A STUDY OF HOLLAND REPRESENTATION THEOREM

and

FREE LATTICE-ORDERED GROUPS

by

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In this study of lattice-ordered groups, we begin with the fundamental properties as found in the book "Lattice Theory" by G. Birkhoff, and then present Holland's fundamental representation of a lattice-ordered group as a group of order preserving permutations of a totally ordered set. Holland's work is essential to the description of a free lattice-ordered group as given by P. Conrad. P. Conrad's results on free lattice-ordered groups are also reviewed. This work constitutes the major portion of this thesis.

If $G$ is an l-group, then $G$ is l-isomorphic to a subdirect product of l-groups $\{B_g : 0 \neq g \in G\}$ such that each $B_g$ is a transitive l-subgroup of the l-group of automorphisms of a totally ordered set $S_g$, where $S_g$ is the set of all right cosets of a convex l-subgroup $M_g$ of $G$ which is maximal with respect to not containing $g$. Furthermore, it then follows that $G$ is also l-isomorphic to an l-subgroup of the l-group of all automorphisms of a totally ordered set.

On the other hand, every free group admits a total order. From this, we have that for a free group $G$, there is a free l-group $F$ generated by $G_\pi$, where $\pi$ is an o-isomorphism, that is, for every o-homomorphism $\tau$ from $G$ into an l-group $H$, there exists a unique l-homomorphism $\sigma$ from $F$ into $H$ such that the following diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\pi} & G_\pi \\
\downarrow & & \downarrow \\
F & \xrightarrow{\sigma} & H
\end{array}
$$

iii
commutes.

Finally, for a po-group $G$, the following are equivalent:

1. There exists a free $l$-group over $G$.
2. There exists an o-isomorphism of $G$ into an $l$-group.
3. $G^+ = \{g : g \geq 0\}$ is the intersection of right orders.
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INTRODUCTION

The purpose of this study is to review some of the developments in the theory of lattice ordered groups closely related to the Holland representation for lattice ordered groups and P. Conrad's paper on free lattice ordered groups. In Chapter 1, a detailed discussion of the properties of regular and prime subgroups of an l-group is presented. A subgroup \( H \) of a lattice ordered group \( G \) is regular if and only if there is an element \( g \in G \) such that \( H \) is a maximal convex \( l \)-subgroup of \( G \) with respect to not containing \( g \); a subgroup \( M \) of a lattice ordered group \( G \) is prime subgroup if \( M \) is the intersection of a chain of regular subgroups of \( G \) or \( M \) is a convex \( l \)-subgroup of \( G \) and \( a, b \in G \setminus M \) implies \( a \wedge b \neq 0 \).

Prime subgroups are of particular importance in obtaining representation of lattice-ordered groups. If \( M \) is prime subgroup of a lattice ordered group \( G \), then the set of cosets of \( M \) can be endowed with total order, where \( a+M > b+M \), if and only if there exists \( m \in M \) such that \( a+m > b \). It follows that if \( M \) is both prime and normal (prime subgroup need not be normal, see Example 1.22), then the set of cosets of \( M \) is a totally ordered group. The properties of prime subgroups were utilized by C. Holland in his representation theorem which is discussed in Chapter 2. Observe that if \( M \) is regular then \( M \) is prime (Corollary 1.19), and so we have that every \( l \)-group always contains prime subgroups. In Chapter 2, we present the Holland representation of a lattice-ordered group \( G \) as a subdirect product.
of \( \Pi K_\beta \), where each \( K_\beta \) is a transitive 1-subgroup of the lattice ordered group of order-preserving permutations on some totally ordered set, where each total order set is the set of cosets of some prime subgroup of \( G \). This answered a problem originally posed by Birkhoff in the second edition of his book on lattice theory, it is an invaluable tool in the study of the nature and occurrence of lattice-ordered groups. In Chapter 3, our main concern is with the groups admitting a linear order, which we shall call O-groups. We prove that all free groups are O-groups, the method is similar to the proof of Simbireva, Neumann Theorem: If a group \( G \) has a transfinite central series ending with \( C_\tau = \{0\} \) such that all factor groups \( C_\alpha/C_{\alpha+1} \) are torsion free, then \( G \) is an O-group. ( A descending chain \( G=G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_\alpha \supset G_{\alpha+1} \supset \ldots \), with \( \alpha \) a variable over ordinals less than a fixed \( \tau \) is called a transfinite series of \( G \) if \( G_{\alpha+1} \) is a (normal) subgroup of \( G_\alpha \) such that the commutator \([G; G_\alpha]\) is contained in \( G_{\alpha+1} \) and for a limit ordinal \( \alpha \), \( G_\alpha \) is the intersection of all \( G_\beta \) with \( \beta < \alpha \). Clearly, the \( G_\alpha \) are normal in \( G \). In the proof, a theorem of Magnus-Witt is used; that is, the lower central series of a free group \( G \) terminates at \( \{0\} \) after \( \omega \) steps, where \( \omega \) denotes the first infinite ordinal, and the factor groups are torsion free. In Chapter 4, let \( \pi \) be an O-isomorphism of a p o-group \( G \) into an l-group \( F \). This means that both \( \pi \) and \( \pi^{-1} \) preserve order. Then \( (F, \pi) \) is a free l-group over \( G \) if (i) \( G\pi \) is a set of generators of the l-group \( F \) (that is, no proper l-subgroup of \( F \) contains \( G\pi \)), and (ii) if \( \sigma \) is an O-homomorphism of \( G \) into an l-group \( L \), then there exists an l-homomorphism \( \tau \) of \( F \) such that
The following two theorems are the main results.

Theorem 4.14. For a p o-group \( G \) the following are equivalent:

1. There exists a free \( l \)-group over \( G \).
2. There exists an \( 0 \)-isomorphism of \( G \) into an \( l \)-group.
3. \( G^+ = \{ g \in G; g > 0 \} \) is the intersection of right orders.

In particular, there exists a free \( l \)-group over a trivially ordered group \( G \) (that is, \( G^+ = \{ 0 \} \)) if and only if \( G \) admits a right order and this is equivalent to \( G \) being a subgroup of an \( l \)-group. Thus if \( G \) is a free group with \( S \) as a free set of generators and a trivial order, then there exists a free \( l \)-group \(( F, \pi )\) over \( G \). Moreover \( F \) is a free \( l \)-group with \( S\pi \) as a free set of generators. That is, \( S\pi \) generates the \( l \)-group \( F \) and each mapping of \( S\pi \) into an \( l \)-group \( L \) has a unique extension to an \( l \)-homomorphism of \( F \) into \( L \).

Suppose that \( G \) is a p o-group such that \( G^+ = \bigcap_{\lambda \in \Lambda} P_\lambda \), where \( \{ P_\lambda | x \in \Omega \} \)
is the set of all right orders of $G$ such that $P_\lambda \supseteq G^\dagger$. For each $\lambda \in \Omega$ let $G_\lambda$ be the right $o$-group $(G, P_\lambda)$ and let $g \leftrightarrow g_\lambda$ be the natural isomorphism of $G$ into the $1$-group $A(G_\lambda)$ of all order preserving permutations of the totally ordered set $G_\lambda$, where

$$xg_\lambda = x + g$$

for all $x \in G$

The direct product $\Pi A(G_\lambda)$ of the $1$-group $A(G_\lambda)$ with component-wise order is an $1$-group, the cardinal product of the $A(G_\lambda)$'s. Now let $\pi$ be the natural map of $G$ onto the subgroup of long constants of the $1$-group $\Pi A(G_\lambda)$

$$g \xrightarrow{\pi} (\ldots g_\lambda \ldots)$$

and let $F$ be the $1$-subgroup of $\Pi A(G_\lambda)$ that is generated by $G_\pi$. It then can be shown that

Theorem 4.13: $(F, \pi)$ is the free $1$-group over $G$.

Note that in our presentation of the Conrad's Generalization of the Method used by Weinberg to construct free abelian $1$-groups is used to construct free $1$-groups. The generalization is quite natural and identical with Weinberg's method if we restrict our attention to abelian group. Proposition 4.15 is a generalization of one of the P. Conrad's propositions.

In our construction of the free $1$-group over a $p$ $o$-group the key concept is that of a right $o$-group. In the papers of P. Cohn, Groups of order automorphisms of ordered sets, Mathematika 4 (1957) 41-50 and P. Conrad, Introduction à la théorie des groups rétinles, Secretariat Mathematiques Paris (1967) there are necessary and sufficient
conditions given a group $G$ to admit a right order. In D. Smirnov
On right ordered groups (Russian), Akad, Nauk, SSSR Siberian Dept.
Algebra and Logic 5 (1966) various right orders of a free group
are investigated.

Notation: Throughout the whole thesis, in general we use additive
notation as group operation, except in permutation group, we use
multiplicative notation.
Chapter One

In this preliminary chapter, we study mainly prime subgroups of an l-group. The reader is referred to Birkhoff [1] and Fuchs [5] for a general theory of lattice-ordered groups. Here we may also assume that the reader is familiar with the basic properties of groups.

Definition: By a partially ordered set is meant a system \( X \) in which a binary relation \( \geq \) is defined, which satisfies

(i) For all \( x \), \( x \geq x \). (Reflexive)

(ii) If \( x \geq y \) and \( y \geq x \), then \( x = y \). (Antisymmetric)

(iii) If \( x \geq y \) and \( y \geq z \), then \( x \geq z \). (Transitive)

Definition: A totally ordered set \( X \) is a partially ordered set in which either \( x \geq y \) or \( y \geq x \), for every \( x, y \in X \).

Definition: A partially ordered group \( (G, \leq, +) \) is such that

(i) \( (G, \leq) \) is a partially ordered set.

(ii) \( (G, +) \) is a group.

(iii) \( x \geq y \) implies \( a + x + b \geq a + y + b \), for all \( a, b, x, y \in G \).

We define totally ordered group similarly.

Definition: \( (G, +, \leq) \) is an right ordered group if and only if

(i) \( (G, +) \) is a group.

(ii) \( (G, \leq) \) is totally ordered.

(iii) \( x \leq y \) implies \( x + a \leq y + a \), for every \( x, y, a \in G \).

We define left ordered group similarly.

Definition: A lattice is a partially ordered set \( P \) any two of whose elements have a greatest lower bound or "meet" \( x \wedge y \), and least upper bound or "join" \( x \vee y \).
Definition: An l-group \((G, \leq, +)\) is such that

(i) \((G, \leq)\) is a lattice.

(ii) \((G, +)\) is a group.

(iii) \(x \leq y\) implies \(a + x + b \leq a + y + b\) for all \(x, y, a, b \in G\).

Theorem 1.1 Let \(G\) be a group with identity \(o\). Let \(P \subseteq G\) be such that

(i) \(o \in P\)

(ii) \(P + P \subseteq P\)

(iii) \(P \cup (\neg P) = G\)

(iv) \(P \cap (\neg P) = \{o\}\)

(A) Let \(\leq\) be defined on \(G\) by \(g \leq h\) if and only if \(h - g \in P\).

(B) Let \(\alpha\) be defined on \(G\) by \(g \iddot h\) if and only if \(-g + h \in P\).

Then the order in (A) is a right total order and the order in (B) is a left total order. Conversely, let \(\leq\) and \(\alpha\) be a right total order and a left total order on \(G\) respectively, then \(P = \{g; o \leq y, g \in G\}\) and \(P' = \{g; o \alpha g, g \in G\}\) satisfy (i) to (iv).

Proof: For A. (i) \(x - x = o \in P\), for every \(x \in G\), implies \(x \leq x\) for every \(x \in G\).

(ii) If \(x \leq y\) and \(y \leq x\), then \(y - x, x - y \in P\). But \(x - y = -(y - x) \in -P\). Hence \(x - y \in P \cap (\neg P) = \{o\}\). Therefore \(x - y = o\). That is \(x = y\).

(iii) If \(x \leq y\) and \(y \leq z\), where \(x, y, z \in G\), then \(y - x\) and \(z - y \in P\). Hence \((z - y) + (y - x) \in P + P \subseteq P\). Therefore \(z - x \in P\). Consequently, \(x \leq z\).

(iv) Let \(x, y \in G\). Then \(x - y \in G\). Therefore \(x - y \in P\) or \(x - y \in (\neg P)\). So \(x \leq y\) or \(y \leq x\).

We have thus shown that "\(\leq\)" is a total order on \(G\). Now suppose \(x \leq y\) for some \(x, y \in G\), consider \(x + a\) and \(y + a\) for any \(a \in G\), we have

\((y + a) \leq (x + a) = y + a - a = y \leq x \in P\). Hence \(x + a \leq y + a\), for any \(a \in G\). Consequently
(G, +, \leq) is a right ordered group.

Similarly, the order in (B) is a left order on G.

Conversely, let "\leq" be a right order on G and \( P = \{ g; \ 0 \leq g, g \in G \} \). Then

(i) \( 0 \leq o \) implies \( o \in P \).

(ii) For any \( g_1, g_2 \in P \), we have \( 0 \leq g_1 \), therefore \( g_1 + g_2 > 0 + g_2 = g_2 > 0 \).

That is \( g_1 + g_2 > 0 \). Hence \( P + P \subseteq P \).

