A UNIVERSAL SUPERCOMPACTIFICATION

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Douglas Bishop Super
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Name: Douglas Bishop Super
Degree: Master of Science
Title of Thesis: A Universal Supercompactification

Examin ing Committee:

Chairman: Edgar Pechlaner

S. K. Thomason
Senior Supervisor

D. Ryeburn

A. R. Freedman

H. Gerber
External Examiner

Date Approved: April 6th, 1977
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Author:

Douglas B. Super

April 21, 1977

(signature)
(name)
ABSTRACT

Supercompactness and supercompactifications are related to types of linked families in the same manner as compactness and compactifications are related to types of filters. The Stone-Čech compactification and Wallman compactification are known to be universal constructions in the sense of category theory. The main result is a supercompactification that likewise is universal. To obtain this result the superextension construction of deGroot is modified by the introduction of $T_1$-sub-base spaces and the kinds of morphisms allowed are reduced from continuous functions to sure functions.
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TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title Page</td>
<td>(i)</td>
</tr>
<tr>
<td>Approval</td>
<td>(ii)</td>
</tr>
<tr>
<td>Abstract</td>
<td>(iii)</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>(iv)</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>(v)</td>
</tr>
<tr>
<td>CHAPTER I</td>
<td>INTRODUCTION</td>
</tr>
<tr>
<td>CHAPTER II</td>
<td>EPIREFLECTIVE SUBCATEGORIES</td>
</tr>
<tr>
<td>CHAPTER III</td>
<td>SUPEREXTENSIONS</td>
</tr>
<tr>
<td>CHAPTER IV</td>
<td>THE SURE CATEGORY</td>
</tr>
<tr>
<td>CHAPTER V</td>
<td>JENSEN'S RESULT</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

Prior to the formal presentation of our topic, which is deferred until the next chapter, we feel obliged to discuss briefly and informally the background and significance of our results. Although the succeeding chapters are self-contained, the present one presupposes a familiarity with general topology and some category theory. An effort has been made throughout to use standard notation whenever possible.

Recall that a pair $(Y,h)$ is called a compactification of a topological space $X$ whenever $Y$ is a compact space and $X$ is homeomorphic with a dense subspace of $Y$ via the function $h$. By a compactification for $C$, where $C$ is a category of topological spaces, we mean a collection $\{(\hat{C},r_C) : C \in C\}$, where each $(\hat{C},r_C)$ is a compactification of $C$. Several methods for constructing compactifications are known which seem relatively "natural" and "uniform". More precisely, we say a compactification of $C$ is a universal construction whenever each $C$ function $f : C \rightarrow B$ with $B$ a compact space has associated with it a unique $C$ function $g : \hat{C} \rightarrow B$ such that $gr_C = f$.

It is well known that the Stone-Čech compactification is a universal construction for the category of Tychonoff spaces [9, p. 137] whereas the one-point compactification [9, p. 136] is not one for the more restricted category of locally compact Hausdorff spaces.

Applications of the concept of universal construction extend beyond compactifications. In fact, MacLane [7] and others have defined and studied this concept within the realm of category theory and shown that instances of it occur throughout mathematics. So ubiquitous and...
desirable is the universal property, that when a construction for a particular instance exists which is not universal, often steps are taken to study the situation further. Traditionally, this might involve a search for a different construction for the same instance which is more "natural" and "uniform" and thus possibly universal. Lately, several investigations, notably by Harris [4],[5] and Bentley and Naimpally [1], have appeared which use a different approach. Since our main result also employs this new approach, which we call a *fur strategy*, an example of its use is mentioned next.

Let \( \text{TOP}_1 \) denote the category of all \( T_1 \)-spaces and continuous functions between these spaces, and \( \text{COMP \ TOP}_1 \) be the full subcategory of \( \text{TOP}_1 \) containing the compact \( T_1 \)-spaces. A Wallman compactification \( \{ (X, w_X) : X \in \text{TOP}_1 \} \) for \( \text{TOP}_1 \) exists which seems relatively "natural" and "uniform", but which is not universal. It was discovered by Harris [5] that, by considering a fur subcategory of \( \text{TOP}_1 \) (one which has the same spaces but fewer functions), the problem could be "bypassed". Specifically, Harris found that the Wallman compactification for the category \( \text{WOSEP} \), a fur subcategory of \( \text{TOP}_1 \), is a universal construction.

Supercompactness is a type of compactness which depends upon the subbase of a topological space. Suppose \( \text{SUPERCOMP \ TOP}_1 \) is the full subcategory of \( \text{TOP}_1 \) containing just supercompact \( T_1 \)-spaces. A superextension construction, \( \{ (X_S, s_X) : X \in \text{TOP}_1 \} \), was discovered by deGroot [3] such that each \( X_S \) is a supercompact \( T_1 \)-space and \( X \) is homeomorphic to a subspace of \( X_S \) via the map \( s_X \). Several map extension situations have been investigated by Jensen [8, pp. 54 - 57].
Our main result is to define a reformulation of the superextension construction which is a universal construction for the category SURE, a fur subcategory of a category related to TOP₁.

Chapter II contains formal definitions of an epireflective subcategory (a notion stronger than that of universal construction) and of the fur strategy. In Chapter III is presented a restatement of supercompactness and of the superextension construction in a form which allows in Chapter IV our proof that a universal situation exists with respect to a suitable fur strategy. Chapter V reviews the results of Jensen mentioned above.
CHAPTER II

EPIREFLECTIVE SUBCATEGORIES

In this chapter, we introduce formally the category-theoretic notion epireflective subcategory, which is needed later in our discussion of the superextension construction of supercompact $T_1$-subbase spaces.

Definition 2.1 [6, p. 16] A category $C$ is an ordered six-tuple $C = \langle \text{Obj } C, \text{Mor } C, \text{dom}, \text{cod}, \circ, \text{id} \rangle$ where

(i) $\text{Obj } C$ is a class whose members are called objects,
(ii) $\text{Mor } C$ is a class whose members are called morphisms,
(iii) $\text{dom}$ and $\text{cod}$ are functions from $\text{Mor } C$ to $\text{Obj } C$,
(iv) $\circ$ is a partial function from $\text{Mor } C \times \text{Mor } C$ to $\text{Mor } C$, whose value at the pair $<f, g>$ (if this is defined) is denoted by $fg$ and is called the composition of $g$ and $f$, and
(v) $\text{id}$ is a function from $\text{Obj } C$ to $\text{Mor } C$ whose value at an object $X$ is denoted by $\text{id}_X$ and is called the identity morphism of $X$

such that the following conditions are satisfied:

1) $fg$ is defined iff $\text{cod}(g) = \text{dom}(f)$;
2) if $fg$ is defined, then $\text{dom}(fg) = \text{dom}(g)$ and $\text{cod}(fg) = \text{cod}(f)$;
3) if $fg$ and $hf$ are defined, then $h(fg) = (hf)g$;
4) for each object $X$, $\text{cod}(\text{id}_X) = \text{dom}(\text{id}_X) = X$;
5) if $\text{cod}(f) = X = \text{dom}(g)$, then $\text{id}_X f = f$ and $g \text{id}_X = g$;
(6) for any pair of objects \((X,Y)\), the class 
\[
\text{hom}_C(X,Y) = \{ f \in \text{Mor}_C : \text{dom}(f) = X \text{ and cod}(f) = Y \}
\]
is a set (rather than a proper class).
For example, \(\text{SET}\) denotes the category for which

(1) \(\text{Obj SET}\) is the class of all sets,

(2) \(\text{Mor SET}\) is the class of all functions between sets,

(3) for each function \(f \in \text{Mor SET}\), the sets \(\text{dom}(f)\) and \(\text{cod}(f)\) are respectively the domain and codomain of \(f\),

(4) for each set \(X\), the function \(\text{id}_X\) is the identity function for \(X\), and

(5) the partial function \(\circ\) corresponds with the usual composition of functions.

It is customary to define a particular category, when little chance for confusion is possible, by merely describing its class of objects and class of morphisms.

The morphisms (objects) of any category \(C\) will be denoted by lower (upper) case letters. Thus, by \(f \in C\) and \(X \in C\), we will mean \(f \in \text{Mor}_C\) and \(X \in \text{Obj}_C\). For \(f, X, Y \in C\), both \(f : X \rightarrow Y\) and \(X \rightarrow Y\) signify that \(f \in \text{hom}_C(X,Y)\).

**Definition 2.2** [6, p. 23] We call \(S\) a subcategory of category \(C\) whenever \(S\) is a category such that

(1) \(\text{Obj } S \subseteq \text{Obj } C\),

(2) \(\text{Mor } S \subseteq \text{Mor } C\), and

(3) the functions \(\text{dom}, \text{cod}, \text{id},\) and \(\circ\) for \(S\) are restrictions of the corresponding functions for \(C\).
Definition 2.3 [6, p. 24] A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called a full subcategory of $\mathcal{C}$ if and only if, for any objects $x, y \in \mathcal{S}$, $\hom_{\mathcal{S}}(x, y) = \hom_{\mathcal{C}}(x, y)$.

