AN EXPOSITION OF A THEOREM OF GOLOD AND SAFAREVIC
WITH APPLICATIONS TO
NIL ALGEBRAS AND PERIODIC GROUPS

by
Tasoula Michael Berggren
B. A., University of Washington, 1965

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics

© TASOULA MICHAEL BERGGREN 1970
SIMON FRASER UNIVERSITY
APRIL, 1970
APPROVAL

Name: Tasoula Michael Berggren
Degree: Master of Science
Title of Thesis: An Exposition of a Theorem of Golod and Šafarević
With Applications to Nil Algebras and Periodic Groups

Examining Committee:

__________________________
(T. C. Brown)
Senior Supervisor

__________________________
(A. J. Das)
Examining Committee

__________________________
(A. L. Stone)
Examining Committee

Date Approved: 15 April 70
TO THE MEMORY OF MY FATHER

MICHALAKIS SAPARILLAS

(1891 – 1959)
ABSTRACT

A theorem by Golod and Šafarevič with application to nil algebras and periodic groups is clearly proved in this thesis. The applications settle negatively Kuroš's question: Is a finitely generated algebraic algebra, finite-dimensional? and Burnside's question: Is a finitely generated periodic group finite?

Remarks and theorems on subjects related to the main theorem are in Chapter 1, the proof of the theorem is in Chapter 2, and the applications of it are in Chapter 3.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approval</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>vi</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter 1: Free-Semigroup Algebras</td>
<td></td>
</tr>
<tr>
<td>Over a Field $F$</td>
<td>3</td>
</tr>
<tr>
<td>Chapter 2: Proof of the Theorem</td>
<td>15</td>
</tr>
<tr>
<td>Chapter 3: Applications</td>
<td>46</td>
</tr>
<tr>
<td>Bibliography</td>
<td>58</td>
</tr>
</tbody>
</table>
I would like to thank the following people: Dr. T.C. Brown for suggesting the topic and supervising the research, my husband Dr. J.L. Berggren, for his help during the absence of Dr. T.C. Brown, and Mrs. Arlene Blundell for the typing of the entire thesis.

The financial assistance of the National Research Council of Canada is deeply appreciated.
INTRODUCTION

The purpose of this thesis is to give a clear exposition of a theorem of Golod and Šafarevič [6] and some of its consequences. The theorem, published in 1964, is a remarkable result. Its proof is rather short, but it provides the answer to many questions. Two of the questions which this paper will discuss are more closely related than originally appeared. The problems referred to are the Kuroš Problem [15], and the general Burnside Problem [1] and the construction of examples which solve these problems is fairly straight-forward (given the main theorem of Golod-Šafarevič).

This thesis has an example of an infinite dimensional nil algebra with a finite number of generators over a countable field. This is a negative answer to the Kuroš question which was asked in 1941: Let $A$ be a finitely generated, algebraic algebra. Is $A$ finite-dimensional (as a vector space)? The history of the question is very interesting. Kuroš discussed several special cases [15], all with affirmative answers. Jacobson and Levitzki [13],[16],[17] settled the question affirmatively for algebras of bounded degree. In the meantime, many special cases had been studied. Then, in 1964, Golod announced that the answer to the Kuroš question was negative. At the same time, he gave a negative answer to the Burnside problem: Let $G$ be a finitely generated periodic group. Is $G$ finite?
Burnside [1] considered the following three cases with affirmative answers.

1. $G$ of exponent 2,
2. $G$ of exponent 3,
3. $G$ of exponent 4, and $G$ with two generators.

In 1940, Sanov [22] obtained an affirmative answer for exponent 4 and an arbitrary (but finite) number of generators. Marshall Hall Jr. [9] gave an affirmative answer for exponent 6. The answer is still unknown for $G$ of exponent 5.

Then Novikov, in 1959, announced [20] that the answer is no, if the exponent of $G \geq 72$ and the number of generators is at least 2. (The proof of [20] appeared in 1968 by P.S. Novikov and S.I. Adyan [21], where $n \geq 72$ has been replaced by odd $n \geq 4381$.)

In 1964, Golod constructed a finitely generated group which is periodic and infinite, which settled negatively the original Burnside problem.
CHAPTER 1.

This chapter is to make the reader familiar with a few terms and some symbols which are closely related to the main part of this thesis. In addition, some definitions will be given, while it will be assumed that the reader is acquainted with the most basic ones.

1.1 Free Semigroups and Generators

Let $X = \{x_1, x_2, \ldots, x_d\}$ be a set of $d$ noncommuting indeterminates, and let $S_X$ consist of all finite sequences of elements of $X$,

$$S_X = \{x_{i_1} x_{i_2} \ldots x_{i_n} | x_{i_k} \in X\}.$$

Define a binary operation, that is, a multiplication on $S_X$, as follows: For any two elements of $S_X$, say $s_1 = x_{i_1} x_{i_2} \ldots x_{i_n}$ and $s_2 = x_{j_1} x_{j_2} \ldots x_{j_m}$, their product $s_1 \cdot s_2$ is the product obtained by juxtaposition of $s_1$ and $s_2$:

$$s_1 \cdot s_2 = x_{i_1} x_{i_2} \ldots x_{i_n} x_{j_1} \ldots x_{j_m}.$$

For example, if $s_1 = x_1 x_2 x_3$ and $s_2 = x_3 x_2$, then $s_1 \cdot s_2 = x_1 x_2 x_3 x_3 x_2 = x_1 x_2 x_3 x_2$. With this definition of multiplication $S_X$ becomes a semigroup; we call it the free semigroup on $X$. Note that the binary operation which we just defined is associative. For example:

$$(x_1 x_2 x_3 x_2 \cdot x_1 x_4 x_3) \cdot x_1 x_4 x_3 = (x_1 x_2 x_3 x_2) \cdot x_1 x_4 x_3.$$
The elements of \( S_X \) are often called "words" but in this thesis, they will be called monomials. We may say that the element \( x_i \) of \( X \) has length 1 if we consider \( x_i \) as a word. However, talking in terms of monomials \( x_i \) has degree 1. Now we are ready to define the degree of a monomial which is simply the number of occurring \( x_i \)'s. For example, the monomial \( x_2 x_3 x_5 x_4 \) is of degree 4.

If in a word we have a succession of indeterminates all the same, say \( x_1 x_1 \ldots x_1 \), (\( m \) times), then we write \( x_1^m \).

We let 1 be a symbol not in \( X \) (we call 1 the "empty word" or the "monomial of degree 0"), and define \( 1 \cdot s = s \cdot 1 = s \) for all \( s \in S_X \). Thus we have a semigroup \( \{1\} \cup S_X \) the free semigroup with identity on \( X \).

Remark: 1.1.1 The number of distinct monomials of a given degree \( n \) is the number of ways of choosing (in order) \( n \) indeterminates from the set \( X \). This number in this case is \( d^n \).

Example: Assume that \( X = \{x_1, x_2, x_3, x_4\} \) is the set of four non-commuting indeterminates. Then the number of monomials of degree 3 is \( 4^3 = 64 \).

The monomials of degree 2 are 16 in number and they are the following:

\[
\begin{array}{cccccccc}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{41} & x_{42} & x_{43} & x_{44}
\end{array}
\]
The elements of $S_X$, that is the monomials, are of the form

$$\prod_{k=1}^{n} x_{i_k}^{k} = x_{i_1}^{i_1} \cdots x_{i_n}^{i_n} \quad x_{i_k} \in X.$$

We say that $X$ is a set of generators of $S_X$. It is often convenient to work with $S^1_X = \{1\}_X$ rather than $S_X$. We index $S^1_X$ by the index set $\Omega$: $S^1_X = \{s_w | w \in \Omega\}$.

### 1.2 Vector Spaces Over a Field $F$ and Algebras Over a Field $F$

Let $T$ be the vector space over a field $F$ with a basis $S^1_X$. Denote $T$ by $F[x_1, x_2, \ldots, x_d]$. Then $T = F[x_1, x_2, \ldots, x_d] = \{ \sum_{w \in \Omega} a_w s_w | a_w \in F \}$ and $a_w \neq 0$ for only finitely many $w \in \Omega$. Each element of $T$ is uniquely expressed as a linear combination of elements of $S^1_X$ over the field $F$. (Note that $s_i \neq s_j$ if $i \neq j$).

Define addition in $T$ by

$$\sum_{w \in \Omega} a_w s_w + \sum_{w \in \Omega} b_w s_w = \sum_{w \in \Omega} (a_w + b_w) s_w \quad a_w, b_w \in F.$$

Addition is obviously well defined since $a_w + b_w \in F$.

Define scalar multiplication by

$$a(\sum_{w \in \Omega} a_w s_w) = \sum_{w \in \Omega} a a_w s_w \quad a, a_w \in F.$$

Note that $\sum_{w \in \Omega} a_w s_w = \sum_{w \in \Omega} b_w s_w$ if and only if $a_w = b_w$ for all $w \in \Omega$.

Then $0 = \sum_{w \in \Omega} a_w s_w$ and $\sum_{w \in \Omega} a_w s_w = 0$ implies $a_w = 0$ for all $w \in \Omega$.

**Example:** Let $S^e_X$ be the semigroup $\{e, a, a^2\}$, where $ea = ae = a$, $ea^2 = a^2 e = a^2$, $aa^2 = a^2 a = e$. Then $T = \{ x_e + y_a + za^2 | x, y, z \in F \}$ is a vector space over the field $F$. Let $x'e + y'a + z'a^2$ and $x''e + y''a + z''a^2$ be any two elements of $T$. Then it is natural to write:
\[(xe + ya + za^2)(x'e + y'a + z'a^2) = xx'ee + xy'ea + xz'ea^2 + yx'ae + yy'aa + yz'aa^2 + zx'a^2e + yz'a^2e + zz'a^2a^2\]
\[= xx'e + xy'a + xz'a^2 + yz'e + yz'a + yy'a + zy'e + zz'a + zx'a^2\]
\[= (xx' + yz' + zy')e + (xy' + yx' + zz')a + (xz' + yy' + zy')a^2\]

Definition 1.2.1 An Algebra \( A \) is a ring which is a vector space over a field \( F \). In addition, the following holds:
\[a(uv) = (au)v = u(av) \text{ for all } a \in F, u, v \in A.\]

Now let us define multiplication on \( T \) over \( F \). Let \( u, v \in T \), where
\[u = \sum_{i \in \Omega} a_is_i \text{ and } v = \sum_{j \in \Omega} b_js_j.\]
Then
\[uv = \sum_{i \in \Omega} a_is_i \cdot \sum_{j \in \Omega} b_js_j = \sum_{i, j \in \Omega} (a_ib_j)(s_is_j)\]

The above multiplication is clearly well defined since \( s_i, s_j \) are elements in \( S^1_X \) where multiplication is already defined.

Theorem 1.2.2 With the multiplication defined in \((*)\), \( T \) is a ring with identity.

Proof: Let \( u = \sum a_is_i, v = \sum b_js_j, w = \sum c ks_k \) be elements of \( T \). Then the multiplication \((*)\) is associative, since
\[(uv)w = (\sum a_is_i \cdot \sum b_js_j) \cdot \sum c ks_k = \sum a_ib_j (s_is_j) \cdot \sum c ks_k\]
\[= \sum (a_ib_j)c_k(s_is_j)s_k = \sum (bkc_k)(s_is_j)\]
\[= \sum a_is_i \cdot \sum bjc_k(s_js_k) = \sum a_is_i \cdot (\sum bjs_j \cdot \sum cks_k) = u(vw).\]

\((**) \text{ since } s_i, s_j, s_k \in S^1_X.\)
The distributive law holds also, since

\[ u(v + w) = \Sigma a_i s_i (\Sigma b_j s_j + \Sigma c_j s_j) = \Sigma a_i s_i (\Sigma (b_j + c_j) s_j) \]

\[ = \Sigma a_i (b_j + c_j) s_i s_j = \Sigma (a_i b_j + a_i c_j) s_i s_j \]

\[ \Sigma (a_i b_j s_i s_j + a_i c_j s_i s_j) = \Sigma a_i b_j s_i s_j + \Sigma a_i c_j s_i s_j \]

\[ = \Sigma a_i s_i (\Sigma b_j s_j + \Sigma a_i c_j s_i s_j) = \Sigma a_i s_i \Sigma b_j s_j = \Sigma a_i s_i (\Sigma c_j s_j) = \Sigma u \Sigma w \]

Similarly \((u + v)w = uw + vw\).

The identity of \(T\) is the monomial of degree 0 denoted by 1. Hence \(T\) is a ring with identity.

Theorem 1.2.3 \(T\) is an algebra over \(F\), called the free semigroup algebra on \(S_X^1\) over \(F\).

Proof: Let \(a, a_i, b_j \in F, \ u = \Sigma a_i s_i\) and \(v = \Sigma b_j s_j \in T\). Then:

\[ a(uv) = a \Sigma a_i s_i (\Sigma b_j s_j) = a \Sigma a_i b_j s_i s_j = \Sigma (a a_i) b_j s_i s_j \]

\[ = \Sigma a_i s_i (\Sigma b_j s_j) = (a \Sigma a_i s_i) \Sigma b_j s_j = (au)v \]

\[ = \Sigma a_i s_i (\Sigma b_j s_j) = \Sigma (a a_i) b_j s_i s_j = \Sigma a_i (\Sigma b_j s_i s_j) \]

\[ = \Sigma a_i s_i (\Sigma b_j s_j) = \Sigma a_i s_i (\Sigma b_j s_j) = u(au) \]

Hence \(T\) is an algebra over \(F\).

It is worthwhile to observe that the elements of \(X\) do not commute with each other, but they do commute with elements of \(F\).

