NAME OF AUTHOR/NOM DE L'AUTEUR: WOODROW, Robert Edward

TITLE OF THESIS/TITRE DE LA THÈSE: Theories with a finite number of countable models and a small language

UNIVERSITY/UNIVERSITÉ: SIMON FRASER UNIVERSITY

DEGREE FOR WHICH THESIS WAS PRESENTED/GRADÉ POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE: Ph.D.

YEAR THIS DEGREE CONFERRED/ANNÉE D'OBTENTION DE CE GRADÉ: 1976

NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE: Dr. A. Lachlan

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

DATED/DATÉ: December 1, 1976

SIGNED/SIGNÉ: ________________________________

PERMANENT ADDRESS/RÉSIDENCE FIXÉ: 619 Pembroke Rd, S.E.

Calgary, Alberta

L'authorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de microfilm cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur se réserve les autres droits de publication, ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.
The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formulaires d'autorisation qui accompagnent cette thèse.

LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU
THEORIES WITH A FINITE NUMBER OF COUNTABLE MODELS AND A SMALL LANGUAGE

by

Robert E. Woodrow
B.Sc. (Hons), The University of Calgary

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
in the Department of Mathematics

© Robert E. Woodrow 1976
Simon Fraser University
October 1976

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.
Name: Robert E. Woodrow

Degree: Doctor of Philosophy (Mathematics)

Title of Thesis: Theories with a finite number of countable models and a small language.

Examining Committee:

Chairman: Dr. S.K. Thomason

Alistair H. Lachlan
Senior Supervisor

R. Harrop

T. Brown

H. Gerber

Michael Makkai
External Examiner
Associate Professor
McGill University
Montreal, Quebec

Date Approved: October 29, 1976
PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis or dissertation (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this thesis for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis/Dissertation:

Theories with finite number of countable models and a small language

Author:

(signature)

Robert E. Woodrow

(name)

December 1, 1976

(date)
ABSTRACT

This work treats countable complete theories having a finite number of countable models and satisfying restrictions on complexity of formulae. Most of the emphasis is on theories with more than one countable model, but some related results on the nature of complete theories which admit elimination of quantifiers and have one binary relation symbol are given. It is shown that there cannot be two incomparable definable equivalence relations if such a theory has infinite models and one 1-type. Tournaments whose theories admit elimination of quantifiers are considered, and it is shown that there is one finite example and there are two countable examples where the set of successors of a member are linearly ordered. It is shown that there are only four countable undirected graphs which omit the complete graph on three vertices and whose theories admit elimination of quantifiers.

It is shown that if a countable complete theory has three countable models and admits elimination of quantifiers in a language with one binary relation and constant symbols then the theory is "essentially" the Ehrenfeucht example $\langle \mathbb{Q}, <, n >_{n \in \omega}$.

Some conjectures regarding the complexity of theories satisfying restrictions on language and number of countable models are formulated and discussed. A theory $T_0$ is constructed which has nine countable models and a nonprincipal 1-type containing infinitely many 2-types. A theory $T_1$ is constructed which has four countable models and an inessential extension $T_2$ having infinitely many countable models.
The author wishes first and foremost to express his thanks for the guidance and encouragement given by his senior supervisor Professor A.H. Lachlan, without which very little progress would have been made. The National Research Council of Canada provided financial support for most of the time the author was engaged in this work.

Thanks are also due to the author's office mates over the years, Robert Lebeuf, Jim Dukarm and Ron Morrow. They endured many hours of discussion of the difficulties of the day. Particular thanks are due to Ron who suffered through the writing up and provided a willing ear and friendly support throughout that time.

Special thanks go to Dolly Rosen who performed the near miracle of transforming my handwritten manuscript into something legible.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approval</td>
<td>(ii)</td>
</tr>
<tr>
<td>Abstract</td>
<td>(iii)</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>(iv)</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter 1. Notation and Preliminaries</td>
<td>13</td>
</tr>
<tr>
<td>Chapter 2. Ehrenfeucht-like theories</td>
<td>17</td>
</tr>
<tr>
<td>§1 Definitions</td>
<td>17</td>
</tr>
<tr>
<td>§2 The Theorem</td>
<td>20</td>
</tr>
<tr>
<td>§3 Proof of Lemma 2.2</td>
<td>21</td>
</tr>
<tr>
<td>§4 Proof of Lemma 2.3</td>
<td>28</td>
</tr>
<tr>
<td>§5 Summary</td>
<td>31</td>
</tr>
<tr>
<td>Chapter 3. $\Sigma$-generic structures</td>
<td>32</td>
</tr>
<tr>
<td>Chapter 4. Quantifier eliminable graphs</td>
<td>40</td>
</tr>
<tr>
<td>§1 Examples</td>
<td>40</td>
</tr>
<tr>
<td>§2 Definable equivalence relations</td>
<td>43</td>
</tr>
<tr>
<td>§3 Tournaments</td>
<td>50</td>
</tr>
<tr>
<td>§4 Undirected graphs</td>
<td>56</td>
</tr>
<tr>
<td>§5 Finite ultrahomogeneous graphs</td>
<td>70</td>
</tr>
<tr>
<td>Chapter 5. The Theory $T_0$</td>
<td>73</td>
</tr>
<tr>
<td>Chapter 6. $C_1$, $C_4$ and the Theory $T_1$</td>
<td>83</td>
</tr>
<tr>
<td>§1 $C_1$ and $C_4$</td>
<td>83</td>
</tr>
<tr>
<td>§2 The Theory $T_1$</td>
<td>86</td>
</tr>
<tr>
<td>Conclusion</td>
<td>98</td>
</tr>
</tbody>
</table>
This work is concerned with those countable complete theories which have a finite number of countable models. An elegant characterization of those theories which have one countable model is the theorem of Engeler [3], Ryll-Nardzewski [9] and Svenonius [11]: a theory is $\omega_0$-categorical just in case for each $n$ there are a finite number of $n$-types. The work of Vaught [12, p.320] shows that no countable complete theory can have exactly two countable models. Work of Baldwin and Lachlan [2] and Lachlan [6] shows that a countable complete theory with a finite number of countable models but more than one cannot be superstable. In a letter to Professor Lachlan, Shelah conjectured that no such theory could be stable. To our knowledge the conjecture remains open. Further progress on the general problem appears to be very difficult. We shall restrict ourselves to theories which are simple in complexity of language. The main emphasis will be on theories with more than one countable model, but we also present some results which are strictly concerned with $\omega_0$-categorical theories. We shall first give a brief account of the history of theories with more than one model.

For fourteen years the only widely known examples of theories with a finite number of countable models but more than one were those due to Ehrenfeucht [12, §6]. The archetype of the example is $< Q, <, \sigma, D_m >_{m \in \omega}$ where $< Q, < >$ is the rational numbers under the usual order, and the $D_m$ are disjoint dense subsets of $Q$ whose union is $Q$. The theory of this structure will have $(K-1)+3$ countable models. In a like manner certain other countable
order types can be distinguished by constants in a model of dense linear order to give a structure whose theory will have finitely many countable models.

Certain techniques can then be applied to known examples to yield others. For example the constants in Ehrenfeucht's example may be replaced by unary predicates which determine initial segments of the order. At the expense of elimination of quantifiers an example with a finite language can be given [12, §6]. For this example one takes an equivalence relation \( E \) which is a congruence relation with respect to a binary relation \( R \). The equivalence classes under \( E \) are densely ordered by \( R \), and for each \( n \) there is exactly one class with \( (n+1) \) members. For \( n < m \) the equivalence class with \( (n+1) \)-members precedes that with \( (m+1) \) in the ordering induced by \( R \). Another way of forming a new theory is to construct a disjoint union. Given two theories \( U_0, U_1 \) which have \( n, m \) countable models, respectively we can form the disjoint union \( W \) having \( n \cdot m \) countable models in the following manner. Let \( P_0, P_1 \) be new unary predicate symbols and \( c \) a new constant symbol. We may assume that \( U_0 \) and \( U_1 \) have no nonlogical symbols in common. Then the language of \( W \) is the union of the languages of \( U_0 \) and \( U_1 \) with \( \{ P_0, P_1, c \} \). The nonlogical axioms of \( W \) are the following:

1) \((P_0 v_0 v P_1 v_0 v v_0 = c) \land \lnot P_0 c \land \lnot P_1 c\)

2) \(R v_{0,i}, \ldots, v_{n,j} \rightarrow \land_{j \leq n} P v_{i,j} \) for \( R \) an \( (n+1) \)-ary predicate symbol of \( U_i \), \( i = 0, 1 \)
3) \( \neg P_i \vee_j f v_0, \ldots, v_n = c \) for \( f \) an \((n+1)\)-ary function symbol of \( U_i \) and \( j \leq n \), \( i = 0,1 \)

4) The relativization to \( P_i \) of each nonlogical axiom of \( U_i \) for \( i = 0,1 \).

With "linking" of the copies in the disjoint union some further control on the number of countable models may be exerted. A link between an \( m \)-type \( p \) and an \( n \)-type \( q \) is an \((m+n)\)-type \( r \) such that \( r \supseteq p \) and for every formula \( \phi \)

\[
\phi \in q \rightarrow \phi_{v_0, \ldots, v_{n-1}}^{v_0, \ldots, v_{n-1}} \in r \ . \quad [1]
\]

Linking then refers to the addition of suitable links. Yet another way to proceed is to form the product \( U_0 \cdot U_1 \) of two theories \( U_0, U_1 \) which have a finite number of countable models, say \( m, n \) respectively. Essentially its models are obtained from models \( M, N \) of \( U_0 \) and \( U_1 \) respectively by replacing each member of \( M \) by a copy of \( N \).

Formally we assume that \( U_0, U_1 \) have no nonlogical symbols in common. A new constant \( c \), two unary predicates \( P_0, P_1 \) and two unary function symbols \( p_0, p_1 \) are added. The nonlogical axioms of \( U_0 \cdot U_1 \) are:

1) The relativization to \( P_i \) of the nonlogical axioms of \( U_i \) for \( i = 0,1 \)

2) Axioms assuring that outside \( P_i \) the nonlogical symbols of \( U_i \) have trivial interpretation.

a) \( R v_0, \ldots, v_n \rightarrow \bigwedge_{j \leq n} P_i v_j \) for \( R \) an \( n \)-ary relation symbol of \( U_i \) and \( i = 0,1 \).
b) $\forall j \leq n \exists_i P^j_i \rightarrow f v_0, \ldots, v_n = c$ for $f$ an $(n+1)$-ary function symbol of $U_i$, and $i = 0, 1$

3) $\neg \exists v \left( P^0 v_0 \land P^1 v_1 \right) \land \neg P^0 c \land \neg P^1 c$

4) $P^0, P^1$ are coordinate projections onto $P^0, P^1$

   a) $\forall v_0 (P^0 v_0 \lor P^1 v_0 \lor v_0 = c \lor P^1 v_0 = c)$ for $i = 0, 1$

   b) $\forall v_0 (v_0 \neq c \land \neg P^0 v_0 \land \neg P^1 v_0 \rightarrow P^1 v_0 \land P^0 v_0)$ for $i = 0, 1$

   c) $\forall v_0 \forall v_1 (P^0 v_0 \land P^1 v_1 \rightarrow \exists! v_2 (P^0 v_2 \lor P^1 v_2 = v_2) \land P^0 v_2 = v_2 \lor P^1 v_2 = v_2)$.  

It is not difficult to see that $U^0_0 \cup U^1_1$ will have $m \cdot n$ countable models; and if $m > 1$ or $n > 1$, $U^0_0 \cup U^1_1$ will have infinitely many non-principal $k$-types for some $k$. As far as we know no more sophisticated techniques than those discussed above have been mentioned in the literature.

The restricted nature of the Ehrenfeucht examples and others closely related to them led to the following conjectures.

(C1): If $p$ is a nonprincipal $1$-type in a complete theory with a finite number of countable models, and $M$ the countable saturated model then the relation on $p(M)$: "$a$ is prime over $b$" is a total partial order.

(C2): Let $T$ be a countable complete theory with a finite number of countable models and let $p$ be a nonprincipal $1$-type in $T$. Let $M$ be the countable saturated model of $T$ and $p(M)$ the set of individuals of $M$ which realize $p$. Then there is a definable linear order on an infinite subset of $p(M)$. The subset need not be definable.
(C3): A nonprincipal 1-type in a countable complete theory with a finite number of countable models can contain at most a finite number of 2-types.

(C4): If T is a complete theory with a finite number of countable models then every inessential extension of T has a finite number of countable models.

(C5): Every model of a complete recursive theory with at most four countable models is totally recursive.