(iii) Clearly \( P \cup (-P) \subseteq G \). Let \( g \in G \). Then \( g > 0 \) or \( g > 0 \), since "\leq" is a total order on G. Therefore \( g \in P \) or \( g \in -P \). Hence \( g \in P \cup (-P) \).

That is \( G \subseteq P \cup (-P) \) and hence \( P \cup (-P) = G \).

(iv) Let \( g \in P \cap (-P) \). Then \( g > 0 \) and \( g \leq 0 \). Hence \( g = 0 \) (by antisymmetry), so \( P \cap (-P) = \{ 0 \} \). The rest is clear.

Similarly,

Theorem 1.2. Let G be a p o-group and let \( G^+ = \{ g; g \in G, g \geq 0 \} \) be the set of all positive elements in G. The following three conditions are equivalent.

(1) \( a \leq b \)

(2) \( b - a \in G^+ \)

(3) \( -a + b \in G^+ \)

Moreover,

(i) \( o \in G^+ \)

(ii) \( G^+ + G^+ \subseteq G^+ \)

(iii) \( a + G^+ = G^+ + a \), for all \( a \in G \)

(iv) \( G^+ \cap (-G^+) = \{ 0 \} \)

On the other hand, suppose that \( G^+ \) is a subset of G which possesses The properties (i) — (iv). Then it is easy to see that G becomes a p o-group if one defines a relation \( \leq \) on G by \( f \leq g \) if and only if \( g - f \) and \(-f + g \in G^+ \).
Proposition 1.3. A p o-group is an l-group if and only if for all \( a \in G \), \( aVo \) exists.

Proof: If \( G \) is an l-group, then obviously \( aVo \) exists for all \( a \in G \). Conversely, let \( G \) be any p o-group in which \( aVo \) exists, for all \( a \). Then \((a-b)V+) exists, it is not hard to see that \((a-b)V+)\) is the least upper bound of \( a \) and \( b \). Similarly \( aA \) always exists in \( G \).

Theorem 1.2 and proposition 1.3 gives

Theorem 1.4. For any l-group \( G \), let \( G^+ \) be the set of its positive elements, that is \( G^+ = \{g; g \geq 0, g \in G\} \). The following three conditions are equivalent.

1. \( a \leq b \).
2. \( b - a \in G^+ \).
3. \( -a + b \in G^+ \).

Moreover,

1. \( 0 \in G^+ \).
2. \( G^+ + G^+ \subseteq G^+ \).
3. \( a + G^+ = G^+ + a \), for all \( a \in G \).
4. \( G^+ \cap (-G^+) = \{0\} \).
5. \( aVo \in G^+ \), that is \( aVo \) exists, for all \( a \in G \).

On the other hand, it is easy to see that if \( G^+ \) is a subset of \( G \) and has the properties (i) — (v), then \( G \) becomes an l-group if one defines a relation \( \leq \) on \( G \) by \( f \leq g \) if and only if \( g - f \) and \( -f + g \in G^+ \).

Notation: Thereafter, a partial order \( \hat{P} \) or an right order \( \hat{P} \) on \( G \) is meant that \( \hat{P} = \{g \geq 0, g \in G\} \) under that particular order.

Definition: A subgroup \( C \) of \( (G, \leq, +) \) is convex, provided \( C \) contains along with \( x \geq 0 \) also all \( y \)'s such that \( x \geq y \geq 0 \). Hence \( C \) is convex if
and only if for any \( c_1, c_2 \in C \), \( c_1 \leq c \leq c_2 \), we have \( c \in C \).

Lemma 1.5. Let \( C \) be a convex subgroup of a partially ordered group \( G \). Let \( R(C) = \{ C \cdot g \mid g \in G \} \) be the set of all right cosets of \( C \) in \( G \).

If we define \( C \cdot g \leq C \cdot h \) to mean that there exists \( c \in C \) with \( c \cdot g \leq h \), then this defines a partial order on the set \( R(C) \).

Proof: (i) \( C \cdot g \leq C \cdot g \), for every \( g \in G \). (Reflexive)

(ii) If \( C \cdot g_1 \leq C \cdot g_2 \) and \( C \cdot g_2 \leq C \cdot g_1 \). Then there exist \( c_1, c_2 \in C \), such that \( c_1 \cdot g_1 \leq g_2 \) and \( c_2 \cdot g_2 \leq g_1 \). Therefore \( c_2 \cdot c_1 \cdot g_1 \leq c_2 \cdot g_2 \leq g_1 \). Hence \( c_2 + c_1 = c_2 \cdot g_2 - g_1 \cdot c_2 \leq g_1 - c_2 = -c_1 - c_2 \). Consequently \( c_2 + g_2 - g_1 - c_2 \in C \), by convexity. So \( g_2 - g_1 \in C \). Hence \( C \cdot g_2 - g_1 = C \). It follows that \( C \cdot g_2 = C \cdot g_1 \) (antisymmetric).

(iii) If \( C \cdot g_1 \leq C \cdot g_2 \) and \( C \cdot g_2 \leq C \cdot g_3 \). Then there exist \( c_1, c_2 \in C \) such that \( c_1 \cdot g_1 \leq g_2 \) and \( c_2 \cdot g_2 \leq g_3 \). Therefore \( c_2 \cdot c_1 \cdot g_1 \leq c_2 \cdot g_2 \leq g_3 \). So \( c_2 + c_1 + g_1 \leq g_3 \).

Hence \( C \cdot g_1 \leq C \cdot g_3 \) (Transitive). Consequently \( \leq \) is a partial order on \( R(C) \).

In an \( l \)-group, the following properties are trivially verified:

(i) \( a + (x \land y) = (a + x) \land (a + y) \) and \( (x \land y) + b = (x + b) \land (y + b) \).

(ii) \( a + (x \lor y) = (a + x) \lor (a + y) \) and \( (x \lor y) + b = (x + b) \lor (y + b) \).

(iii) \( -(a \land b) = (-a) \land (-b) \) and \( -(a \lor b) = (-a) \lor (-b) \).

Definition: A subgroup of an \( l \)-group \( G \) which is also a sublattice, is an \( l \)-subgroup. A convex \( l \)-subgroup is a convex subgroup and also a sublattice.

Lemma 1.6: In lemma 1.5, if \( G \) is an \( l \)-group and \( C \) is a sublattice, then \( R(C) \) is a lattice with \( (C \cdot x) \lor (C \cdot y) = C \cdot x \lor y \) and \( (C \cdot x) \land (C \cdot y) = C \cdot x \land y \).

Proof: Note that \( C \cdot x \lor y \geq C \cdot x \), \( C \cdot y \). Now suppose \( C \cdot g \geq C \cdot x \), \( C \cdot y \). Then
there exist $c_1, c_2 \in C$, such that $c_1 + x \leq g$, $c_2 + y \leq g$. Hence $x \leq -c_1 + g$ and $y \leq -c_2 + g$. Therefore $xVy \leq (-c_1 + g)V(-c_2 + g) = (-c_1)V(-c_2) + g = c + g$, where $c = (-c_1)V(-c_2) \in C$, since $C$ is a sublattice. Hence $xVy \leq c + g$. Therefore $C+xVy \leq C+g$. Consequently $C+xVy = (C+x)V(C+y)$. Dually $C+x\Lambda y = (C+x)\Lambda (C+y)$.

Hence $R(C)$ is a lattice.

**Example 1.7:** Let $C$ be the set of all complex numbers, then $(C, \leq, +)$ is a partially ordered group in which $a+bi \leq c+di$ if and only if $b=d$ and $a \leq c$. But $(C, \leq, +)$ is not an $l$-group.

**Example 1.8:** Let $R$ be the set of all real numbers, then $(R, \leq, +)$ is an $l$-group with usual order.

**Example 1.9:** Let $Z$ be the set of all integers, then $(\mathbb{Z}, \leq, +)$ is an $l$-group with partial order defined by $(a, b) \leq (c, d)$, if and only if $a \leq c$ and $b \leq d$.

The following proposition 1.10 is important in Chapter 2, and also is an example of non-abelian $l$-group.

**Definition:** Let $S$ be a totally ordered set, and let $G$ be the group of all functions $f : S \to S$ such that $f$ is one to one, onto and the inequality $x < y$ ($x, y \in S$) implies that $xf < yf$. We call such a function an automorphism of $S$.

**Proposition 1.10:** If $f$ and $g$ are automorphisms of $S$, then define $f \leq g$ if $xf \leq xg$ for all $x \in S$. Then this defines a partial order on $G$ under which $G$ is a lattice-ordered group.

**Proof:** Let $G$ be the group of all automorphisms of $S$. Then

(i) $f \leq f$, for every $f \in G$.

(ii) If $f \leq g$ and $g \leq f$, then $xf \leq xg$ and $xg \leq xf$, for every $x \in S$. Hence $xf = xg$, for every $x \in S$; since $S$ is totally ordered. Therefore $f = g$. 
(iii) Suppose \( f \preceq g \), and \( g \preceq h \). Then \( xf \preceq xg \) and \( xg \preceq xh \), for every \( x \in S \).
Therefore \( xf \preceq xh \), for every \( x \in S \). Consequently \( f \preceq h \).
Therefore "\( \preceq \)" is a partial order on \( G \).

Now given any \( f, g \in G \), we define \( F \) by \( xF = xfVxg \), for every \( x \in S \).
Suppose \( x > y, x, y \in S \), then \( xF = xfVxg \), since \( S \) is totally ordered.
Therefore \( F \) is order-preserving, and hence \( F \) is one-one. Furthermore, if \( y \in S \), then there exist \( x_1, x_2 \in S \), such that \( x_1f = y, x_2g = y \).
Without loss of generality, say \( x_1 > x_2 \), we get \( x_2f < x_1f = y = x_2g \).
That is \( x_2g > x_2f \). Hence \( x_2F = x_2fVx_2g = x_2g = y \). Therefore \( F \) is onto.
Consequently \( F \in G \). Note that \( F \geq f, g \). Now let \( h \geq f, g \). Then
\( h(x) > f(x) \) and \( h(x) > g(x) \), for every \( x \in S \). Therefore \( xh \geq xfVxg \), for every \( x \in S \). Hence \( h \geq F \). Consequently \( F = fVg \). Similarly \( f \geq g \in S \).
Hence \( G \) is a lattice. Finally, let \( f > g \in G \). Consider \( h_1 f h_2 \) and
\( h_1 gh_2 \), where \( h_1, h_2 \in G \). Then \( xh_1 = xh_1 \) for every \( x \in S \). Therefore
\( xh_1f > xh_1g \), for every \( x \in S \). So \( xh_1f > xh_1gh_2 \), for every \( x \in S \). That
is \( h_1f > h_1gh_2 \). Hence \( G \) is an \( l \)-group.

Definition: A subgroup \( H \) of \( G \) is transitive, if and only if for every \( x, y \in S \), there exists \( h \in H \), such that \( xh = y \).

Notation: \( F \) or \( x \in G \), and \( G \) is an \( l \)-group. \( |x| = xV(-x) \).

Lemma 1.11: Let \( G \) be an \( l \)-group and a \( \in G \), \( x, y \in G^+ \).

(i) \( a = aVo + a\ominus \).

(ii) \( (aVo) \Lambda ((-a)Vo) = o \).

(iii) \( |a| = aVo + (-a)Vo \).

(iv) \( x\ominus y = o \) implies \( x + y = y + x \).

(v) Let \( b, c \in G^+ \) be such that \( b\Lambda c = o \) and \( a = b - c \). Then \( b = aVo \), \( c = (-a)Vo \).
In other words, \( b = aVo \), \( c = (-a)Vo \) are the unique elements of \( G \) such
that \( b+c=a, \ a=b-c. \)

**Proof:**

(i) \( a-a\lambda_0=a+(-a)\lambda_0=(a-a)V(a+o)=oVa. \) Hence \( a=a\lambda_0+a\lambda_0. \)

(ii) \( (a\lambda_0)\land((-a)\lambda_0)=(a+oV(-a))\land((-a)\lambda_0)=((-a-(a\lambda_0))\land(-(a\lambda_0)))=a\lambda_0-a\lambda_0=o. \)

(iii) \( a\lor(-a)=(a\land(-a))\lor+(a\lor(-a))\lambda_0 \) by (i)

\[
= a\lor(-a)\lor(-(a\land a))\lor_0 \geq o. \text{ Hence } a\lor+(-a)\lor_0=(a+(a\lor_0)\lor((o+(-a)\lor_0)
= ((a-a)\lor_0)\lor((-a)\lor_0)=oVaV(-a)\lor_0=a\lor(-a)=|a|.
\]

(iv) \( x\lor y=o \) implies \( x+y=x\lor x\land y+y=x(-x)V(-y)+y=(x-x+y)V(x-y+y) \)

\[
yVx=xV_0+y+x, \text{ since } x\land y=y\land x=o. \text{ Hence } x+y=y+x, \text{ if } x\land y=o.
\]

(v) If \( a=b-c, \) then \( b=a+c. \) Hence \( o=b\lambda c=(a+c)\lambda c=a\lambda_0+c. \) Therefore \( c=-(a\lambda_0)=(-a)\lambda_0, \) and \( b=a+c=a+(-a)\lambda_0=oVa. \)

**Corollary 1.12:** If \( G \) is an \( 1 \)-group. Let \( H \) be a subgroup of \( G, \) such that \( H\supseteq G. \) Then \( H=G. \)

**Proof:** For every \( a\in G, \) we have \( a\lor_0, -(a\lambda_0)\in H. \) Hence \( a=a\lor_0+a\lor_0\in H. \)

That is, \( H=G. \)

**Lemma 1.13:** Let \( G \) be an \( 1 \)-group, \( x, y, z\in G. \) Then \( x\land(y+z)<x\land y+x\land z. \)

**Proof:** \( x\land y+x\land z=(x+x\land z)\land(y+x\land z)=\{(x+x)\land(x+z)\}\land\{(y+x)\land(y+z)\} \)

\[
=\{(x+x)\land(x+z)\land(y+x)\}\land(y+z)>x\land(y+z).
\]

To prove Lemma 1.14, we need the result from Birkhoff [1] P.134 that "A lattice \( G \) is distributive if and only if \( a\land x=a\land y \) and \( a\lor x=a\lor y \)

implies \( x=y, \) for every \( a, x, y \in G. \)

**Lemma 1.14:** Any \( 1 \)-group is a distributive lattice.

**Proof:** Let \( G \) be an \( 1 \)-group, \( a, x, y\in G \) and \( a\land x=a\land y, \) \( a\lor x=a\lor y. \)

We know that \( a-a\land x+x=a+(-a)V(-x)+x=(a-a+x)V(a-x+x)=xVa. \) Therefore \( x=a\land x-a+xVa=a\land y-a+yVa \) by assumption

\[
=y.
\]
The following lemma is useful.