1.3 Homogeneous Polynomials and Subvector-Spaces of \(T\) Over \(F\).

In this section, we will call the elements of \(T\) polynomials. This is why the \(x_i\) are called non-commuting indeterminates.
Definition 1.3.1 A homogeneous polynomial of degree $n$ is a linear combination of distinct monomials each of degree $n$.

If $u$ is a homogeneous polynomial of degree $i$ we denote the degree $i$ by $\partial(u) = i$. Here is an example:

Let $x_1x_2 + x_2x_2 = u$. Then $u$ is a homogeneous polynomial and $\partial(u) = 2$.

Let $x_1 + x_1x_2x_1 = v$. Clearly $v$ is not a homogeneous polynomial.

Theorem 1.3.2 Let $T_n$ be the set of all homogeneous polynomials of $T$ of degree $n$. Then $T_n$ is a subvector space of $T$ over $F$.

Proof: At first note that $T_n$ is a subset of $T$. We need to show that $T_n$ is itself a vector space over $F$. Also observe that $T_n \neq \emptyset$ since $x^n \in T_n$.

Now let $u = \sum a_is_i$ and $v = \sum b_is_i \in T_n$ where $a_i$, $b_i$, $c$ are in $F$.

It is clear that $\partial(s_i) = n$ for each $s_i$ that appears in $u$ or in $v$.

Then $cu + v = c\Sigma a_is_i + \Sigma b_is_i = \Sigma ca_is_i + \Sigma b_is_i = \Sigma (ca_i + b_i)s_i$

is in $T_n$.

Note that a basis for $T_n$ consists of all distinct monomials of degree $n$, hence $\dim T_n = d^n$.

Example: Let $T = F[x_1, x_2, x_3]$. Then

$$\dim T_0 = 3^0 = 1, \quad \dim T_1 = 3^1 = 3, \quad \dim T_2 = 3^2 = 9$$

$$\dim T_3 = 3^3 = 27, \quad \dim T_4 = 3^4 = 81 \quad \text{and so on.}$$

$T_0$ has basis $\{1\}$ (the identity of $S_X$), and may be identified with the field $F$. 
$T_1$ has basis \( \{ x_1, x_2, x_3 \} \)

$T_2$ has basis \( \{ x_1x_1, x_1x_2, x_1x_3, x_2x_1, x_2x_2, x_2x_3, x_3x_1, x_3x_2, x_3x_3 \} \)

and $T_3$ has basis

\[
\begin{align*}
&\{ x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_3x_2, x_1x_2x_1, x_1x_3x_1, x_1^2x_2, x_1x_2, x_1x_3 \} \\
&\{ x_2^3, x_2^2x_1, x_2^2x_3, x_2x_1x_3, x_2x_3x_1, x_2x_1x_2, x_2x_3x_2, x_2x_1, x_2x_3 \} \\
&\{ x_3^3, x_3^2x_1, x_3^2x_2, x_3x_1x_2, x_3x_2x_1, x_3x_1x_3, x_3x_2x_3, x_3^2, x_3x_2 \}
\end{align*}
\]

The polynomial \( x_1^2x_2 + x_2^3 + x_2x_3^2 + x_2x_1^2 \) is a homogeneous polynomial of degree 4 and hence is in $T_4$.

**Definition 1.3.3** Let $W_1, W_2, \ldots, W_k, \ldots$ be subspaces of the vector space $W$. We shall say that $W$ is the direct sum of $W_1, W_2, \ldots, W_k, \ldots$ and we write $W = W_1 \oplus W_2 \oplus \ldots \oplus W_k \oplus \ldots$ if any of the following equivalent conditions hold:

(i) $W = W_1 + W_2 + \ldots + W_k + \ldots$ and $W_1, \ldots, W_k, \ldots$ are independent. (That is if $\alpha_1 + \alpha_2 + \ldots + c_k = 0$, $\alpha_1 \in W_1$ implies that each $\alpha_1 = 0$ (for any $k$).)

(ii) Each vector $\alpha \neq 0$ in $W$ can be uniquely expressed in the form $\alpha = \alpha_1 + \ldots + \alpha_k$ with $\alpha_j \in W_j$ (for some $k$) where the indices are distinct and $\alpha_j \neq 0$, $1 \leq j \leq k$.

(iii) $W = W_1 + W_2 + \ldots + W_k + \ldots$ and, for each $j \geq 1$, the subspace $W_j$ is disjoint from (has intersection \{0\} with) the sum $(W_1 + \ldots + W_{j-1} + W_{j+1} + \ldots)$. 
Theorem 1.3.4  Let $T = F[x_1, x_2, \ldots, x_d]$ be the vector space over a field $F$ and let $T_n$ be the subspace of $T$ of all homogeneous polynomials of degree $n$, for $n = 0, 1, 2, \ldots$. Then $T$ is the direct sum of $T_0, T_1, \ldots, T_n, \ldots$ and we write $T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$

Proof: Let $T_j$ be the subspace of $T$ of all homogeneous polynomials of degree $j$. Then

$$T_j \cap (T_0 + T_1 + \ldots + T_{j-1} + T_{j+1} + \ldots) = \{0\}$$

because the subspaces $T_0, T_1, \ldots, T_{j-1}, T_{j+1}, \ldots$ have only homogeneous polynomials of degrees $0, 1, \ldots, j-1, j+1, \ldots$ respectively. Also, clearly $T = T_0 + T_1 + \ldots + T_n + \ldots$. Hence

$$T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$$

Corollary 1.3.5 Each element $u \in T$ can be uniquely expressed as a sum of homogeneous polynomials.

Proposition 1.3.6 $T_n = T_{n-1}x_1 \oplus T_{n-1}x_2 \oplus \ldots \oplus T_{n-1}x_d$

Proof: The elements in $T_{n-1}x_i$, $(i = 1, 2, \ldots, d)$ are of degree $n$, and clearly

$$T_n = T_{n-1}x_1 + T_{n-1}x_2 + \ldots + T_{n-1}x_d.$$  

Moreover

$$T_{n-1}x_1 \cap (T_{n-1}x_1 + \ldots + T_{n-1}x_{i-1} + T_{n-1}x_{i+1} + \ldots + T_{n-1}x_d) = \{0\}$$

because the $x_i$'s do not commute. Hence

$$T_n = T_{n-1}x_1 \oplus T_{n-1}x_2 \oplus \ldots \oplus T_{n-1}x_d.$$
Example 1.3.7 Let \( T = F[x_1, x_2, x_3] \). The basis elements for \( T \) are grouped into the following sets:

\[
\{x_1x_1, x_2x_1, x_3x_1\}, \{x_1x_2, x_2x_2, x_3x_2\}, \{x_1x_3, x_2x_3, x_3x_3\}
\]

where

- \( T_1x_1 = \) subspace spanned by \( \{x_1x_1, x_2x_1, x_3x_1\} \)
- \( T_1x_2 = \) subspace spanned by \( \{x_1x_2, x_2x_2, x_3x_2\} \)
- \( T_1x_3 = \) subspace spanned by \( \{x_1x_3, x_2x_3, x_3x_3\} \)

Hence \( T_2 = T_1x_1 \oplus T_1x_2 \oplus T_1x_3 \).

1.4 Ideals and Quotient Algebras

The following section consists of some definitions and results which are actually part of (used for) the proof of the main result in Chapter 2, but are put here to get the reader even more familiar with the basic structure we shall be working with.

Let \( H \) be a subset of \( T \) which consists of nonzero homogeneous polynomials \( f_1, f_2, \ldots \) such that \( 2 \leq \deg(f_1) \leq \deg(f_2) \leq \ldots \) and let \( \deg(f_1) = n_1 \). Now rewriting, we have \( 2 \leq n_1 \leq n_2 \leq \ldots \).

Let the number of all those \( f_j \)'s which have degree \( i \) be denoted by \( r_i \). This number is assumed to be finite (for each \( i \)) and is possibly zero.

Let \( \mathfrak{M} \) be the intersection of all ideals of \( T \) which contain \( H \). \( \mathfrak{M} \) is then an ideal, which is in fact the smallest ideal of \( T \) containing the set \( H \). This ideal is called the ideal generated by \( H \).

In what follows, the subset \( H \) which generates the ideal \( \mathfrak{M} \) will always be as described above. In particular, \( H \) contains only homogeneous polynomials of degree \( \geq 2 \).
Theorem 1.4.1 Let $T$ be an algebra, $H = \{ f_1, f_2, \ldots \}$ and $\mathfrak{M}$ the ideal generated by $H$ in $T$. Then the elements of $\mathfrak{M}$ are all elements of $T$ which may be represented in the form $\sum_{i \in I} a_i f_i b_i$, where $I$ is a finite set, $a_i$ and $b_i$ are elements of $T$ and the $f_i$ are elements of $H$.

Proof: Let $B = \{ \sum_{i=1}^{n} a_i f_i b_i | a_i, b_i \in T, f_i \in H, n = 1, 2, 3, \ldots \}$.

Then $\mathfrak{M} \subseteq B$.

Now since $\mathfrak{M}$ is the intersection of all the ideals of $T$ which contain $H$, to get $\mathfrak{M} \subseteq B$ it is sufficient to show that $B$ is an ideal of $T$. But this is obvious. Hence $B \subseteq \mathfrak{M}$ and $\mathfrak{M} \subseteq B$, so $\mathfrak{M} = B$.

Remark 1.4.2 Consider the element $\sum_{i \in I} a_i f_i b_i$ of $\mathfrak{M}$. By expressing each $a_i$ and each $b_i$ as a sum of homogeneous polynomials, and then multiplying out, we see that in fact, $\mathfrak{M}$ is the set of all elements of this form $\sum_{j \in J} a'_j g_j b'_j$, where $J$ is a finite set, $a'_j$ and $b'_j$ are homogeneous elements of $T$, and the $g_j$ are elements of $H$.

Corollary 1.4.3 Let $r = u_0 + u_1 + u_2 + \ldots + u_g \in \mathfrak{M}$, where $u_i \in T_i$;

then each $u_i \in \mathfrak{M}$.

Proof: By preceding remark,

$$r = \sum_{j \in J} a_j g_j b_j,$$

where each $g_j \in H$ and the $a_j, b_j$ are homogeneous polynomials. By collecting summands of equal degree, we get

$$r = v_0 + v_1 + \ldots + v_t,$$

where $v_i \in T_i \cap \mathfrak{M}$. 
But by corollary 1.3.5, \( r \) can be expressed uniquely as a sum of homogeneous polynomials. Hence (since we assume \( u_{s} \neq 0 \neq v_{t} \)) we must have \( s=t \) and \( u_{0}=v_{0}, \ u_{1}=v_{1}, \ldots, \ u_{s}=v_{s} \). But \( v_{0}, \ldots, v_{s} \in \mathcal{U}, \) hence \( u_{0}, \ldots, u_{s} \in \mathcal{U}. \) (Note that since \( \mathcal{H} \) contains only polynomials of degree \( \geq 2 \), this gives us \( u_{0} = u_{1} = 0. \))

Remark 1.4.4 Let \( A_{i} \) be the quotient vector space \( (T_{i} + \mathcal{U})/\mathcal{U} \) over \( F \); then \( A_{i} \) is a vector subspace of the quotient vector space \( T/\mathcal{U} \) over \( F. \)

Theorem 1.4.5 Let \( A \) be the quotient vector space \( T/\mathcal{U} \) over \( F. \) Then as a vector space, \( A = A_{0} \oplus A_{1} \oplus A_{2} \oplus \ldots \oplus A_{n} \oplus \ldots, \) where \( A_{i} = (T_{i} + \mathcal{U})/\mathcal{U}. \)

**Proof:** Let \( a \in A \). Then \( a = u + \mathcal{U} \) where \( u \in T. \) But then \( u \) can be written uniquely as the sum of \( u_{i}'s, \) i.e. \( u = u_{0} + u_{1} + \ldots + u_{n}, \) where \( u_{i} \in T_{i} \) and

\[
a = (u_{0} + u_{1} + \ldots + u_{n}) + \mathcal{U} = (u_{0} + \mathcal{U}) + (u_{1} + \mathcal{U}) + \ldots + (u_{n} + \mathcal{U})
\]

where each \( u_{k} + \mathcal{U} \in (T_{k} + \mathcal{U})/\mathcal{U} = A_{k}. \) Hence \( A = A_{0} + A_{1} + \ldots + A_{n} + \ldots. \)

Now we need to show that \( a \) can be written in only one way as a sum of elements of the different \( A_{i}. \) Hence, suppose that

\[
(1) \quad a = (u_{0} + \mathcal{U}) + \ldots + (u_{n} + \mathcal{U}) = (v_{0} + \mathcal{U}) + \ldots + (v_{m} + \mathcal{U})
\]

\( u_{i}, v_{i} \in T_{i} \)

We want to show that \( u_{i} + \mathcal{U} = v_{i} + \mathcal{U}. \) From (1) we have that

\[
(u_{0} + \ldots + u_{n}) + \mathcal{U} = (v_{0} + \ldots + v_{m}) + \mathcal{U}
\]
hence \[ u_0 + \ldots + u_n \equiv (v_0 + \ldots + v_m) \pmod{\mathfrak{y}} \]

Now if \( m \geq n \)
\[ (u_0 - v_0) + (u_1 - v_1) + \ldots + (u_n - v_n) + (0 - v_{n+1}) + \ldots + (0 - v_m) \equiv 0 \pmod{\mathfrak{y}} \]

Hence \( (u_0 - v_0) + \ldots + (u_n - v_n) + (0 - v_{n+1}) + \ldots + (0 - v_m) \in \mathfrak{y} \)
where \( u_i - v_i \in T_i \) and therefore \( u_i - v_i \in \mathfrak{y}, \forall i \), by corollary 1.4.3.

