A PROOF OF HANF'S THEOREM ON
ISOMORPHIC FIRST ORDER LANGUAGES

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ABSTRACT

In this paper we present a proof of a theorem on isomorphisms of first order languages. Both the definition of isomorphism between two first order languages and the theorem, which states sufficient conditions for two first order languages to be isomorphic, are due to W. Hanf.

The methods of proof used are all basic techniques of first order logic and elementary set theory.

In chapter 1 the necessary and basic definitions are given and the notation is explained. A definition of isomorphism between first order languages and a statement of the main result are given in chapter 2, as well as an outline of the proof of the main result. Most of the necessary preliminary results are obtained in chapter 3, and are combined in chapter 4 to yield the main theorem. Chapter 4 concludes with several remarks on certain features of the proof and with some references to Hanf's more general statement of his result.
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INTRODUCTION

In this thesis we prove a theorem by Hanf concerning formal languages. Hanf's theorem applies to languages which are specified by designating a finite number of symbols, say $p_1, \ldots, p_n$, to be predicate symbols with $p_i$ having $k_i$ argument places, where $k_i$ is finite. By convention it is understood that $p_i$ is $=$ for some $i$.

If $L$ is such a language, then we associate with it a class of structures such that $L$ may be thought of as referring to any of them. Let $L$ be such a language, then a structure $\mathcal{L}$ for $L$ consists of a nonempty set $|\mathcal{L}|$, called the universe of $\mathcal{L}$, together with a rule which assigns to each $p_i$ a $k_i$-place predicate on $|\mathcal{L}|$.

For example, if $n = 2$ and $k_1 = 2 = k_2$, then we can define a structure $\mathcal{N}$ for $L$ by letting $|\mathcal{N}|$ be the set of natural numbers and assigning the usual relations of equality and order, denoted by $=$ and $<$, to $p_1$ and $p_2$ respectively.

With each language $L$ we associate the set of all sentences of $L$. A precise definition of $L$ is given in chapter 1. However, we can briefly describe $L$ as follows. To the predicate symbols $p_1, \ldots, p_n$ add a countable list of variables $x_1, x_2, \ldots$ and the logical symbols $\&$, $\lor$, $\neg$, $\forall$, $\exists$ which intuitively mean "and", "or", "not", "for all", and "there exists" respectively.
For convenience we also add parentheses (, ) at this point. Then \( L \) consists of all finite sequences of the above symbols which can be regarded as making meaningful assertions about a structure \( \mathcal{L} \) for \( L \) when the variables are regarded as ranging through the universe of \( \mathcal{L} \).

Let us return to the example above where \( n = 2 \) and \( k_1 = 2 = k_2 \). Then the sequences \( x_1 = \exists \ell \) and \( p_2x_1x_2 \) are not sentences of \( L \); \( x_1 = \exists \ell \) is clearly not a sentence of \( L \) because it makes no meaningful assertion about \( \eta \) and \( px_1x_2 \) is not a sentence of \( L \) because there is no way of knowing whether it is supposed to be an assertion about all natural numbers or just some of them and hence is not meaningful. On the other hand the sequences \( \forall x_1 \exists x_2 (\exists p_1x_1x_2 \land p_2x_1x_2) \) and \( \exists x_1 \forall x_2 (p_1x_1x_2 \lor p_2x_2x_1) \) are sentences of \( L \); \( \forall x_1 \exists x_2 (\exists p_1x_1x_2 \land p_2x_1x_2) \) can be regarded as asserting that for every natural number \( x \) there is another distinct natural number \( y \) which is greater than \( x \) and \( \exists x_1 \forall x_2 (p_1x_1x_2 \lor p_2x_2x_1) \) can be regarded as asserting that there is a natural number \( x \) such that for every natural number \( y, x \) equals \( y \) or \( y \) is less than \( x \).

Now each sentence \( \Lambda \) of \( L \) makes a meaningful assertion about \( \mathcal{L} \); this assertion says something about \( \mathcal{L} \) which is either true or false, but not both. Therefore, to each sentence \( \Lambda \) of \( L \) we can assign a truth-
value $T$ or $F$, with respect to $L$, denoted by $L(A) = T$ or $L(A) = F$. A precise definition of truth-values is given in chapter 1.

Let $L$ and $M$ be two languages which are specified by designating a finite number of predicate symbols. Then Hanf defines $L$ and $M$ to be isomorphic just in case there are bijections $F$ and $G$ such that

(i) $F$ maps $L$ the class of all structures of $L$ to $M$ the class of all structures of $M$;
(ii) $G$ maps $L$ the class of all sentences of $L$ to $M$ the class of all sentences of $M$;
(iii) for any sentence $A$ of $L$ and any structure $X$ of $L$, $X(A) = T$ just in case $F(X)(G(A)) = T$.

If $A$ is a sentence of $L$ and $X$ is a structure of $L$, then we write $X \models A$ just in case $L(A) = T$. Let $A \times B$ denote the Cartesian product of the classes $A$ and $B$.

Using the above notation we see that we can consider $\models$ as a binary relation on, say $L \times L$ such that for any $A$ in $L$ and $X$ in $L$, $\langle X, A \rangle$ is in $\models$ just in case $L(A) = T$. Therefore we can consider $\langle L \times L, \models \rangle$ as a mathematical structure in the same way that we can consider, say $\langle G, \circ \rangle$ as the group $G$ with the operation $\circ$. Then we see that if $L$ is isomorphic to $M$ via the bijections $F$ and $G$ as above, then $\langle L \times L, \models \rangle$ and $\langle M \times L, \models \rangle$ are isomorphic in the usual algebraic
sense via the bijection $H$, where $H$ is defined by: for any $<\mathcal{L},A>$ in $\mathcal{L} \times \mathcal{L}$, $H(<\mathcal{L},A>) = <F(\mathcal{L}), G(A)>$.  

We now state Hanf's result:

**Theorem:** If $L$ and $M$ are both languages of the kind described above and they both have a predicate symbol of two or more places which is not $=$, then $L$ is isomorphic to $M$.

Before proceeding we point out that the languages considered above are all first order languages. A precise definition of a first order language is given in chapter 1. The study of first order languages has been central to the study of many axiom systems used in mathematics and has often yielded important results. Hence our interest in Hanf's theorem.

Throughout this paper the notations $[ ]$, ( ), $\Box$ are used to denote references to the bibliography, footnotes, and the ends of proofs respectively.

We refer the reader to [3], chapter 1, for Hanf's discussion of isomorphism and his result.

In chapter 1 we give precise definitions of a first order language and its related notions as well as any set-theoretic or functional notation which is used. Unless otherwise stated, we refer the reader to [1] for any definitions or results concerning ordinals, cardinals, orderings, etc. Chapter 2 is devoted to formulating a slightly stronger form of Hanf's result and outlining
its proof. The necessary preliminary results appear in chapter 3 and in chapter 4 these results are tied together in two theorems which yield the main result. The final section of chapter 4 consists of several remarks on certain features of our proof as well as a short discussion of Hanf's more general formulation of his theorem and related results.

Unless otherwise stated we shall use the following notations:

(i) \( L, M, N \) denote first order languages and \( L, M, N \) denote their respective classes of all sentences;
(ii) \( \mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{N} \) denote structures for first order languages and \( \mathcal{A}, \mathcal{B}, \) etc., denote classes of structures for first order languages;
(iii) \( \alpha, \beta, \xi, \chi, \xi, \sigma \) denote cardinals;
(iv) \( \psi, \mu, \phi, \theta, \tau \) denote functions between universes of structures;
(v) any primed and/or subscripted variations of the above are allowed.
1.1 General Description of a First Order Language

Let $L$ be a first order language; then $L$ has the following symbols:

(i) variables $x_1, x_2, \ldots$;

(ii) for each non-negative integer $n$, the $n$-ary function symbols and the $n$-ary predicate symbols;

(iii) $\exists, \forall, \neg$.

A constant is a 0-ary function symbol and for each non-negative integer $n$ there are any non-negative number of function and predicate symbols. By convention we include, for $n=2$, the binary predicate symbol $\equiv$ (equality). There are countably many variables. The above listing of the variables of $L$ is called the alphabetical listing of the variables of $L$.

In all subsequent definitions of syntactical variables it is assumed that all subscripted, superscripted, or primed variations, as well as any combination thereof, are allowed.

We use $x, y, z$ as syntactical variables which range through the variables of $L$; we use $f$ and $e$ as syntactical variables which range through the function symbols and constants of $L$; furthermore we use $p$ and $q$ as syntactical variables which range through the predicate symbols of $L$.

A function symbol or predicate symbol other than
is called a nonlogical symbol; other symbols are called logical symbols.

An expression of $L$ is any finite sequence of symbols of $L$ and the number of symbols appearing in an expression is the length of the expression.

We use $u$ and $v$ as syntactical variables which range through the expressions of $L$.

If $u$ and $v$ are arbitrary syntactical variables, then $uv$ is the expression obtained by writing down $u$ and then writing down $v$ immediately after it. For example, if $u$ represents the expression $7$ and $v$ represents the expression $3x_1$, then $uv$ represents the expression $73x_1$.

We inductively define terms by:

(i) a variable is a term;
(ii) if $u_1, \ldots, u_n$ are terms and $f$ is an $n$-ary function symbol, then $fu_1 \ldots u_n$ is a term.

Whenever context makes our usage clear we shall often not underline the syntactical variables being used unless they are being defined. For example, we usually write $px_1 \ldots x_n$ instead of $px_1 \ldots x_n$.

We use $a$, $b$, and $c$ as syntactical variables which range through the terms of $L$.

An atomic formula is an expression of the form $pa_1 \ldots a_n$ where $p$ is $n$-ary.

The definition of formula is given inductively
(i) an atomic formula is a formula;
(ii) if \( u \) is a formula, then \( \forall u \) is a formula;
(iii) if \( u \) and \( v \) are formulas, then \( \forall uv \) is a formula;
(iv) if \( u \) is a formula, then \( \exists xu \) is a formula.

Definition 1: A first order language is a language whose symbols and formulas are as described above.

It is worth noting that a first order language is completely determined by its nonlogical symbols.

The following convention is used. If a symbol is used as an n-ary function symbol in one first order language then it will not be used in any other first order language except as an n-ary function symbol (n is the same in all cases). The same convention is used with regard to the predicate symbols.

It is now evident that all first order languages are uniquely determined by their nonlogical symbols.

1.2 More Notation and Conventions

An appearance of the symbol \( u \) in the expression \( v \) is called an occurrence of \( u \) in \( v \); a designator is an expression which is either a term or a formula.

We shall use \( A, B, C, \) and \( D \) as syntactical variables which range through the formulas of \( L \).

It is easily seen that every designator has the form \( uv_1 \ldots v_n \) where \( u \) is a symbol and \( n \) is a non-negative integer determined by \( u \). We call \( n \) the index
of $u$.

The expressions $u$ and $v$ are **compatible** just in case one can be obtained from the other by adding some expression, possibly the empty expression, to the right of the other. Some immediate consequences of the definition of compatible are:

(i) if $uv$ and $u_1v_1$ are compatible, then $u$ and $u_1$ are compatible;

(ii) if $uv_1$ and $uv$ are compatible, then $v_1$ and $v$ are compatible.

See [5], page 15, for a proof of the following result.

**Proposition 1.** If $u_1, \ldots, u_n$ and $u_1', \ldots, u_n'$ are designators and $u_1 \ldots u_n$ is compatible with $u_1' \ldots u_n'$, then $u_i$ is $u_i'$ for $i = 1, 2, \ldots, n$.

As a consequence of proposition 1, we have the next result; see [5], page 16, for a proof.

**Proposition 2.** Every designator can be written in the form $uv_1 \ldots u_n$, where $u$ is a symbol of index $n$ and $v_1, \ldots, v_n$ are designators, in exactly one way.

Proposition 2 tells us that given an expression we can determine the proper grouping of its parts in a unique way, to ensure the existence of exactly one way of reading the expression.

An occurrence of $x$ in $A$ is bound in $A$ if it occurs
in a part of A of the form $\exists x B$; otherwise, it is free in A. We say that x is free (bound) in A if some occurrence of x in A is free (bound) in A. The reader should note that x may be both free and bound in A.

If A has at most the free variables $x_1, \ldots, x_n$, then we write $A(x_1, \ldots, x_n)$. We use $b_x[a]$ to denote the expression obtained by replacing each occurrence of x in b by a, and $A_x[a]$ is the expression obtained by replacing each free occurrence of x in A by a.

If we induct on the length of b, we see that $b_x[a]$ is a term; likewise if we induct on the length of A we see that $A_x[a]$ is a formula.

Usually $A_x[a]$ says the same thing about the individual denoted by a that A says about the individual denoted by x; sometimes this does not happen. For example, if A is $\exists y(x = 2y)$, x is x and a is $y + 1$, then A says that x is even, and $A_x[a]$, which is $\exists y(y + 1 = 2y)$ no longer says that $y + 1$ is even. The change has occurred because the y occurring in a was bound in A. To avoid such difficulties we use the following convention.

We say that a is substitutable for x in A if, for each variable y occurring in a, no part of A of the form $\exists y B$ contains a free occurrence of x in A. Whenever $A_x[a]$ is written it is understood that a is substitutable for x in A.
The above definitions are extended as follows:

(i) \( b_{x_1, \ldots, x_n} [a_1, \ldots, a_n] \) is used to denote the term obtained from \( b \) by replacing each occurrence of \( x_i \) by \( a_i \) for \( i = 1, 2, \ldots, n \);

(ii) \( A_{x_1, \ldots, x_n} [a_1, \ldots, a_n] \) is the formula obtained from \( A \) by replacing each free occurrence of \( x_i \) in \( A \) by \( a_i \) for \( i = 1, 2, \ldots, n \).

