Since we are dealing with orthogonal polygons, \( l_{k+1} \) contains the segment that will be the backward extension of \( e_{k+1} \) through \( u_{k+1} \), when \( u_{k+1} \) becomes a reflex vertex.

By Theorem 4.3, we may slide the edge \( \overline{v_2v_3} \) in the positive \( x \)-direction any amount and still have an orthogonal realization of \( E_k \). By appropriately moving \( \overline{v_2v_3} \), we can ensure that when we add the rectangle that introduces a reflex vertex at \( u_{k+1} \), we get an orthogonal realization of \( E_{k+1} \).
To see why this will work, recall that $\mathcal{E}_k$ was obtained by removal of $k + 1$ from $\mathcal{E}_{k+1}$. Since $\mathcal{E}_{k+1}$ is orthogonally valid, $k + 1$ occurs strictly between $k - 1$ and $k$. Consider the set of equivalence classes that occur strictly between $k - 1$ and $k$ in $\mathcal{E}_k$; call this set $\Sigma$. In $\mathcal{E}_{k+1}$, there are only four possibilities for the location of the equivalence class of $k + 1$:

1. $k + 1$ becomes a member of one of the equivalence classes in $\Sigma$
2. the equivalence class is located between two members of $\Sigma$
3. the equivalence class is located before the members of $\Sigma$ and after the equivalence class of $k - 1$
4. the equivalence class is located after the members of $\Sigma$ and before the equivalence class of $k$.

In the latter three cases $k + 1$ is the only member of the equivalence class.

For every $E_i \in \Sigma$, let $l_i$ correspond to its last forward extension. In $P'_k$, each $l_i$ must intersect the convex chain after an extension of $e_{k-1}$ and before an extension...
of $e_k$, when moving clockwise along the convex chain. Also, since we are considering the last forward extensions, this must occur in the right end region, clockwise of the forward extension of $e_{k-1}$ and counterclockwise of the degenerate forward extension of $e_k$. Thus, these last forward extensions must pass through $u_k$ and intersect $v_2v_3$, and sliding $v_2v_3$ in the positive $x$-direction will decrease the $y$-coordinate at which these intersection points occur (See Figure 5.6). By our construction, initially each $l_i$ intersects $v_2v_3$ above $l_{k+1}$. As we slide $v_2v_3$ in the positive $x$-direction, eventually each $l_i$ will intersect $l_{k+1}$ before it intersects $v_2v_3$. Thus, by moving $v_2v_3$, we can control where the backward extension point of $e_{k+1}$ lies on the convex chain, relative to the other extension points, and hence, control the equivalence class of $k + 1$.

To decide where to move $v_2v_3$ to, we must consider each of the four cases (See Figure 5.7). In the first case, if $E_i$ is the equivalence class of $E_k$ that $k + 1$ becomes a member of, then we slide $v_2v_3$ so that it intersects the intersection point of $l_{k+1}$ and $l_i$. For the second case, let $E_a$ and $E_b$ be the equivalence classes that the equivalence class of $k + 1$ lies between. Let $p_a$ and $p_b$ be the points at which $l_{k+1}$ intersects $l_a$ and $l_b$, respectively. We place $v_2v_3$ so that it passes through the midpoint of $p_ap_b$. In the third case, if $E_b$ is the member of $\Sigma$ immediately after the equivalence class of $k - 1$.

Figure 5.7: Modifying $P'_k$ to obtain $P''_k$. 

```latex

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Figure 5.8: Modifying $P''_k$ to obtain $P_{k+1}$.

in $E_k$, then we may place $\overline{v_2v_3}$ one unit to the right of the intersection point of $l_b$ and $l_{k+1}$. The fourth case is satisfied by our construction of $P'_k$. Let $P''_k$ be the polygon resulting from our modifications to $P'_k$, if any.

To obtain an orthogonal spiral having $E_{k+1}$ as its extension sequence, we need to add a reflex vertex to $P''_k$. Let $R$ be the rectangle having the vertices $v_1$, $u_{k+1}$, $(x(v_5) - 2, y(u_{k+1}))$, and $(x(v_5) - 2, y(v_1))$. Let $P_{k+1}$ be the polygon such that $P_{k+1} = P''_k \cup R$ (See Figure 5.8).

