SUMMABILITY-TOPOLOGICAL METHODS

by

Che-Young Lee

Dip. of Sc., Chung Chi College, 1966.

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in the Department
of
Mathematics

© CHE-YOUNG LEE 1969

SIMON FRASER UNIVERSITY

June, 1969
EXAMINING COMMITTEE APPROVAL

J. J. Sembar, Senior Supervisor

T. C. Brown, Examining Committee

David M. Eaves, Examining Committee
ABSTRACT

This thesis is a survey of applications of topological methods to summability. We also review and discuss some of the results obtained by A. Wilansky and K. Zeller.

Chapters 1 and 2 are of introductory nature. In Chapter 3 we discuss the classification of conservative matrices as co-null and co-regular matrices. In Chapter 4, we study the inclusion relations of $c$ and $l_A$ and give a detailed proof of a result due to Wilansky and Zeller. In Chapter 5, we study perfectness and type M for different classes of matrices.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>v</td>
</tr>
<tr>
<td>A list of symbols</td>
<td>vii</td>
</tr>
<tr>
<td>Chapter 1: Some definitions and general results</td>
<td>1</td>
</tr>
<tr>
<td>Chapter 2: Continuous linear functionals on $c_A$</td>
<td>20</td>
</tr>
<tr>
<td>Chapter 3: Co-null and co-regular matrices</td>
<td>23</td>
</tr>
<tr>
<td>Chapter 4: $c$ as a subset of $c_A$</td>
<td>30</td>
</tr>
<tr>
<td>Chapter 5: Perfectness and matrices of type M</td>
<td>37</td>
</tr>
<tr>
<td>Bibliography</td>
<td>48</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENT

The author wishes to express his thanks to Dr. J. J. Sember who has made many helpful suggestions and patiently read the manuscript of this thesis.

The financial assistance of the National Research Council of Canada is deeply appreciated.
INTRODUCTION

This thesis is a survey of applications of topological methods to the theory of infinite matrix summability. We also review and discuss some of the results obtained by A. Wilansky and K. Zeller. Most of the materials are from [7], [8], [9] and [10].

Chapter 1 is of introductory nature; it consists of results of the theory of topological vector spaces, a sketch of the theory of FK spaces and some important results on infinite matrices (Theorem 1.22., Theorem 1.23., Theorem 1.25., and Proposition 1.27.). An example is given to show that multiplication of infinite matrices is, in general, not associative.

In Chapter 2, the general form of continuous linear functionals on the summability field $c_A$ is given. This identification has numerous applications to the theory.

In Chapter 3 we define co-null and co-regular matrices in terms of the matrix entries. Also, we point out that co-nullity and co-regularity can be regarded as properties of the summability field rather than the matrix. In the second part of this chapter, we study the 'size' of summability fields of co-null and co-regular matrices. The construction of the matrix in Example 3.e. is based on the proof of Theorem 3 of [10].

In Chapter 4, we consider the inclusion relation be-
between \( c \) and \( l_A \). In the second part of this chapter, a detailed proof of part of Theorem 1 in [10] is given; the original proof in that paper is very precise. Theorem 4.3 and Proposition 1.12 assure that a co-null matrix must sum a bounded divergent sequence. This result was also obtained originally by K. Zeller.

In Chapter 5, perfectness and type \( M \) are studied for different classes of matrices in terms of different subsets of their summability fields. Concrete examples are given to show that these conditions are in general not equivalent.
A LIST OF SYMBOLS

A, B, C, ... infinite matrices with complex entries

\((a_{nk})\) the infinite matrix whose element at the nth row and kth column is \(a_{nk}\)

\(x, y, z\) sequences of complex numbers

\(\{x^n\}\) sequence of sequences

\(i\) the sequence \((1, 1, 1, \cdots)\)

\(\delta^k\) the sequence whose kth coordinate is 1 and others are zero

\(F\) \(\{i\} \cup \{\delta^k|k=1, 2, \cdots\}\)

\(s\) the space of all sequences

\(c\) the space of all convergent sequences

\(c_0\) the space of all sequences converging to zero

\(l_1\) sequences such that \(\sum_n|x_n|<\infty\)

\(X\) the closure of the subset \(X\) in some topological space

\(c_A\) the set of all continuous linear functionals on \(c_A\)

\(a, b, c, \ldots\) complex numbers

\(v\) vectors in some linear space

\(V\) linear spaces over the complex numbers
CHAPTER I
SOME DEFINITIONS AND GENERAL RESULTS

1.1. Topological vector spaces

In what follows we will state some definitions and results from the theory of topological vector spaces. The details may be found in [3] and [7].

Definition: A seminorm on a vector space $V$ is a map $q$ from $V$ to the non-negative real numbers satisfying
i) $q(av) = |a|q(v)$, for all complex numbers $a$ and $v \in V$.
ii) $q(v_1 + v_2) \leq q(v_1) + q(v_2)$.

It is known that given a family $(q_\xi)_{\xi \in I}$ of seminorms on $V$, a locally convex linear topology can be defined on $V$ with the sets $\bigcap_{k=1}^{n} E_k V_{\epsilon_k}$ as a fundamental system of neighborhoods of $0$, where $E_k \ni 0$ and $V_{\epsilon_k} = \{v| q_{\epsilon_k}(v) < 1\}$.

When the family of seminorms $(q_\xi)_{\xi \in I}$ is countable and total, that is, if $v \neq 0$, there exists $q_\xi$ such that $q_\xi(v) \neq 0$, we have the following result:

Theorem 1.1. If the locally convex topology on a vector space $V$ is generated by a countable and total family of seminorms $(q_\xi)_{\xi \in N}$, then

$$q(v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{r_n(v)}{1 + r_n(v)}$$  \hspace{1cm} (1.1)

where $r_n(v) = \max_{1 \leq \xi \leq n} q_\xi(v)$, satisfies
a) $|a| \leq 1 \implies q(av) \leq q(v)$.

b) $a_k \to 0 \implies q(a_k v) \to 0$.

c) $q(v) = 0$ if and only if $v = 0$.

d) $q(-v) = q(v)$.

e) $q(v_1 + v_2) \leq q(v_1) + q(v_2)$.

Furthermore, the metric $d(v_1, v_2) = q(v_1 - v_2)$ defines the linear topology on $V$.

Proof: See Theorem 1 on p.111 and Proposition 2 on p.114 of [1].

Definition: A paranorm on a linear space $V$ is a non-negative real function $P$ satisfying

i) $P(0) = 0$.

ii) $P(-v) = P(v)$.

iii) $P(v_1 + v_2) \leq P(v_1) + P(v_2)$.

iv) If $\{a_n\}$ is a sequence of scalars with $a_n \to a$ and $\{v_n\}$ is a sequence of vectors such that $p(v_n - v) \to 0$, then $P(a_n v_n - av) \to 0$.

A paranorm is total if $P(v) = 0$ implies that $v = 0$. It can easily be seen that $q(v)$ in Theorem 1.1. is a total paranorm where iv) is justified by

$$q(a_n v_n - av) \leq q(a_n v_n - av_n) + q(av_n - av) = q((a_n - a)v_n) + q(a(v_n - v)).$$

The term $q((a_n - a)v)$ tends to zero as $n$ increases by b). As for $q(a(v_n - v))$, we may assume $|a| > 1$, otherwise a) assures that $q(a(v_n - v)) \to 0$, as $n \to \infty$; now $|a| > 1$ implies that $q(a(v_n - v))$

$$\leq \sum_{n=1}^{\infty} \frac{|a|}{2} \frac{r_n(v_n - v)}{1 + r_n(v_n - v)} = |a| q(v_n - v),$$

hence $q(a(v_n - v))$ tends to zero as $n$ increases.
Definition: A linear metric space is a linear topological space, the topology being generated by a metric $d$ that arises from a total paranorm, that is, $d(x,y)=P(x-y)$ for some total paranorm $P$.

For linear metric spaces, we say that $\sum \nu_n$ converges to $v$ if for any $\epsilon > 0$, there exists integer $N$ such that $n_0 > N$ implies $P\left(\sum_{n=1}^{n_0} \nu_n - v\right) < \epsilon$.

Definition: A sequence of vectors $\{\nu_n\}$ is said to be a Schauder basis for a linear metric space $V$ if, for every vector $v$ in $V$, there is a unique sequence of scalars $\{a_n\}$ such that $v = \sum a_n \nu_n$.

If the locally convex topology on $V$ is defined by a countable and total family of seminorms $(q_{\ell})_{\ell \in N}$, then it clearly is a linear metric space.

Theorem 1.2. If the locally convex topology on $V$ is generated by a countable and total family $(q_{\ell})_{\ell \in N}$ of seminorms and $\{\nu_n\}$ is a sequence of vectors in $V$ such that for every $v$ in $V$ there is a unique sequence of scalars $\{a_n\}$ such that $q_{i}\left(\sum_{k=1}^{n} a_k \nu_k - v\right) \to 0$, for $i=1,2,3,\ldots$, then $\{\nu_n\}$ is a Schauder basis.

Proof: For any $\epsilon > 0$, choose integer $N_1$ so that $\sum_{j=N_1+1}^{\infty} \frac{1}{2^j} < \frac{\epsilon}{2}$. Consider $q_1,\ldots,q_{N_1}$. Choose $N_2$ so that for $n > N_2$ we have $q_1(v - \sum_{k=1}^{n} a_k \nu_k) < \frac{\epsilon}{2^n}$, $\ldots$.
Theorem 1.3. If the locally convex linear topology on a vector space V is generated by a countable and total family of seminorms \( (q_I)_{i \in N} \) and \( q \) is a seminorm not in \( (q_I)_{i \in N} \), then the following conditions are equivalent.

i) \( q \) is discontinuous at the origin.

ii) \( q \) is discontinuous on \( V \).

iii) The topology generated by \( \{ q \mid q \in (q_I)_{i \in N} \} \) is strictly stronger than the topology generated by \( \{ q \mid q \in (q_I)_{i \in N} \} \).

iv) For any positive real number \( M \), any \( (l_1, \ldots, l_n) \), there exists \( v \) in \( V \) such that \( q(v) > M \max_{1 \leq k \leq n} q_{l_k}(v) \).

v) For all \( \varepsilon > 0 \), and integer \( N_1 \), there exists \( v \in V \) such that \( q(v) = 1 \) and \( q_{l}(v) < \varepsilon \) for all \( l \leq N_1 \).

Proof: i) clearly implies ii). If \( q \) is continuous at the origin and \( v_1 \) is any vector in \( V \), then

\[
v_1 + \{ v \mid q(v) < \varepsilon \} = \{ v_1 + v \mid q(v) < \varepsilon \} \subseteq q^{-1}(N(q(v_1), \varepsilon))
\]

implies that \( q \) is continuous at \( v_1 \). Hence i) and ii) are equivalent. If \( q \) is discontinuous at the origin, then for some \( \varepsilon > 0 \), \( \{ v \mid q(v) < \varepsilon \} \) does not contain any open set of the topology generated by \( (q_I)_{i \in N} \). Hence the topology generated by \( \{ q \mid q \in (q_I)_{i \in N} \} \) is strictly stronger. iii)
clearly implies i). The fact that iii) and iv) are equivalent is proved on p.98 of [3]. Now suppose iv) holds and for any $\xi > 0$ let $M = \frac{1}{\xi}$, then there exists $v_1$ in $V$ such that $q(v_1) > M \max_{l \leq k \leq n} q_{l,k}(v_1)$ for any $(l_1, \cdots, l_k)$. Let $v = \frac{v_1}{q(v_1)}$, then $q(v) = 1$ and $q_{l,k}(v) < \xi$. Conversely assume v). For any positive real number $M$ and any $(l_1, \cdots, l_k)$, choose integer $N_1$ so that $N_1 > \max(l_1, \cdots, l_n)$ and let $\xi = \frac{1}{M}$, then there exists $v_1$ in $V$ such that $q(v_1) = 1$ and $q_{l_1}(v_1), \cdots, q_{l_n}(v_1)$ are all smaller than $\xi$. Hence $q(v_1) = 1\frac{1}{\xi} \max_{l \leq k \leq n} q_{l,k}(v_1) = M \max_{l \leq k \leq n} q_{l,k}(v_1)$.

**Theorem 1.4.** (Hahn-Banach) Let $V_1$ be a subspace of a linear space $V$, $q$ be a seminorm defined on $V$ and $f$ a linear functional defined on $V_1$ such that $|f(v)| \leq q(v)$ for all $v$ in $V_1$, then there is an extension $F$ of $f$ which is a linear functional on $V$ and $|F(v)| \leq q(v)$ for all $v$ in $V$.

**Proof:** See p.65 [7].

The following is a corollary of the Hahn-Banach extension theorem.

**Theorem 1.5.** Let $V$ be a seminormed linear space, $V_1 \subseteq V$ be a linear subspace and $v$ be a vector which does not belong to the closure of $V_1$, then there is a continuous linear functional $f$ which vanishes on $V$, and $f(v) \neq 0$.