Again the convention that \( a_i \) be substitutable for \( x_i \), for \( i = 1, 2, \ldots, n \), is followed. When the \( x_1, \ldots, x_n \) are immaterial or context makes their usage clear they will often be dropped from the notations \( b_{x_1, \ldots, x_n} [a_1, \ldots, a_n] \) and \( A_{x_1, \ldots, x_n} [a_1, \ldots, a_n] \).

The following defined symbols will be used as shown below:

(i) \((A \lor B)\) is an abbreviation of \( \lor AB \);

(ii) \((A \rightarrow B)\) is an abbreviation of \( \neg A \lor B \);

(iii) \((A \land B)\) is an abbreviation of \( \neg (A \rightarrow \neg B) \);

(iv) \((A \leftarrow \rightarrow B)\) is an abbreviation of \( (A \rightarrow B) \land (B \rightarrow A) \);

(v) \( \forall x A \) is an abbreviation of \( \exists x \forall A \)

(vi) if \( u \) is a binary predicate or function symbol, then \( \{aub\} \) is an abbreviation of \( u a b \);

(vii) if \( u \) is a binary predicate symbol, then \( (a \land b) \) is an abbreviation of \( \neg \{aub\} \).

In each case the defined formulas represent those expressions obtained when the defined symbols are
eliminated, according to the rules given above.

Parentheses are often omitted providing no ambiguity arises. For example, we write \( x = y \rightarrow y = x \) instead of \((x = y) \rightarrow (y = x)\). Extra parentheses and commas are sometimes added for increased ease in reading an expression. We use the convention that \( \rightarrow \) and \( \leftrightarrow \) take precedence over \( \lor \) and \&. For example, \( A \rightarrow B \lor C \) is read as \( A \rightarrow (B \lor C) \). The convention of association to the right for omission of parentheses is used whenever we have a sequence of formulas all connected by \( \lor \), \& or \( \rightarrow \). For example, \( A \lor B \lor C \) is read as \( A \lor (B \lor C) \).

Also, if \( A_1, \ldots, A_n \) are formulas we will sometimes write \( \bigvee_{i=1}^{n} A_i \) instead of \( A_1 \lor \ldots \lor A_n \).

1.3 Structures for First Order Languages

Definition 2. Let \( L \) be a first order language, then a structure \( \mathfrak{L} \) for \( L \) consists of the following:

(i) a nonempty set \( |\mathfrak{L}| \) which is called the universe of \( \mathfrak{L} \); the elements of \( |\mathfrak{L}| \) are called the individuals of \( \mathfrak{L} \);

(ii) for each \( n \)-ary function symbol \( f \) of \( L \), an \( n \)-ary function \( f_{\mathfrak{L}} \) which goes from \( |\mathfrak{L}| \) into \( |\mathfrak{L}| \); in particular, for each constant \( e \) of \( L \), \( e_{\mathfrak{L}} \) is an individual of \( \mathfrak{L} \);

(iii) for each \( n \)-ary predicate symbol \( p \) of \( L \) other than \( = \), an \( n \)-ary predicate \( p_{\mathfrak{L}} \) in \( |\mathfrak{L}| \).
We say that $\mathcal{L}$ is an $L$-structure just in case it is a structure for $L$.

Let $\mathcal{L}$ be an $L$-structure. For each individual $a$ of $\mathcal{L}$, we choose a new constant, called the **name** of $a$. It is understood that different individuals have different names. The first order language obtained from $L$ by adding a name for each individual of $\mathcal{L}$ is denoted by $L(\mathcal{L})$. We use $i$, $j$, and $k$ as syntactical variables which range through names, and if $a$ is an element of $|\mathcal{L}|$, then $i_a$ denotes the name of $a$ in $L(\mathcal{L})$. We often write $a$ for $i_a$ whenever no ambiguity arises. Also, if $\mathcal{Q}$ and $\mathcal{B}$ are both $L$-structures and $\phi$ maps $|\mathcal{Q}|$ one-one into $|\mathcal{B}|$ and $a$ in $|\mathcal{Q}|$ has the name $i$ in $L(\mathcal{Q})$, then $i^\phi$ denotes the name of $\phi(a)$ in $L(\mathcal{B})$. We often write $a^\phi$ instead of $i^\phi$ whenever no ambiguity arises.

An expression of $L$ is said to be **variable-free** just in case it contains no variables. We now define an individual $\mathfrak{z}(a)$ of the $L$-structure $\mathcal{L}$ for each variable-free term $a$ of $L(\mathcal{L})$ by means of induction on the length of $a$. If $a$ is a name, then $\mathfrak{z}(a)$ is the element of $|\mathcal{L}|$ of which $a$ is the name. If $a$ is not a name, then $a$ must have the form $f_a a_1 \ldots a_n$ with $f$ an $n$-ary function symbol of $L$ and we define $\mathfrak{z}(a)$ to be $f_{\mathfrak{z}} (\mathfrak{z}(a_1), \ldots, \mathfrak{z}(a_n))$.

In the second part of the above definition use was made of proposition 2 when we assumed that $a$ could
be written as $fa_1 \ldots a_n$ in exactly one way. Future definitions in terms of induction on the length of a term or a formula will often use proposition 2 in the same manner.

A formula $A$ is said to be **closed** if no variable is free in $A$. A closed formula is often called a **sentence** or a **statement**. If no bound variables occur in $A$, then $A$ is **open**.

Let $\mathfrak{L}$ be an $L$-structure. We now define a **truth-value** $\mathfrak{L}(A)$, which is exactly one of $\top$ or $\bot$, for each closed formula $A$ of $L(\mathfrak{L})$ as follows. If $A$ is $a = b$, then $a$ and $b$ must be variable-free because $A$ is closed and we let $\mathfrak{L}(A) = \top$ just in case $\mathfrak{L}(a) = \mathfrak{L}(b)$. If $A$ is $p_{a_1} \ldots a_n$, where $p$ is not $=$, then we let $\mathfrak{L}(A) = \top$ just in case $p_{\mathfrak{L}}(\mathfrak{L}(a_1), \ldots, \mathfrak{L}(a_n))$ is true in $\mathfrak{L}$. If $A$ is $\neg B$, then $\mathfrak{L}(A) = \bot$ just in case $\mathfrak{L}(B) = \top$. If $A$ is $B \lor C$, then $\mathfrak{L}(A) = \top$ just in case at least one of $\mathfrak{L}(B)$ and $\mathfrak{L}(C)$ is equal to $\top$. If $A$ is $\exists x B$, then $\mathfrak{L}(A) = \top$ just in case $\mathfrak{L}(B_{x[i]}) = \top$ for some $i$ in $L(\mathfrak{L})$. In each case $\mathfrak{L}(A) = \bot$ just in case $\mathfrak{L}(A)$ does not equal $\top$.

What we have done above is to define a function from the set of all closed formulas of $L(\mathfrak{L})$ onto the set $\{\top, \bot\}$. With reference to the first part of the above definition we recall that $\mathfrak{L}(a) = \mathfrak{L}(b)$ just in case $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ are the same individual
of $\mathcal{L}$.

From the definition of truth-value we have the following immediate results. If $A$ and $B$ are closed formulas of $L(\mathcal{L})$, then $\mathcal{L}(A \rightarrow B) = T$ just in case $\mathcal{L}(A) = F$ or $\mathcal{L}(B) = T$ and $\mathcal{L}(A \& B) = T$ just in case $\mathcal{L}(A) = T$ and $\mathcal{L}(B) = T$, and $\mathcal{L}(A \leftrightarrow B) = T$ just in case $\mathcal{L}(A) = \mathcal{L}(B)$. If $A_1, \ldots, A_n$ are closed formulas of $L(\mathcal{L})$, then $\mathcal{L}(A_1 \lor \ldots \lor A_n) = T$ just in case $\mathcal{L}(A_i) = T$ for at least one $i$ ($1 \leq i \leq n$) and $\mathcal{L}(A_1 \land \ldots \land A_n) = T$ just in case $\mathcal{L}(A_i) = T$ for $i = 1, 2, \ldots, n$. Lastly, if $\forall x A$ is a closed formula of $L(\mathcal{L})$, then $\mathcal{L}(\forall x A) = T$ just in case $\mathcal{L}(A_i \mid i) = T$ for every $i$ in $L(\mathcal{L})$.

We will say the two sentences $A$ and $B$ are equivalent just in case $\mathcal{L}(A \leftrightarrow B) = T$ for any $L$-structure $\mathcal{L}$.

A formula $A$ is in prenex form if it has the form $Qx_1 \ldots Qx_n B(x_1, \ldots, x_n)$, where each $Qx_i$ ($1 \leq i \leq n$) is either $\exists x_i$ or $\forall x_i$; $x_1, \ldots, x_n$ are distinct and $B$ is open. We allow the case for $n = 0$.

In [5], pages 37 and 38 we see that any formula $A$ may be converted into a formula $A'$ in prenex form. Any such formula $A'$ is called a prenex form of $A$. We also see that if $A$ is any sentence and $A'$ is a prenex form of $A$, then $A'$ is a sentence and $A$ is equivalent to $A'$.

We now extend the notion of truth with respect to a given $L$-structure $\mathcal{L}$ to the formulas of $L$. Let
A(x₁, ..., xₙ) be an L-formula and let Ł be an L-structure, then an Ł-instance of A(x₁, ..., xₙ) is a closed formula of L(Ł) of the form A[i₁, ..., iₙ]. The L-formula A is valid in Ł just in case Ł(A') = T for every Ł-instance A' of A.

Another concept connected with structures is that of isomorphism. The L-structures Ł and Ô are isomorphic if there is a one-one function φ from |Ł| onto |Ô| such that for any n-ary function symbol f of L we have for any a₁, ..., aₙ in |Ł|, φ(fₜ(a₁, ..., aₙ)) = fₛ(φ(a₁), ..., φ(aₙ)) and for any n-ary predicate symbol p of L we have for any a₁, ..., aₙ in |Ł|, pₜ(a₁, ..., aₙ) just in case pₛ(φ(a₁), ..., φ(aₙ)).

We usually write Ł ≅ Ô if Ł is isomorphic to Ô.

A very important consequence of the above definition is the fact that if A is any formula of L and Ł ≅ Ô, then A is valid in Ł just in case A is valid in Ô.

In the case where Ł and Ô are familiar algebraic structures, we see that Ł ≅ Ô just in case Ł is isomorphic to Ô in the usual algebraic sense.

Ł is an elementary or arithmetic class of L-structures if Ł consists of all L-structures in which some L-formula, say A, is valid. We say that Ł is defined by A.

Let Ł be an L-structure; the class of all L-struct-
ures which are isomorphic to $L$ is the isomorphism-type of $L$. We say that $L$ and $L'$ have the same isomorphism-types just in case $L \cong L'$. The class of all isomorphism-types of $L$-structures is denoted by $L/\cong$. Since the members of $L/\cong$ are disjoint we define a complete class of isomorphism-types of $L$ to be a class consisting of exactly one element from each member of $L/\cong$.

We refer the reader to [5] for further reference to the material in sections 1.1 to 1.3.

1.4 Set-Theoretic Background and Definitions

In the course of the paper we shall deal with some very large objects. For example, the class of all structures for a given first order language is such an object. We must be careful not to introduce set-theoretic paradoxes when we deal with such objects. Fortunately our main theorem can easily be stated and proven in the familiar Von Neumann-Bernays-Godel set theory plus the Axiom of Choice (strong form). Therefore we shall assume the axioms of this theory for the remainder of the paper.

During most of our discussions of structures, sets, and related notions, we shall use the symbols $\forall, \exists, \forall, \&,$, $\Rightarrow$, and $\Leftrightarrow$ in their usual informal sense. Context will always make such usage unambiguous. If $A$ and $B$ are classes, then the following notations have their
usual meanings: $A \cap B$, $A \cup B$, $A \subseteq B$, $A \subset B$, $A = B$, $A \neq B$, $A - B$, and $\{A \mid A$ satisfies some property $\}$.

For $n$ a positive integer we give the following inductive definition of ordered $n$-tuples:

(i) $<x_1> = x_1$;
(ii) $<x_1, x_2> = \{\{x_1\}, \{x_1, x_2\}\}$;
(iii) for $n \geq 3$, $<x_1, \ldots, x_n> = <x_1, <x_2, \ldots, x_n>>$.

If $A_1, \ldots, A_n$ are classes, then the Cartesian product of $A_1, \ldots, A_n$ is defined as $A_1 \times \ldots \times A_n = \{<a_1, \ldots, a_n > \mid a_i \in A_i$ for $i = 1, 2, \ldots, n\}$ and in the case where $A_i = A$ for $i = 1, 2, \ldots, n$ we write $A^n$ instead of $A_1 \times \ldots \times A_n$.

We say that $R$ is an $n$-ary relation on the class $A$ just in case $R \subseteq A^n$.

We also say that $F$ is a function just in case there exist classes $A$ and $B$ with $F \subseteq A \times B$ and $\forall x, y, z, <x, y> \in F \land <x, z> \in F \Rightarrow y = z$. We usually write $F(x) = y$ instead of $<x, y> \in F$.