(C1), (C2), (C3) arose when we were first attempting to characterize theories with three countable models in 1974. As far as we know (C2) remains open to this day. The conjecture is given some support by Lemma 2.1 Chapter 2 where it is shown that a nonprincipal 1-type in a complete theory with three countable models contains a 2-type q such that if (a,b) realizes q in a model then b is prime over a while a is not prime over b. (For precise definitions refer to Chapter 1). C3 would confirm C2 for theories with three countable models by an application of Ramsey's theorem. C4 derives from work by Benda [1], and C5 is due to M. Morley (unpublished). The conjecture C1 never seemed very plausible but does focus attention on an interesting class of theories.

The state of knowledge described above was drastically changed when we learned by letter of the example of Peretyiatkin [8, §5]. He introduced a dense binary tree. With thanks to A.H. Lachlan we reproduce the following model of the theory in question, 

\[ M = < M, \land > \] where \( \land \) is a meet operation on \( M \). \( < Q, < > \) is the rational numbers under the usual order.
For \( f, g \in M \) let \( \alpha \) be the largest rational such that \( \alpha \) belongs to the domains of \( f \) and of \( g \) and

\[
\forall \beta \leq \alpha (f(\beta) = g(\beta)) .
\]

Define \( f \land g = f \upharpoonright \{ \beta : \beta \leq \alpha \} . \)

It is then not difficult to show that the class of finite substructures of \( M \) is just the class of finite binary trees. Also by a back and forth argument one can show that any isomorphism of two finite substructures of \( M \) can be extended to an automorphism of \( M \). Using the techniques of Chapter 3 we then know that the theory of \( M \), \( \text{Th}(M) \), admits elimination of quantifiers and also that any countable binary tree can be embedded in \( M \). These facts ensure that \( \text{Th}(M) \) is equivalent to the theory \( T_0 \) of \( \S 2 \) of [8]. If a binary tree which is a subtree of the complete binary tree \( \omega_2 \) and which has only one infinite branch is distinguished in \( M \) by constants the theory of the structure will have three countable models. Since there is a recursively enumerable tree with one infinite branch and that branch is nonrecursive Peretyiatkin was able to give a counterexample to C5. That C5 could not be strengthened by replacing four by six was known because of an example of Lachlan (unpublished) where the order type \( \omega^* + \omega \) was distinguished by constants in a model of dense linear order.
Peretyiatkin's example also shows that C1 is false. As non-principal 1-type we take the type of an upper bound for the members of the infinite branch. For the 2-type \( q \) we take the 2-type in \( p \) which contains \( (v_0 \land v_1 \neq v_0) \land (v_0 \land v_1 \neq v_1) \) i.e. the 2-type of two incomparable elements realizing \( p \). Those theories for which C1 is true will be considered in Chapter 6.

A comparison of Ehrenfeucht's example with that of Peretyiatkin raises the question of whether the presence of a function symbol in the latter is really essential. We may formulate the question precisely as follows. Call the language of a theory (or structure) \textit{small} if the language contains a finite number of nonlogical symbols other than unary predicate symbols or constant symbols and the theory (of the structure) admits elimination of quantifiers. The question can then be phrased:

\textbf{Q1}: Let \( T \) be a countable complete theory in a small language which has no \( n \)-ary function symbols for \( n > 0 \). Does \( T \) satisfy C1 through C4?

A second question which naturally arises is:

\textbf{Q2}: Which of C2 through C3 hold for theories with small languages?

In Chapter 2 we give a partial answer to Q1. There we define precisely what we mean when we say that a structure is like the Ehrenfeucht example. Roughly it means that the model can be viewed as the union of a finite number of "dense orders" and an \( \omega_0 \)-categorical set. Each "order" is actually the induced relation with respect to a definable equivalence relation and it densely
orders the equivalence classes. In each "order" there are infinitely many principal 1-types which are "almost" arranged in an $\omega$-sequence.

We conjecture that:

C6: Any complete theory with exactly three countable models in a small language without function symbols other than constants is like the Ehrenfeucht example.

In Chapter 2 it is shown that any complete theory which has only constant symbols, which admits elimination of quantifiers and which has exactly three countable models is like the Ehrenfeucht example.

The examples of Ehrenfeucht and Peretyiatkin show that it is fruitful to examine $\omega_0$-categorical theories in small language in order to build examples of theories with other finite numbers of countable models. Of particular interest are those having one binary relation symbol. In Chapter 3, methods of Jonsson [5] and Morley and Vaught [7] are redeveloped for the purpose of constructing and characterizing examples of theories with small languages. Given a class $\Sigma$ of finite structures for a finite language such that $\Sigma$ has the amalgamation property and has a prime structure, a denumerable $\Sigma$-homogeneous $\Sigma$-universal structure $M$ is constructed such that each finite subset of $M$ is contained in a member of $\Sigma$. When $\Sigma$ contains arbitrarily large finite structures $M$ is countable and when $\Sigma$ is closed under substructure and there are uniform bounds on generated substructures, $\text{Th}(M)$ admits elimination of quantifiers. This result is given a partial converse. Let $M$ be a countable structure in a finite language such that the theory of $M$ admits
elimination of quantifiers and every finite subset of $M$ is contained in a finite substructure. Let $\Sigma$ be the class of finite substructures of $M$. Then $\Sigma$ has the amalgamation property and $M$ is the denumerable $\Sigma$-homogeneous $\Sigma$-universal structure.

Using the results of Chapter 3 we turn our attention to those countable complete theories which admit elimination of quantifiers and have one binary relation symbol. There are two approaches which may be used to classify this class of theories. From the work of Chapter 3 we see that each such theory is determined by the class of finite graphs which can be embedded in its countable model. A natural question is then:

Q3: Is there a proper subclass of the class of finite graphs which serves to determine every countable graph whose theory admits elimination of quantifiers?

Another approach uses the notion of neighbourhood. Let $H = (F, G)$ be such that $F$ and $G$ are finite classes of finite graphs. Then $N(H)$ is the set of isomorphism types of those countable graphs whose theories admit elimination of quantifiers, in which each graph in $F$ can be embedded, and in which no graph in $G$ can be embedded. $N(H)$ is a neighbourhood of $K$ just in case the isomorphism type of $K$ is in $N(H)$.

The examples of Peretyiatkin [8, §4] show that not all graphs have even a countable neighbourhood and also show that the first approach is little help in classification. He builds $2^{\omega_0}$ graphs whose theories admit elimination of quantifiers as follows. For each subset $A$ of $\omega$ he shows that the class $\Sigma(A)$ of finite graphs in
which the undirected \((n+3)-\)cycle \(C_n\) cannot be embedded for \(n \notin A\) has the amalgamation property. By the results of Chapter 3 there is a graph \(G(A)\) whose theory admits elimination of quantifiers and such that \(\Sigma(A)\) is the class of finite graphs which can be embedded in \(G(A)\). Thus there are many neighbourhoods which contain uncountably many members. Several natural questions are immediate.

Q4: Are there countably infinite neighbourhoods? Are there finite neighbourhoods? Are there nontrivial empty neighbourhoods?

Q5: Does every neighbourhood have a finite subneighbourhood?

An examination of the examples of Peretyiatkin reveals that all possible 2-types occur. Let \(\varnothing\) be the graph with one element \(a\) and relation \(\{(a,a)\}\). Let \(\bullet\) be the graph with universe \(\{a,b\}\) and relation \(\{(a,b)\}\). Let \(\circ\circ\) be the graph with universe \(\{a,b\}\) and empty relation and let \(\circ\circ\) be the graph with universe \(\{a,b\}\) and relation \(\{(a,b), (b,a)\}\). We conjecture

C7: Each of \(H(\phi,\{\varnothing,\bullet\})\), \(H(\phi,\{\varnothing,\circ\circ\})\) and \(H(\phi,\{\varnothing,\circ\circ\})\) are countable.

In Chapter 4 we give examples which show that the answers to Q4 are "yes" and to Q5 "no". Also we provide some arguments which lend support to C7. In this chapter three basic situations are considered, and the theories are assumed to have only one 1-type, and indeed it is assumed that their models are irreflexive. In the second section we show that such a theory cannot have two incomparable definable equivalence relations if it has an infinite model. This provides us
with a nontrivial empty neighbourhood. In the third section we give partial results for those theories whose models are tournaments, i.e. which omit $\sigma, \rightarrow, \cdots$. It is shown that there is only one finite tournament whose theory admits elimination of quantifiers, and that there are only two such countable tournaments which have the property that the successors of an element are linearly ordered by the relation. There is also the countable tournament homogeneous and universal for the class of finite tournaments. We do not know what other examples exist. In the fourth section we turn to undirected graphs whose theories admit elimination of quantifiers. Work of A. Gardiner [4] neatly classifies the finite examples. We show that for each $n$ the class $\Sigma^n$ of finite undirected graphs which omit the complete graph with $n+2$ vertices $K_{n+2}$ has the amalgamation property and thus obtain infinitely many countable graphs. The conjecture that $H(\phi, \langle \sigma, \rightarrow \rangle)$ is countable is supported by the main result of this section where it is shown that there are only four countable graphs which omit the triangle and whose theories admit elimination of quantifiers - i.e. $H(\phi, \langle \sigma, \rightarrow, K_3 \rangle)$ has four members.

In Chapter 5 we use the results of Chapter 3 to furnish a counterexample to C2. The example is related to that of Peretyiatkin. We produce a dense tree with infinite branching - the subtrees above a node being indexed by the rationals. Using this a complete theory $T_0$ with nine countable models is produced. The nonlogical symbols of $T_0$ are two binary function symbols, a unary predicate symbol, a binary predicate symbol and countable many constant symbols. $T_0$ admits
elimination of quantifiers and has a nonprincipal 1-type which contains infinitely many 2-types. However we conjecture:

(C2'): No nonprincipal 1-type in a theory with three countable models can contain infinitely many 2-types.

Another way to focus the attack on theories with a finite number of countable models is to consider the implications of C1 and C4. In Chapter 6 we investigate these conjectures at greater depth. It is shown that if every complete inessential extension of a theory has a finite number of countable models then the theory has a universal model which is prime over a finite set, thus improving a result of Benda [1, Thm 2]. In the second section we consider a complete theory $T$ with a nonprincipal 1-type $p$ with the property that in the countable saturated model $M$ the relation "$a$ is prime over $b$" is a linear order of $p(M)$. We show that in fact it is a dense order defined by a formula, thus echoing the result of Chapter 2. Finally the results of Chapter 3 are applied to produce a complete theory $T_1$ with a binary relation symbol, a 3-ary function symbol and infinitely many unary predicate symbols. $T_1$ admits elimination of quantifiers and has four countable models. In the saturated model of $T_1$ the relation "$a$ is prime over $b$" is a linear order of the set of elements realizing each nonprincipal 1-type. $T_1$ has a complete inessential extension with infinitely many countable models. An interesting unresolved question is whether an example like $T_1$ can be found with three countable models.
Chapter 1

Notation and Preliminaries

In this chapter we set down the basic notation, definitions and preliminary results which we shall employ throughout. Generally we shall follow the conventions of Shoenfield [10] but in certain important respects our notation will differ from his.

Let \( L \) be a first order language. We use upper case Latin letters to denote structures for \( L \). If \( M \) is a structure for \( L \) its underlying set will be denoted \( |M| \); if confusion is possible and otherwise more simply by \( M \). We shall write \( a \in M \) for \( a \subseteq |M| \) and \( \bar{a} \in M \) for \( \bar{a} \subseteq |M|^{n} \) when \( \bar{a} \) is an \( n \)-tuple of elements of \( M \). \( L(A) \) will denote the language \( L \) augmented by constant symbols for names of elements of \( A \). In general we shall abuse the notation by using an element as its own name. If \( u \) is a symbol of \( L \) and \( M \) is a structure for \( L \) then \( u_{M} \) is the interpretation of \( u \) in \( M \). For ease in reading we shall also write \( u(M) \) or simply \( u \) when it is clear from context that \( u_{M} \) is meant. If \( ACM (M,A) \) is the expansion of \( M \) to the theory with names for individuals \( a \in A \) with \( a_{M} = a \). \( (M,a) \) is \( (M,Rng a) \).

We assume a fixed list of variables \( \{v_{n} : n \in \omega \} \). We shall let \( x,x_{n},y,y_{n},z,z_{n} \) range among variables and constants of the language. For a formula \( \varphi \) and a sequence \( \bar{x} = <x_{0},...,x_{n-1}> \) we write \( \varphi(\bar{x}) \) for the formula \( \varphi_{v_{0},...,v_{n-1}}[x_{0},...,x_{n-1}] \). It is assumed that no clash of quantifiers results, and unless the context implies otherwise that the free variables of \( \varphi(\bar{x}) \) occur among \( x_{0},...,x_{n-1} \).
If \( x \), \( y \) are two finite sequences then \( \varphi(x,y) \) is \( \varphi(x \cap y) \) where \( \cap \) is the operation of concatenation of finite sequences. We write \( x \) for \( <x> \). \( \text{lh}(x) \) is the length of \( x \).