Lemma 1.15 (A. H. Clifford [4]): Let G be an 1-group. Let M be a
convex 1-subgroup of G. Let a, b ∈ G. For any x, let G(M, x) denote
the smallest convex 1-subgroup of G containing M and x. If x > 0,
then G(M, x) = \{g; |g| ≤ m_1 + m_2 + x + ... + m_n + x + m_{n+1},
for some m_i ∈ \mathbb{M}^+\}.
Moreover G(M, a) ∩ G(M, b) = G(M, aAb).

Proof: Let \( A = \{g; |g| ≤ m_1 + m_2 + x + ... + m_n + x + m_{n+1},
m_i ∈ \mathbb{M}^+\} \). Since
\( G(M, x) \) is a subgroup and contains M and x, it is clear that \( G(M, x) \)
contains all expressions of the form \( m_1 + m_2 + x + ... + m_n + x + m_{n+1} \).
Moreover \( o < |g|, \) for every \( g ∈ G \), hence \( G(m, x) ⊆ A \), by convexity.

Now if \( |g_1| ≤ m_1 + m_2 + x + ... + m_p + x + m_{p+1}, \) and \( |g_2| ≤ m_1 + m_2 + x + ... + m_q + x + m_{q+1}, \)
m_1, n_j ∈ \mathbb{M}^+ . Then \( (g_1Vg_2)V(-g_1Vg_2) = g_1Vg_2V((-g_1)A(-g_2)) ≤ g_1Vg_2V(-g_1Vg_2)\)
\( = (g_1V(-g_1)Vg_2V(-g_2)) = |g_1|V|g_2| ≥ m_1 + m_2 + x + ... + m_n + x + m_{n+1}, \)
since \( m_1n_j > m_1, n_1, \) and assuming that \( q > m \). This implies \( g_1Vg_2 ≤ A \).

Furthermore, suppose \( g_1, g_2 ∈ A, \) and \( g_1 ≤ g_2 \), then \( g_1Vg_2 ≤ A \). Hence
\( |g| = gV(-g) ≤ g_2V(-g_1) ≤ A, \) since \( -g_1 ≤ A \). Therefore \( g ∈ A \). Moreover,
\( |g_1 - g_2| = (g_1 - g_2)V = g_1V(-g_1) + g_2V(-g_2) = |g_1| + |g_2| \). Therefore \( g_1 - g_2 ∈ A \).

Consequently, \( A \) is a convex 1-subgroup of G containing M and x. That
is \( G(M, x) = A \). Now suppose that a, b ∈ G; we get \( o ≤ aAb ≤ a \) and \( o ≤ aAb ≤ b \).

Hence \( G(M, a) ∩ G(M, aAb) \) and \( G(M, b) ∩ G(M, aAb) \), by convexity, that is,
\( G(M, a) ∩ G(M, b) ∩ G(M, aAb) \). Let \( g G(M, a) ∩ G(M, b) \). Then
\( |g| ≤ m_1 + a + m_2 + a + ... + m_p + a + m_{p+1} \) and \( |g| ≤ n_1 + b + n_2 + b + ... + n_q + b + n_{q+1} \),
where \( m_i, n_j ∈ \mathbb{M}^+ \), therefore
\( |g| ≤ m_1 \Lambda (m_1 + a + m_2 + a + ... + m_p + a + m_{p+1}) \Lambda (n_1 + b + n_2 + b + ... + n_q + b + n_{q+1}) \)
\( ≤ m_1 \Lambda (n_1 + b + n_2 + b + ... + n_q + b + n_{q+1}) + a \Lambda (n_1 + b + n_2 + b + ... + n_q + b + n_{q+1}) + ... + m_{p+1} \Lambda (n_1 + b + n_2 + b + ... + n_q + b + n_{q+1}) + a \Lambda (n_1 + b + n_2 + b + ... + n_q + b + n_{q+1}) \).
We now consider some properties of prime subgroups of an 1-group, which are important in the representation of an 1-group as a transitive 1-subgroup of the 1-group of automorphisms of an ordered set to be observed in Chapter 2.

Definition: Let H be a convex 1-subgroup of an 1-group G. If there is an element \( g \in G \) such that H is a maximal convex 1-subgroup of G with respect to not containing g, then H is called a regular subgroup.

Theorem 1.16: For a convex 1-subgroup M of an 1-group G, the following are equivalent.

(i) If A and B are convex 1-subgroups of G such that \( A \cap B \leq M \), then either \( A \leq M \) or \( B \leq M \).

(ii) If A, B are convex 1-subgroups of G, \( A \not\subseteq M \) and \( B \not\subseteq M \), then \( A \cap B \not\subseteq M \).

(iii) If \( a, b \in G^+ \setminus M \), then \( aAb \in G^+ \setminus M \).

(iv) If \( a, b \in G^+ \setminus M \), then \( aAb \neq o \).

(v) The lattice \( R(M) \) of right cosets of M is totally ordered.

(vi) The set of convex 1-subgroups of G which contain M form a totally ordered set (under inclusion).

(vii) M is the intersection of a chain of regular subgroups.

Proof: We prove that (i) \( \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (i) \).
(i) \(\Rightarrow\) (ii). If \(A \cap B = M\), then, by (i), \(A \subseteq M\) or \(B \subseteq M\), contradiction. Hence \(A \cap B \not\subseteq M\).

(ii) \(\Rightarrow\) (iii). Since \(G(M, a) \not\subseteq M\), \(G(M, b) \not\subseteq M\), we have by (ii) \(G(M, a) \cap G(M, b) = G(M, aB) \not\subseteq M\). Therefore \(aB \not\subseteq M\), and so \(aB \subseteq G^+ \subseteq M\).

(iii) \(\Rightarrow\) (iv) Let \(a, b \in G^+ \subseteq M\). Then \(aB \subseteq G^+ \subseteq M\), by (iii). Hence \(aB \not\subseteq o\).

(iv) \(\Rightarrow\) (v). By way of contradiction, if neither \(M + g \leq M + f\) nor \(M + f \leq M + g\), then \(o < g - gAf \subseteq M\), since otherwise \(M + g = M + gAf \leq M + f\). Likewise \(o < f - gAf \subseteq M\). But \((g - gAf) \land (f - gAf) = gAf - gAf = o\), a contradiction. Hence \(R(M)\) is totally ordered.

(v) \(\Rightarrow\) (vi). Let \(M_1, M_2\) be convex \(1\)-subgroups of \(G\) which contain \(M\). Suppose \(M_1\) and \(M_2\) are incomparable, then there exist \(m_1, m_2 \in G^+\), such that \(m_1 \in M_1 \not\subseteq M_2\) and \(m_2 \in M_2 \not\subseteq M_1\) by Corollary 1.12. Without lost of generality, we may assume that \(M + m_1 \geq M + m_2 > M\), then there exist \(m, m' \in M\), such that \(m + m_1 \geq m_2 > m'\). This implies \(m_2 \in M_1\), by convexity, a contradiction. Hence the set of convex \(1\)-subgroup of \(G\) which contain \(M\) form a totally ordered set.

(vi) \(\Rightarrow\) (vii). If \(M\) is a maximal convex \(1\)-subgroup of \(G\) with respect to not containing some \(g \in G\), we are done. For every \(o \neq g \in G\) and \(g \not\subseteq M\), there exists a maximal convex \(1\)-subgroup of \(G\) with respect to not containing \(g\) in the set of (vi). For every \(g \not\subseteq o\), \(g \not\subseteq M\), we denote one of such subgroups by \(C_g\). Then \(\bigcup_{o \neq g \in G \cap \not\subseteq M} C_g \supset M\). Clearly, the set of all \(C_g\)'s is totally ordered, since \(C_g\)'s are in the set of (vi). Let \(g \not\subseteq M, g \not\subseteq G\). Then \(g \not\subseteq C_g\). Hence \(g \not\subseteq \bigcup_{o \neq g \in G \cap \not\subseteq M} C_g\). That is,
Suppose $M$ is regular. If there exist $a$, $b$ such that $a \in A \setminus M$ and $b \in B \setminus M$, then $a \neq b$, by hypothesis. Now $G(M, a) \cap G(M, b) = G(M, a \wedge b) = M$, since $a \wedge b \in A \cap B$ and $A \cap B \subseteq M$. Moreover $G(M, a) \supseteq M$ and $G(M, b) \supseteq M$, but then $M$ is not regular. Because $g \not\in M$ implies that $g \not\in G(M, a)$ or $g \not\in G(M, b)$. Consequently, $A \subseteq M$ or $B \subseteq M$.

Now suppose $M$ is an intersection of a chain of a regular subgroups $C$ where $C_g$ is a regular subgroup with respect to not containing $g \in G$. That is, suppose that $A \cap B \subseteq M = \bigcap_{g \in G} C_g$. Then $A \cap B \subseteq C_g$ for every such $C_g$. Therefore $A \subseteq C_g$ or $B \subseteq C_g$ by previous argument.

If there exist $C_{g_1}$ and $C_{g_2}$ of such type such that $A \subseteq C_{g_1}$, $B \subseteq C_{g_2}$, and $B \subseteq C_{g_1}$, $A \subseteq C_{g_2}$, then from $A \subseteq C_{g_1}$ and $A \subseteq C_{g_2}$, we have $C_{g_2} \subseteq C_{g_1}$, since the set of such $C_g$'s are totally ordered. Similarly, $C_{g_2} \subseteq C_{g_1}$, a contradiction. Hence $A \subseteq C_g$ for every such $C_g$ or $B \subseteq C_g$ for every such $C_g$. That is $A \subseteq \bigcap_{g \in G} C_g$ or $B \subseteq \bigcap_{g \in G} C_g$. Hence $A \subseteq M$ or $B \subseteq M$.

Definition: (a) Let $G$ and $H$ be $p$-groups. Then an $o$-homomorphism $\theta$ from $G$ into $H$ is an isotone homomorphism: that is to say $\theta$ is a group homomorphism such that for any $x, y \in G$, if $x \leq y$, then $x\theta \leq y\theta$.

(b) If $G$ and $H$ are $l$-groups, then an $l$-homomorphism $\theta$ from $G$ into $H$ is an $o$-homomorphism such that for any $x, y \in G$,

\[(x \lor y)\theta = x\theta \lor y\theta \ldots (i)\]
\[(x \land y)\theta = x\theta \land y\theta \ldots (ii)\]

(d) An $l$-isomorphism $\theta$ from $l$-group $G$ into $l$-group $H$ is an $o$-isomorphism from $G$ into $H$ such that (i) and (ii) of (c) hold.
Remark: An o-homomorphism need not be an l-homomorphism. Consider the l-group $G$ of all continuous functions from $[0, 1]$ into reals. Let $H$ be the l-group of all linear functions from $[0, 1]$ into reals. Then $H$ is a subgroup of $G$. The inclusion mapping from $H$ into $G$ is an o-homomorphism but not an l-homomorphism.

Definition: The Cardinal product of l-groups $A_i$, $i \in I$, denoted by $\prod_{i \in I} A_i$ is the direct product $\prod_{i \in I} A_i$ with the partial order defined by $a \geq o$ if and only if $a_i \geq o_i$, for every $i \in I$, where $a \in \prod_{i \in I} A_i$ and $a_i \in A_i$.

To verify that $\prod_{i \in I} A_i$ is an l-group is routine.

Definition: $G$ is l-isomorphic to a subdirect product of l-groups $\{A_i; i \in I\}$ if and only if there is an l-group monomorphism $k: G \rightarrow \prod_{i \in I} A_i$ such that $k \pi_i$ is an epimorphism for all $i \in I$, where $\pi_i: A \rightarrow A_i$ is the $i$th projection.

Corollary 1.17: Every abelian l-group is a subdirect product of abelian o-groups.