We now introduce a series of functional notations and definitions. If $F$ and $G$ are functions and $A$ and $B$ are classes, we say that:

(i) $\text{dom}(F) = \{x \mid \exists y \ (F(x) = y)\}$;
(ii) $\text{rng}(F) = \{y \mid \exists x \ (F(x) = y)\}$;
(iii) $F(A) = \{F(x) \mid x \in A \land \text{dom}(F)\}$;
(iv) $F^{-1}(A) = \{x \mid x \in \text{dom}(F) \land F(x) \in A\}$;
(v) $F \upharpoonright A$ is the function with $\text{dom}(F \upharpoonright A) = \text{dom}(F) \cap A$ and
\( \forall x \in \text{dom}(F \upharpoonright A), F \upharpoonright A(x) = F(x); \)

(vi) \( F: A \to B \) just in case \( \text{dom}(F) = A \) and \( \text{rng}(F) \subseteq B \);

(vii) \( F: A \to B \) or \( F \) is a one-one function from \( A \) to \( B \) just in case \( F: A \to B \) and \( \forall x, y \in A, \ x \neq y \Rightarrow F(x) \neq F(y) \);

(viii) \( F: A \leftrightarrow B \) just in case \( F: A \to B \) and \( \text{rng}(F) = B \);

(ix) \( F: A \leftrightarrow B \) or \( F \) is a bijection from \( A \) to \( B \) just in case \( F: A \to B \) and \( F: A \to B \);

(x) \( F \circ G \) is the function called the composition of the functions \( G \) and \( F \); \( F \circ G \) is such that \( \text{dom}(F \circ G) = \{x \mid x \in \text{dom}(G) \land G(x) \in \text{dom}(F)\} \) and \( \forall x \in \text{dom}(F \circ G), F \circ G(x) = F(G(x)). \)

We will often modify some of the above notation as follows; if \( \text{dom}(F) \supseteq A \) and \( F(A) \subseteq B \) we will sometimes write \( F: A \to B \) instead of \( F \upharpoonright A: A \to B \).

If \( A \) is a class, then \( \aleph \) is the cardinality of \( A \), that is \( \aleph \) is the least cardinal such that there exists a bijection between \( \aleph \) and \( A \).\(^{(1)} \)

As mentioned in the introduction we refer the reader to \([1]\) for definitions and notations concerning ordinal and cardinal numbers. In particular, we refer the reader to \([1]\), page 21, for a definition of \( \gamma \) well-orders \( X \). We use the notation \( < \) for a well-ordering (w.o.) on a class \( X \). If \( < \) is a w.o. on \( X \) and \( < a, b \leq < \), we write \( a < b \). One important consequence of the strong form of the Axiom of Choice which we will often use is the result that any class may be well-ordered.

\(^{(1)} \) This applies only if \( A \) is a set. If \( A \) is a proper class then \( \aleph = \text{On} \) (the class of all ordinals).
2.1 Isomorphisms of First Order Languages and Statement of Main Result

Definition 3: Let $L$ and $M$ be first order languages with $\mathcal{L}$ and $\mathcal{M}$ as their respective classes of all sentences, and $\mathcal{L}_M$ and $\mathcal{M}_L$ as their respective classes of all structures, then $L \cong M$ just in case there exist bijections $F$ and $G$ such that $F: \mathcal{L} \rightarrow \mathcal{M}$, $G: \mathcal{M} \rightarrow \mathcal{L}$, and for any $A$ in $\mathcal{L}$ and any $\mathcal{S}$ in $\mathcal{L}$, $\mathcal{S}(A) = T \iff F(G(A)) = T$.

We now restate Hanf's theorem for first order languages.

Theorem: Let $L$ and $M$ both be first order languages with a nonzero finite number of predicate symbols, at least one of which has two or more places, as their only nonlogical symbols. Then $L \cong M$.

Since $\cong$-isomorphism is clearly symmetric and transitive, the following is equivalent to the above result.

Theorem: If $L$ is a first order language with a binary predicate symbol as its only nonlogical symbol and $M$ is a first order language with a finite nonzero number of predicate symbols, one of which is at least binary, as its only nonlogical symbols, then $L \cong M$.

Now we introduce a slightly stronger form of isomorphism which, as we see in section 2.2, implies
\( \simeq_{\omega} \)-isomorphism.

**Definition 4:** The first order languages \( L \) and \( M \) are \( \simeq'_{\omega} \)-isomorphic if all the conditions of definition 3 are satisfied except that now \( \mathcal{L} \) and \( \mathcal{M} \) are the complete classes of isomorphism-types of \( L \) and \( M \) respectively.

Now we can state our main result.

**Theorem:** If \( L \) and \( M \) are first order languages which satisfy all of the hypotheses of Hanf's theorem, then \( L \not\simeq'_{\omega} M \).

2.2 \( \simeq'_{\omega} \)-isomorphism Implies \( \simeq_{\omega} \)-isomorphism

First we show that if \( \mathcal{H} \) is a structure for a first order language \( N \) and \( \mathcal{K} \) is the isomorphism-type of \( \mathcal{H} \), then \( \mathcal{K} \) is a proper class. To see this we assume that \( \mathcal{K} \) is not a proper class and therefore that \( \mathcal{K} \neq \text{On} \) (the class of all ordinals). Let \( \mathcal{K} = \alpha \) and \( \alpha^+ \) be the cardinal successor of \( \alpha \). Since \( |\mathcal{H}| \) is a set, we know that \( |\mathcal{K}| = \gamma \) for some cardinal \( \gamma \), and we can list the elements of \( |\mathcal{K}| \) as follows:

\[
\begin{align*}
&n_0, n_1, \ldots, n_\beta, \ldots, \text{where } \beta < \gamma.
\end{align*}
\]

Then for each \( \xi < \alpha^+ \) we define an \( N \)-structure \( \mathcal{H}_\xi \) as follows:

1. Let \( |\mathcal{H}_\xi| = \gamma \) and \( \delta \) the least cardinal such that \( \delta \) is infinite and \( \delta \in \{ n_\rho | \rho < \omega \} \), so that \( |\mathcal{H}_\xi| = \delta \) and we can
list the elements of \( |\mathcal{M}_\xi| \) as follows:
\[ \mathcal{M}_\xi, \ldots, \mathcal{M}_{\xi+\delta}, \ldots, \]
where \( \delta < \xi \);
(ii) for any \( n \)-ary function symbol \( f \) of \( N \), we define \( f_{\mathcal{M}_\xi} \) as follows:
for any \( \mathcal{M}_{\xi+\delta}, \ldots, \mathcal{M}_{\xi+\delta+n} \in |\mathcal{M}_\xi| \), where \( \delta < \xi \),
\[ f_{\mathcal{M}_\xi}(\mathcal{M}_{\xi+\delta}, \ldots, \mathcal{M}_{\xi+\delta+n}) = \mathcal{M}_{\xi+\delta+n} \iff \]
\[ f_\mathcal{M}(n_\delta, \ldots, n_{\delta+n-1}) = n_{\delta+n} ; \]
(iii) for any \( n \)-ary predicate symbol \( p \) of \( N \), we define \( p_{\mathcal{M}_\xi} \) as follows:
for any \( \mathcal{M}_{\xi+\delta}, \ldots, \mathcal{M}_{\xi+\delta+n-1} \), where \( \delta < \xi \),
\[ p_{\mathcal{M}_\xi}(\mathcal{M}_{\xi+\delta}, \ldots, \mathcal{M}_{\xi+\delta+n-1}) \iff p_\mathcal{M}(n_\delta, \ldots, n_{\delta+n-1}) . \]
It is immediate that \( \mathbb{N} \supseteq \alpha^T \), because \( \mathcal{M}_\xi \notin \mathbb{N} \) for each \( \xi < \alpha^T \) and \( \delta \neq \rho < \alpha^T \) implies that \( \mathcal{M}_\xi \neq \mathcal{M}_\rho \). But this contradicts our assumption that \( \mathbb{N} \) is not a proper class and hence \( \mathbb{N} \) must be a proper class.

We now assume that \( L^{\omega_1} \mathcal{M} \) and use the notation of definition 4; furthermore, we let \( \mathcal{L} \) and \( \mathcal{M} \) be the classes of all structures of \( L \) and \( M \) respectively. Then we let \( \mathcal{L}_0, \ldots, \mathcal{L}_\omega, \ldots \) be a listing of the elements of \( \mathcal{L} \) and using \( F \) and \( G \) as in definition 4 we define a bijection \( F_1: \mathcal{L} \to \mathcal{M} \), such that for any \( \lambda \in \mathcal{L} \) and any \( \mathcal{L} \in \mathcal{L}' \),
\[ \mathcal{L}(A) = T \iff F_1(\mathcal{L})(G(A)) = T, \]
as follows. For any \( \alpha \).
\[ F_1(\mathcal{L}_\alpha) = \text{the least member of } \mathcal{B} - \bigcup_{\gamma < \alpha} \{ F_1(\mathcal{L}_\gamma) \} , \] where \( \mathcal{B} \) is the isomorphism-type of \( F(\mathcal{M}) \) and \( \mathcal{M} \) is the unique member of \( \mathcal{L} \) such that \( \mathcal{M} \cong \mathcal{L}_\alpha \).
class \( \mathcal{B} = \bigcup_{\mathcal{A} \in \mathcal{B}} \{ F_1(\mathcal{A}) \} \) is nonempty. We also know that any sentence is true in a structure \( \mathcal{M} \) just in case it is true in all members of the isomorphism-type of \( \mathcal{M} \). Therefore it is easily seen that \( F_1 \) is a one-one function from \( \mathcal{L}' \) to \( \mathcal{M}' \) which preserves the truth-values of sentences under the function \( G \). The function \( F_1 \) is also onto. Because if not, then since the image of an isomorphism-type under \( F_1 \) is a subclass of an isomorphism-type and \( F_1 \) is defined on all of \( \mathcal{L}' \), there must be an isomorphism-type \( \mathcal{A} \) of \( L \)-structures and an isomorphism-type \( \mathcal{B} \) of \( M \)-structures with \( F_1(\mathcal{A}) \not\subseteq \mathcal{B} \).

But \( F_1 \) is defined so that \( \mathcal{A}_1 < \mathcal{A}_2 \) and \( F_1(\mathcal{A}_1) \not\subseteq F_1(\mathcal{A}_2) \) implies that \( F_1(\mathcal{L}_1) < F_1(\mathcal{L}_2) \). Therefore there must be some element, say \( \mathcal{B}_\alpha \), of \( \mathcal{B} \) such that all the elements of \( F_1(\mathcal{A}) \) are \( < \mathcal{B}_\alpha \). This implies that \( \mathcal{A} < \mathcal{A} \) because \( F_1 \) is a one-one function and hence we have a contradiction.

It is now immediate that \( L \not\subseteq \mathcal{M} \) since \( F_1 \) and \( G \) satisfy the conditions of definition 3.

2.3 Outline of the Proof of the Main Result

The results of the last section point out that a proof of our main result is also a proof of Hanf's theorem.

For the remainder of this paper \( L \) is the first order language with the binary predicate symbol \( p \) as
its only nonlogical symbol and \( M \) is a fixed first order language which has only finitely many nonlogical symbols, namely the \( n \) predicate symbols \( q_1, \ldots, q_n \), where \( n \geq 1 \), \( q_i \) is \( k_i \)-ary, and \( k_1 \geq \ldots \geq k_n \) with \( k_1 \geq 2 \). We denote the classes of all sentences of \( L \) and \( M \) by \( \mathcal{L} \) and \( \mathcal{M} \) respectively and the complete classes of isomorphism-types of \( L \) and \( M \) by \( \mathcal{K} \) and \( \mathcal{M} \) respectively. Finally, we denote the classes of all \( L \)-structures and \( M \)-structures by \( \mathcal{L}' \) and \( \mathcal{M}' \) respectively.

From now on a subclass \( \mathcal{B} \) of a complete class of isomorphism-types \( \mathcal{K} \) is elementary just in case \( \mathcal{B} \) is the intersection of \( \mathcal{K} \) and an elementary class.

We now define a certain subclass \( \mathcal{M}_1 \) of \( \mathcal{M} \). Let \( B_2 \) be the \( M \)-sentence 
\[
\exists x_1 \cdots \exists x_{k_2} q_2 x_1 \cdots x_{k_2} \lor \cdots \lor \exists x_1 \cdots \exists x_{k_2} q_2 x_1 \cdots x_{k_2} \lor \exists x_1 \cdots \exists x_{2k_1-2} (q_1 x_1 x_2 x_3 \cdots x_{k_1} \land q_1 x_1 x_2 x_{k_1+1} \cdots x_{2k_1-2}).
\]
Then we define \( \mathcal{M}_1 \) to consist of all \( M \in \mathcal{M} \) such that \( \mathcal{M}(B_2) = T \). Clearly \( \mathcal{M}_1 \) is well defined and, with our new usage of the term elementary, is defined by \( B_2 \). Less formally we see that \( \mathcal{M}_1 \) consists of all \( M \in \mathcal{M} \) such that \( (q_1)_M \neq \emptyset \) for some \( r(2 \leq r \leq n) \) or \( (q_1)_M \) cannot be considered as a binary relation \((q_1)_M \) is not wholly dependent on its first two places). Of course if \( k_1 = 2 \), then the sentence 
\[
\exists x_1 \cdots \exists x_{2k_1-2} (q_1 x_1 \cdots x_{k_1} \land q_1 x_1 \cdots x_{2k_1-2})
\]
becomes 
\[
\exists x_1 \exists x_2 (q_1 x_1 x_2 \land \neg q_1 x_1 x_2)
\]
which must always be false and hence \( \mathcal{M}_1 \) becomes the class of all \( M \in \mathcal{M} \) such that
An L-structure \(\mathcal{L}\) is connected if \(\forall a_1, a_2 \in |\mathcal{L}|\)

one of the following is true in \(\mathcal{L}\):

(i) \(p_L(a_1, a_2)\) or \(p_L(a_2, a_1)\);

(ii) \(p_L(a_1, b_1) \& \ldots \& p_L(b_m, a_2)\) or \(p_L(a_2, b_1) \& \ldots \& p_L(b_m, a_1)\) for some \(m \geq 1\) and \(b_1, \ldots, b_m \in |\mathcal{L}|\);

(iii) \(p_L(a_1, b_1) \& \ldots \& p_L(b_{i-1}, b_i) \& \ldots \& p_L(b_{i-1}, b_i) \& \ldots \& p_L(b_{m}, b_{i+1}) \& \ldots \& p_L(b_{m}, a_2)\) for some \(m \geq 1\) and \(i (1 \leq i \leq m)\).

The individual \(c\) of \(\mathcal{L}\) is isolated if \(\forall b \in |\mathcal{L}|\), both

\(p_L(b, c)\) and \(p_L(c, b)\) are false in \(\mathcal{L}\).