Since $v_5$ was the bottommost leftmost vertex of $P'_k$, none of the newly added rectangle intersects $P''_k$ other than along $\overline{u_{k+1}v_1}$ and $P_{k+1}$ is a valid polygon. Also, since the edges of $R$ were all parallel to either the $x$-axis or $y$-axis, $P_{k+1}$ is still orthogonal. Furthermore, $P_{k+1}$ has only a single reflex chain since the only new reflex vertex in $P_{k+1}$ is $u_{k+1}$ which is immediately adjacent to $u_k$. Therefore, $P_{k+1}$ is an orthogonal spiral and we may assume that we have labelled its vertices according to our conventions for spiral polygons.

In $P_{k+1}$, the edge $\overline{v_2v_3}$ is intersected by $\overline{v_5v_6}$. Also, notice that since the edge
$\overline{v_1u_{k+1}}$ of $P_{k+1}$ is horizontal, it is collinear with $l_{k+1}$, and the backward extension of $e_{k+1}$ will intersect $\overline{v_3v_4}$ (which was $\overline{v_2v_3}$ in $P''_k$) at the same point as $l_{k+1}$. Thus, the position that we chose for $\overline{v_2v_3}$ when creating $P''_k$ determined the equivalence class of $k+1$ in $P_{k+1}$. So, the $P''_k$ constructed ensures that $P_{k+1}$ is an orthogonal realization of $E_{k+1}$.

Since $v_5$ was the bottommost leftmost vertex in $P''_k$, $v_2$ will be the bottommost leftmost vertex in $P_{k+1}$.

Thus, we have constructed an orthogonal realization of $E_{k+1}$ having $v_2$ as an extreme vertex. We may repeat this process until $k + 1 = r$, giving an orthogonal realization of $E_r$. This completes the proof of Theorem 5.2.

**Theorem 5.3** Let $E_r$ be an orthogonally valid extension sequence. We can construct an orthogonal realization of $E_r$ in $O(r^2)$ time using the real RAM computational model.

**PROOF** Theorem 5.2 assures that we may construct an orthogonal realization of $E_r$ and we consider the algorithm described in the sufficiency proof of this theorem. This algorithm behaves very similar to the banana construction algorithm analyzed in Theorem 4.8. Suppose that we have already constructed $P_i$ and are constructing $P_{i+1}$. The time required to determine the two equivalence classes that $i + 1$ will lie between (or the one equivalence class it joins) will again be $O(r)$. Similarly, the time required to slide $\overline{v_1v_2}$ or $\overline{v_2v_3}$ will be at most $O(i)$.

However, the computation of the forward extensions will again be the dominant step, and will require $O(i)$ time for each $P_i$. Thus, the total time required will be $O(r^2)$. \qed
Chapter 6

Conclusions

We have considered the problem of polygon isomorphism, defining two polygons to be isomorphic if their point visibility graphs are isomorphic. The general problem seems to be very complex, so we restricted our scope to spiral polygons. We have satisfactorily answered this case: two spiral polygons are isomorphic if and only if their cell decompositions are isomorphic. No other results of this kind are known other than trivial results on convex polygons.

Open Problem 6.1 *Given two simple polygons P and P', can we determine if P is isomorphic to P'?*

Shermer conjectures that when two polygons are isomorphic, there exists a homeomorphic isomorphism between the polygons [22] (a one-to-one mapping between two points sets is homeomorphic if both the mapping and its inverse are continuous). O'Rourke has suggested altering the definition of polygon isomorphism to always require that the isomorphism be a homeomorphism[18]. This enables the simple derivation of some interesting properties such as the boundaries of the two polygons mapping to each other. It is possible that this alternate definition will provide enough extra information to help solve the general polygon isomorphism problem.

A satisfactory definition for a cell decomposition of a general polygon has not even been established. One possibility is to define the first cell decomposition as for spiral polygons; i.e. extend the two edges adjacent to each reflex vertex through that vertex
Figure 6.1: A polygon whose cell decomposition never terminates.