**Proof:** See p.67 of [7].

It follows from the above theorem that if every continuous linear functional $f$ that vanishes on $V_1$ is identically zero, then $V_1$ must be dense in $V$. This
argument will be applied very frequently in the following chapters.

The following theorem contains two forms of the Banach-Steinhaus theorem.

**Theorem 1.6.** i) Let \((q_L)_{\ell \in \mathbb{I}}\) be a pointwise bounded family of continuous seminorms on a complete seminormed space, then \(\{\|q_L\| : \ell \in \mathbb{I}\}\) is uniformly bounded.

ii) Let \(\{f_n\}\) be a sequence of pointwise convergent continuous linear functions from a complete seminormed space to a normed space, then \(f(x) = \lim_n f_n(x)\) defines a continuous linear function \(f\).

**Proof:** See p.117 of [7].

1.2. Sequence Spaces and FK Spaces.

For every sequence \(x=(x_1, \ldots, x_n, \ldots)\) in \(c\), define

\[ \|x\| = \sup_n |x_n|, \]

and for every sequence \(x=(x_1, \ldots, x_n, \ldots)\) in \(l_1\), define \(\|x\| = \sum_n |x_n|\). It is well-known that \(c\) and \(l_1\) become Banach spaces with these norms. Also \(c\) has \(F\) as Schauder basis where each \(x \in c\) is represented by

\[ x = (\lim x)i + \sum_n (x_n - \lim x)\delta^n \quad (1.2) \]

If \(f\) is a continuous linear functional on \(c\), then

\[ f(x) = (\lim x)f(i) + \sum_n (x_n - \lim x) t_n \quad (1.3) \]

where \(t_n = f(\delta^n), n=1,2,\ldots\) \( (1.4) \)

and \(\sum_n |t_n| < \infty\) \( (1.5)\)

For an arbitrary infinite matrix \(A=(a_{ij})\) of complex numbers, denote a sequence \(x=(x_1, \ldots, x_n, \ldots)\) by a column
vector

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{pmatrix}
\]

By \( Ax \) we mean the column vector

\[
\begin{pmatrix}
  \xi^a_{1k} x_k \\
  \xi^a_{2k} x_k \\
  \vdots \\
  \xi^a_{nk} x_k \\
\end{pmatrix}
\]

if \( \xi^a_{nk} x_k \) exists for all \( n \).

For an arbitrary matrix \( A \), let \( d_A = \xi | Ax \) exists, that is, \( \xi^a_{nk} x_k \) exists for all \( n \). On \( d_A \) define

\[
| P_n | (x) = | x_n | \quad \text{and} \quad h_n (x) = \sup \left\{ | \xi^a_{nk} x_k | \mid r = 1, 2, \ldots \right\}
\]

for \( n = 1, 2, 3, \ldots \). From the triangular inequality

\[
| P_n | (x+y) = |x_n+y_n| \leq |x_n| + |y_n| = |P_n | (x) + |P_n | (y).
\]

Also,

\[
| P_n | (ax) = |a| \| x_n \| = |a| | P_n | (x).
\]

Hence \( | P_n | \) is a seminorm for any \( n \). For any \( n \), \( h_n (ax) = |a| \sup \left\{ | \xi^a_{nk} x_k | \mid r = 1, 2, \ldots \right\} \) and

\[
\sup \left\{ | \xi^a_{nk} x_k | \mid r = 1, 2, \ldots \right\} \leq \sup \left\{ | \xi^a_{nk} x_k | \mid r = 1, 2, \ldots \right\} + \sup \left\{ | \xi^a_{nk} x_k | \mid r = 1, 2, \ldots \right\}
\]

implies that \( h_n \) is a seminorm.

Throughout this paper, the linear topology on \( d_A \) is defined to be the locally convex topology generated by

\[
\{ h_n \mid n = 1, 2, \ldots \} \cup \{ | P_n | \mid n = 1, 2, \ldots \}.
\]

Proposition 1.7. For an arbitrary matrix \( A \), \( \{ \xi^k \mid k = 1, 2, \ldots \} \)

is a Schauder basis for \( d_A \).

Proof: Let \( x = (x_1, \ldots x_n, \ldots) \in d_A \). For any \( n \), \( | P_n | \left( x - \frac{1}{k} \sum_{k=1}^{\infty} x_k \delta^k \right) = 0 \)
\[ if \, i > n. \text{ Hence } \lim_{i \to \infty} |P_n(x - \sum_{k=1}^{n} x_k e_k^i)| = 0. \] For any \( n \), \[ \sum_{k=1}^{n} a_k x_k \] exists since \( x \in d_A \). Given any \( \varepsilon > 0 \), choose integer \( K \) so that \( \left| \sum_{k=1}^{K} a_k x_k \right| < \varepsilon \) for any \( k \geq K \). Let \( i > K \), then \[ h_n(x - \sum_{k=1}^{i} x_k e_k) = \sup_{r=1}^{i} \left| \sum_{k=1}^{r} a_k x_k \right| \] for any \( k \geq K \). Hence \[ \lim_{i \to \infty} h_n(x - \sum_{k=1}^{i} x_k e_k) = 0. \] By theorem 1.2, \( \{ x^k \}_{k=1}^{\infty} \) is a Schauder basis.

The particular type of topological space known as an FK space and introduced by Zeller has played an increasingly important role in summability. As examples of FK spaces we mention the spaces \( c_A \) and \( d_A \). The general form of continuous linear functionals on \( c_A \) can be obtained from the general theory of FK spaces and this has numerous applications in summability theory. The details can be found in 11.3 and 12.4 of [7].

Definition: A complete linear metric space is called a Fréchet space.

Definition: Let \( H \) be a Hausdorff space and a linear space. An FH space is a Fréchet space such that

1) \( X \) is a linear subspace of \( H \).

2) The topology of \( X \) is stronger than that of \( H \).

The special kind of FH spaces when \( H = \mathbb{R} \) with the norm \[ \|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{1+|x_n|} \] are called FK spaces.

Definition: Let \( X, Y \) be topological spaces and \( f : X \to Y \) be a function, then \( f \) is said to be closed if the graph \[ \{(x, f(x)) \mid x \in X\} \] is closed in \( X \times Y \) with the product topology.

Theorem 1.8. Let \( X, Y \) be topological spaces, \( f : X \to Y \) be
continuous and \( Y \) be Hausdorff, then \( f \) is closed.

Proof: See p.195 of [7].

It is clear that if \( f \) is closed and the topology on \( Y \) is replaced by a stronger topology then \( f \) remains closed.

Theorem 1.9. (The Closed-Graph Theorem) Let \( X, Y \) be Fréchet spaces and \( f:X \to Y \) be a closed linear map, then \( f \) is continuous.

Proof: See p.200 of [7].

Theorem 1.10. Let \( X \) be a Fréchet space, \( Y \) be an FH space with respect to some \( H \) and \( f:X \to Y \) a linear function, then \( f \) is continuous if and only if it is continuous as a function from \( X \) to \( H \).

Proof: If \( f:X \to Y \) is continuous, then the topology of \( Y \) is stronger and \( f(X) \subseteq Y \) imply that \( f:X \to H \) is continuous. Conversely, if \( f:X \to H \) is continuous then \( f \) is closed, by Theorem 1.8., hence \( f:X \to Y \) is closed, by Theorem 1.9., it is continuous.

Corollary 1.11. Let \( X,Y \) be FH spaces with respect to the same \( H \), \( X \subseteq Y \), then the topology of \( X \) is stronger than that of \( Y \), in particular a linear space of \( H \) has at most one topology that makes it an FH space.

Proof: Let \( i \) be the inclusion map \( i:X \to H \), then \( i \) is continuous since the topology on \( X \) is stronger than the subspace topology on it. Hence by Theorem 1.10., \( i:X \to Y \) is continuous and the result follows.

Proposition 1.12. In Corollary 1.11. the topology on \( X \)
is strictly stronger than the subspace topology if and only if \( X \) is not closed in \( Y \).

**Proof:** If \( X \) is closed in \( Y \) then the subspace topology is complete, hence \( X \) is an FH space with the subspace topology. By Corollary 1.11, the topology on \( X \) is the same as the subspace topology. Conversely, suppose the two topologies on \( X \) are the same, then the subspace topology is complete and hence \( X \) is closed in \( Y \).

**Corollary 1.13.** Let \( X \) be a Fréchet space, \( Y \) an FK space, \( f:X \to Y \) a linear function, then \( f \) is continuous if and only if \( f(x) = \{ f_n(x) \} \) where each \( f_n \) is a continuous linear functional on \( X \).

**Proof:** Recall that the norm on \( s \) is defined by

\[
\|x\|_s = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{1+|x_n|}
\]

hence the coordinate projections \( P_n(x) = x_n \) are continuous. Now if \( f \) is continuous as a mapping from \( X \) to \( Y \) then \( f:X \to s \) is continuous hence if we let \( f_n(x) = P_n f(x) \), we have \( f(x) = \{ f_n(x) \} \) with each \( f_n \) continuous. Conversely let \( d(u,v) \) be the metric of \( X \), if \( d(u_n,u) \to 0 \) then \( f_i(u_n,u) \to 0 \) \((i=1,2,3,\ldots)\), hence

\[
\sum_{i=1}^{\infty} \frac{1}{2} \frac{|f_i(u_n-u)|}{1+|f_i(u_n-u)|} \to 0,
\]

hence \( f:X \to s \) is continuous.

**Corollary 1.14.** Let \( A \) be an infinite matrix and \( X,Y \) be FK spaces. If for every \( x \in X \), \( Ax \) exists and belongs to \( Y \), then \( A \), considered as a mapping from \( X \) to \( Y \), is continuous.

**Proof:** Consider \( \sum_{k=1}^{\infty} a_{nk} x_k \), the \( n \)th coordinate of \( Ax \), by Corollary 1.13, it suffices to show that \( \sum_{k=1}^{\infty} a_{nk} x_k \) is a continuous linear functional on \( X \). For this we define
Define \( f: X \rightarrow \mathbb{C} \) by \( f(x) = (f_1(x), \ldots, f_m(x), \ldots) \). Now \( X \) is an FK space, hence convergence in \( X \) implies coordinate-wise convergence, thus it also implies convergence in \( s \), therefore \( f: X \rightarrow s \) is continuous. By Theorem 1.10, \( f: X \rightarrow \mathbb{C} \) is continuous. Now in (1.3) let \( t_n = o, (n=1,2,\ldots) \) and \( f(i) = 1 \), then it follows that for every \( (x_1, \ldots, x_n, \ldots) \) in \( c \), 
\[
\lim_{n \to \infty} x_n \text{ is a continuous linear functional on } c.
\]
Thus \( \sum_{k=1}^{\infty} a_k x_k \) is a continuous linear functional on \( X \) since it is the composite of \( f \) and \( \lim_{n \to \infty} x_n \).

**Theorem 1.15.** Let \( X,Y \) be FK spaces with their topologies generated by the families of seminorms \( (q_L)_{l \in \Lambda} \) and \( (r_{\lambda})_{\lambda \in \Lambda} \) respectively. Let \( f: X \rightarrow s \) be a continuous linear map. Then \( f^{-1}(Y) \) with the linear topology generated by \( (q_L)_{l \in \Lambda} \) and \( (r_{\lambda} \circ f)_{\lambda \in \Lambda} \) is an FK space and \( f: f^{-1}(Y) \rightarrow Y \) is continuous.

**Proof:** \( f^{-1}(Y) \) is clearly a linear subspace of \( s \) and the topology generated by \( (q_L)_{l \in \Lambda} \cup (r_{\lambda} \circ f)_{\lambda \in \Lambda} \) is stronger than the subspace topology relative to \( X \), hence stronger than that relative to \( s \). Now let \( \{ x^n \} \) be a Cauchy sequence in \( f^{-1}(Y) \), then it is a \( (q_L)_{l \in \Lambda} \) Cauchy sequence in \( X \), hence \( x^n \rightarrow x \) in \( X \), on the other hand \( \{ f(x^n) \} \) is a Cauchy sequence in \( Y \) hence \( f(x^n) \rightarrow y \) in \( Y \), but \( f \) is continuous as a mapping from \( X \) to \( s \), hence \( f(x^n) \rightarrow f(x) \) in \( s \), but the topology on \( Y \) is stronger than the subspace topology, hence \( f(x^n) \rightarrow y \) in \( s \), therefore \( f(x) = y \), so \( x \in f^{-1}(Y) \), hence the space \( f^{-1}(Y) \) is complete.
Proposition 1.16. Under the assumption of Theorem 1.15, if \( f \) is one-one onto \( Y \), then the linear topology generated by \( (r_{\lambda} \circ f)_{\lambda \in \Lambda} \) alone is an FK space.

Proof: If \( f \) is one-one onto \( Y \), then \( f: f^{-1}(Y) \rightarrow Y \) is a congruence onto where \( f^{-1}(Y) \) has the topology generated by \( (r_{\lambda} \circ f)_{\lambda \in \Lambda} \). Now \( Y \) is an FK space, hence \( f^{-1}(Y) \) is an FK space.