Definition 2.4 A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called a full subcategory of $\mathcal{C}$ if and only if $\text{Obj}_\mathcal{S} = \text{Obj}_\mathcal{C}$.

Definition 2.5 [6, p. 35] A morphism $f \in \mathcal{C}$ is a $\mathcal{C}$-isomorphism if and only if there exists a morphism $g \in \mathcal{C}$ such that $fg = \text{id}_{\text{cod}(f)}$ and $gf = \text{id}_{\text{dom}(f)}$.

Definition 2.6 [6, p. 40] A morphism $f \in \mathcal{C}$ is a $\mathcal{C}$-epimorphism if and only if, for all morphisms $g, h \in \mathcal{C}$, $gf = hf$ implies $g = h$.

Definition 2.7 [6, p. 275][5] A full subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is an epireflective subcategory of $\mathcal{C}$ with respect to a class $\{(\underline{r}_X, X)\}_{X \in \mathcal{C}}$ if and only if

1. for each $X \in \mathcal{C}$, $X_R \in \mathcal{S}$ and $r_X : X \to X_R$ is a $\mathcal{C}$-epimorphism,
2. if $X \in \mathcal{S}$, then $r_X$ is an $\mathcal{S}$-isomorphism, and
3. for any objects $Y, Z \in \mathcal{C}$ and morphism $f \in \hom_{\mathcal{C}}(Y, Z)$,
   there exists a morphism $f_R \in \hom_{\mathcal{S}}(Y_R, Z_R)$ such that $r_Z f = f_R r_Y$.

\[
\begin{array}{ccc}
Y & \xrightarrow{r_Y} & Y_R \\
\downarrow f & & \downarrow f_R \\
Z & \xrightarrow{r_Z} & Z_R
\end{array}
\]
A full subcategory $S$ of a category $C$ is called an epireflective subcategory of $C$ whenever a class $\{(r, x, x, x)\}_{x \in C}$ exists which satisfies the three previous conditions.

**Definition 2.8** Let $S$ be a full subcategory of a category $C$.

Then a category $R$ is called a successful fur strategy (with respect to the class $\{(r, x, x, x)\}_{x \in C}$) if and only if

1. $R$ is a fur subcategory of $C$ and
2. $R \cap S$ is an epireflective subcategory of $R$ (with respect to the class $\{(r, x, x, x)\}_{x \in C}$).
The superextension construction and the property of super-
compactness, close relatives respectively of the Wallman-type compacti-
fication and the property of compactness, have been studied extensively
by deGroot [3], Császár [2], Verbeek [8], and others. In this chapter,
we introduce a formulation of these concepts which permits a successful
fur strategy.

Definition 3.1 [9, p. 24] A topology for a set \( X \) is a collection \( \tau \)
of subsets of \( X \) such that:

1. any intersection of members of \( \tau \) is a member of \( \tau \),
2. any finite union of members of \( \tau \) is a member of \( \tau \),
3. the sets \( X \) and \( \phi \) both belong to \( \tau \).

We call the members of a topology closed sets. A topological space
is a pair \((X, \tau)\) where \( \tau \) is a topology for \( X \). We often will
abbreviate \((X, \tau)\), when no confusion is likely, to \( X \).

Definition 3.2 [9, pp. 38-39] Suppose \((X, \tau)\) is a topological space.
A base for \( \tau \) is a collection \( \beta \subseteq \tau \) such that \( \tau = \{ \bigcap \alpha : \alpha \subseteq \beta \} \).
A subbase for \( \tau \) is a collection \( \sigma \subseteq \tau \) such that \( \beta = \{ \bigcup \lambda \subseteq \lambda \} \) is a finite subcollection of \( \tau \) is a base for \( \tau \).

Proposition 3.3 [9, p. 39] Any collection \( \sigma \) of subsets of set \( X \)
is a subbase for a topology for \( X \).
Definition 3.4 Suppose \( \alpha \) is a collection of subsets of \( X \). The members of \( \alpha \) are said to meet whenever \( \bigcap A \neq \emptyset \). We call \( \alpha \) an ip (intersection property) family if its members meet, a fip (finite intersection property) family if each finite subcollection of it consists of members which meet, and a linked family [8, p.1] if every two of its members meets.

Definition 3.5 [9, p. 118] A topological space \( (X, \tau) \) is compact if and only if each collection \( \alpha \subseteq \tau \) which is a fip family is also an ip family.

Definition 3.6 [8, p. 48] A topological space \( (X, \tau) \) is supercompact if and only if there exists a subbase \( \sigma \) for the topology \( \tau \) such that each collection \( \alpha \subseteq \sigma \) which is a linked family is also an ip family.

Proposition 3.7 [8, p. 48] Each supercompact topological space is compact.

This result is easily deduced from the Alexander subbase theorem [9, p. 129]: A topological space \( (X, \tau) \) is compact if and only if there exists a subbase \( \sigma \) for the topology \( \tau \) such that each collection \( \alpha \subseteq \sigma \) which is a fip family is also an ip family.

Definition 3.8 [9, p. 86] A topological space is a \( T_{1} \)-space if and only if each of its singletons is a closed set.

Remark 3.9 [8, p. 48] It is known that every compact metrizable space is supercompact. Although compact \( T_{1} \)-spaces exist which are not supercompact, it is an open question whether there exists a compact \( T_{2} \)-space which is not supercompact.
In 1967, J. deGroot [3, p. 90] introduced the superextension construction by which a supercompact $T_1$-space can be constructed from any $T_1$-space. A formulation of the superextension construction is discussed in detail below.

**Definition 3.10** Suppose $f : X_1 + X_2$ is a function and $\alpha_2$ is a collection of subsets of $X_2$. Then we let $f^{-}[\alpha_2] = \{f^{-}(A) : A \in \alpha_2\}$.

For topological spaces $(X_1, \tau_1)$ and $(X_2, \tau_2)$, a function $f : X_1 + X_2$ is a continuous function if and only if $f^{-}[\tau_2] \subseteq \tau_1$ [9, p. 44]. The function $f$ is an embedding if and only if $f$ is one-one and continuous and $f^{-}$ is continuous.

**Remark 3.11** Suppose $TOP_1$ denotes the category whose class of spaces consists of all $T_1$-spaces and whose class of functions consists of all continuous functions between these spaces, and $SUPERCOMP TOP_1$ is the full subcategory of $TOP_1$ whose class of spaces consists of all supercompact $T_1$-spaces. For the categories $TOP_1$ and $SUPERCOMP TOP_1$ we have not found a satisfactory successful fur strategy with respect to the superextension construction mentioned above. We have, however, developed a closely related formulation of this problem for which a successful fur strategy does exist.

**Definition 3.12** [8, p. 44] A collection $R_X$ is a $T_1$-subbase for a topological space $(X, \tau)$ whenever

1. $R_X$ is a subbase for $\tau$ which contains $X$ and $\phi$,
2. for each $x \in X$, $\{x\} = \cap \{R : x \in R$ and $R \in R_X\}$,
3. $x \in X$, $R \in R_X$, and $x \not\in R$ imply there is a $T \in R_X$ such that $x \in T$ and $R \cap T = \phi$.
Proposition 3.13  [8, p. 44] Each $T_1$-space has a $T_1$-subbase. If a topological space has a $T_1$-subbase, then it is a $T_1$-space.

The topology of each $T_1$-space is a $T_1$-subbase. Conversely, from (2) above it follows that $\{x\}$ is closed for each $x \in X$.

Definition 3.14 A $T_1$-subbase space is a pair $(X,R_X)$ where $X$ is a topological space and $R_X$ is a $T_1$-subbase for $X$. A supercompact $T_1$-subbase space is a $T_1$-subbase space $(X,R_X)$ such that $X$ is supercompact with respect to the subbase $R_X$.

Let $T_1$-SUBBASE denote the category consisting of all $T_1$-subbase spaces and all continuous functions between these spaces. $\text{SUPERCOMPT}_T$-SUBBASE denote the full subcategory of $T_1$-SUBBASE consisting of all supercompact $T_1$-subbase spaces.

Remark 3.15 $T_1$-SUBBASE and $\text{SUPERCOMPT}_T$-SUBBASE are the categories to which we will eventually apply the fur strategy. The functions considered will be more directly related to the $T_1$-subbases than are continuous functions in general.

Definition 3.16 [2, p. 57] Let $R_X$ be a $T_1$-subbase for $X$.