Proof: Let $C$ be a maximal convex l-subgroup with respect to not containing $g \in G$, then $C \triangle G$ and $G/C = R(C)$. Hence $G/C$ is an abelian o-group. Define $f: G \rightarrow R = \prod_{0 \neq g \in G} R(C)$ by $xf = (\ldots, Cx, \ldots)$, for every $x \in G$.

Clearly $f$ is an l-homomorphism. Furthermore, $0 \neq g \in G$, then $C + g \neq C$. Hence $g \not\in \ker f$. It follows that $f$ is one to one. Moreover $f_{p_g}$ is an epimorphism for all $g \neq 0 \in G$, where $p_g$ is the projection from $R$ onto $R(C)$. Consequently, $G$ is a subdirect product of $G/C_g$, for every $0 \neq g \in G$.

Definition: Any convex l-subgroup of an l-group $G$ which satisfies one of the conditions in Theorem 1.10 is called prime.
Corollary 1.18: If $A$ and $B$ are primes, then $A \cap B$ is prime, if and only if $A$ and $B$ are comparable.

Proof: $A \cap B$ is prime, then if $A \nmid A \cap B$ and $B \nmid A \cap B$, then $A \cap B \nmid A \cap B$, by (ii) of Theorem 1.16, a contradiction. Therefore, $A = A \cap B$ or $B = A \cap B$. Hence $A \supset B$ or $B \subset A$. Conversely, if $A \supset B$ or $B \subset A$, then $A \cap B = \text{Bor } A$ which is prime.

Corollary 1.19: If the convex 1-subgroup $M$ is regular, then $M$ is prime.

Proof: By (vii) of Theorem 1.16.

Corollary 1.20: If $A$, $B$, $C$ are regular subgroups with respect to not containing some $g \notin \text{ocG}$, and $A$, $B$, $C$ are distinct, then $A \cap B \nmid C$.

Remark: From Corollary 1.18, we know that the p. o. set of prime subgroups looks like the roots of a tree in picture, we call it a root system. The following diagram is a root system.

```
  G
  |
  A
  |
  B
  |
  B1B2
  |
  C
  |
  D
```

```
where $B_1$ and $B_2$ are regular, and $B$ covers $B_1$, $B_2$. That is there is no prime subgroup between $B$, $B_1$ and $B$, $B_2$ in the diagram.

**Proposition 1.21** For any prime subgroup $A$ of an $l$-group $G$, there exists a minimal prime subgroup $M$ with $M \subseteq A$.

**Proof:** Let $B = \bigcap_{i \in I} A_i$, where $\{A_i : i \in I\}$ is a chain of prime subgroups. Then if $a, b \in G \uparrow B$ we have $a, b \in G \uparrow A_i$ for some $i$. Hence $a \wedge b \in G \uparrow A_i$

That is $a \wedge b \notin A_i$. Hence $a \wedge b \notin B$. Consequently $a \wedge b \notin B$.

That is, every chain of prime subgroups is bounded below. Hence by Zorn's Lemma, for any prime subgroup $A$, there exists a minimal prime $M$ with $M \subseteq A$.

**Example 1.22:** Let $G$ be the stabilizer of "$x$" in $A(X)$, the $l$-group of all automorphisms of totally ordered set $X$. Then if $a, b \in A(X) \uparrow G$, we have $xa \neq x \neq xb$. Hence $x(a \wedge b) = x a \wedge x b = x a$ or $x b \neq x$, since $X$ is totally ordered. That is $a \wedge b \neq i$. Therefore $G$ is prime.

We claim that if $X$ is actually the set of real numbers then $G$ is also regular. Now suppose $g \notin G$, then $xg \neq x$. For any $x \neq a \in X$, we have

(i) If $x > xg > a$, then there exists $g_0 \in G$ such that $xg_0 = a$.

(ii) If $x > a > xg$, then there exists $g_0 \in G$, such that $xg_0 = a$.

(iii) If $a > x > xg$, then either $a > xg^{-1} > x$ or $xg^{-1} > a > x$.

In either case, there exists $g_0 \in G$, such that $xg^{-1} = a$.

(iv) Note that the remaining case is $xg > x > a$. In this case, we have either $x > xg^{-1} > a$ or $x > a > xg^{-1}$. Hence, there exists $g_0 \in G$, such that $xg^{-1} = a$. Therefore, $A(X) = G \cup_{g_0 \in G} \cup_{g_0 \in G} g_0 G \cup_{g_0 \in G} g_0^{-1} g_0 G$

$$= G \cup_{g \in G} \cup_{g^{-1} \in G} \langle G, g \rangle$$

That is, $A(X) = \langle G, g \rangle$. Consequently, $G$ is regular.
Remark: Let \( g \in G, f \in \langle X \rangle \backslash G \). Then \( xf \neq x \). Hence \( xfg \neq x \). Therefore \( xfgf^{-1} \neq x \). That is \( fgf^{-1} \notin G \). Hence \( G \) is not normal.

Proposition 1.23. If every minimal prime subgroup of \( G \) is normal, then \( G \) is a subdirect product of \( o \)-groups.

Pf: The proof is similar to that of Corollary 1.17. Note that for every \( g \in G \), there exists regular subgroup \( C_g \) of \( G \) with respect to not containing \( g \). Hence there exists a minimal prime subgroup \( M_g \subseteq C_g \) and \( M_g \neq M \).

Lemma 1.24: If \( G \) is an \( l \)-group, the following are equivalent:

1. \( G \) is totally ordered.
2. Every convex \( l \)-subgroup of \( G \) is prime.
3. The set of all prime subgroups of \( G \), \( \Gamma(G) \), is totally ordered.

Pf: Suppose \( G \) is totally ordered and \( M \) is a convex \( l \)-subgroup of \( G \). Let \( a, b \in G \uparrow M \). Then \( a \Lambda b = a \lor b \). Hence \( a \Lambda b \in G \uparrow M \). Therefore \( M \) is prime. Conversely, if every convex \( l \)-subgroup of \( G \) is prime, then \( \{0\} \) is prime, implies \( G \) is totally ordered. This converse also implies \( \Gamma(G) \) is totally ordered. Now suppose \( M' \neq \{0\} \) is a minimal prime subgroup of \( G \), let \( C_g' \) be regular in \( M' \) with respect to not containing \( g \), \( g \in M' \). Then if \( C_g' \) is not regular in \( G \), we have \( C_g \) regular in \( G \) with respect to not containing \( g \), hence \( C_g \supseteq M' \), contradiction. That is \( C_g' \) is regular in \( G \), consequently, \( C_g' \) is prime, contradiction. Therefore \( M' = \{0\} \). Hence every convex \( l \)-subgroup of \( G \) is prime.

Example 1.25: Let \( H = \bigcap_{r \in \mathbb{R}} Z_r \), \( Z_r = \mathbb{Z} \). Let \( G = \{ h \in H, \text{ such that the support of } h \text{ satisfies ascending chain condition } \} \cup \{0\} \).

Let \( g, h \in G \), \( h \neq g \), \( D = \{ r; rg \neq rh \} \subseteq \text{Support of } g \cup \text{Support of } h \)

Clearly, \( \text{Support of } g \cup \text{Support of } h \) satisfies ascending chain condition (A.C.C.). Hence \( D \) satisfies A.C.C.
Therefore D has a maximum member r. Then define g > h if and only if \( rg > rh \). Now \( 0 \in G \). Let \( f, g \in G \), then \( D = \{ r, rg \neq rf \} \) satisfies A.C.C. we have \( f - g \in G \). G is a group. Furthermore

(i) \( g \geq g \) for every \( g \in G \). Reflexive.

(ii) Suppose \( g > h, h > g \), then it is immediate that \( g = h \) Antisymmetric.

(iii) Suppose \( g > h \) and \( h > f \), then \( r_1 g > r_1 h \) and \( r_2 h > r_2 f \) where
\[
\begin{align*}
  r_1 &= \max \{ r; \ rg \neq rh \} \\
  r_2 &= \max \{ r; \ rh \neq rf \}
\end{align*}
\]
Let \( r_3 = \max \{ r; \ rf \neq rg \} \). To show that \( r_3 = \max \{ r_1, r_2 \} \) is routine.

Hence \( g > f \).

Consider any two elements of G which are comparable, that is \( g > h \) and \( g, h \in G \), let \( r_0 = \max \{ r; \ rg \neq rh \} \). Then \( r_0 g > r_0 h \), and note that
\[
\begin{align*}
  r_0 &= \max \{ r; \ r(f+g+k) \neq r(f+g+k) \} \text{ for every } f, k \in G.
\end{align*}
\]
Hence
\[
\begin{align*}
  r_0(f+g+k) &= r_0 f + r_0 g + r_0 k \\
  &\geq r_0 f + r_0 h + r_0 k \\
  &= r_0(f+g+k)
\end{align*}
\]
Consequently \( f+g+k \geq f+h+k \). Therefore, G is totally ordered abelian group. This implies \( \Gamma(G) \) is totally ordered by Lemma 1.24.

Now we are going to discuss \( \Gamma(G) \). For every \( r \in R \), let \( G_r = \{ h; \text{ max. element of support of } h \prec r, h \in G \} \cup \{ 0 \} \). Then \( 0 \in G_r \), and if \( f, g \in G_r \), clearly \( f - g \in G_r \) and \( fV g, fAg c G_r \). If \( f \leq g, \leq g \), then \( g \in F c G_r \). Consequently, \( Gr \) is a regular subgroup of G with respect to not containing k where k is any element whose max. element of support is r. On the other hand \( r_1 \geq r_2 \) if and only if \( G_{r_1} \geq G_{r_2} \) under inclusion. Conversely, if \( G_g \) is regular with respect to not containing g, let \( r \) be the max. element of support of g. Then for every element \( f \) in \( G_g \), the max. element of support of f is less than r. Hence we have thus shown that there
exists a one-one order preserving mapping from R onto the set of all regular subgroup of G. Let (0,r) denote Gr and (1,r) denote the intersection of all regular subgroups Gr'. Then (1,r') is prime and (1,r)>(0,r) for every r. Hence Γ(G) = {(i,r), i = 0,1} ∪ {G,{0}}.

where (i_1,r_1)>(i_2,r_2) if and only if r_1>r_2 or r_1=r_2 and i_1>i_2 and G>(i,r), (i,r)>{0} for every i, r. The picture of Γ(G) is as follow:

```
(1,r)
(0,r)

(1,r_n)
(0,r_n)

(1,r_m)
(0,r_m)

