THE USE OF COMPLEX VARIABLE THEORY
IN SEVERAL AREAS OF PLANE ELASTICITY

by

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ABSTRACT

In this paper the use of complex variable theory in solving problems in plane elasticity and thermoelasticity is discussed. Two approaches to solving the plane elasticity problem are presented. One is that developed by N.E. Muskhelishvili using integral equations and the other utilizing orthogonal polynomials by Stippes-Shadman. A method of solution for the plane thermoelastic problem, again using complex variable theory and developed by B.E. Gatewood is then set forth.

The paper concludes with the application of complex variable theory to the solution of the problem of torsion of polygonal bars. This is done through the use of orthogonal polynomials and illustrates how readily this method is adaptable to numerical computations.
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INTRODUCTION

G. Kolossoff [1], as early as 1909, proposed that complex variable theory be used in the solution of problems in plane elasticity. It wasn't until nearly forty years later that this proposal was brought to a successful conclusion by the work of N.I. Muskhelishvili [2].

The purpose of this paper is to show not only how Muskhelishvili applied complex variable theory to plane elasticity problems, but also to illustrate one alternate method to Muskhelishvili's, and how complex variable theory was applied to thermoelasticity by B.E. Gatewood [3].

In addition the torsion problem of an hexagonal beam in two-dimensional elasticity is solved through the use of complex potentials and orthogonal polynomials.
A. COMPLEX VARIABLES IN TWO DIMENSIONAL ELASTICITY

1. Muskhelishvili's Method

We will first discuss the approach used by Muskhelishvili [5] in the solution of the plane elastostatic problem* in linear elasticity.

In the above case the relevant differential equations and boundary conditions take the form

\[
\begin{align*}
\tau_{\alpha \beta, \beta} &= 0^{**} \quad \text{in } D \\
v^2(\tau_{11} + \tau_{22}) &= 0 \quad \text{in } D
\end{align*}
\]

and

\[
\tau_{\alpha \beta} v_\beta = T_\alpha(S) \quad \text{on } C
\]

where \(T_\alpha(S)\) are known functions of the arc parameter \(S\) and \(v_\beta\) is the rectangular cartesian component of the exterior unit normal to \(C\), the boundary of some plane region \(D\).

* In plane stress the displacements and stresses are average ones and \(\lambda\) also changes its value.

** Repeated indices mean summation with Greek letters ranging over 1,2 and the Roman over 1,2,3.
G.B. Airy [6] noted that there exists a function \( U(x_1, x_2) \) such that if

\[
\begin{align*}
\tau_{11} &= U_{,22} \\
\tau_{22} &= U_{,11} \\
\tau_{12} &= U_{,12}
\end{align*}
\]

then equations (1), (2), and (3) reduce to

\[
\begin{align*}
\nabla^4 U &= 0 & \text{in } D \\
U_{,\alpha} &= f_{\alpha}(s) + C_{\alpha} & \text{on } C
\end{align*}
\]

E. Goursat [7] first obtained a representation for the biharmonic function \( U \) in terms of two analytic functions of a complex variable. In particular we will represent \( U \) in the form [10]

\[
2U = z Q(z) + z \bar{Q}(z) + \chi(z) + \bar{\chi}(z)
\]

(6)

where \( Q(z) \) and \( \psi(z) \) are analytic functions. Equation (6) implies that

\[
U_{,1} + iU_{,2} = Q(z) + z \bar{Q}'(z) + \bar{\psi}(z)
\]

(7)

where we have replaced \( \chi'(z) \) by \( \psi(z) \). Using this representation the stresses become
\[ \tau_{11} + \tau_{22} = 4 \text{Re}[Q^r(z)] \]  
\[ \tau_{22} - \tau_{11} + 2i\tau_{12} = 2[\bar{z} Q^f(z) + \psi(z)] \]  
\[ 2\mu(u_1 + i u_2) = \kappa Q(z) - z \bar{\phi}(z) - \overline{\psi}(z) \]

where \( \kappa = (\lambda + 3\mu)/(\lambda + \mu) \).

When complex potentials are used the boundary condition (3) for the first boundary value problem becomes,

\[ \mathcal{Q}(t) + t \overline{Q^r(t)} + \overline{\psi(t)} = i \int_{t_0}^{t} \left[ T_1(s) + i T_2(s) \right] ds \]  

while for the displacement boundary value problem we have

\[ -\kappa Q(t) + t \overline{Q^r(t)} + \overline{\psi(t)} = -2\mu(u_1(t) + iu_2(t)) \]  

where \( T_\alpha \), and \( u_\alpha \) are the cartesian components of the surface tractions and displacements, respectively, on the boundary \( C \).
We will first use the method proposed by Muskhelishvini and solve the problem for a simply connected domain.

Let the region $D$ be mapped conformally onto the unit circle $|\zeta|<1$ by the analytic function

$$z = \omega(\zeta)$$  \hspace{1cm} (13)

If $D$ is finite $\omega(\zeta)$ can be written in the form of a power series

$$z = \omega(\zeta) = \sum_{n=1}^{\infty} \frac{k_n}{\zeta^n} \quad |\zeta| \leq 1$$  \hspace{1cm} (14)

while for $D$ infinite

$$z = \omega(\zeta) = \frac{c}{\zeta} + \sum_{n=0}^{\infty} k_n \zeta^n \quad |\zeta| \leq 1$$  \hspace{1cm} (15)

Substituting (13) in (7) and (10) we get

$$U_1 + iU_2 = Q_1(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)} \bar{Q}_1(\zeta) + \bar{\psi}_1(\zeta) \quad |\zeta| \leq 1$$  \hspace{1cm} (16)

$$2\mu(u_1 + iu_2) = \kappa Q_1(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)} \bar{Q}_1(\zeta) - \bar{\psi}_1(\zeta) \quad |\zeta| \leq 1$$  \hspace{1cm} (17)
and into the boundary conditions (11) and (12) to get

\[ Q_1(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)} \tilde{Q}_1(\zeta) + \tilde{\psi}_1(\zeta) = F(\theta) \text{ on } |\zeta| = 1 \]  

\[ \kappa Q_1(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)} \tilde{Q}_1(\zeta) - \tilde{\psi}_1(\zeta) = G(\theta) \text{ on } |\zeta| = 1 \]  

where \( Q_1(\zeta) = Q(\omega(\zeta)) \)

\[ \psi_1(\zeta) = \psi(\omega(\zeta)) \]

and \( F(\theta), G(\theta) \) represents the transformed value of the right hand sides of equations (11) and (12) respectively.