until it intersects the polygon boundary. Then, as before, we let $B$ be the set of boundary points corresponding to the endpoints of the extensions through the reflex vertices. However, we now add to $B$ each point that was the intersection point of two such extensions. Then for the higher order cell decompositions, we extend each member of $B$ through every reflex vertex seen (instead of through just the highest or lowest numbered reflex vertex seen) such that the extension through the reflex vertex initially stays interior to the polygon. The new set of extension points (i.e. the points to be extended in the next higher order cell decomposition) will be each point that was either the endpoint of an extension or the intersection point of two extensions in the current cell decomposition.

Unfortunately, with this definition, it is possible that the cell decomposition construction process never terminates; that is, every cell decomposition will have at least one extension point that was not in the previous cell decomposition. Consider Figure 6.1: the first cell decomposition (under this definition) of the polygon is given, along with an additional segment $ab$ that passes through the two reflex vertices. The extension points of the polygon will asymptotically approach $a$ and $b$.

Open Problem 6.2 Define a cell decomposition for general polygons such that it captures all of the visibility structure of the polygons.
A $k$-spiral polygon is a polygon in which the boundary contains at most $k$ maximal reflex chains (a reflex chain here does not include the convex vertices adjacent to it). Every polygon is a $k$-spiral for some $k$. One approach to the general problem of polygon isomorphism is to consider specific cases of $k$; in this thesis, we have restricted our attention to 1-spiral polygons. In terms of $k$-spirals, an alternate statement of Open Problem 6.1 is: given two $k$-spiral polygons $P$ and $P'$, can we determine if $P$ is isomorphic to $P'$?

Considering the isomorphism of 2-spiral polygons is a logical extension of the work presented in this thesis. This problem again seems quite difficult, but the structure of 2-spiral polygons may be restricted enough to make this problem solvable. One potential approach would be to coordinatize the points in a 2-spiral using four coordinates, a forward and backward coordinate for each reflex chain. It may be enlightening to initially consider restricted classes of two spirals such as stars and funnels (a funnel is a 2-spiral having exactly three convex vertices, one of which is shared by both reflex chains).

We have characterized the point visibility graphs of spiral polygons as interval graphs, and shown that the point visibility graphs of general polygons are not chordal. Shermer has found some forbidden induced subgraphs (some of which are bipartite) of point visibility graphs [21]. However, no complete characterization of point visibility graphs is known.

**Open Problem 6.3** Characterize point visibility graphs.

Again, it may be informative to initially consider this problem for 2-spirals.

We have determined a canonical representation for the visibility structure of spiral polygons and shown that it conveys the same amount of visibility information as the polygon and as its cell decomposition. The existence of such a descriptor for $k$-spirals, $k > 1$, or any other class of polygons is unknown.

In Theorem 4.10 we showed that for every spiral polygon $P$, there exists a banana spiral $P_B$ whose vertex visibility graph is isomorphic to that of $P$. However, our proof does not guarantee that $P$ and $P_B$ are isomorphic, only their vertex visibility graphs. Thus, an interesting extension would be, given a spiral polygon $P$, determine
a banana polygon $P'$ that is isomorphic to $P$ and such that $VVG(P)$ is isomorphic to $VVG(P')$. This problem seems like it will require that the construction algorithm for the banana spirals be modified so that it uses some of the information in the vertex visibility graph of $P$.

We have shown how to construct a banana spiral or a parallel banana spiral from a canonical representation. However, one could consider the other extreme case for the reflex chain; i.e. what is the maximum number of times we can force a realization of a canonical representation to wrap around itself (the idea of wrapping around would have to be more precisely defined). We have also established necessary and sufficient conditions on a canonical representation for there to be an orthogonal spiral having this canonical representation and we provide a construction algorithm for generating such an orthogonal spiral. Further work may be done using different constraints on the angles of the reflex chain or constraints on the edges of the polygon. For example, for a given canonical representation can we always construct a spiral such that: 1) the exterior angle between two consecutive edges of the reflex chain is some constant value; 2) the exterior angle between two consecutive edges of the reflex chain is at most $180^\circ - \alpha$, for some angle $\alpha$; 3) all the edges have the same length; 4) the reflex chain is chain of chords of some semicircle.