So \( u_1 + \mathfrak{y} = v_1 + \mathfrak{y} \) which shows the uniqueness i.e. \( u_1 \equiv v_1 \pmod{\mathfrak{y}} \). Consequently, \( A = A_0 \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots \).

**Theorem 1.4.6** If \( \mathfrak{y}_n = T_n \cap \mathfrak{y} \), then \( A_n \cong T_n / \mathfrak{y}_n \), where \( \mathfrak{y}_n \) and \( T_n \) are regarded as vector spaces over \( F \).

**Proof:** Consider the vector space \( T \) over \( F \) as an additive group. Then the ideal \( \mathfrak{y} \) is a normal subgroup of \( T \), and by the Second Isomorphism Theorem of group theory, we have the following diagram and isomorphisms.

\[
\begin{array}{ccc}
T + \mathfrak{y} & \cong & T_n / \mathfrak{y}_n \\
| & & | \\
\downarrow & & \downarrow \\
T_n & \cong & T_n / \mathfrak{y}_n \\
| & & | \\
T_n \cap \mathfrak{y} & \cong & \mathfrak{y} \\
\end{array}
\]

\[
A_n = \frac{T_n + \mathfrak{y}}{\mathfrak{y}_n} \cong \frac{T_n}{\mathfrak{y}_n} = \frac{T_n}{\mathfrak{y}_n} . (i)
\]

Because \( \mathfrak{y}_n \cap T_0 = \{0\}, \mathfrak{y}_n \cap T_1 = \{0\} \)
we have the following particular cases:

\( A_0 \cong T_0 \cong F \) \hspace{1cm} (ii)

\( A_1 \cong T_1 \) \hspace{1cm} (iii)

**Remark 1.4.6** Note that \( \mathfrak{y} = \mathfrak{y}_0 \oplus \mathfrak{y}_1 \oplus \ldots \oplus \mathfrak{y}_n \oplus \ldots \) where \( \mathfrak{y}_0 = \{0\} = \mathfrak{y}_1 \).
CHAPTER 2.

This chapter is devoted to the proof of the main Golod-Šafarevič theorem. In the first part of the chapter, we will derive some results which will be used to give a short proof of the first theorem. The second part of the chapter deals with the proof of the first theorem. We give two different proofs; one is a re-worked and expanded version of the proof in the paper by Fisher and Struik [5]. The other proof, which has to do with homology, is a re-worked and expanded version of a proof from Herstein's book "Noncommutative Rings" [11]. In the same part, two more theorems follow which are re-worked from the original paper by Golod and Šafarevič [6]. Finally, the last part of this chapter deals with some corollaries and special cases [5], [19].

2.1 Some Subspaces and Their Dimensions

In this section before the derivation of the results necessary for the proof of the main theorem, let us recall the various notations we have introduced up to this point.

We have a field $F$ and $d$ noncommutative indeterminates over $F$, which are $x_1, x_2, \ldots, x_d$. Also,

$$T = F[x_1, x_2, \ldots, x_d]$$

is the free associative algebra over $F$ in the $x_i$ and, moreover:

$$T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$$

where each $T_n$ is the subspace of $T$ consisting of all the homogeneous polynomials of degree $n$. Recall that:

$$H = \{f_1, f_2, \ldots, f_n, \ldots\},$$
where each \( f_i \) is a homogeneous polynomial of \( T \) and
\[
2 \leq \partial(f_i) \leq \partial(f_2) \leq \ldots
\]
(\( \partial(f_i) = n_i \) is the degree of \( f_i \)). \( \mathfrak{H} \) is the ideal of \( T \) generated by \( H \), and \( r_i \) is the number of polynomials \( f_j \) in \( H \) which have degree \( i \). The quotient algebra \( A = T/\mathfrak{H} \) is also of the form:
\[
A = A_0 \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots
\]
where \( A_i = (T_i + \mathfrak{H})/\mathfrak{H} \). We mentioned that \( \dim T_n = d^n \). Now let
\[
\dim A_n = b_n
\]
and observe:
\[
\begin{align*}
2.1.1 & \quad 1 = d^0 = \dim T_0 = \dim A_0 = b_0 \\
2.1.2 & \quad d = \dim T_1 = \dim A_1 = b_1.
\end{align*}
\]

**Proposition 2.1.3** Recall that \( \mathfrak{H}_n = T_n \cap \mathfrak{H} \) (by definition), and that \( A_n \cong T_n/\mathfrak{H}_n \). Let \( S_n \) be a complementary subspace of \( \mathfrak{H}_n \) in \( T_n \); that is, \( T_n = \mathfrak{H}_n \oplus S_n \). Then \( \dim S_n = \dim A_n = b_n \).

**Proof:** \( T_n = \mathfrak{H}_n \oplus S_n \) gives \( \dim T_n = \dim \mathfrak{H}_n + \dim S_n \).

Also \( A_n \cong T_n/\mathfrak{H}_n \) gives \( \dim T_n = \dim \mathfrak{H}_n + \dim A_n \).

These two equalities give the desired result.

**Proposition 2.1.4** \( \dim \mathfrak{H}_2 \leq r_2 \), where \( \mathfrak{H}_2 = T_2 \cap \mathfrak{H} \) and \( r_2 \) is the number of \( f_j \) of degree 2.

**Proof:** Look at \( \mathfrak{H} \) as a vector space. Recall that
\[
\mathfrak{H} = \mathfrak{H}_2 \oplus \mathfrak{H}_3 \oplus \ldots \oplus \mathfrak{H}_n \oplus \ldots
\]
where each \( \mathfrak{H}_n \) is a subvector space of \( \mathfrak{H} \).

Now \( \mathfrak{H}_2 \) has a basis of elements of the form \( mf_jn \), where \( m, n \) are monomials and \( \partial(mf_jn) = 2 \). If the degree of \( mf_jn \) is 2, then \( \partial(f_j) \leq 2 \). But \( \partial(f_j) > 1 \) always. Hence \( \partial(f_j) = 2 \) which implies that \( m, n \) are constants. In other words, a basis for \( \mathfrak{H}_2 \) is a set of linearly independent
f_2 of degree 2. If the f_2 of degree 2 were linearly independent, then \( \dim \mathcal{H}_2 \) would equal \( r_2 \).

Since the number \( r_2 \) does not necessarily denote linearly independent \( f_i \) of degree 2, we have

\[
\dim \mathcal{H}_2 \leq r_2.
\]

**Definition 2.1.5** Let \( J = \mathcal{H}_{n-1} x_1 \oplus \cdots \oplus \mathcal{H}_{n-1} x_d \).

**Note 2.1.6** To prove that the sum \( J \) is direct, we need to show that if \( g_1, g_2, \ldots, g_d \in T_{n-1} \), then \( g_1 x_1 + \ldots + g_d x_d = 0 \) implies

\[
g_1 = \ldots = g_d = 0.
\]

**Proof:** Each \( g_i \) is the sum of distinct monomials of degree \( n-1 \). Therefore, \( g_i x_1 \) is the sum of distinct monomials of degree \( n \). If \( i \neq k \), then the monomials in \( g_i x_1 \) are distinct from the monomials in \( g_k x_k \).

Therefore, the set of all monomials involved in \( g_1 x_1 + \ldots + g_d x_d \) is a set of distinct monomials, hence is a set of linearly independent monomials. Therefore, every coefficient in \( g_1 x_1 + \ldots + g_d x_d = 0 \) must be zero. Therefore:

\[
g_1 = g_2 = \ldots = g_d = 0
\]

**Proposition 2.1.7** \( \dim J = d \dim \mathcal{H}_{n-1} \)

**Proof:** Since \( J = \mathcal{H}_{n-1} x_1 \oplus \cdots \oplus \mathcal{H}_{n-1} x_d \) is a direct sum and

\[
\dim \mathcal{H}_{n-1} x_1 = \dim \mathcal{H}_{n-1},
\]

it follows that:

\[
\dim J = d \dim \mathcal{H}_{n-1}.
\]
Example 2.1.8  Let \( g_1, g_2, \ldots, g_m \) be a basis for \( \mathcal{U}_{n-1} \). Then we have the following bases for each \( \mathcal{U}_{n-1} x_i \) (\( i = 1, 2, \ldots, d \)).

\[
B_1 = \{ g_1 x_1, g_2 x_1, \ldots, g_m x_1 \} \text{ forms a basis for } \mathcal{U}_{n-1} x_1.
\]

\[
B_2 = \{ g_1 x_2, g_2 x_2, \ldots, g_m x_2 \} \text{ forms a basis for } \mathcal{U}_{n-1} x_2.
\]

\[
\vdots
\]

\[
B_d = \{ g_1 x_d, g_2 x_d, \ldots, g_m x_d \} \text{ forms a basis for } \mathcal{U}_{n-1} x_d
\]

The elements of the above sets are linearly independent, for suppose that:

\[
\sum_{j=1}^{n} a_j g_j = g \quad \text{where } g \in \mathcal{U}_{n-1} \subset T_{n-1}, \quad a_j \in F
\]

write

\[
g = \sum_{i=1}^{n} b_i u_i
\]

where the \( u_i \) are distinct monomials of degree \( n-1 \). (We have \( d^{n-1} \) possible \( u_i \)'s.)

Suppose

\[
0 = gx_k = \sum_{i=1}^{n} (b_i u_i) x_k = \sum_{i=1}^{n} b_i (u_i x_k)
\]

Then if the \( u_i x_k \) are distinct (monomials of degree \( n \)), then they are linearly independent, therefore \( b_i \)'s = 0, therefore \( g = 0 \). Let \( u_1, u_2 \) be distinct monomials of degree \( n-1 \). Then \( u_1 x_k \) and \( u_2 x_k \) are distinct since \( T \) is the free associative algebra over \( F \). Thus the set

\[\{g_1 x_1, g_2 x_1, \ldots, g_m x_1\}\]

consists of linearly independent elements. Hence, \( B_1 \) is a basis for \( \mathcal{U}_{n-1} x_1 \).
Proposition 2.1.9  Prove that \( \dim T_n = db_{n-1} + \dim J \)

Proof: We know that \( T_{n-1} = \mathbb{W}_{n-1} \oplus S_{n-1} \), where \( S_{n-1} \) is the complement of \( \mathbb{W}_{n-1} \) in \( T_{n-1} \). Let \( s_1, s_2, \ldots, s_{b_{n-1}} \) be a basis for \( S_{n-1} \) and let \( g_1, g_2, \ldots, g_m \) be a basis for \( \mathbb{W}_{n-1} \). (\( S_{n-1} \) and \( \mathbb{W}_{n-1} \) are considered as vector spaces over \( F \).) The elements \( s_i x_j \) and \( g_k x_j \), where \( i = 1, 2, \ldots, b_{n-1}, j = 1, 2, \ldots, d, \) and \( k = 1, 2, \ldots, m \) form a basis for \( T_n \) for the following reasons: The \( s_i x_j \) and \( g_k x_j \) are all of degree \( n \) and:

\[
\dim T_n = b_{n-1} d + md = (b_{n-1} + m)d = d^{n-1}d = d^n \tag{1}
\]

where \( b_{n-1} + m = \dim T_{n-1} = d^{n-1} \). Finally the set \( \{ s_i x_j \} \cup \{ g_k x_j \} \), (\( i,j,k \) as before) consists of linearly independent vectors, for suppose not, then:

\[
\sum_{i,j} a_{ij} s_i x_j = \sum_{k,l} b_{kl} g_k x_l
\]

implies

\[
\sum_{j} (\sum_{i} a_{ij} s_i) x_j = \sum_{k,l} b_{kl} g_k x_l
\]

Hence for all \( e = 1, 2, \ldots, d \)

\[
\sum_{i} (a_{ie} s_i) x_e = \sum_{k} (b_{ke} g_k) x_e
\]

implies

\[
\sum_{i} (a_{ie} s_i) x_e + \sum_{k} (-b_{ke} g_k) x_e = 0
\]

By example 2.1.8, \( a_{ie} = 0 = -b_{ke} \)
Thus the set \( \{ s_i x_j \} \cup \{ g_k x_j \} \) \((i = 1, 2, \ldots, b_{n-1}; j = 1, 2, \ldots, d; \ and \ k = 1, 2, \ldots, m) \) consists of linearly independent elements, moreover, it has \( d^n \) elements all of degree \( n \), therefore:

\[
\{ s_i x_j \} \cup \{ g_k x_j \}
\]

is a basis for \( T_n \). So

\[
T_n = S_{n-1} x_1 \oplus \ldots \oplus S_{n-1} x_d \oplus U_{n-1} x_1 \oplus \ldots \oplus U_{n-1} x_d
\]

Hence:

\[
\dim T_n = db_{n-1} + d \dim U_{n-1}
\]
or

\[
\dim T_n = db_{n-1} + \dim J
\]

**Definition 2.1.10** Let \( L \) be the vector space spanned by all elements of degree \( n \) of the form \( v_i f_j \), where \( f_j \) is in the set \( H \) and \( \{ v_i \} \) is a set of homogeneous polynomials of degrees up to \( n-2 \), which forms a basis for \( S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2} \)

**Proposition 2.1.11** \( \dim L \leq \sum_{i=2}^{n-1} b_i r_i \) where \( L \) is defined above and

where \( b_{n-1} = \dim A_{n-1} \) and \( r_i \) is the number of those polynomials \( f_i \) in \( H \) of degree \( i \).