At this stage there are two procedures that can be followed. The first is to assume the functions \( Q_1(\zeta) \) and \( \psi_1(\zeta) \) can be represented in the form

\[ Q_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \quad |\zeta| < 1 \]  

\[ \psi_1(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n \]

if \( D \) is finite, while if \( D \) is infinite the following representation must be used.
In the above \( \psi^0(\xi) \) and \( \psi^0(\xi) \) are analytic, \( X_1 \) and \( X_2 \) are surface tractions and \( B^-, B^-, \) and \( C^- \) are related to the stress distribution at infinity. The substitution of these representations in the boundary conditions leads to a system of equations for the coefficients \( a_n \) and \( b_n \).

In general this approach is cumbersome while the following procedure resulting from the conversion of the boundary conditions into functional equations, leads to considerably more effective methods of solution.

Let \( \delta = 1 \) in the case of the stress boundary value problem and \( \delta = \kappa \) for the displacement boundary value problem and similarly \( H(\theta) = F(\theta) \) or \( H(\theta) = G(\theta) \) respectively.

Multiply the boundary condition (18 or 19) by \( \frac{1}{2\pi i} \frac{d\xi}{\xi - \delta} \) and after integrating over the boundary of unit disc we get

\[
Q_1(\xi) = \frac{X_1 + iX_2}{2\pi(1 + \kappa)} \log \xi + (B + iC) \frac{C}{\xi} + Q^0(\xi),
\]

\[
\psi_1(\xi) = \frac{\kappa(X_1 - iX_2)}{2\pi(1 + \kappa)} \log \xi + (B^- + iC^-) \frac{C}{\xi} + \psi^0(\xi).
\]
\[ \delta Q_1(\sigma) + \frac{1}{2\pi i} \int_{c} \frac{\omega(\zeta)}{\omega^*(\zeta)} \frac{\bar{Q}_1^*(\zeta)}{(\zeta - \sigma)} \, d\zeta + \bar{\psi}_1(\sigma) = A(\sigma) \quad (22) \]

where

\[ A(\sigma) = \frac{1}{2\pi i} \int_{c} \frac{H(\zeta)}{\zeta - \sigma} \, d\zeta \quad (23) \]

Using the inequality

\[ \frac{\omega(\sigma)}{2\pi i} \int_{c} \frac{\bar{Q}_1^*(\zeta)}{\omega^*(\zeta)} \frac{d\zeta}{(\zeta - \sigma)} = \frac{\bar{Q}_1^*(\sigma)}{\omega^*(\sigma)} \quad (24) \]

equation (22) becomes

\[ \delta Q_1(\sigma) + \frac{1}{2\pi i} \int_{c} \frac{\omega(\zeta) - \omega(\sigma)}{\omega^*(\zeta)} \frac{Q_1^*(\zeta)}{(\zeta - \sigma)} \, d\zeta + k \, \omega(\sigma) + \bar{\psi}_1(\sigma) = A(\sigma) \quad (25) \]

where \( k = \frac{\bar{Q}_1^*(\sigma)}{\omega^*(\sigma)} \). In the case of an infinite domain \( k = 0 \) since \( \omega^*(\sigma) = \infty \) while if in the case of a finite domain one introduces \( Q_o(\sigma) \) such that
\[ Q_1(\sigma) = \frac{-k}{\delta} \omega(\sigma) + Q_0(\sigma) \]  

(26)

then (25) in both cases becomes

\[ \delta Q_0(\sigma) + \frac{1}{2\pi i} \int_C \frac{\omega(\zeta) - \omega(\sigma)}{\omega^{-}(\zeta)(\zeta - \sigma)} \bar{Q}_0^{-}(\zeta) \, d\zeta + \psi_1(\sigma) = A(\sigma) \]  

(27)

On differentiating with respect to \( \sigma \) and letting \( \sigma \to t \) on \( C \) (27) yields a Fredholm integral equation

\[ \delta Q_0^{-}(t) + \frac{1}{2\pi i} \int_C \frac{\partial}{\partial t} \left[ \frac{\omega(\zeta) - \omega(t)}{\zeta - t} \right] \bar{Q}_0^{-}(\zeta) \, d\zeta = A^{-}(t) \]  

(28)

The existence of the solution \( Q_0(t) \) now follows directly from Fredholm theory. Once \( Q_0(t) \) and thus \( Q_1(t) \) is determined then \( \psi_1(\sigma) \) is given by

\[ \psi_1(\sigma) = \frac{1}{2\pi i} \int_C \bar{H}(\zeta) \frac{Q_1(\zeta)}{\zeta - \sigma} \, d\zeta - \frac{1}{2\pi i} \int_C \frac{\omega(\zeta)}{\omega^{-}(\zeta)} \frac{Q_1^{-}(\zeta)}{\zeta - \sigma} \, d\zeta \]  

(29)

The case of multiple connected domains is more complicated. One approach is that used by S.G. Mikhlin [8, 9] in which he modifies Muskhelishvili's integral equations so that they will be valid for
both multiply and simply connected regions. He used the concept on the complex Green's function for the region which is valid for both multiply and simply connected domains. Some examples of its use can be found in [11, 12].
2. Orthogonal Polynomials

One difficulty that arises in Muskhelishvili's method is finding the required conformal mapping. A way of avoiding this difficulty is through the use of orthogonal polynomials. The idea of using orthogonal polynomials was first advanced by S. Bergman [13].

The following method was later applied to the solutions of plane problems for isotropic elastic bodies [5, 14].