Lemma 1.17. \( d_{\lambda} \) is an FK space for any matrix \( A \).

Proof: For the \( m \)th row of \( A \) define \( D_m = \{ x | \sum_{k=1}^{\infty} a_{mk} x_k \text{ exists} \} \), then \( D_m \) with the seminorms \( \| P_n \| \) and \( h_m = \sup \{ \sum_{k=1}^{\infty} a_{mk} x_k \| \} \) \( r=1,2,\ldots \) is an FK space, for we can let \( X = s, Y = c \) in Theorem 1.15, and let \( f \) be defined by the matrix

\[
A_m = \begin{pmatrix}
a_{m1}^{0} & \cdots \\
a_{m1}a_{m2}^{0} & \cdots \\
a_{m1}a_{m2}a_{m3}^{0} & \cdots \\
\cdots & \cdots \\
a_{m1} & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

then \( f \) is continuous by Corollary 1.14 but clearly \( c_{A_m} = D_m \) and the seminorm \( h_m \) is just the composite of the usual norm on \( c \) and \( f \), hence \( D_m \) is an FK space by Theorem 1.15. Now \( d_A = \bigcap_{m} D_m \), and \( \{ \| P_n \| n=1, \cdots \} \cup \{ h_n \| n=1,2,3,\ldots \} \) generate the topology on \( d_A \) hence the topology on \( d_A \) is clearly stronger than the subspace topology relative to \( s \) since it is stronger than that relative to \( D_m \) for any \( m \). Let
\{x^n\} be a Cauchy sequence in \( d_A \) then \( \{x^n\} \) is a Cauchy sequence in \( D_m \) for each \( m \), let it converge to \( y_m \) in \( D_m \), also \( \{x^n\} \) is a Cauchy sequence in \( s \), hence it converges to \( x \), but then \( x=y_1=y_2=\ldots=y_m \), hence \( x\in \bigcap_{m} D_m \), therefore \( d_A \) is complete, \( d_A \) is clearly a linear subspace of \( s \), hence it is an FK space.

**Theorem 1.18.** Let \( A \) be a matrix, then \( c_A \) with the linear topology generated by the seminorms on \( d_A \) and the seminorm \( P(x)=\sup \{ |\sum_{k=1}^{\infty} a_{nk}x_k| \mid n=1,\ldots\} \) is an FK space.

**Proof:** In Theorem 1.15., let \( X=d_A, Y=c \), \( f \) be defined by \( A \), then \( f \) is continuous by Corollary 1.14., now \( c_A=f^{-1}(c) \) and \( P(x) \) is the composite of \( f \) and the usual norm on \( c \) hence by Theorem 1.15., \( c_A \) is an FK space.

**Definition:** A matrix \( A \) is said to be reversible if it is a one-one onto mapping from \( c_A \) to \( c \).

**Proposition 1.19.** Let \( A \) be reversible, then \( c_A \) is an FK space with the seminorm \( P(x)=\sup \{ |\sum_{k=1}^{\infty} a_{nk}x_k| \mid n=1,2,\ldots\} \).

**Proof:** Follows from Theorem 1.18. and Proposition 1.16.

**Lemma 1.20.** Let \( q_1,q_2 \) be seminorms on a linear space \( V \) and \( f \) be a linear functional on \( V \) such that

\[ |f(v)|\le q_1(v)+q_2(v) \]

then there exist linear functionals \( f_1,f_2 \) on \( V \) such that

\[ |f_1(v)|\le q_1(v), |f_2(v)|\le q_2(v) \quad \text{and} \quad f(v)=f_1(v)+f_2(v). \]

**Proof:** Define \( q:V\times V\rightarrow R^+ \) (the positive reals) by \( q(v_1,v_2)=q(v_1)+q(v_2) \), on the diagonal subspace \( \{(v,v)\mid v\in V\} \) of \( V\times V \), define \( g(v,v)=f(v) \), then \( g(v,v) \) is a linear functional.
and q is a seminorm on $V \times V$, now $g(v, v) = f(v)q_1(v) + q_2(v) = q(v, v)$, hence by Theorem 1.7., $g$ can be extended to $V \times V$ with $|g(v_1, v_2)| \leq q_1(v_1) + q_2(v_2)$, let $f_1(v) = g(v, o), f_2(v) = g(o, v)$, then $|g(v, o)| = |f_1(v)| \leq q_1(v) + o$, similarly $|g(o, v)| = |f_2(v)| \leq q_2(v)$, clearly $f(v) = g(v, o) + g(o, v) = f_1(v) + f_2(v)$.

**Theorem 1.21.** Let $X, Y$ be FK spaces with their topologies generated by the families of seminorms $(q_\iota)_{\iota \in I}$ and $(r_\lambda)_{\lambda \in A}$ respectively. Let $f: X \rightarrow Y$ be a continuous linear map and $f^{-1}(Y)$ has the linear topology generated by $(q_\iota)_{\iota \in I}$ and $(r_\lambda \circ f)_{\lambda \in A}$. If $g$ is a continuous linear functional on $f^{-1}(Y)$, then there exists $F \in X', G \in Y'$ such that $g = F + G \circ f$.

**Proof:** If $g$ is a continuous linear functional, then $|g(x)|$ is a continuous seminorm, hence by Theorem 1.3.iv) there exists $M$ and seminorms in $(q_\iota)_{\iota \in I} \cup (r_\lambda \circ f)_{\lambda \in A}$ such that

$$|g(x)| \leq M \max \{q_1(x), \ldots, q_n(x), r_1 \circ f(x), \ldots, r_m \circ f(x)\}$$

$$\leq M(q_1(x) + \ldots + q_n(x) + r_1 \circ f(x) + \ldots + r_m \circ f(x)).$$

we may assume that $M(q_1 + \ldots + q_n) \in (q_\iota)_{\iota \in I}$ and $M(r_1 \circ f + \ldots + r_m \circ f) \in (r_\lambda \circ f)_{\lambda \in A}$ since adding these seminorms to $(q_\iota)_{\iota \in I}$ and $(r_\lambda \circ f)_{\lambda \in A}$ does not change the topology on $f^{-1}(Y)$, hence $|g(x)| \leq q(x) + r \circ f(x)$ where $q \in (q_\iota)_{\iota \in I}$ and $r \in (r_\lambda \circ f)_{\lambda \in A}$.

By Lemma 1.20, there exist $F \in X'$ and $F_1 \in X'$ such that $g = F + F_1$ and $|F| \leq q, |F_1| \leq r \circ f$. Define $G$ on $f(X) \cap Y$ by $G(y) = F_1(x)$ if $y = f(x)$, if $y = f(x_1) = f(x_2)$, then $|F_1(x_1) - F_1(x_2)| = |F_1(x_1 - x_2)| \leq r \circ f(x_1 - x_2) = r(o) = 0$, hence $G$ is well-defined, by Theorem 1.4., $G$ can be extended to $Y$, by construction
of G we have \( g = F + G \).

1.3. Infinite Matrices.

Definition: A matrix \( A \) is said to be conservative if 
\( c_A \geq c \), that is, it transforms convergent sequences into convergent sequences.

Theorem 1.22. (Kojima-Schur) A matrix \( A \) is conservative if and only if

i) \( \|A\| = \sup\{\sum_{k=1}^{m} |a_{nk}| \mid n = 1, 2, \ldots \} < \infty \) and

ii) \( c_A \geq \{1! \sum_{k=1}^{m} a_k \mid k = 1, 2, \ldots \} \).

Proof: Suppose 1), ii) hold, then \( \lim_{n \to \infty} a_{nk} = \lim_{n \to \infty} A(\delta^k) \) exists for all \( k \). Let \( a_k = \lim_{n} a_{nk} \) and \( \|A\| \leq M \) then

\[ \sum_{k=1}^{m} |a_{nk}| \to \sum_{k=1}^{m} |a_k| \text{ for any finite } m \text{ and } M \sum_{k=1}^{m} |a_{nk}| \text{ imply that} \]

\[ \sum_{k=1}^{m} |a_k| \leq M \quad (1.6). \]

Now if \( x = (x_1, \ldots, x_k, \ldots) \) and \( \lim_{n} x_k = a \), write \( x_k = a + \epsilon_k \), hence for any \( \epsilon > 0 \), \( \exists N(\epsilon) \) such that \( k > N(\epsilon) \) implies \( |\epsilon_k| < \frac{\epsilon}{3M} \)

for \( n > N(\epsilon) \) choose \( N_1 \) great enough so that \( \sum_{k=1}^{N(\epsilon)} (a_{nk} - a_k) \epsilon_k \leq \frac{\epsilon}{3} \) for \( n > N_1 \), then

\[ \left| \sum_{k=1}^{N(\epsilon)} (a_{nk} - a_k) \epsilon_k \right| \leq \sum_{k=1}^{N(\epsilon)} |a_{nk} - a_k| \epsilon_k \leq \frac{\epsilon}{3} + \frac{M}{3} \epsilon = \epsilon, \text{ for } n > N_1 \]

Therefore \( \lim_{n} \sum_{k=1}^{N(\epsilon)} a_{nk} \epsilon_k = \sum_{k=1}^{N(\epsilon)} a_k \epsilon_k \). Now \( (Ax)_n = \sum_{k=1}^{N(\epsilon)} a_{nk} (a + \epsilon_k) = a \sum_{k=1}^{N(\epsilon)} a_{nk} + \sum_{k=1}^{N(\epsilon)} a_{nk} \epsilon_k \), but \( \epsilon \in c_A \), hence \( \lim_{n} \sum_{k=1}^{N(\epsilon)} a_{nk} = b \) exists.

Therefore \( \lim_{n \to \infty} (Ax)_n = ab + \sum_{k=1}^{N(\epsilon)} a_{nk} \epsilon_k \), hence \( c_A \geq c \). To prove the converse, we apply the Banach-Steinhaus Theorem twice. For any \( n \) define a sequence
$\{f_m\}$ of functional on $c$ by
\[ f_m(x) = \sum_{k=1}^{m} a_{nk} x_k \quad , (m=1,2,\ldots) . \]
Then $\{f_m\}$ is a sequence of continuous linear functionals on $c$ since convergence in $c$ implies coordinate-wise convergence. Now by definition $\|f_m\| = \sup\{ \| \sum_{k=1}^{m} a_{nk} x_k \| : \|x\|_1 = 1 \}$, hence $\|f_m\| = \sum_{k=1}^{m} |a_{nk}|$, conversely we can let $x$ be the sequence $(e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_m}, 0,0,0,\ldots)$ where $\theta_1, \ldots, \theta_m$ are the arguments of $a_{n1}, \ldots, a_{nm}$ respectively, then $\|x\|_1 \leq 1$ and $\|f_m(x)\| = \sum_{k=1}^{m} |a_{nk}| \leq \|f_m\|$, thus $\|f_m\| = \sum_{k=1}^{m} |a_{nk}|$. The sequence $\{f_m\}$ is pointwise convergent hence pointwise bounded by Theorem 1.6, $\{\|f_m\| : m=1,\ldots\}$ is bounded, hence $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for any $n$. Now for any $n, \varepsilon_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ defines a continuous linear functional on $c$ by Theorem 1.6, again $\|\varepsilon_n\| = \sum_{k=1}^{\infty} |a_{nk}|$, now $\{\varepsilon_n\}$ considered as seminorms on $c$ is pointwise bounded, hence $\{\sum_{k=1}^{\infty} |a_{nk}| : n=1,2,\ldots\}$ is bounded.

Definition: A matrix $A$ is said to be regular if for all $x \in c$, we have $\lim_{n} (Ax)_n = \lim_{n} x_n$.

Theorem 1.23. (Toeplitz-Silverman) A matrix $A$ is regular if and only if

1) $\|A\| < \infty$,

2) $\lim_{n} a_{nk} = 0$, for each $k$

3) $\lim_{n} \sum_{k=1}^{\infty} a_{nk} = 1$.

Proof: Suppose $A$ is regular then $c_A \supseteq c$ hence 1) follows from Theorem 1.23., $\lim \delta_k = 0$ for each $k$ hence $\lim A \delta_k = \lim a_{nk} = 0$, $\lim i = 1$ hence $\lim A i = \lim \sum_{k} a_{nk} = 1$. The converse follows from (1.7).
Definition: Let $l_A = \{ x \in l_1 : Ax \in l_1 \}$, then a matrix $A$ is said to be an $l_1$-$l_1$ method if $l_A \supseteq l_1$.

In Theorem 1.15, let $X = d_A, Y = l_1$, $f$ be defined by $A$ then $l_A$ becomes an FK space. Theorem 1.25, concerning $l_1$-$l_1$ methods is due to Mears, Knopp and Lorentz. (See Satz 1. of [5]).

Lemma 1.24: The space $l_1$ has $\{ 8^k | k = 1, 2, 3, \ldots \}$ as Schauder basis.