An $R_X$-ultrasieve is a nonempty set $\sigma$ with the properties

1. $\sigma \subseteq R_X \setminus \{\emptyset\}$
2. if $B, D \in \sigma$, then $B \cap D \neq \emptyset$,
3. if $B \in R_X$ and $B \cap D \neq \emptyset$ for each $D \in \sigma$, then $B \in \sigma$.
Definition 3.17 (cf. [1]) The $R_X$-bicontiguity relation, denoted by $c_{R_X}$ or just $c$, is defined by: for any sets $A_1, A_2 \subseteq X$,

$A_1 \ c_{R_X} \ A_2$ if and only if, for each pair $R_1, R_2 \in R_X$ with $A_1 \subseteq R_1$ and $A_2 \subseteq R_2$, $R_1 \cap R_2 \neq \emptyset$. If $A_1 \ c_{R_X} \ A_2$, we say $A_1$ and $A_2$ are $R_X$-bicontiguous. We let $A_1 \not\ c_{R_X} \ A_2$ mean that $A_1 \ c_{R_X} \ A_2$ is not true.

Definition 3.18 (cf. [1]) An $R_X$-bicontiguity cluster is a non-empty collection $\sigma$ of subsets of $X$ such that

1. $\emptyset \not\in \sigma$,
2. if $A, B \in \sigma$, then $A \ c_{R_X} B$,
3. if $A \subseteq X$ and $A \ c_{R_X} B$ for each $B \in \sigma$, then $A \in \sigma$.

Definition 3.19 Let $(X, R_X) \in \mathcal{T}_1$-SUBBASE, $R \in R_X$, and $A \subseteq X$. We define $R^+ = \{\sigma : \sigma \text{ is an } R_X\text{-ultrasieve containing } R\}$ and $R_X^+ = \{R^+ : R \in R_X\}$ [8, p. 45]. In addition, we define $+A = \{\sigma : \sigma \text{ is an } R_X\text{-bicontiguity cluster containing } A\}$ and $+R_X = \{+R : R \in R_X\}$ (cf. [1]).

Theorem 3.20 [8, pp. 46-48] If $(X, R_X) \in \mathcal{T}_1$-SUBBASE then $(X^+, R_X^+) \in \mathcal{X}_1$-SUPERCOMP $\mathcal{T}_1$-SUBBASE (where $X^+$ has the topology generated by the subbase $R_X^+$).

The function $\tilde{\lambda}_{R_X} : (X, R_X) \to (X^+, R_X^+)$ defined so $\tilde{\lambda}_{R_X}(x) = \{R : x \in R \text{ and } R \in R_X\}$ is a $\mathcal{T}_1$-subbase space embedding.

If $(X, R_X) \in \mathcal{X}_1$-SUPERCOMP $\mathcal{T}_1$-SUBBASE, then $\tilde{\lambda}_{R_X}$ is a $\mathcal{X}_1$-SUPERCOMP $\mathcal{T}_1$-SUBBASE isomorphism.
Theorem 3.21 (cf. [1]) If \((X, R_x) \in \text{T}_1\text{-SUBBASE}\), then \((+X, +R_x) \in \text{SUPERCOMP T}_1\text{-SUBBASE}\) (where \(+X\) has the topology generated by the subbase \(+R_x\)).

The function \(\lambda_{R_x} : (X, R_x) \to (+X, +R_x)\) defined so
\[
\lambda_{R_x}(x) = \{A : \{x\} c R_x A \text{ and } A \subseteq X\}
\]
is a \(T_1\)-subbase space embedding.

If \((X, R_x) \in \text{SUPERCOMP T}_1\text{-SUBBASE}\), then \(\lambda_{R_x}\) is a \(\text{SUPERCOMP T}_1\text{-SUBBASE}\) isomorphism.

The function \(\gamma : (+X, +R_x) \to (X^+, R_x^+\) defined by \(\gamma = \sigma \cap R_x\)
for each \(\sigma \in +X\) is a \(\text{SUPERCOMP T}_1\text{-SUBBASE}\) isomorphism such that \(\gamma_{R_x} = \lambda_{R_x}\).

Our proof of these theorems depends upon the following lemma:

Lemma 3.22 [8, pp. 45-46] Suppose \((X, R_x) \in \text{T}_1\text{-SUBBASE}\) and \(S, T \in R_x\). Then,

(1) \(S \subseteq T\) if and only if \(S^+ \subseteq T^+\) and
(2) \(S \cap T = \emptyset\) if and only if \(S^+ \cap T^+ = \emptyset\).

(1) Clearly, for any set \(R \in R_x\) and \(R_x\)-ultrasieve \(\sigma\), \(\sigma \in R^+\) if and only if \(R \in \sigma\). Suppose \(S \subseteq T\) and \(\sigma \in S^+\). Since \(S \in \sigma\) and \(\sigma\) is an \(R_x\)-ultrasieve, \(S\) meets each member of \(\sigma\).

Thus, \(T\) meets each member of \(\sigma\) and by 3.16(3) \(T \in \sigma\). Hence \(\sigma \in T^+\) and, since \(\sigma\) was arbitrary, \(S^+ \subseteq T^+\).

Now suppose there is an \(x \in X\) such that \(x \in S\) and \(x \notin T\). We claim that \(\bar{\lambda}_{R_x}(x) = \{R \in R_x : x \in R\}\) is an \(R_x\)-ultrasieve such that \(\bar{\lambda}_{R_x}(x) \subseteq S^+\) and \(\bar{\lambda}_{R_x}(x) \notin T^+\). Note that this would mean \(S^+ \subseteq T^+\) implies \(S \subseteq T\). Obviously, \(\bar{\lambda}_{R_x}(x) \subseteq R_x-\{\emptyset\}\). If \(R_1, R_2 \in \bar{\lambda}_{R_x}(x)\),
then \( x \in R_1 \cap R_2 \) and hence \( R_1 \cap R_2 \neq \phi \). Since \( R_X \) is a \( T_1 \)-subbase, if \( Q \in R_X \) with \( x \notin Q \), then there is an \( R \in R_X \) with \( x \in R \) such that \( Q \cap R = \phi \). Hence, if \( Q \) meets each member of \( \bar{\lambda}_{R_X}(x) \), then \( x \in Q \) and thus \( Q \in \bar{\lambda}_{R_X}(x) \). Consequently, \( \bar{\lambda}_{R_X}(x) \) is an \( R_X \)-ultrasieve.

Since we know that \( S \in \bar{\lambda}_{R_X}(x) \) and \( T \notin \bar{\lambda}_{R_X}(x) \), then \( \bar{\lambda}_{R_X}(x) \in S^+ \) and \( \bar{\lambda}_{R_X}(x) \notin T^+ \).

(2) If \( \sigma \in S^+ \cap T^+ \), then \( S \in \sigma \) and \( T \in \sigma \). Since \( \sigma \) is an linked family, \( S \cap T \neq \phi \). Thus, \( S \cap T \neq \phi \) implies \( S^+ \cap T^+ \neq \phi \).

Suppose \( S \cap T \neq \phi \). Choose any \( x \in S \cap T \). Since
\[
\bar{\lambda}_{R_X}(x) = \{ R \in R_X : x \in R \}
\]
is an \( R_X \)-ultrasieve with \( \bar{\lambda}_{R_X}(x) \in S^+ \cap T^+ \), clearly \( S^+ \cap T^+ \neq \phi \).

Now, we resume our demonstration of Theorems 3.20 and 3.21.

Since \( R_X^+ \) is a collection of subsets of \( X \), it is a subbase for a topology of \( X^+ \). To prove \( R_X^+ \) is a \( T_1 \)-subbase, first note that \( \phi^+ = \phi \in R_X^+ \) and \( X^+ \in R_X^+ \) since \( \phi \in R_X \) and \( X \in R_X \). For any \( R_X \)-ultrasieve \( \sigma \), surely \( \sigma \in \cap \{ R^+ \in R_X^+ : \sigma \in R^+ \} = \cap \{ R^+ : R \in \sigma \} \).