Let the region \( D \) be simply connected and a compact subset of the complex plane. It follows that the potentials \( Q(z) \) and \( \psi(z) \) are holomorphic in \( D \) and if in addition one assumes they are of the class \( L_2 \) then they belong to a Hilbert space \( H \). If the set of functions \( \{ P_n(z) \} \) form a basis in \( H \) then we can write

\[
f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad a_n = (f, P_n), (P_n, P_m) = \delta_{mn}
\]

where

\[
f(t) = \int_{t_0}^{t} \left[ T_1(s) + i T_2(s) \right] ds
\]
Assuming \([P_n(z)]\) \(n = 0,1,2,...\) is known equation [12] yields

\[
\int_c \kappa Q(\beta) \tilde{P}_n(\beta) \, d\beta + \int_c \beta \tilde{Q}'(\beta) \tilde{P}_n(\beta) \, d\beta = \int_c f(\beta) \tilde{P}_n(\beta) \, d\beta \tag{31}
\]

Using Green's formula we get

\[
\int_D \left[ \kappa Q'(z) + \tilde{Q}'(z) \right] \tilde{P}_n(z) \, d\omega = \frac{i}{2} \int_c f(\beta) \tilde{P}_n(\beta) \, d\beta \tag{32}
\]

Assuming \(Q'(z) \in H\) we write

\[
Q'(z) = \sum_{m=1}^{\infty} p_m(z) \tag{33}
\]

Substituting this in (32) the following result is obtained:

\[
\begin{align*}
\kappa p_n + \sum_{m=0}^{\infty} G_{nm} \tilde{p}_m &= H_n \quad \text{for} \quad n = 0,1,2,\ldots \\
G_{nm} &= \int_D \tilde{p}_n(z) \tilde{p}_m(z) \, d\omega \quad \text{for} \quad n,m = 0,1,2,\ldots \\
H_n &= \frac{i}{2} \int_c f(\beta) \tilde{p}_n(\beta) \, d\beta \quad \text{for} \quad n = 0,1,2,\ldots
\end{align*}
\tag{34}
\]
The solution is thus determined if this infinite system of linear equations can be solved for \( p_n \) \( n=0,1,2,\ldots \). Thus (33) now gives \( Q(z) \) as

\[
Q(z) = \sum_{m=0}^{\infty} p_m \hat{P}_m(z) \quad \hat{P}_m(z) = \int_{0}^{z} p_m(\zeta) \, d\zeta
\]

Similarly

\[
\psi^- (z) = \sum_{n=0}^{\infty} q_n \, P_n(z)
\]

or

\[
\psi^- (z) = i \frac{1}{Z} \left[ \int_{c}^{\infty} \psi(\beta) \, \overline{P}_n(\beta) \, d\beta \right] P_n(z)
\]

Integration yields

\[
\psi(z) = \psi_0 + i \frac{1}{Z} \sum_{n=0}^{\infty} \hat{P}_n(z) \int_{c}^{\infty} \psi(\beta) \, \overline{P}_n(\beta) \, d\beta
\]

where \( \psi(\beta) \), the boundary value of \( \psi(z) \), is given by (11).
For multiply connected bounded domains the approach is the same but more complicated. Suppose \( D \) is a \( (p+1) \)-fold connected domain bounded by a smooth rectifiable Jordan curves \( \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_p \) where \( \gamma_0 \) contains the point at infinity. Also suppose \( z_1, z_2, z_3, \ldots \) are fixed points in the finite components of the compliment of \( D \). Then if \( \left[ X_n, Y_n \right] \) denote the components of the resultant of the external forces acting on the contour \( \gamma_n \) one can write

\[
Q(z) = \frac{1}{2\pi (k+1)} \sum_{n=1}^{p} (X_n + i Y_n) \log (z - z_n) + Q_0(z)
\]

\[
\psi(z) = \frac{1}{2\pi (k+1)} \sum_{n=1}^{p} (X_n - i Y_n) \log (z - z_n) + \psi_0(z)
\]

Where \( Q_0(z) \) and \( \psi_0(z) \) are holomorphic in \( D \) [14]. Here we recognize two cases. The first, when the external forces acting on each boundary component are balanced, in which case \( Q(z) = Q_0(z) \) and \( \psi(z) = \psi_0(z) \). So that from equations (11) and (12)

\[
\delta Q_0(\beta) + \beta \tilde{Q}_0(\beta) + \tilde{\psi}_0(\beta) = f^{(\alpha)}(\beta) \quad \beta \in \gamma_\alpha
\]

In the second case when the external forces of \( \gamma_\alpha \) do not vanish equations (11) and (12) assume the form
\[ \delta Q_0(\beta) + \beta \overline{Q'_0(\beta)} + \overline{\psi}_0(\beta) = f^{(\alpha)}_0(\beta), \quad \beta \in \gamma_\alpha \]

and where \( f^{(\alpha)}_0(\beta) \) has the form

\[
f^{(\alpha)}_0(\beta) = f^{(\alpha)}(\beta) + \frac{1}{2\pi(k+1)} \sum_{n=1}^{p} (X_n + iY_n) \left[ \delta \log (\beta - z_n) \right]
- \log (\overline{\beta - z_n}) + \frac{\beta}{2\pi(k+1)} \sum_{n=1}^{p} \frac{(X_n - iY_n)}{(\beta - z_n)}
\]

Assuming \( (p_n(z)) \) is an orthonormal basis for the Hilbert space \( L_2 \) of holomorphic functions on \( D \), we get

\[
\begin{align*}
\int_{\gamma_{\alpha}} \delta Q_0(\beta) \overline{p}_n(\beta) \, d\beta &+ \int_{\gamma_{\alpha}} \beta \overline{Q'_0(\beta)} \overline{p}_n(\beta) \, d\beta + \\
&+ \int_{\gamma_{\alpha}} \psi_0(\beta) \overline{p}_n(\beta) \, d\beta = \\
&+ \int_{\gamma_{\alpha}} f^{(\alpha)}(\beta) \overline{p}_n(\beta) \, d\beta \\
\end{align*}
\]

\[
\begin{array}{cccc}
n=0,1,2,... \\
\alpha=0,1,...,p \\
\end{array}
\]

where the sense of description of \( \gamma_0 \) is assumed to be counter-
clockwise whereas $\gamma_\alpha$, $\alpha \neq 0$ is assumed to be described in a clockwise direction. Adding the $(p+1)$ equations of the type (38) we obtain

\[
\begin{align*}
\left\{ \delta Q_0(\beta) \bar{P}_n(\beta) d\beta + \beta \bar{Q}_0(\beta) \bar{P}_n(\beta) d\beta \right\} \quad (39) \\
= \sum_{\alpha=0}^{P} \int_{\gamma_\alpha} f^{(\alpha)}(\beta) \bar{P}_n(\beta) d\beta
\end{align*}
\]