Ongoing work involves further investigation into the properties of canonical representations. Work has been done to enumerate the nonisomorphic spirals having a given number of reflex vertices. This is also being done for orthogonal spirals. It will also be interesting to investigate algorithms that answer questions about the class of polygons by considering only a canonical representation and not constructing a member of the class. For example, given a canonical representation $C$, are the members of the class represented by $C$ starshaped? The answer to this problem is yes, if and only if $C = 0 \ 1 \ 2 \ldots \ r$. Pure visibility problems (See Table 1.1) should be solvable like this. Finally, we are also working on building a catalogue of spiral polygons having a small number of reflex vertices ($r \leq 6$).
Appendix A

Glossary

**backward extension** Let \( x \) be a point in a \( P \) and let \( u_i \) be the lowest numbered reflex vertex seen by \( x \). The *backward extension* from \( x \) is the line segment from \( x \) through \( u_i \), extended until it intersects the convex chain at \( x'' \) (See Figure 2.2). If \( x \) is in the left end region, then no backward extension is defined. The point \( x'' \) is called the *backward extension point* of \( x \). The segment \( \overline{ux''} \) is called the *former half* of the extension, while the segment \( \overline{u_i x''} \) is called the *latter half*.

**backward extension chain** Let \( u_i \) be a reflex vertex of a spiral polygon. The backward extension chain of \( u_i \) is the polygonal chain consisting of \( u_i \), the backward extension \( \overline{u_i b_1} \) of \( u_i \), the backward extension \( \overline{b_1 b_2} \) of \( b_1 \), the backward extension \( \overline{b_2 b_3} \) of \( b_2 \), etc. until \( b_j \) lies in the left end region.

**banana spiral** Let \( P \) be a spiral polygon having its vertices numbered according to our conventions for spiral polygons. Let \( l_0 \) and \( l_{r+1} \) be the lines orthogonal to \( \overline{u_0 u_{r+1}} \) that pass through \( u_0 \) and \( u_{r+1} \), respectively. Then, \( P \) is called a *banana spiral* if none of \( l_0 \), \( l_{r+1} \) and \( \overline{u_0 u_{r+1}} \) intersect the interior of \( P \). Notice that this definition allows the edges \( \overline{v_{n-r} v_{n-r-1}} \) and \( \overline{v_1 v_2} \) to lie along \( l_0 \) and \( l_{r+1} \), respectively (remember that \( u_0 = v_{n-r} \) and \( u_{r+1} = v_1 \)) (See Figure 4.1).

**blocking vertex** Assume that a graph \( G \) has a Hamiltonian cycle \( H \) and define the chain from \( v_a \) to \( v_b \), denoted chain(\( v_a, v_b \)), to be the ordered path in \( H \) from \( v_a \)
to \(v_b\), inclusive. Let \(v_i, v_j\) and \(v_k\) be vertices in \(G\) such that \(v_i < v_j < v_k\) with respect to the ordering of the vertices in \(H\). The vertex \(v_j\) is a blocking vertex with respect to \(H\) for the invisible pair \(v_i\) and \(v_k\) if \(i \neq k\) and no two vertices \(v_s \in \text{chain}(v_i, v_{j-1})\) and \(v_t \in \text{chain}(v_{j+1}, v_k)\) are adjacent in \(G\). An invisible pair is minimal with respect to \(v_j\) if \(v_j\) is its only blocking vertex.

canonical representation  The canonical representation of a spiral polygon \(P\), denoted \(CR(P)\), is the lexicographically smaller of the extension sequence of \(P\) and the reflection of the extension sequence. In the lexicographic ordering, we consider sets to occur after single numbers.