If \( \delta \) is an \( R_X \)-ultrasieve contained in \( \cap \{ R^+ : R \in \sigma \} \), then \( \sigma \subseteq \delta \) since \( R \in \sigma \) implies \( R \in \delta \). But, if \( S \in \delta \), then \( S \) meets each member of \( \sigma \) and hence belongs to \( \sigma \). Therefore, \( \{ \sigma \} = \cap \{ R^+ \in R_X^+ : \sigma \in R^+ \} \). Finally, let us next assume \( \sigma \in X^+ \), \( R^+ \in R_X^+ \), and \( \sigma \notin R^+ \). Since \( R \notin \sigma \), there exists an \( S \in \sigma \) such that \( R \cap S = \phi \). As a result, \( R^+ \cap S^+ = \phi \) and \( \sigma \in S^+ \). We conclude that \( R_X^+ \) is a \( T_1 \)-subbase.
To show that \((X^+, R_X^+)\) is a supercompact \(T_1\)-subbase space, consider any linked family composed of members from \(R_X^+\), say \([R_i^+]_{i \in I}\). Clearly, \([R_i]_{i \in I}\) is a linked family composed of members from \(R_X\). But it is easy to show, using Zorn's lemma, that a class \([\delta : \delta \subseteq R_X \text{ and } \delta \text{ is a linked set}]\) has a maximal member \(\sigma\) (with respect to set inclusion). Further, \(\sigma\) is an \(R_X\)-ultrasieve by 3.16; for, if \(R \in R_X\) meets each member of \(\sigma\), then \(R \in \sigma\) since otherwise \(\{V : V \in \sigma \text{ or } V = R\}\) is a linked set properly containing \(\sigma\). In particular then, \([R_i]_{i \in I}\) is contained in an \(R_X\)-ultrasieve, say \(\sigma\). Since each \(R_i \in \sigma\), then \(\sigma \subseteq \bigcap R_i^+\); that is \([R_i^+]_{i \in I}\) is an ip family.

For each \(x \in X\), \(\lambda^{-1}_{R_X}(x) = \{R \in R_X : x \in R\}\) is an \(R_X\)-ultrasieve; so \(\lambda^{-1}_{R_X}\) is indeed a function from \(X\) to \(X^+\). If \(y, z \in X\) and \(y \neq z\), then, since \(R_X\) is a \(T_1\)-subbase, there is an \(R \in R_X\) with \(y \in R\) and \(z \notin R\); but then \(\lambda^{-1}_{R_X}(y) \neq \lambda^{-1}_{R_X}(z)\). Thus \(\lambda^{-1}_{R_X}\) is one-to-one.

From the fact that \(x \in R\) iff \(R \in \lambda^{-1}_{R_X}(x)\) iff \(\lambda^{-1}_{R_X}(x) \subseteq R^+\), we can deduce \(\lambda^{-1}_{R_X}(R) \subseteq R^+\) and \(\lambda^{-1}_{R_X}(R) = R^+ \cap \lambda^{-1}_{R_X}(X)\). Since \(\lambda^{-1}_{R_X}\) is an injection, \(\lambda^{-1}_{R_X}\) is a function. Note \(\lambda^{-1}_{R_X}(R^+ \cap \lambda^{-1}_{R_X}(X)) = R\) and, incidentally, \(\lambda^{-1}_{R_X}(R^+) = R\) and \(\lambda^{-1}_{R_X}(R^+) = R_X\). Since \((X^+ \cap \lambda^{-1}_{R_X}(X), \{R^+ \cap \lambda^{-1}_{R_X}(X) : R \in R_X\})\) is surely a \(T_1\)-subbase space, we can conclude the function \(\lambda^{-1}_{R_X} : (X, R_X) \to (X^+ \cap \lambda^{-1}_{R_X}(X), \{R^+ \cap \lambda^{-1}_{R_X}(X) : R \in R_X\})\) is a \(T_1\)-subbase-isomorphism.
Assume \((X,R_X)\) is a supercompact \(T_1\)-subbase space and \(\sigma \in X^+\).

Since \(\sigma\) is a linked family, it is an ip family. If \(x \in X\) is contained in each member of \(\sigma\), then \(\sigma = \lambda_{R_X}(x)\) by 3.16(3) since \(\sigma\) and \(\lambda_{R_X}(x)\) are \(R_X\)-ultrasieves with \(\sigma \subseteq \lambda_{R_X}(x)\). As a result

\[
\lambda_{R_X}(x) = X^+ , \quad \lambda_{R_X}(R) = R^+ , \quad \lambda_{R_X}[R] = R^+_X , \quad \text{and} \quad \lambda_{R_X}:(X,R_X) \to (X^+,R^+_X)
\]

is a SUPERCOMP \(T_1\)-SUBBASE - isomorphism.

We claim \(\lambda_{R_X}(x) = \{A \in X : \{x\} \subseteq A\}\) is an \(R_X\)-bicontiguity cluster. Since \(\{x\} \in \lambda_{R_X}(x)\), if \(B\) is \(R_X\)-bicontiguous with each member of \(\lambda_{R_X}(x)\), then \(B \in \lambda_{R_X}(x)\). Next, suppose \(A,B \in \lambda_{R_X}(x)\) and \(R_A,R_B \in R_X\) with \(A \subseteq R_A\) and \(B \subseteq R_B\). Since \(R_X\) is a \(T_1\)-subbase, \(R_A \subseteq R_X\{x\}\) implies \(x \subseteq R_A\). Clearly, \(x \subseteq R_A \cap R_B\) and hence \(A \subseteq R_X\).

Consequently, \(\lambda_{R_X}: X \to X^+\) is a function.

Assume \(\sigma \in +X\) and \(\sigma = \sigma \cap R_X\). Obviously, \(X \subseteq \sigma\). If \(R_1,R_2 \in \sigma\), then \(R_1 \cap R_2 \neq \emptyset\) since \(R_1 \subseteq R_X\). Suppose \(S \subseteq R_X\) meets each \(R \in \sigma\). For any \(A \in \sigma\), \(S \subseteq A\) because each set \(R \subseteq R_X\) containing \(A\) is a member of \(\sigma\). Thus \(S \in \sigma\) by 3.18(3) and hence \(\sigma \in X^+\) by definition 3.16. Thus \(\sigma : +X \to X^+\) is a function.

Next, we show there is a function \(\sigma : X^+ \to \sigma^\dagger\) defined by \(\sigma = \{A \subseteq X : \exists R \in \sigma \quad \text{for each} \quad R \in \sigma\}\). Obviously, \(X \subseteq \sigma\). Next suppose \(A,B \in \sigma^\dagger\) and \(R_A,R_B \in R_X\) with \(A \subseteq R_A\) and \(B \subseteq R_B\). Since \(A\) is bicontigous with every \(R \in \sigma\), \(R_A\) meets every \(R \in \sigma\). Likewise \(R_B\) meets every \(R \in \sigma\). Hence \(R_A\) and \(R_B\) each belong to \(\sigma\) and thus meet. Consequently, \(A \subseteq \sigma\); condition 3.18(2) is satisfied. Since \(\sigma \subseteq \sigma\), if a set \(D\) is bicontiguous with all members of \(\sigma\), then \(D \in \sigma\). Thus \(\sigma \in X^+\) and \(\sigma : X^+ \to X^+\) is a function.
In fact, the function $\rightarrow$ is the inverse of $\leftarrow$. For if

\[\sigma \in X^+\), then $\sigma = \leftarrow o\sigma\] by 3.18 since $\sigma \subseteq \leftarrow o\sigma$ and if $\sigma \in X^+$, then

\[\sigma = \leftarrow o\sigma\] by 3.16 since $\sigma \subseteq \leftarrow o\sigma$. For any $R \in R_X$, consider $\rightarrow(\rightarrow R)$. If

\[\sigma \in ^+R,\] then $R \in \sigma$, $R \in \rightarrow o\sigma$, and finally $\rightarrow o\sigma \in R^+$. Similarly, if

\[\sigma \in R^+,\] then $R \in \sigma$, $R \in \rightarrow o\sigma$, and so $\rightarrow o\sigma \in ^+R$. Thus if $\sigma \in R^+$, then $\sigma = \rightarrow o\sigma \in ^+\rightarrow(\rightarrow R)$ and consequently $\rightarrow(\rightarrow R) = R^+$. Likewise if

\[\sigma \in ^+R,\] then $\sigma = \rightarrow o\sigma \in ^+\rightarrow(R^+)$ and consequently $\rightarrow(R^+) = ^+R$. Now one can easily deduce the remainder of Theorem 3.21.

Remark 3.23 From the previous results, we conclude that the constructions $(\lambda_{R^+_X}, (X^+, R^+_X))$ and $(\lambda_{R^+_X}, (X^+, R^+_X))$ are essentially the same for our purposes. We call either construction the superextension of the

$T_1$-subbase space $(X, R_X)$. In addition, we say the morphisms $\lambda_{R^+_X}$ and $\lambda_{R^+_X}$ are superextension maps. Our construction $(\lambda_{R^+_X}, (X^+, R^+_X))$, which is similar to the Wallman-type compactification investigated by Bentley and Naimpally [1], has been introduced, as in [1], to simplify certain aspects of the presentation of a successful fur strategy.