If \( L \) is a first order language and \( n \in \omega \) then \( L_n \) is the set of formulae whose free variables occur among \( v_0, \ldots, v_{n-1} \). \( L_0 \) is thus the set of sentences for \( L \). \( L_n(A) \) is \( (L(A))_n \).

We shall use lower case Greek letters \( \chi, \phi, \psi, \theta, \lambda \) to range among formulae and \( \sigma, \rho, \xi \) among mappings. For a formula \( \psi, \psi^1 \) denotes \( \neg \psi \) and \( \psi^0 \) denotes \( \psi \).

Let \( T \) be a complete theory with language \( L \) and let \( \vec{x} \) be a sequence of \( n \) new constant symbols. A set \( \Gamma \) of formulae in \( L_n \) is an \( n \)-type in \( T \) just in case \( T[\Gamma] = T \cup \{ \varphi(x) : \varphi \in \Gamma \} \) is a consistent extension of \( T \). \( \Gamma \) is a complete \( n \)-type just in case \( T[\Gamma] \) is complete. We reserve the letter \( p \) to denote a complete \( 1 \)-type in \( T \). Upper case Greek \( \Lambda, \Gamma, \Phi \) will normally denote complete \( n \)-types. Unless otherwise specified \( n \)-type will mean complete \( n \)-type.

If \( M \) is a structure for \( L \) and \( \varphi \in L(M) \) then \( M \models \varphi \) means that \( \varphi \) is valid in \( M \). If \( \Gamma \subseteq L \) then \( M \models \Gamma \) means \( M \models \varphi \) for all \( \varphi \in \Gamma \). \( M \models T \) has the usual meaning. If \( \vec{a} \) is an \( n \)-tuple in \( M \) then \( \text{tp}(\vec{a}) = \{ \varphi \in L_n : M \models \varphi(\vec{a}) \} \), \( \text{tp}(\vec{a}) \) is an \( n \)-type in \( \text{Th}(M) \). \( \text{tp}(\vec{a},\vec{b}) = \text{tp}(\vec{a} \cap \vec{b}) \). We say that \( \vec{a} \) realizes \( q \) if \( q \in \text{tp}(\vec{a}) \). If \( \varphi \in L_n(M) \) then \( \varphi(M) = \{ \vec{a} \in M : M \models \varphi(\vec{a}) \} \).

\( I(T, \kappa) \) is the cardinality of the set of isomorphism types of models of \( T \) of power \( \kappa \). If \( S \) is a set, \( |S| \) denotes its cardinality. Countable means infinite and countable.
If $T$ is a theory then the language of $T$ is $L(T)$. When no confusion is likely we shall write $L$ for $L(T)$.

$T'$ is an inessential extension of the complete theory $T$ if $T'$ is complete and an extension of $T$ in a language which differs from $L$ by finitely many constant symbols.

Let $T$ be a complete theory. We shall say that an $(m+n)$-type $\Delta$ is principal over an $m$-type $\Lambda$ just in case $\Delta \subseteq \Lambda$ and there is some $\theta \in \Delta$ such that whenever $M \models T \vdash \bar{a} \in M$ and $\bar{a}$ realizes $\Delta$ we have for each $X \in \Lambda$, $M \models (\theta \to X) (\bar{d} \cap \langle v_m, \ldots, v_m+n-1, v_{m+n}, \ldots, v_{m+n-1} \rangle)$. In this case $\theta$ is said to generate $\Delta$ over $\Lambda$. If $M \models T$ and $\bar{a}, \bar{b} \in M$ then $\bar{b}$ is prime over $\bar{a}$ just in case $\text{tp}(\bar{a}, \bar{b})$ is principal over $\text{tp}(\bar{a})$.

If $m,n < \omega$ are given let the transposition operator $T_{m,n}$ be defined for each $\theta$ by

$$T_{m,n}(\theta) = \theta'_{v_0, \ldots, v_{m+n-1}, v_m, \ldots, v_{m+n-1}, v_{m+n-1}, \ldots, v_{m+n-1}, v_m, \ldots, v_{m+n-1}}$$

where $\theta'$ is a variant of $\theta$ chosen so that no $v_i$ for $i < m+n$ occurs bound in $\theta'$. We extend $T_{m,n}$ to $(m+n)$-types in the obvious way. Note that if $\bar{a}$ is an $n$-tuple and $\bar{b}$ is an $m$-tuple then $\text{tp}(\bar{b}, \bar{a}) = T_{m,n}(\text{tp}(\bar{a}, \bar{b}))$.

We shall make frequent use of the following well-known result.

Lemma 1.1 Let $M$ be a model of the complete theory $T$ and let $\bar{a}, \bar{b}, \bar{c} \in M$ be such that $\bar{b} \cap \bar{c}$ is prime over $\bar{a}$ and $\bar{a}$ is prime over $\bar{b}$. Then $\bar{b} \cap \bar{c}$ is prime over $\bar{b}$.

Proof. Let $lh(\bar{a}) = m$ $lh(\bar{b}) = n$ $lh(\bar{c}) = k$. Let $\theta$ generate the type of $\bar{a} \cap \bar{b} \cap \bar{c}$ over $\text{tp}(\bar{a})$ and let $\psi$ generate
If $\Gamma$ is a 2-type, $p, q$ 1-types with $p = \Gamma$ and $q \subseteq T_{1,1}(\Gamma)$ then $\Gamma$ is said to be in $pxq$.

A formula $\psi$ with one free variable $v_0$ is said to be $\omega_0$-categorical just in case for each $m \in \omega$ there are a finite number of $m$-types $\Gamma$ which contain $\psi(v_i)$ for $i < m$.

A graph is a structure for the language with one binary relation symbol $R$. If $G$ is a graph $|G|$ is the vertex set of $G$ and members are called vertices while $R_G$ is the edge set and members are called edges.
Chapter 2

Ehrenfeucht-like theories

1. Definitions

In this chapter we give some results which indicate a distinction between two types of theories in a small language - those with only relation symbols and constant symbols and those which allow function symbols. These differences are related to conjectures C1, C2, and C3. The main theorem uses severe restrictions on the number of countable models and the language. Before introducing these however we shall present some more general results, and the notion of being "like" an Ehrenfeucht structure.

Vaught's argument of [12; p.320] that no countable complete theory has exactly two isomorphism types of countable models may be modified using his Theorem 3.5 [12; p.311] on the existence of prime models to give the following observation.

If $I(T,\omega) = 3$ then $T$ has countable models $M_0$, $M_1$, and $M$ where $M_0$ is prime, $M_1$ is saturated, and for each nonprincipal n-type $A$ there is a sequence $\bar{a} \in M$ such that $tp(\bar{a}) = A$ and $M$ is prime over $\bar{a}$. The modification is to note that $T$ and any complete inessential extension of $T$ have prime and saturated countable models. Thus given a nonprincipal n-type $A$ there is a countable model $N$ of $T$ and $\bar{b} \in N$ such that $N$ is prime over $\bar{b}$ and $\bar{b}$ realizes $A$. But then $N$ cannot be saturated and it cannot be prime. Thus if $M_0$ is a prime model of $T$, $M_1$ is a countable saturated model of $T$ and $M$ is the third countable model of $T$, $M$ must be isomorphic to $N$. We call $M$ the middle model of $T$. 
Another result which was discovered independently by Benda [1]
and the author is:

Lemma 2.1 Let \( U \) be a countable complete theory with
\( I(U, \omega) = 3 \). Then if \( \Lambda \) is any nonprincipal \( m \)-type there is a \( 2m \)-type
\( \Gamma \) with \( \Lambda \subseteq \Gamma \) and \( \Lambda \subseteq T_{m,m}(\Gamma) \) such that \( \Gamma \) is principal over \( \Lambda \)
while \( T_{m,m}(\Gamma) \) is not.

Proof. Let \( M \) be prime over \( \bar{a} \) realizing \( \Lambda \). There are
infinitely many \( m \)-types since \( \Lambda \) is nonprincipal and hence infinitely
many \( m \)-types over \( \bar{a} \), i.e. in \( \text{Th}(M, \bar{a}) \). Thus there is a \( 2n \)-type
\( \Gamma' \) which is not principal over \( \Lambda \). Choose \( \bar{a}' \) realizing \( \Lambda \)
and \( \bar{b} \) such that \( \bar{a}' \cap \bar{b} \) realizes \( \Gamma' \) and \( M \) is prime over
\( \bar{a}' \cap \bar{b} \). Consider the type \( \Gamma = \text{tp}(\bar{a} \cap \bar{a}') \). \( \Gamma \) is principal over \( \Lambda \)
since \( M \) is prime over \( \bar{a} \) but \( T_{m,m}(\Gamma) \) cannot be principal over \( \Lambda \)
for otherwise \( \Gamma' \) would be principal over \( \Lambda \) by Lemma 1.1. This
establishes the lemma.

Let \( T \) be a countable complete theory which has a binary relation
symbol \( R \). For \( \phi \in L_1 \) define
\[
\lambda^\phi xy = ((Rxy \land Ryx) \lor (\lnot Rxy \land \lnot Ryx)) \land \phi(x) \land \phi(y).
\]

We say that property \( E \) holds of \( \phi \) if the following seven
conditions are satisfied:

(i) \( \models_T \lambda^\phi xy \land \lambda^\phi yz \to \lambda^\phi xz \)
i.e. \( \lambda^\phi \) is an equivalence relation on \( \phi \).

(ii) \( \models_T \lambda^\phi xx_1 \land \lambda^\phi yy_1 \land Rxy \land \lnot Ryx \to Rx_1 y_1 \land \lnot Ry_1 x_1 \)
i.e. \( \lambda^\phi \) is a congruence relation with respect to \( Rv_0 v_1 \land \lnot Rv_1 v_0 \)
on \( \phi \).
(iii) $\vdash_T \phi(x) \land \phi(y) \land \phi(z) \land (R_{xy} \land \lnot R_{yx}) \land (R_{yz} \land \lnot R_{zy}) \Rightarrow$

$R_{xz} \land \lnot R_{zx}$

i.e. $R_{01}v \land \lnot R_{01}v_0$ is transitive on $\phi$.

(iv) $\vdash_T \phi(x) \land \phi(y) \land \lnot R_{xv} \land \lnot R_{xy} \Rightarrow \exists z(\phi \land (R_{xz} \land \lnot R_{zx}) \land \lnot R_{zy}))$ where $z$ is a new variable

i.e. $(R_{01}v \land \lnot R_{01}v_0)$ is dense on $\phi/\lambda^\phi$.

(v) $\vdash_T \phi(x) \Rightarrow \exists z(\phi(z) \land (R_{xz} \land \lnot R_{zx}))$ where $z$ is a new variable.

(vi) $\vdash_T \phi(x) \Rightarrow \exists z(\phi(z) \land (R_{xz} \land \lnot R_{zx}))$ where $z$ is a new variable.

i.e. $R_{01}v \land \lnot R_{01}v_0$ is "without endpoints" on $\phi/\lambda^\phi$.

and:

(vii) Either for each principal 1-type $p$ containing $\phi$ there are at most finitely many 1-types $q$ containing $\phi$ such that there is a 2-type $\Delta$ in $p_1q$ containing

$$((R_{01}v \land \lnot R_{01}v_0) \lor \lambda^\phi v_0 v_1),$$

or for each principal 1-type $p$ containing $\phi$ there are at most finitely many 1-types $q$ containing $\phi$ such that there is a 2-type $\Delta$ in $p_1q$ containing

$$((\lnot R_{01}v \land R_{01}v_1) \lor \lambda^\phi v_0 v_1).$$

Roughly speaking property $E$ holds of $\phi$ if in each model $N$ of $T$ the structure obtained by restriction to $\phi$ resembles the Ehrenfeucht example: $R_N$ is a dense ordering of equivalence classes, and principal types are almost arranged in a sequence.
A model $M$ of the complete theory $T$ whose language includes the binary relation symbol $R$ is said to be $E$-like if there are a finite number of formulae $\phi_0', \ldots, \phi_n \in L_1$ such that:

(i) property $E$ holds of $\phi_i$ for $i \leq n$

(ii) $\models_T \neg \exists \psi_0 (\phi_i \land \phi_j)$ for $i \neq j$

(iii) $\forall \psi_i$ is $\omega_0$-categorical.

In this case we also say that $T$ is $E$-like.

2. The Theorem

For the remainder of this chapter we assume that $T$ is a complete theory in the language with one binary relation symbol $R$ and constant symbols $\{a_i : i \in \omega\}$. We also assume that $T$ admits elimination of quantifiers and that it has three countable models.

Our main result is the following:

Theorem 2.1 There is $n \in \omega$ and there are formulae $\phi_0', \ldots, \phi_n \in L_1$ such that $\forall \psi_i$ is $\omega_0$-categorical and for each $i \leq n$ property $E$ holds of $\phi_i$.

The proof of this theorem rests on the following two lemmas whose proofs are deferred to §3 and §4.