...
...
(0)
```

We have thus shown that G contains prime subgroups that are not regular such as (1,r') for every r'∈R.
Lemma 2.1: Let C be a convex 1-subgroups of an 1-group G, and let
$O \subseteq a \in G$. Define $C^*(a) = \{ x \in G : a \lor |x| \in C \}$. Then $C^*(a)$ is a convex 1-subgroup of G and $C \subseteq C^*(a)$.

Proof: $a \land |0| = 0 \in C$. Hence $0 \in C^*(a)$. Consider $d_1, d_2 \in C^*(a)$, then
$0 \leq a \land |d_1 - d_2| \leq a \land (|d_1| + |d_2|) \leq (a \land |d_1|) + (a \land |d_2|) \in C$. Hence $a \land |d_1 - d_2| \in C$. That is, $d_1 - d_2 \in C^*(a)$. Consequently $C^*(a)$ is a subgroup. Furthermore,
$(a \land |d_1|) V (a \land |d_2|) = a \land (|d_1| V |d_2|) \geq a \land |d_1| V |d_2| \geq 0$, implies $a \land |d_1| V |d_2| \in C$. Hence $d_1 V d_2 \in C^*(a)$. Therefore $d_1 \land d_2 \in C^*(a)$ dually. Consequently, $C^*(a)$ is a sublattice. Now if $d \geq 0 \in C^*(a)$ and $x \in G$ such that $d \geq x \geq 0$, then
$a \land |d| \geq a \land |x| \geq 0$ implies $a \land |x| \in C$, that is $x \in C^*(a)$. Hence $C^*(a)$ is a convex 1-subgroup. Finally, if $g \in C$, then $|g| \in C$, but $0 \leq a \land |g| \leq |g|$. Hence $a \land |g| \in C$, and therefore $g \in C^*(a)$. Consequently $C \subseteq C^*(a)$.

Lemma 2.2: Let C be a convex subgroup of 1-group G, and suppose $R(C)$ is totally ordered. Then each $g \in G$ induces an automorphism $\beta(g,C)$ of $R(C)$ defined by $(C+x) \beta(g,C) = C+x+g$.

Proof: Straight-forward.

If C is a convex 1-subgroup of G and if $R(C)$ is totally ordered, we let $AR(C)$ denote the 1-group of all automorphisms of $R(C)$.

Lemma 2.3: If C is a convex 1-subgroup of G and if $R(C)$ is totally ordered, then the mapping $\alpha(C) : G \rightarrow AR(C)$ defined by $g \alpha(C) = \beta(g,C)$ is an 1-group homomorphism of G onto a transitive 1-subgroup of AR(C).

Proof: The only non-trivial part of the proof is to show that the lattice operation are preserved. We must show that $(g V 0) \alpha(C) = \beta(g,C) V i$, where $i$ denotes the identity function in $AR(C)$. In other words, if we can show
that for any right coset \( C+x \), \((C+x)(gV0) = (C+x+g)V(C+x)\), we are done. But this follows immediately from Lemma 1.6.

Example 2.4: Let \( G = \mathbb{R} \oplus \mathbb{R} \), then if \( H \) is a convex 1-subgroup of \( G \), then \( H \in \{ \{0\} \oplus \{0\}, \{0\} \oplus \mathbb{R}, \mathbb{R} \oplus \mathbb{R}\} \). Note that \( \mathbb{R} \oplus \mathbb{R} \supset \{0\} \supset \{0\} \oplus \{0\} \) and \( \mathbb{R} \oplus \mathbb{R} \supset \{0\} \supset \mathbb{R} \oplus \{0\} \oplus \{0\} \). Moreover \( \mathbb{R} \oplus \{0\} \) and \( \{0\} \oplus \mathbb{R} \) are maximal convex 1-subgroups with respect to not containing \((0,a),(b,0)\), respectively, where \( a \neq 0 \neq b \).

Theorem 2.5 (Holland [6]): If \( G \) is an 1-group, then \( G \) is 1-isomorphic to a subdirect product of 1-groups \( \{ B_g : 0 \neq g \in G \} \) such that each \( B_g \) is a transitive 1-subgroup of the 1-group of automorphisms of a totally ordered set \( S_g \).

Proof: For each \( 0 \neq g \in G \), by Zorn's Lemma, there exists a convex 1-subgroup \( C_g \) of \( G \) which is maximal with respect to not containing \( x \). By Theorem 1.16 and Corollary 1.19, the set of convex 1-subgroups of \( G \) which contains \( C_g \) form a tower. Let \( S_g = \mathbb{R}(C_g) \). Then by Theorem 1.16, \( S_g \) is totally ordered.

By Lemma 2.3, the mapping \( \alpha_g : G \rightarrow A(S_g) \), where \( (C_g+x)\alpha_g(k) = C_g + x + k, k \in G \), is an 1-homomorphism of \( G \) onto a transitive 1-subgroup \( B_g \) of \( A(S_g) \). Let \( \sigma : G \rightarrow \prod B_g \) defined by \( \sigma(k) = \alpha_g(k); k \in G \). Clearly, since each \( \alpha_g \) is an 1-homomorphism, \( \sigma \) is an 1-homomorphism. Furthermore, if \( 0 \neq g \in G \), then \( C_g + x \neq C_g \). Hence \( g \notin \text{Ker} \alpha_g \), that is \( g \notin \bigcap_{0 \neq x \in G} \text{Ker} \alpha_x = \text{Ker} \sigma \). Consequently \( \sigma \) is one to one. Note that \( \sigma \pi_g \) is an epimorphism for all \( 0 \neq g \in G \), where \( \pi_g : B_g \rightarrow B_g \) is a projection. Hence the proof is complete.

Example 2.6: The maximal convex 1-subgroups of \( \mathbb{Z} \oplus \mathbb{Z} \) without \((a,b)\), where \( a \neq 0 \) or \( b \neq 0 \) are \( \{0\} \oplus \mathbb{Z}, \mathbb{Z} \oplus \{0\} \). A transitive 1-subgroup \( B_1 \) of \( \text{AR}(\{0\} \oplus \mathbb{Z}) \) is \( \{ \alpha_z : z \in \mathbb{Z}, (\{0\} \oplus \mathbb{Z}) \alpha_z = \{z\} \oplus \mathbb{Z} \} \) and a transitive 1-group \( B_2 \) of \( \text{AR}(\mathbb{Z} \oplus \{0\}) \) is \( \{ \beta_z : z \in \mathbb{Z}, (\mathbb{Z} \oplus \{0\}) \beta_z = \mathbb{Z} \oplus \{z\} \} \). Define \( f : \mathbb{Z} \oplus \mathbb{Z} \rightarrow B_1 \times B_2 \) by...
\((z_1, z_2)f = (\alpha_{z_1}, \beta_{z_2})\). Then \(f\) is an \(1\)-monomorphism and \(f\pi_i\) is an epimorphism, where \(\pi_i\) is the \(i\)th projection of \(B_1 \times B_2\).

Example 2.7: The convex \(1\)-subgroup of \((Z \oplus Z) \hat{\times} Z\) are:

(i) \((0) \oplus (0) \times (0)\)  
(ii) \((Z \oplus \{0\}) \times \{0\}\)

(iii) \((\{0\} \oplus Z) \times \{0\}\)  
(iv) \((Z \oplus Z) \times \{0\}\)

(v) \((Z \oplus Z) \hat{\times} \{0\}\)

The regular subgroups are:

(1) \(Z \oplus Z \times \{0\}\)  
(ii) \(\{0\} \oplus Z \times \{0\}\)

(iii) \(Z \oplus \{0\} \times \{0\}\)

A transitive \(1\)-subgroup \(B_1\) of \(AR(\{0\} \oplus Z \times \{0\})\) is

\[\{\alpha_{z_1z_3} : z_1, z_2, z_3 \in Z, (\{0\} \oplus Z \times \{0\})\alpha_{z_1z_2} = (\{0\} \oplus Z) \hat{\times} \{\{z_3\}\}\}\]

which is \(\sigma\)-isomorphic to \(Z \hat{\times} Z\), and \(B_2\) of \(AR(Z \oplus \{0\} \times \{0\})\) is

\[\{\beta_{z_2z_3} : z_2, z_3 \in Z, (Z \oplus \{0\} \times \{0\})\beta_{z_2z_3} = (Z \oplus \{z_2\}) \hat{\times} \{\{z_3\}\}\}\]

which is also \(\sigma\)-isomorphic to \(Z \hat{\times} Z\), and \(B_3\) of \(AR(Z \oplus Z \times \{0\})\) is

\[\{\gamma_{z_3} : z_3 \in Z, (Z \oplus Z \times \{0\})\gamma_{z_3} = (Z \oplus Z) \hat{\times} \{\{z_3\}\}\}\]

which is \(\sigma\)-isomorphic to \(Z\).

Define \(f : (Z \oplus Z) \hat{\times} Z \rightarrow (B_1 \oplus B_2) \hat{\times} B_3\) by \(f(z_1, z_2, z_3) = (\alpha_{z_1z_3}, \beta_{z_2z_3}, \gamma_{z_3})\).

Clearly \(Z \oplus Z \hat{\times} Z\) is \(1\)-isomorphic to a subdirect product of \(B_1, B_2, B_3\).

If \(H\) is the direct product of \(1\)-groups \(B_\alpha\) and if \(B_\alpha\) is the \(1\)-group of automorphisms of an ordered set \(S_\alpha\) where \(S_\alpha \cap S_\beta = \emptyset\) for \(\alpha \neq \beta\), then we may totally order the set \(\bigcup S_{\alpha}\) as follows: first order the collection of sets \(S_{\alpha}\) in any way; for example, it may be well-ordered. Then for \(x, y \in \bigcup S_{\alpha}\), let \(x < y\) if \(x, y \in S_{\alpha}\) and \(x < y\) as elements of \(S_{\alpha}\), or if \(x \in S_{\alpha}\) and \(y \in S_{\beta}\) where
If $\phi \in H$, then $\phi$ induces an automorphism of the ordered set $\bigcup S_\alpha$ in the following way: $x^{\phi^\prime} = x^{\phi_\alpha}$, where $x \in S_\alpha$ and $\phi_\alpha$ is the $\alpha$th component of $\phi$.

From this and Theorem 2.5 we have the following theorem.

**Theorem 2.8:** If $G$ is an $L$-group, $G$ is $L$-isomorphic to an $L$-subgroup of the $L$-group of all automorphisms of an ordered set.

**Definition:** By an $L$-ideal of an $L$-group $G$ is meant a normal subgroup of $G$ provided it contains any $a$, then also all $x$ with $|x| \leq |a|$.

**Theorem 2.9:** An $L$-group $G$ is $L$-isomorphic to a transitive $L$-subgroup of the $L$-group of all automorphisms of an ordered set if and only if there exists a convex $L$-subgroup $C$ of $G$ such that;

1. the set of convex $L$-subgroups of $G$ containing $C$ is totally ordered under inclusion, and
2. the only $L$-ideal of $G$ contained in $C$ is $\{0\}$.

**Proof:** If $G$ is a transitive $L$-subgroup of the $L$-group of automorphisms of an ordered set $L$, and if $x \in L$, then $C = \{g \in G : xg = x\}$ is clearly a convex $L$-subgroup of $G$. $C$ contains no $L$-ideals of $G$. For if $0 \neq g \in C$, then $yg \neq y$, for some $y \in L$. Hence, as $G$ is transitive, there exists $f \in G$ such that $xf = y$. Therefore $xfgf^{-1} = ygf^{-1} \neq yf^{-1} = x$ and so $fgf^{-1} \notin C$. The convex $L$-subgroups of $G$ containing $C$ form a tower, for otherwise, by Theorem 1.16, there exists $a, b \notin C$, such that $a \land b = 1$. That is, $xa \neq x \neq xb$, and yet $x = xl = x(a \land b) = xa \land xb$, which is impossible since $L$ is totally ordered.

Conversely, if $C$ is such a subgroup of $G$, then by Theorem 1.16, $R(C)$ is totally ordered, and by Lemma 2.3, the mapping $\alpha(C)$ is an $L$-homomorphism of $G$ onto a transitive subgroup of $AR(C)$. If $g$ is in the kernel of $\alpha(C)$, then $C + g = C$; thus the kernel is contained in $C$. As the kernel is an $L$-ideal of $G$, the kernel is $\{0\}$ and $\alpha(C)$ is one-to-one.
Corollary 2.10: If there exists an $1$-ideal $K \neq \{0\}$ of $G$ such that every
$1$-ideal ($\neq \{0\}$) of $G$ contains $K$, then $G$ is a transitive $1$-group of auto-
morphisms of an ordered set.

Proof: Let $0 \neq g \in K$, and let $C_g$ be a convex $1$-subgroup of $G$ maximal without
$g$. Then $C_g$ satisfies conclusions (1) and (2) of the above theorem.

Corollary 2.11: A simple $1$-group (without proper $1$-ideal) is a transitive
$1$-group of automorphisms of an ordered set.

Corollary 2.12: If $G$ is abelian and is a transitive $1$-group of auto-
morphisms of an ordered set, then $G$ is totally ordered.

Proof: Any such $C$ in Theorem 2.8 is an $1$-ideal. Hence $C=\{0\}$, and $G$ is
isomorphic as an ordered set to $R(C)$, which is totally ordered.
Chapter Three

Let $S(a_1,a_2,\ldots,a_n)$ denote the normal subsemigroup of a group $G$ that is generated by the elements $a_1,a_2,\ldots,a_n \in G$, and define $S'(a_1,a_2,\ldots,a_n)$ as $S(a_1,a_2,\ldots,a_n)$ with 0 adjoined. These normal subsemigroups will play an important role in dealing with extensions partial orders $P$, that is, for some partial order "$\leq"$ on $G$, $P=\{g \geq 0, g \in G\}$. This is due to the fact that they obey the following rules:

(a) $a \in P$ implies $S'(a) \subseteq P$;
(b) $a \in P, a \neq 0$, implies $P \cap S(-a) = \emptyset$;
(c) $S'(a_1,a_2,\ldots,a_n) = S'(a_1) + S'(a_2) + \ldots + S'(a_n)$;
(d) $-S(a_1,a_2,\ldots,a_n) = S(-a_1,-a_2,\ldots,-a_n)$.

The next result has numerous consequences.

Theorem 3.1 [Fuchs (5)]. A partial order $P$ of a group $G$ can be extended to a full order of $G$, if and only if, it has the property:

(*) for every finite set of elements $a_1,a_2,\ldots,a_n$ in $G$, $(a_i \neq 0)$, the signs $\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_n (\varepsilon_1=0, or 1)$ can be chosen such that

$P \cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots,(-1)^{\varepsilon_n}a_n) = \emptyset$.