Proof: See p.86 of [7].

Theorem 1.25: A matrix $A$ is an $l_1$-$l_1$ method if and only if there exists $M$ such that

$$\sum_n |a_{nk}| \leq M, \quad k = 1, 2, 3, \ldots$$

Proof: Suppose $l_A \supseteq l_1$, then considering $A$ as a matrix transformation from $l_1$ to $l_1$, it is continuous by Corollary 1.14., hence there exists $M$ such that $\| Ax \| \leq M \| x \|$, where the norm is the usual norm on $l_1$, hence $\| A 8^k \| = \sum_n |a_{nk}| \leq M \| 8^k \| = M$ for all $k$.

Conversely, let $x = (x_1, \ldots, x_n, \ldots) \in l_1$, then $x = \sum_n x_n 8^n$, by Lemma 1.24. Now if $A$ is column bounded then $\sum_n x_n A(8^n)$ is convergent in $l_1$. For given $\varepsilon > 0$, we may choose $N(\varepsilon)$ so that $\sum_{n=N(\varepsilon)}^{\infty} |x_n| < \frac{\varepsilon}{M}$ where $M$ is the bound of the columns, then for $i, j \geq N(\varepsilon)$ $\| x_i A(8^i) + \cdots + x_j A(8^j) \| \leq |x_i| M + \cdots + |x_j| M < \varepsilon$. Hence the partial sum of $\sum_n x_n A(8^n)$ form a Cauchy sequence, thus it is convergent in $l_1$. But the $n$th partial sum is just$(\sum_{k=1}^n a_1 k x_k, a_2 k x_k, \ldots, \ldots)$, therefore the limit of $\sum_n x_n A(8^n)$ must be $(\sum_{k=1}^\infty a_1 k x_k, \sum_{k=1}^\infty a_2 k x_k, \ldots)$ which is $Ax$, hence $Ax \in l_1$. 


For any two infinite matrices $A=(a_{ij}), B=(b_{ij})$, the product $AB$ is defined to be $(c_{ij})$ where $c_{ij} = \sum_k a_{ik} b_{kj}$, if each $c_{ij}$ exists. With this definition multiplication is not associative in general, this can be seen as follows, let $\sum_n b_n$ be a convergent series which has a rearrangement $\sum_n r_n$ that converges to a different limit, let $b_n = cf(n)$, where $f(n)$ is the rearrangement. Now let $B$ be the matrix $b_{ij}$ where $b_{ij} = b_i$ if $f(i) = j$, $b_{ij} = 0$ if $f(i) \neq j$, let $A$ and $C$ be the matrix whose elements are all equal to one, then the elements of $(AB)C$ are all equal to $\sum_n r_n$, whereas all elements of $A(BC)$ are $\sum_n b_n$.

**Definition**: A matrix $A$ is called a lower semi-matrix if for $j \geq i$ $a_{ij} = 0$.

**Proposition 1.26**: Lower semi-matrices are associative.

**Proof**: Let $(AB)C = (d_{ij})$, $A(BC) = (e_{ij})$, $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$, then for $j \geq i$ clearly $d_{ij} = e_{ij} = 0$ since both $(AB)C$ and $A(BC)$ are again lower semi-matrices. For $i < j$, we have $d_{ij} = \left( \prod_{k=j}^i a_{ik} b_{kj} \right) c_{jj} + \left( \prod_{k=j+1}^i a_{ik} b_{kj} \right) c_{j+1,j} + \ldots + a_{ij} c_{ij}$, this can be re-grouped to form $a_{ii} b_{jj} c_{jj} + a_{i2} (b_{j+1,j} c_{jj} + b_{j+1,j+1} c_{j+1,j}) + \ldots + a_{ij} (b_{ij} c_{jj} + \ldots + b_{ij} c_{ij}) = e_{ij}$, hence $(AB)C = A(BC)$.

**Definition**: A matrix is said to be row-bounded if there exists $M$ such that $\sum_{k=1}^\infty |a_{nk}| \leq M$ for all $n$.

**Proposition 1.27**: Row-bounded matrices are associative.

**Proof**: Let $A, B, C$ be row-bounded matrices, $(AB)C = (d_{ij})$, $A(BC) = (e_{ij})$, without loss of generality consider $d_{11}$ and
be carried on for any n hence we have
e_{11}=c_{11}(\sum_k a_{1k}b_{k1})+\cdots+c_{n1}(\sum_k a_{1k}b_{kn})+\cdots,e_{11}=
\sum_k b_{k1}c_{k1}+\cdots+a_{1m}(\sum_k b_{mk}c_{k1})=a_{11}(\sum_k b_{1k}c_{k1})+
\sum_k b_{mk}c_{k1}+\cdots, \text{ now } |b_{ml}c_{11}| m=1,2,\ldots\}
is bounded and \sum_m |a_{1m}|<\infty, therefore e_{11}=c_{11}(\sum_k a_{1k}b_{k1})+
a_{11}(\sum_k b_{1k}c_{k1})+\cdots+a_{1m}(\sum_k b_{mk}c_{k1})+\cdots, \text{ this step can}
be carried on for any \ n \ hence \ we \ have \ e_{11}=c_{11}(\sum_k a_{1k}b_{k1})+
\cdots+c_{n1}(\sum_k a_{1k}b_{kn})+a_{11}(\sum_{k=n+1} b_{1k}c_{k1})+\cdots+a_{1m}(\sum_{k=n+1} b_{mk}c_{k1})+\cdots\).
The last term tends to zero since all three matrices are
row-bounded, to see this we can choose \ m \ great enough so
that |a_{1m}(\sum_k b_{mk}c_{k1})+\cdots|<\frac{\epsilon}{2} \ for \ any \ n, \ then \ choose \ n
great enough using row-boundedness of \ B \ so \ that
|a_{11}(\sum_{k=n+1} b_{1k}c_{k1})+\cdots+a_{1m-1}(\sum_{k=n+1} b_{m-1,k}c_{k1})|<\frac{\epsilon}{2}. \ Therefore
\ e_{11}=d_{11}.

Definition: A matrix \ A \ is said to be normal if \ A \ is a
lower semi-matrix with non-zero diagonal elements.

Proposition 1.28. If \ A \ is normal then the equation
\Ax=\y \ with \ \x \ as \ unknown \ has \ a \ unique \ solution.

Proof: We have
\begin{align*}
a_{11}x_1 &= y_1 \\
a_{21}x_1 + a_{22}x_2 &= y_2
\end{align*}

\begin{align*}
\therefore x_1 &= \frac{y_1}{a_{11}} \\
x_2 &= \frac{y_2-a_{21}x_1}{a_{22}}
\end{align*}

Theorem 1.29. If the terms of a series \ \Sigma \ \rn \ are defined
by series, with \ \rn=\sum_k a_{nk} \ and \ \sum_k a_{nk}=s_k \ for \ each \ k, \ then
\sum_k |a_{nk}|=t_n \ and \ \sum t_n \ is \ convergent \ imply \ that \ \sum r_n = \sum s_k.

Proof: See p.241 of [4].
CHAPTER II
CONTINUOUS LINEAR FUNCTIONALS ON $\mathcal{C}_A$

**Lemma 2.1.** Let $g$ be a continuous linear functional on $d_A$ for an arbitrary matrix $A$, then $g((x_1,\ldots,x_n,\ldots)) = \sum_n^{\infty} x_n g(\delta^n)$ for all $(x_1,x_2,\ldots,x_n,\ldots)$ in $d_A$.

**Proof:** By Proposition 1.7, $\{\delta^n | n=1,2,\ldots\}$ is a Schauder basis for $d_A$. Hence $\sum_{n=1}^m x_n \delta^n \to x$ as $m \to \infty$, therefore $\sum_{n=1}^m x_n g(\delta^n) = g(\sum_{n=1}^m x_n \delta^n) = g(x)$ as $m \to \infty$, hence $g(x) = \sum_n^{\infty} x_n g(\delta^n)$.

**Theorem 2.2.** Let $A$ be a conservative matrix, $f \in \mathcal{C}_A$. Then $f$ may be expressed as

$$f(x) = \lim A x + \sum \beta x + \sum \beta x$$

where $\sum |\beta| < \infty$ and $\sum \beta x$ converges for all $x \in \mathcal{C}_A$.

**Proof:** By Theorem 1.8, and Lemma 1.17, $\mathcal{C}_A$ and $d_A$ are FK spaces. In Theorem 1.21, let $X = d_A$, $Y = c$, then by the same theorem every continuous linear functional $f$ on $\mathcal{C}_A$ can be expressed as $f = G \circ A + F$ with $G \in \mathcal{C}'$ and $F \in d_A'$. By (1.3) and Lemma 2.1, we may take $G(x) = \sum x t + \sum x \beta t$ and $F(x) = \sum x \beta t$, where $\beta = F(\delta_n)$, hence $G \circ A = \lim A x + \sum t \beta x$ and the result follows.

In (2.1) let $x = \delta^k, (k=1,2,\ldots)$, then $f(\delta^k) = \sum a_k + \sum x t a r k + \beta k$ where $a_k = \lim a_n k$. Hence $\beta = f(\delta^k) - \sum a_k - \sum t a r k$ and

$$f(x) = \lim A x + \sum t \beta x + \sum [f(\delta^k) - \sum a_k - \sum t a r k] x_k$$

If $A$ is conservative, by Theorem 1.22, $\sum |a_k| < \infty$, hence $\sum a_k$ is convergent. We define

$$\chi(A) \equiv \lim a_k^{\infty} - \sum a_k = \lim a_n k - \sum a_n k$$

In Chapter 3 we will classify the conservative matrices.
by means of this number.

In Theorem 2.2, if A is also reversible, then by Proposition 1.19, c_A and c are congruent. Let A^{-1} be the inverse map of A, then f_{oA^{-1}} is a continuous linear functional on c since A^{-1}: c \rightarrow c_A is continuous. By (1.3), let f_{oA^{-1}} = \lim x + \sum_n x_n t_n$, hence

$$f_{oA^{-1}} = \lim x + \sum_n (Ax)_n t_n$$

(2.4).

A is a continuous linear transformation from c_A to c by Corollary 1.14, and the functional f(x) = \lim x is a continuous linear functional on c, hence their composite lim_{oA} x is a continuous linear functional on c_A. We also have the following result.

**Theorem 2.3.** If c_B \supseteq c_A, then \lim_B x is a continuous linear functional on c_A.

**Proof:** If c_B \supseteq c_A then we can consider B as a matrix transformation from c_B to c, it is linear and continuous by Corollary 1.14. Now lim x is a continuous linear functional on c, hence so is the composite lim_B x. The topology of c_A is not weaker than the subspace topology relative to c_B, hence lim_B x \subseteq c_A.

**Definition:** A conservative matrix A is said to be multiplicative m if for any x \in c, \lim_A x = m \lim x.

**Proposition 2.4.** A matrix A is multiplicative m if and only if a_k = \lim_n a_{nk} = 0 for all k.

**Proof:** By Theorem 2.3, \lim_A x is a continuous linear functional on c where c is considered as c_T, then by (1.3)
\[ \lim_{A} x = (\lim_{A}(i) - \sum_{k} \lim_{A}(g^k)) \lim_{x} + \sum_{k} x_{k} \lim_{A}(g^k) = \chi(A) \lim_{x} + \sum_{k} x_{k} a_{k}, \]

but \( \chi(A) = m \) if \( A \) is multiplicative \( m \), hence \( a_{k} = 0 \) for all \( k \).

Conversely if \( a_{k} = 0 \) for all \( k \) then \( \lim_{A} x = (\lim_{A} i) \lim_{x} \) hence \( A \) is multiplicative.

For any continuous linear functional \( f \) on \( c_{A} \) where \( A \) is conservative, \( \sum_{k} f(g^k) \) is convergent because we can consider \( f \) as a continuous linear functional on \( c \) then

\[ \sum_{k} |f(g^k)| < \infty \text{ by (1.5), we define} \]

\[ \chi(f) \equiv f(i) - \sum_{k} f(g^k) \]

(2.5).

**Proposition 2.5.** If \( A \) is conservative, \( f \) is in \( c_{A} \), and \( f \) is represented as in (2.1), then \( \chi(f) = \chi(A) \).

**Proof:** \( f(i) = \lim_{A} i + \sum_{r} t_{r} (\sum_{k} a_{r k}) + \sum_{r} \beta_{r} \), \( \sum_{n} f(g^n) = \sum_{n} \lim_{A} g^n + \sum_{k} (\sum_{r} t_{r} a_{r k}) + \sum_{r} \beta_{r} \) hence \( f(i) - \sum_{n} f(g^n) = \chi(A) + \sum_{r} t_{r} (\sum_{k} a_{r k}) - \sum_{k} (\sum_{r} t_{r} a_{r k}), \) now \( A \) is a conservative matrix hence row-bounded by Theorem 1.23., \( \sum_{r} |t_{r}| < \infty \), hence by Theorem 1.29.

\[ \sum_{r} t_{r} (\sum_{k} a_{r k}) = \sum_{k} (\sum_{r} t_{r} a_{r k}), \] therefore \( \chi(f) = \chi(A) \).
Definition: A conservative matrix $A$ is said to be co-regular if $\chi(A) \neq 0$, co-null if $\chi(A) = 0$.