**Proof:** Let \( \Theta(v_{i,j}) = i \). Then:

\[
v_0, v_1, v_1, \ldots, v_1
\]

forms a basis for \( S_0 \)

\[
v_1, v_1, \ldots, v_1
\]

form a basis for \( S_1 \)

\[
v_1, v_2, \ldots, v_b
\]

\[
v_{n-2}, v_{n-2}, \ldots, v_{n-2b}
\]

form a basis for \( S_i \).
The elements \( v_0, v_1, \ldots, v_{b_1}, v_{2_1}, \ldots, v_{b_2}, \ldots \)

\[ v_{n-2}, \ldots, v_{n-2b_{n-2}} \]

form a basis for \( S \oplus S \oplus \ldots \oplus S_{n-2} \).

Let

\[ H = \{ f_1, f_2, \ldots, f_{r_1}, f_{r_2}, \ldots, f_{r_3}, \ldots, f_{r_{n}}, \ldots, f_{r_{n}} \} \]

where \( f_{r_j} \) has degree \( r_j \) (and for some \( r_j \)s, perhaps there are no \( r_j \)s.) Let \( \deg(G_i) \) denote the degree of the elements of \( G_i \).

\[ \{ f_1, f_{r_2}, \ldots, f_{r_n} \} = G_2 \]

where \( \deg(G_2) = 2 \)

\[ \ldots \]

\[ \ldots \]

\[ \ldots \]

\[ f_{r_n}, f_{r_n}, \ldots, f_{r_n} = G_n \]

where \( \deg(G_n) = n \)

Now we observe that the following elements are homogeneous polynomials of degree \( n \).

\[ v_{(n-i)} f_{i_1}, v_{(n-i)} f_{i_1}, \ldots, v_{(n-i)} f_{i_1} \]

\[ v_{(n-i)} f_{i_2}, v_{(n-i)} f_{i_2}, \ldots, v_{(n-i)} f_{i_2} \]

\[ \ldots \]

\[ \ldots \]

\[ \ldots \]

\[ v_{(n-i)} f_{i_n}, v_{(n-i)} f_{i_n}, \ldots, v_{(n-i)} f_{i_n} \]
Also they span

\[ S_{n-i}f_{i_1} + S_{n-i}f_{i_2} + \ldots + S_{n-i}f_{i_r} = S_{n-i}G_i \]

and their number is \( b_{n-i}r_1 \). By summing \( S_{n-i}G_i \) over \( i = 2, \ldots, n \) we get

\[ L = S_{n-2}G_2 + S_{n-3}G_3 + \ldots + S_1G_{n-1} + S_0G_n \]

and hence

\[ \dim L \leq \dim (S_{n-2}G_2) + \dim (S_{n-3}G_3) + \ldots + \dim (S_0G_n) \]

\[ \leq b_{n-2}r_2 + b_{n-3}r_3 + \ldots + b_1r_{n-1} + b_0r_n \]

\[ = \sum_{j=2}^{n} b_{n-j}r_j \]

Hence \( \dim L \leq \sum_{j=2}^{n} b_{n-j}r_j \)

**Proposition 2.1.12** \( \dim \mathcal{U}_n \leq \dim J + \dim L \).

**Proof:** Here \( J \) and \( L \) are as they have been previously defined by definitions 2.1.5 and 2.1.10 respectively.

Now we take \( w \in \mathcal{U}_n \), and we wish to show that:

\[ u = w + v \]

where \( w \in J \) and \( v \in L \). By proposition 1.4.1, \( u \) is the sum of polynomials of the form

\[ a_{i_j}b_k \]

where \( a_{i_j}, b_k \) are homogeneous polynomials and \( \partial(a_{i_j}b_k) = n \). (We know nothing about \( \partial(a_{i_j}), \partial(b_k), \) and \( \partial(f_j) \)).

Now if we get simply \( a_{i_j}b_k \in J \), we have **case I**, while if we get

\[ a_{i_j}b_k = w'f_j + v'f_j \]

where \( w'f_j \in J \) and \( v'f_j \in L \), we have the **case II**.
Thus each polynomial $a_i f_j b_k \in J + L$, and hence, the sum of a lot of them, namely $u$, also $\in J + L$.

Case I: Assume $\partial(b_k) \geq 1$. We can write $b_k = b'_k x_n (b'_k \in T_{j-1}, i f b_k \in T_j)$.

Now
\[
a_{ij} f_j b'_k = a_{ij} f_j b'_k x_n = (a_{ij} f_j b'_k) x_n \in \mathfrak{g}_{n-1} x_n \subset J
\]
since
\[
a_{ij} f_j b'_k \in \mathfrak{g}_{n-1} \quad (n = 1, 2, \ldots, d).
\]

Case II: Assume that $\partial(b_k) = 0$, that is $a_{ij} f_j b_k$ is a homogeneous polynomial of the form $a_{ij} f_j$, where $\partial(a_{ij} f_j) = n$. Now $a_i$ is homogeneous and say has degree $k$, so:
\[
a_i \in T_k = \mathfrak{g}_i \ominus S_k;
\]
let
\[
a_i = w' + v', \quad w' \in \mathfrak{g}_k, v' \in S_k.
\]
therefore
\[
a_{ij} f_j = w' f_j + v' f_j
\]
First look at $w' f_j$. Now $\partial(w') = k = \partial(a_i)$, and $\partial(a_{ij} f_j) = n$.

Say $\partial(f_j) = k'$, so $k + k' = n$. Note $k' \geq 2$. Now $w' \in \mathfrak{g}_k \subset \mathfrak{g}$.

Since $\partial(f_j) \geq 2$, we can write $f_j = hx_m$, where $x_m \in \{x_1, x_2, \ldots, x_d\}$.

Then $w' \in \mathfrak{g} = w'h \in \mathfrak{g}$, but $w'h$ is homogeneous and
\[
\partial(w'h) = \partial(w'f_j) - 1
\]
\[
= k + k' - 1
\]
\[
= n - 1
\]
Therefore $w'h \in \mathfrak{g} \cap T_{n-1} = \mathfrak{g}_{n-1}$. Therefore:
\[
w' f_j = (w'h)x_m \in \mathfrak{g}_{n-1} x_m \subset J
\]
Next, look at $v^i f_j$. Now $v^i \in S_k \subset T_k$, so $v^i$ is a homogeneous polynomial of degree $k$. We still have $k + k' = n$ and $k' \geq 2$. So

$$k \leq n - 2$$

i.e. $v^i \in S_k \subset S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2}$. Since $\{v_1, v_2, \ldots\}$ is a basis of homogeneous polynomials for $S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2}$, we can write $v^i$ as a linear combination of some of these basis elements, say $v^i$ is a linear combination of

$$v_{i_1}, v_{i_2}, \ldots, v_{i_p}.$$  

Since $v^i$, $v_{i_1}$, ..., $v_{i_p}$ are all homogeneous, and the $v_{i_1}$, ..., $v_{i_p}$ are linearly independent, it must be the case that

$$k = \deg(v^i) = \deg(v_{i_1}) = \ldots = \deg(v_{i_p})$$

Thus

$$v^i f_j = c_{i_1}(v^i f_j) + c_{i_2}(v^i f_j) + \ldots + c_{i_p}(v^i f_j);$$

But

$$v_{i_1} f_j, \ldots, v_{i_p} f_j$$

are among the polynomials which span $L$. (Each has degree $n$, and is of the proper form.) Thus:

$$v^i f_j \in L.$$ 

2.2 Golod–Safarevič Theorem and its Proofs

**Theorem 2.2.1** (Golod and Safarevič)

(i) \[ b_n \geq d b_{n-1} - \prod_{i=1}^{n-1} b_{n-1} \]

(ii) \[ b_n \geq d b_{n-1} - \sum_{i=1}^{n-1} b_{n-1} \]

n \geq 2
Note: The same notation used previously holds here. Also (i) and (ii) are the same. Because some of the \( n_i \)'s may be equal, (ii) may be written as follows:

\[
\begin{align*}
  b_n &\geq d b_{n-1} - \sum_{i \mid n_i \leq n} b_{n-n_i} \\
  \end{align*}
\]

Proof of (i): (Fisher and Struick) This proof is using the dimensionality of the various subspaces.

Clearly we know that

\[
T_n = \mathcal{U}_n \oplus S_n
\]

And therefore

\[
\text{dim } T_n = \text{dim } \mathcal{U}_n + \text{dim } S_n.
\]

(1) Let \( n = 2 \). Then we have that

\[
\begin{align*}
  \text{dim } T_2 &= \text{dim } \mathcal{U}_2 + \text{dim } S_2 \\
  d^2 &\leq r_2 + b_2 \quad \text{(by 2.1.4 and 2.1.3)} \\
  b_2 &\geq d^2 - r_2 \\
  b_2 &\geq d - r_2 + 1 \\
  b_2 &\geq d - r_2 + b_0 \quad \text{(by 2.1.2 and 2.1.1)}
\end{align*}
\]

which is statement (i) for \( n = 2 \).

(2) Let \( n \geq 2 \). Then

\[
\begin{align*}
  \text{dim } T_n &= \text{dim } \mathcal{U}_n + \text{dim } S_n \\
  \leq \text{dim } J + \text{dim } L + b_n \quad \text{(*)} \\
  \leq \text{dim } J + \sum_{i=2}^{n} i b_{n-i} + b_n \quad \text{(***)}
\end{align*}
\]

(* ) by 2.1.12 and 2.1.3, (** ) by 2.1.11.
But since, by 2.1.9, we have that
\[ \dim T_n = db_{n-1} + \dim J \]
we have
\[ db_{n-1} \leq \sum_{i=2}^{p} r_i b_{n-1} + b_n \]
and therefore
\[ b_n \geq db_{n-1} - \sum_{i=2}^{p} r_i b_{n-1} \quad n \geq 2. \]

Proof of (ii) (Herstein) This proof is using homology which was used in the original proof by Golod and Šafarevič.

Suppose that we can exhibit linear mappings, \( \phi, \psi \) so that the following sequence is exact
\[
\begin{array}{ccc}
\cdots & \Phi & \cdots \\
A_{n-n_1} \oplus \cdots \oplus A_{n-n_k} & \Phi & A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \\
& \Phi & A_n \\
\end{array}
\]

Then
\[ \dim(A_{n-1} \oplus \cdots \oplus A_{n-1}) = \text{rank } \psi + \text{nullity } \psi, \]
or
\[ db_{n-1} = \dim \text{Im } \psi + \dim \text{ker } \psi \\
= b_n + \dim \text{Im } \phi \]

by exactness, so
\[ db_{n-1} \leq b_n + \sum_{i} \Sigma_{n-n_i} b_{n-n_i} \quad n \geq 2. \]

Hence:
\[ b_n \geq db_{n-1} - \sum_{i} \Sigma_{n-n_i} b_{n-n_i} \quad n \geq 2. \]
Now our objective is that of defining the $\phi$ and $\psi$. First we shall define mappings $\phi$ and $\psi$ for the following sequence

$$\begin{align*}
T_{n_{-1}} \oplus \cdots \oplus T_{n_{-k}} \oplus \cdots \rightarrow T_{n_{-1}} \oplus T_{n_{-1}} \oplus \cdots \oplus T_{n_{-1}} \rightarrow T_{n} + 0
\end{align*}$$

(2)

where $\phi$ and $\psi$ are linear. We are not interested if the sequence is exact or not at the $T$-level. However, we want to induce the proper $\phi$ and $\psi$ from the $\phi$ and $\psi$ so the sequence will be exact at the $\Lambda$-level.

Define $\psi$ by:

$$\psi : t_1 \oplus \cdots \oplus t_d \rightarrow t_1 x_1 + t_2 x_2 + \cdots + t_d x_d \quad (\text{for } t_i \in T_{n_{-1}})$$

If $u \in T_{n_{-1}} \oplus \cdots \oplus T_{n_{-1}}$, then $u$ may be written uniquely as

$$u = t_1 \oplus \cdots \oplus t_d$$

where $t_i$ is in the $i$th $T_{n_{-1}}$ in the above direct sum. Hence if we define

$$\psi(u) = t_1 x_1 + \cdots + t_d x_d$$

$\psi$ is well defined and obviously, if $a,b \in F$ and $u,v \in T_{n_{-1}} \oplus \cdots \oplus T_{n_{-1}}$, we have

$$\psi(au + bv) = a\psi(u) + b\psi(v)$$

and hence $\psi$ is linear.

Define $\phi$ by:

$$\phi : s_{n_{-1}} \oplus \cdots \oplus s_{n_{-k}} \oplus \cdots \rightarrow u_1 \oplus u_2 \oplus \cdots \oplus u_d$$

where

$$s_{n_{-1}} \oplus \cdots \oplus s_{n_{-k}} \oplus \cdots \in T_{n_{-1}} \oplus \cdots \oplus T_{n_{-1}}$$

and $u_1 \oplus \cdots \oplus u_d \in T_{n_{-1}} \oplus \cdots \oplus T_{n_{-1}}$. 
The way we follow to get $\phi$ is the following: If
\[ s_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \ldots \in T_{n-n_1} \oplus \ldots \oplus T_{n-n_k} \oplus \ldots \]
then, recalling that $\partial(f_i) = n_i$, we see that
\[ \sum_{n_i \leq n} s_{n-n_i} f_i \in T_n. \]
($\partial(s_{n-n_i}) + \partial(f_i) = n - n_i + n_i = n$). As an element in $T_n$, we can write
\[ \sum_{n_i \leq n} s_{n-n_i} f_i = \sum_{i=1}^{d} u_i x_i \]
where the $u_i$ are uniquely determined elements in $T_{n-1}$. Hence $\phi$ is well defined and like $\psi$, $\phi$ is linear.