Note that a superextension $(\lambda_{R^+_X}, (X^+, R^+_X))$ is not, in general, a compactification of $(X, R_X)$ since $\lambda_{R^+_X}(X)$ is seldom dense in $X^+$. As an example, Verbeek [8, p. 47] has shown that, for any topological space $(X, \tau)$, which is a Hausdorff space containing at least three elements, if we consider $(X, \tau) \in T_1-$SUBBASE, then the set $\lambda_{\tau}(X)$ is not dense in $(X^+, \tau)$.
Example 3.24 Let the set $X = \{a, b\} \cup [0,1]$ be endowed with a subbase $R_X$ consisting of $\emptyset, X$, and \{a\} $\cup [0,x)$, \{a\} $\cup [x,1]$, \{b\} $\cup [0,x]$, and \{b\} $\cup [x,1]$ where $x \in (0,1)$. It is easy to see $R_X$ is a $T_1$-subbase for $X$.

Let $Y = [0,1] \times [0,1]$ be endowed with a $T_1$-subbase $S_Y$ consisting of $\emptyset, Y$, and $[0,x] \times [0,1]$, $[x,1] \times [0,1]$, $[0,1] \times [0,x]$, and $[0,1] \times [x,1]$ where $x \in (0,1)$.

Cszázár [2, p. 64] has shown there exists a SUPERCOMP $T_1$-SUBBASE - isomorphism $v : (X^+, R_X^+) \to (Y, S_Y)$ such that

$$v\lambda_{R_X}^X(x) = \begin{cases} (x,x) & \text{if } x \in [0,1] \\ (1,0) & \text{if } x = a \\ (0,1) & \text{if } x = b \end{cases}$$

Note that $S_Y$ is a subbase for the usual topology on $Y$.

Clearly, there are an infinite number of continuous functions from $(Y, S_Y)$ to $(Y, S_Y)$ that are invariant on the set $v\lambda_{R_X}^X(X)$. We can conclude $\text{id}_X : (X, R_X^+) \to (X, R_X^+)$ is a morphism in $T_1$-SUBBASE which does not have a unique corresponding morphism $g : (X^+, R_X^+) \to (X^+, R_X^+)$ in $T_1$-SUBBASE such that $\lambda_{R_X}^X \circ \text{id}_X = g \lambda_{R_X}^X$.

Considerations of this example influenced the various fur strategies which were tried. In the next chapter, we discuss the most suitable successful fur strategy that was found.

-18-
CHAPTER IV
THE SURE CATEGORY

In this chapter, a fur subcategory of \( T_1 \)-SUBBASE is defined and shown to be a successful fur strategy with respect to the superextension construction.

Definition 4.1 For \( T_1 \)-subbase spaces \((X, R_X)\) and \((Y, S_Y)\), let \( f : (X, R_X) \to (Y, S_Y) \) be a function. We will call \( f \) a sure map if and only if \( f^{-1}[S_Y] \subseteq R_X \) and, for any sets \( A \subseteq X \) and \( S \in S_Y \), \( f(A) \subseteq S_Y \) implies \( A \subseteq f^{-1}(S) \).

Proposition 4.2 Every identity map in \( T_1 \)-SUBBASE is a sure map. The composition of sure maps is a sure map. Each sure map is continuous.

The proof of the first statement is trivial.

Let \( f : (X, R_X) \to (Y, S_Y) \) and \( g : (Y, S_Y) \to (Z, T_Z) \) be sure maps. Clearly, \( (gf)^{-1}[T_Z] \subseteq R_X \) since \( g^{-1}[T_Z] \subseteq S_Y \) and \( f^{-1}[S_Y] \subseteq R_X \). Suppose \( gf(A) \subseteq T_Z \) for sets \( A \subseteq X \) and \( T \in T_Z \). Since \( g \) is a sure map, \( f(A) \subseteq S_Y \). Further, since \( f \) is a sure map and \( g^{-1}(T) \subseteq S_Y \), \( A \subseteq f^{-1}(g^{-1}(T)) \). Since \( f^{-1}(g^{-1}(T)) = (gf)^{-1}(T) \), this shows that \( gf \) is a sure map.

Quite obviously, each sure map is continuous. For, suppose \( f : (X, R_X) \to (Y, S_Y) \) is a sure map and \( S \) is a closed set in \( Y \).

Since \( S_Y \) is a subbase for \( Y \), there exists a subcollection \( \{S_{jn} \}_{j \in J} \) of \( S_Y \), where each \( N_j \) is a finite index set dependent on \( j \), such that \( S = \bigcup_{n \in N_j} S_{jn} \). Thus, \( f^{-1}(S) = \bigcap_{n \in N_j} f^{-1}(S_{jn}) \). \( j \in J \) is a closed set in space \( X \) as each \( f^{-1}(S_{jn}) \subseteq R_X \).
Definition 4.3 Let SURE be the full subcategory of T$_1$-SUBBASE which has as its functions the sure maps and let SUPERCOMP SURE be the full subcategory of SURE which has as its spaces the supercompact T$_1$-subbase spaces.

Theorem 4.4 SUPERCOMP SURE is an epireflective subcategory of SURE.

Suppose $f : (X, R_X) \to (Y, S_Y)$ is a sure function and $(T, T_Z)$ is a T$_1$-subbase space. Define $^+f$ so that, for each $R_X$-bicontiguity cluster $\sigma$, $^+f(\sigma) = \{C : \text{for each } D \in \sigma, C \in S_Y f(D)\}$. Then, our proof proceeds as follows: first, it is shown that the superextension map $\lambda_{T_Z}$ is a sure map and that, if $(T, T_Z)$ is a supercompact T$_1$-subbase space, then $\lambda_{T_Z}$ is a SURE-isomorphism; next, it is proven that $^+f : ^+X \to ^+Y$ is a set map such that $^+f \lambda_{R_X} = \lambda_{S_Y} f$; then, it is demonstrated that $^+f : (^+X, ^+R_X) \to (^+Y, ^+S_Y)$ is a sure map; and finally, it is shown that $\lambda_{T_Z}$ is a SURE-epimorphism. By the definition of epireflective subcategory, completing these steps finishes the proof.

For each T$_1$-subbase space $(Z, T_Z)$, recall $^+T_Z = \{^+T : T \in T_Z\}$, $\lambda_{T_Z}^{-1}(^+T) = T$ for each $^+T \in ^+T_Z$, and thus $\lambda_{T_Z}^{-1}[^+T_Z] = T_Z$. For sets $A \subseteq Z$ and $^+T \in ^+T_Z$, suppose $A \not\subseteq ^+T_Z$, suppose $A \not\subseteq ^+T_Z$, and so $^+T_Z = \lambda_{T_Z}^{-1}(^+T)$ or equivalently

$A \not\subseteq ^+T_Z$. Hence, there is a set $T_a \in T_Z$ such that $A \subseteq T_a$ and $T_a \cap T = \emptyset$. Recall this implies $^+T_a \cap ^+T = \emptyset$. Since

$\lambda_{T_a}(A) \subseteq \lambda_{T_Z}(T_a) \subseteq ^+T_a$, then $\lambda_{T_Z}(A) \not\subseteq ^+T_Z$ and so $\lambda_{T_Z}$ is a sure map.

Now, assume $(Z, T_Z)$ is a supercompact T$_1$-subbase space. It is already known $\lambda_{T_Z}$ is a T$_1$-SUBBASE-isomorphism, $\lambda_{T_Z}(T) = ^+T$ for
each $T \in T_Z$, and $\lambda_{T_Z}^{T} = +T_Z$. For $B \subseteq +T$ and $T \in T_Z$, suppose $B \not\subseteq +T_Z^T$ (T) or equivalently $B \not\subseteq +T_Z^T$. Thus, there exists a set $+T_b \subseteq T_Z$ such that $B \subseteq +T_b$ and $+T_b \cap +T = \emptyset$. Since $+T_b \cap +T = \emptyset$ implies $T_b \cap T = \emptyset$, then $\lambda_{+T_Z}^T(B) \subseteq \lambda_{+T_Z}^T(+T_b) = T_b$

implies $\lambda_{+T_Z}^T(B) \not\subseteq T_Z$. In other words, $\lambda_{+T_Z}^T$ is sure map and so $\lambda_{+T_Z}^T$ is a SURE isomorphism.

Note that if $A, B \subseteq Y$, and $A \not\subseteq B$, then $f^{-}(A) \not\subseteq f^{-}(B)$.

For suppose that $A \subseteq S_a \subseteq S_Y$, $B \subseteq S_b \subseteq S_Y$, and $S_a \cap S_b = \emptyset$.

Then $f^{-}(A) \subseteq f^{-}(S_a)$, $f^{-}(B) \subseteq f^{-}(S_b)$, and $f^{-}(S_a) \cap f^{-}(S_b) = \emptyset$. Since $f^{-}(S_a) \subseteq R_X$, then $f^{-}(S_a) \not\subseteq f^{-}(S_b)$ and hence $f^{-}(A) \not\subseteq f^{-}(B)$.

In particular, if $C, D \subseteq X$ and $f(C) \not\subseteq f(D)$ then $C \not\subseteq f^{-}(D)$, for $C \subseteq f^{-}(f(C))$, $D \subseteq f^{-}(f(D))$, and $f^{-}(f(C)) \not\subseteq f^{-}(f(D))$.