Lemma 2.2 Let $\phi' \notin L_1$ be contained in a nonprincipal $1$-type $p$. Then there is a formula $\phi \in p$ such that $\models_T \phi \land \phi'$ and property $E$ holds of $\phi$.

Lemma 2.3 There are only finitely many nonprincipal $1$-types in $T$. The following is immediate:
Lemma 2.4: If \( \bar{b} \) is an \( n \)-tuple in a model \( N \) of \( T \) then \( \text{tp}(\bar{b}) \) is the unique \( n \)-type \( \Lambda \) such that \( T_{n-j,j}(\Lambda) \supset \text{tp}(b_j) \),
\[
v_i = v_j \in \Lambda \iff b_i = b_j \land Rv_iv_j \in \Lambda \iff N \models Rb_ib_j \text{ for } i, j < n.
\]

Proof of the theorem from the lemmas. By Lemma 2.4 and Ryll-Nardzewski's theorem there must be a nonprincipal \( 1 \)-type in \( T \).

By Lemma 2.3 there are finitely many nonprincipal \( 1 \)-types, say \( p_1, \ldots, p_n \). Choose \( \psi_i \) such that \( p_i \) is the only nonprincipal \( 1 \)-type that contains \( \psi_i \) for \( i \leq n \). By Lemma 2.2 we may choose \( \phi_i \in p_i \) such that \( T \phi_i \rightarrow \psi_i \) and property \( E \) holds of \( \psi_i \) for \( i \leq n \). There can be no nonprincipal \( 1 \)-type which contains \( \bigwedge_{i \leq n} \phi_i \).

From Lemma 2.4 it is easy to see that \( \bigwedge_{i \leq n} \phi_i \) is \( \omega_0 \)-categorical. This completes the proof.

3. Proof of Lemma 2.2

Recall that we must show that if \( T \) satisfies the conditions of this chapter, \( p \) is a nonprincipal \( 1 \)-type of \( T \) and \( \phi \in p \) then there is a formula \( \phi \in p \) such that \( T \phi \rightarrow \phi' \) and \( \phi \) has property \( E \).

Let \( p \) be a nonprincipal \( 1 \)-type of \( T \). Let \( M \) be the middle model of \( T \).

Consider now those \( 2 \)-types \( \Delta \) in \( pxp \) which contain \( v_0 \neq v_1 \). By Lemma 2.4 we see that each such \( \Delta \) is specified by the pair \((i,j) \in \{0,1\}^2 \) such that \( (Rv_0v_1)^i \land (Rv_1v_0)^j \in \Delta \). By Lemma 2.1 there is some type \( \Gamma \) in \( pxp \) which is principal over \( p \) while \( T_{1,1}(\Gamma) \) is not. Let \( \theta \) generate \( \Gamma \) with respect to \( p \). By elimination of quantifiers without loss of generality we may take \( \theta \) to be
\[
\phi_0 \land \phi_0(v_1) \land Rv_0v_1 \land \neg Rv_1v_0 \land v_0 \neq v_1 \text{ where } \phi_0 \in p.
\]
Note that $\theta$ cannot be symmetric in $v_0, v_1$ since $T_{1,1}(\Gamma)$ is not principal over $p$.

Claim. $T_{1,1}(\Gamma)$ is the only 2-type in $pxp$ which is nonprincipal over $p$.

For proof by contradiction assume that $\Lambda$ is a 2-type in $pxp$ which is nonprincipal over $p$ and different from $T_{1,1}(\Gamma)$. Let $a$ be a member of the middle model $M$ which realizes $p$ and over which $M$ is prime. Let $M$ be prime over $(c,d)$ realizing $\Delta$. Now $(a,c)$ and $(a,d)$ realize types principal over $p$ and from Lemma 1.1 neither $(c,a)$ nor $(d,a)$ realizes a type principal over $p$. Therefore $\Gamma = tp((a,c)) = tp((a,d))$ by Lemma 2.4. The type $\Lambda$ of $(a,c,d)$ is principal over $p$ and $T_{1,2}(\Lambda)$ is principal over $\Lambda$. Since $d$ realizes $p$ there is a pair $(e,f)\in M$ such that $(d,e,f)$ realizes $\Lambda$. We shall show that $(a,e,f)$ also realizes $\Lambda$. Now $(e,f)$ realizes $\Lambda$ and as before by Lemmas 1.1 and 2.4 $\Gamma = tp(a,e) = tp(a,f)$ since $M$ is prime over $a$ but not over $e$ or $f$. For $M$ prime over $e$ or $f$ would imply by Lemma 1.1 that $M$ was prime over $d$, a contradiction. But then by Lemma 2.4 the type of $(a,e,f)$ is $\Lambda$. By homogeneity of prime models (see Vaught [12, p.310]) there is an automorphism of $M$ fixing $a$ and taking $(c,d)$ onto $(e,f)$. Thus $M$ is also prime over $(e,f)$. But $(d,e,f)$ realizes a type principal over $p$. Hence by Lemma 1.1 $M$ is prime over $d$ whence $(d,a)$ realizes a type principal over $p$, a contradiction. This verifies the claim.
From the definition of $\Gamma, \Gamma \neq T_{l, 1}(\Gamma)$. Thus from the claim we may choose $\phi \in p$ such that for $i, j < 2$

either $\vdash_T \forall v_0 \exists v_1 (\phi \land \phi(v_1) \land (Rv_0 v_1)^i \land (Rv_1 v_0)^j \land v_0 \neq v_1)$
or there is a unique 2-type $\Delta_{i,j}$ in $pxp$ containing

$$\lambda_{i,j}^\phi = \phi \land \phi(v_1) \land (Rv_0 v_1)^i \land (Rv_1 v_0)^j \land v_0 \neq v_1$$

and $\lambda_{i,j}^\phi$ generates $\Delta_{i,j}$ over $p$ or $T_{l, 1}(\lambda_{i,j}^\phi)$
generates $T_{l, 1}(\Delta_{i,j})$ over $p$.

We call this the basic restriction on $\phi$, and assume that $\phi$ satisfies the basic restriction in the remainder of this section.

We may also suppose without loss that

$$\vdash_T \forall v_0 (\phi \to \neg \neg Rv_0 v_0) \lor \forall v_0 (\phi \to \neg Rv_0 v_0)$$

For if $\psi \in p$ and $\vdash_T \psi \to \phi$ and $\phi$ satisfies the basic restriction then $\psi$ does as well.

The proof is now reduced to a number of claims which permit one to choose a formula $\phi \in p$ such that $\vdash_T \phi \to \phi'$ and $\phi$ is E-like.

Claim 2.1 Given a formula $\phi \in p$ there is a formula $\psi$ in $p$ such that if $\phi^*$ is any formula in $p$ implying $\phi \land \psi$ then $\phi^*$ satisfies $E(i)$.

Let $\psi$ be the formula

$$\psi = df \forall v_1 \forall v_2 (\lambda_{v_1}^\phi v_1 \land \lambda_{v_2}^\phi v_2 : \to : \lambda v_0 v_2)$$

We shall show first that $\psi \in p$. Let $M$ be prime over a realizing $p$ and let $b, c$ be elements such that

$$M \models \lambda_{ab}^\phi \land \lambda_{bc}^\phi$$
From the definition of $\lambda^\varphi$ either $\lambda^\varphi_{0,0}$ or $\lambda^\varphi_{1,1}$ belongs to the type realized by $(a,b)$. Suppose that $M \models \lambda^\varphi_{0,0} (a,b)$ without loss of generality. There is a unique 2-type $\Delta_{0,0}$ in pxp containing $\lambda^\varphi_{0,0}$ from Lemma 2.4. Since $\lambda^\varphi_{0,0}(v_0^1,v_0^1) = \lambda^\varphi_{0,0}(v_0^1,v_1^1)$, $\Delta_{0,0} = T_{1,1}(\Delta_{0,0})$ whence $\Delta_{0,0}$ is generated over $p$ by $\lambda^\varphi_{0,0}$. Thus $(b,a)$ realizes $\Delta_{0,0}$ and $M$ is prime over $b$ by Lemma 1.1.

Similarly, $M$ is prime over $c$. It follows that the types realized by $(a,c)$ and $(c,a)$ are both in pxp and principal over $p$. Neither of these types can contain $\lambda^\varphi_{0,1}$ whence $M \not\models \lambda^\varphi(a,c)$ as required.

If $\varphi^*$ is any formula in $p$ implying $\varphi \land \psi$ it is clear that

$$\models T_{1,2} \lambda^\varphi_{xy} \land \lambda^\varphi_{yz} \rightarrow \lambda^\varphi_{xz}$$

since by inspection $\lambda^\varphi_{v_0^1,v_1^1}$ is equivalent to $\varphi^* \land \varphi^*(v_1^1) \land \lambda^\varphi_{v_0^1,v_1^1}$. Thus any $\varphi^*$ in $p$ which implies $\varphi \land \psi$ will serve to satisfy $E(i)$. Thus claim 2.1 is verified.

Let $\Theta^X$ be the formula $X \land X(v_1^1) \land Rv_0^1v_1 \land \neg Rv_1v_0$

where $X$ has at most $v_0$ occurring free.

Claim 2.2 Let $\varphi \in p$. There is a formula $\psi \in p$ such that any $\varphi^* \in p$ which implies $\psi$ will satisfy $E(ii)$.

Let $\psi$ be the formula

$$\forall v_1 \forall v_2 \forall v_3 (\lambda^\varphi v_0^1 v_1 \land \Theta^\varphi(v_0^1,v_2) \land \lambda^\varphi v_2 v_3 \rightarrow \Theta^\varphi(v_1^1,v_3))$$

We argue that $\psi \in p$. Let $a,b,c,d$ be elements of $M$ such that $M$ is prime over $a$, $a$ realizes $p$ and $M \models \lambda^\varphi ab \land \Theta^\varphi(a,c) \land \lambda^\varphi cd$. It is easy to see that $b,c$ and $d$ realize $p$ and that $(a,c)$ realizes $\Gamma$ because of the basic restriction on $\varphi^-$. Also if
(b,d) does not realize \( \Gamma \) then \( M \) is prime over \( d \) and thus also over \( c \) contradicting the fact that \( (a,c) \) realizes \( \Gamma \). Arguing as for Claim 2.1 we see that any \( \varphi^* \) in \( p \) which implies \( \psi \) will satisfy \( E(ii) \).

Claim 2.3 Let \( \varphi \in p \) satisfy \( E(ii) \). Then there is \( \psi \in p \) such that if \( \varphi^* \in p \) and \( T \varphi^* \rightarrow \varphi \land \psi \) then \( \varphi^* \) satisfies \( E(iii) \).

Let \( \psi \) be the formula

\[
\forall v_1 \forall v_2 (\theta^0(v_0,v_1) \land \theta^0(v_1,v_2) \rightarrow \theta^0(v_0,v_2))
\]

We show that \( \psi \in p \). If \( M \) is prime over \( a \) realizing \( p \) and \( b,c \) are such that \( M \models \theta^0(a,b) \land \theta^0(b,c) \), then \( b,c \) realize \( p \) from the basic restriction on \( \varphi \). The type of \( (a,c) \) is determined by the unique pair \((i,j)\in\{0,1\}^2\) such that \( M \models \lambda_{i,j}^\varphi(a,c) \). If \((i,j)\) is \((0,0)\) or \((1,1)\) then \( M \models \lambda_{0}^\varphi ac \) which gives \( M \models \theta^0(c,b) \) since \( \varphi \) satisfies \( E(ii) \). This contradicts \( M \models \theta^0(b,c) \). If \((i,j)\) is \((1,0)\) then \( (c,a) \) realizes \( T \), which means that \( (a,c) \) realizes \( T_{1,1}^{(1)}(\Gamma) \) contradicting the fact that \( M \) is prime over \( a \).

Thus \((i,j)\) is \((0,1)\) which yields \( M \models \theta^0(a,c) \) as required.

Arguing as for Claim 2.1 any \( \varphi^* \) in \( p \) which implies \( \varphi \land \psi \) will serve to satisfy \( E(iii) \).

Claim 2.4 Let \( p \) be the only nonprincipal \( 1 \)-type which contains \( \varphi \). Then if \( \varphi^* \in p \) and \( T \varphi^* \rightarrow \varphi \) we have that \( \varphi^* \) satisfies \( E(vii) \).

Let \( \varphi \in p \) be such that \( p \) is the unique nonprincipal \( 1 \)-type containing \( \varphi \). Let \( q \) be a principal \( 1 \)-type which contains \( \varphi \).
Notice that

\[ \{q': \text{ there exists } \Delta \text{ in } q' \times q \text{ such that } \lambda^\theta \nu_0 v_1 \lor \theta^\phi \in \Delta \} \]

is finite. Otherwise, by the compactness theorem there exists non-principal \( q' \) and \( \Delta \) in \( q' \times q \) such that \( \lambda^\theta \nu_0 v_1 \lor \theta^\phi \in \Delta \). Since \( p \) is the only nonprincipal \( 1 \)-type containing \( \phi \), we have \( q' = p \). But \( (\lambda^\theta \nu_0 v_1 \lor \theta^\phi) \in \Delta, \Delta \supset p \) and \( \theta_{1,1}(\Delta) \supset q \) imply that \( q = p \) by the basic restriction on \( \phi \). Thus \( q \) is nonprincipal, a contradiction. This shows that \( \phi \) satisfies \( E(\text{vii}) \) and the claim follows easily.