Employing Green's formula the above equation reduces to

\[
\begin{align*}
\left\{ \left[ \delta Q_0^-(z) + \bar{Q}_0^-(z) \right] \bar{P}_n(z) d\omega \right\} \\
= \sum_{\alpha=0}^{P} \int_{\gamma_\alpha} f^{(\alpha)}(\beta) \bar{P}_n(\beta) d\beta d\omega
\end{align*}
\]

\[
d\omega = dx \, dy 
\]

\[
(40)
\]

As before assume $Q_0^-(z) \in L_2$ and expand it in the series

\[
Q_0^-(z) = \sum_{n=0}^{\infty} p_n \, P_n(z) 
\]

\[
(41)
\]
Substituting this equation in equation (40) yields

\[ \delta p_n + \sum_{m=0}^{\infty} \gamma_{nm} \overline{p}_m = \sum_{\alpha=0}^{p} \left\{ f^{(\alpha)}(\beta) \overline{P}_n(\beta) d\beta \right\} \quad n=0,1,2,\ldots \]

\[ G_{nm} = \left\{ \overline{p}_n(z) \overline{p}_m(z) d\omega \right\} \]

Equations (42) form an infinite system of equations for the unknown \( p_n \). If this system of equations can be solved for \( p_n \) then the stresses and displacements are readily determined. The substitution of \( p_n \) in equation (41) determines \( Q_0(z) \) and when the boundary values of \( Q_0(z) \) are known, \( \psi_0(z) \) can be evaluated by an argument similar to the one used in the previous case.

For the first boundary value problem

\[ f^{(\alpha)}(\beta) = f_1^{(\alpha)}(\beta) + i f_2^{(\alpha)}(\beta) + c_{\alpha} \quad \alpha=0,1,2,\ldots p \]

*For the displacement boundary value problem this system of eqs. also contains additional unknowns \( X \) and \( Y \).
where \( f_{1}^{(a)}(\beta) \) and \( f_{2}^{(a)}(\beta) \) are obtained from the components of the tractions on the boundary components \( \gamma_{\alpha} \) so that (42) reduces to

\[
\begin{align*}
    p_{n} + \sum_{m=0}^{\infty} G_{mn} \bar{p}_{m} &= \sum_{\alpha=0}^{\infty} \left\{ f_{1}^{(a)}(\beta) \bar{p}_{n}(\beta) \, d\beta \right\}_{\gamma_{\alpha}} \\
    &+ \sum_{\alpha=1}^{p} c_{\alpha} \left\{ \bar{p}_{n}(\beta) \, d\beta \right\}_{\gamma_{\alpha}}
\end{align*}
\]

Only one of the constants \( c_{\alpha} \) may be assigned arbitrarily; the rest must be determined from the conditions of the problem.

Until now nothing has been said of the construction of the orthogonal polynomials \( P_{n}(z) \). It can be shown that the set of functions \( 1, z, z^{2}, z^{3}, \ldots \) form a complete set with respect to a class of \( L^{2} \) functions on a finite simply connected domain whose compliment is a closed domain [15]. Similarly

\[
1, \frac{1}{(z-\alpha_{v})^{m}} \quad v=1, \ldots, n, \quad m=1, 2, \ldots
\]
from a complete set with respect to $L^2(D)$ where $D$ is a multiply connected domain bounded by simple closed analytic curves $c_1 \ldots c_n$ and $a_v$ is a point inside the hole of $D$ which is surrounded by $c_v$. The set of orthonormal polynomials is easily found now by using the Gram-Schmidt orthogonalization process on your complete set (15).
3. **Thermoelasticity**

Another major area of elasticity to which the theory of complex variables can be applied is thermoelasticity. We will solve the problem of a cylindrical body, whose length is large compared to the greatest linear dimension of its cross-section, subject to a temperature change independent of the length of the cylinder (z axis). The cross-sections are assumed to remain plane and the strain in the z-direction is a constant of such value that the resultant normal force over the cross-section is zero. Stresses near the ends will not be considered.

Let \( T_0(x,y) = T(x,y,t_0) \) be the initial temperature and let \( T(x,y,t) \) be the temperature at some later time \( t \) diminished by \( T_0(x,y) \). The Duhamel-Neuman Law connecting stress, strains, and displacements is as follows*:

\[
e_{ij} = 1/2(u_{i,j} + u_{j,i})
\]

\[
= 1/26 \begin{vmatrix} 2\sigma_{ij} + (2GaT - \sigma_{ij} - S/m+1) \delta_{ij} \end{vmatrix}
\]

where

\[
S = \sigma_x + \sigma_y + \sigma_z
\]

\[
G = \frac{Em}{2(m+1)}
\]

*No summation over repeated indicies.
Under the conditions of the body \( \gamma_{xz} = 0, \gamma_{yz} = 0, \) and \( e_z = \) constant; also if the body forces are zero the equilibrium equation becomes

\[
\sigma_{ij,j} = 0 \quad i, j = 1, 2
\]  (45)

The boundary condition is

\[
\sigma_{ij} \cos(n, x_i) = X_i \quad \text{on } \Gamma
\]  (46)

where \( X_i \) are the applied surface tractions and \( n \) the unit outward normal. A further condition

\[
\int_D \sigma_z \, dx \, dy = 0
\]  (47)

which follows from the assumption that the resultant force in the \( z \) direction is zero, must be satisfied. Finally the compatibility equation takes the form

\[
\frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]  (48)
Solve in (44) for $\sigma_z$ $(i=j=3)$ and resubstituting in equation (44) $(i,j=1,2)$ then (44) is divided into two parts

\begin{align*}
e_x &= \frac{1}{2G} \left( \frac{\sigma_x + \sigma_y}{m} \right) - \frac{e_z}{m} + \frac{(m+1)}{m} \alpha T \\
e_y &= \frac{1}{2G} \left( \frac{\sigma_x + \sigma_y}{m} \right) - \frac{e_z}{m} + \frac{(m+1)}{m} \alpha T \\
\gamma_{xy} &= \frac{1}{G} \sigma_{xy}
\end{align*}

and

\begin{align*}
\sigma_z &= E(e_z - \alpha T) + \frac{1}{m} (\sigma_x + \sigma_y) \\
\tau_{xz} &= \tau_{yz} = 0
\end{align*}

Now if (45), (46), (48) and (49) are solved then equations (47) and (50) can also be solved.