For each $R_X$-bicontiguity cluster $\sigma$, define $f\sigma = \{f(C) : C \in \sigma\}$. We can conclude $f\sigma \subseteq +f(\sigma)$ since, for any sets $C, D \in \sigma$, we know $C \subseteq f^{-}(D)$ and hence $f(C) \subseteq f(D)$.

To show $+f : +X \rightarrow +Y$ is a set map, it is necessary to prove that, for any $R_X$-bicontiguity cluster $\sigma$, the set $+f(\sigma)$ is an $S_Y$-bicontiguity cluster. Suppose the set $A \subseteq Y$ is $S_Y$-bicontiguous with each member of $+f(\sigma)$. Then, since $f\sigma \subseteq +f(\sigma)$, $A$ is $S_Y$-bicontiguous with each member of $f\sigma$ and, by the definition of $+f(\sigma)$, $A \subseteq +f(\sigma)$.

Next, suppose $A, B \subseteq +f(\sigma)$. For any sets $S_a, S_b \subseteq S_Y$ such that $A \subseteq S_a$ and $A \subseteq S_b$, clearly $S_a, S_b \subseteq +f(\sigma)$. If we assume $f^{-}(S_a), f^{-}(S_b) \in \sigma$, then $f^{-}(S_a) \subseteq f^{-}(S_b)$ and hence $S_a \subseteq S_b$. In other words, $A, B \subseteq +f(\sigma)$ would imply $A \subseteq S_Y$. Consequently, $+f(\sigma)$ would be an
an $S_Y$-bicontiguity cluster. Thus, we must demonstrate that, for any
set $S \in S_Y$, $S \in \downarrow \downarrow f(\sigma)$ implies $f^-(S) \in \sigma$. Given $S \in \downarrow \downarrow f(\sigma)$, then
$S \subseteq S_Y f(C)$, for each $C \in \sigma$. Because $f$ is a sure map, $f^-(S) \subseteq \downarrow \downarrow R_X C$.
Thus, $f^-(S) \subseteq \sigma$ since $\sigma$ is an $R_X$-bicontiguity cluster.

For each $x \in X$, $\lambda_{R_X}(x) = \{A \subseteq X : \{x\} \subseteq A\}$ and
$\lambda_{S_Y}(f(x)) = \{B \subseteq Y : \{f(x)\} \subseteq B\}$. Since $\{x\} \subseteq \lambda_{R_X}(x)$, then
$\{f(x)\} \subseteq \downarrow \downarrow \lambda_{R_X}(x) \subseteq \downarrow \downarrow f(\lambda_{R_X}(x))$. Because the only $S_Y$-bicontiguity cluster
containing $\{f(x)\}$ is $\lambda_{S_Y}(f(x))$, clearly $\downarrow \downarrow f(\lambda_{R_X}(x)) = \lambda_{S_Y}(f(x))$.

Therefore, $\downarrow \downarrow f_{X} \lambda_{R_X} = \lambda_{S_Y} f$.

To prove that $\downarrow \downarrow f : \downarrow \downarrow (X, +_{R_X}) \to \downarrow \downarrow (Y, +_{S_Y})$ is a sure map suppose,
for the moment, that we know $\downarrow \downarrow f(\downarrow \downarrow B) \subseteq \downarrow \downarrow f(\downarrow \downarrow B)$ for each $B \subseteq X$ and
$\downarrow \downarrow f(\downarrow \downarrow S) = (\downarrow \downarrow f)^-(\downarrow \downarrow S)$ for each $S \in S_Y$. Since $f^-(S) \subseteq \downarrow \downarrow R_X$ and
$\downarrow \downarrow f(\downarrow \downarrow S) = (\downarrow \downarrow f)^-(\downarrow \downarrow S)$ for each $S \in S_Y$, obviously $\downarrow \downarrow f(\downarrow \downarrow S_Y) = \downarrow \downarrow R_X$.

Now, assume $A \subseteq \downarrow \downarrow X$ and $\downarrow \downarrow S \in \downarrow \downarrow S_Y$ so that $A \not\subseteq \downarrow \downarrow (\downarrow \downarrow f)^-(\downarrow \downarrow S)$ or
equivalently $A \not\subseteq \downarrow \downarrow (f(S))$. Thus, there exists a set $R \in \downarrow \downarrow R_X$ such
that $A \subseteq \downarrow \downarrow R_X R \cap \downarrow \downarrow f(\downarrow \downarrow S) = \phi$. Hence, $R \cap \downarrow \downarrow f(\downarrow \downarrow S) = \phi$ and so
$\downarrow \downarrow R_X R \cap \downarrow \downarrow f(\downarrow \downarrow S)$. Since $f$ is a sure map, then $f(R) \not\subseteq \downarrow \downarrow S_Y S$. In other
words, there is a set $S \in S_Y$ such that $f(R) \subseteq S$ and $S \cap S = \phi$.

Since $\downarrow \downarrow S \cap \downarrow \downarrow S = \phi$, $\downarrow \downarrow f(R)) \subseteq \downarrow \downarrow S$ would imply $\downarrow \downarrow f(\downarrow \downarrow f(R)) \not\subseteq \downarrow \downarrow S_Y S$. But, for any $S_Y$-bicontiguity cluster $\sigma$, if $f(R) \subseteq \sigma$ then $S \subseteq \sigma$. Hence,
$\downarrow \downarrow f(R)) = \{\sigma \in \downarrow \downarrow Y : f(R) \subseteq \sigma\} \subseteq \{\sigma \in \downarrow \downarrow Y : S \subseteq \sigma\} = \downarrow \downarrow S$. By supposition
$\downarrow \downarrow f(R) \subseteq \downarrow \downarrow f(\downarrow \downarrow f(R))$. We conclude $\downarrow \downarrow f(A) \not\subseteq \downarrow \downarrow S_Y S$ since $\downarrow \downarrow f(\downarrow \downarrow A) \subseteq \downarrow \downarrow f(\downarrow \downarrow R)$.
For any set \( B \subseteq X \) and \( R_X \)-bicontiguity cluster \( \sigma \), recall \( \sigma \in ^+B \) implies \( B \in \sigma \). But if \( B \in \sigma \), \( f(B) \in f\sigma \subseteq ^+f(\sigma) \). This means \( ^+f(\sigma) \subseteq ^+(f(B)) \) and consequently \( ^+f(\sigma) \subseteq ^+(f(B)) \).

Consider \( S \in S_Y \). Notice \( ^+(f^{-}(S)) \subseteq (^+f)^{-}(^+S) \) if and only if \( ^+f(^+f^{-}(S)) \subseteq ^+S \). But, since \( f(f^{-}(S)) \subseteq S \), then \( ^+f(\delta) = \sigma \). Then, it has been demonstrated above that \( f^{-}(S) \subseteq \delta \) since \( S \in \sigma \cap S_Y \) and \( f \) is a sure map. Since \( \delta \in ^+(f^{-}(S)) \), we have shown \( ^+(f^{-}(S)) = (^+f)^{-}(^+S) \). In conclusion, if \( f \) is a sure map, then \( ^+f \) is a sure map also.

The proof that each superextension map is a \textbf{SURE} - epimorphism requires two stages. In the first, we demonstrate, if \( f : (X,R_X) \rightarrow (Y,S_Y) \) and \( g : (^+X,^+R_X) \rightarrow (^+Y,^+S_Y) \) are sure maps so that \( g \lambda_X = \lambda_Y f \), then \( g = ^+f \).

Assuming \( S \in S_Y \), since \( g^{-}(^+S) \subseteq ^+R_X \) and \( g \lambda_X = \lambda_Y f \), then \( f^{-} \lambda_X^{-}(^+S) = \lambda_Y^{-} g^{-}(^+S) \subseteq R_X \). Recalling that \( \lambda_Y^{-}(^+S) = S \) and \( ^+(\lambda_X^{-}(^+R)) = ^+R \) for each \( R \in R_X \), clearly \( ^+(f^{-}(S)) = g^{-}(^+S) \).

Now, if \( \sigma \) is an \( R_X \)-bicontiguity cluster such that \( S \in g(\sigma) \), then \( g(\sigma) \subseteq ^+S \). This implies \( \sigma \subseteq g^{-}(^+S) = ^+(f^{-}(S)) \) and so \( f^{-}(S) \subseteq \sigma \).