Claim 2.5 If \( \phi \) satisfies \( E(\text{i}), E(\text{ii}), E(\text{iii}) \) and \( p \) is the unique nonprincipal \( 1 \)-type containing \( \phi \) then there is a formula \( \chi \) in \( p \) such that \( \models_{\chi} \chi \lor \phi \) and \( \chi \) satisfies \( E(\text{iv}) \).

Let \( \psi \) be

\[ \forall v_1 \exists v_2 (\theta^\phi(v_0, v_1) \rightarrow \theta^\phi(v_0, v_2) \land \theta^\phi(v_2, v_1)) \]

We have \( \psi \in p \) since \( \phi \) satisfies \( E(\text{iii}) \) and \( \theta^\phi \) generates \( \Gamma \) over \( p \). For if \( M \) is prime over a realizing \( p \) there are then \( b, c \) such that \( M \models \theta^\phi(a,b) \land \theta^\phi(b,c) \). By \( E(\text{iii}) \) \( M \models \theta^\phi(a,c) \) whence \( \exists v_2 (\theta^\phi(v_0, v_2) \land \theta^\phi(v_2, v_1)) \) belongs to \( \Gamma \). By choice of \( \phi \), \( \phi \land \neg \psi \) is \( \omega_0 \)-categorical. Let \( p_0, \ldots, p_m \) enumerate those \( 1 \)-types containing \( \phi \land \neg \psi \). We claim that there is a finite list of \( 1 \)-types \( q_0, \ldots, q_n \) extending \( p_0, \ldots, p_m \) such that for each \( i \leq n \) \( \phi \in q_i \) and

\[ (*) \forall q (q \text{ a } 1 \text{-type } \land \exists \Delta (\Delta \text{ a } 2 \text{-type } \Delta \supset q \land \theta_{1,1}(\Delta) \supset q \land (\lambda^\theta \nu_0 v_1 \lor \theta^\phi) \in \Delta) \implies \exists j \leq n (q = q_j)) \]

\( (*) \) says that \( \{q_0, q_1, \ldots, q_n\} \) is closed under "predecessors" with respect to \( (\theta^\phi \lor \lambda^\phi) \). The list \( <q_i : i \leq n> \) exists since \( \phi \) already satisfies \( E(\text{i}), E(\text{ii}), E(\text{iii}) \) and \( E(\text{vii}) \): one need only extend the list \( p_0, \ldots, p_m \) by adding those \( q \) for which
\begin{itemize}
\item[(\exists \xi \leq m)] \end{itemize}
\exists \Delta (\Delta \text{ in } q \times p_i \text{ and } \chi_{\psi q \xi} \in \theta (\Delta)).

Let \( \chi = df \varphi \wedge \lambda \rightarrow \chi_i \). Now \( \chi \) satisfies \( E(i), E(ii), E(iii) \) and (vii).

and (vii). Also since the list \( q_0', \ldots, q_n \) extends \( p_0', \ldots, p_m \) we have
\[
\vdash_T \forall \nu \forall \nu \exists \nu_2 (\vartheta_X \rightarrow \vartheta (\nu_0, \nu_2) \wedge \vartheta (\nu_2, \nu_1)).
\]

From (*) we can deduce
\[
\vdash_T \forall \nu \nu \nu_1 (\vartheta (\nu_0, \nu_1) \rightarrow \chi (\nu_1)).
\]
Thus \( \chi \) satisfies \( E(iv) \).

Claim 2.6 If \( \varphi \in p \) and \( p \) is the unique nonprincipal 1-type containing \( \varphi \) then \( \vdash_T \forall \nu \exists \nu_1 (\varphi \rightarrow \vartheta (\nu_0, \nu_1)) \) and \( \varphi \) satisfies \( E(v) \). For (p) be prime over a realizing \( p \) and let \( b \in M \) be such that \( M \models \varphi (b) \). If \( b \) realizes a nonprincipal 1-type then that type is \( p \) and since \( \vartheta (\nu_0, \nu_1) \) generates \( \Gamma \) over \( p \), \( M \models \exists \nu_1 \vartheta (b, \nu_1) \).

Otherwise \( b \) realizes a principal type and again because of the basic restriction on \( \varphi \), \( M \models \vartheta (b, a) \). Thus \( \varphi \) satisfies \( E(v) \).

Now we shall apply Claims 2.1, 2.2, 2.3. Choose \( \varphi_0 \in p \) such that \( \varphi \) satisfies the basic restriction, \( \vdash_T \varphi_0 \rightarrow \varphi \), \( p \) is the unique nonprincipal 1-type containing \( \varphi_0 \) and if \( \varphi^* \in p \) and \( \vdash_T \varphi^* \rightarrow \varphi_0 \) then \( \varphi^* \) satisfies \( E(i), E(ii) \) and \( E(iii) \). By Claim 2.5 we may assume that \( \varphi_0 \) satisfies \( E(iv) \). Now let \( \varphi \) be \( \exists \nu_1 (\vartheta (\nu_0, \nu_1)) \). Clearly \( \varphi \in p \). Also \( \varphi \) satisfies \( E(i), E(ii) \) and \( E(iii) \) and by Claim 2.4 \( E(vii) \) and by Claim 2.6 \( E(v) \). \( \varphi \) satisfies \( E(iv) \) and \( E(vi) \) because \( \varphi_0 \) satisfies \( E(iv) \).

Thus \( \varphi \) has property \( E \) and the lemma is proved.
4. Proof of Lemma 2.3

Under the conditions of the theorem we prove that there are only finitely many nonprincipal 1-types.

Assume there are infinitely many. Then there is a formula \( \phi \) which is contained in infinitely many nonprincipal 1-types and is "minimal" among such formulae: i.e. for no formula \( \psi \) is it the case that \( \phi \land \psi \) and \( \phi \land \neg \psi \) are each contained in infinitely many nonprincipal 1-types. Otherwise there would be \( 2^{\omega_0} \) 1-types and thus \( 2^{\omega_0} \) countable models. [12, §5.1]

It is easy to see that there are exactly \( \omega_0 \) nonprincipal 1-types \( \psi' \) containing \( \phi \) such that for some \( \psi \) \( \psi' \) is the unique nonprincipal 1-type containing \( \psi \). Such 1-types we call isolated nonprincipal 1-types. Let \( <p_i : i \in \omega> \) be an enumeration without repetition of all isolated nonprincipal 1-types containing \( \phi \) and let \( \psi'_i \) be chosen such that \( p_i \) is the unique nonprincipal type containing \( \psi'_i \).

Let \( \psi_0 \) be \( \psi'_0 \land \phi \) and for each \( i > 0 \) let \( \psi_i \) be
\[
\psi'_i \land (\neg \lor \psi_i) \land \phi.
\]

Then the sequence \( <\psi_i : i \in \omega> \) satisfies for each \( i, j < \omega \)

(i) \( T \psi_i \rightarrow \phi \)
(ii) \( \psi_i \) belongs to exactly one nonprincipal 1-type namely \( p_i \)
(iii) \( T \neg \exists \psi_0 (\psi_i \land \psi_j) \) if \( i \neq j \)

and

(iv) if \( \phi \) belongs to an isolated nonprincipal 1-type \( \psi \) there is \( i \in \omega \) with \( \psi_i \in \psi \).
By the compactness theorem there is some nonprincipal \(1\)-type 
\[ p \supseteq \{ \phi \} \cup \{ \neg \phi_i : i \in \omega \} . \]
By the choice of \( \phi \) and \( \{ \phi_i : i \in \omega \} \therefore \) there is only one such \( p \).

By Lemma 2.1 let \( \Gamma_i \) be a 2-type in \( p \times p_i \) which is principal over \( p_i \) but such that \( T_{1,1}(\Gamma_i) \) is not principal over \( p_i \) and let \( \Gamma \) be a type in \( p \times p \) with the same property. Let \( M \) be prime over \( c_i \) realizing \( p_i \) for each \( i \in \omega \) and also prime over \( c \) realizing \( p \).

Let \( c_i', i \in \omega \), and \( c' \) be elements of \( M \) such that \( (c_i, c_i') \) realize \( \Gamma_i \) and \( (c, c') \) realizes \( \Gamma \). The contradiction will be obtained by showing that there are five distinct 2-types in \( p \times p \).

From Lemma 2.4 there can be at most four.

We claim that there is a strictly increasing sequence of positive integers \( i_j : j \in \omega \) and there are pairs \( \langle k_0, k_1 \rangle \), \( \langle \ell_0, \ell_1 \rangle \) and \( \langle m_0, m_1 \rangle \) in \( \{0,1\} \) such that for each \( j \in \omega \).

\[
\begin{align*}
(i) \quad & (R_{v_0} v_1)^{k_0} \land (R_{v_1} v_0)^{k_1} \in tp(c_0, c_i) \\
(ii) \quad & (R_{v_0} v_1)^{\ell_0} \land (R_{v_1} v_0)^{\ell_1} \in tp(c_0, c_i') \\
(iii) \quad & (R_{v_0} v_1)^{m_0} \land (R_{v_1} v_0)^{m_1} \in tp(c_0, c_i') .
\end{align*}
\]

To find \( \langle k_0, k_1 \rangle \) and a subsequence \( i_j : j \in \omega \) which satisfies (i), we need only observe that some one of the four possible pairs, say \( \langle k_0, k_1 \rangle \) must be such that for infinitely many \( i \in \omega \)
\[
(R_{v_0} v_1)^{k_0} \land (R_{v_1} v_0)^{k_1} \in tp(c_0, c_i) .
\]

By thinning \( i_j : j \in \omega \) twice we can find \( i_j : j \in \omega \), \( \langle \ell_0, \ell_1 \rangle \) and \( \langle m_0, m_1 \rangle \) such that (i), (ii) and (iii) are all satisfied.
From Lemma 1.1 \( tp(c_0, c_i) \) is principal over \( p_0 \) and \( T_{1,1}(tp(c_0, c_i)) \) is principal over \( p_1 \). Also \( tp(c_0', c_i') \) is principal over \( p_0 \) and \( T_{1,1}(tp(c_0', c_i')) \) is not principal over \( p_1 \); while \( tp(c_0', c_i') \) is not principal over \( p_0 \) and \( T_{1,1}(tp(c_0', c_i')) \) is principal over \( p_1 \). Thus the pairs \( <k_0, k_1>, <l_0, l_1> \) and \( <m_0, m_1> \) are distinct by Lemma 2.4. By the compactness theorem there are distinct 2-types \( \Lambda_0, \Lambda_1, \Lambda_2 \) in \( p_0 \times p \) such that

\[
(Rv_0v_1)_{k_0} \land (Rv_1v_0)_{k_1} \in \Lambda_0
\]

\[
(Rv_0v_1)_{k_0} \land (Rv_1v_0)_{l_1} \in \Lambda_1
\]

\[
(Rv_0v_1)_{m_0} \land (Rv_1v_0)_{m_1} \in \Lambda_2
\]

Notice that none of \( \Lambda_0, \Lambda_1, \Lambda_2 \) is principal over \( p_0 \). Suppose, for example, that \( \Lambda_0 \) is generated by \( \theta \) over \( p_0 \). By Lemma 2.4 and elimination of quantifiers we may take \( \theta \) to be

\[
(Rv_0v_1)_{k_0} \land (Rv_1v_0)_{k_1} \land \psi(v_1) \] where \( \psi \in p \). Every formula in \( p \) is contained in all but finitely many of the types \( p_i \).

Thus there is some \( j \in \omega \) such that \( M \models \psi(c_i_j) \). But then \( (c_0, c_i_j) \) realizes \( \Lambda_0 \), an impossibility. Thus we have three 2-types in \( p_0 \times p \) none of which is principal over \( p_0 \). At the same time there are at least two 2-types in \( p_0 \times p \), namely those realized by \( (c_0', c) \) and \( (c_0', c') \) which are principal over \( p_0 \). This is the desired contradiction.

Thus there are only a finite number of nonprincipal 1-types in \( T \).
5. Summary

Our analysis shows that under considerable restrictions $I(T,\omega) = 3$ implies that any model of $T$ is like the Ehrenfeucht example. Benda has pointed out that we may not have used the full force of the assumption $I(t,\omega) = 3$ since we only use the existence of the middle model $M$. This remark suggests the following question.

Q6: If a countable complete theory has a middle model, does it have three countable models?

An analysis of the proofs of Lemmas 2.2 and 2.3 shows that $T$ satisfies (C1), (C2) and (C3) of the introduction.