**Proposition 2.2.2** Let $\psi$ be defined as above. Then sequence (2) is exact at $T_n$.

**Proof:** To show exactness at $T_n$, we need to show that $\psi$ is an onto homomorphism. Now if $w \in T_n$, then $w$ can be written uniquely as follows:
\[ w = t_1 x_1 + \ldots + t_d x_d \]
where $t_1, t_2, \ldots, t_d$ are in $T_{n-1}$ and such that
\[ \psi(t_1 \oplus \ldots \oplus t_d) = t_1 x_1 + \ldots + t_d x_d = w \]
Hence $\psi$ is onto and since $\psi$ is linear, the sequence
\[ \underbrace{T_{n-1} \oplus \ldots \oplus T_{n-1}}_{d \text{ times}} \to T_n \to 0 \]
is exact.

Recall that $\mathcal{U}_{n-1} = \mathcal{U} \cap T_{n-1}$. Since $\mathcal{U}_{n-1} \subseteq T_{n-1}$, obviously we have:
\[ \underbrace{\mathcal{U}_{n-1} \oplus \mathcal{U}_{n-1} \oplus \ldots \oplus \mathcal{U}_{n-1}}_{d \text{ times}} \subseteq T_{n-1} \oplus \ldots \oplus T_{n-1} \]
If we can show that
\[ \psi(\mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}) = \{ \psi(t_1 \oplus \ldots \oplus t_d) \mid t_1 \in \mathfrak{U}_{n-1} \} \subseteq \mathfrak{U}_n \]
then we can induce a new homomorphism
\[ \psi : \frac{T_{n-1} \oplus \ldots \oplus T_{n-1}}{\mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}} \rightarrow \frac{T_n}{\mathfrak{U}_n} \]
given by
\[ \psi(t_1 \oplus \ldots \oplus t_d + \mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}) = \psi(t_1 \oplus \ldots \oplus t_d) + \mathfrak{U}_n \]
\[ = t_1 x_1 + \ldots + t_d x_d + \mathfrak{U}_n \]

**Proposition 2.2.3** \( \psi(\mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}) \subseteq \mathfrak{U}_n. \)

**Proof:** Take \( t_1, \ldots, t_d \) such that \( t_i \) is in the \( i \)th \( \mathfrak{U}_{n-1} \); then
\[ \psi(t_1 \oplus \ldots \oplus t_d) = t_1 x_1 + \ldots + t_d x_d \]
t_1, t_2, \ldots, t_d are in \( \mathfrak{U}_{n-1} \) and hence in \( \mathfrak{U} \); since \( \mathfrak{U} \) is an ideal then \( t_1 x_1, t_2 x_2, \ldots, t_d x_d \) are in \( \mathfrak{U} \) and therefore their sum is in \( \mathfrak{U} \).
But it is also in \( T_n \). Hence
\[ t_1 x_1 + \ldots + t_d x_d \in \mathfrak{U}_n, \]
and so
\[ \psi(\mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}) \subseteq \mathfrak{U}_n. \]

Now if we show that
\[ \frac{T_{n-1} \oplus \ldots \oplus T_{n-1}}{\mathfrak{U}_{n-1} \oplus \ldots \oplus \mathfrak{U}_{n-1}} = \frac{T_{n-1}}{\mathfrak{U}_{n-1}} \oplus \ldots \oplus \frac{T_{n-1}}{\mathfrak{U}_{n-1}} \]
then we can induce the required mapping
\[ \psi : A_{n-1} \oplus \ldots \oplus A_{n-1} \rightarrow A_n \]
Proposition 2.2.4

\[
\begin{align*}
\mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1} &= \mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1} \\
\mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1} &= \mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1}
\end{align*}
\]

Proof: We need to find an onto map \( \gamma \) such that

\[
\gamma : \mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1} \rightarrow \mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1}
\]

and such that \( \ker \gamma = \mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1} \). So define

\[
\gamma(t_1 \oplus \ldots \oplus t_d) = \bar{t}_1 \oplus \ldots \oplus \bar{t}_d = (t_1 + \mathbb{U}_{n-1}) \oplus \ldots \oplus (t_d + \mathbb{U}_{n-1}).
\]

Let \( t_1 \oplus \ldots \oplus t_d \in \ker \gamma \), then

\[
\gamma(t_1 \oplus \ldots \oplus t_d) = (t_1 + \mathbb{U}_{n-1}) \oplus \ldots \oplus (t_d + \mathbb{U}_{n-1})
\]

\[
= (0 + \mathbb{U}_{n-1}) \oplus \ldots \oplus (0 + \mathbb{U}_{n-1}).
\]

Since we are working with direct sum, this holds if and only if:

\[
t_i + \mathbb{U}_{n-1} = 0 + \mathbb{U}_{n-1},
\]

that is

\[
t_i \in \mathbb{U}_{n-1}
\]

so

\[
(t_1 \oplus \ldots \oplus t_d) \in \mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1},
\]

therefore

\[
\ker \gamma = \mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1}.
\]

Now if \( \Theta \) is the natural map such that

\[
\Theta : \mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1} + \frac{\mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1}}{\mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1}}
\]

then there exists an isomorphism \( \sigma \) such that

\[
\sigma : \frac{\mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1}}{\mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1}} \rightarrow \frac{\mathbb{T}_{n-1} \oplus \ldots \oplus \mathbb{T}_{n-1}}{\mathbb{U}_{n-1} \oplus \ldots \oplus \mathbb{U}_{n-1}}.
\]
Thus the mapping $\psi$ induces
$$\psi : A_{n-1} \oplus \ldots \oplus A_{n-1} \rightarrow A_n$$
given by
$$\psi(t_1 + \mathfrak{m}_{n-1}) \oplus \ldots \oplus (t_d + \mathfrak{m}_{n-1}) = (t_1x_1 + \ldots + t_dx_d) + \mathfrak{m}_n$$
where
$$\psi(t_1 + \ldots + t_d) = t_1x_1 + \ldots + t_dx_d.$$

We can now consider $\phi$. Suppose that $s_{n-n_1}, s_{n-n_2}, \ldots, s_{n-n_k}$
are in $\mathfrak{m}_{n-1}, \mathfrak{m}_{n-2}, \ldots, \mathfrak{m}_{n-n_k}$
respectively. We must show that $u_1, u_2, \ldots, u_d$
defined by $\Sigma s_{n-n_i} f_i = \Sigma u_i x_i$ are in $\mathfrak{m}_{n-1}$. Since $\phi$
is linear it suffices to do so for each $s_{n-n_i}$ in $\mathfrak{m}_{n-n_i}$.
Note that
$$\partial(s_{n-n_i} f_i) = n-n_i + n_i = n.$$ Since $\partial(f_i) = n_i$ implies that
$$f_i = j^{\partial_i} g_{ij} x_j$$
where $g_{ij} \in T_{n_i-1}$. Therefore:
$$s_{n-n_i} f_i = s_{n-n_i} j^{\partial_i} g_{ij} x_j = j^{\partial_i} (s_{n-n_i} g_{ij}) x_j = j^{\partial_i} u_j x_j$$
where $u_j = s_{n-n_i} g_{ij}$ and $\partial(u_j) = n - n_i + n_i - 1 = n - 1$. Thus
$u_j \in T_{n-1}$. But $u_j = s_{n-n_i} g_{ij} \in \mathfrak{m}$, as $s_{n-n_i}$
is in the ideal $\mathfrak{m}$. Therefore, $u_j \in \mathfrak{m} \cap T_{n-1} = \mathfrak{m}_{n-1}$. Therefore $\phi$ induces a map:
$$\phi : A_{n-n_1} \oplus \ldots \oplus A_{n-n_k} \rightarrow A_{n-1} \oplus \ldots \oplus A_{n-1}$$
given by
$$\phi'(s_{n-n_1} t_{n-n_1} \oplus \ldots \oplus (s_{n-n_k} t_{n-n_k}) \oplus \ldots \oplus s_{n-n_k} t_{n-n_k}) = (u_1 + \mathfrak{m}_{n-1} \oplus \ldots \oplus u_d + \mathfrak{m}_{n-1}),$$
where
$$\phi(s_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \ldots) = u_1 \oplus \ldots \oplus u_d.$$
Proposition 2.2.5  The sequence

\[
\begin{array}{c}
A_n \oplus \cdots \oplus A_{n-k} \oplus \cdots \oplus A_{n-1} \\
\oplus A_{n_1} \oplus A_{n_2} \oplus \cdots \oplus A_{n_d} \psi A_n \to 0
\end{array}
\]  \hspace{1cm} (1)

\[n_1 \leq n \hspace{1cm} \text{d times}\]

is exact.

Proof: To show the exactness of (1), we must prove exactness at \(A_n\) and exactness at \(A_{n-1} \oplus \cdots \oplus A_{n-1}\). To show exactness at \(A_n\), we need to show that \(\psi\) is a homomorphism onto. So let \(t + \mathfrak{y}_n\) be in \(A_n\), where \(t \in T_n\).

We want to find some \((t_1 + \mathfrak{y}_{n-1}) \oplus \cdots \oplus (t_d + \mathfrak{y}_{n-1}) \in A_{n-1} \oplus \cdots \oplus A_{n-1}\), where \(t_1 \oplus \cdots \oplus t_d \in T_{n-1} \oplus \cdots \oplus T_{n-1}\), such that,

\[
t + \mathfrak{y}_n = \psi((t_1 + \mathfrak{y}_{n-1}) \oplus \cdots \oplus (t_d + \mathfrak{y}_{n-1}))
\]

\[
= t_1 x_1 \oplus \cdots \oplus t_d x_d + \mathfrak{y}_{n-1}
\]

where \(\psi(t_1 \oplus \cdots \oplus t_d) = t_1 x_1 \oplus \cdots \oplus t_d x_d\). But \(\psi\) is onto by 2.2.2.

Hence, \(\psi\) is onto and the sequence (1) is exact at \(A_n\).

Now we need to show that the sequence (1) is exact at \(A_{n-1} \oplus \cdots \oplus A_{n-1}\). That is, we need to show that \(\text{Im} \phi = \ker \psi\).

(a) \(\text{Im} \phi \subseteq \ker \psi\), that is \(\phi \psi = 0\). So, if \(s_{n_1}, s_{n_2}, \ldots, s_{n_k}\) are elements of \(T_{n_1}, T_{n_2}, \ldots, T_{n_k}\) respectively, so

\[
(s_{n_1} \oplus s_{n_2} \oplus \cdots \oplus s_{n_k}) \phi \psi = u_1 x_1 + u_2 x_2 + \cdots + u_d x_d.
\]

where \(\Sigma_{i=1}^{d} u_i x_i = \Sigma_{i=1}^{n} s_{n_i} f_i\); but the \(f_i\)'s generate \(\mathfrak{y}\), thus \(\mathfrak{y} \subseteq \mathfrak{y}\). And so \(\Sigma_{i=1}^{d} u_i x_i \in \mathfrak{y}\). But \(\Sigma_{i=1}^{d} u_i x_i \in T_n\) too.
So $\Psi$ maps $T_{n-n_1} \oplus T_{n-n_2} \oplus \ldots \oplus T_{n-n_k}$ into $\mathcal{U}_n = T_n \cap \mathcal{U}$ and so $A_{n-n_1} \oplus A_{n-n_2} \oplus \ldots \oplus A_{n-n_k}$ is mapped into 0 by $\phi\psi$, as follows:

Let $\bar{s}_{n-n_1} \oplus \ldots \oplus \bar{s}_{n-n_k}$ be an element of $A_{n-n_1} \oplus \ldots \oplus A_{n-n_k}$

Then:

$$(\bar{s}_{n-n_1} \oplus \ldots \oplus \bar{s}_{n-n_k}) \phi\psi = (s_{n-n_1} \oplus \mathcal{U}_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \mathcal{U}_{n-n_k}) \phi\psi$$

$$= (\phi(s_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \mathcal{U}_{n-n_1} \oplus \ldots \oplus \mathcal{U}_{n-k-1})) \phi\psi$$

$$= (u_1 \oplus \ldots \oplus u_d \oplus \mathcal{U}_{n-1} \oplus \ldots \oplus \mathcal{U}_{n-k-1}) \phi\psi$$

$$= \Psi(u_1 \oplus \ldots \oplus u_d) + \mathcal{U}_n$$

$$= (u_1x_1 + \ldots + u_dx_d) + \mathcal{U}_n$$

$$= 0 + \mathcal{U}_n,$$

since $u_1x_1 + \ldots + u_dx_d \in \mathcal{U}_n$. Hence $\text{Im } \phi \subseteq \ker \psi$.