Since \( f^{-}(S) \subseteq R_X \), \( f^{-}(S) \subseteq \sigma \cap R_X \). In summary, that \( S \in g(\sigma) \cap S_X \) implies \( f^{-}(S) \subseteq \sigma \cap R_X \) requires only the assumptions \( g \lambda_X = \lambda_Y f \) and \( g^{-}[^+S_Y] \subseteq ^+R_X \).
Suppose \( C \in \sigma \) and \( S \in g(\sigma) \cap S_Y \). Since \( f^{-}(S) \in \sigma \cap R_X \), clearly \( C \subseteq f^{-}(S) \). Recalling \( f \circ \xi^+ = f(\sigma) \), obviously
\[
f(C) \subseteq f(f^{-}(S)) \quad \text{and hence} \quad f(C) \subseteq S_Y \cdot \]
Since \( f(C) \) is \( S_Y \)-bincontiguous with each set \( S \in g(\sigma) \cap S_Y \), we can conclude \( f(C) \in g(\sigma) \). In other words, \( f \circ \xi^+ \subseteq g(\sigma) \). But, this means \( g(\sigma) \subseteq +f(\sigma) \), since, if \( A \in g(\sigma) \), then \( A \) is \( S_Y \)-bicontiguous with each set \( D \in f \sigma \). Because \( g(\sigma) \) and \( +f(\sigma) \) are \( S_Y \)-bicontiguity clusters, \( g(\sigma) = +f(\sigma) \). In summary, the sure map \( +f \) is the unique sure map \( g \) for which \( g \lambda_{R_X} = \lambda_S f \).

The second part of the proof that each superextension map is a \textbf{SURE} epimorphism follows by a strictly categorical argument from the result just proven. Let \( \lambda_{R_X} : (X,R_X) \to (\text{Id}^+_X, \text{Id}^+_Y) \) be a superextension map and \( f,g : (\text{Id}^+_X, \text{Id}^+_X) \to (Y,S_Y) \) be sure maps so that \( f \lambda_{R_X} = g \lambda_{R_X} \).

\[
\begin{array}{ccc}
(X,R_X) & \xrightarrow{\lambda_{R_X}} & (\text{Id}^+_X, \text{Id}^+_X) \\
\downarrow f & & \downarrow f \\
(Y,S_Y) & \xrightarrow{\lambda_{S_Y}} & (\text{Id}^+_Y, \text{Id}^+_Y)
\end{array}
\]

Since \( f \lambda_{R_X} = g \lambda_{R_X} \) is a sure map, we now know that \( +f \lambda_{R_X} \) is the unique sure map such that \( \lambda_S(f \lambda_{R_X}) = +f \lambda_{R_X} \lambda_{R_X} = \lambda_S(g \lambda_{R_X}) \).

But obviously \( \lambda_{S_Y} f \) and \( \lambda_{S_Y} g \) are also sure maps such that
\[
(\lambda_{S_Y} f) \lambda_{R_X} = \lambda_{S_Y} (f \lambda_{R_X}) \quad \text{and} \quad (\lambda_{S_Y} g) \lambda_{R_X} = \lambda_{S_Y} (g \lambda_{R_X}) .
\]
Therefore,
\[ \lambda_{S_Y} f = + (f_{R_X}^+) = \lambda_{S_Y} g. \] Since the superextension map \( \lambda_{S_Y} \) is one-to-one, it is easy to see \( f = g \). Thus, for each space \((X, R_X)\), the map \( \lambda_{R_X} \) is a sure epimorphism.

It should be noted the previous categorical result is subsumed by the following simple proposition [6, p. 276]: If \( S \) is a full mono-reflective subcategory of \( C \), then \( S \) is an epireflective subcategory of \( C \).

**Remark 4.5** The proof of 3.4 can, of course, be argued in terms of ultrasieves rather than bicontiguity clusters. For example, we would call \( f : (X, R_X) \rightarrow (Y, S_Y) \) a "sure map" whenever \( f^{-}[S_Y] \subseteq R_X \) and, for sets \( R \in R_X \) and \( S \in S_Y \) such that \( R \cap f^{-}(S) = \phi \), there exists a set \( S_X \in S_Y \) such that \( R \subseteq f^{-}(S_X) \) and \( S_X \cap S = \phi \). Also, for each \( R_X \)-ultrasieve \( \sigma \), we would define the map \( f^+ : (X^+, R_X^+) \rightarrow (Y^+, S_Y^+) \) by \( f^+(\sigma) = \{ S \in S_Y : f^{-}(S) \in \sigma \} \). Further details may be verified by the interested reader. As justification of our approach, we note the natural formulation of the sure maps and the function \( +f \) and cite the methods used by Bentley and Naimpally in [1].

**Proposition 4.6** If \( f : (X, R_X) \rightarrow (Y, S_Y) \) is a set map, \( g : (X^+, R_X^+) \rightarrow (Y^+, S_Y^+) \) is a sure map, and \( g^\lambda_{R_X} = \lambda_{S_Y} f \), then \( f \) is a sure map and so \( g = +f \).

In the previous proof, we have shown that \( g^{-}[S_Y] \subseteq R_X \) and \( g^\lambda_{R_X} = \lambda_{S_Y} f \) imply \( f^{-}[S_Y] \subseteq R_X \) and, for each \( S \in S_Y \), \( g^{-}(S) = +(f^{-}(S)) \).

Assume \( A \subseteq X \), \( S \in S_Y \), and \( A \not\subseteq f^{-}(S) \). Then, \( \lambda_{R_X}^{-}(A) \not\subseteq R_X(+)f^{-}(S) \)

or equivalently \( \lambda_{R_X}^{-}(A) \not\subseteq R_X^+(f^{-}(S)) \). Since \( g \) is a sure map,

\[ g^\lambda_{R_X}^{-}(A) \not\subseteq S_Y^+ \] and so \( \lambda_{S_Y} f(A) \not\subseteq S_Y^+ \). Finally, we have \( f(A) \not\subseteq S_Y \).
Thus, $f$ is a sure map. Since $f$ is a sure map, we know from 3.4 that $g = +f$.

**Proposition 4.7** Let $f : (X, R_X) \rightarrow (Y, S_Y)$ be a sure map. The map $f$ is a SURE-epimorphism if and only if $+f$ is a SURE-empimorphism. If $f$ is a SURE-isomorphism, then $+f$ is a SURE-isomorphism. Also, if $f$ is a surjection, then $+f$ is a surjection.

Suppose $f$ is a SURE-epimorphism and $g, h : (Y, S_Y) \rightarrow (Z, T_Z)$ are sure maps such that $g + f = h + f$. Then, $g \lambda_{S_Y} = g + f \lambda_{R_X} = h + f \lambda_{R_X} = h \lambda_{S_Y}$. Since $f$ and $\lambda_{S_Y}$ are SURE-epimorphisms, $g = h$.

Thus, $+f$ is also a SURE-epimorphism.

Next, suppose $+f$ is a SURE-epimorphism and $g, h : (Y, S_Y) \rightarrow (Z, T_Z)$ are sure maps such that $gf = hf$. Since $f$, $g$, and $h$ are sure maps, then $+f$, $+g$, and $+h$ are sure maps such that $+f \lambda_{R_X} = \lambda_{S_Y} f$, $+g \lambda_{S_Y} = \lambda_{T_Z} g$, and $+h \lambda_{S_Y} = \lambda_{T_Z} h$. Thus, $+g + f$ and $+h + f$ are sure maps such that $+g + f \lambda_{R_X} = \lambda_{T_Z} gf = \lambda_{T_Z} hf = +h + f \lambda_{R_X}$.

Since $\lambda_{R_X}$ and $+f$ are SURE-epimorphisms, then $+g = +h$. This means $\lambda_{T_Z} g = \lambda_{T_Z} h$ and so $g = h$. Thus, $f$ is a SURE-epimorphism. Notice that the preceding proof uses a strictly categorical argument. Likewise, via a trivial categorical proof, it can be shown that $+f$ is a SURE-isomorphism if $f$ is.

Finally, assume $f$ is a surjection and $\sigma \in +Y$. Consider $d = \{f^{-1}(S) : S \in \sigma \cap S_X\}$. Since each pair of members from $d$ has a nonempty intersection, clearly $d$ is contained in some $R_X$-bicontiguity cluster, say $\delta$. For each $D \in \delta$ and $S \in \sigma$, recall $D \subset R_X f^{-1}(S)$ implies
\( f(D) \subset_S S \) since \( f(f^{-1}(S)) \subset S \). Hence, \( f(D) \in \sigma \) or equivalently \( \delta \in \sigma \). We can conclude \( f(\delta) = \sigma \) since \( f(\delta) \) and \( \sigma \) are \( S_Y \)-bicontiguity clusters.
CHAPTER V
JENSEN'S RESULT

In this chapter, we present a mapping result due to G. A. Jensen [8, p. 56]. The situation Jensen describes was not motivated by nor is it particularly compatible with the notion of epireflective subcategories.