We now define a one-one map which takes \(L \in \mathcal{L}\) to \(\mathcal{M} \in \mathcal{M}\) which has only one nonempty predicate, namely

\((q_1)_\mathcal{M}\), which is also reducible to a binary predicate in a natural way. The canonical map \(J: \mathcal{L} \rightarrow \mathcal{M}\) is defined as follows:

For any \(L \in \mathcal{L}\), let \(|M'| = |L|\) and define \((q_1)_{M'}\) by,

\(\forall a_1, \ldots, a_k \in |M'|, (q_1)_{M'}(a_1, a_2, \ldots, a_k) \iff p_L(a_1, a_2)\), and for \(2 \leq i \leq n\) we define \((q_i)_{M'} = \emptyset\). It is apparent that we have defined a unique \(M' \in \mathcal{M}\); so let \(J(L) = M\), where \(M\) is the unique element of \(\mathcal{M}\) which is isomorphic to \(M'\).

If \(\mathcal{A} \subseteq \mathcal{L}\), then \(\mathcal{A}^n\) (\(n\) a nonnegative integer) is the subclass of \(\mathcal{L}\) obtained by adding \(n\) isolated points to each \(A \in \mathcal{A}\), and \(\mathcal{A}^\infty\) is the subclass of \(\mathcal{L}\) obtained by adding, for each infinite cardinal \(\alpha\), \(\alpha\) isolated points to each \(A \in \mathcal{A}\).
A final function is now defined before we proceed to the outline. Let $\mathcal{A}$ be a subclass of $\mathbb{L}$ consisting only of connected structures. Then we define the bijection $H: \bigcup_{n=1}^{\infty} \mathcal{A}^n \cup \mathcal{A}^\infty \rightarrow \bigcup_{n=0}^{\infty} \mathcal{A}^n \cup \mathcal{A}^\infty$ as follows:

For any $\mathcal{A} \in \mathcal{A}^n (n \geq 0)$ we know that there is a unique $\mathcal{A}' \in \mathcal{A}^{n-1}$ such that $\mathcal{A}'$ plus an isolated point is isomorphic to $\mathcal{A}$; so let $H(\mathcal{A}) = \mathcal{A}'$. Then it is immediate that $H: \mathcal{A}^n \rightarrow \mathcal{A}^{n-1}$ is a bijection. For any $\mathcal{A} \in \mathcal{A}^\infty$ we know that $\mathcal{A}$ minus an isolated point is still isomorphic to $\mathcal{A}$; so we define $H$ as above and we see that $H(\mathcal{A}) = \mathcal{A}$. Therefore $H$ is the identity function on $\mathcal{A}^\infty$. Finally, since $0 \leq m \neq n < \infty$ implies that $\mathcal{A}^n \cap \mathcal{A}^m = \emptyset$ and $\mathcal{A}^\infty \cap \mathcal{A}^m = \emptyset$, we see that $H$ is a bijection from $\bigcup_{n=1}^{\infty} \mathcal{A}^n \cup \mathcal{A}^\infty$ to $\bigcup_{n=0}^{\infty} \mathcal{A}^n \cup \mathcal{A}^\infty$.

We now begin the outline of the proof of the main result.

First we construct a one-one function $F_1: \mathcal{M} \rightarrow \mathbb{L}$ such that $F_1(\mathcal{M})$ is defined by a single sentence which we denote by $A_1$, the elements of $F_1(\mathcal{M})$ are all connected, and there exist one-one functions $H_1: \mathbb{L} \rightarrow \mathcal{M}$ and $G_1: \mathcal{M} \rightarrow \mathbb{L}$ such that $\forall \mathcal{A} \in \mathbb{L}, \forall \mathcal{F} \in F_1(\mathcal{M}), \mathcal{F}(\mathcal{A}) = T \iff F_1^{-1}(\mathcal{F})(H_1(\mathcal{A})) = T$ and $\forall \mathcal{M} \in \mathcal{M}, \forall \mathcal{A} \in \mathbb{L}, \mathcal{M}(\mathcal{A}) = T \iff F_1(\mathcal{M})(G_1(\mathcal{A})) = T$. Since the sentence $B_2$ defines $\mathcal{M}$, we easily show that the sentence $B_1$, defined to be $G_1(B_2)$ & $A_1$, defines $F_1(\mathcal{M})$.

We now denote $F_1(\mathcal{M})$ by $\mathbb{L}$, and define $\mathbb{L}_2$ as
Using the functions \( J \) and \( H \) mentioned on pages 25 and 26, we define \( m_\lambda \) to be \( J \circ H ( \mathcal{X}_\lambda ) \). It is immediate that \( \mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset \), \( m_\lambda \cap m_\mu = \emptyset \), and \( J \circ H \) is a bijection from \( \mathcal{X}_\lambda \) to \( m_\lambda \). Then we construct a sentence \( C_1 \) which defines \( \mathcal{X}_\lambda \) and from \( C_1 \) we easily construct a sentence \( C_2 \) which defines \( m_\lambda \). After this the existence of one-one functions \( H_2 : L \to M \) and \( G_2 : M \to L \) is proven. They have the same properties as \( H_1 \) and \( G_1 \) have, except that now we are concerned with \( \mathcal{X} \in \mathcal{X}_\lambda \) and \( m \in m_\lambda \) instead of \( \mathcal{X} \in F_1 ( m ) \) and \( m = m_\lambda \).

Two final subclasses of \( \mathcal{X} \) and \( m \) are defined. We let \( \mathcal{X}_3 = \mathcal{X} - ( \mathcal{X}_1 \cup \mathcal{X}_2 ) \) and we let \( m_3 = M - ( m_1 \cup m_2 ) \). It follows that \( \varnothing ( B_1 \cup C_1 ) \) and \( \varnothing ( B_2 \cup C_2 ) \), which are denoted by \( D_1 \) and \( D_2 \), define \( \mathcal{X}_3 \) and \( m_3 \) respectively.

Next we note that \( J \) is a bijection from \( \mathcal{X}_3 \) to \( m_3 \) because \( J \) is a one-one function, \( J ( \mathcal{X} ) = m - m_\lambda = m_\lambda \cup m_\lambda \) and \( m_\lambda \) is \( J \circ H ( \mathcal{X}_2 ) = J ( \mathcal{X}_1 ) \cup J ( \mathcal{X}_2 ) \). Then we easily prove the existence of one-one functions \( H_3 \) and \( G_3 \) which are similar to \( H_1 \) and \( G_1 \) except that now we are concerned with \( \mathcal{X} \in \mathcal{X}_3 \) and \( m \in m_3 \) instead of \( \mathcal{X} \in F_1 ( m ) \) and \( m = m_\lambda \).

Now we define \( F : \mathcal{X} \cup m_\lambda \) such that \( F \circ \mathcal{X}_1 = F_1^{-1} \), \( F \mathcal{X}_3 = J \circ H \), and \( F \mathcal{X}_\lambda = J \). A pictorial representation of \( F \) is given below.
For any $A \in \mathcal{L}$, let $5(A)$ be the sentence $(H_1(A) \& B_1)$ \[ \lor (H_2(A) \& C_2) \lor (H_3(A) \& D_2). \] We define a similar function $N: \mathcal{L} \to \mathcal{M}$ in an analogous manner using the functions $G_1, G_2,$ and $G_3$ and the sentences $B_1, C_1,$ and $D_1$.

Using the functions $K_L$ and $K_M$, the effective listings $A_0, \ldots, A_n, \ldots$ and $B_0, \ldots B_n, \ldots$ of $L$ and $M$, and the fact that any sentence $A$ is true in a structure just in case any sentence consisting of a finite number of conjunctands of $A$ is true in the same structure we construct a bijection $G: \mathcal{L} \to \mathcal{M}$ such that $\forall A \in \mathcal{L}, \forall \xi \in \mathcal{L}, \xi(A) = T \iff F(\xi)(G(A)) = T.$
Then we note that we have now shown that $L$ is $\text{iso}_\omega'$-isomorphic to $M$ and restate the main result.

This concludes the outline.
CHAPTER 3

3.1 Results Concerning $M_1$ and $F_1(M_1)$

First we introduce the notion of a specific type of pictorial representation of an L-structure.

A directed graph of $L$ is defined as follows:

Let points $O_a$ represent the elements $a$ of $|L|$ and if $O_a$ and $O_b$ represent $a$ and $b$ in $|L|$ we write $O_b^O_a$ just in case $p^L(a, b)$ is true in $L$. The line between $O_b$ and $O_a$ can be nonlinear and/or nonvertical, but it must not touch any other points and $O_b$ must be positioned above $O_a$ on the page.

Now we construct a corresponding $L$ in $L'$ from a given $M$ in $M$ such that $p^L$ is the usual $\in$ relation. Assume that we are given $M$ in $M$, then all information regarding $M$ is known to us. Of course $n$, $k_1$, $\ldots$, $k_n$ have all been fixed and known since the language $M$ was defined. Then $|L|$ is defined to be the least class which satisfies the following conditions:

(i) let $\alpha, \alpha+1, \ldots, \alpha+n = \beta$ be the least $n+1$ successive ordinals not in $|M|$, then $\alpha, \alpha+1, \ldots, \alpha+n$ are in $|L|$;
(ii) for any $j$ ($1 \leq j \leq k_1$), if $a_1, \ldots, a_j$ are in $|M|$, then $\langle \{a_1, \beta\}, \ldots, \{a_j, \beta\} \rangle$ is in $|L|$;
(iii) for any $j$ ($2 \leq j \leq k_1$), if $a_1, \ldots, a_j$
are in |M|, then \( \{a_1, \beta_1, \ldots, a_j, \beta_j\} \) is in \(|L|\);

(iv) for any \( j \) (1 \( \leq \) \( j \leq \) \( n \)), there is a unique element, say \( \langle i \rangle \), in \(|L|\) such that \( \alpha + (j-1) \) is in \( \langle i \rangle \) and for any \( b \) in \(|L|\), \( b \) is in \( \langle i \rangle \) just in case \( b \) is \( \alpha + (j-1) \) or \( b \) is \( \langle a_1, \beta_1, \ldots, a_k \rangle \),

It is important to realize that the function \( \phi: \{a | a \in M\} \longrightarrow \{\{a, \beta | \{a, \beta\} \in L\}\} \), defined by \( \phi(a) = \{a, \beta\} \), is a bijection. This function will play a major role in proving the existence of the one-one function \( G_1: \mathbb{I} \rightarrow \mathbb{L} \) described on page 26.

If \( L \) is defined from \( M \) in \( N \) as above then we can construct a directed graph of \( L \).

For example, assume \( n = k_1 = 2, k_2 = 1, |M| = 3 \), \( \langle a, b, c \rangle, (q_1)_N = \{\langle a, b \rangle, \langle a, c \rangle\} \), and \( (q_2)_N = \{\langle a, b \rangle\} \). Let \( \alpha, \alpha + 1, \alpha + 2 = \xi \) be the least three successive ordinals not in \(|M|\). Then the following is a directed graph of the \( L \) corresponding to \( M \).
Let $a'$, $b'$, and $c'$ denote $\{a, \phi\}$, $\{b, \beta\}$, and $\{c, \eta\}$ respectively.
It is evident that the $L$ defined from a given $M$ in $M$ as above is uniquely determined; furthermore if $M \neq M'$, $L$ and $L'$ are defined from $M$ and $M'$ as above, then $L \neq L'$.

The next definition is immediate.

**Definition 5:** We define the one-one function $F_1 : M \rightarrow L$ as follows: For any $M$ in $M$, let $L'$ be the $L$-structure defined from $M$ as above and let $L$ be the unique member of $L$ which is isomorphic to $L'$, then let $F_1(M) = L$.

We now introduce abbreviations for certain $L$-expressions.

The abbreviation $L^i x$ ($1 \leq i \leq 2k_1 + n + 1$) is inductively defined as follows:

1. $L^1 x$ is an abbreviation of $\forall x_1 \neg px_1 x \& \exists x_1 pxx_1$;
2. for $2 \leq j \leq 2k_1 + n - 1$, $L^j x$ is an abbreviation of $\exists x_1 \exists x_2 (L^{j-1} x_1 \& px_1 x \& pxx_2)$;
3. $L^{2k_1 + n} x$ is an abbreviation of $\exists x_1 (L^{2k_1 + n-1} x_1 \& px_1 x)$;
4. $L^{2k_1 + n+1}$ is an abbreviation of $\exists x_1 (L^1 x_1 \& \ldots \lor x_{n} x_1 \& px_1 x) \& \forall x_1 \neg pxx_1$.

When $L$ is in $F_1(M)$ and $a$ is in $\{L\}$ we say that $a$ is in, of, or from level $i$ just in case $L(L^i a) = T$.

If the reader will refer to the figure on page 32 the usage of this terminology will become more apparent,
as will the fact that each element of $|\mathcal{L}|$ is in exactly one of levels 1 to $2k_1+n+1$.

From now on if $\mathcal{L} = F_1(\mathcal{M})$ for some $\mathcal{M}$ in $\mathcal{M}$, then $\theta$ denotes the isomorphism from $\mathcal{L}$ to $\mathcal{L}'$, where $\mathcal{L}'$ is defined from $\mathcal{M}$, and $\phi$ denotes the bijection from the universe of $\mathcal{M}$ to the level $n+2$ elements of $|\mathcal{L}'|$. The one-one function $\phi^{-1}\theta$ is denoted by $\tau$; in particular, $\tau$ is a bijection from the level $n+2$ individuals of $\mathcal{L}$ to the universe of $\mathcal{M}$.