It is well known that the Airy stress function $F$ satisfies (45). Substituting $F$ in (49) and the result in (48) we get
\[ \psi^4 F + \frac{E\alpha}{m-1} \psi^2 T = 0 \]  

The boundary conditions in terms of \( F \) become

\[
\begin{align*}
\frac{\partial^2 F}{\partial y^2} \cos(n,x) - \frac{\partial^2 F}{\partial x \partial y} \cos(n,y) &= X_1 \\
- \frac{\partial^2 F}{\partial x \partial y} \cos(n,x) + \frac{\partial^2 F}{\partial x^2} \cos(n,y) &= X_2
\end{align*}
\]

Note that \( \sigma_z \) and \( e_z \) can be determined from equations (47) and (50) and can be written as

\[ \sigma_z = E(e_z - \alpha T) + \frac{\psi^2 F}{m} \]  

\[ \int_D \left[ E(e_z - \alpha T) + \frac{\psi^2 F}{m} \right] \, dx \, dy = 0 \]

It is convenient to make a further simplification of the differential equation (51).

Let

\[ F = U-V \]
where \( V \) is any solution of the differential equation

\[
\nabla^2 V = k T
\]

(56)

then (51) is equivalent to a system of two differential equations

\[
\begin{align*}
\nabla^2 V &= k T \\
\n\nabla^4 U &= 0
\end{align*}
\]

(57)

where \( k = \frac{E m c}{m-1} \).

The boundary conditions (52) now become

\[
\begin{align*}
\frac{\partial^2 U}{\partial x_i^2} \cos(n, x_j) - \frac{\partial^2 U}{\partial x_i \partial x_j} \cos(n, x_i) &= 0 \\
\quad &\text{for } i, j = 1, 2 \text{ and } i \neq j \\
X_j + \frac{\partial^2 V}{\partial x_i^2} \cos(n, x_j) - \frac{\partial^2 V}{\partial x_i \partial x_j} \cos(n, x_i) &= 0
\end{align*}
\]

(58)

Now the problem has been reduced to solving (57) subject to (58).

It is necessary to first find a particular solution of the equation

\[
\nabla^2 V = k T
\]
since then the boundary conditions (58) for $U(x,y)$ are determined.

Therefore assume $V(x,y)$ is known. Then the function $U(x,y)$ can be obtained by solving $\nabla^4 U = 0$ subject to the boundary condition (52) which can be written in the form

$$\frac{d}{ds} \frac{\partial U}{\partial x} = -x_2 + \frac{d}{ds} \frac{\partial V}{\partial x},$$

$$\frac{d}{ds} \frac{\partial U}{\partial y} = x_1 + \frac{d}{ds} \frac{\partial V}{\partial y},$$

or if these derivatives are single valued, in the form

$$\frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = i \int_{s_0}^{s} (x_1 + i x_2) \, ds + \frac{\partial V}{\partial x} + i \frac{\partial V}{\partial y}$$

(59)

where $s$ is the length on the boundary $c$ of the cross-section $D$ of the body, with $c$ a simple closed curve enclosing the region $D$. This part of the problem is now exactly that problem discussed earlier which has been solved by Muskhelishvili using complex potentials.

Thus it remains only to determine $V(x,y)$ in terms of its first
derivatives, since it is only those functions which appear in the boundary conditions (59), from
\[ \nabla^2 V = kT. \quad (60) \]

A particular integral of (60) is
\[ V(x, y) = \frac{k}{2\pi} \int_D T(x_1, y_1) \log p \, dx_1 \, dy_1 \quad (61) \]

where \((x, y)\) is a point in \(D\), \((x_1, y_1)\) is a variable point in \(D\), \(p^2 = (x-x_1)^2 + (y-y_1)^2\), and \(D\) is the area of the cross-section under consideration. If \(z = \omega(s)\) maps \(D\) conformally on the unit disc of the \(s\)-plane then in terms of the variables of the \(s\)-plane (61) becomes
\[ V(s, \eta) = \frac{k}{2\pi} \int_{S'} T(s_1, \eta_1) \omega'(t) \bar{\omega}'(\bar{t}) \log p \, ds_1 \, d\eta_1 \quad (62) \]

where \(s = s + i\eta = re^{i\theta}\), \(t = s_1 + i\eta_1 = Re^{i\beta}\), \(S'\) the unit disc and \(p^2 = (s-t)(s-\bar{t})\). Differentiate (62) to obtain
\[ \frac{\partial V}{\partial x} - i \frac{\partial V}{\partial y} = \frac{-k}{2\pi \omega'(s)} \int_0^{2\pi} \int_0^R \frac{T \omega'(t) \bar{\omega}'(\bar{t})}{t-s} R \, dR \, d\beta \]
Since both $V(x,y)$ and $U(x,y)$ are now known the displacements and stress can be found with little trouble.

In the preceding discussion it should be noted that knowledge of the temperature function $T$ has been assumed.

The most important restriction on the theory that has been developed above is that the mapping function be rational. This restriction was introduced to provide a system of equations which could be solved explicitly for the desired functions. This restriction can be dropped with the result that the mapping function can be written in the form

$$\omega(s) = \frac{a}{s} + \sum_{k=0}^{n} a_k s^k$$

which maps any simply-connected region (sufficiently regular) onto the unit disc to any desired degree of accuracy. Hence for all practical purposes any thermoelastic problem for simply connected regions can be solved.
B. TORSION OF AN HEXAGONAL BEAM (2D)

All the techniques illustrated in the preceding section have, in their approach, one aspect in common. They all make use of complex potentials.

In all cases it is necessary to assume some form for these potentials. The most obvious one is a power series. It is also possible to express these potentials in another form. That is in terms of a set of orthogonal polynomials which form a basis in the region under consideration.