Definition 5.1 [8, p. 51] A $T_1$-subbase $R_X$ for space $X$ is normal if, for any disjoint sets $R_1, R_2 \in R_X$, there exist sets $R_1^C, R_2^C \in R_X$ such that $R_1 \cap R_2^C = \phi$, $R_2 \cap R_1^C = \phi$, and $R_1^C \cup R_2^C = X$. A $T_1$-subbase space $(X, R_X)$ is called normal if $R_X$ is normal.

Proposition 5.2 [8, p. 52] If $R_X$ is a normal $T_1$-subbase for $X$, then $R_X$ generates a Hausdorff topology.

Let $x_1$ and $x_2$ be different points in $X$. Since $R_X$ is a $T_1$-subbase, we know there exist sets $R_1, R_2 \in R_X$ such that $x_1 \in R_1$, $x_2 \in R_2$, and $R_1 \cap R_2 = \phi$. Thus, there exist sets $R_1^C, R_2^C \in R_X$ such that $R_1^C \cup R_2^C = X$, $R_1 \cap R_1^C = \phi$, and $R_2 \cap R_2^C = \phi$. Clearly, $X - R_1^C$ and $X - R_2^C$ are open sets such that $(X - R_1^C) \cap (X - R_2^C) = \phi$, $x_1 \in X - R_1^C$, and $x_2 \in X - R_2^C$.

Lemma 5.3 [8, p. 13] Suppose $(X, R_X) \in T_1$-SUBBASE and $S, T \in R_X$. Then, $S \cup T = X$ if and only if $S^+ \cup T^+ = X^+$.

Assume $x \in X$ and $S^+ \cup T^+ = X^+$. Obviously, $\lambda_{R_X}^R(x) \in X^+$ and so, without loss of generality, $\lambda_{R_X}^R(x) \in S^+$. Hence, $S \in \lambda_{R_X}^R(x)$ and $x \in S$. Thus, $S \cup T = X$.
Next, suppose $S \cup T = X$ and $\sigma \in X^+$. If $\sigma \notin S^+ \cup T^+$, then $S \cup T \notin \sigma$. Hence, there exist $R_S, R_T \in \sigma$ such that

$$R_S \cap S = R_T \cap T = \emptyset.$$ Since $R_S \cap R_T \not\subseteq$ complement $S \cap$ complement $T = \emptyset$, we have a contradiction. Thus, $S \cup T = X$ implies $S^+ \cup T^+ = X^+$.

Theorem 5.4 [8, p. 55] Let $(X, R_X)$ be a $T_1$-subbase space and $(Y, S_Y)$ be a normal $T_1$-subbase space. If $f : (X, R_X) \rightarrow (Y, S_Y)$ is a map for which $f^{-1}[S_Y] \subseteq R_X$, then there exists a continuous closed map $g : (X^+, R_X^+) \rightarrow (Y^+, S_Y^+)$ such that $g^\lambda_{R_X} = \lambda_{S_Y}^f$.

For each $R_X$-ultrasieve $\sigma$, consider the sets

$f^\sigma = \{ S \in S_Y : f^-(S) \subseteq \sigma \}$ and $\bar{f}(\sigma) = \{ S \in S_Y : \text{for each } S_\sigma \in f^\sigma, S \cap S_\sigma \neq \emptyset \}$. Clearly, $f^\sigma$ is nonempty since at least $Y \in f^\sigma$.

Also, if $S_a, S_b \in f^\sigma$, then $S_a \cap S_b \neq \emptyset$ since $f^-((S_a) \cap f^-((S_b)) \neq \emptyset$.

Thus, $f^\sigma \subseteq \bar{f}(\sigma)$. Consequently, if a set $S \in S_Y$ meets each member of $\bar{f}(\sigma)$, then $S \in \bar{f}(\sigma)$.

Now consider sets $S_1, S_2 \in \bar{f}(\sigma)$. Let us suppose $S_1 \cap S_2 = \emptyset$.

Since $S_Y$ is normal, we have sets $S_1^c, S_2^c \in S_Y$ such that $S_1^c \cap S_1 = \emptyset, S_2^c \cap S_2 = \emptyset$, and $S_1^c \cup S_2^c = Y$. Thus, $f^-(S_1^c) \cup f^-(S_2^c) = X$.

Recall this and $f^-(S_1^c), f^-(S_2^c) \subseteq R_X$ imply that either $f^-(S_1^c) \subseteq \sigma$ or $f^-(S_2^c) \subseteq \sigma$. But, if $f^-(S_1^c) \subseteq \sigma$, then $S_1^c \subseteq f^\sigma$ and so $S_1 \cap S_1^c \neq \emptyset$.

Similarly, if $f^-(S_2^c) \subseteq \sigma$, then $S_2 \cap S_2^c \neq \emptyset$. The contradiction is resolved only if $S_1 \cap S_2 \neq \emptyset$. Thus $f(\sigma)$ is an $S_Y$-ultrasieve and $\bar{f} : X^+ \rightarrow Y^+$ is a set map.

Let $z \in X$. Recall $\lambda_{R_X}(z) = \{ R : z \in R \subseteq R_X \}$ and $\lambda_{S_Y}(f(z)) = \{ S : f(z) \in S \subseteq S_Y \}$. But, if $S \in \lambda_{S_Y}(f(z))$, then $f^-(S) \subseteq \lambda_{R_X}(z)$. Thus, since $\lambda_{S_Y}(f(z)) \subseteq f^-(z)$, clearly

$\lambda_{S_Y}(f(z)) = \bar{f}(\lambda_{R_X}(z))$. 

-29-
To prove \( \tilde{f} \) is a continuous map, it is sufficient to show \( \tilde{f}^-(S_1^+) \) is a closed set in space \( X \) for any \( S_1 \subseteq S_Y \). Suppose \( \sigma \in X^+ \) and \( \sigma \notin \tilde{f}^-(S_1^+) \). Then \( \tilde{f}(\sigma) \notin S_1^+ \). This implies there is a set \( S_2 \subseteq \tilde{f}(\sigma) \) such that \( S_1 \cap S_2 = \emptyset \). Since \((Y,S_Y)\) is normal, there exist sets \( S_1^C, S_2^C \subseteq S_Y \) such that \( S_1^C \cap S = \emptyset \), \( S_2^C \cap S = \emptyset \), and \( S_1^C \cup S_2^C = Y \). We claim the set \( (f^-((S_2^C)^+)) \) has the properties that \( \tilde{f}^-(S_1^+) \subseteq (f^-((S_2^C)^+)) \) and \( \sigma \notin (f^-((S_2^C)^+)) \). Since \( f^-[S_Y] \subseteq R_X \), this would imply \( \tilde{f}^-(S_1^+) \) is a closed set in \( X \).

First, note since \( S_2 \subseteq \tilde{f}(\sigma) \) and \( S_2^C \cap S_2 = \emptyset \), then \( S_2^C \notin \tilde{f}(\sigma) \). In particular, \( \tilde{f}^-(S_2^C) \notin \sigma \) and hence \( \sigma \notin (f^-((S_2^C)^+)) \).

Next, \( S_1^C \cap S_1 = \emptyset \) implies \( (S_1^C)^+ \cap (S_1^+)^+ = \emptyset \) and \( \tilde{f}^-((S_1^C)^+) \cap \tilde{f}^-((S_1^+)^+) = \emptyset \). Then, \( (f^-((S_1^C)^+) \cap \tilde{f}^-((S_1^+)^+) = \emptyset \) if we can show \( (f^-((S_1^C)^+) \subseteq \tilde{f}^-((S_1^C)^+) \). But, \( \delta \in (f^-((S_1^C)^+) \iff \tilde{f}(\delta) \in (S_1^C)^+ \iff \delta \in \tilde{f}^-((S_1^C)^+) \). Moreover, since \( S_1^C \cup S_2^C = Y \), then \( f^-((S_1^C) \cup f^-((S_2^C)^+) = X^+ \) and hence \( (f^-((S_1^C))^+ \cup (f^-((S_2^C))^+ = X^+ \). Consequently, \( \tilde{f}^-((S_1^+)^+) \subseteq (f^-((S_2^C))^+) \).

Recall, for any \( S_a, S_b \subseteq S_Y \), that \( S_a \subseteq S_b \implies S_a^+ \subseteq S_b^+ \), \( S_a \cap S_b = \emptyset \) implies \( S_a^+ \cap S_b^+ = \emptyset \), and \( S_a \cup S_b = Y \) implies \( S_a^+ \cup S_b^+ = Y^+ \). Hence, since \((Y,S_Y)\) is normal, clearly \((Y^+,S_Y^+)\) is normal. Thus, \( \tilde{f} : (X^+,R_X^+) \to (Y^+,S_Y^+) \), being a continuous map from a compact space into a Hausdorff space, is a closed map [9, p. 123].

Letting \( g = \tilde{f} \), the theorem is proven.
BIBLIOGRAPHY