Proof: If $P$ can be extended to a full order $Q$, then let $\varepsilon_i$ be chosen such that $-((-1)^{\varepsilon_i}a_i) \in Q$. Now $-S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots,(-1)^{\varepsilon_n}a_n)$

$=S((-1)^{\varepsilon_1}a_1,,-(-1)^{\varepsilon_2}a_2,\ldots,-(-1)^{\varepsilon_n}a_n) \subseteq Q$, and so

$P \cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots,(-1)^{\varepsilon_n}a_n) \subseteq Q \cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots,(-1)^{\varepsilon_n}a_n)$

$\neq \emptyset$.

For the proof of the sufficiency we need the following lemma.

Lemma 3.2. If $P$ satisfies (*) and $a \in G$, then either $P + S'(a)$ or $P + S'(-a)$
defines a partial order \( P' \) in \( G \) which again satisfies (*)
Proof: Suppose that \( G \) contains elements \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \) \((\neq 0)\) such that for every choice of the signs \( \epsilon_i, \eta_j \) one has
\[
P \cap S(a, (-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n) \neq \emptyset, \text{ and}
\]
\[
P \cap S(-a, (-1)^{\eta_1} b_1, (-1)^{\eta_2} b_2, \ldots, (-1)^{\eta_m} b_m) \neq \emptyset, \text{ then the intersection of}
P \text{ with } S((-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n,(-1)^{\eta_1} b_1, (-1)^{\eta_2} b_2, \ldots, (-1)^{\eta_m} b_m)
is never void, contrary to (*). Thus either (i) to every finite set
\[a_1, a_2, \ldots, a_n (\neq 0) \text{ in } G \text{ there are signs } \epsilon_1, \epsilon_2, \ldots, \epsilon_n \text{ such that}
P \cap S(a, (-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n) = \emptyset; \text{ we then put } P' = P + S'(-a);
\]
or (ii) to every finite set \( a_1, a_2, \ldots, a_n (\neq 0) \text{ in } G \text{ there are signs}
\[\epsilon_1, \epsilon_2, \ldots, \epsilon_n \text{ such that } P \cap S(-a, (-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n) = \emptyset; \text{ in this case we put } P' = P + S'(a). \text{ (If both (i) and (ii) are true, we can choose either.) Now in case (i) for example, } P'
\text{ is evidently a normal subsemigroup with "0", which moreover satisfies (*); for}
(P + S'(-a)) \cap S((-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n) \neq \emptyset \text{ implies}
P \cap S(a, (-1)^{\epsilon_1} a_1, (-1)^{\epsilon_2} a_2, \ldots, (-1)^{\epsilon_n} a_n) \neq \emptyset.
\]
Property (*) of \( P' \) shows that, for all \( b (\neq 0) \text{ in } G, P' \cap S((-1)^{\epsilon} b) = \emptyset \) for \( \epsilon = 0 \text{ or } 1, \) that is, either \( b \not\in P' \text{ or } -b \not\in P'. \) Thus \( P' \) is a partial order of \( G. \)
To complete the proof of Theorem 3.1, let \( Q \) be a maximal element in the set \( B \) of all partial orders of \( G \) which are extensions of \( P \) and satisfies (*). Such \( Q \) exists, by Zorn Lemma, for (*) is satisfied by union of an ascending chain of partial orders provided it is satisfied by the member of the chain. By the Lemma 3.2, for every \( a \in G, \) either \( Q + S'(a) \) or \( Q + S'(-a) \) also belong to \( B. \) Therefore \( Q + S'(a) \) or \( Q + S'(-a) \) coincides with \( Q, \) that is \( a \in Q \) or \( -a \in Q, \) proving that \( Q \) defines a full order on \( G. \)
Our main concern now is with the group admitting a linear order. Following Neumann [5] we shall call these groups 0-groups (orderable groups). A necessary and sufficient condition for having this property can read directly from Theorem 3.1.

Theorem 3.3 [Los', Ohnishi (5)]. A group G is an 0-group if, and only if, given $a_1, a_2, \ldots, a_n$ in G with $a_i \neq 0$, for at least one choice of the sign $\varepsilon_i = 0$ or 1, one has

$$0 \notin S((-1)^{\varepsilon_1} a_1, (-1)^{\varepsilon_2} a_2, \ldots, (-1)^{\varepsilon_n} a_n).$$

In a group G, the intersection of the $2^n$ subsemigroups $S((-1)^{\varepsilon_1} a_1, (-1)^{\varepsilon_2} a_2, \ldots, (-1)^{\varepsilon_n} a_n)$ with fixed $a_1, a_2, \ldots, a_n$ and varying the signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is either a subgroup or void, therefore another formulation of the Theorem 3.3 is

Theorem 3.4 [Lorenzen(5)]. A necessary and sufficient condition for a group G to be an 0-group is that, for every finite set $a_1, a_2, \ldots, a_n$ in G ($a_i \neq 0$), the intersection of the $2^n$ subsemigroup $S((-1)^{\varepsilon_1} a_1, (-1)^{\varepsilon_2} a_2, \ldots, (-1)^{\varepsilon_n} a_n)$ taken from all choices of signs $\varepsilon_i = 0$ or 1 is void.

Corollary 3.5 [Neumann (5)]. In order that G be an O-group it is necessary and sufficient that every finitely generated subgroup of G be an O-group.

Assume that H is a finitely generated abelian group. If H is an O-group, then it must be torsion-free. If it is torsion-free, then it is a direct sum of n copies of its generators, hence it can be given a lexicographic order, that is, it is an O-group.
Corollary 3.5 implies

Corollary 3.6 [Levi (5)] An abelian group is an 0-group if and only if it is torsion free.

Definition: $F$ is free abelian group on\{x<sub>k</sub>\} in case $F$ is a direct sum of infinite cyclic group $Z_k$, where $Z_k=[x_k]$.

Definition: Let $X$ be a set and $F$ a group containing $X$; $F$ is free on $X$ if, for every group $G$, every function $f:X \rightarrow G$ has a unique extension to a homomorphism of $F$ into $G$.

![Diagram]

Proposition 3.7 For every free group $F$, $F/\langle F,F \rangle$ is a free abelian group.

Pf: Let $M$ be the free set of generators of free group $F$ and free abelian group $G$, that is

![Diagram]

commutes, where $f_2$ is the unique homomorphism from $F$ to $G$ such that $xif_2=xf_1$, for every $x \in M$. Since $F$ and $G$ have the same free set of generators, it is immediate that $f_2$ is onto. Therefore there exists $N \triangleleft F$, such that $F/N=G$. We know that $x+y+N=y+x+N$, for every $x,y \in F$. Hence $-x-y+x+y+N=N$, so $-x-y+x+y+N$, for every $x,y \in F$. Therefore $[F,F] \subseteq N$. Now consider the
where $\Theta$ is the natural mapping, and $f_3$ is a homomorphism from $G$ to $F/[F,F]$, such that $gf_2f_3 = (g+N)f_3 = g+[F,F]$ for every $g \in F$, and $g \not\in N$.

Note that $mf_1 = mif_2$, $mf_1f_3 = mi\Theta$. Hence $mif_2f_3 = mf_1f_3 = mi\Theta$, that is $gf_2f_3 = g\Theta$ for every $g \in F$. Consequently $f_2f_3 = \Theta$. Therefore $[F,F] = \text{Ker } \Theta = \text{Ker } f_2f_3 \supseteq N$. Hence $[F,F] \supseteq N$, that is $[F,F] = N$.

Recall that the members of the lower central series of group $G$ are defined by $G_0 = G$, $G_{n+1} = [G,G_n]$.

Theorem 3.8: Let $G$ be a free group, for each total order of the free abelian group $G/[G,G]$, there is a total order of $G$ so that the natural map of $G$ onto $G/[G,G]$ is an $o$-homomorphism.

Proof: Note that all the factor $G_n/G_{n+1}$, $n = 0,1,2,\cdots$ of the lower central chain of a free group $G$ are free abelian groups [9], and hence $G_n/G_{n+1}$ is an $O$-group. Thus by corollary 3.6, there exists a total order $P_n$ on $G_n/G_{n+1}$ . For $0 \neq g \in G$, if $n$ is the integer defined by $g \in G_n \cap G_{n+1}$ (such an integer exists, because by a theorem of Magnus-Witt[8], the lower central series $\{G_n\}$ of a free group $G$ is such that $\bigcap_{n=1}^\omega G_n = \{ 0 \}$, where $\omega$ denotes the first infinite ordinal), let $P_0$ be the given order on $G/[G,G]$, we define $P$ in $G$, $P_0 \subseteq P$ and $P$ consists of $0$ and all such $g \in G$, where $g+G_{n+1} \in P_n$. Then for each $g \in G$, either $g \in P$ or $-g \in P$, but not
both unless \( g = 0 \). If \( g, h \in P \) and \( m, n \) are the integers with \( g \in G_m \) \( \leq G_{m+1} \) and \( h \in G_n \) \( \leq G_{n+1} \), then \( g + G_{m+1} \in P_m \), \( h + G_{n+1} \in P_n \). Without loss of generality, we assume that \( m \geq n \). Then we have \( g + h \in G_{n+1} \) \( \leq G_{n+1} \) and \( g + h + G_{n+1} \in P_n \). Hence \( g + h \in P \). In the same way, we also get \( h + g \in P \). Finally, if \( g \) is as before and \( x \in G \) is arbitrary, then \( -x + g + x = g + [g, x] \in G_{n+1} \), that is \( -x + g + x \) again belongs to \( P \). Consequently, \( P \) defines a full order on \( G \).

Now consider the natural map \( f \) from \( G \) onto \( G/G[G, G] \) under order \( P \). Then if \( x, y \in G \) and \( x > y \), then \( -y + x > 0 \). Therefore there exists an \( n \), such that \( -y + x + G_{n+1} \in P_n \). Hence \( -y + x + [G, G] \geq [G, G] \), for otherwise if \( -y + x + [G, G] < [G, G] \), then \( -x + y + [G, G] \geq [G, G] \), that is \( -x + y > 0 \), in other words, \( y > x \), a contradiction. Hence we have \( x + [G, G] \geq y + [G, G] \). We have thus shown that \( f \) is an \( o \)-homomorphism.

Let \( G \) be a free group, and \( K \triangleleft G \), let \( G/K \) be totally ordered. We define a total order on \( K \) as in theorem 3.8, that is, let \( G_0, G_1, G_2, \ldots \) be lower central series of free group \( G \), say \( k \in P_k \) if \( k \in G_n \) \( \leq G_{n+1} \) and \( k \in G_{n+1} \in P_n \), where \( P_n \) is some fixed order on \( G_n/G_{n+1} \). Now define \( g > h \) if and only if \( g + K > h + K \) in \( G/K \) or \( gK = hK \) and \( -h + g \in P_k \). Then

(i) Let \( P \) consists of \( 0 \) and all \( g \) with the property: \( g \in P_k \) or \( g + K \geq K \) in \( G/K \). Then \( g \in P \) or \( -g \in P \), for every \( g \in G \), but not both, unless \( g = e \).

(ii) Let \( h, g \in P \).

(A) If \( h, g \in K \), then \( h + g, g + h \in P_k \). Hence \( h + g, g + h \in P \).

(B) Without loss of generality, let \( h \in K \) and \( g \notin K \), then \( g + h + K = g + K \geq K \).

Hence \( g + h \in P \) and \( h + g + K = g + K \geq K \). (by normality) Hence \( h + g \in P \).

(C) If \( h, g \notin K \), without loss of generality, we may assume that \( h + K > g + K \).

Hence \( g + h + K > g + K > g + K \), that is \( g + h \in P \) and \( g + K \geq K \) implies
h+g+K>h+K>h, that is h+g+K>h. Hence h+g∈P.

(iii) Let g∈P, for any x∈G, if g∈K implies g∈P_k and x+g−x∈K, but in theorem 3.8, x+g−x is a positive element, hence x+g−x∈P_k. That is x+g−x∈P. If g∉K, then g+K>K. Hence x+g−x+K>K. Consequently x+g−x∈P.

We have thus shown that G is totally ordered. Furthermore, g>h implies g+K>h+K, for otherwise if g+K<h+K, then −h+g−K<K implies −h+g<0 in P, that is g<h. Therefore the natural map of G onto G/K is an o-homomorphism.
Definition: Let \( \pi \) be an \( o \)-isomorphism of a \( p \)-group \( G \) into an \( l \)-group \( F \). This means that both \( \pi \) and \( \pi^{-1} \) preserve order. Then \((F,\pi)\) is a free \( l \)-group over \( G \) if

1. \( G\pi \) is a set of generators of the \( l \)-group \( F \) (that is, no proper \( l \)-subgroup of \( F \) contains \( G\pi \)), and

2. if \( \sigma \) is an \( o \)-homomorphism of \( G \) into an \( l \)-group \( L \), then there exists an \( l \)-homomorphism \( \tau \) of \( F \) into \( L \) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & F \\
\downarrow{\sigma} & & \downarrow{\tau} \\
L & & L
\end{array}
\]

commutes.

Proposition 4.1: In any distributive lattice and hence in any \( l \)-group

\[
\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij} = \bigwedge_{j \in J} \bigvee_{i \in I} a_{ij},