Another abbreviation $T^i_x x_1 \ldots x_i$ ($1 \leq i \leq k_1$) is defined inductively as follows:

(i) $T^i_x x_1$ is an abbreviation of $L^{n+2} x_1 \& x = x_1$;
(ii) for $2 \leq i \leq k_1$, $T^i_x x_1 \ldots x_i$ is an abbreviation of $L^{n+2i} x \& \exists y_1 \exists y_2 \exists y_3 \forall y_4 (L^{n+2} x_1 \& T^{i-1} y_1 x_2 \ldots x_i \& L^{n+2i-1} y_2 \& py_1 y_2 \& px_1 y_2 \& py_2 x \& L^{n+3} y_3 \& (py_4 y_3 \rightarrow y_4 = x_1) \& py_3 x)$. 

Then we define the abbreviation $P^i_x$ ($1 \leq i \leq n$) by:
for $1 \leq i \leq n$, $P^i_x$ is an abbreviation of $L^{2k_1+n+1} x \& \exists x_1 (L^i x_1 \& px_1 x)$.

The usage of these last two abbreviations becomes more apparent when we consider $\mathcal{L} = F_1(\mathcal{M})$ for some $\mathcal{M}$ in $\mathcal{M}$. Then, for any $b$, $a_1$, ..., $a_i$ ($1 \leq i \leq k_1$) in $|\mathcal{L}|$. $\mathcal{L}(a^i b a_1 \ldots a_i) = \tau$ just in case $a_1$, ..., $a_i$ are all in level $n+2$, $b$ is in level $n+2i$, and $\theta(b) =$
\[\langle \theta(a_1), \ldots, \theta(a_i) \rangle.\] Also, if we recall the definition of \( L^{2k_1+n+1}x \) we see that for any \( a \) in \(|L|, L(F^i a) = T \) just in case \( a \) is in level \( 2k_1+n+1 \) and for any \( a_1, \ldots, a_{k_i} \) in \(|L|\), there is a \( b \) in \(|L|\) of level \( n+2i \) such that \( \theta(b) = \langle \theta(a_1), \ldots, \theta(a_{k_i}) \rangle \) if and only if \((q_i)_m(a_1), \ldots, \tau(a_{k_i}) \) happens in \( m \). That is, if \( a \) is in \(|L|\) and \( L = L(\mathcal{M}) \) for some \( \mathcal{M} \) in \( \mathcal{M} \), then \( L(F^i a) = T \) just in case we can think of \( a \) as being the element of \(|L|\) which corresponds in a certain sense to the predicate \((q_i)_m\) on \(|\mathcal{M}|\).

The reader may find it interesting to notice that given \( L \) in \( L(\mathcal{M}) \) we can determine the preimage of \( L \) under \( L \) by using the function \( \tau \). We merely let \(|\mathcal{M}| = \tau (|L|a \in |L| \& L(L^{n+2} a) = T \})\), and for any \( i \) (\( 1 \leq i \leq n \)) we define \((q_i)_m\) on \(|\mathcal{M}|\) by: for any \( a_1, \ldots, a_{k_i} \) in \(|\mathcal{M}|\), \((q_i)_m(a_1, \ldots, a_{k_i}) \) just in case \( \exists b_1, b_2 \) in \(|L|\) such that \( L(\tau^{k_i} b_1 a_1 \tau^{-1} \ldots a_{k_i} \tau^{-1} \& p_1 b_2 \& p_1 b_2) = T \). It follows that \( \mathcal{M} \) is an \( M \)-structure and that \( L(\mathcal{M}) = L \).

The abbreviation \( C^i x \) (\( 1 \leq i \leq 2k_1+2n \)) is defined as follows:

(i) for \( 1 \leq i \leq 2k_1+n \), \( C^i x \) is just \( L^i x \);

(ii) for \( 2k_1+n+1 \leq i \leq 2k_1+2n \), \( C^i x \) is an abbreviation of \( p^{i-2k_1-n} x \).

This last abbreviation is just a refinement of the \( L^i \)-abbreviation. It can easily be shown that if

(1) \( \ldots \) and \( \theta(b) \) is in \( \theta(a) \)\ldots
\(L\) is in \(F_1(M)\), then for any \(a \in |L|\), \(L(c^i a) = T\) for exactly one \(i\) between 1 and \(2k_1 + 2n\).

Let \(L\) be in \(F_1(M)\) and \(a \in |L|\). Then \(a\) is in category \(i\) just in case \(L(c^i a) = T\). Let \(A(x_1, \ldots, x_n)\) be an \(L\)-formula. Then \(x_1, \ldots, x_n\) are in categories \(1_1, \ldots, 1_n\) respectively just in case we only consider \(L\)-instances \(A[a_1, \ldots, a_n]\) of \(A(x_1, \ldots, x_n)\) such that

\[L(A[a_1, \ldots, a_n]) = T \iff L(A[a_1, \ldots, a_n] \land c^1 a_1 \land \cdots \land c^n a_n) = T.\]

If \(L\) is in \(F_1(M)\) we associate each \(a\) in category \(i\) (\(1 \leq i \leq 2k_1 + 2n\) and \(i \neq n+3\)) in \(|L|\) with a sequence of elements of \(|F_1^{-1}(L)|\) which has length between 0 and \(k_1\). If \(a\) in \(|L|\) is in category \(n+3\) we associate \(a\) with two sequences of length 2.

The association is defined as follows:

(i) for a in category \(j\) (\(1 \leq j \leq n+1\) or \(2k_1 + n + 1 \leq j \leq 2k_1 + 2n\)) we associate \(a\) with the empty sequence;

(ii) for a in category \(2s+n\) (\(1 \leq s \leq k_1\)) there exist unique \(a_1, \ldots, a_s\) in \(|L|\) from level \(n+2\) such that \(\theta(a) = \langle \theta(a_1), \ldots, \theta(a_s) \rangle\) and \(a\) is associated with the sequence \(\tau(a_1), \ldots, \tau(a_s)\);

(iii) for a in category \(2s+n-1\) (\(3 \leq s \leq k_1\)) there exist unique \(a_1, \ldots, a_s\) of level \(n+2\) such that \(\theta(a) = \{\theta(a_1), \langle \theta(a_2), \ldots, \theta(a_s) \rangle\}\) and \(a\) is associated with \(\tau(a_1), \ldots, \tau(a_s)\);
(iv) for a in category n+3 there exist unique $a_1, a_2$ of level n+2 in $|\mathcal{L}|$ such that $\vartheta(a) = \{\vartheta(a_1), \vartheta(a_2)\} = \{\vartheta(a_2), \vartheta(a_1)\}$ and we associate the sequences $\gamma(a_1), \gamma(a_2)$ and $\gamma(a_2)\gamma(a_1)$ with a.

It is clear that the length of a sequence of elements associated with a depends only upon which category a is in. For a in category i let $m(i)$ denote the length of an associated sequence.

Let I be the set of all elements of $|\mathcal{L}|$ in category i $(1 \leq i \leq 2k_1+2n)$. Then for each a in I there is at least one sequence of length $m(i)$ associated with a and for each sequence of length $m(i)$ there is a unique a in I associated with it.

The categories in which the elements of $|\mathcal{L}|$ occur can provide information about how they are related. For example, if $a_1$ and $a_2$ in $|\mathcal{L}|$ are in category i $(1 \leq i \leq n+1)$, then $a_1 = a_2$. Also, if $a_1$ is in category 1 and $a_2$ is in category 2, then $p_{\mathcal{L}}(a_1, a_2)$ and $a_1 \neq a_2$ but not $p_{\mathcal{L}}(a_2, a_1)$.

Let $A(x_1, \ldots, x_n)$ be an L-formula with $x_1, \ldots, x_n$ in categories $l_1, \ldots, l_n$ respectively and let $\mathcal{L}$ be in $F_1(\mathcal{M})$ and $A[a_1, \ldots, a_n]$ be an $\mathcal{L}$-instance of $A(x_1, \ldots, x_n)$. Then $a_i$ $(1 \leq i \leq n)$ has at least one sequence $a_{i,1}, a_{i,2}, \ldots, a_{i,m(l_i)}$ associated with it.

For the remainder of this section $x_1, \ldots, x_n$ will denote the first n variables in the alphabetical listing of the variables of L, and will be restricted to categories
l_1, ..., l_n respectively unless otherwise stated. For each i we associate the fixed variables x_{i,1'}, ..., x_{i,m(l_i)}', with x_i such that x_{i,j} and x_{i',j'} are distinct unless i = i' and j = j'. We say that x_{i,1'}, ..., x_{i,m(l_i)} are determined by x_i in category l_i.

The following lemma is a necessary prelude to setting up a mapping from L-sentences to M-sentences which preserves truth-values under F_{l}^{-1}.

**Lemma 1:** Let A(x_1, ..., x_n) be an L-formula and let \mathcal{L} be an element of F_1(\mathcal{M}). Then there is an M-formula B(x_1,1', ..., x_n,m(l_n)'), where x_{i,1'}, ..., x_{i,m(l_i)} (1 \leq i \leq n) are determined by x_i in category l_i, such that for any a_1, ..., a_n in |\mathcal{L}|, where a_{i,1}, ..., a_{i,m(l_i)} (1 \leq i \leq n) is a sequence associated with a_i,

\mathcal{L}(A[a_1, ..., a_n]) = T just in case

F_{l}^{-1}(\mathcal{L})(B[a_{1,1}', ..., a_{n,m(l_n)'},]) = T.

**Proof:** We use induction on the length of A(x_1, ..., x_n).

(i) A(x_1, ..., x_n) is x_i = x_j (1 \leq i, j \leq n). We consider two subcases.

(a) If l_i \neq l_j, then it is apparent that for a_i, a_j \in |\mathcal{L}| with \mathcal{L}(c^{l_i}a_i \& c^{l_j}a_j) = T, we must have
Therefore it is sufficient to define $B(x_1, l_1', ..., x_n, m(l_n))$ as the formula $∃x(x \neq x)$.

(b) The subcase for $l_i = l_j$ is split into three parts.

1. If $1 \leq l_i = l_j \leq n+1$ or $2k_1+n+1 \leq l_i = l_j \leq 2k_1+2n$, then we must have $A(a_i = a_j) = T$ and so $B(x_1, l_1', ..., x_n, m(l_n))$ is defined to be the formula $∃x(x = x)$.

2. If $n+4 \leq l_i = l_j \leq 2k_1+n$ or $l_i = l_j = n+2$, then it is obvious that $0 < m(l_i) = m(l_j)$ and hence $B(x_1, l_1', ..., x_n, m(l_n))$ is the formula $x_1, l_1 = x_2, l_2, ..., x_1, m(l_i) = x_2, m(l_j)$.

3. If $l_i = l_j = n+3$, then $m(l_i) = m(l_j) = 2$ and $B(x_1, l_1', ..., x_n, m(l_n))$ is the formula $(x_1, l_1 = x_2, l_2 & x_1, 2 = x_2, 2) \lor (x_1, l_1 = x_2, 2 & x_1, 2 = x_2, l_2)$.

(ii) $A(x_1, ..., x_n)$ is $p x_i x_j$. This case is split into three subcases.

(a) $l_i$ and $l_j$ are such that $∀a_i, a_j \in \mathcal{L}$, if $A(c_i a_i \& c_j a_j) = T$, then $A(a_i, a_j) = T$. We let $B(x_1, l_1', ..., x_n, m(l_n))$ be the formula $∃x(x \neq x)$.

(b) $l_i$ and $l_j$ are such that $∀a_i, a_j \in \mathcal{L}$, if $A(c_i a_i \& c_j a_j) = T$, then $A(a_i, a_j) = T$. We let $B(x_1, l_1', ..., x_n, m(l_n))$ be the formula $∃x(x = x)$.
(c) The subcase for \( l_i \) and \( l_j \) such that no information is given about the truth-value of \( A[a_i, a_j] \) is split into five parts.

1. If \( l_j = 2k_1 + n + r \) (\( 1 \leq r \leq n \)), then we must have \( l_i = n + 2k_r \) and \( m(l_i) = k_r \). Hence we let

\[
B(x_1, 1, \ldots, x_n, m(l_n)) \text{ be the formula } q_x x_1, 1, \ldots, x_i, k_r.
\]

2. If \( l_j = 2s + n \) (\( 3 \leq s \leq k_1 \)), then we have:

1. \( l_i = n + 3 \) and so \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is the formula \( x_1, 1 = x_1, 2 = x_2, 1 \);

2. \( l_i = 2s + n - 1 \) (\( 3 \leq s \leq k_1 \)) and so \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is the formula \( x_1, 1 = x_2, 1 \) \& \ldots \& \( x_1, s, s = x_2, s \).

3. If \( l_j = 2s + n - 1 \) (\( 3 \leq s \leq k_1 \)) we have:

1. \( l_i = n + 2 \) and so \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is the formula \( x_1, 1 = x_2, 1 \);

2. \( l_i = 2s + n - 2 \) and so \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is the formula \( x_1, 1 = x_2, 2 \) \& \ldots \& \( x_1, s - 1, s = x_2, s \).

4. If \( l_j = n + 3 \), then \( l_i \) is \( n + 2 \) and so \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is the formula \( x_1, 1 = x_2, 1 \lor x_1, 1 = x_2, 2 \).

5. If \( l_j = n + 4 \), then \( l_i = n + 3 \) and \( B(x_1, 1, \ldots, x_n, m(l_n)) \) is \( (x_1, 1 = x_1, 2 = x_2, 1) \lor (x_1, 1 = x_2, 1 \land x_1, 2 = x_2, 2 \lor x_1, 1 = x_2, 2 \land x_1, 2 = x_2, 1) \).

(iii) \( A(x_1, \ldots, x_n) \) is \( \neg A_1(x_1, \ldots, x_n) \). By induction we obtain \( B_1(x_1, 1, \ldots, x_n, m(l_n)) \) from \( A_1(x_1, \ldots, x_n) \). Define \( B(x_1, 1, \ldots, x_n, m(l_n)) \) as \( \neg B_1(x_1, 1, \ldots, x_n, m(l_n)) \).
(iv) \( A(x_1, \ldots, x_n) \) is \( A_1(x_1, \ldots, x_n) \lor A_2(x_1, \ldots, x_n) \).