As an illustration of how orthogonal polynomials can be used in two dimensional elasticity, the torsion problem for an hexagonal beam will be solved. The approach used here is due to S. Bergman [16].

1. Basic Equations for the Torsion of a Bar

Consider a set of rectangular cartesian coordinates $x, y, z$ with the $z$ axis being perpendicular to the cross-section of the bar.

According to Saint Venant the components of the displacement vector $u, v, w$ are given by the expressions
where $\alpha$ is the angle of twist per unit length and $G(x,y)$ is harmonic over the cross-section, $D$, of the bar. Using Hooke's Law the only non-zero components of the stress tensor are found to be

\[
\tau_{xz} = \mu \alpha \left( \frac{\partial G}{\partial x} - y \right)
\]

\[
\tau_{yz} = \mu \alpha \left( \frac{\partial G}{\partial y} + x \right)
\]

where $\mu$ is the modules of rigidity.

Now let $H(x,y)$ be the harmonic conjugate of $G(x,y)$. Then the non-zero components of the stress tensor can be written as

\[
\tau_{xz} = \mu \alpha \left( \frac{\partial H}{\partial y} - y \right)
\]

\[
\tau_{yz} = \mu \alpha \left( x - \frac{\partial H}{\partial x} \right)
\]
The boundary condition on the lateral surface of the bar is given by

\[
\frac{\partial G}{\partial \mathbf{n}} = y \cos(x, \mathbf{n}) - x \cos(y, \mathbf{n}) \text{ on } C,
\]

where \( \mathbf{n} \) is a unit outer normal to the boundary \( C \) of the cross-section \( D \) and \( \frac{\partial G}{\partial \mathbf{n}} \) denotes the normal derivative of \( G(x, y) \) with respect to \( \mathbf{n} \). The quantities \( \cos(x, \mathbf{n}) \) and \( \sin(x, \mathbf{n}) \) are the cosines of the angle between the \( x \) axis and \( \mathbf{n} \), and the \( y \) axis and \( \mathbf{n} \) respectively. In terms of \( H(x, y) \) the above boundary condition assumes the following form

\[
H(x, y) = \frac{1}{2}(x^2 + y^2)
\]

Therefore, the torsion problem reduces to the determination of the function \( H(x, y) \) from

\[
\begin{align*}
\nabla^2 H(x, y) &= 0 \quad \text{in } C \\
H(x, y) &= \frac{1}{2}(x^2 + y^2) \quad \text{on } C
\end{align*}
\]

(63)

We note that \( G(x, y) \) can be uniquely determined, apart from an arbitrary constant from \( H(x, y) \) by the use of the Cauchy-Riemann equations.
2. **Method of Solution**

Let the bar have an hexagonal cross-section and let \( C \) be its boundary (see Fig. 1).

Since \( G(x,y) \) is harmonic in \( D \) the function \( F(z) \)

\[
F(z) = H(x,y) + i \, G(x,y)
\]

defines an analytic function in \( D \).
Now let \( \{P_n\} \) be a complete system of orthonormal functions in \( D \). Then \( F'(z) \) can be expanded in the series

\[
F'(z) = \sum_{n=0}^{\infty} a_n P_n(z)
\]  

(64)

where \( a_n \) is given by

\[
a_n = \int_D F'(z) P_n(z) \, dx \, dy
\]

It can be shown that the above series converges absolutely on \( D \) and uniformly on any closed subregion of \( D \) [15].

Integrating equation (64)

\[
F(z) = \sum_{n=0}^{\infty} T_n(z) \int_D F'(z) \overline{T'_n(z)} \, dx \, dy + c
\]  

(65)

where

\[
T_n(z) = \int_0^z P_n(z) \, dz
\]
\[ C = c_1 + i c_2 \quad \text{where } c_1, c_2 \text{ real} \]

Now write \( T_n(z) \) in the form

\[ T_n(z) = f_n(x,y) + i g_n(x,y) \]

with \( f_n(x,y) \) and \( g_n(x,y) \) being real functions.

Taking the real and imaginary parts of equation (65) we get

\[
\begin{align*}
H(x,y) = & c_1 \\
& + \sum_{n=0}^{\infty} f_n(x,y) \left[ \frac{\partial H}{\partial x} \frac{\partial f_n}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial f_n}{\partial y} \right] \, dx \, dy \\
& + \sum_{n=0}^{\infty} g_n(x,y) \left[ \frac{\partial H}{\partial x} \frac{\partial g_n}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial g_n}{\partial y} \right] \, dx \, dy
\end{align*}
\]

(66)
\[ G(x, y) = c_2 \]

\[ + \sum_{n=0}^{\infty} g_n(x, y) \left[ \left( \frac{\partial H}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial f}{\partial y} \right) \right] \, dx \, dy \]

\[ - \sum_{n=0}^{\infty} f_n(x, y) \left[ \left( \frac{\partial H}{\partial y} \frac{\partial g_n}{\partial x} + \frac{\partial H}{\partial x} \frac{\partial g_n}{\partial y} \right) \right] \, dx \, dy \]  \hspace{1cm} (67)

where we have used the Cauchy-Riemann equations and the fact that if \( B(z) \) is an analytic function of \( z \) and

\[ B(z) = B_1(x, y) + i B_2(x, y) \]

then

\[ \frac{\partial B(z)}{\partial z} = \frac{\partial B_1}{\partial x} + i \frac{\partial B_2}{\partial x} \]

Converting the areal integrals in equations (66) and (67) to line integrals by the use of Green's theorem we get

\[ H(x, y) = c_1 + \sum_{n=0}^{\infty} \left\{ A_n f_n(x, y) - B_n g_n(x, y) \right\} \]  \hspace{1cm} (68)
\[ G(x,y) = c_2 + \sum_{n=0}^{\infty} \left\{ A_n g_n(x,y) + B_n f_n(x,y) \right\} \]  

(69)

with

\[ A_n = \int_{c} H(s) \, d g_n(s) \]

\[ B_n = \int_{c} H(s) \, d f_n(s) \]

where \( H(s), f_n(s), g_n(s) \) denote the boundary values of \( H(x,y) \), \( f_n(x,y), g_n(x,y) \) respectively on \( c \).