cell decomposition  Let \(P\) be a spiral polygon. The first order cell decomposition of \(P\), denoted \(CD_1(P)\), is the diagram consisting of the boundary of \(P\) together with the forward and backward extensions of the vertices of the reflex chain (See Figure 2.3). Let \(B\) be the set of extension points of the extensions of \(CD_1(P)\). The second order cell decomposition of \(P\), denoted \(CD_2(P)\), is obtained by adding the forward and backward extensions of each member of \(B\) to \(CD_1(P)\) (See Figure 2.4). Notice that for each member of \(B\), either the forward or backward extension is already in \(CD_1(P)\); this extension is not added again to \(CD_2(P)\). The third and higher order cell decompositions are determined in a similar manner, by adding the forward and backward extensions of each extension point new to the previous cell decomposition. A cell decomposition is well defined for each possible order. However, it is possible that for a given order \(i\), \(CD_i(P)\) is identical to \(CD_k(P)\) for all \(k > i\). Such a cell decomposition is called a full cell decomposition (or just the cell decomposition of \(P\)) and is denoted \(CD(P)\) (See Figure 2.5).

chordal graph  A continuous graph is a chordal graph if it contains no chordless induced cycles of length four or greater.

chordless cycle  A cycle \([v_0, v_1, \ldots, v_k, v_0]\) is chordless if \(v_iv_j \notin E\) for \(i\) and \(j\) differing by more than one, except the edge \(v_kv_0\).
**chromatic number** The *chromatic number* of $G$, denoted $\chi(G)$, is the fewest number of colours needed to properly colour the vertices of $G$.

**clear visibility** Let $x$ and $y$ be two points in a polygon $P$. Then $x$ is said to be *clearly visible to $y$* if the line segment $\overline{xy}$ does not intersect the boundary of $P$, except possibly at $x$ or $y$.

**clique** A subset of size $k$ of the vertex set $V(G)$ is called a $k$-clique if it induces a complete subgraph.

**clique cover number** The *clique cover number* of $G$, denoted $k(G)$, is the fewest number of complete subgraphs such that the union of their vertex sets equals the vertex set of $G$.

**clique number** The *clique number* of $G$, denoted $\omega(G)$, is the size of the largest complete subgraph of $G$.

**colouring** A *$c$-colouring* of $G$ is a partition of the vertices into $c$ sets such that no two vertices in the same set are adjacent.

**complete graph** A graph $G$ is called a *complete graph* if every pair of distinct vertices is adjacent.

**connectedness** The graph $G$ is called $k$-connected if $k$ is the minimum number of vertices whose removal can result in a disconnected graph.

**convex chain of a spiral polygon** The chain of consecutive vertices in a spiral which includes all of the convex vertices and none of the reflex vertices.

**convex vertex** A vertex of a polygon is called *convex* if its interior angle is less than $180^\circ$.

**corridor** A *corridor* of a spiral polygon is the region visible to an open edge of the reflex chain (See Figure 2.8).

**cover** A collection of subsets of $P$ is said to cover $P$ if their union is exactly $P$; $P$ is said to be covered by the collection.
cycle A cycle in a graph $G$ is a path in which the first and last vertices are identical.

end regions of a spiral polygon Let $P$ be a spiral polygon having $r$ reflex vertices and whose vertices are labelled as is described in the definition of a spiral polygon. Then $P$ contains two end regions (See Figure 2.1):

- the subpolygon of $P$ that contains $u_0$ and is one of the two subpolygons formed by cutting $P$ along the extension of $u_0u_1$ through $u_1$, extended until it intersects the convex chain. Call this the left end region.
- the subpolygon of $P$ that contains $u_{r+1}$ and is one of the two subpolygons formed by cutting $P$ along the extension of $u_ru_{r+1}$ through $u_r$, extended until it intersects the convex chain. Call this the right end region.

equivalence class Each label in the extension sequence of $P$ is considered to be an equivalence class of reflex edges. An equivalence class is a maximal set of reflex edges that all have identical extension chains.

extension chain Let $e_i = u_iu_{i+1}$ be an edge in the reflex chain of a spiral. The extension chain of $e_i$ is the union of the forward extension chain of $u_i$ and the backward extension chain of $u_{i+1}$ (See Figure 3.1).

extension sequence The prefix of a long extension sequence, up to, but not including the second 0 label.