The above definition is due to Wilansky (p. 61 of [9]).

Example 3a. The process of taking the arithmetic mean or Cesàro mean can be represented by the conservative matrix

$$A = \begin{pmatrix}
1 & 0 & 0 & \ldots & \ldots \\
\frac{1}{2} & \frac{1}{2} & 0 & \ldots & \ldots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \ldots \\
\frac{1}{n} & \ldots & \frac{1}{n} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}$$

$A$ is co-regular since $\chi(A) = 1$. In fact, $A$ is regular, and it follows from ii) and iii) of Theorem 1.23. that regular matrices are co-regular.

Example 3b. The conservative matrix

$$A = \begin{pmatrix}
1 & 0 & 0 & \ldots & \ldots \\
0 & \frac{1}{2} & 0 & \ldots & \ldots \\
0 & -1 & \frac{1}{3} & 0 & \ldots \\
0 & -1 & -\frac{1}{4} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}$$
is co-null since \(\lim_{n} \sum_{k} a_{nk} = 0\) and \(\lim_{n} a_{nk} = 0\) for each \(k=1,2,\ldots\).

If \(A\) is multiplicative zero, then \(\lim_{A} A^k = 0\) and \(\lim_{A} A^k = 0\) for all \(k\). Hence \(A\) is co-null. Thus every multiplicative zero matrix is co-null.

For a conservative matrix \(A\), we define
\[
W_{A} = \{x \in c_{A} | f(x) = \sum_{n} x_{n} f(\varepsilon^{n}), \text{ for all } f \in c_{A}' \}
\]

**Proposition 3.1.** A conservative matrix \(A\) is co-null if and only if \(i \in W_{A}\).

**Proof:** If \(i \in W_{A}\), consider \(f(x) = \lim_{A} Ax\). \(f(i) = \sum_{n} f(\varepsilon^{n})\) implies that \(A\) is co-null. Conversely, if \(A\) is co-null, then every \(f \in c_{A}'\) we have \(f(i) = \lim_{A} f(\varepsilon^{n}) = 0\), hence \(f(i) = \sum_{n} f(\varepsilon^{n})\), thus \(i \in W_{A}\).

**Corollary 3.2.** A conservative matrix \(A\) is co-null if and only if for every \(f \in c_{A}\), \(f(\sum_{n=1}^{k} \varepsilon^{n}) \rightarrow f(i)\) as \(k \rightarrow \infty\), that is, \(\sum_{n=1}^{k} \varepsilon^{n}\) converges weakly to \(i\) in \(c_{A}\).

**Proof:** This follows immediately from Proposition 3.1.

From Corollary 3.2, it follows that we can regard coregularity as a property of \(c_{A}\) rather than the matrix \(A\). This was done by Snyder, A.K. (Math. Z. 90, 1965, 376-381)

**Proposition 3.3.** If \(c\) is closed in \(c_{A}\), then \(A\) is co-regular.

**Proof:** If \(c\) is closed in \(c_{A}\), by Proposition 1.12., the subspace topology and the usual topology on \(c\) are equivalent, if \(A\) is co-null then every continuous linear functional that vanishes on \([\varepsilon^{n}]_{n=1,2,\ldots}\) must vanish at \(i\). In \(c\) consider the subspace \(V_{1}\) generated by
\{e_n \mid n=1, \ldots \} \text{ and the vector } i, \text{ clearly } d(i, V_1) = \inf \{ \| i - v \| \mid v \in V_1 \} < 1, \text{ hence } i \notin V_1 \text{ (in } c_A). \text{ Thus by Theorem 1.5, there is a continuous linear functional } f \in c_A \text{ satisfying the condition } f(V_1) = 0 \text{ and } f(i) \neq 0. \text{ This is a contradiction, hence } A \text{ cannot be co-null.}

The converse of the above Proposition is not true, to see this we consider the arithmetic mean in Example 3.1. This matrix is a reversible matrix by Proposition 1.28. Hence \( c_A \text{ and } c \text{ are congruent under } A. \) Now let \( \{x^n\} = \{(-1,0,0,\ldots),(-1,1,0,0,\ldots),(-1,1,-1,0,\ldots),\ldots\} \), then \( \{x^n\} \subseteq c, \text{ let } x = \{(-1)^n\}, \text{ then } Ax^n \to Ax \text{ but } x \notin c \text{ hence } Ax \notin A(c). \) Therefore \( A(c) \) is not closed in \( c \) hence \( c \) is not closed in \( c_A. \)

**Theorem 3.4.** If \( A, B \) are conservative matrices and \( c_A = c_B, \) then both \( A \) and \( B \) are co-regular or both are co-null.

**Proof:** By Theorem 2.3., \( \lim_{A \in c_B} x \in c_A \) and \( \lim_{B \in c_A} x \in c_A \), by Proposition 2.5. \( \chi(A) = d_1 \chi(B) \) and \( \chi(B) = d_2 \chi(A) \) for some \( d_1, d_2, \) hence \( \chi(A) \) and \( \chi(B) \) are both non-zero or both zero. This completes the proof.

The above theorem shows that co-regularity is a property that depends on the summability field \( c_A \) alone and not the matrix \( A. \)

**Proposition 3.5.** If \( A, B \) are conservative matrices and \( c_A \subseteq c_B, \) then \( A \) is co-null implies that \( B \) is also co-null.

**Proof:** By Theorem 2.3., \( \lim_{c_A} x \in c_A \), by Proposition 2.5. \( \chi(B) = d \chi(A) \) hence the result follows.
We now turn to the study of the "size" of the summability field \( c_A \). We will first assume that \( A \) is co-regular.

**Theorem 3.6.** (Steinhaus) If \( A \) is a regular matrix, then \( c_A \neq \mathbb{M} \).

**Proof:** By Theorem 1.23, we have 1) \( \sum_j |a_{ij}| < M \) for some \( M \) and for all \( i \), ii) \( \lim_{i \to \infty} a_{ij} = 0 \) for all \( j \) and iii) \( \sum_j a_{ij} = A_i \to 1 \).

We will construct a sequence \( x \) that consists of o's and 1's such that \( A x \) is not convergent. By iii) choose \( i_1 \) so that
\[
|\sum_j a_{i_1,j}| > \frac{3}{4},
\]
by i) choose \( j_1 \) so that \( \sum_{j=j_1}^{j_1+1} |a_{i_1,j}| < \frac{1}{12} \), for \( 1 \leq j \leq j_1 \) let \( x_n = 1 \) then \( (Ax)_1 = \sum_{j=1}^{j_1} a_{i_1,j} + \sum_{j=j_1+1}^{\infty} a_{i_1,j} x_j \)
\[
= \sum_{j=j_1+1}^{\infty} a_{i_1,j} x_j - \sum_{j=1}^{j_1} a_{i_1,j} + \sum_{j=j_1+1}^{\infty} a_{i_1,j},
\]
\[
> \frac{3}{4} - \frac{1}{12} = \frac{2}{3}.
\]
Now choose \( i_2 > i_1 \) by ii) so that \( \sum_{j=j_1}^{j_1+1} |a_{i_2,j}| < \frac{1}{6} \), choose \( j_2 > j_1 \) so that \( \sum_{j=j_2+1}^{\infty} |a_{i_2,j}| < \frac{1}{6} \) by i) for \( j_1 < n < j_2 \) let \( x_n = 1 \) then \( (Ax)_2 = \sum_{j=j_1}^{j_1+1} a_{i_2,j} + \sum_{j=j_1+1}^{\infty} a_{i_2,j} x_j \)
\[
= \sum_{j=j_1+1}^{\infty} a_{i_2,j} x_j - \sum_{j=1}^{j_1} a_{i_2,j} + \sum_{j=j_1+1}^{\infty} a_{i_2,j},
\]
\[
> \frac{2}{3} - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}.
\]
Continuing in this way we can construct \( x_n \) so that \( \{(Ax)_n\} \)
is divergent, hence \( \{x_n\} \) must be divergent.

**Theorem 3.7.** If \( A \) is co-regular then \( c_A \neq \mathbb{M} \).

**Proof:** Let \( A = (a_{nk}) \), consider \( B = (a_{nk} - a_k) \) where \( a_k = \lim_n a_{nk} \), then \( B \) is a multiplicative matrix, since \( \sum_k |a_k| < \infty \) by (1.6), we have \( c_A \neq \mathbb{M} = c_B \neq \mathbb{M} \), hence it suffices to show \( c_B \neq \mathbb{M} \). Now
\[
\lim_B i = \lim_n \sum_k (a_{nk} - a_k) = \lim_n \sum_k a_{nk} - \sum_k a_k = \rho(A) \neq 0
\]
hence \( B \) is
multiplicative \( P(A) \), thus \( \frac{1}{P(A)} \) B is regular and \( c_B = c_{\frac{1}{P(A)}} B \) by Theorem 3.6.

Example 3.c. Consider the arithmetic mean and the bounded sequence defined by the following rules

\[
\begin{align*}
    x_1 &= 1 \\
    x_n &= 0, & 1 < n \leq 3 \\
    x_n &= 1, & 3 < n \leq 3^2 \\
    x_n &= 0, & 3^2 < n \leq 3^3
\end{align*}
\]

The sequence is clearly bounded, but \((Ax)_1 = 1, (Ax)_3 = \frac{1}{3}\), \((Ax)_{3^2} \geq \frac{2}{3}\), \((Ax)_{3^3} \leq \frac{1}{3}\), ..., \((Ax)_{3^{2n}} \geq \frac{2}{3}\), \((Ax)_{3^{2n+1}} \leq \frac{1}{3}\), hence the sequence \( x \) is not in \( c_A \).

For a co-regular matrix \( A, c_A \) may be a proper subset of \( m \), for example if \( A = I \), the identity matrix. However, the next main result (Theorem 3.10.) tells us that whenever a co-regular matrix sums a divergent bounded sequence, \( c_A \) is not a subset of \( m \).

Lemma 3.8. If \( A \) is a co-regular matrix, then in \( c_A, c_{\geq c_A} \cap m \).

Proof: Consider \( c \) as a linear subspace of \( c_A \cap m \), by Theorem 1.5, it suffices to show that every continuous linear functional that vanishes on \( c \) must vanish on \( c_A \cap m \). Let \( f \in c_A \) and \( f(c) = 0 \), then in the representation (2.1), \( x = 0 \), because \( \Lambda(A) \neq 0 \), \( \Lambda(f) = 0 \) and \( \Lambda(f) = x(\Lambda(A)) \), also \( f(\delta^k) = 0 \) for all \( k \), hence by (2.2).

\[
f(x) = \sum_{r} t_r(A_{r}) = \sum_{k} (\sum_{r} t_r a_{rk}) x_k
\]

But \( \sum_{r} t_r(A_{r}) \) may be considered as \( t(Ax) \) where \( t \) is the
matrix whose first row is \((t_1, \ldots, t_r, \ldots)\) and other rows are zero, \(x\) may be considered as the matrix whose first column is
\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_r
\end{pmatrix}
\]
and other columns are zero and \(\sum_k (\sum_T t_{rk} a_{rk}) x_k\) may be considered as \((TA)x\) in the same way. Now all three matrices \(T, A\) and \(x\) are row bounded if \(x \in C A \cap m\), hence \(t(Ax) = (TA)x\) by Proposition 1.27, hence \(f(CA \cap m) \ni 0\).

**Lemma 3.9.** If \(CA \subseteq m\), then \(CA\) is closed in \(m\).

**Proof:** Recall that \(\sup_n |x_n|\) is the norm on \(m\). Let \(x \in \overline{CA}\) in \(m\), to show that \(x \in CA\) in \(m\) it suffices to show that \(Ax\) is a Cauchy sequence. For any \(\varepsilon > 0\), consider \(N(x, \varepsilon / 4M)\) where \(M = \sup_{i} |a_{ik}|\) (for if \(MA = 0\), \(A\) is the zero matrix, then \(s = CA \subseteq m\)), let \(y \in CA \cap N(x, \varepsilon / 4M)\) and \(N(\varepsilon)\) be an integer such that for \(m, n > N(\varepsilon)\) we have \(|A(y)_n - (Ay)_n| = |\sum_k (a_{mk} - a_{nk}) y_k| < \frac{\varepsilon}{2M}\). Let \(x_k = y_k + c_k\) where \(|c_k| < \frac{\varepsilon}{4M}\) by the choice of \(y\). Thus we have \(|(Ax)_n - (Ax)_n| = |\sum_k (a_{mk} - a_{nk}) (y_k + c_k)| \leq |\sum_k (a_{mk} - a_{nk}) y_k| + |\sum_k (a_{mk} - a_{nk})||c_k| < \frac{\varepsilon}{2M} < 2M, \frac{\varepsilon}{4M} = \varepsilon\).