By induction we have \( B_1(x_1, l', \ldots, x_n, m(l_n)) \) and
\( B_2(x_1, l', \ldots, x_n, m(l_n)) \) of the desired type. Therefore
it is sufficient to define \( B(x_1, l', \ldots, x_n, m(l_n)) \) as
\( B_1(x_1, l', \ldots, x_n, m(l_n)) \lor B_2(x_1, l', \ldots, x_n, m(l_n)) \).

(v) \( A(x_1, \ldots, x_n) \) is \( \exists x A'(x, x_1, \ldots, x_n) \). With no
loss of generality we assume that \( x \) is none of \( x_1, \ldots, x_n \).
Also, we consider the case where \( n = 0 \) because the
presence of \( x_1, \ldots, x_n \) in \( A(x, x_1, \ldots, x_n) \) makes no real
difference to the argument.

For the rest of this proof \( x_i \) is in category \( i \). Define
\( A^i \) to be \( A_x'[x_i] \) and define \( B^i(x_1, l', \ldots, x_i, m(i)) \) to be
the formula \( B(x_1, l', \ldots, x_n, m(l_n)) \) given by the induction
hypothesis when \( A(x_1, \ldots, x_n) \) is \( A^i \). Then we have:
\[
\mathcal{L}(A) = \top
\]
\[
\mathcal{L}(\exists x A'(x)) = \top
\]
\[
\mathcal{L}(A_x'\{b\}) = \top \text{ for some } b \in \mathcal{L}
\]
\[
\mathcal{L}(A_x'\{b\} \land C^i) = \top \text{ for some } b \in \mathcal{L} \text{ and } i \text{ between }\]
\[
1 \text{ and } 2k_1 + 2n
\]
\[
\mathcal{L}(\bigvee_{i=1}^{2k_1+2n} A^i x_1\{b\}) = \top \text{ for some } b \in \mathcal{L}
\]
\[
\mathcal{L}(\bigvee_{i=1}^{2k_1+2n} B^i x_1, l', \ldots, x_i, m(i) \{b_i, l', \ldots,
\]
\[
b_i, m(i)\}) = \top \text{ for some } i(1 \leq i \leq 2k_1 + 2n) \text{ and some}
\]
sequence \( b_i, l', \ldots, b_i, m(i) \) of elements of \( |F_1^{-1}(\mathcal{L})|\)
\[
\mathcal{L}(\bigvee_{i=1}^{2k_1+2n} \exists x_i, l', \ldots, \exists x_i, m(i) B^i(x_i, l', \ldots,
\]
\[
x_i, m(i))) = \top
\]
Therefore it is sufficient to define $B$ as the formula
\[
\bigvee_{i=1}^{2^{k_1}+n} \exists x_{i,1} \ldots \exists x_{i,m(i)} B^i(x_{i,1}, \ldots, x_{i,m(i)}).
\]

The next result is immediate.

**Lemma 2.** Let $A$ be an $L$-sentence and let $\mathfrak{L}$ be an element of $F_1(\mathfrak{M})$. Then there is an $M$-sentence $B$ such that $\mathfrak{L}(A) = T$ just in case $F_1^{-1}(\mathfrak{L})(B) = T$.

Now we can easily prove the existence of $H_1$.

**Lemma 3.** There is a one-one function $H_1: L \to M$ such that for any $A$ in $L$ and any $\mathfrak{L}$ in $F_1(\mathfrak{M})$, $\mathfrak{L}(A) = T$ just in case $F_1^{-1}(\mathfrak{L})(H_1(A)) = T$.

**Proof:** Let $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ be effective listings of $L$ and $M$ respectively. For any $n$, we know that lemma 2 guarantees the existence of an $M$-sentence $B$ which is true in $F_1^{-1}(\mathfrak{L})$ just in case $A_n$ is true in $\mathfrak{L}$. Let $B'_n$ be the least such $B$ and let $B'_n = \{B^1_n, B^2_n, \ldots\}$, where $B^j_n (1 \leq j) \text{ is } B'_n \& \ldots \& B'_n$ ($j$ conjunctands). Then we let $H_1(A_0) = B'_0$ and for $n \geq 1$ we let $H_1(A_n) = \text{the least member of } B'_n = \bigcup_{i=0}^{n-1} \{H_1(A_i)\}$.

Clearly, for any $n$, $B'_n = \bigcup_{i=0}^{n-1} \{H_1(A_i)\}$ is nonempty, since $B'_n$ is countable. Therefore $H_1$ is a one-one function from $L$ to $M$ and since, for any $j$ and $n$, $F_1^{-1}(\mathfrak{L})(B'_n) = T \iff F_1^{-1}(\mathfrak{L})(B^j_n) = T$, we see that for any $A$ in $L$ and $\mathfrak{L}$ in $F_1(\mathfrak{M})$, $\mathfrak{L}(A) = T \iff$
The convention that $x_1, \ldots, x_n$ denote the first $n$ variables of the alphabetical listing of the variables of $L$ and that $x_i$ is in category $l_i$ is no longer used.

Now we proceed to show the existence of $G_1$ as mentioned in the outline. First we must set up a suitable mapping from $M$-formulas to $L$-formulas. This is done in:

**Lemma 4.** Let $A(x_1, \ldots, x_n)$ be an $M$-formula and let $\mathcal{M}$ be an element of $\mathcal{M}$. Then there is an $L$-formula $B(x_1, \ldots, x_n)$ such that for any $a_1, \ldots, a_n$ in $|\mathcal{M}|$, $\mathcal{M}(\mathcal{A}[a_1, \ldots, a_n]) = T$ just in case $F_1(\mathcal{M})(B[a_1^{-1}, \ldots, a_n^{-1}]) = T$. (1)

**Proof:** We use induction on the length of $A(x_1, \ldots, x_n)$.

(i) $A(x_1, \ldots, x_n)$ is $x_i = x_j$. It is sufficient to define $B(x_1, \ldots, x_n)$ as $x_i = x_j$.

(ii) $A(x_1, \ldots, x_n)$ is $q_i y_1 \ldots y_{k_i}$ for some $i$ between 1 and $n$, where $\{y_1, \ldots, y_{k_i}\} \subseteq \{x_1, \ldots, x_n\}$. In this case $B(x_1, \ldots, x_n)$ is defined to be the formula $\exists z_1 \exists z_2 (P^i z_2 \& p z_1 z_2 \& T^{k_i} z_1 y_1 \ldots y_{k_i})$. Then for any $a_1, \ldots, a_{k_i} \in |\mathcal{M}|$, $\mathcal{M}(\mathcal{A}[a_1, \ldots, a_{k_i}]) = T$.

$\Leftrightarrow (q_i)_{\mathcal{M}}(a_1, \ldots, a_{k_i})$

$\Leftrightarrow F_1(\mathcal{M})(P^i b \& p a b \& T^{k_i} a_1^{-1} \ldots a_{k_i}^{-1}) = T$ for

$(1)$ $\tau^{-1}$ is the bijection from $|\mathcal{M}|$ to the level $n+2$ elements of $F_1(\mathcal{M})$. 
some $a, b \in |F_1(\mathcal{M})|$

$\iff F_1(\mathcal{M})(\exists z_1 \exists z_2 (P^1z_2 \& \rho z_1z_2 \& T^kz_1a_1^{-1} \ldots a_{k_1}^{-1})) = T$

$\iff F_1(\mathcal{M})(B[a_1^{-1}, \ldots, a_{k_1}^{-1}]) = T.$

(iii) $A(x_1, \ldots, x_n)$ is $\forall A_1(x_1, \ldots, x_n)$. From the induction hypothesis we have $B_1(x_1, \ldots, x_n)$ of the desired type with respect to $A_1(x_1, \ldots, x_n)$. Therefore it is sufficient to define $B(x_1, \ldots, x_n)$ as the formula $\forall B_1(x_1, \ldots, x_n)$.

(iv) $A(x_1, \ldots, x_n)$ is $\forall A_1(x_1, \ldots, x_n) \lor A_2(x_1, \ldots, x_n)$ and again the induction hypothesis yields the formulas $B_1(x_1, \ldots, x_n)$ and $B_2(x_1, \ldots, x_n)$ of the desired types. Therefore we define $B(x_1, \ldots, x_n)$ to be the formula $B_1(x_1, \ldots, x_n) \lor B_2(x_1, \ldots, x_n)$.

(v) $A(x_1, \ldots, x_n)$ is $\exists x A_1(x, x_1, \ldots, x_n)$. With no loss of generality we assume that $x \notin \{x_1, \ldots, x_n\}$. We only consider the case for $n = 0$ because the presence of $x_1, \ldots, x_n$ in $A_1(x, x_1, \ldots, x_n)$ makes no real difference to the argument. By induction we have $B_1(x)$ of the desired type with respect to $A_1(x)$. It is sufficient to define $B$ as the formula $\exists x B_1(x)$.

The next result is immediate.

**Lemma 5**: Let $A$ be an $M$-sentence and let $\mathcal{M}$ be an element of $\mathcal{M}$. Then there is an $L$-sentence $B$ such that for any $A$ in $M$ and any $\mathcal{M}$ in $\mathcal{M}$, $\mathcal{M}(A) = T$ just in case
\[ F_1(\eta_1)(B) = T. \]

The proof of the next result is analogous to that of lemma 3.

**Lemma 6:** There is a one-one function \( G_1 : \mathcal{M} \to \mathcal{L} \) such that for any \( A \) in \( \mathcal{M} \) and any \( \eta \) in \( \mathcal{M} \), \( \eta(A) = T \) just in case \( F_1(\eta_1)(G_1(A)) = T \).

Now we show that \( F_1(\eta_1) \) is an elementary class.

**Lemma 7:** \( F_1(\eta_1) \) is defined by an L-sentence \( A_1 \).

**Proof:** We shall define \( A_1 \) to be the conjunction of a finite number of L-sentences, only some of which will be explicitly shown.

First we construct an L-sentence \( B_1 \) which says that any directed graph of \( L \) will have no cycles of length less than \( 2k_1+n+2 \). We define \( B_1 \) to be the sentence

\[
\forall x_1 \forall x_2 (p_{x_1x_2} \rightarrow \neg p_{x_2x_1}) \land \forall x_1 \forall x_2 \forall x_3 (p_{x_1x_2} \land p_{x_2x_3} \rightarrow \neg p_{x_3x_1}) \land \ldots \land \forall x_1 \forall x_2 \forall x_3 \ldots \forall x_{2k_1+n+1} (p_{x_1x_2} \land \ldots \land p_{x_{2k_1+n+1}x_{2k_1+n+1}} \rightarrow \neg p_{x_{2k_1+n+1}x_{2k_1+n+1}}).
\]

Let \( B_2 \) be an L-structure which says that there is an element in level \( j \) (\( 1 \leq j \leq 2k_1+n+1 \)) and that every element occurs in one of these levels. That is, we define \( B_2 \) to be the sentence

\[
\exists x_1 \exists x_2 \ldots \exists x_{2k_1+n+1} (Lx_1 \lor \ldots \lor Lx_{2k_1+n+1}).
\]

Let \( B_3 \) be an L-sentence which says that there are exactly \( n \) distinct elements in level \( 2k_1+n+1 \).

Let \( B_4 \) be an L-sentence which says that each ele-
ment occurs in a unique level and that levels 1 to n+1 contain exactly one element each.

For the remainder of this proof we use the expression "the element x contains the element y" and variations of it to mean 'pyx'.

Let $B_5$ be an L-sentence which says that each level $2k_1+n+1$ element contains exactly one element from levels 1 to n and that no two distinct level $2k_1+n+1$ have any common members; furthermore, $B_5$ says that if $x$ is a level $2k_1+n+1$ level element and $x$ contains $y$ from level $j$ ($1 \leq j \leq n$), then any $z$ in $x$ which is not equal to $y$ must be in level $2k_j+n$. (1)

Let $B_6$ be an L-sentence which says that each level $n+3$ element contains one or two elements of level $n+2$ and no other elements and that every possible such combination of level $n+2$ elements is a level $n+3$ element.

Let $B_7$ be an L-sentence which says that any element $x$ of level $n+4$ contains two elements $y$ and $z$ of level $n+3$ and no other elements, and that $y$ contains just one element $w$ and $w$ is also in $z$, and that all possible such combinations of level $n+3$ elements are level $n+4$ elements. That is, we define $B_7$ to be the sentence

$$\forall x_1 (L^{n+4}x_1 \leftrightarrow \exists x_2 \exists x_3 \exists x_4 (L^{n+3}x_2 \land L^{n+3}x_3 \land L^{n+2}x_4 \land \forall x_5 (px_5x_2 \leftrightarrow x_5 = x_4) \land px_4x_3 \land \forall x_5 (px_5x_1 \leftrightarrow x_5 = x_1 \lor x_5 = x_2)))$$

For $4 \leq s \leq k_1+1$ we let $B_{2s}$ be an L-sentence which

(1) Let $B_5$ also say that no element contains itself and that elements are equal just in case they contain exactly the same elements.
says that all elements of level \( n+2s-3 \) contain exactly two elements, one from level \( n+2 \) and one from level \( n+2s-4 \), and that all such possible combinations of level \( n+2 \) and level \( n+2s-4 \) elements are level \( n+2s-3 \) elements.

For \( 4 \leq s \leq k_1+1 \) we let \( B_{2s+1} \) be an L-sentence which says that all elements of level \( n+2s-2 \) contain exactly two elements \( x \) and \( y \), with \( x \) in level \( n+3 \) and \( y \) in level \( n+2s-3 \) such that \( x \) has exactly one member \( w \) and \( w \) is also in \( y \), and that all possible such combinations of level \( n+3 \) and level \( n+2s-3 \) elements are level \( n+2s-2 \) elements.