forward extension Let $x$ be a point in a $P$ and let $u_i$ be the highest numbered reflex vertex seen by $x$. The forward extension from $x$ is the line segment from $x$ through $u_i$, extended until it intersects the convex chain at $x'$ (See Figure 2.2). If $x$ is in the right end region, then no forward extension is defined. The point $x'$ is called the forward extension point of $x$. The segment $ux'$ is called the former half of the extension, while the segment $x'x''$ is called the latter half.

forward extension chain Let $u_i$ be a reflex vertex of a spiral polygon. The forward extension chain of $u_i$ is the polygonal chain consisting of $u_i$, the forward extension $u_if_1$ of $u_i$, the forward extension $f_1f_2$ of $f_1$, the forward extension $f_2f_3$ of $f_2$, etc. until $f_j$ lies in the right end region.
guard Let $P$ be a polygon and let $G$ be a set of points whose visibility polygons cover $P$. Then $G$ is called a guard set of $P$ and $G$ is said to guard $P$.

Hamiltonian cycle A Hamiltonian cycle in a graph is a cycle which goes through each vertex of the graph once and only once.

interval graph A graph $G$ is called an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals of a linearly ordered set (i.e. the real line) such that two vertices are connected by an edge if and only if their corresponding intervals have nonempty intersection.

invisible pair An invisible pair of vertices in a graph $G$ is a pair of vertices that have no edge between them.

kernel Let $P$ be a star shaped polygon. Then the kernel of $P$, denoted $K(P)$, is that set of points in $P$ that see all of $P$. That is $K(P) = \{x \in P \mid \forall p \in P, x \neq p \}$. 

left end chain Let $p$ be the forward extension point of $u_0$. Then the left end chain of $P$ is the portion of the boundary of $P$ from $u_0$ clockwise to $p$.

link-$j$ visibility Two points in a polygon are called link-$j$ visible if there exists a path between them composed of $j$ or fewer line segments that lie entirely inside the polygon.

long extension sequence The long extension sequence of a spiral polygon $P$ is the list, read clockwise along the convex chain, of labels of the vertices of $CD(P)$, where each label corresponds to the edge that the extension chain producing the vertex is associated with (See Figure 3.2).

neighbourhood Let $v$ be a node in a graph $G$. Then the neighbourhood of $v$, denoted $N(v)$, is the set of all nodes of $G$ that are adjacent to $v$.

occurs strictly between Let $E$ be an extension sequence of a spiral polygon having $k$ equivalence classes. Thus, we may let $E = E_1E_2\ldots E_k$ where each $E_i$ represents an equivalence class. We say that $i$ occurs strictly between $h$ and $j$ if one of the following is true:
1. $h \in E_s$ and $j \in E_t$ with $s < t-1$ and $i$ belongs to one of $E_{s+1}, E_{s+2}, \ldots, E_{t-1}$

2. $h \in E_s$ and $j \in E_t$ with $s > t$ and $i$ belongs to one of $E_{s+1}, E_{s+2}, \ldots, E_k$
or $E_1, E_2, \ldots, E_{t-1}$

ordered cycle A cycle in a graph $G$ is ordered with respect to a Hamiltonian cycle $H$ if, for all vertices $v_i$ and $v_j$ in the cycle, $v_i$ precedes $v_j$ in the cycle implies that $v_i$ precedes $v_j$ in $H$.

orthogonally realizable An extension sequence $E$ is called orthogonally realizable if there exists an orthogonal spiral $P$ having $E$ as its extension sequence. The polygon $P$ is said to be an orthogonal realization of $E$.

orthogonally valid Let $E$ be an extension sequence for a polygon having $r \geq 2$ vertices. We say that $E$ is orthogonally valid if $i$ occurs strictly between $i - 2$ and $i - 1$ for every $i = 2, 3, \ldots, r$.

path A path in a graph $G$ is an alternating sequence of vertices and edges that starts and ends with a vertex and such that no two vertices in the sequence are identical; the number of edges in a path is called its length.

perfect graph A perfect graph is a graph such that for every subset $A$ of $V(G)$, $\omega(G_A) = \chi(G_A)$ and $\alpha(G_A) = k(G_A)$, where $G_A$ denotes the subgraph induced by the vertices of $A$.