Hence \(Ax\) is a Cauchy sequence and \(x \in CA\).

**Theorem 3.10.** If a co-regular matrix sums a bounded divergent sequence, it must sum an unbounded sequence.

**Proof:** Suppose \(CA \subseteq m\), then by Lemma 3.9, and Proposition 1.12, the usual topology on \(CA\) is the same as the subspace topology. But \(C\) is closed with respect to the usual topology.
of $m$, hence $c$ is closed in $c_A$. By Lemma 3.8., $c = \mathcal{C} = c_A \setminus m$, that is, $c$ is all the bounded sequences in $c_A$, this contradicts the assumption that $A$ sums a bounded divergent sequence, therefore $c_A \setminus m$ and the result follows.

Example 3.d. The arithmetic mean sums the bounded divergent sequence $\{(-1)^n\}$, it also sums the unbounded sequence $(1, -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ...)$.

There exist matrices that sum unbounded sequences but do not sum any bounded divergent sequence. It will be seen in the next chapter that such matrices must be co-regular. We give now an example of such a matrix.

Example 3.e. We will define a matrix $A$ whose diagonal elements are all equal to one. Construct a one-one correspondence $k$ from the positive integers into themselves by the following rules

\[ k(1) = 2^2, \quad k(2) = 2^3, \quad k(3) = 2^5, \ldots, \quad k(n+1) = 2^n k(n), \ldots \]

Let $A$ be the matrix whose diagonal elements are one and $a_n, k(n) = -\frac{n}{k(n)}$, the other elements are zero, then $A$ sums the sequence $(1, 2, 3, 4, \ldots)$ since $(Ax)_n = n - n = 0$. Now if $x$ is a bounded divergent sequence then $Ax$ is divergent, for otherwise $\lim_n (Ax)_n = \lim_n x_n - (\frac{n}{k(n)}) x_k(n)$ exists but $x$ is bounded and $\lim_n \frac{n}{k(n)} = 0$ hence $\lim_n x_n$ exists, this is a contradiction.

As for co-null matrices, we will see that every co-null matrix must sum a bounded divergent sequence hence an unbounded one in the next chapter.
CHAPTER 4

c AS A SUBSET OF cₐ

In the first part of this chapter we will study the conservative matrices that are also 1-1 matrices and relationships between c and 1ₐ. In the second part we will assume that c is closed in cₐ and study the consequences.

For a conservative matrix A, the conditions 1ₐ ⊆ c and c ⊆ 1ₐ may or may not hold. For example if A=I, the identity matrix, then 1ₐ = 1 ⊆ c = cₐ, but c ⊈ 1ₐ. If

\[
A = \begin{pmatrix}
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & \frac{1}{n} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

then (1, 2, 3, ...) ∈ 1ₐ but it is not in c, hence 1ₐ ⊈ c. Also, it is easy to see that c ⊈ 1ₐ.

Definition: A conservative matrix is said to be perfect if c is dense in cₐ.

Theorem 4.1. If A is perfect, an lₐ method and A(cₐ) = c, then 1ₐ ⊈ c.

Proof: The matrix A considered as a mapping from cₐ to c is continuous by Corollary 1.15., hence c = cₐ implies \(\overline{A(c)} = c\) since A is onto. Now if 1ₐ ⊈ c, then \(\overline{A(1ₐ)} = c\) with respect to the norm of c, but A(1ₐ) ⊈ 1ₐ since A is an
\textbf{l}_{1}-l_{1} \text{ method, hence } l_{1} \text{ is dense in } c. \text{ The last statement is not true because if we let } \xi = \frac{1}{2}, \text{ } x=(1,1,\ldots,1,\ldots), \text{ then } N(x,\varepsilon) \text{ contains no element of } l.

It is obvious that, for an arbitrary matrix } A, \text{ if } c \subseteq l_{A}, \text{ then } A \text{ is conservative and multiplicative zero. However for a conservative matrix } A \text{ which is also an } l_{1}-l_{1} \text{ method, } A \text{ multiplicative zero does not imply } c \subseteq l_{A}.

Consider

\begin{align*}
A &= \begin{pmatrix}
1 & 0 & \cdots \\
\frac{1}{2} & 1 & 0 & \cdots \\
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\end{align*}

the matrix } A \text{ is row-bounded, } \lim_{k \to \infty} A^{k} = 0 \text{ for all } k \text{ and } \lim_{k \to \infty} A^{1} = 0, \text{ hence by Theorem 1.22, } A \text{ is conservative. Also, } A \text{ is column-bounded, hence it is an } l_{r}-l_{1} \text{ method by Theorem 1.25. Now } j \in c \text{ and } \sum_{n=1}^{\infty} |(Ai)_{n}| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty, \text{ hence } j \notin l_{A}.

\textbf{Theorem 4.2.} If } A \text{ is an } l_{1}-1 \text{ method, then a necessary condition for } l_{A} \subseteq c \text{ is that for any subsequence } (r_{1}, r_{2}, \ldots, r_{i}\ldots) \text{ of the sequence } (1, 2, 3, \ldots) \text{ with } r_{i+1} < r_{i+1} \text{ for infinitely many } r_{i}:

\begin{equation}
\sum_{n=1}^{\infty} \left( \sum_{k=r_{1}, r_{2}, \ldots}^{\infty} a_{nk} \right) = \infty
\end{equation}

\textbf{Proof :} Suppose } l_{A} \subseteq c, \text{ for any such sequence } (r_{1}, r_{2}, \ldots). \text{ Construct a sequence } x \text{ whose } r_{i} \text{th term is } 1 \text{ and others are zero, then } x \text{ is a divergent sequence since } r_{i+1} < r_{i+1} \text{ for }
infinitely many i. Hence $x \notin l_A$, that is, (4.1) holds.

The condition (4.1) is not sufficient, for example, let

$$
A = \begin{pmatrix}
-1 & 0 & 0 & \cdots \\
-2 & 1 & 0 & \cdots \\
0 & -2 & 1 & 0 \\
\cdots & \cdots & \cdots & \ddots \\
\cdots & \cdots & \cdots & -2 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \ddots \\
\end{pmatrix}
$$

then $A$ is column-bounded hence an $l_\infty$ method, let $x = (1,2,4,8,...)$, then $Ax = (-1,0,0,0,\ldots)$, thus $x \in l_A$, but $x \notin c$.

In what follows we will study the condition that $c$ is closed in $c_A$, for this we arrange the seminorms that generate the linear topology on $c_A$ in the following manner:

$$q_0(x) \equiv p_0(x) = \sup_n \left| \sum_{k=1}^{m} a_{nk} x_k \right|$$

$$q_{2n-1}(x) \equiv h_n(x) = \sup_m \left| \sum_{k=1}^{m} a_{nk} x_k \right|$$

$$q_{2n}(x) \equiv |p_n(x)| = |x_n|$$

Recall that the locally convex linear topology on $c_A$ is generated by

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1+p_n(x)}$$

where $p_n(x) = \max_{0 \leq i \leq n} q_i(x)$.

Also recall that $c_A$ is complete; hence a series is convergent if the partial sums form a Cauchy sequence. The following interesting result is due to Wilansky and Zeller [10].

**Theorem 4.3.** For a conservative matrix $A$, $c$ is closed in $c_A$ if and only if $A$ sums no bounded divergent sequence,
that is, \( c_A \cap m \subseteq c \).

Proof: Suppose \( c \) is not closed in \( c_A \) and consider the subspaces

\[
V_K = \left\{ x \in c \mid x_k = 0, \ k \leq K \right\}, \ K = 0, 1, 2, \ldots
\]

These subspaces are not closed in \( c_A \); for suppose \( V_{K_0} \) is closed for some \( K_0 \) and let \( \{x^m\} \subseteq c \) converge to \( x \) in \( c_A \), furthermore, for each \( x^m = (x_1^m, x_2^m, \ldots, x_{K_0-1}^m, x_{K_0}^m, \ldots) \), let \( y^m = (x_1^m, \ldots, x_{K_0-1}^m, 0, 0, \ldots) \) and \( z^m = (0, \ldots, 0, x_{K_0}^m, x_{K_0+1}^m, \ldots) \), then \( x^m = y^m + z^m \) and \( z^m \in V_{K_0} \). Now \( q_i(x^m-x) \to 0 \) for all \( i \) by assumption, hence \( q_{2n}(y^m-y) \to 0 \); but each \( y^m \) has zero coordinate after the \( K_0 \)-th coordinate, hence \( q_0(y^m-y) \to 0 \) and \( q_{2n-1}(y^m-y) \to 0 \). Now let \( x = (x_1, \ldots, x_{K_0-1}, x_{K_0}, \ldots) = (x_1, \ldots, x_{K_0-1}, 0, 0, \ldots) + (0, \ldots, 0, x_{K_0}, x_{K_0+1}, \ldots) \) and let \( y = (x_1, \ldots, x_{K_0-1}, 0, 0, \ldots), z = (0, \ldots, 0, x_{K_0}, x_{K_0+1}, \ldots) \), then \( x = y + z \).

For any \( i \), \( q_i(z^m-z) = q_i(x^m-x) + q_i(y^m-y) \); hence \( q_i(z^m-z) \to 0 \). If \( V_{K_0} \) is closed in \( c_A \), then \( z \notin V_{K_0} \), hence \( x \notin c \) and \( c \) is closed in \( c_A \) — contradiction.

By Proposition 1.12., the usual topology on \( V_K \subseteq c_A \) is strictly stronger than the subspace topology relative to \( c_A \), hence the seminorm \( q(x) = \sup_n |x_n| \) is discontinuous with respect to the subspace topology by definition. By v) of Theorem 1.3., for any \( \xi > 0 \), any integers \( b, K \), there exists \( x \in V_K \) such that

\[
q(x) = 1 \quad (4.2)
\]

\[
p_K(x) < \xi \quad \text{for } k < b \quad (4.3)
\]
Case I. If $A$ is a co-regular matrix. We may assume $\mathcal{C}(A) = 1$, for otherwise we may consider $\frac{1}{\mathcal{C}(A)} A$; this matrix has the same summability field as $A$, also the identity map is a homeomorphism. Consider $\lim_A x$ as a continuous linear functional on $c$, and let $\lim_A x = \lim_{x \in c} x + \sum_k a_k x_k$, where $a_k = \lim_{n} a_{nk}$; since $\mathcal{C}(A) = 1$, by Proposition 2.5., $\mathcal{C} = 1$. By (1.6) $\sum_k |a_k| < \infty$. For any $\varepsilon > 0$, choose $K$ great enough so that $\sum_{k=K}^\infty |a_k| < \varepsilon$; by the preceding part, there exists $x \in V_K$ such that (4.2) and (4.3) hold; hence $|\lim_A x| < \varepsilon$, because $p_0(x) < \varepsilon$. Therefore $|\lim_{x \in c} x| = |\lim_A x + \sum_{K}^\infty a_k x_k| < |\lim_A x| + \sup_n |x_n| (\sum_{K}^\infty |a_k|)$, but $q(x) = \sup_n |x_n| = 1$, hence $|\lim_{x \in c} x| < |\lim_A x| + \sum_{K}^\infty |a_k| < \varepsilon + \varepsilon = 2\varepsilon$. If $\varepsilon < \frac{1}{2}$, then $|\lim_{x \in c} x| < 1$. Now $|x_n| < 1$ for $n$ sufficiently large, $x_n = 0$ for $n = 1, 2, \ldots, K-1$ and $\sup_n |x_n| = 1$, therefore there is a finite interval $N(x)$ of natural numbers such that $|x_n| < 1$ for $n \notin N(x)$ and $|x_n| = 1$ for some $n \in N(x)$.