Thus the problem has been reduced to the computation of \( H(x,y) \) and \( G(x,y) \) from equations (68) and (69) which depend only on the boundary values of \( H(x,y) \) and on the orthogonal polynomials \( P_n(z) \). Obviously in any particular calculation the infinite series in (68) and (69) must be truncated after a finite number of terms, depending upon the degree of accuracy required.

Therefore, the solution requires only the completion of the following steps:
(1) Determination of \( P_n(z) \) \( n=0,1,2,\ldots \) and thus the functions \( f_n(x,y) \) and \( g_n(x,y) \)

(2) Evaluation of

\[
A_n = \int_c H(s) \, d \, g_n(s) \quad n=0,1,2,\ldots
\]

\[
B_n = \int_c H(s) \, d \, f_n(s) \quad n=0,1,2,\ldots
\]

(3) Computation of the sums in equations (68) and (69) and of the partial derivatives \( \frac{\partial H}{\partial x} \) and \( \frac{\partial H}{\partial y} \).
3. Determination of the Orthogonal Polynomials $P_n(z)$

It is well known that the set of functions $1, z, z^2, z^3, \ldots$ form a complete system in $D$ [15]. Using the Gram-Schmidt orthogonalization process we form an orthonormal system $(P_n(z))$ given by [15].

$$P_n(z) = D_n(z) \left[ E_{n-1} E_n \right]^{-\frac{1}{2}}$$

(70)

where $D_n(z)$ and $E_n$ are determinants.

$$D_n(z) = \begin{vmatrix}
L_{00} & L_{01} & \cdots & L_{0n-1} & 1 \\
L_{10} & & & & \vdots \\
L_{20} & & & & \vdots \\
\vdots & & & & \vdots \\
L_{n0} & & & & L_{nn-1}
\end{vmatrix}_n=0,1,2,\ldots$$

(71)

$$E_n^* = \begin{vmatrix}
L_{00} & L_{01} & \cdots & L_{0n} \\
L_{10} & & & \vdots \\
\vdots & & & \vdots \\
L_{n0} & & & \vdots \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
L_{nn}
\end{vmatrix}_n=0,1,2,\ldots$$

(72)

$* E_{-1} = 1$
The elements, $L_{nm}$, $m,n=0,1,2,\ldots$, of these determinants are given by areal integrals

$$L_{nm} = \int \int_{D} z^n z^{-m} \, dx \, dy \quad n,m=0,1,2,\ldots \quad (73)$$

Letting $z = r e^{i\theta}$ equation (73) becomes

$$L_{nm} = \int_{0}^{2\pi} \int_{0}^{r(\theta)} r^{n+m+1} \cos((n-m)\theta) \, dr \, d\theta$$

$$= \frac{2\pi}{n+m+2} \left( \int_{0}^{2\pi} r^{n+m+2}(\theta) \cos((n-m)\theta) \, d\theta \right)$$

$$+ \frac{i}{n+m+2} \left( \int_{0}^{2\pi} r^{n+m+2}(\theta) \sin((n-m)\theta) \, d\theta \right) \quad (74)$$
Let $\text{Re}$ denote "the real part of" and $\text{Im}$ denote "the imaginary part of", then by making use of the symmetry of the cross-section we can easily show that

$$\text{Re} \ L_{nm} = \frac{4}{n+m+2} \left\{ \int_0^\frac{\pi}{2} r^{n+m+2}(\theta) \cos(n-m)\theta \ d\theta \right\}$$

$$= 0 \quad , |n-m|=0,2,4,...$$

and

$$\text{Im} \ L_{nm} = 0$$

Therefore

$$L_{nm} = \frac{4}{n+m+2} \left\{ \int_0^\frac{\pi}{2} r^{n+m+2}(\theta) \cos(n-m)\theta \ d\theta \right\}$$

$$= 0 \quad , |n-m|=0,2,4,...$$

$$= 0 \quad , |n-m|=1,3,5,...$$

From the geometry of the hexagon it is clear that
Substituting equation (77) in equation (76) we finally get

\[
\begin{align*}
L_{nm} &= \frac{4}{n+m+2} \int_0^{\pi/6} \sec^{n+m+2}(\theta) \cos(n-m)\theta \, d\theta \\
&\quad + \frac{4}{n+m+2} \int_{\pi/6}^{\pi/2} \sec^{n+m+2}(\theta - \pi/3) \cos(n-m)\theta \, d\theta \\
&\quad , |n-m|=0,2,4,\ldots \\
&\quad , |n-m|=1,3,5,\ldots \\
&= 0 
\end{align*}
\]

These integrals in equation (78) are evaluated using the Romberg Method of Integration [17]. The values of \(L_{nm}\) for \(n,m=0,1,2,\ldots,19\) are given in Table II.

Next we expand the right hand side of equation (70) in powers of \(z\), taking only the first twenty terms of \(P_n(z)\). However, some of
the coefficients of $z^n$ in this expansion vanish and we get

$$P_n(z) = a_n z^n + b_n z^{n-2} + ... + s_n z^{n-18} \quad (79)$$

where $n=0,1,2,3,...,19$ and

$$
\begin{align*}
    a_n &= 0 & n < 0 \\
    b_n &= 0 & n < 2 \\
    \vdots & & \vdots \\
    \vdots & & \vdots \\
    s_n &= 0 & n < 18
\end{align*}
$$

More explicitly the polynomials $P_n(z)$ take the form

$$
\begin{align*}
P_0(z) &= a_0 \\
P_1(z) &= a_1 z \\
P_2(z) &= a_2 z^2 + b_2 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
P_{19}(z) &= a_{19} z^{19} + b_{19} z^{17} + ... + s_{19} z
\end{align*}
\quad (80)$$
The numerical values of the constants $a_n$, $s_n$, $n=0,1,2,...,19$ are given in Table III.