Finally we define \( A_1 \) to be \( B_1 \land \cdots \land B_{2k_1+3} \). Then it is immediate that if \( \mathcal{L} \) is in \( F_1(M) \), then \( \mathcal{L}(A_1) = T \). Also if \( \mathcal{L}(A_1) = T \) we see that \( \mathcal{L} \models \mathcal{L}' \) where \( \mathcal{L}' \) is in \( F_1(M) \), but this means that \( \mathcal{L} \) must be \( \mathcal{L}' \). Therefore, for any \( \mathcal{L} \) in \( \mathcal{L} \), \( \mathcal{L}(A_1) = T \) just in case \( \mathcal{L} \) is in \( F_1(M) \).

\( \square \)

It is now easy to show that \( F_1(M_1) \) is elementary.

Lemma 8: \( F_1(M_1) \) is defined by a single L-sentence \( B_1 \).
Proof: We use the function \( G_1 \) and the fact that \( M_1 \) is defined by the M-sentence \( B_2 \) and \( F_1(M) \) is defined by the L-sentence \( A_1 \). The sentence \( B_1 \) is defined to be the sentence \( A_1 \land G_1(B_2) \). Then for any \( \mathcal{L} \) in \( \mathcal{L} \) we have:
3.2 Results Concerning $L_2$ and $M_2$

For the rest of this paper $F_1(M_1)$ is denoted by $L_1$ and $\bigcup_{n=1}^{\infty} L_1^n \cup \bigcup_{n=1}^{\infty} L_2^n$ is denoted by $L_2$. It is immediate that $L_1 \cap L_2$ is empty. This fact will be quite important later on. The one-one maps $J$ and $H$, as defined on pages 25 and 26 are used to define $M_2$ as $J \circ H(L_2)$. It is immediate that $M_2 \subset M - M$, $M_2 = \bigcup_{n=1}^{\infty} J(L_1^n) \cup J(L_2^n)$, and that $J \circ H$ is a bijection from $L_2$ to $M_2$.

Now we show that $L_2$ is elementary.

Lemma 9: $L_2$ is defined by an L-sentence $C_1$.

Proof: The proof is straightforward once we notice that $L$ is a member of $L_2$ just in case $L$ is isomorphic to some member of $L_1$, plus a nonzero number of isolated points.

We start with the sentence $B_1$ of lemma 8 which defines $F_1(M_1)$ and let $Q_1 x_1 \ldots Q_m x_m B$ be a prenex form of $B_1$. Then we define $U_x$ to be an abbreviation of $\exists y (pxy \lor pyx)$. That is, $L(Ua) = T$ just means that $a$ is not isolated in $L$. Next we form the sentence
B^i from Q_{1}x_{1}...Q_{m}x_{m}B^{m+1} as follows:

(i) If Q_{m}x_{m} is \exists x_{m} replace Q_{m}x_{m}B^{m+1} with \exists x_{m}(Ux_{m} & B^{m+1}) and if Q_{m}x_{m} is \forall x_{m} replace Q_{m}x_{m} with \forall x_{m}(Ux_{m} \rightarrow B^{m+1}); let the resulting part be denoted by B^{m};

(ii) for 1 \leq i \leq m-1, let B^{i+1} be the result of making the appropriate replacements for Q_{m}x_{m}B^{m+1}, ..., Q_{i+1}B^{i+2}, then if Q_{i}x_{i} is \exists x_{i} replace Q_{i}x_{i}B^{i+1} with \exists x_{i}(Ux_{i} & B^{i+1}) and if Q_{i}x_{i} is \forall x_{i} replace Q_{i}x_{i}B^{i+1} with \forall x_{i}(Ux_{i} \rightarrow B^{i+1}); let the resulting part be B^{i}.

It is easily seen that B^{i} is an L-sentence which says that the nonisolated elements of \mathcal{L} form a structure which is isomorphic to a member of \mathcal{P}_{1}.

Then let B'' be the sentence \exists x\forall Ux and define C_{1} to be the sentence B^{1} & B''. It is immediate that C_{1} defines \mathcal{L}_{2}.

From lemma 9 we easily obtain the next result.

Lemma 10: \mathcal{M}_{2} is defined by a single M-sentence C_{2}.

Proof: First we notice that \mathcal{M}_{2} = \bigcup_{n=0}^{\infty} J(\mathcal{L}_{n}) \cup J(\mathcal{L}_{\omega}).

Then we form the M-sentence C_{2} from the L-sentence C_{1} of lemma 9 which defines \mathcal{L}_{2} by deleting the conjunctand B'' which says that there is an isolated element and by replacing every part of the form pxy with
a part of the form $q_1xy...y$ and lastly, by adding a
conjunct and which says that $(q_i)_m$ is empty for each
$i$ between 2 and $n$.

It is not difficult to see that for any $m \in M$
$m(C_2) = T \iff m \in M_\lambda$.

Now we proceed to prove the existence of the one-
one function $H_2$ which was mentioned in the outline.
The method of proof used is somewhat like the method
used to prove the existence of $H_1$.

First we introduce some new notation. For every
$L$ in $L_\lambda$ we know that there is a unique $L'$ in $L_1 \cup
L_\lambda$ such that for some isolated point $a^*$ of $L$ we have
$L - \{a^*\} \cong L'$ via some bijection $\theta_1$. We also know that
$J(L') = J_0H(\lambda)$ and that $L'$ is isomorphic, in the usual
algebraic sense, to $J(L') = J_0H(\lambda)$ via some bijection
$\theta_2$. Let $\mu$ denote $\theta_2 \circ \theta_1$. Then $\mu$ is an algebraic isomorphism from $L - \{a^*\}$ to $J_0H(\lambda)$. Let $E^1x$ be an abbreviation of $x = a^*$ and let $E^2x$ be an abbreviation of
$x \neq a^*$. Now the term category is used in a different
sense than before. We say, for $L$ in $L_\lambda$ and $a$ in $\lambda$, that $a$ is in category $i$ ($i = 1, 2$) if $L(E^i a) = T$.

Clearly every individual of $L$ is in exactly one category
and $\mu$ is a bijection from the category 2 elements of
$|\mathcal{L}|$ to the universe of $J_0H(\lambda)$. We also say that the
variable $x$ is in category $i$ ($i = 1, 2$) just in case
for any $L$-formula $A$, if $A'$ is an $\mathcal{L}$-instance of $A$ in which $a$ is substituted for $x$ in $A$, then $a$ is in category $i$.

During the statement and proof of the next lemma $x_1, \ldots, x_n$ will denote the first $n$ variables of the alphabetical listing of all the variables of $L$.

**Lemma 11.** Let $A(x_1, \ldots, x_n)$ be an $L$-formula with $x_1, \ldots, x_n$ in categories $l_1, \ldots, l_n$ respectively and let $\mathcal{L}$ be an element of $\mathcal{L}_1$. Also let $y_1, \ldots, y_k$ be all of the $x_i$ $(1 \leq i \leq n)$ with $l_i = 2$ in the order of their appearance in $x_1, \ldots, x_n$. Then there is an $\mathcal{L}$-formula $B(y_1, \ldots, y_k)$ such that for any $a_1, \ldots, a_n$ in $\mathcal{L}$ with $(E^{l_1}a_1 & \ldots & E^{l_n}a_n) = T$, $\mathcal{L}(A[a_1, \ldots, a_n]) = T$ just in case $J \circ H(I)(B[a'_1\mu, \ldots, a'_k\mu]) = T$, where $a'_i$ is $a_j$ just in case $y_i$ is $x_j$ $(1 \leq i \leq k$ and $1 \leq j \leq n)$.

**Proof:** Again we induct on the length of $A(x_1, \ldots, x_n)$.

(i) $A(x_1, \ldots, x_n)$ is $x_i = x_j$. Three subcases are considered.

(a) If $l_i = l_j$, then we let $B(y_1, \ldots, y_k)$ be $\exists x(x = x)$.

(b) If $l_i = l_j = 2$, then we let $B(y_1, \ldots, y_k)$ be $x_i = x_j$.

(c) If $l_i \neq l_j$, then we let $B(y_1, \ldots, y_k)$ be $\exists x(x \neq x)$.

(ii) $A(x_1, \ldots, x_n)$ is $px_i x_j$. Two subcases arise.

(a) If $l_i = l_j = 1$ or $l_i \neq l_j$, then we let
If $l_i = l_j = 2$, then we let $B(y_1, \ldots, y_k)$ be $q_1 x_1 x_j \ldots x_j$.

(iii) $A(x_1, \ldots, x_n)$ is $\neg A_1(x_1, \ldots, x_n)$. By induction we have $B_1(y_1, \ldots, y_k)$ with the desired properties with respect to $A_1(x_1, \ldots, x_n)$. Therefore we let $B(y_1, \ldots, y_k)$ be $\neg B_1(y_1, \ldots, y_k)$.

(iv) $A(x_1, \ldots, x_n)$ is $A_1(x_1, \ldots, x_n) \lor A_2(x_1, \ldots, x_n)$. By induction we have $B_1(y_1, \ldots, y_k)$ and $B_2(y_1, \ldots, y_k)$ of the desired type with respect to $A_1(x_1, \ldots, x_n)$ and $A_2(x_1, \ldots, x_n)$. Therefore we let $B(y_1, \ldots, y_k)$ be $B_1(y_1, \ldots, y_k) \lor B_2(y_1, \ldots, y_k)$.

(v) $A(x_1, \ldots, x_n)$ is $\exists x A'(x, x_1, \ldots, x_n)$. With no loss of generality we assume that $x$ is none of $x_1, \ldots, x_n$. Also, we just treat the case for $n = 0$ because the presence of $x_1, \ldots, x_n$ makes no real difference to the argument. Let $A^1$ be $A'_x(x_1)$ and let $A^2$ be $A'_x(x_2)$ with $l_1 = 1$ and $l_2 = 2$. Then let $B^i(x_i)$ be the formula given by the induction hypothesis when $A(x_1, \ldots, x_n)$ is $A^i (i = 1, 2)$. Notice that $B^1(x_1)$ is a sentence because the only free variable of $A^1$ is in category $1$.

Now we have the following:

\[
\mathcal{L}(A) = T
\]

\[\iff \mathcal{L}(\exists x A'(x)) = T \]

\[\iff \mathcal{L}(A_1^1 x_1 [a] \lor A_2^2 x_2 [a]) = T \text{ for some } a \in |\mathcal{L}| \]

\[\iff J^{\mathcal{L}}(A_1^1 x_1 [a] \lor B^2 x_2 [a]) = T \text{ for some } a \in |J^{\mathcal{L}}| \]
The last bi-implication above is true because \( \mu \) is a bijection from the category 2 elements of \( \mathcal{L} \) to \( J \circ H(\mathcal{L}) \) and because \( B^1(x_1) \) is a sentence.

Now we see that it is sufficient to define \( B \) as the formula \( B^1(x_1) \lor \exists x_2 B^2(x_2) \).

The next result is immediate.

**Lemma 12:** If \( A \) is an \( L \)-sentence and \( \mathcal{L} \) is in \( \mathcal{L}_2 \), then there is an \( M \)-sentence \( B \) such that \( \mathcal{L}(A) = T \) just in case \( J \circ H(\mathcal{L})(B) = T \).

From lemma 12 we have:

**Lemma 13:** There is a one-one function \( H_2 : L \rightarrow M \) such that for any \( A \) in \( L \) and any \( \mathcal{L} \) in \( \mathcal{L}_2 \), \( \mathcal{L}(A) = T \) just in case \( J \circ H(\mathcal{L})(H_2(A)) = T \).

Let \( \mu \) be the one-one function defined on page 50. Then we have the next result.

**Lemma 14:** Let \( A(x_1, \ldots, x_n) \) be an \( M \)-formula and let \( \mathcal{M} \) be in \( \mathcal{M}_2 \). Then there is an \( L \)-formula \( B(x_1, \ldots, x_n) \) such that for any \( a_1, \ldots, a_n \) in \( |\mathcal{M}| \), \( \mathcal{M}(A[a_1, \ldots, a_n]) = T \) just in case \( (J \circ H)^{-1}(\mathcal{M})(B[a_1 \mu^{-1}, \ldots, a_n \mu^{-1}]) = T \).

**Proof:** Once again we induct on the length of \( A(x_1, \ldots, x_n) \).

(i) \( A(x_1, \ldots, x_n) \) is \( x_i = x_j \). We let \( B(x_1, \ldots, x_n) \) be \( x_i = x_j \).
(ii) \( A(x_1, \ldots, x_n) \) is \( q_1 y_1 \ldots y_k \), where \( \{y_1, \ldots, y_k\} \subseteq \{x_1, \ldots, x_n\} \), for some \( i \) between 1 and \( n \). We consider two subcases.

(a) If \( i \geq 2 \), then we let \( B(x_1, \ldots, x_n) \) be \( y_1 \neq y_1 \).

(b) If \( i = 1 \), then we let \( B(x_1, \ldots, x_n) \) be \( p y_1 y_2 \).

(iii) \( A(x_1, \ldots, x_n) \) is \( \neg A_1(x_1, \ldots, x_n) \). By induction we have \( B_1(x_1, \ldots, x_n) \) of the desired type. Therefore we let \( B(x_1, \ldots, x_n) \) be \( \neg B_1(x_1, \ldots, x_n) \).

(iv) \( A(x_1, \ldots, x_n) \) is \( A_1(x_1, \ldots, x_n) \lor A_2(x_1, \ldots, x_n) \). By induction we have \( B_1(x_1, \ldots, x_n) \) and \( B_2(x_1, \ldots, x_n) \) of the desired type. Therefore we let \( B(x_1, \ldots, x_n) \) be \( B_1(x_1, \ldots, x_n) \lor B_2(x_1, \ldots, x_n) \).

(v) \( A(x_1, \ldots, x_n) \) is \( \exists x A_1(x, x_1, \ldots, x_n) \). We assume with no loss of generality that \( x \notin \{x_1, \ldots, x_n\} \). Also, we treat only the case for \( n = 0 \) because the presence of \( x_1, \ldots, x_n \) in \( A(x_1, \ldots, x_n) \) makes no real difference to the argument used. Let \( B_1(x) \) be the formula which the induction formula yields when \( A(x_1, \ldots, x_n) \) is \( A_1(x) \). Then we define \( B \) to be the formula \( \exists x B_1(x) \).