Let $\varepsilon_r = 2^{-r-3}$, $r = 1, 2, 3, \ldots$ and $b = r$; for each $r$ choose $x^r \in V_{K_r}$ satisfying (4.2) and (4.3), furthermore, for each $r, r+1$, $K_{r+1}$ is chosen in such a way so that $N(x^r) r = 1, 2, \ldots$ are pairwise disjoint and that infinitely many natural numbers are not in any $N(x^r)$. We claim that $\sum x^r$ is a convergent series in $c_A$. For any $\varepsilon > 0$, choose $r$ so that $\sum_{n=r}^\infty \frac{1}{2^r} < \frac{\varepsilon}{2}$, then for $j > i > r$,

$$
\|x^i + \ldots + x^j\| = \sum_{n=0}^{i-1} \frac{1}{2^r} \frac{p_n(x^i + \ldots + x^j)}{1 + p_n(x^i + \ldots + x^j)} +
$$
therefore the partial sums form a Cauchy sequence, hence

\[ \sum_{n=1}^{\infty} \frac{1}{2} \sum_{n=0}^{i-1} \frac{1}{2n(2n+1)} \leq \sum_{n=0}^{\infty} \frac{1}{2} \left[ \frac{1}{p_n(x^i + \ldots + x^j)} \right] \]

\[ \xi \leq \sum_{n=0}^{1-1} \frac{1}{2(2-1)} \sum_{n=0}^{2-1} \frac{1}{2n(2n+1)} \]

\[ \xi \leq \frac{1}{2} \left( \sum_{n=0}^{1-1} \frac{1}{2^n} \right) 2^{-i+1} \]

\[ \xi \leq 2^{i-1} + \frac{\xi}{2} \leq i \]

\[ \xi \leq \frac{\xi}{2} \]

\[ \xi \leq \frac{\xi}{2} \]

\[ \xi \leq \frac{\xi}{2} \]

\[ \xi \leq \frac{\xi}{2} \]

therefore the partial sums form a Cauchy sequence, hence

\[ \sum_{n=1}^{\infty} x^n \] is convergent to, say, \( x \) in \( c_A \). The sequence \( \sum_{n=1}^{\infty} x^n \) is bounded by construction; in fact, \( |x_n| \leq 1 + \sum_{r=1}^{\infty} 2^{-r-j} \) for all \( n \), furthermore, it has a subsequence tending to \( 1 \) and a subsequence tending to zero, hence \( \sum_{n=1}^{\infty} x^n \) is a divergent sequence; this completes the proof for co-regular matrices.

Case II. \( A \) is co-null. We first notice that \( c_\infty \) cannot be closed in \( c_A \), for otherwise there exists \( f \in c_A \) such that \( f(c_\infty) = 0 \) and \( f(1) \neq 0 \) by the Hahn-Banach Theorem. But \( A \) is co-null. Hence \( f(c_\infty) = 0 \) implies \( f(1) = 0 \), therefore such \( f \) does not exist. Hence \( c_\infty \) cannot be closed in \( c_A \). In the first part of this proof if we consider

\[ V'_K = \{ x \in c_\infty : x_k = 0, k < K \} \]

instead of \( V_K \), then the \( V'_K (K=1,2,\ldots) \) are not closed. The proof is exactly the same as the preceding one. Hence by \( v \) of Theorem 1.3, for every \( \epsilon > 0 \), positive integers \( b, K \), there exists \( x \in V'_K \) such that (4.2) and (4.3) are satisfied. Now \( \lim x = 0 \), hence the argument used in Case 1 can be applied to show that there is a bounded divergent sequence in \( c_A \).
Corollary 4.4. A co-null matrix must sum a bounded divergent sequence.

Proof: By Proposition 3.3., c is not closed in c_A if A is co-null; by Theorem 4.3., A must sum a bounded divergent sequence.

Corollary 4.5. A co-null matrix must sum an unbounded sequence.

Proof: Suppose c_A \subseteq m, then c_A is closed in m by Lemma 3.9., hence by Proposition 1.12., the topology of c_A is the same as the subspace topology relative to m. c is complete with the usual topology, hence c is closed in c_A, thus A is co-regular—contradiction. Therefore c_A \not\subseteq m, hence A sums an unbounded sequence.

Corollary 4.6. If A sums a bounded divergent sequence, then c is not closed in c_A.
CHAPTER V

PERFECTNESS AND MATRICES OF TYPE M

Definition: Let \( A = (a_{nk}) \) be an arbitrary matrix. Any sequence \( \{a_n\} \) in \( l_1 \) satisfying

\[
\sum_{n} a_{nk} = 0 \quad \text{for } k=1,2,\ldots \quad (5.1)
\]

is said to be orthogonal to \( A \). If the only sequence orthogonal to \( A \) is the zero sequence, \( A \) is said to be of type \( M \).

All diagonal matrices with non-zero diagonal elements are of type \( M \). For certain classes of matrices, perfectness and type \( M \) are closely related. In this chapter we will study these concepts for different classes of matrices. The concept of type \( M \) will be applied to consistency.

Definition: Let \( \{a_n\} \) be orthogonal to a conservative matrix \( A \) and let \( f(x) = \sum_{n} a_n (Ax) \). We call \( f(x) \) an orthogonal functional on \( c_A \).

Proposition 5.1. If \( A \) is conservative, then every orthogonal functional vanishes on \( c_A \).

Proof: Let \( f = \sum_{n} a_n (Ax) = \alpha (Ax) \) be an orthogonal functional, then by Proposition 1.27, \( \alpha (Ax) = (\alpha A)x \) for every \( x \in c_A \), hence \( \alpha (Ax) = (\alpha A)x = 0x = 0 \).

Proposition 5.2. Let \( A = (a_{nk}) \) be conservative and reversible, then \( c_A \cap m = c_A \) implies that \( A \) is of type \( M \).

Proof: Suppose \( c_A \cap m = c_A \), then by Theorem 1.5., any continuous linear functional that vanishes on \( c_A \cap m \) is identi-
cally zero on $c_A$. Let $\lambda=(d_1,d_2,\ldots)\in E_1$ and $\sum_n a_{nk}=0$ for all $k$. Suppose $d_{n_0}\neq 0$ for some $n_0$ and let $Ay=\delta_{n_0}^0$. Such $y$ exists because $A$ is reversible, and clearly $y\in c_A$. Now the continuous linear functional $\sum_n (Ax)_n$ is identically zero on $c_A$ by Proposition 1.27, hence $\sum_n (Ax)_n$ is identically zero on $c_A$ by assumption. But $\sum_n (Ay)_n=d_{n_0}\neq 0$ and this is a contradiction. Hence $\lambda=0$ and $A$ is of type $M$.

**Proposition 5.3.** If $A$ is co-regular, then $\overline{c}=c_A^{\infty}$.

Proof: Clearly $\overline{c}\subseteq c_A^{\infty}$. To show that $\overline{c}=c_A^{\infty}$ it suffices to prove that $\overline{c}\subseteq c_A^{\infty}$, for then $c=\overline{c}=c_A^{\infty}$. Let $f$ be a continuous linear functional on $c_A$ that vanishes on $c$, we show that $f(c_A^{\infty})=0$. By (2.2), $f(x)=\lim_{n} t_n(Ax)_n+\sum_k [f(\delta^k)-A_k-\sum_n a_{nk}]x_k$ and recall that $\chi(f)=A\chi(A)$. Now $f(1)=0$ and $f(\delta^k)=0$ for all $k$, hence $\chi(f)=0$, but $A$ is co-regular thus $\chi(A)\neq 0$, hence $\lambda=0$. Also, the representation of $f(x)$ is reduced to $f(x)=\sum_n t_n(Ax)_n-\sum_k (\sum_n a_{nk})x_k$, hence $f(x)=t(Ax)-(tA)x$, where $t=(t_1,t_2,\ldots,t_n,\ldots)$. By Proposition 1.27, $f(x)$ vanishes on $c_A^{\infty}$.

**Theorem 5.4.** A reversible, co-regular matrix $A$ is perfect if and only if it is of type $M$.

Proof: If $A$ is perfect, then $\overline{c}=c_A$. Thus, by Proposition 5.3, $c_A^{\infty}=c_A^{\infty}$. Hence by Proposition 5.2, $A$ is of type $M$.

Conversely, suppose $A$ is of type $M$. It suffices to show that every continuous linear functional that vanishes on $c$ must vanish on $c_A$. By (2.4), $f(x)=\lim_{n} t_n(Ax)_n$. 


In exactly the same way as in the proof of Proposition 5.3., we obtain \( f(x) = \sum_{n} t_n (Ax)_n \). Now \( f \) vanishes on \( c \) and \( f(\delta^k) = \sum_{n} t_n a_{nk} = 0 \), hence by assumption \( t = (t_1, t_2, \ldots, t_n, \ldots) \) \( = 0 \), thus \( f \equiv 0 \). This completes the proof.

In general, perfectness and type M are not equivalent conditions. For example the matrix

\[
A = \begin{pmatrix}
  0 & 0 & 0 & \cdots \\
  0 & 1 & 0 & \cdots \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

is not of type M, since \( (1,0,\ldots)A = 0 \), but \( c_A = c \), hence it is perfect. On the other hand, consider the matrix in Example 3.e., that is, the matrix \( A \) whose diagonal elements are 1, \( a_n, k(n) = \frac{n}{k(n)} \), where \( k(1) = 2^2 \), \( k(2) = 2^3 \), \ldots, \( k(n+1) = 2^nk(n) \) and other elements are zero. This matrix does not sum any bounded divergent sequence, hence \( \bar{c} = c \) in \( c_A \), but \( (1,2,3,\ldots) \in c_A \), hence \( c_A \neq c \), therefore the matrix is not perfect. The matrix is of type M. This can be seen as follows: Suppose \( d = (d_1, \ldots, d_n, \ldots) \) and \( dA = 0 \), then \( d_1 = d_2 = d_3 = 0 \), also \( d_{k(1)} = d_{k} = 0 \), because \( d_{k(1)} + d_1 \cdot a_{1k(1)} = 0 \) but \( d_1 = 0 \), hence \( d_{k(1)} = d_{k} = 0 \). Similarly, we have \( d_1 = d_2 = \ldots = d_{k(2)} = d_7 = 0 \) and \( d_8 + d_2 \cdot a_{28} = 0 \) hence \( d_8 = 0 \). Continuing in this way we have \( d = (0,0,\ldots,0,\ldots) \), hence the matrix is of type M.

Now we will consider a different class of matrices,
that is, the reversible and multiplicative matrices. It is a different class from the reversible co-regular matrices because the matrix

\[
\begin{pmatrix}
2 & 0 & 0 & \ldots \\
1 & 2 & 0 & 0 & \ldots \\
1 & 0 & 2 & 0 & 0 & \ldots \\
1 & 0 & 0 & 2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & 0 & 0 & \ldots & 2 & 0 & 0 \\
\end{pmatrix}
\]

is co-regular, reversible but not multiplicative since the first column does not tend to zero. On the other hand, the matrix in Example 3.b. is reversible, multiplicative and co-null.

Definition: A maximal subspace of a linear space is a subspace whose complementary subspace has dimension one.

Lemma 5.5. Let \( V_1 \) be a linear subspace of a linear space \( V \). If there exist two independent linear functional \( f_1, f_2 \) such that \( f_1(V_1) = f_2(V_1) \Xi_0 \), then \( V_1 \) is not a maximal subspace.

Proof: Suppose \( V_1 \) is maximal in \( V \) and let \( v \) span the complementary subspace. Let \( f_1(v) = d_1 \) and \( f_2(v) = d_2 \), then

\[-d_1 \frac{d_1}{d_2} f_2(V) + f_1(V) \Xi_0 \text{ and this contradicts the assumption, hence } V_1 \text{ is not maximal.}

Theorem 5.6. Let \( A \) be reversible and multiplicative. Then \( A \) is of type \( M \) if and only if \( c_0 \) is a maximal subspace of \( c_A \).
Proof: Suppose $A$ is of type $M$. It suffices to show that \( \mathcal{C}_0 \) is the kernel of some linear functional on $\mathcal{C}_A$. Let $x_1 \in \mathcal{C}_0$, by Theorem 1.5., there exists $f \in \mathcal{C}_A$ such that $f(c_0) = 0$ and $f(x_1) \neq 0$. By (2.4), we may let $f(x) = \lim_A x + \sum_n t_n (Ax)_n$. Now $f$ is multiplicative and of type $M$, hence $f(\xi^n) = 0 + \sum_n t_n (A\xi)_n$. Therefore $t_n = 0$ for all $n$, hence $f(x) = \lim_A x$.

By assumption $f(x_1) \neq 0$, hence $\lim_A x_1 \neq 0$. Now consider the continuous linear functional $h(x) = \lim_A x$ on $\mathcal{C}_A$. Since $A$ is multiplicative, we have $\lim_A x = 0$ for all $x \in \mathcal{C}_0$, hence $\ker h(x) \cong \mathcal{C}_0$. By the preceding part of this proof, $h(x) \neq 0$ for all $x \in \mathcal{C}_0$, hence $\ker h(x) = \mathcal{C}_0$, thus $\overline{\mathcal{C}_0}$ is a maximal subspace.

Conversely, suppose $A$ is not of type $M$. Let $t = (t_1, \ldots, t_n, \ldots) \in 1$ be non-zero and $\delta A = 0$. Consider $f_1(x) = \delta (Ax)$ and $f_2(x) = \lim_A x$; both $f_1(x)$ and $f_2(x)$ vanish on $\mathcal{C}_0$. By Lemma 5.5., if $f_1(x)$ and $f_2(x)$ are independent, then $\overline{\mathcal{C}_0}$ is not a maximal subspace. Let $a_1, a_2$ be two scalars such that $f(x) = a_1 \lim_A x + (a_2 \cdot t)(Ax) \neq 0$ on $\mathcal{C}_A$ and suppose $t_n \neq 0$. Let $x_1 \in \mathcal{C}_A$ be such that $Ax_1 = \xi^n$, then $f(x_1) = 0 + a_2 t_n = 0$, hence $a_2 = 0$. Let $x_2 \in \mathcal{C}_A$ satisfy $Ax_2 = \xi^n$, then $f(x_2) = a_1 \cdot 1 = 0$, hence $a_1 = 0$, thus $f_1$ and $f_2$ are independent.