Recall

$$T_n(z) = \int_0^z P_n(z) \, dz = f_n(x,y) + i \, g_n(x,y) \quad (81)$$

Substituting equation (79) in equation (81), integrating and replacing $z$ by $r e^{i\theta}$, we get the following expressions for $f_n(x,y)$ and $g_n(x,y)$

$$f_n(x,y) = \frac{a_n}{n+1} \, r^{n+1} \cos(n+1)\theta + \frac{b_n}{n-1} \, r^{n-1} \cos(n-1)\theta$$

$$+ \ldots + \frac{s_n}{n-17} \, r^{n-17} \cos(n-17)\theta \quad (82)$$

$$g_n(x,y) = \frac{a_n}{n+1} \, r^{n+1} \sin(n+1)\theta + \frac{b_n}{n-1} \, r^{n-1} \sin(n-1)\theta$$

$$+ \ldots + \frac{s_n}{n-17} \, r^{n-17} \sin(n-17)\theta \quad (83)$$
4. Determination of $H(r, \theta)$

In order to determine $H(r, \theta)$ the constants $A_n$ and $B_n$ must be determined. Recall that these constants are given as line integrals.

\[
A_n = \int_C H(s) \, d g_n(s) \quad (84)
\]

\[
B_n = \int_C H(s) \, d f_n(s) \quad (85)
\]

and that the boundary values of $H(x,y)$ on $C$ are given by

\[
H(x,y) = \frac{1}{2} (x^2 + y^2) \quad (86)
\]

In polar coordinates this becomes

\[
H(r, \theta) = \frac{1}{2} r^2(\theta) \quad (87)
\]

It follows then that
Again because of the symmetry of the problem under consideration

\[ A_n = \frac{1}{2} \int_c r^2(\theta) \, d\, g_n(\theta) \quad n=0,2,\ldots,20 \]  \hspace{1cm} (88)

\[ B_n = \frac{1}{2} \int_c r^2(\theta) \, d\, f_n(\theta) \quad n=0,2,\ldots,20 \]  \hspace{1cm} (89)

Again because of the symmetry of the problem under consideration

\[ B_n \equiv 0 \quad n=0,1,2,\ldots,19 \]

\[ A_n = 2 \int_0^{\pi/2} r^2(\theta) \, d\, g_n(\theta) \quad n=1,3,5,\ldots,19 \]

\[ \equiv 0 \quad n=0,2,4,\ldots,18 \]  \hspace{1cm} (90)

The above integral representations can be approximated by the summation

\[ A_n = 2 \sum_{j=1}^{90} r^2(\theta_j) \left[ g_n(\theta_j) - g_n(\theta_{j-1}) \right] \]  \hspace{1cm} (91)
The numerical values of $A_n$ may be found in Table IV.

The function $r_0$ is the exact boundary value of $r_0$.

From equations (68) and (90)

$$H_0(r, \theta) = c_0 + \sum_{n=0}^{\infty} A_n f_n(r, \theta)$$

where $H_0(r, \theta)$ denotes our approximation of $H(r, \theta)$ and $c_0$ the constant $c_1$.

On the boundary $C$ of $D$ equation (93) implies that

$$C_0 = \frac{1}{2} r_0^2 - \sum_{n=0}^{\infty} n^2 A_n f_n(r, \theta)$$

where $r_0$ is the exact boundary value of $r_0$.

Since the infinite sum in equation (68) has been replaced by a finite sum

we cannot expect the right hand side of equation (93) to remain a constant. Therefore we compute the difference

$$\theta_j = \frac{\pi}{10}$$

where
\[ \frac{1}{2} r^2(0) - \sum_{n=0}^{19} A_n f_n(r, \theta) \]

at a number of points on the boundary (in our case at 36 points) and then choose the average of these values for \( C_{20} \). Hence we take

\[ C_{20} = \frac{1}{36} \sum_{j=1}^{36} \left[ \frac{1}{2} r^2(0) - \sum_{n=0}^{19} A_n f_n(r_{i_j}, \theta_{i_j}) \right] \] (94)

where

\[ \theta_{i_j} = 10i^0 \]

Using this procedure the following "average" value for \( C_{20} \) is obtained

\[ C_{20} = .54049265 \]

Finally we get

\[ H_{20}(r, \theta) = .54059265 + \sum_{n=0}^{19} A_n f_n(r, \theta) \] (95)
Since we have already found the values of $A_n$ and the polynomials $f_n(r, \theta)$, this expression completely determines the harmonic function $H(x, y)$. The numerical values of $H_{20}(x, y)$ on the boundary $C$ are given in Table I. The values of $H_{20}(x, y)$ in $D$ are listed in Table V.
5. **Determination of the Stresses**

The determination of the stresses requires the evaluation of the derivative of $H(x,y)$.

The above method yields only function $H_{20}(x,y)$ approximating the required function. Inside the domain $D$ the derivative $H_{20}(x,y)$ will in general approximate the corresponding exact solution quite satisfactorily. Near the boundary or on the boundary itself the approximation obtained for the derivatives will, in many cases, not be satisfactory. In particular at sharp corners where the radius of curvature of the boundary is no longer continuous, it is necessary to apply special methods for the evaluation of the derivatives.
6. Conclusion

In conclusion, the torsion problem of a bar of hexagonal cross-section has been considered in order to illustrate the use of orthogonal polynomials to solve this problem in particular but, more generally, the Dirichlet problem as given by equation (63).

It should be noted that no attempt was made to predetermine the degree of accuracy of the approximations made. However, we note that due to the nature of the problem the function $H(x,y)$ must take both its maximum and minimum values on the boundary. From the geometry of the cross-section one easily sees that

$$H_{\text{max}} = \frac{1}{2} r^2 = .666$$

$$H_{\text{min}} = \frac{1}{2} r^2 = .5$$

The approximate values of $H(x,y)$ on the boundary vary from .5000873 to .6537620 (see Table I). The approximate values of $H(x,y)$ inside D fall within the exact values for $H_{\text{min}}$ and $H_{\text{max}}$ (see Table V).
C. RESULTS

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<th>$\frac{1}{2} r^2(\theta)$</th>
<th>$H_{20}[r(\theta), \theta] - C_{20}$</th>
<th>$H_{20}[r(\theta), \theta]$</th>
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TABLE II

The values of the non-zero constants $L_{AM}$

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TABLE III

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\[
A_n = \int_c H(s) \, d \, g_n(\theta)
\]
TABLE V

The values of $H_{20}(p, \theta)$ in D along several rays in the sector $0 < \theta < \frac{\pi}{6}$

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