(ii) $\ker \psi \subseteq \text{Im } \phi$. Here we want to show that if $\bar{t}_1 \oplus \ldots \oplus \bar{t}_d \in \ker \psi$, then $\bar{t}_1 \oplus \ldots \oplus \bar{t}_d \in \text{Im } \phi$. That is we want to find some $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_d$ in $A_{n-1}$, where $\bar{u}_1 \oplus \ldots \oplus \bar{u}_d \in \text{Im } \phi$ and such that:

$$(\bar{t}_1 \oplus \ldots \oplus \bar{t}_d) - (\bar{u}_1 \oplus \ldots \oplus \bar{u}_d) = 0.$$ 

That is

$$\bar{t}_1 - \bar{u}_1 \oplus \ldots \oplus \bar{t}_d - \bar{u}_d = 0,$$

or

$$\bar{t}_i - \bar{u}_i = 0$$

($i = 1, 2, \ldots, d$) (by the direct sum), or

$t_1 - u_1 + \mathcal{U}_{n-1} = 0$, or $t_1 - u_1 \in \mathcal{U}_{n-1} = \mathcal{U} \cap T_{n-1}$, or $t_1 - u_1 \in \mathcal{U}$. 
Also $\bar{u}_1 \oplus \ldots \oplus \bar{u}_d \in \text{Im } \phi$ implies that there exist 

$\bar{s}_{n_{-1}}, \bar{s}_{n_{-2}}, \ldots, \bar{s}_{n_{-k}}, \ldots$ in $A_{n_{-1}}, A_{n_{-2}}, \ldots, A_{n_{-k}}, \ldots$

such that:

$$\bar{u}_1 \oplus \ldots \oplus \bar{u}_d = \phi(\bar{s}_{n_{-1}} \oplus \ldots \oplus \bar{s}_{n_{-k}} \oplus \ldots)$$

$$= \phi((s_{n_{-1}} + \mathcal{U}_{n_{-1}}) \oplus \ldots \oplus (s_{n_{-k}} + \mathcal{U}_{n_{-k}}) \oplus \ldots)$$

$$= u_1 \oplus \ldots \oplus u_d + \mathcal{U}_{n_{-1}} \oplus \ldots \oplus \mathcal{U}_{n_{-1}},$$

$$= (u_1 + \mathcal{U}_{n_{-1}}) \oplus \ldots \oplus (u_d + \mathcal{U}_{n_{-1}}),$$

where $\frac{d}{i=1} u_i x_i = n_{-1} \sum s_{n_{-1}} f_{i}$, for some $s_{n_{-1}}, \ldots, s_{n_{-k}}, \ldots$ in $T_{n_{-1}}, \ldots, T_{n_{-k}}, \ldots$, and $u_i \in \text{ith } T_{n_{-1}}$.

Moreover, if $\bar{t}_1 \oplus \bar{t}_2 \oplus \ldots \oplus \bar{t}_d \in \ker \psi$, means that

$$\psi(\bar{t}_1 \oplus \bar{t}_2 \oplus \ldots \oplus \bar{t}_d) = 0$$

which implies that

$$\psi(t_1 \oplus t_2 \oplus \ldots \oplus t_d) \in \mathcal{U}_n = T_n \cap \mathcal{U}_n$$

Hence

$$\psi(t_1 \oplus t_2 \oplus \ldots \oplus t_d) \in \mathcal{U}_n$$

**Conclusion:** So we need to show that if $\psi(t_1 \oplus \ldots \oplus t_d) \in \mathcal{U}_n$, then we can find elements $u_1, u_2, \ldots, u_d$ in $T_{n_{-1}}$ such that

$$t_i - u_i \in \mathcal{U} \quad \text{for } i = 1, 2, \ldots, d$$

and such that $\sum u_i x_i = n_{-1} \sum s_{n_{-1}} f_{i}$ for some $s_{n_{-1}}$ in the appropriate $T_{n_{-1}}$.

Suppose then, that $\psi(t_1 \oplus \ldots \oplus t_d) = \frac{d}{i=1} t_i x_i \in \mathcal{U}$. Since $\mathcal{U}_n$ is a two-sided ideal generated by the $f_j$, we have that the elements in $\mathcal{U}$ can
be written in the following form and hence:

\[ \sum_{i=1}^{d} t_i x_i = \sum_{k,q} a_{kq} f b_{kq} + \sum_{q} c_{q} f q \]

where the \( a_{kq}, b_{kq}, c_{q} \) are homogeneous and where the degree of \( b_{kq} \) is at least 1. On comparing degree on both sides, we may even assume that the \( a_{kq} f b_{kq}, c_{q} f q \) are all in \( T_n \). Since the \( b_{kq} \) are of degree at least 1,

\[ b_{kq} = \sum_{m=1}^{d} d_{kqm} x_m \]

where \( d_{kq} \) is any homogeneous polynomial or constant. Then

\[ \sum_{k,q} a_{kq} f b_{kq} = \sum_{k,q,m=1}^{d} a_{kq} f d_{kqm} x_m = \sum_{m=1}^{d} d_{m} x_m \]

where

\[ d_{m} = \sum_{k,q} a_{kq} f d_{kqm} \]

But since \( f_{q} \in \mathbb{M} \) we have that \( d_{m} \in \mathbb{M} \). If we write

\[ \sum_{q} c_{q} f q = \sum_{i=1}^{d} u_{i} x_{i} \]

we then have that

\[ \sum_{i=1}^{d} t_{i} x_{i} = \sum_{i=1}^{d} d_{i} x_{i} + \sum_{i=1}^{u_{i}} u_{i} x_{i} \]

implies

\[ t_{i} = d_{i} + u_{i} \quad i = 1, 2, \ldots, d \]

hence

\[ t_{i} - u_{i} = d_{i} \in \mathbb{M} . \]

But \( \phi(c_{1} \oplus \cdots \oplus c_{k} \oplus \cdots) = u_{1} \oplus \cdots \oplus u_{d} \) by the definition of \( \phi \); hence we have proved (ii).
The two inclusions (i) and (ii) give us the desired result, and hence we have proved exactness of (1) at $A_{n-1} \oplus \ldots \oplus A_{n-1}$. This proves proposition 2.2.5, and hence also Theorem 2.2.1(ii).

**Definition 2.2.6** The power series

\[ P_A(t) = \sum_{n=0}^{\infty} b_n t^n \]

is called the Poincare function of the algebra $A$.

The following two theorems and corollary 2.3.1 are reworked from the original paper by Golod and Safarevic.

**Theorem 2.2.7**

\[ P_A(t)(1 - dt + \sum_{i=2}^{\infty} r_i t^i) \geq 1, \]

where inequality between power series is understood coefficient-wise.

**Proof:** Recall that

\[ A = A_0 \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots \quad (1) \]

and that the numbers $b_n = \dim A_n$, $n \geq 0$ are all finite. For the dimensions of the subspaces of $A$ we obtained the inequality:

(Theorem 2.2.1 (ii))

\[ b_n \geq d b_{n-1} - \sum_{i=1}^{n} b_{n-i} \quad (n \geq 1) \quad (2) \]

Multiplying this inequality by $t^n$ and adding up for all $n \geq 1$, we obtain an inequality for the series:

\[ \sum_{n=1}^{\infty} b_n t^n \geq \sum_{n=1}^{\infty} d b_{n-1} t^n - \sum_{n=1}^{\infty} \sum_{i=1}^{n} b_{n-i} t^n \quad (3) \]

If we set in the last sum $n - n_i = m$, and from the definition of $r_i$, we see that:
\[ \sum_{n=1}^{\infty} \left( \sum_{i \leq n} t^{n \_n} \right) = \sum_{i=1}^{\infty} \left( \sum_{n=1}^{\infty} t^{n \_n} \right) = \sum_{i=1}^{\infty} t^{n \_n} \left( \sum_{m=0}^{n} t^{m \_m} \right) = \sum_{i=1}^{\infty} t^{n \_n} P_A(t) \]

\[ = \left( \sum_{n=1}^{\infty} t^{n \_n} \right) P_A(t) = \left( \sum_{i=2}^{\infty} r_1 t^{i \_1} \right) P_A(t) \] (4)

On the other hand

\[ \sum_{n=1}^{\infty} b^n t^n = \sum_{n=0}^{\infty} b^n t^n = P_A(t) - 1 \] (5)

since \( b_0 = 1 \), and

\[ \sum_{n=1}^{\infty} db^n t^n = \sum_{n=1}^{\infty} db t^{n \_1} = dt P_A(t) \] (6)

Therefore, the inequality (3) yields:

\[ P_A(t) - 1 \geq dt P_A(t) - \left( \sum_{i=2}^{\infty} r_1 t^{i \_1} \right) P_A(t), \] (7)

hence

\[ P_A(t)(1 - dt + \sum_{i=2}^{\infty} r_1 t^{i \_1}) \geq 1 \] (8)

This proves theorem 2.2.7.

**Theorem 2.2.8** (Golod and Safarevic) If the coefficients of the power series

\[ (1 - dt + \sum_{i=2}^{\infty} r_1 t^{i \_1})^{-1} \]

are non-negative, then

\[ P_A(t) \geq (1 - dt + \sum_{i=2}^{\infty} r_1 t^{i \_1})^{-1} \] (9)

and the algebra \( A \) is infinite-dimensional.

**Proof:** The inequality (9) is obtained from (8) by multiplying both sides by the power series

\[ f(t) = (1 - dt + \sum_{i=2}^{\infty} r_1 t^{i \_1})^{-1}, \] (10)
which by assumption has non-negative coefficients. It remains to show that the algebra \( A \) is infinite-dimensional. For this purpose, it is sufficient to show that \( b_n > 0 \) for an infinite number of values of \( n \), and this follows from (10) if we can show that the power series \( F(t) \) is not a polynomial in \( t \). We set

\[
1 + \sum_{i=2}^{\infty} r_i t^i = U(t)
\]  

(11)

Then

\[
F(t)(U(t) - dt) = 1
\]

i.e.

\[
F(t)U(t) = 1 + dtF(t)
\]  

(12)

Since both \( F(t) \) and \( U(t) \) have non-negative coefficients, and \( U(t) \) is not a polynomial, then clearly \( F(t)U(t) \) is not a polynomial. Hence the left hand side of (12) is not a polynomial. Hence the right hand side of (12) is not a polynomial. Hence \( F(t) \) is not a polynomial.

2.3 Conditions on \( r_i \)

Corollary 2.3.1 If the numbers \( r_i \) satisfy the inequalities \( r_i \leq s_i \), and all the coefficients of the power series:

\[
(1 - dt + \sum_{i=2}^{\infty} s_i t^i)^{-1}
\]

are non-negative, then \( A \) is infinite dimensional.

Proof: Let

\[
F = 1 - dt + \sum_{n=2}^{\infty} r_n t^n,
\]

\[
G = 1 - dt + \sum_{n=2}^{\infty} s_n t^n,
\]

\[
U = G - F = \sum_{n=2}^{\infty} (s_n - r_n) t^n.
\]
We have then:

\[ F = G - U = G (1 - U G^{-1}) \text{, and } G^{-1} \geq 0, \ U \geq 0, \text{ from which we find:} \]

\[ F^{-1} = G^{-1} (1 - U G^{-1})^{-1}. \]

Now since \( U \geq 0 \) and \( G^{-1} \geq 0 \), we have \( U G^{-1} \geq 0 \), which implies \(-UG^{-1} \leq 0\), which implies \( 1 - U G^{-1} \leq 1 \), which implies

\[ (1 - U G^{-1})^{-1} \geq 1, \]

(for if \( 1 - U G^{-1} = 1 - a_1 t - a_2 t^2 - \ldots \)

and \( (1 - U G^{-1})^{-1} = 1 + b_1 t + b_2 t^2 + \ldots \)

then \( (1 - a_1 t - a_2 t^2 \ldots)(1 + b_1 t + b_2 t^2 + \ldots) = 1; \)

computing, we get

\[
\begin{align*}
1 &= 1 \\
- a_1 + b_1 &= 0 \Rightarrow b_1 = a_1 \geq 0 \\
b_2 - a_1 b_1 - a_2 &= 0 \Rightarrow b_2 = a_1 b_1 + a_2 \geq 0 \\
& \quad \vdots \\
& \quad \vdots \\
& \quad \vdots \\
& \quad \vdots \\
& \quad \vdots \\
b_n - a_1 b_{n-1} - a_2 b_{n-2} - \ldots - a_n &= 0 \Rightarrow b_n = a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n \geq 0.
\end{align*}
\]

Hence:

\[ F^{-1} = G^{-1} (1 - U G^{-1})^{-1} \geq 0 \]

But \( F^{-1} = (1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1} \). Hence, by Theorem 2.2.8, \( A \) is infinite-dimensional.

**Corollary 2.3.2** If for each \( i = 2, 3, \ldots, r_i \leq \left( \frac{d-1}{2} \right)^2 \), then the algebra \( A \) is infinite-dimensional.
Proof: Since \( r_i \leq \left( \frac{d-1}{2} \right)^2 \), we need to examine the coefficients of

\[
\left( 1 - dt + \sum_{i=1}^{\infty} a_i (\frac{d-1}{2})t^i \right)^{-1}
\]

and apply Corollary 2.3.1. So we have

\[
1 - dt + \sum_{i=2}^{\infty} \left( \frac{d-1}{2} \right)^2 t^i = 1 - dt + \left( \frac{d-1}{2} \right)^2 (-1-t+t^2+t^3+...)
\]

But

\[
-(1+t)+1+t^2+t^3+... = -(1+t) + \frac{1}{1-t} = \frac{-(1+t)(1-t)+1}{1-t}
\]

\[
= \frac{-1+t^2+1}{1-t} = \frac{t^2}{1-t}
\]

To continue the above we have

\[
1 - dt + \sum_{i=2}^{\infty} \left( \frac{d-1}{2} \right)^2 t^i = 1 - dt + \left( \frac{d^2-2d+1}{4} \right) \left( \frac{t^2}{1-t} \right)
\]

\[
= \frac{(1-dt)(4-4t) + (d^2-2d+1)t^2}{4(1-t)}
\]

\[
= \frac{4-4dt-4t^2+4dt^2+2t^2-2dt^2+t^2}{4(1-t)}
\]

\[
= \frac{4-4(d+1)t + (d+1)^2 t^2}{4(1-t)}
\]

\[
= \frac{(2-(d+1)t)^2}{4(1-t)}
\]

Taking the inverse of the above, we have

\[
\left( 1 - dt + \sum_{i=2}^{\infty} \left( \frac{d+1}{2} \right)^2 t^i \right)^{-1} = \frac{4(1-t)}{2-(d+1)t)^2} = \frac{(1-t)}{1 - \frac{d+1}{2}(t)^2}
\]

\[
= (1-t) \left( \sum_{n=1}^{\infty} \left( \frac{d+1}{2} t \right)^{n-1} \right)
\]

\[
= (1-t) \left( 1 + \sum_{n=1}^{\infty} (n+1)\left( \frac{d+1}{2} t \right)^{n} \right)
\]
Now since \( d \geq 1 \), we have that \( \frac{d+1}{2} \geq 1 \), and also \((n+1)d - (n-1)\geq 2\). So (*) has non-negative coefficients. Hence, by Corollary 2.3.1, \( A \) is infinite dimensional.