The most important concept involved is that of a class of structures having the amalgamation property. Let \( \Sigma \) be a class of structures for a language \( L \). We say that \( \Sigma \) has the amalgamation property (AP) provided that whenever \( G, H, H_0', H_1' \in \Sigma \), \( e_0 : G \to H_0 \) and \( e_1 : G \to H_1 \) are embeddings then there is \( H \in \Sigma \) and there are embeddings \( f_0, f_1 \) where \( f_0 : H_0 \to H, f_1 : H_1 \to H \) and \( f_0 \circ e_0 = f_1 \circ e_1 \).

For the remainder of this chapter let \( L \) be a fixed finite language. The reader should bear in mind that \( L \) may contain function symbols.

Let \( \Sigma \) be a class of finite structures for \( L \) and let \( M \) be a denumerable structure for \( L \).

\( M \) is said to be \( \Sigma \)-universal provided that every structure in \( \Sigma \) can be embedded in \( M \).

\( M \) is said to be \( \Sigma \)-homogeneous provided that whenever \( G, H \) are substructures of \( M \) which are isomorphic to members of \( \Sigma \) and when \( f : G \to H \) is an isomorphism then there is an automorphism \( g \) of \( M \) which extends \( f \) (i.e. \( f \circ g \)).

\( M \) is said to be \( \Sigma \)-generic provided that \( M \) is \( \Sigma \)-homogeneous, \( \Sigma \)-universal and every finite subset of \( M \) is contained in a substructure of \( M \) isomorphic to a member of \( \Sigma \).
The following lemma provides a convenient test to determine whether a model \( M \) is \( \Sigma \)-homogeneous.

**Lemma 3.1** Assume that \( \Sigma \) is closed under isomorphism and that every finite subset of \( M \) is contained in a substructure of \( M \) belonging to \( \Sigma \). Then \( M \) is \( \Sigma \)-homogeneous just in case whenever \( H_0, K_0, H \) are substructures of \( M \) belonging to \( \Sigma \), \( H_0 \subseteq H \) and \( g : H_0 \to K_0 \) is an isomorphism there is a substructure \( K \) of \( M \) with \( K_0 \subseteq K \) and an isomorphism \( \hat{g} : H \to K \) such that \( g \subseteq \hat{g} \).

**Proof.** Necessity is trivial. The proof of sufficiency uses a form of back and forth argument. Let \( H_0, K_0 \) be substructures of \( M \) and let \( g_0 : H_0 \to K_0 \) be an isomorphism. Let \( \{a_k : k < \omega\} \) enumerate \( M \). We extend \( g_0 \) to an automorphism as follows. Construct by recursion sequences \( \langle H_n : n \in \omega \rangle \), \( \langle K_n : n \in \omega \rangle \) and \( \langle g_n : n \in \omega \rangle \) where for \( n \in \omega \), such that \( M = \bigcup H_n = \bigcup K_n \) and for \( n \in \omega \)

\[
H_n \subseteq H_{n+1}, \quad K_n \subseteq K_{n+1} \quad \text{and} \quad g_n \subseteq g_{n+1}.
\]

For \( n \) even we may choose \( k \) minimal such that \( a_k \notin H_n \) and choose \( H_{n+1} \supseteq H_n \cup \{a_k\} \) such that \( H_{n+1} \in \Sigma \) and \( H_{n+1} \subseteq M \). Then we may take \( K_{n+1} \supseteq K_n \) and \( g_{n+1} \supseteq g_n \) so that \( g_{n+1} : H_{n+1} \to K_{n+1} \) is an isomorphism. For \( n \) odd we may choose \( k \) minimal such that \( a_k \notin K_n \), \( K_{n+1} \supseteq K_n \cup \{a_k\} \) with \( K_{n+1} \in \Sigma \) and \( K_{n+1} \subseteq M \). We may now apply the condition to \( K_n, H_n, K_{n+1} \) and \( g^{-1} \) to obtain \( H_{n+1} \) and \( g_{n+1} \).

In the next lemma we provide a sufficient condition for the existence of a \( \Sigma \)-generic structure.
Lemma 3.2 Let $\Sigma$ be closed under isomorphism and assume that $\Sigma$ has AP and that there is a structure $S$ in $\Sigma$ which can be embedded in every member of $\Sigma$. Then there is a $\Sigma$-generic structure $M$.

Proof. If there is a bound on the cardinality of members of $\Sigma$ let $M$ be a member of $\Sigma$ of largest cardinality. Without loss of generality we may take $S \subseteq M$. Let $G \in \Sigma$. There is no loss in assuming that $S \subseteq G$ and $G \cap M = S$. Since $\Sigma$ has AP there is $N$ in $\Sigma$ and there are embeddings $f_0 : M \rightarrow N$ and $f_1 : G \rightarrow N$. But then $f_0$ is the identity and $f_1$ is an embedding of $G$ in $M$.

Thus $M$ is $\Sigma$-universal. Now assume that $H,K$ are substructures of $M$ which belong to $\Sigma$ and let $f : H \rightarrow K$ be an isomorphism. If $H = M$ we are done. For an application of AP consider the identity mapping $1_H : H \rightarrow M$ and $f : H \rightarrow M$. There are therefore $N \in \Sigma, g_1 : M \rightarrow N, g_2 : M \rightarrow N$ such that $g_1, g_2$ are embeddings and $g_1 \circ 1_H = g_2 \circ f$. But then $g_1, g_2$ are isomorphisms and there is no loss of generality in assuming $g_1$ is the identity mapping on $M$.

The required automorphism of $M$ is then $g_2^{-1}$. Thus $M$ is $\Sigma$-generic.

If there is no bound on the cardinality of members of $\Sigma$ we proceed as follows. The collection of isomorphism types of structures in $\Sigma$ is countable so we may choose a sequence $<S_n, n \in \omega>$ of members of $\Sigma$ such that $S_0 = S$ and if $G \in \Sigma$ then $G \cong S_n$ for some $n$. Now construct an increasing sequence $<M_n : n \in \omega>$ of members of $\Sigma$ such that for all $n \in \omega, M_n \subset M_{n+1}$ and also
(i) \( M_0 = S_0 \)

(ii) \( \forall n \forall H \forall k \exists g ( H \subseteq M_n \land f : H \rightarrow S_k \) is an embedding such that there is no embedding \( g : S_k \rightarrow M_m \) making \( g \circ f = i_H \).

At stage \( m+1 \) choose \( n \leq m, H \subseteq M_n \), and an embedding \( f : H \rightarrow S_k \) such that there is no embedding \( g : S_k \rightarrow M_m \) making \( g \circ f = i_H \).

Further, make these choices so as to minimize \( n+k \). By an application of AP we may choose \( M_{m+1} \) such that \( M_m \subseteq M_{m+1} \) and there is an embedding \( g : S_k \rightarrow M_{m+1} \) making \( g \circ f = 1_H \).

To see that the sequence \( \{ M_n : n \in \omega \} \) satisfies (ii) assume that \( n+k \) is minimal such that there are \( H \subseteq M_n \) and \( f : H \rightarrow S_k \) with no embedding \( g : S_k \rightarrow M_m \) with \( g \circ f = 1_H \) for any \( m \). There are a finite number, say \( \ell \), of pairs \( (H', f') \) where \( H' \subseteq M_n \) and \( f' : H' \rightarrow S_{k'} \) is an embedding and \( n'+k' \leq n+k \). Thus we may choose \( m \leq n \) such that for each such \( (H', f') \) with \( n'+k' < n+k \) there is an embedding \( g' : S_{k'} \rightarrow M_m \) with \( g' \circ f' = 1_H \). The construction ensures that there must be an embedding \( g : S_k \rightarrow M_{m+\ell} \) making \( g \circ f = 1_H \), a contradiction.

Let \( M = \bigcup_{n \in \omega} M_n \). Since \( S_0 \) may be embedded in each member of \( \Sigma \)

(ii) ensures that \( M \) is \( \Sigma \)-universal. It is easy to see from (ii) and Lemma 3.1 that \( M \) is \( \Sigma \)-homogeneous.

Thus \( M \) is \( \Sigma \)-generic and countable and the lemma is proved.

A back and forth argument which closely resembles the proof of Lemma 3.1 provides us with a uniqueness result for \( \Sigma \)-generic structures.
Lemma 3.3 Any two $\Sigma$-generic structures are isomorphic.

Proof. Let $M, N$ be two $\Sigma$-generic structures. It suffices to show that if $H, K \in \Sigma$, $H \subseteq M$, $K \subseteq N$, and if $f : H \rightarrow K$ is an isomorphism then there is an embedding $g : H \rightarrow N$ such that $g \upharpoonright H_0 = f$. But there is an embedding $h$ of $H$ into $N$. Also $K_0$ and the image of $H_0$ under $h$ are isomorphic by $h \circ f^{-1}$. Since $N$ is $\Sigma$-homogeneous there is an automorphism $k'$ of $N$ which extends $h \circ f^{-1}$. We may take $g = (k')^{-1} \circ h$.

The next lemma establishes that construction of a class of structures $\Sigma$ satisfying the conditions of Lemma 3.2 and a condition on substructures is a way of generating $\omega_0$-categorical theories.

Lemma 3.4 If $M$ is $\Sigma$-generic and if there is a function $f : \omega \rightarrow \omega$ such that whenever $A$ is a subset of a structure in $\Sigma$ there is a member of $\Sigma$ containing $A$ whose cardinality is at most $f(|A|)$ then $\text{Th}(M)$ is $\omega_0$-categorical. Moreover if $\Sigma$ is closed under substructure then $\text{Th}(M)$ admits elimination of quantifiers.

Proof. Let $M$ be $\Sigma$-generic and let $f : \omega \rightarrow \omega$ be such that whenever $A \subseteq H \in \Sigma \exists K \in \Sigma A \subseteq K$ and $|K| \leq f(|A|)$. If $M$ is finite there is nothing to prove. Otherwise by Lemma 3.3 it suffices to show that any countable model of $\text{Th}(M)$ is $\Sigma$-generic. Since $L$ is finite the isomorphism type of any member of $\Sigma$ can be described by a formula and thus every model of $\text{Th}(M)$ is $\Sigma$-universal. We claim that any finite subset of a model of $\text{Th}(M)$ is contained in a substructure which belongs to $\Sigma$. Let $n$ be fixed. There are a finite number of
isomorphism types of structures which belong to $\Sigma$ and have cardinality at most $f(n)$. For each $\bar{a}$ a sequence of $n$ distinct elements from a structure $H$ in $\Sigma$ there is a formula $\phi \in L_n$ which fixes the isomorphism type of $H$ over $\bar{a}$; that is we have $K \models \phi(\bar{b})$ where $K$ is an $L$-structure and $\bar{b} \in K$ iff there is an embedding of $H$ into $K$ which takes $\bar{a}$ onto $\bar{b}$. It is easy to see that a finite disjunction of such formulas is valid in $M$ and the claim follows.

From Lemma 3.1 any countable model of $\text{Th}(M)$ is also $\Sigma$-homogeneous. The extension property in the criterion of Lemma 3.1 for each $H_0, K_0, H$ and $\sigma$ can be encoded in a formula which is valid in $M$.

Thus $\text{Th}(M)$ is $\omega_0$-categorical.

Now if $\Sigma$ is closed under substructure we may employ the following well-known proposition which we do not prove, to see that $\text{Th}(M)$ admits elimination of quantifiers.

**Proposition 3.1** Let $T$ be a countable complete theory. Then $T$ admits elimination of quantifiers just in case for each denumerable model $N$ of $T$ and for each pair of sequences $\bar{a}, \bar{b}$ of the same length, $n$ say, such that for each open formula $\psi \in L_n$ $N \models \psi(\bar{a}) \leftrightarrow \psi(\bar{b})$, there is a denumerable elementary extension $N'$ of $N$ with an automorphism carrying $\bar{a}$ to $\bar{b}$.

When $\Sigma$ is closed under substructure $\Sigma$-homogeneity ensures that the conditions of Proposition 3.1 are fulfilled and therefore $\text{Th}(M)$ admits elimination of quantifiers.

This completes the proof of Lemma 3.4.
The following simple example shows that the uniform bound on the cardinality of the smallest member of \( \Sigma \) containing a subset \( A \) is really needed in Lemma 3.4.

Example 3.1 Let \( L \) be the language with one unary function symbol \( P \) and let \( \Sigma \) be the class of structures for \( L \) isomorphic to one of \( <n,P_n> \) for \( n \geq 1 \) where \( n = \{ m : m < n \} \) and \( P_n(m) = m-1 \) if \( m > 0 \) and \( P_n(0) = 0 \). The \( \Sigma \)-generic model is then \( <\omega,P_\omega> = \bigcup_{n<\omega} <n,P_n> \).

The following lemma provides a partial converse to Lemma 3.4 and strengthens the connection between classes of structures with AP and theories which admit elimination of quantifiers.