We have seen that in general the concepts of perfectness and type $M$ are not equivalent. In what follows, we will look at some subsets of $\mathcal{C}_A$ and study some sufficient conditions on these subsets for $A$ to be perfect or to be of type $M$.

For a conservative matrix $A$, we define
\[ B_A = \{ x \in c_A \mid \text{there exists } M > 0 \text{ depending on } x \text{ such that } \left| \sum_{k=1}^{m} a_{nk} x_k \right| < M, \text{for } m, n = 1, 2, \ldots \} \]

\[ L_A = \{ x \in c_A \mid (tA)x = \sum_{n=1}^{\infty} t_n a_{nk} x_k \text{ exists for all } t \in L_1 \} \]

\[ P_A = \{ x \in c_A \mid (tA)x = t(Ax) \text{ for all } t \in L_1 \text{ such that } (tA)y \text{ exists for all } y \in c_A \} \]

In general the subset \( B_A \) does not fill up \( c_A \). For example, let \( A \) be the matrix in Example 3.b. and \( x = (1, 1+ \frac{1}{2}, 1+ \frac{1}{2} + \frac{1}{3}, \ldots, 1+ \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \ldots) \), then \((Ax)_n = (1+ \frac{1}{2} + \ldots + \frac{1}{n-2}) - \left( 1 + \frac{1}{n-2} + \frac{1}{n} (1+ \frac{1}{2} + \ldots + \frac{1}{n}) = \frac{1}{n-1} + \frac{1}{n^2} (1+ \ldots + \frac{1}{n}) \right)\), hence \( \lim_{n \to \infty} (Ax)_n = 0 \) and \( x \in c_A \), but \( x \notin B_A \) because \( \left| \sum_{k=1}^{m} a_{nk} x_k \right| = \left| 1 + \ldots + \frac{1}{n-2} \right| \) which tends to infinity as \( n \) increases.

**Theorem 5.7.** If \( A \) is co-regular then \( P_A = \overline{c} \).

**Proof:** Let \( f \) be a continuous linear functional vanishing on \( c \). In the proof of Theorem 5.3, we proved that \( f(x) \) is of the form \( \langle t(Ax) - (tA)x, \rangle \), hence by the definition of \( P_A \) we have \( f(x) \) vanishes on \( P \), therefore \( P_A = \overline{c} \).

Conversely, it is clear that \( c \subseteq P_A \), hence it suffices to show that \( P_A \) is closed. Let \( F = \{ t \in L_1 \mid (tA)x \text{ exists for all } x \in c_A \} \) and for every \( t \in F \) define \( f_t = (tA)x - t(Ax) \). Each \( f_t \) is a continuous linear transformation from \( c_A \) to \( s \) by Corollary 1.14., hence the kernel of \( f_t \) is closed. Now \( P_A = \bigcap_{t \in F} \ker f_t \), hence \( P_A \) is closed. Therefore \( P_A = \overline{c} \) and hence \( P_A = \overline{c} \).

The above theorem characterizes \( c \) in case \( A \) is co-regular. Notice that \( P \supseteq \overline{c} \) does not depend on the co-regularity of \( A \). The following corollary follows trivially.
Corollary 5.8. A co-regular matrix $A$ is perfect if and only if $P_A = c_A$.

Corollary 5.8. is not true for co-null matrices, for example, consider the matrix

$$
\begin{pmatrix}
1 & 0 & \cdots & \\
0 & \frac{1}{2} & 0 & \\
0 & 0 & \frac{1}{3} & 0 \\
\vdots & \vdots & \vdots & \\
0 & 0 & \frac{1}{n} & 0 \\
\vdots & \vdots & \vdots & \\
\end{pmatrix}
$$

Clearly $P_A = c_A$ and the sequence $x_0 = (1, 2, 3, \cdots) \in c_A$.

Recall that for normal matrices the topology is defined by the norm $\|x\| = \sup_n \|Ax\|$ for all $x \in c_A$. Let $\varepsilon = \frac{1}{2}$, then $N(x_0, \varepsilon)$ does not contain any element of $c$, thus $A$ is not perfect.

Proposition 5.9. $B_A = L_A$

Proof: Let $x \in B_A$ and $t = (t_1, t_2, \cdots, t_n, \cdots) \in \ell_1$. For any $k$,

$$(\sum_{n=1}^k a_{nk})x_k$$

exists because $t \in \ell_1$ and $\{a_{nk} | n = 1, 2, \cdots\}$ is bounded. Let

$$S_1 = (\sum_{n=1}^k a_{n1})x_1, S_2 = (\sum_{n=1}^k a_{n1})x_1 + (\sum_{n=1}^k a_{n2})x_2 = \frac{1}{n}t_n(a_{n1}x_1 + a_{n2}x_2), \cdots$$

$$\cdots, S_k = (\sum_{n=1}^k a_{n1})x_1 + \cdots + (\sum_{n=1}^k a_{nk})x_k = \frac{1}{n}t_n(a_{n1}x_1 + \cdots + a_{nk}x_k), \cdots$$

Let $S = t_1(\sum_k a_{1k}x_k) + t_2(\sum_k a_{2k}x_k) + \cdots$. We claim that $S_k$ tends to $S$. For any $\varepsilon > 0$, choose $N(\varepsilon)$ so that $\sum_{n=N(\varepsilon)+1}^{\infty} |t_n| < \varepsilon$ and for $n = 1, 2, \cdots, N(\varepsilon)$, choose $K$ great enough so that $k_0 > K$ implies

$$|t_1(a_{n1}x_1 + \cdots + a_{nk_0}x_{k_0}) - (\sum_{n=1}^k a_{1k}x_k)| + \cdots + |t_{N(\varepsilon)+1}(a_{n1}x_1 + \cdots + a_{nk_0}x_{k_0})|$$
Then for \( k > K \), we have
\[
\| S_k - S \| < \frac{\varepsilon}{2} \sum_{n=N(\varepsilon)}^{\infty} |a_{nk}x_k| + \frac{\varepsilon}{4} \sum_{n=1}^{N(\varepsilon)} |a_{nk}x_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\]

Hence \( \sum_{k=1}^{\infty} \frac{1}{n} a_{nk}x_k = \sum_{k=1}^{\infty} \frac{1}{n} a_{nk}x_k \) and thus \( (tA)x \) exists.

Conversely, let \( x = (x_1, x_2, \ldots) \in \ell^1 \). Define a sequence of linear functionals \( \{ f_m \} \) on \( l_1 \) by
\[
f_m(t) = t_1(a_1x_1 + \cdots + a_mx_m) + t_2(a_2x_1 + \cdots + a_mx_m) + \cdots.
\]
Each \( f_m \) is well-defined since \( A \) is conservative. Recall that the norm on \( l_1 \) is defined by \( \|t\| = \sum |t_n| \), hence it is easy to see that each \( f_m \) is a continuous linear functional on \( l_1 \).

Now \( \|f_m\| = \sup_n \left( \sum |a_{nk}x_k| \right) \| t_n | \right) \leq \sup_n |a_{nk}x_k| \). On the other hand, let \( t = e_n \), then \( \|f_m\| \geq \sup_n |a_{nk}x_k| \). Hence we have \( \|f_m\| = \sup_n |a_{nk}x_k| \). By definition of \( f_m \) we also have
\[
f_m(t) = (t_1a_1 + t_2a_2 + \cdots) x_1 + \cdots + (t_1a_1 + t_2a_2 + \cdots) x_m.
\]
Since \( x \in \ell_A \), \( \lim_m f_m(t) = \sum_k \frac{1}{n} a_{nk}x_k \) exists for each \( t \in \ell_1 \). By Theorem 1.6, \( \{ \|f_m\| \}_{m=1,2,\ldots} \) is uniformly bounded, hence there exists \( M \) such that \( \sup_n |a_{nk}x_k| \leq M \) for all \( m \), hence \( x \in B_A \).

Theorem 5.10. If a conservative matrix \( A \) has a right inverse whose columns belong to \( B_A \) except for a finite number of them, then \( A \) is of type \( M \).

Proof: Recall that in the proof of Proposition 5.10, we actually proved that \( (tA)x = t(Ax) \) for all \( x \in B_A \) and \( t \in \ell_1 \).
Suppose \( x \) is the \( n \)th column of \( A^{-1} \) belonging to \( B_A \), \( t \in 1 \) and \( t \) is orthogonal to \( A \), then \( (tA)x = t(Ax) = t(\delta^n) = t_n = 0 \), but all except a finite number of the columns of \( A^{-1} \) belong to \( B_A \), hence \( t_n = 0 \) except for a finite number of them.

Let \( t = (t_1, \ldots, t_n, 0, o, o, o, \ldots) \) and let \( u_1, u_2, \ldots u_n \) be the first \( n \) columns of \( A^{-1} \), then \( (tA)u_1 = 0 = (t_1, o, o, \ldots) \) hence \( t_1 = o \), similarly \( t_2 = t_3 = \ldots = t_n = o \).

**Definition:** A conservative matrix \( A \) is said to have the mean value property if \( B_A = c_A \).

**Corollary 5.11.** A reversible matrix that has the mean value property is of type \( M \).

**Proof:** Since \( A \) is reversible, there exists \( x^k \) such that \( Ax^k = s^k \). Let \( D \) be the matrix whose \( k \)th column is \( x^k \), then \( D = A^{-1} \). If \( A \) has the mean value property, then \( x^k \in c_A = B_A \).

By Theorem 5.10, \( A \) is of type \( M \).

**Proposition 5.12.** A co-regular matrix that has the mean value property is perfect.

**Proof:** In the proof of Proposition 5.9, we proved that for all \( x \in B_A \) and \( t \in 1 \), \((tA)x = t(Ax)\), hence \( B_A \subseteq P_A \). By Theorem 5.7, when \( A \) is co-regular \( P_A = \bar{c} \), hence if \( A \) has the mean value property \( B_A = c_A \subseteq P_A = \bar{c} \), thus \( c_A = \bar{c} \). Therefore \( A \) is perfect.

**Definition:** Two matrices \( A \) and \( B \) are said to be consistent if \( \lim A x = \lim B x \) for all \( x \in c_A \cap c_B \).

**Lemma 5.13.** Let \( A \) be a reversible conservative matrix,
then \( f \in c_A \) if and only if \( f(x) = \lim_B x \) for some \( B \) such that \( c_B \supseteq c_A \).

**Proof:** If \( c_B \supseteq c_A \), then by Theorem 2.3., \( \lim_B x \in c_A \). Conversely, if \( f \in c_A \), let \( f(x) = \lim_A x + \sum_n t_n(Ax)_n \) as in (2.4).

Define a matrix \( B = (b_{nk}) \) where \( b_{nk} = t_1 a_{1k} + t_2 a_{2k} + \cdots + t_{n-1} a_{n-1,k} + a_{nk} \), then

\[
(Bx)_m = d \left( \sum_k a_{mk} x_k \right) + \sum_{n=1}^{m-1} t_n(Ax)_n,
\]

hence \( \lim_B x = f(x) \).

**Theorem 5.14.** Let \( A \) be reversible and co-regular, then a necessary and sufficient condition for \( A \) to be type \( M \) is that \( A \) is consistent with every matrix \( B \) such that

1) \( c_B \supseteq c_A \n
ii) \( \lim_B x = \lim_A x \) for all \( x \in \{(k^k)_{k=1,2,\cdots} \cup \{i\} = F \). \n
**Proof:** We will first prove that \( A \) is consistent with every \( B \) satisfying i) and ii) is equivalent to the condition that \( A \) is perfect. Then the theorem will follow from Theorem 5.4. Suppose \( A \) is consistent with every \( B \) satisfying i) and ii). Let \( f \in c_A \) be consistent \( f(F) = 0 \) and consider \( f + \lim_A x = f_1 \). Obviously \( f_1 \in c_A \), by Lemma 5.13., we can let \( f_1(x) = \lim_B x \), then i) and ii) are satisfied.

Hence \( \lim_A x = f_1(x) \) for all \( x \in c_A \) and \( f(A) = 0 \). Now \( F \in c \), thus any continuous linear functional that vanishes on \( c \) must be identically zero on \( c_A \). Hence by Theorem 1.5., \( c \) is dense in \( c_A \), thus \( A \) is perfect. Conversely, let \( A \) be perfect and \( B \) be a matrix satisfying i) and ii), then \( f = \)
lim_{x \to x_0} x is in c_A and f vanishes on F. Now F is a Schauder basis for c with the usual topology and this topology is stronger than the subspace topology relative to c_A, hence f also vanishes on c. But A is perfect, hence c = c_A and thus f(c_A) = 0. Therefore \lim_{x \to x_0} x = \lim_{x \to x_0} x for all x \in c_A and A, B are consistent.
BIBLIOGRAPHY