An even stronger condition on \( r_1 \) is the following due to Golod.

**Corollary 2.3.3** Let \( r_1 \) and \( A \) be as previously defined. If

\[ r_1 \leq \varepsilon^2 (d-2\varepsilon)^{i-2} \]

where \( \varepsilon \) is any positive number such that \( d-2\varepsilon > 0 \), then \( A \) is infinite dimensional.

**Proof:** It is sufficient to examine the coefficients of

\[ \left( 1 - dt + \sum_{i=2}^{\infty} \varepsilon^2 (d-2\varepsilon)^{i-2} t^i \right)^{-1} \]  

We have that

\[ 1 - dt + \sum_{i=2}^{\infty} \varepsilon^2 (d-2\varepsilon)^{i-2} t^i = 1 - dt + \varepsilon^2 t^2 \left( 1 + (d-2\varepsilon)t + (d-2\varepsilon)^2 t^2 + \ldots \right) \]

\[ = 1 - dt + \varepsilon^2 t^2 \left( \frac{1}{1 - (d-2\varepsilon)t} \right) \]

\[ = \frac{(1-dt)(1 - dt + 2\varepsilon t) + \varepsilon^2 t^2}{1 - (d-2\varepsilon)t} \]
Taking the inverse of (2), we have (1) which is equal to

$$\frac{1-(d-2\varepsilon)t}{(1-(d-\varepsilon)t)^2} = \left(1-(d-2\varepsilon)t\right)\frac{1}{(1-(d-\varepsilon)t)^2} = \left(1-(d-2\varepsilon)t\right)\left(\sum_{n=1}^{\infty} n(d-\varepsilon)^{n-1} t^{n-1}\right)$$

$$= \left(1-(d-2\varepsilon)t\right)\left(1 + \sum_{n=1}^{\infty} (n+1)(d-\varepsilon)^n t^n\right)$$

$$= 1 + \sum_{n=1}^{\infty} (n+1)(d-\varepsilon)^n t^n - (d-2\varepsilon)t - (d-2\varepsilon)t \sum_{n=1}^{\infty} (n+1)(d-\varepsilon)^n t^n$$

$$= 1 + \sum_{n=1}^{\infty} (n+1)(d-\varepsilon)^n t^n - (d-2\varepsilon) \sum_{n=1}^{\infty} n(d-\varepsilon)^{n-1} t^n$$

$$= 1 + \sum_{n=1}^{\infty} (d-\varepsilon)^{n-1} \left((n+1)(d-\varepsilon) - (d-2\varepsilon)n\right) t^n$$

$$= 1 + \sum_{n=1}^{\infty} (d-\varepsilon)^{n-1} (nd + d - n\varepsilon - \varepsilon - nd + 2n\varepsilon) t^n$$

$$= 1 + \sum_{n=1}^{\infty} (d-\varepsilon)^{n-1} \left(d + (n-1)\varepsilon\right) t^n$$

Since $d-2\varepsilon > 0 = \frac{d}{2} > \varepsilon = d - \varepsilon > \varepsilon > 0$.

Hence all the coefficients of (1) are nonnegative and, by corollary 2.3.1, $A$ is infinite dimensional.

Corollary 2.3.4 Let $d = 2$ and $r_i = 0$ for $i = 2, 3, \ldots, 9$ and $r_i = 0$ or $1$ for $i \geq 10$. Then $A$ is infinite dimensional.
Proof: Here corollary 2.3.2 does not apply for

(1) \( r_i = 0 \leq \left( \frac{2-1}{2} \right)^{2} = \frac{1}{4} \) but (2) \( r_i = 1 > \left( \frac{2-1}{2} \right)^{2} = \frac{1}{4} \)

So we use corollary 2.3.3 and we choose \( \varepsilon = \frac{1}{4} \). Then

\[
d - 2\varepsilon > 0 \quad \text{i.e.} \quad 2 - \frac{1}{2} > 0
\]

Clearly for \( i = 2, 3, \ldots, 9 \) \( r_i \leq \frac{1}{16} \left( 2 - \frac{2}{2} \right)^{i} \). Now suppose that \( i \geq 10 \). Then

\[
r_i \leq 1 \leq \frac{\left( 2 - \frac{1}{2} \right)^{8}}{16} \leq \frac{\left( 2 - \frac{1}{2} \right)^{i-2}}{16}
\]

Expanding \( \left( 2 - \frac{1}{2} \right)^{8} \) using the binomial theorem, we find that the first four terms add to 19, so \( \left( 2 - \frac{1}{2} \right)^{8} > 16 \). Hence \( \varepsilon^{2}(d-2\varepsilon)^{8}>1 \).

Since

\[
(d - 2\varepsilon)^{i} < (d - 2\varepsilon)^{i+1}
\]

if \( d - 2\varepsilon > 1 \), this is sufficient to prove corollary 2.3.4.

Corollary 2.3.6. below is re-worked from a paper due to Newman.[19]

Lemma 2.3.5 The following two conditions are equivalent

(i) There exists \( 0<\varepsilon<d/2 \) such that

\[
r_i \leq \varepsilon^{2}(d - 2\varepsilon)^{i-2} \quad \text{for} \quad i = 2, 3, \ldots.
\]

(ii) There exists \( 0 < k < d \) such that

\[
r_i \leq \left( \frac{d-k}{2} \right)^{k-2} \quad \text{for} \quad i = 2, 3, \ldots.
\]

Proof: Set \( d - k = 2\varepsilon \). Then \( 0<\varepsilon<d/2 \) if and only if \( 0<k<d \), and

\[
\varepsilon^{2}(d-2\varepsilon)^{i-2} = \left( \frac{d-k}{2} \right)^{k-2}
\]
Corollary 2.3.6 There is a positive integer $N$ such that, if $r_i = 0$ for $i < N$ and $r_i \leq (d-1)^i$ for $i \geq N$, then $A$ is infinite dimensional.

Proof: Let $N$ be an integer satisfying $N > 4d$ and $\left(1 + \frac{1}{2d}\right)^{N-2} \geq N^2$. Put $k = \frac{(N-2)d}{N}$, then for $i \geq 2$, by Lemma 2.3.5 (ii), we have:

\[
\left(\frac{d-k}{2}\right)^{k+1} = \left(\frac{d}{2} - \frac{(N-2)d}{2N}\right)^{1-2}\left(\frac{(N-2)d}{N}\right)
\]

\[
= \left(\frac{dN-Nd+2d}{2N}\right)^{1-2}\left(\frac{(N-2)d}{N}\right)
\]

\[
= \frac{d^2(N-2)^{1-2}}{N^2} \text{ for } i \geq 2
\]

Since $\left(1 + \frac{1}{2d}\right)^{N-2} \geq N^2$, it follows that

\[
\frac{1}{N^2} \geq \left(1 + \frac{1}{2d}\right)^{N-2} \geq N^2
\]

Note that since $N > 4d$ implies $\frac{2}{N} \leq \frac{2}{4d}$. Hence

\[
\left(\frac{N-2}{N}\right)^{1-2} = \left(1 - \frac{2}{N}\right)^{1-2} \geq \left(1 - \frac{2}{4d}\right)^{1-2}
\]

Substituting (2) and (3) in (1), we have that

\[
\geq \frac{1}{\left(1 + \frac{1}{2d}\right)^{N-2}} \left(1 - \frac{2}{4d}\right)^{1-2} \text{ for } i \geq 2
\]

(4)

\[
\geq (1 - \frac{1}{d})^{i-2} \text{ for } i \geq 2
\]

(5)

We have (5) because when $i \geq N$, then

\[
\frac{(1 - \frac{1}{2d})^{i-2}}{(1 + \frac{1}{2d})^{N-2}} \geq \frac{(1 - \frac{1}{2d})^{i-2}}{(1 + \frac{1}{2d})^{i-2}} \geq (1 - \frac{1}{d})^{i-2}
\]

provided that

\[
\frac{1 - \frac{1}{2d}}{1 + \frac{1}{2d}} \geq 1 - \frac{1}{d}
\]
So

\[ 1 - \frac{1}{2d} \geq (1 - \frac{1}{d})(1 + \frac{1}{2d}) = 1 - \frac{1}{d} + \frac{1}{2d} - \frac{1}{2d^2} = 1 - \frac{1}{2d} - \frac{1}{2d^2} \]

That is

\[ 0 \geq -\frac{1}{2d^2} \]

Now since \( d^2 \geq (d-1)^2 \) we have that

\[ (1 - \frac{1}{d})^{i-2}d^i = \left(\frac{d - 1}{d}\right)^{i-2}d^i = (d - 1)^{i-2}d^2 \]

\[ \geq (d - 1)^{i-2}(d - 1)^2 = (d - 1)^i \geq r_i \]

hence \( A \) is infinite dimensional.
In this chapter we will construct some examples of nil algebras and periodic groups. Before this, however, we will state clearly the Kuroš problem and the Burnside question adding all the definitions necessary to understand them.

3.1 Algebraic and Nil Algebras

Definition 3.1.1 An algebra, A, is finitely-generated if there is a finite subset \( a_1, \ldots, a_r \) (called its generators) such that every element of A can be obtained from the generators by a finite number of additions, multiplications, and/or scalar multiplications.

Definition 3.1.2 Let A be an algebra over a field F; \( a \in A \) is said to be algebraic over F if there is a non-zero polynomial \( p(x) \in F[x] \) such that \( p(a) = 0 \). That is

\[
(1) \quad p(a) = k_n a^n + k_{n-1} a^{n-1} + \ldots + k_1 a + k_0 = 0
\]

where \( k_i \in F \). The equation (1) may differ for different \( a \in A \).

Definition 3.1.3 An algebra A over F is said to be algebraic over F if every \( a \in A \) is algebraic over F.

The following theorem is a very interesting one and we will see soon that it gives us the converse of the Kuroš problem.

Theorem 3.1.4 If A is a finite-dimensional (as a vector space) algebra over F, then it is algebraic over F.
Proof: Let \( a \in A \), and let \( n = \dim A \). Then the \( n+1 \) elements 
\( a, a^2, a^3, \ldots, a^n, a^{n+1} \), are linearly dependent over \( F \). Thus there exist scalars \( \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \) in \( F \) such that they are not all zero and such that

\[
\alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n + \alpha_{n+1} a^{n+1} = 0
\]

Thus \( p(a) = 0 \), where \( p(x) \) is the non-zero polynomial

\[
p(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n+1} x^{n+1}
\]

in \( F[x] \).

Hence \( a \) is algebraic over \( F \). But since \( a \) was any element of \( A \), we can conclude that every element of \( A \) is algebraic and therefore \( A \) is algebraic over \( F \).

Definition 3.1.5 Let \( A \) be an algebra over \( F \); \( a \in A \) is said to be \textbf{nilpotent} if there exists a positive integer \( n \) such that \( a^n = 0 \).

Definition 3.1.6 Let \( A \) be an algebra over \( F \) such that \( A^n = (0) \) for some positive integer \( n \); then \( A \) is said to be a \textbf{nilpotent} algebra over \( F \).

Definition 3.1.7 An algebra \( A \) over \( F \) is \textbf{nil} if every element of \( A \) is nilpotent.

Theorem 3.1.8 If \( A \) is a nil algebra over \( F \), then \( A \) is algebraic over \( F \).

Proof: Since \( A \) is nil, this implies that for \( a \in A \), there exists a positive integer \( n \) such that \( a^n = 0 \). Clearly \( a \) is algebraic over \( F \) since it satisfies the following polynomial.
\[ lx^n + Ox^{n-1} + \ldots + Ox + 0 \]

i.e.
\[ la^n + Oa^{n-1} + \ldots + Oa + 0 = 0 \]

Hence A is algebraic over F.

3.2 Kuroš's Problem

First we will define the locally finite algebras. Then we will discuss the Kuroš Problem.

Definition 3.2.1. An algebra A over a field F is \textbf{locally finite} if every finite subset of A generates a finite dimensional subalgebra.

We have seen that any finite dimensional algebra is algebraic (Theorem 3.1.4), hence any locally finite algebra is algebraic. Now the following question (an analog to the Burnside's Problem on groups), was raised by Kuroš in 1941.

Problem 3.2.2 Is every algebraic algebra locally finite?

In other words, if A is an algebraic algebra over F, does a finite number of elements of A generate a finite dimensional subalgebra of A? Or, is a finitely generated algebraic algebra finite dimensional?

As Jacobson says, "A number of interesting open questions on algebraic algebras seem to hinge on the answer to this problem."

Some of these are the following:

Question 3.2.3 If A and B are algebraic, then is A\otimes B algebraic?
It is easy to see that if $A$ and $B$ are locally finite, then $A \otimes B$ is locally finite. Hence an affirmative answer to Kuroš's problem would provide an affirmative answer to 3.2.3.

In the coming sections, we shall give some examples of infinite dimensional algebras.

Also, we like to mention that Kuroš's question has an affirmative answer for algebras with a polynomial identity (PI - algebras) and hence for algebras of bounded degree. The results are due to Kaplansky which generalize earlier results by Jacobson and by Malcev.