Lemma 3.5 Let \( M \) be a denumerable structure such that \( \text{Th}(M) \) admits elimination of quantifiers, and such that every finite subset of \( M \) is contained in a finite substructure of \( M \). Let \( \Sigma \) be the class of finite structures which can be embedded in \( M \). Then \( \Sigma \) has the amalgamation property, and \( M \) is \( \Sigma \)-generic.

Proof. Let \( A, B_0, B_1 \in \Sigma \) and let \( f_i : A \rightarrow B_i \) be embeddings for \( i = 0,1 \). There is no loss of generality in assuming that \( A, B_0, B_1 \) are substructures of \( M \) and that \( f_0 \) is the inclusion of \( A \) in \( B_0 \). Since \( A \) and \( B_1 \) are finite and \( L \) is finite there are open formulas which fix their isomorphism types as substructures of \( M \). But \( A, f_1(A) \) satisfy the same open formula and so \( f_1 \) is in fact elementary since \( \text{Th}(M) \) admits elimination of quantifiers. But then \( f_1^{-1} \) can be extended to an isomorphism of \( B_1 \) and an extension \( B_1' \) of
A. Now $(B_0 \cup B_1)$ generates a member of $\Sigma$. This argument also shows that $M$ satisfies the conditions of Lemma 3.1 and is $\Sigma$-homogeneous. Therefore $\Sigma$ has AP and $M$ is $\Sigma$-generic.
Chapter 4

Quantifier eliminable graphs

1. Examples

In this chapter we apply the results of Chapter 3 to graphs whose theories admit elimination of quantifier. Theories are assumed to be complete theories with one binary relation symbol $R$, and to admit elimination of quantifiers. We also stipulate that each model be irreflexive, i.e. model $\forall v_0 \neg R_0 v_0$. This ensures that there is only 1-type.

In this section we shall present several simple examples and some basic definitions.

We first introduce abbreviations for some basic formulas. These abbreviations are $I, \Gamma, \Delta_0, \Delta_1, \Delta_2, U$ defined as follows:

\[
I_{xy} = \text{df } x = y
\]

\[
\Gamma_{xy} = \text{df } R_{xy} \land \neg R_{yx}
\]

\[
\Delta_0_{xy} = \text{df } (R_{xy} \land R_{yx}) \lor x = y
\]

\[
\Delta_1_{xy} = \text{df } (\neg R_{xy} \land \neg R_{yx}) \lor x = y
\]

\[
\Delta_2_{xy} = \text{df } (\Gamma_{xy} \lor \Gamma_{yx}) \lor x = y
\]

\[
U_{xy} = \text{df } x = x
\]

Several well-known examples are the following:

**DO** - The theory of $\langle \mathbb{Q}, \langle \rangle \rangle$ the rationals under the usual ordering.

**$E^n_k$** - The theory of an equivalence relation with $n$ equivalence classes of power $k$, where $1 \leq n$, $k \leq \omega$.

**$G^\Sigma_A$** - The $\Sigma(A)$ generic structure for the class $\Sigma(A)$ of finite graphs in which the $(n+3)$ cycle cannot be embedded for $n \in A \subset \omega$. 
The theory of the direct product of a model of DO and the complete graph on n points where n ∈ \(\omega\).

Given T we may form its dual \(\tilde{T}\) from the denumerable model \(G\) of \(\tilde{G}\) is the graph with universe \(G\) and relation:

\[R_G = (G \times G) \setminus \{(g,g) : g \in G\}\] . \(\tilde{T}\) is the theory of \(\tilde{G}\).

Clearly \(\tilde{T}\) also admits elimination of quantifiers. DO is self dual, while the duals of the other theories are not included in the list.

The following proposition gives a way of constructing new examples from known ones.

Proposition 4.1 Let i,j,k be 0,1,2 in some order and let \(T_0,T_1\) be such that \(\models_{T_0} (\Delta_{i=0} \vee \Delta_{j=0} \vee \Delta_{k=0})\) and \(\not\models_{T_1} (\Delta_{i=0} \vee \Delta_{j=0} \vee \Delta_{k=0})\).

Let \(A,B\) be denumerable models of \(T_0,T_1\) respectively. Define the structure \(C\) by \(|C| = |A| \times |B|\) and \(R_C = \{((a_0,b_0),(a_1,b_1)) : (a_0 = a_1 \wedge (b_0,b_1) \in R_B) \vee ((a_0,a_1) \in R_A)\}\). Define the structure \(D\) by \(|D| = |C|\) and \(R_D = \{((a_0,b_0),(a_1,b_1)) : ((b_0,b_1) \in R_B) \vee ((a_0,a_1) \in R_A \wedge b_0 = b_1)\}\). Then \(\text{Th}(C)\) and \(\text{Th}(D)\) admit elimination of quantifiers.

Proof: We present the proof that \(\text{Th}(C)\) admits elimination of quantifiers. The proof for \(D\) is similar. The main idea is that from \(C\) we can retrieve \(A\) and \(B\) in order to construct enough automorphisms in the Wreath product of the automorphism groups. It suffices to show that if \(c,d\) are two sequences of n-elements of \(C\) such that...
$\Gamma \vdash \Gamma \Delta c_m \leftrightarrow \Gamma \Delta d_m$

and

\[
\begin{align*}
C &\models \Delta \mu \Delta c_m \leftrightarrow \Delta \mu \Delta d_m, \\
C &\models \Gamma \Delta c_m \leftrightarrow \Gamma \Delta d_m
\end{align*}
\]

for $l, m < n$ and $\mu < 2$ then there is an automorphism $h$ of $C$ which carries $\vec{c}$ onto $\vec{d}$. The conditions ensure that there are distinct $a_0, \ldots, a_{k-1} \in A$ and distinct $a'_1, \ldots, a'_{k-1} \in A$ such that

\[
\{m : a_p \text{ is the first coordinate of } c_m\} = \{m : a'_p \text{ is the first coordinate of } d_m\}
\]

for $p < l$. and $\vec{a}$ realizes the same type as $\vec{a}'$ in $T_0$. We may choose an automorphism $f$ of $A$ which carries $\vec{a}$ onto $\vec{a}'$. Let $p < l$ and consider now $I_p = \{m : a_p \text{ is the first coordinate of } c_m\}$. Let $I_p = \{m_0, \ldots, m_{r-1}\}$ and consider the $r$-tuples $\vec{b}$ and $\vec{b}'$ from $B$ with $\vec{b}_s$ the second coordinate of $a_m$ and $\vec{b}'_s$ the second coordinate of $d_m$. Then $\vec{b}$ and $\vec{b}'$ realize the same type in $T_1$ and there is an automorphism $g_a$ of $B$ which sends $\vec{b}$ onto $\vec{b}'$. For $a \not\in \{a_0, \ldots, a_{k-1}\}$ let $g_a$ be the identity on $B$. Consider now the automorphism $h$ of $C$ defined by

\[
h((a,b)) = (f(a), g_a(b))
\]

which carries $\vec{c}$ onto $\vec{d}$.

Note that in the construction above $\{(c,d) : C \models \Delta k cd\}$ is an equivalence relation on $C$. 
2. Definable equivalence relations

In this section we examine the structure of the lattice of definable equivalence relations of a model of a fixed theory $T$. The main result is the following:

Theorem 4.1 If $T$ has an infinite model then no model has two incomparable definable equivalence relations.

The main theorem will follow easily from the lemmas of this section.

Fix a theory $T$, which has an infinite model. Note first that if $\Delta$ is a formula in $L_2$ which defines an equivalence relation in $T$ then $\Delta$ is equivalent to a disjunction of some of $\Delta_0 v_1, \Delta_{10} v_1, \Delta_{20} v_1$ and $\Delta_{01} v_1$.

The following lemma deals with the most difficult case in the proof of Theorem 4.1.

Lemma 4.1 Assume that $T \models \Delta_i v_1 \land \Delta_{i1} v_2 \rightarrow \Delta_i v_2$ for $i = 0, 1$. Then either $\Delta_0 v_1$ or $\Delta_{10} v_1$ is equivalent to $v_0 = v_1$ in $T$.

Proof. Let $A \models T$ and let $A \models \Delta_0 a b \land a \neq b$. Note that

$$A \models \forall v_0 (\Delta_1 b v_0 \land v_0 \neq b \rightarrow (\Gamma a v_0 \lor \Gamma v_0 a)) .$$

Claim. $A \models \exists v_0 (\Delta_1 b v_0 \land \Gamma a v_0)$

Otherwise we have

$$A \models \forall v_0 (\Delta_1 b v_0 \land b \neq v_0 \rightarrow \Gamma v_0 a) .$$

Let $A \models \Delta_0 a b \land \Delta_1 b c \land \Gamma c a \land \Delta_1 a d \land a \neq d$.
By I we see that $A \models \neg \Delta_0 \text{cd}$. Otherwise $(c,d)$ realizes the same type as $(a,b)$ yielding $A \models \Gamma \text{ac}$, contradiction. Also it is trivial that $A \models \neg \Delta_1 \text{cd} \land c \neq d$ since $\Delta_1 v v_1(A)$ is an equivalence relation. Thus $A \models (\Gamma \text{cd} \lor \Gamma \text{dc})$. Suppose $A \models \Gamma \text{cd}$, then $(c,d)$ and $(c,a)$ realize the same type and we have $e \in A$ with $A \models \Delta_1 \text{ce} \land \Delta_0 \text{de}$. It is easy to see that $e$ is none of $a,b,c,d$. Since $\Delta_1 v v_1(A)$ is an equivalence relation and I holds $A \models \Gamma \text{ea}$. Comparing the triples $(a,b,c)$ and $(e,d,a)$ we see that $A \models \Gamma \text{ae}$ since I must hold with $e,d$ for $a,b$ respectively. This contradiction leaves only the possibility that $A \models \Gamma \text{dc}$. Now $(c,a)$ and $(d,c)$ realize the same type whence there exists $f$ such that $A \models (\Delta_1 \text{fd} \land \Delta_0 \text{fc})$. Comparing the triples $(a,b,c)$ and $(c,f,a)$ we get $A \models \Gamma \text{ac}$, a contradiction. This establishes the claim.

Clearly the above argument can be modified by exchanging the roles of $\Delta_0$ and $\Delta_1$ to establish that

$$A \models \exists v_0 (\Delta_0 \text{ab} \land \Delta_1 v_0 v_0 \land \Gamma v_0 a) .$$

Thus

$$\exists v_0 \exists v_1 \exists v_2 (\Delta_0 v_0 v_1 \land \Delta_1 v_1 v_2 \land \Gamma v_0 v_2) \quad (1)$$

and

$$\exists v_0 \exists v_1 \exists v_2 (\Delta_0 v_0 v_1 \land \Delta_1 v_1 v_2 \land \Gamma v_2 v_0) \quad (2)$$

are both theorems of $\mathcal{T}$.

Fix $a,b,c \in A$ such that $A \models \Delta_0 \text{ab} \land \Delta_1 \text{bc} \land \Gamma \text{ac}$ for the remainder of the proof. By (2) and elimination of quantifiers there exists $d$ such that $A \models \Delta_1 \text{ad} \land \Delta_0 \text{dc}$. Note that since $\Delta_0 v_0 v_1(A)$ and $\Delta_1 v_0 v_1(A)$ are both equivalence relations
\[ a \models \exists ! v_0 (\Delta_1 a v_0 \land \Delta_0 v_0 c) \tag{3} \]

Now it will be established that
\[ T \models \exists v_0 \exists v_0 \exists v_0 \exists v_0 (\Delta_0 0 v_0 2 \land \Delta_1 0 v_0 1 \land \Delta_2 0 v_0 3 \land \Delta_0 v_0 3 \land \Delta_1 v_0 1 \land \Delta_2 v_0 2). \tag{4} \]

We require the following lemma which will be proved later.

Lemma 4.2 Suppose that the hypothesis of Lemma 4.1 holds and that the conclusion fails, then for each \( n \leq \omega \)
\[ T \models \forall v_0 \exists \geq n v_1 (\Delta_0 0 v_0 1) \land \forall v_0 \exists \geq n v_1 (\Delta_1 0 v_1). \]

If (4) fails, then
\[ a \models \forall v_0 (\Delta_1 a v_0 \land \Delta_0 c v_0 \rightarrow \Gamma b v_0). \tag{5} \]

But there is a unique \( d \) such that \( a \models \Delta_1 ad \land \Delta_0 cd \). The idea is to exploit the existence of this unique point and the condition (5) to define a partition of \( \{ e : a \models \Delta_1 be \} \) with parameters \( a, b \), which is independent of \( a \) in the sense that if \( a \models \Delta_1 a' a \) and \( a' \) is substituted for \( a \) the same partition will result. The existence of many points in the \( \Delta_1 v_0 v_1(A) \) equivalence class of \( b \) will then furnish a contradiction. Let \( \varphi(z) \) be
\[ \Delta_1 b z \land \exists v_1 (\Delta_1 a v_1 \land \Delta_0 v_1 z \land \Gamma b v_1). \]

From Lemma 4.2 \( S = \{ e : a \models \Delta_1 be \} \) is infinite and without loss of generality \( S' = \{ e : a \models \varphi(e) \} \) is also infinite. This is because otherwise we may apply a similar argument to \( \{ e : a \models \neg \varphi(e) \land e \neq b \land \Delta_1 eb \} \). This corresponds to replacing \( \Gamma b v_1 \) by \( \Gamma v_1 b \) in the definition of \( \varphi(z) \).
Let $e_0$ be such that $A \models \varphi(e_0)$ and let $f_0$ be such that
$A \models \Delta_0 f_0 \land \Delta_0 e_0 f_0 \land \Gamma f_0$. Then $A \models \Gamma e_0$ since (5) holds.