\( \square \)

Now we have:

**Lemma 15:** Let \( A \) be an \( \mathcal{M} \)-sentence and let \( \mathcal{M} \) be an element of \( \mathcal{M}_2 \). Then there is an \( \mathcal{L} \)-sentence \( B \) such that \( \mathcal{M}(A) = \top \) just in case \( (J \circ H)^{-1}(\mathcal{M})(B) = \top \).

Then from lemma 15 we obtain:
Lemma 16: There is a one-one function \( G_2 : \mathcal{M} \to \mathcal{L} \) such that for any \( A \) in \( \mathcal{M} \) and any \( \mathcal{M} \) in \( \mathcal{M}_2 \), \( \mathcal{M}(A) = T \) just in case \( (J \circ H)^{-1}(\eta)(G_2(A)) = T \).

This completes the necessary preliminary results concerning \( \mathcal{L}_2 \) and \( \mathcal{M}_2 \).

3.3 Results Concerning \( \mathcal{L}_3 \) and \( \mathcal{M}_3 \)

We define \( \mathcal{L}_3 \) as \( \mathcal{L}_3 = (\mathcal{L}_1 \cup \mathcal{L}_2) \) and \( \mathcal{M}_3 \) as \( \mathcal{M}_3 = (\mathcal{M}_1 \cup \mathcal{M}_2) \). Let us recall that the sentences \( B_1, C_1, B_2, \) and \( C_2 \) define the classes \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1, \) and \( \mathcal{M}_2 \) respectively. Then it is immediate that the sentences \( \neg(B_1 \lor C_1) \) and \( \neg(B_2 \lor C_2) \), which we denote by \( D_1 \) and \( D_2 \), define \( \mathcal{L}_3 \) and \( \mathcal{M}_3 \) respectively. If the reader refers to comments on the function \( J \) on page 27 it is easy to see that \( J \) is a bijection from \( \mathcal{L}_3 \) to \( \mathcal{M}_3 \).

Now we prove the existence of a one-one function \( H_3 : \mathcal{L} \to \mathcal{M} \) which preserves truth-values under \( J | \mathcal{L}_3 \).

First we recall that for any \( \mathcal{L} \) in \( \mathcal{L}_3 \), \( \mathcal{L} \) is isomorphic in the usual algebraic sense to \( J(\mathcal{L}) \). Let \( \psi \) be the isomorphism from \( \mathcal{L} \) in \( \mathcal{L}_3 \) to \( J(\mathcal{L}) \). Then we have the following immediate result.

Lemma 17: Let \( \Lambda(x_1, \ldots, x_n) \) be an \( \mathcal{L} \)-formula and let \( \mathcal{L} \) be a member of \( \mathcal{L}_3 \). Then there is an \( \mathcal{M} \)-formula \( B(x_1, \ldots, x_n) \) such that for any \( a_1, \ldots, a_n \) in \( \mathcal{L}_1 \), \( \mathcal{L} \Lambda(a_1, \ldots, a_n) = T \) just in case \( J(\mathcal{L})B(a_1 \psi, \ldots, a_n \psi) = T \).
From the above lemma we have:

**Lemma 18:** Let $A$ be an $L$-sentence and let $\mathcal{L}$ be in $\mathcal{L}_3$. Then there is an $M$-sentence $B$ such that $\mathcal{L}(A) = T$ just in case $J(\mathcal{L})(B) = T$.

Finally we obtain:

**Lemma 19:** There is a one-one function $H_3: L \rightarrow M$ such that for any $A$ in $L$ and any $\mathcal{L}$ in $\mathcal{L}_3$, $\mathcal{L}(A) = T$ just in case $J(\mathcal{L})(H_3(A)) = T$.

It is just as easy to prove the existence of the one-one function $G_3: M \rightarrow L$ which preserves truth-values under $J^{-1}|M_3$ as we see in the next three results.

If $\psi$ is the isomorphism used in lemma 17, then the next result has the same proof as lemma 14, with $\psi^{-1}$ in place of $\mu^{-1}$.

**Lemma 20:** Let $A(x_1, \ldots, x_n)$ be an $M$-formula and let $\mathcal{M}$ be in $\mathcal{M}_3$. Then there is an $L$-formula $B(x_1, \ldots, x_n)$ such that for any $a_1, \ldots, a_n$ in $|\mathcal{M}|$, $\mathcal{M}(A[a_1, \ldots, a_n]) = T$ just in case $J^{-1}(\mathcal{M})(B[a_1\psi^{-1}, \ldots, a_n\psi^{-1}])$.

From lemma 20 we have:

**Lemma 21:** Let $\Lambda$ be an $M$-sentence and let $\mathcal{M}$ be in $\mathcal{M}_3$. Then there is an $L$-sentence $B$ such that $\mathcal{M}(\Lambda) = T$ just in case $J^{-1}(\mathcal{M})(B) = T$.

Therefore we obtain:

**Lemma 22:** There is a one-one function $G_3: M \rightarrow L$ such
that for any \( A \) in \( M \) and any \( \mathcal{M} \) in \( \mathcal{M}_3 \), \( \mathcal{M}(A) = T \) just in case \( J^{-1}(\mathcal{M})(G_3(A)) = T \).

This completes all of the necessary preliminary results.
CHAPTER 4

4.1 Proof of the Main Result

For the following theorem the reader may find it convenient to refer to the pictorial representation of the function F given on page 28. By combining some of the results of chapter 3, we have the following result.

**Theorem 1:** There is a bijection $F: \mathcal{L} \rightarrow \mathcal{M}$ such that:

(i) $F|_{\mathcal{L}_1} = F_1^{-1}$;

(ii) $F|_{\mathcal{L}_2} = J \circ H$;

(iii) $F|_{\mathcal{L}_3} = J$.

**Proof:** First we recall that $F_1^{-1}: \mathcal{L}_1 \rightarrow \mathcal{M}_1$, $J \circ H: \mathcal{L}_2 \rightarrow \mathcal{M}_2$, and $J: \mathcal{L}_3 \rightarrow \mathcal{M}_3$ are all bijections. Then we recall that $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are pairwise disjoint with their union equal to $\mathcal{L}$ and that similar results hold for $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$. Then we define $F$ as follows:

(i) for $\mathcal{L} \in \mathcal{L}_1$, $F(\mathcal{L}) = F_1^{-1}(\mathcal{L})$;

(ii) for $\mathcal{L} \in \mathcal{L}_2$, $F(\mathcal{L}) = J \circ H(\mathcal{L})$;

(iii) for $\mathcal{L} \in \mathcal{L}_3$, $F(\mathcal{L}) = J(\mathcal{L})$.

Now we consider the one-one functions $K_L$ and $K_M$ mentioned on page 28. Recall that $B_2$, $C_2$ and $D_2$ are the $M$-sentences which define the classes $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ respectively and that $H_1$, $H_2$, and $H_3$ are the one-one functions yielded by lemmas 3, 13, and 19 respectively. Then we define a one-one function
$K_L : L \rightarrow M$ as follows: for any $A$ in $L$, $K_L(A)$ is the $M$-sentence $(H_1(A) \land B_2) \lor (H_2(A) \land C_2) \lor (H_3(A) \land D_2)$. It is evident that $K_L$ is a one-one function; furthermore for any $A$ in $L$ and any $x$ in $M$, $L(A) = T$ just in case $F(L)(K_L(A)) = T$.

Next we recall that the $L$-sentences $B_1$, $C_1$, and $D_1$ define the classes $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ respectively, and that $G_1$, $G_2$, and $G_3$ are the one-one functions given by lemmas 6, 16, and 22 respectively. Then we define $K_M : M \rightarrow L$ in the same manner as $K_L$ is defined. Again we see that $K_M$ is a one-one function and that for any $A$ in $M$ and any $x$ in $M$, $x(A) = T$ just in case $F^{-1}(x)(K_M(A)) = T$.

Now we use $K_L$ and $K_M$ to define the desired bijection from $L$ to $M$.

**Theorem 2:** There is a bijection $G : L \rightarrow M$ such that for any $A$ in $L$ and any $x$ in $L$, $L(A) = T$ just in case $F(L)(G(A)) = T$.

**Proof:** Since $L$ and $M$ are both countable we let $A_0$, $A_1$, ... and $B_0$, $B_1$, ... be effective listings of their elements. For $A_n \in L$ we define $A^i_n$ ($1 \leq i < \infty$) to be $K_L(A_n)$ & ... & $K_L(A_n)$ (i conjunctands), and for $B_n \in M$ we define $B^i_n$ ($1 \leq i < \infty$) to be $K_M(B_n)$ & ... & $K_M(B_n)$ (i conjunctands).

Then we define $G$ as follows:

(i) $G(A_0) = K_L(A_0)$;

(ii) $G^{-1}(B_0) = A_0$ if $G(A_0)$ is $B_0$. 

for the least $i$ such that $B^i_0$ is not $A_0$ if $G(A_0)$ is not $B_0$;

(iii) for $n \geq 1$,

$$G(A_n) = \begin{cases} B_m & \text{if } G^{-1}(B_m) \text{ is } A_n \text{ for some } m \\
A_n & \text{for the least } i \text{ such that}
\end{cases}$$

between 0 and $n-1$,

$$A_n \bigcup_{m=0}^{n-1} G(A_m) \bigcup \bigcup_{m=0}^{n-1} B_m$$

if $A_n$ is not $G^{-1}(B_m)$ for all $m$ between 0 and $n-1$;

(iv) for $n \geq 1$,

$$G^{-1}(B_n) = \begin{cases} A_m & \text{if } G(A_m) \text{ is } B_n \text{ for some } m \\
B_n & \text{for the least } i \text{ such that}
\end{cases}$$

between 0 and $n$,

$$B_n \notin \bigcup_{m=0}^{n-1} G^{-1}(B_m) \bigcup \bigcup_{m=0}^{n-1} A_m$$

if $B_n$ is not $G(A_m)$ for all $m$ between 0 and $n$.

It is now immediate that $G$ satisfies the hypothesis of the theorem.

$\square$

Theorem 3: $L \cong' M$

Proof: This result follows from theorems 1 and 2 and the definition of $\cong'$ - isomorphism.

If we recall the definitions of $L$ and $M$ on page the next result is immediate.
Theorem 4: If $L$ is a first order language with a binary predicate symbol as its only nonlogical symbol and $M$ is a first order language with a finite nonzero number of predicate symbols, one of which has two or more places, as its only nonlogical symbols, then $L \equiv_\omega M$.

4.2 Concluding Remarks

The bijection $F$ of theorem 1 has the following property: If $F'$, a bijection from the class of all $L$-structures to the class of all $M$-structures, is derived from $F$ as in section 2.2, then for any $L$-structures $L$ and $L'$, $L \equiv L'$ just in case $F'(L) \equiv F'(L')$.

In particular, $F'$ takes isomorphism-types onto isomorphism-types and, for any $L$-sentence $A$ and any class $L$ of $L$-structures, $A$ defines $L$ just in case $G(A)$ defines $F'(L)$, where $G$ is the bijection of theorem 2.

We also point out that the bijection $G$ of theorem 2 is recursive. (1)

In [2], Hanf stated a version of his theorem and pointed out that instead of one 2-ary predicate symbol we would have considered two 1-ary function symbols.

In his thesis [3], Hanf was working with languages which were a natural extension of the usual first order languages. For an infinite cardinal $\alpha$ he defined an $L_\alpha$ language to be a language which was first order with the following exceptions:

(i) there are $\alpha$ distinct variables;
(ii) there are no function symbols;
(iii) for each $\beta < \alpha$, the predicate symbols may have $\beta$ places;
(iv) for each $\beta < \alpha$, disjunctions of length $\beta$ are allowed;
(v) for each $\beta < \alpha$, quantification over $\beta$ variables is allowed.

As Hanf pointed out, if $\alpha$ is $\aleph_0$, then an $L_\alpha$ language is an ordinary first order language.

His definition of isomorphism is unchanged except that now $L$ and $M$ are $L_\alpha$ languages for some infinite cardinal $\alpha$.

The general form of his theorem, which appears in [3], can be stated as follows:
Let $L$ and $M$ be two $L_\alpha$ languages such that they both have less than $\alpha$ predicate symbols and they both have a predicate symbol which has two or more places and is not $\equiv$, then $L$ and $M$ are isomorphic.

The methods of this paper can be extended to include the general form of Hanf's theorem.

Let us think of a similarity-type $\nu = \langle \nu_0, \nu_1, \ldots, \nu_\delta, \ldots \rangle$ as an ordered-tuple of arbitrary size whose entries are cardinal numbers. Let the number of entries in $\nu$ be $\delta$. Then we can let $\nu$ represent a well ordered class of predicate symbols with $\delta$ members such that the least member has $\nu_0$ places, etc., then we speak of
the language $L_\alpha(\mu)$.

In [3] Hanf sometimes constructed sets of sentences in $L_\alpha(\mu)$ with certain properties. Since the construction only involved the similarity-type $\mu$ it was natural to ask whether sets of sentences with the same properties could be constructed using other similarity-types. As Hanf states in [3], the above theorem provides a partial answer to this question.

Hanf also defined isomorphism of two first order theories $T$ and $T'$ by considering models of $T$ and $T'$ instead of structures of $L(T)$ and $L(T')$ and by requiring the bijection $G$ between classes of sentences to be recursive. We refer the reader to [4] for Hanf's discussion of this definition and related results.

To the best of our knowledge, except for Hanf's works which we have mentioned above, nothing has been published on isomorphisms of first order languages. This is probably because in one sense the definition of isomorphism is so general that not enough information is gained by showing that two first order languages are isomorphic, and because in another sense only languages which are quite similar can be isomorphic.
BIBLIOGRAPHY