3.3 PI - Algebras and Bounded Algebras

**Definition 3.3.1** An algebra $A$ over a field $F$ is said to satisfy a polynomial identity if there is an $f \neq 0$ in $F[x_1, \ldots, x_d]$, the free algebra over $F$ in the noncommuting variables $x_1, x_2, \ldots, x_d$ for some $d$, such that $f(a_1, \ldots, a_d) = 0$ for all $a_1, \ldots, a_d$ in $A$. An algebra $A$ which satisfies a polynomial identity is called a PI - algebra.

**Example 3.3.2** Let $A$ be a nil algebra of bounded index of nilpotency. That is, $x^k = 0$ holds for every $x$ for some fixed $k$. Then $A$ is a PI - algebra.

**Example 3.3.3** Any commutative algebra $A$ over $F$ is a PI - algebra, for it satisfies the polynomial identity $f(x_1, x_2) = 0$, where $f(x_1, x_2) = x_1 x_2 - x_2 x_1$. 
We mention the following results to give an idea of what was known regarding the Kuroš problem prior to the work of Golod and Šafarevič. If $A$ is finite dimensional over $F$, of dimension $n$, then every element in $A$ satisfies a polynomial of degree $n+1$ over $F$. This defines the notion of an algebraic algebra of bounded degree over $F$.

**Definition 3.3.4** A is said to be an algebraic algebra of bounded degree over $F$ if there exists an integer $n$ such that given $a \in A$, there exists a polynomial $x^n + \alpha_1x^{n-1} + \ldots + \alpha_n \in F[x]$ satisfied by $a$.

i.e. $a^n + \alpha_1a^{n-1} + \ldots + \alpha_n = 0$.

**Lemma 3.3.5** If $A$ is algebraic of bounded degree over $F$, then $A$ is a PI algebra. [11]

**Theorem 3.3.6** If $A$ is an algebraic algebra over $F$ satisfying a polynomial identity, then $A$ is locally finite. [11]

**Theorem 6.4.4** If $A$ is an algebraic algebra of bounded degree over $F$, then it is locally finite. [11]

### 3.4 Periodic Groups and Locally Finite Groups

**Definition 3.4.1** A group $G$ is said to be a periodic or torsion group if every element in $G$ is of finite order.

**Definition 3.4.2** The order of an element $b$ is the smallest positive integer $n$ such that $b^n = 1$, if it exists. If there is such an $n$, we say that $b$ has finite order.
Definition 3.4.3: If \( b^n = 1 \), with \( n \) fixed, for all \( b \in G \), and \( n \) is the smallest positive integer for which this is true, then \( n \) is called the **exponent** of \( G \).

Definition 3.4.4: A group \( G \) is said to be **locally finite** if every finitely generated subgroup of \( G \) is finite.

Definition 3.4.5: \( G \) is a **finitely generated** group if \( G \) contains a finite set of elements \( g_1, g_2, \ldots, g_r \) (called its generators) such that every element can be expressed as a finite product of the generators and their inverses.

Theorem 3.4.5: Every locally finite group is a torsion group.

Proof: Let \( G \) be a locally finite group. We want to show that every element of \( G \) has finite order. That is, the subgroup generated by that element is finite. But the subgroup generated by a given element is certainly finitely generated, hence is finite, which implies that the given element has finite order.

Hence \( G \) is a torsion group.

Example 3.4.6: The group \( \mathbb{Z}^+ \) of integers is not a torsion group since a single element does not have finite order. Hence \( \mathbb{Z}^+ \) is not locally finite.

Example 3.4.7: This is an example of an infinite group which is locally finite. Take an infinite dimensional vector space \( V \) over the field of integers modulo \( p, \mathbb{Z}_p \). Then \( V \) is an abelian group. Now take any finite subset of \( V, a_1, a_2, \ldots, a_n \), then the subgroup
generated by this subset is just the set of all \( \bigoplus_{i=1}^{n} \xi_i a_i \), where \( \xi_i \in \mathbb{Z}_p \).

There are only finitely many choices of each \( \xi_i \). Hence, only a finite number of elements of the subgroup generated by the \( a_i \).

Hence that finitely generated subgroup is finite. Hence the group is locally finite.

3.5 Burnside Problem

The converse to Theorem 3.4.5 is the Burnside Problem which originally was asked in 1904. We state two versions of the Burnside Problem.

1. Original Burnside Problem. Is every torsion group locally finite? An equivalent version of this question is: Is a finitely generated periodic group finite?

2. Burnside Problem for Exponent \( N \). Let \( G \) be a torsion group in which \( x^N = 1 \) for all \( x \in G \), \( N \) a fixed positive integer. Is \( G \) then locally finite?

These problems have answers now and they are as follows:

1. As a result of the work of Golod and Šafarevič, the original Burnside problem is answered in the negative. In the following section, we will exhibit a finitely generated periodic group which is infinite.

However, for matrix groups, Burnside himself settled the original Burnside Problem in the affirmative, by the following:
Theorem 3.5.1 (Burnside) A torsion group of matrices over a field is locally finite.

2. Novikov in 1959, announced the existence of an infinite group $G_N$ generated by two elements in which $x^N = 1$ holds for all $x \in G$. This is true for any odd $N \geq 4381$. The proof done by induction appeared in 1968 in paper nearly 300 pages long, which gives an actual construction.

Regardless of the answer to the Burnside Problem, for exponent $N$, the following problem is still an interesting one.

Restricted Burnside Problem for Exponent $N$: Among all the finite groups on $K$ generators with exponent $N$, is there a largest one?

The answer is "Yes", if $N$ is prime, done by Kostrikin.

If $N$ is prime and $\geq 4381$, we have two results:

(a) There is a largest finite group of exponent $N$ in two generators (Kostrikin).

(b) There is an infinite group of exponent $N$ in two generators (Novikov and Adyan).

3.6 Setting the Kuroš Problem and the Original Burnside Problem in Negative

In this Section, we are ready to apply the Golod–Safarevič theorem to construct a finitely generated nil algebra which is infinite-dimensional and a finitely generated infinite periodic group.

This settles the Kuroš and Burnside problems negatively.
Theorem 3.6.1 If $F$ is any countable field, there exists an infinite dimensional nil algebra over $F$ generated by two elements.

Proof: Let $T = F[x_1, x_2]$. Then

$$T = F \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$$

where the elements of $T_i$ are homogeneous of degree $i$. Let

$$T' = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$$

$T'$ is an ideal, since if $u \in T'$ and $r \in T$, then $ru \in T'$ and $ur \in T'$, because $\partial(ru)$ and $\partial(ur)$ are always $\geq 1$ since $\partial(u) \geq 1$. Also, $T'$ is a vector space with a countable basis since the basis of each $T_i$ is finite. Hence, by Lemma 3.6.2, $T'$ is countable. Now let

$$s_1, s_2, \ldots, s_n, \ldots$$

be the elements of $T'$. Pick $m_1 = 10$ and raise $s_1$ to the $m_1$ power so

$$s_1 = s_{11} \oplus s_{12} \oplus \ldots \oplus s_{1, k_1}$$

Choose $m_2 > 0$ so that

$$s_2 = s_{2, k_1+1} \oplus s_{2, k_1+2} \oplus \ldots \oplus s_{2, k_2}$$

and

$$s_1 \in T_{10} \oplus T_{11} \oplus \ldots \oplus T_{k_1}.$$
Clearly \( k_1 < k_2 < \ldots < k_n < \ldots \).

Now let \( \mathcal{U} \) be the ideal of \( T \) generated by all the \( s_{ij} \).

Notice that for that choice of the \( s_{ij} \)'s, we have \( r_k = 0 \), \( 2 \leq k \leq 9 \), and \( r_k = 0 \) or 1 for \( k \geq 10 \), by construction. Hence, by corollary 2.3.4, we have that \( T/\mathcal{U} \) is infinite dimensional. Now since \( \mathcal{U} \subseteq T \), we form the quotient algebra \( T'/\mathcal{U} \) which is obviously infinite dimensional. But \( T'/\mathcal{U} \) is a nil algebra by construction, for if \( \bar{s}_i \in T'/\mathcal{U} \) then \( \bar{s}_i = s_i + \mathcal{U} \), and \( \bar{s}_i^{m_i} = (s_i + \mathcal{U})^{m_i} = s_i^{m_i} + \mathcal{U} = \mathcal{U} \), hence, \( \bar{s}_i^{m_i} = 0 \). Hence the algebra \( T'/\mathcal{U} \) is the required finitely generated algebraic algebra (in fact, a nil algebra) which is infinite dimensional.

**Lemma 3.6.2** Let \( V \) be a vector space with a countable basis over a countable field \( F \). Then \( V \) is countable.

**Proof:** Let \( B = \{ v_1, v_2, \ldots, v_n, \ldots \} \) be a countable basis for \( V \), and let \( B_n = \{ v_1, v_2, \ldots, v_n \} \) be a subset of \( B \). Now let \( \overline{B}_n \) be the subspace of \( V \) spanned by \( B_n \). Then \( \overline{B}_n \) is countable since there is a natural one to one correspondence between \( \overline{B}_n \) and \( F \times F \times \ldots \times F \) \( n \) times.

But then

\[
V = \bigcup_{n \in \mathbb{N}} \overline{B}_n
\]

is the countable union of countable sets and hence countable.

Let \( F \) be a finite field with \( p \) elements and let \( \mathcal{U} \) be the ideal in \( T = F[x_1, x_2] \) as in Theorem 3.6.1 and let \( T' = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots \)

If \( A = T/\mathcal{U} \) then \( a_1 = x_1 + \mathcal{U} \) and \( a_2 = x_2 + \mathcal{U} \) is the generating set for \( T'/\mathcal{U} \).
**Definition 3.6.3** A group $G \neq \{1\}$ is a $p$-group if every element of $G$ except the identity has order a power of the prime $p$.

**Lemma 3.6.4** Let $G$ be the multiplicative semigroup in $A$ generated by $1 + a_1, 1 + a_2$. Then $G$ is a group, and is in fact, a $p$-group.

**Proof:** Obviously $G$ is the subset of $A$ consisting of all finite power products of the elements $1 + a_1, 1 + a_2$, (with non-negative exponents). Hence:

$$G \subset \{1 + a| \text{ for some } a \in T'/\mathbb{U}\}.$$ 

But the algebra $T'/\mathbb{U}$ is a nil algebra (Theorem 3.6.1) and therefore, each $a \in T'/\mathbb{U}$ is nilpotent, i.e. for some $n$ we have $a^n = 0$.

Now take $n$ large enough that $p^n > n$. Then

$$a^n = a^n p^n - n = 0$$

and

$$(1 + a)^{p^n} = 1 + p^n a + \binom{p^n}{2} p^{n-1} a^2 + \ldots + p^n a^{p^n-1} a^p = 1 + a^{p^n} = 1.$$ 

This is because all the coefficients are 0, since they are divisible by $p$ and $F$ is the finite field with $p$ elements. Hence $G$ contains a multiplicative identity 1. Hence $G$ is a semigroup with identity. Also, since powers of the same element commute, we have

$$1 = (1 + a)^{p^n} = (1 + a)(1 + a)^{p^n-1} = (1 + a)^{p^n-1}(1 + a);$$

that is $1 + a$ has a multiplicative inverse $(1 + a)^{p^n-1}$, which is clearly in $G$.

Therefore, $G$ is a group. Moreover, $G$ is a $p$-group.
Lemma 3.6.5 Let $A$ be an algebra over a field $F$ and let $G$ be a finite subset of $A$ which is a group under multiplication. Then the linear combinations of the elements of $G$ form a finite dimensional subalgebra $B$ over $F$.

Proof: Let $G = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $A$ and moreover, let $G$ be a multiplicative group. Then the elements of the subalgebra generated by $G$ are of the form

$$n \sum_{i=1}^{n} \xi_i a_i$$

($\xi_i \in F$)

The subalgebra looked at as a vector space is spanned by $a_1, \ldots, a_n$. Therefore, it has a finite basis and hence is finite-dimensional.

Theorem 3.6.6 If $p$ is any prime, there is an infinite group $G$ generated by two elements in which every element has finite order a power of $p$.

Proof: Let $G$ be the group in Lemma 3.6.4. Then $G$ is a $p$-group, and it remains to show that $G$ is infinite. Assume that $G$ is finite. Since $G$ is finite, the linear combinations of the elements of $G$ form a finite dimensional algebra $B$ over $F$, as in Lemma 3.6.5. Since $1, 1 + a_1, 1 + a_2$, are in $G$, then the elements

$$a_1 = (1 + a_1) - 1$$

$$a_2 = (1 + a_2) - 1$$

$$1 = (1 + a_1) - a_1 = (1 + a_2) - a_2$$

are in $B$. Observing that $1, a_1, a_2$ generate the algebra $A$, we get $A = B$, contradicting that $A$ is infinite-dimensional over $F$. Therefore, $B$ is infinite dimensional and hence $G$ is infinite.
BIBLIOGRAPHY


Association of America, 1968.

wood Cliffs, N.J.,


[14] Jans, James P., Rings and Homology, ed. by Holt, Rinehart, and
Winston,


[16] Levitzki, Jakob, On the radical of a general ring, Ball,
49(1943) 462 – 466.

[17] Levitzki, Jakob, On a problem of A. Kuroš, Ball AMS, 52(1946)
1033 – 1035.


[19] Newman, M.F., A theorem of Golod and Šafarevič and an appli-
cation in group theory, Canberra, A.C.T., Australia.

127 (1959) 4, 749 – 752. AMS Translations, Ser. 2, 45
(1965) 19 – 22.