\]

for \( I \) and \( J \) finite sets and dually.

Proof: Let \( J \) be arbitrary finite set, we prove the statement by induction on \(|I|\). If \(|I|=1\), the proposition holds clearly. Suppose it is true if \(|I|=n\). Now if \(|I|=n+1\), then

\[
\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij} = \bigwedge_{j \in J} \bigvee_{i \in I} a_{ij} \bigvee \bigl( \Lambda_{i \in I} a_{i+1} \bigr),
\]

where \( I'=I-\{i_{n+1}\} \).

Thus

\[
\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij} = \bigwedge_{j \in J} \bigvee_{i \in I} a_{ij} \bigvee \bigl( \Lambda_{i \in I} a_{i_{n+1}} \bigr),
\]

by induction

\[
= \bigwedge_{j \in J} \bigvee_{i \in I'} \left( \bigvee_{i \in I} a_{if(i)} \bigwedge_{j \in I} a_{i_{n+1}f(i)} \right),
\]

\[
= \bigwedge_{j \in J} \bigvee_{i \in I'} \left( \bigwedge_{j \in J} \left( a_{i_{n+1}f(i)} \right) \bigvee_{j \in I} a_{if(i)} \right),
\]

\[
= \bigwedge_{j \in J} \bigvee_{i \in I'} \bigwedge_{i_{n+1}} a_{if(i)}
\]
Hence, \( \bigwedge_{i=1}^{n} a_i \) and dually.

**Proposition 4.2:** If \( S \) is a subgroup of an \( l \)-group \( L \), then

\[ T = \{ V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} : S_{\alpha \beta} \in S, \alpha \in A, \beta \in B, \text{ and } A \text{ and } B \text{ are finite sets} \} \]

is the \( l \)-subgroup of \( L \) that is generated by \( S \). If \( S \) is abelian, then so is \( T \). Now either \( S = \{0\} \) or \(|S| \) is infinite and so \( S \) and \( T \) have the same cardinality.

**Proof:** Clearly, \( T \) is a sublattice. On the other hand, \( 0 \in T \). Let \( t_1, t_2 \in T \), that is,

\[ t_1 = V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta}, \quad t_2 = V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta}' \]

Then \( t_1 - t_2 = V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} - V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta}' \). Thus

\[ t_1 - t_2 = V \bigwedge_{\alpha \in A, \beta \in B} (S_{\alpha \beta} - S_{\alpha \beta}') \]

Thus \( t_1 - t_2 \in T \), by Proposition 4.1.

Hence \( T \) is a subgroup. Finally for any \( x, y \in T \) and \( x \geq y \), we have

\[ t_1 + x + t_2 \geq t_1 + y + t_2 \]

in \( L \), for any \( t_1, t_2 \in T \). Hence \( t_1 + x + t_2 \geq t_1 + y + t_2 \) in \( T \), for any \( t_1, t_2 \in T \). We have thus shown that \( T \) is an \( l \)-subgroup of \( L \). Now if \( S \) is abelian, then

\[ V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} + V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} = V \bigwedge_{\alpha \in A, \beta \in B} (S_{\alpha \beta} + S_{\alpha \beta}) \]

\[ = V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} + V \bigwedge_{\alpha \in A, \beta \in B} S_{\alpha \beta} \]
Let \((F,T)\) be a free 1-group over the p o-group \(G\).

**Proposition 4.3:** If \(G\) is abelian, then so is \(F\). If \(S\) is a set of generators for the group \(G\), then \(ST\) is a set of generators for the 1-group \(F\).

**Proof:** \(G\) is abelian, implies \(F\) is abelian by Proposition 4.2. Let \(S\) be a set of generators of \(G\). Then \(ST\) is a set of group generators for \(G\) and \(G\) generates \(F\) (as an 1-group). Hence \(S\) generates \(F\).

**Proposition 4.4:** If \(\sigma\) is an o-homomorphism of the group \(G\) into an 1-group \(L\), then there exists a unique 1-homomorphism \(T\) of \(F\) into \(L\) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & F \\
\downarrow & & \downarrow \\
G & \xrightarrow{T} & L
\end{array}
\]

commutes.

**Proof:** Suppose that \(T_1\) and \(T_2\) are two such 1-homomorphisms and consider \(f = \sum_{\alpha \in A, \beta \in B} (g_{\alpha\beta})\pi \in F\), where \(\alpha \in A, \beta \in B\) and \(A\) and \(B\) are finite sets. Then

\[
f_{T_1} = \sum_{\alpha \in A, \beta \in B} (g_{\alpha\beta} \pi T_1) = \sum_{\alpha \in A, \beta \in B} (g_{\alpha\beta} \pi) = \sum_{\alpha \in A, \beta \in B} (g_{\alpha\beta} \pi T_2)
\]

Proposition 4.5: If \((F_1, \pi_1)\) and \((F_2, \pi_2)\) are free 1-groups over the p o-group \(G\), then there exists a unique 1-isomorphism of \(F_1\) onto \(F_2\) such that the diagram
commutes.

Proof: By Proposition 4.4, there exist 1-homomorphisms $T_1$ and $T_2$ such that

Consider $f = \bigwedge_{A, B} \pi_A \in \mathcal{F}$ where $A$ and $B$ are finite sets, $\alpha \in A$, $\beta \in B$.

$$fT_1T_2 = \left( \bigwedge_{A, B} \pi_A \right) T_2 = \left( \bigwedge_{A, B} \pi_A \right) = \bigwedge_{A, B} \pi_A = f.$$

Thus $T_1T_2$ is the identity on $F_1$ and similarly $T_2T_1$ is the identity on $F_2$.

Therefore $T_1$ is an 1-isomorphism.

Proposition 4.6: If $(F, \pi)$ is the free 1-group over the trivially ordered free group $G$ and $S$ is a free set of generators for $G$, then $F$ is a free 1-group with $ST$ as a free set of generators.

Proof: By Proposition 4.3, $ST$ is a set of generators of the 1-group $F$.

Let $\eta$ be a mapping of $S\pi$ into an 1-group $L$. Then $\pi\eta$ is a map of $S$ into $L$ and since $S$ is a free set of generators of $G$ there exists a unique homomorphism $\sigma$ of $G$ into $L$ such that $s\pi\eta = s\sigma$ for $s \in S$. Since $G$ is trivially ordered $\sigma$ is an o-homomorphism and hence there exists a unique 1-homomorphism $\tau$ of $F$ into $L$ such that $s\pi\eta = s\sigma = s\pi\tau$ for all $s \in S\pi$.

Proposition 4.7: If $L$ is an 1-group, then $L^+ = \{g \in L : g \alpha \}$ is the intersection
or right orders on $L$, that is, $L^+ = \bigcap_{\lambda \in \Omega} P_\lambda$, for some $\Omega$, where $P_\lambda$ is the set of positive elements of some right order on $L$.

Proof: By Theorem 2.8, we may assume that $L$ is an $1$-subgroup of the $1$-group $A(T)$ of all $\alpha$-permutations of a totally ordered set $T$. Let $e$ denote the identity in $A(T)$. Well order $T$ and for each $e \neq \alpha \in A(T)$, let $f(\alpha)$ be the first element in this well ordering such that $f(\alpha) \alpha \neq f(\alpha)$. Define $\alpha$ to be positive if $f(\alpha) \alpha \geq f(\alpha)$. Let $B(T)'$ be the set of all such positive elements then $B(T)^+ = B(T)' \cup \{e\}$ is a subset of $A(T)$ satisfying;

(i) $e \in B(T)^+$

(ii) If $x, y \in B(T)^+$, then clearly $f(xy) = \text{Min.}\{f(x), f(y)\}$, since $\text{Min.}\{f(x), f(y)\}$ is moved by $xy$, and also if $z < f(x)$ and $f(y)$ in this well order, then $zxy = zy = z$.

Therefore

$$f(xy) = \begin{cases} f(xy)y = f(xy) & \text{if } f(x) < f(y) \\ f(xy)y > f(xy) & \text{if } f(x) > f(y) \end{cases}$$

Hence $xy > e$. That is $xy \in B(T)^+$ and consequently $B(T)^+ \subseteq B(T)^+$. 

(iii) Clearly, $B(T)^+ \cup (B(T)^+)^{-1} \subseteq A(T)$. Now consider $e \neq g \in A(T)$, if $g \in B(T)^+$, fine; if not, then $f(g)g < f(g)$ by hypothesis. But $f(g) = f(g^{-1})$, thus $f(g^{-1}) < f(g^{-1})g^{-1}$. Hence $g^{-1} \in B(T)^+$ and therefore $g \in (B(T)^+)^{-1}$, with the consequence that $B(T)^+ \cup (B(T)^+)^{-1} = A(T)$. 

(iv) Consider $e \neq g \in B(T)^+ \cap (B(T)^+)^{-1}$. That is $f(g)g < f(g)$ and $f(g)g > f(g)$. This is impossible, hence $B(T)^+ \cap (B(T)^+)^{-1} = \{e\}$. Furthermore, let "$\leq"$ be defined on $A(T)$ by $g \leq h$ if and only if $hg^{-1} \in B(T)^+$. Then "$\leq"$ is a right total order on $A(T)$ by Theorem 1.1. Finally, let $\alpha \in A(T)^+$, then $t\alpha \geq t$, for all $t$. Then $\alpha \in B(T)^+$, for every such $B(T)^+$. Hence $\alpha \in \bigcap$ right order of this type on
A(T). That is $A(T)^+ \subseteq \bigcap$ right order of this type on $A(T)$. Now let
\[ a \in \bigcap \text{right order of this type on } A(T). \]
Suppose there exist $t \in T$ such that $a < t$, then there exist a right order $B(T)^+$ of this type such that $a \notin B(T)^+$, a contradiction. (For instance, we can let such $t$ to be the first element of $T$ in this particular well order). Hence $t \geq t$, for every $t \in T$. That is $a \in A(T)^+$. Therefore, $A(T)^+ = \bigcap$ right order of this type on $A(T)$. Thus
\[ L^+ = L \cap A(T)^+ = L \cap (\bigcap \text{right orders of this type on } A(T)) \]
\[ = \bigcap (L \cap \text{right orders of this type on } A(T)) \]
\[ = \bigcap \text{right orders on } L. \]

Example 4.8: Consider the 1-group $G = R \oplus R$, the cardinal sum of reals. Let
\[ G_1 = R \oplus R \quad \text{and} \quad G_2 = R \oplus R \]
with lexicographic orders. Then $G^+ = G_1^+ \cap G_2^+$. Furthermore, let $P \lambda_3 = G^+ \cup \{(x,y) : x \leq 0, y \geq 0, |x| \leq y\} \cup \{(x,y) : x \geq 0, y \leq 0, |y| < x\}.

Define a new relation in $G$, let $a \leq b$ in $G$ if and only if $b - a \in P \lambda_3$. Then $G$ is also a totally ordered group under $P \lambda_3$. Note that $G^+ = G_1^+ \cap G_2^+ \cap G_3^+$ holds also.

Proposition 4.9: A p o-group $G$ is o-isomorphic to a subgroup of an 1-group if and only if $G^+$ is the intersection of right orders.

Proof: We may assume that $G$ is a subgroup of an 1-group $L$ with the order of $G$ induced by the order of $L$. We know that $L^+ = \bigcap P \lambda$, where $P \lambda$ are right orders of $L$. Thus $G^+ = G \cap L^+ = G \cap (\bigcap P \lambda) = (G \cap P \lambda)$ and each $G \cap P \lambda$ is a right order for $G$.

Conversely, if $G^+ = \bigcap P \lambda$, where $P \lambda$'s are right orders for $G$, let $G_\lambda$ be the right ordered group under the right order $P \lambda$ on $G$, and let $g \rightarrow g^\lambda$ be the right regular representation of $G$ in $A(G_\lambda)$, which is a one-one homomorphism, where $xg^\lambda = x + g$, for every $x \in G$. If $g \in G^+$, then $x + g \geq x$, for every $x \in G$. Hence $x + g \geq x$ for every $x$ in $G_\lambda$ and for each $\lambda$. Therefore $xg^\lambda \geq x$ for
every x in G and for each λ. That is g^λ is positive, for each λ. If each g^λ is positive, then e \in seg^λ = g in G, and so g^λ \cap P^{+}_x = g^+_. Thus the map g \mapsto (\ldots, g^λ, \ldots) is an o-isomorphism of G into the l-group \Pi A(G^\lambda).

Proposition 4.10: If \Lambda A^\lambda B a^\alpha \beta \neq 0 in the l-group L, where A and B are finite sets, \alpha \in A, \beta \in B, then there is a right order of L which extends the given lattice-order and such that \Lambda A^\lambda B a^\alpha \beta \neq 0 in this right o-group L.

Proof: We may, by the Holland embedding theorem, assume that L is an l-subgroup of an l-group \Pi B(G) of all o-permutations of a totally ordered set T and \Lambda A^\lambda B a^\alpha \beta \neq e in \Pi B(G).

Case I: There exists an \alpha such that \Lambda B a^\alpha \beta e. Then t < t(\Lambda B a^\alpha \beta) = \min \{t \alpha \beta : \beta \in B\} for some t \in T. Now well order T so that this t is the first element in the well ordering. This determines a right order of \Pi B(G), (see the proof of Proposition 4.7), that extends the given lattice order and so that \Lambda B a^\alpha \beta > e and hence \Lambda A^\lambda B a^\alpha \beta > e in the right o-group \Pi B(G).