point visibility graph Let $P$ be a simple polygon. We define the point visibility graph of $P$ denoted $PVG(P)$ as follows:

- $V(PVG(P)) = \{x \mid x \in P\}$
- $E(PVG(P)) = \{(x, y) \mid xy \subseteq P\}$

reflex chain of a spiral polygon The chain of consecutive vertices in a spiral that includes all of the reflex vertices and the two convex vertices immediately adjacent to the reflex vertices.

reflex vertex A vertex of a polygon is called reflex if its interior angle exceeds $180^\circ$. 
removal Let $\mathcal{E}_{r+1} = E_1 E_2 \ldots E_k$ be a valid extension sequence for a spiral polygon having $r + 1$ reflex vertices and $k$ equivalence classes. Let $E_j$ be the equivalence class that $r + 1$ belongs to and let $E'_j = E_j \setminus \{r + 1\}$. If $E'_j = \emptyset$, then let $\mathcal{E}_r = E_1 E_2 \ldots E_{j-1} E_{j+1} \ldots E_k$ otherwise, let $\mathcal{E}_r = E_1 E_2 \ldots E_{j-1} E'_j E_{j+1} \ldots E_k$. We say that $\mathcal{E}_r$ is obtained from $\mathcal{E}_{r+1}$ by removal of $r + 1$ (See Table 4.1).

replacement chain Let $P'$ be the object resulting from replacing the right end chain of $P$ with some new polygonal chain $c'$. Then $c'$ is a replacement chain if:

- $P'$ is a spiral polygon having $r$ reflex vertices
- the last segment of $c'$ is a non-trivial portion of $u_{r+1} u_r$

right end chain Let $p_{r+1}$ be the backward extension point of $u_{r+1}$. Then the right end chain of $P$ is the portion of the boundary of $P$ from $p_{r+1}$ clockwise to $u_r$.

semikernel A semikernel in a polygon $P$ is a maximal set of points that all have the same visibility polygon. If the set has cardinality greater than one, it is considered to be a nontrivial semikernel.

separable invisible pairs Two invisible pairs $\{v_i, v_k\}$ and $\{v_s, v_t\}$ are separable with respect to $v_j$ if, when $H$ is traversed from $v_j$, either $v_i$ and $v_k$ are encountered before $v_s$ and $v_t$ or vice versa.

simplicial node A graph node $v$ is called simplicial if its neighbourhood is a complete graph.

simplicial point A point $p$ in a polygon $P$ is called simplicial if its visibility polygon is convex, or equivalently, if the node of $PVG(P)$ that corresponds to $p$ is a simplicial node.

spiral polygon Let $P$ be a polygon having $n$ vertices, exactly $r$ of which are reflex. Then, $P$ is called spiral if in a counterclockwise traversal of the vertices of $P$ starting from some convex vertex, the $r$ reflex vertices occur consecutively. The vertices of $P$ may be labelled so that $u_1, u_2, \ldots, u_r$ are reflex and consecutive.
in the order given and \( v_1, v_2, \ldots, v_{n-r} \) are convex and consecutive in the order given (See Figure 1.4).

**starshaped polygon** A polygon \( P \) is called *star shaped* if some non-empty subset of \( P \) has all of \( P \) as its visibility polygon.

**valid extension sequence** Let \( \mathcal{L} \) be a partitioning of the numbers \( 0, 1, \ldots, r \) into \( k \) sets. Let \( \mathcal{E} \) be an ordered list of the members of \( \mathcal{L} \) that starts with the set containing 0. If no set in \( \mathcal{E} \) contains both \( i \) and \( i + 1 \), \( i = 0, 1, \ldots, r - 1 \), then we call \( \mathcal{E} \) a *valid extension sequence*.

**visibility** Let \( x \) and \( y \) be two points in a polygon \( P \). Then \( x \) is said to be *visible to* \( y \) if the line segment \( xy \) does not intersect the exterior of \( P \). We say that \( x \) sees \( y \).

**visibility polygon** Let \( P \) be a polygon and let \( x \in P \). Then the *visibility polygon* of \( x \), denoted \( VP(x) \), is the set of points of \( P \) that are visible to \( x \).
Bibliography