Case II: For each \alpha, \Lambda B a^\alpha \beta \leq e. Then \Lambda A^\lambda B a^\alpha \beta < e, since \Lambda A^\lambda B a^\alpha \beta \neq e, and so t > t(\Lambda B a^\alpha \beta) for some t \in T. Thus for each \alpha, t > t(\Lambda B a^\alpha \beta) = \min \{t \alpha \beta : \beta \in B\}. Let t be the first element in a well ordering of T. Then in the corresponding right order of \Pi B(G) we have \Lambda A^\lambda B a^\alpha \beta < e in the right o-group \Pi B(G).

Proposition 4.11: If G is a right o-group, \Lambda A^\lambda B a^\alpha \beta \neq 0 in G, where A and B are finite sets, \alpha \in A, \beta \in B, and g \mapsto g^\gamma is the right regular representation of G in \Pi B(G), then \Lambda A^\lambda B a^\alpha \beta \neq 0 in the l-group \Pi B(G).

Proof: Case I; \Lambda A^\lambda B a^\alpha \beta > 0 in G. Then for some \alpha, 0 < \Lambda A^\lambda B a^\alpha \beta = \min \{a^\alpha \beta : \beta \in B\}. Thus 0(\Lambda A^\lambda B a^\alpha \beta) = \min \{a^\alpha \beta : \beta \in B\} > 0 and so 0(\Lambda A^\lambda B a^\alpha \beta) is the largest element in a finite subset of G, where at least one element in this subset is strictly positive. Thus \Lambda A^\lambda B a^\alpha \beta \neq e.

Case II; \Lambda A^\lambda B a^\alpha \beta < 0 in G. Then for each \alpha, \Lambda A^\lambda B a^\alpha \beta < 0 in G, \min \{a^\alpha \beta : \beta \in B\} < 0.
Thus $0(V_{A,B}^\Lambda a_{\alpha\beta})$ is the largest element in a finite set of strictly negative element in $G$ and so it is strictly negative, since $V_{A,B}^\Lambda a_{\alpha\beta} \neq 0$ in $G$.
Therefore $V_{A,B}^\Lambda a_{\alpha\beta} \neq e$.

Throughout the following let $G$ be a p o-group with $G^+ = \bigcap_{\lambda \in \Omega} P^\lambda$, $\Omega$ the set of all $\lambda$, such that $P^\lambda$ is a right order on $G$, and $P^\lambda \supseteq G^+$. For each $\lambda \in \Omega$ let $G^\lambda$ be the right o-group $(G, P^\lambda)$ and let $g \mapsto g^\lambda$ be the right regular representation of $G$ as a subgroup of the l-group $A(G^\lambda)$, where $xg^\lambda = x + g$ for all $x \in G$. Let $L^\lambda$ be the 1-subgroup of $A(G^\lambda)$ generated by the image of $G$ under this isomorphism.

Lemma 4.12: If $\sigma$ is an o-homomorphism of $G$ into an l-group $L$ and $V_{A,B}^\Lambda a_{\alpha\beta} \neq 0$ for $\{ a_{\alpha\beta} \in G : \alpha \in A, \beta \in B$ and $A$ and $B$ are finite set$, then $V_{A,B}^\Lambda a_{\alpha\beta} \neq e$ in some $L^\lambda$.

Proof: By Proposition 4.10 we can extend the lattice-order of $L$ to a right order so that $V_{A,B}^\Lambda a_{\alpha\beta} \neq 0$ in right o-group $L$. Now $G/K(\sigma) \simeq$ subgroup of $L$, where $K(\sigma)$ is the kernel of $\sigma$, $K(\sigma) + g \mapsto g\sigma$ and this isomorphism defines a right order on $G/K(\sigma)$. Let $P^\alpha$ be one of the right orders of $G$ such that $P^\alpha \supseteq G^+$. Then $K(\sigma) \cap P^\alpha$ is a right order for $K(\sigma)$ that extends the given partial order of $K(\sigma)$. Define $g \in G$ to be positive if $g \in K(\sigma)$ and $K(\sigma) + g$ is positive in the right o-group $G/K(\sigma)$, or $g \in K(\sigma)$ and $g$ is positive in the right o-group $K(\sigma)$. By using Theorem 1.1, it is not hard to prove that this is a right order for $G$ that extends the given partial order and hence it is one of the $P^\alpha$. Also the natural map of $G$ into $G/K(\sigma)$ is an o-homomorphism with respect to this right order $P^\alpha$. Thus $K(\sigma) \neq V_{A,B}^\Lambda a_{\alpha\beta}(K(\sigma) + a_{\alpha\beta}) = K(\sigma) + V_{A,B}^\Lambda a_{\alpha\beta}$ and hence $V_{A,B}^\Lambda a_{\alpha\beta} \neq 0$ in $G^\lambda$. Now embed $G$ in $A(G^\lambda)$. Then by Proposition 4.11, $V_{A,B}^\Lambda a_{\alpha\beta} \neq e$ in $L^\lambda$. 

Now let \( \pi \) be the natural map of the \( p \)-o-group \( G \) onto the subgroup of long constants of \( \prod_{\lambda} \Lambda(\mathcal{G}(\lambda)), g -+ (\ldots, g^\lambda, \ldots) \). Then \( \pi \) is an o-isomorphism (see the proof of Proposition 4.8).

**Theorem 4.13:** The 1-subgroup \( F \) of \( \prod_{\lambda} \Lambda(\mathcal{G}(\lambda)) \) generated by \( G \pi \) is the free 1-group over the partially ordered group \( G \).

**Proof:** Suppose that \( \sigma \) is an o-homomorphism of \( G \) into an 1-group \( L \). Consider \( V_{A:B} k_{\alpha \beta} \pi e F \) and define \( (V_{A:B} (a_{\alpha \beta})) = V_{A:B} (a_{\alpha \beta} \sigma) \). If \( V_{A:B} k_{\alpha \beta} \sigma \neq V_{C:D} g_{\gamma \delta} \sigma \), then \( 0 \neq V_{A:B} k_{\alpha \beta} \sigma - V_{C:D} g_{\gamma \delta} \sigma = V_{A:B} k_{\alpha \beta} \sigma + V_{C:D} (-g_{\gamma \delta} \sigma) = V_{A:B} V_{C:D} (k_{\alpha \beta} - g_{\gamma \delta} f(\gamma)) \) \( \sigma \) \( \in V_{A:B} \cup (D^{\lambda}) \). By Lemma 4.12 there exists an \( L \) such that in \( L \), \( k_{\alpha \beta} \pi \neq V_{C:D} g_{\gamma \delta} \pi \). Therefore \( \pi \) is single-valued. Next consider \( k = V_{A:B} k_{\alpha \beta} \pi \) and \( g = V_{C:D} g_{\gamma \delta} \pi \) in \( F \). Then \( (k-g) \pi = (V_{A:B} k_{\alpha \beta} \pi - V_{C:D} g_{\gamma \delta} \pi) \pi \) \( \in V_{A:B} \cup (D^{\lambda}) \). By Proposition 4.12, (2) and (3) are equivalent and clearly (1) implies (2). It follows from Theorem 4.13 that (3) implies (1).

**Theorem 4.14:** For a \( p \)-o-group \( G \), the following are equivalent:

1. There exists a free 1-group over \( G \).
2. There exists an o-isomorphism of \( G \) into an 1-group.
3. \( G^+ = \{ g \in G : g \geq 0 \} \) is a intersection of right orders.

**Proof:** By Proposition 4.9, (2) and (3) are equivalent and clearly (1) implies (2). It follows from Theorem 4.13 that (3) implies (1).

**Proposition 4.15:** If \( K \) is a free 1-group with \( S \) a free set of generators,
then $S$ is a free set of generators for the subgroup $[S]$ of $K$ generated by $S$.

Proof: Let $G$ be a free group with $S$ as a free set of generators. Let $H = \Pi A(G)$ and let $g \mapsto (\ldots, g^\lambda, \ldots)$ be the $\omega$-isomorphism of $G$ into the long constants of $H$. Then the 1-subgroup $F$ of $H$ generated by $G\pi$ is the free 1-group with $S\pi$ as a free set of generators. Clearly, there exists an 1-isomorphism $\tau$ of $F$ onto $K$ such that $s\pi\tau = s$ for all $s \in S$ and hence $(G\pi)\tau = [S]$. Since $S\pi$ freely generates $G\pi$ it follows that $S$ freely generates $[S]$.

Proposition 4.16: Let $G$ be a free group with $S$ as a free set of generators and let $H = \Pi A(G\lambda)$. Let $K_n$ be the 1-subgroups of $H$ generated by the long constants from $G_n$ (where $G_0 = G, G_{n+1} = [G, G_n]$). Let $C_n$ be a free set of generators for $G_n$. Then $K_n$ is the free 1-group (really $(K_n, \pi | G_n)$) over $G_n$ with $C_n \pi$ as a free set of generators.

Proof: Let $-a-b+a+b\in [G, G_{n-1}], b\in G_{n-1}, a\in G$ and let $x\in G$. From $x-a-b+a+b-x = (x-a-x) + (x-b-x) + (x+a-x) + (x+b-x)$, we have $G \subset G_n$. Furthermore, $x+y+[G_n, G_{n-1}] = y+x+[G_{n-1}, G_n]$ for every $x, y \in G_{n-1}$. Hence $x+y+[G_n, G_{n-1}] = y+x+[G_n, G_{n-1}]$, since $[G_{n-1}, G_{n-1}] \subset [G_n, G_{n-1}]$. It can then be shown that $G_n/[G_n, G_{n-1}]$ is free abelian, by a theorem of Magnus-Witt. Now suppose that $\sigma$ is a homomorphism of $[G_n, G_{n-1}]$ into an 1-group $L$ and $V_{A_B}^\Lambda (a_{\alpha\beta}^\sigma) \neq 0$ in $L$. By Lemma 4.12, there exists a right order for $[G_n, G_{n-1}]$ so that in the 1-group $A(G_n), V_{A_B}^\Lambda (a_{\alpha\beta}^\nu) \neq 0$, where $x \rightarrow x^\nu$ is the right regular representation of $G_n$ in $A(G_n)$. Pick a total order for $G_n/[G_n, G_{n-1}]$, then the lexicographic extension of the right $\omega$-group $G_n$ by the $\omega$-group $G_n/[G_n, G_{n-1}]$ is a right order for $G_n$. Therefore by induction we get a right order for $G$ (which induces the right order on $G_n$), say $P$. For each $a_{\alpha\beta} \in G_n, a_{\alpha\beta}^\lambda$ maps $G_n$ onto itself. Thus there is a $t \in G_n$ such that $t V_{A_B}^\Lambda (a_{\alpha\beta}^\lambda) \neq 0$ and so
VA B e \in the \textbf{1-subgroup of } P(G) \text{ generated by } G_n^\lambda. \text{ Thus by the proof of Theorem 4.13 it follows that } K_n \text{ is the free 1-group on the free set of generators } C_n \pi.

We are now going to give some conditions for an 1-group G to be free. Let (F, \pi) be the free 1-group over the 1-group G constructed as in Theorem 4.12. If G is not an o-group, then there exists right orders \( P_{\lambda_1} \not\subseteq P_{\lambda_2} \) of G such that \( P_{\lambda_1} \cap P_{\lambda_2} \supsetneq G^+ \). If \( g \in P_{\lambda_1} \setminus P_{\lambda_2} \), then \( 0 g_{\lambda_1} = g > 0 \) and \( 0 g_{\lambda_2} = g < 0 \). Thus

\[
0 (g \lambda_1 V e) = g \text{ and } 0 (g \lambda_2 V e) = 0 \text{ and so } \pi g V e \pi = (\ldots, g \lambda V e, \ldots) \not\subseteq (g V O) \pi \text{ because } 0 (g V O)_{\lambda_1} = g V O > 0 \text{ and } 0 (g V O)_{\lambda_2} = g V O > 0. \text{ Thus } \pi \text{ is an o-isomorphism of } G \text{ into } F, \text{ but not an 1-isomorphism and so } G \pi \subsetneq F.

Theorem 4.17: Let \( (F, \pi) \) be the free 1-group over 1-group G. Then the following are equivalent;

(1) \( G \pi = F \).

(2) G is an o-group.

(3) Each o-homomorphism of G into an 1-group is an 1-homomorphism.

Proof: We have shown that (1) implies (2) and clearly (2) implies (3).

(3) \implies (1). \pi \text{ is an o-isomorphism of } G \text{ into } 1-group F \text{ and so } \pi \text{ is an 1-isomorphism. Thus } G \pi \text{ is an 1-subgroup of } F \text{ and hence } G \pi = F.

Proposition 4.18: Let G be an 1-group and \( (F, \pi) \) be the free 1-group over G. For every element \( a, b \in G, (a V b) \pi = a \pi V b \pi \) if and only if a, b are comparable.

Proof: Clearly if a, b are comparable, \( (a V b) \pi = a \pi V b \pi \).

Conversely, it is trivial if G is an o-group. Now suppose G is not an o-group. Then \( G^+ = \bigcap_{I \in I} P_{\lambda I}^+ \) for some index I, and \( P_{\lambda I}^+ \) is a right order on G, such that \( P_{\lambda I}^+ \supsetneq G^+ \). Let \( h = a - b \). (i) If \( h \| 0 \), then \( h \cap P_{\lambda I}^+ \), that is,
If \( h \in P_{\lambda_i} \) for some \( i \in I \). If \( h \in P_{\lambda_j} \) for some \( j \in I \), then \( h \in P_{\lambda_j} \). Therefore, \( \lambda_j = h > 0 \) in \( G_{\lambda_j} \) and \( \lambda_i = h < 0 \) in \( G_{\lambda_i} \). Thus \( 0(\lambda_j Ve) = h \), and \( 0(\lambda_i Ve) = 0 \) and so \( hVe = (\ldots, \lambda_j Ve, \ldots) \neq (hVe)^\pi \), because \( 0(hVe)^\lambda = hVe > 0 \) for all \( \lambda \). Hence \((aVb)\pi \neq _0^aVb\pi \) in this case. On the other hand, if \( h \in P_{\lambda_i} \) for all \( i \in I \), then \( -h \in P_{\lambda_i} \) for all \( i \in I \). Hence \(-h \in G^+, \) that is \(-h \geq 0 \). Therefore \( h \leq 0 \), contradiction. Consequently, if \( h \mid 0 \), then \((aVb)\pi \neq _0^aVb\pi \). The proof is complete.
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8. Magnus, W.: Math. Ann 111 (1935), 259-280, and E, Witt, Joun. reine u. angew, Math. 177(1937), 152-160. In the case of free groups, the factor groups in the lower central series are free abelian groups and the $\omega$th member is $\{0\}$.