Let $\varphi'$ result from $\varphi$ by replacing $a$ by $f_0$. Then

$$A \models \forall v_0 (\varphi v_0 \leftrightarrow \varphi' v_0).$$

But $T$ admits elimination of quantifiers and there is an open formula $\theta$ with parameters $b, f_0$ such that

$$A \models \varphi' v_0 \leftrightarrow \theta.$$

Hence either

$$\forall e (e \in S' \land e \neq e_0 \leftrightarrow (A \models \Gamma f_0 e))$$

or

$$\forall e (e \in S' \land e \neq e_0 \leftrightarrow (A \models \Gamma e f_0)).$$

This is because there is a member of $S'$ other than $e_0$, and there is an element other than $b$ which satisfies $\Delta_1 b v_0 \land \neg \varphi(v_0)$.

Clearly there are $e_1, e_2$ in $S'$ such that for the corresponding $f_1, f_2$ either (6) holds when the subscript 0 is replaced by either 1 or 2, or (7) holds whichever of 1 and 2 replaces 0. In the first case $A \models \Gamma f_1 e_2 \land \Gamma f_2 e_1$ and in the second $A \models \Gamma e_1 f_2 \land \Gamma e_2 f_1$.

Thus (4) holds in either case.

This argument does not use the full strength of Lemma 4.2. One needs only that
$T \models \exists y_1 \exists y_3 (\Delta_0 y_0 y_1 \land \exists y_3 (\Delta_1 y_0 y_1 \land \Delta_1 y_1 y_3 \land \Gamma v_3 y_3 \land \Gamma v_1 y_1))$.

The argument used for (4) when applied to $T$ gives

$$T \models \exists y_0 \exists y_3 \exists y_2 \exists y_3 (\Delta_0 y_0 y_2 \land \Delta_0 y_1 y_3 \land \Delta_1 y_0 y_1 \land \Delta_1 y_1 y_3 \land \Gamma y_0 y_3 \land \Gamma y_1 y_3) \land (8)$$

From (4) and (8) however we easily derive a contradiction for we must have

$$A \models \exists y_1 (\Delta_0 y_1 c \land \Delta_1 a y_1 \land \Gamma b y_1) \land \exists y_1 (\Delta_0 y_1 c \land \Delta_1 a y_1 \land \Gamma y_1 b).$$

This contradicts uniqueness in (3), and completes the proof of Lemma 4.1.
We now turn to the proof of Lemma 4.2.

Assume the hypothesis of Lemma 4.1 holds and that the conclusion fails. Let \( A \models T \) and let \( a \in A \). It will suffice to show that

\[
S_0 = \{a' : A \models A_0a a'\}
\]

is infinite, for the same argument applied to \( T \) shows that \( S_1 = \{b : A \models A_1ab\} \) is also infinite. Assume for contradiction that \( A \) is finite and let \( A = \{a = a_0, a_1, \ldots, a_m\} \).

One may now establish as follows that \( S_1 \) is infinite. It is easy to see that at least one of \( \{c : A \models \Gamma ac\} \) and \( \{c : A \models \Gamma ca\} \) must be infinite. Assume that \( \{c : A \models \Gamma ac\} \) is infinite, the argument given below can easily be adapted for the other eventuality.

In the proof of Lemma 4.1 we showed that (1) holds, i.e.

\[
T \models \exists v_0 \exists v_1 \exists v_2 (A_0 v_0 \land A_1 v_1 \land \Gamma v_0 \land \Gamma v_1).
\]

Because of elimination of quantifiers this ensures that for each \( c \) with \( A \models \Gamma ac \) there is \( j \leq m \) such that \( A \models A_1 a_j c \). But by the pigeon-hole-principle there is some \( j \leq m \) with \( A \models A_1 a_j c \) for infinitely many \( c \). But then \( S_1 \) is infinite since \( a_0 \) and \( a_j \) realize the same 1-type in \( T \).

Consider now the equivalence relation \( \Theta \) defined on \( S_1 \) by the formula \( \Theta_0 v_1 \)

\[
A_1 a v_0 \land \forall v_2 (A_0 a v_2 \iff (Rv_0 v_2 \iff Rv_1 v_2) \land (Rv_2 v_0 \iff Rv_2 v_1)).
\]

Given \( j \leq m \), we can define an equivalence relation \( \Theta_j \) on \( S_1 \) where two elements of \( S_1 \) are equivalent iff they realize the same 1-type over \( a_j \). \( \Theta \) is then the intersection of the \( \Theta_j \).
Consider now the partition of $S_1$ induced by $\Theta$. Clearly $\{a\}$ is one equivalence class. Since $T$ admits elimination of quantifiers the classes are either $\{a\}$ and $S_1 \setminus \{a\}$ or all singletons $\{e\}$, $e \in S_1$. The latter is impossible because it would make $\Theta$ the identity on $B$ yet from the form of $\Theta$ the number of classes in the partition is $\leq 2^m + 1$.

Consider the equivalence relation $\Lambda$ defined on $S_0$ by $\lambda v_0 v_1 :$

$$\Delta_0 a v_0 \land \Delta_1 a v_1 \land \forall v_2 (\Delta_1 a v_2 \rightarrow (Rv_0 v_2 \leftrightarrow Rv_1 v_2) \land (Rv_0 v_0 \leftrightarrow Rv_1 v_1))$$

Again $\{a\}$ is an equivalence class under $\Lambda$. Also the equivalence classes are either $\{a\}$ and $\{a_1, \ldots, a_m\}$ or $\{a_0, \ldots, a_m\}$. The latter can only occur if $m + 1 \leq 2^m + 1 = 3$. Thus either

$$\Delta_0 a v_0 \land \Delta_1 a v_1 \rightarrow (\Gamma v_0 v_1 \lor a = v_0)$$

or

$$\Delta_0 a v_0 \land \Delta_1 a v_1 \rightarrow (\Gamma v_1 v_0 \lor a = v_0)$$

is valid in $A$, or $m = 2$ and without loss of generality

$$A \models \forall v_1 (\Delta_1 a v_1 \rightarrow (\Gamma v_1 a_1 \land \Gamma a_2 v_1) \lor a = v_1).$$

The first two cases occur only when the $\Lambda$ equivalence classes are $\{a\}$ and $\{a_1, \ldots, a_m\}$ and are impossible from (1) and (2) which were established in the proof of Lemma 4.1. The third case is also impossible because $a_1$ would be definable from $a$. This completes the proof of Lemma 4.2.

Thus we see that if $T$ has an infinite model then $\Delta_0 v_0 v_1$ and $\Delta_2 v_0 v_1$ cannot both be nontrivial equivalence relations. In the next lemma we show that neither both of $\Delta_0 v_0 v_1$ and $\Delta_2 v_0 v_1$ nor both of $\Delta_1 v_0 v_1$ and $\Delta_2 v_0 v_1$ can be nontrivial equivalence relations.
Lemma 4.3 Let \( i \in \{0, 1\} \). If \( \Delta_i v_0 v_1 \) and \( \Delta_2 v_0 v_1 \) both define equivalence relations in \( T \) and \( T \models \exists v_0 \exists v_1 (v_0 \neq v_1 \land \Delta_i v_0 v_1) \) then \( \Delta_2 v_0 v_1 \) is equivalent to \( v_0 = v_1 \).

Proof. Assume for contradiction that \( T \models \exists v_0 \exists v_1 (v_0 \neq v_1 \land \Delta_2 v_0 v_1) \). Let \( A \models T \) and let \( a, b, c \) be distinct such that

\[
A \models \Delta_1 ab \land \Delta_2 bc.
\]

Then \( A \models \Delta_1 ac \) follows from the hypothesis of the lemma. Without loss of generality suppose that \( A \models \Gamma bc \). There exists \( c' \) such that \( A \models \Gamma c'b \). From the hypothesis \( A \models \Delta_1 ac' \). Since \( (a, c) \) and \( (a, c') \) realize the same type and \( A \models \exists v_0 (\Gamma c'v_0 \land \Delta_1 av_0) \) there exists \( b' \) such that \( A \models \Gamma cb' \land \Delta_1 ab' \). Then we have \( A \models \Delta_2 bb' \) and \( A \models \Delta_1 ab \land \Delta_1 ab' \land b \neq b' \), contradiction. This proves Lemma 4.3.

From Lemmas 4.1 and 4.3 no two of \( \Delta_0 v_0 v_1 \), \( \Delta_1 v_0 v_1 \) and \( \Delta_2 v_0 v_1 \) can be nontrivial equivalence relations in \( T \). We use this observation to establish the theorem, that is to show that the definable equivalence relations of \( T \) form a chain. For this it suffices to show that the following two statements are true:

(I) Let \( i, j, k \) be \( 0, 1, 2 \) in some order and that

\[
(\Delta_i v_0 v_1 \lor \Delta_j v_0 v_1) \text{ and } (\Delta_k v_0 v_1)
\]

define equivalence relations in \( T \). Then one of these formulas is equivalent to \( v_0 = v_1 \).

(II) Let \( i, j, k \) be \( 0, 1, 2 \) in some order and that

\[
T \models \exists v_0 \exists v_1 (v_0 \neq v_1 \land \Delta_{\lambda} v_0 v_1) \text{ for } \lambda = 0, 1, 2. \text{ Then}
\]

not both of \( (\Delta_i v_0 v_1 \lor \Delta_j v_0 v_1) \) and \( (\Delta_i v_0 v_1 \lor \Delta_k v_0 v_1) \)

can define an equivalence relation in \( T \).
To see I let $A \models T$ and $a,b,c$ be distinct such that $A \models \Delta \kappa ab$ and $A \models \Delta \iota ac \lor \Delta \jmath ac$ then either $A \models \Delta \kappa bc$ or $A \models (\Delta \iota \lor \Delta \jmath)bc$ a contradiction.

For II let $a,b,c$ be distinct members of $A \models T$ such that $A \models \Delta \iota ac \land \Delta \kappa ab$. Then $A \models (\Delta \iota bc \lor \Delta \jmath bc)$ and $A \models (\Delta \jmath bc \lor \Delta \kappa bc)$ contradiction.

But this establishes Theorem 4.1. Lemmas 4.1 and 4.2 give interesting examples of nontrivial empty neighbourhoods, namely $N(\{E_0,E_{1,2}\}, \{\emptyset, D_{i,j}, D_{i,k}, D_{j,i}, D_{j,k}\})$ where $i,j,k$ are distinct in $\{0,1,2\}$ and for $\lambda, \mu$ distinct members of $\{0,1,2\}$ $D_{i,j}$ is the structure with universe $\{a,b,c\}$ which models $\Delta \mu ab \land \Delta \mu bc \land \Delta \mu ac$, and where $E_i$ is the structure with universe $\{a,b\}$ which models $\Delta \iota ab$.

3. Tournaments

In this section we turn our attention to those $T$ for which $\forall \nu \forall \nu_1 (\Delta_2 \nu \nu_1)$ is a theorem. A model of $T$ with two or more members is a tournament. In the first part of this section we characterize those $T$ which have a finite model. In so doing we present a lemma which will be useful in Section 5 where we make a beginning on the problem of characterizing all directed graphs without loops whose theories admit elimination of quantifiers.

Lemma 4.4 Let $\theta \in L_{n+1}$ generate an $(n+1)$-type in $T$ such that $T \models \theta \rightarrow v_0 \neq v_1$ for $1 \leq i \leq n$. Let $G \models T$ and let $a$ be an $n$-tuple from $G$ for which $G \models \exists v_0 \theta(v_0,a)$. Let $H$ be the substructure of $G$ determined by $\theta(v_0,a)(G)$. Then $\text{Th}(H)$ also
admits elimination of quantifiers.

Proof. It suffices to show that if \( H_0 \) and \( H_1 \) are finite substructures of \( H \) and if \( f : H_0 \to H_1 \) is an isomorphism then there is an automorphism of \( H \) which extends \( f \). But consider the substructures of \( G \) determined by \( H_0 \cup \text{Rng } \bar{a} \) and \( H_1 \cup \text{Rng } \bar{a} \) and the extension of \( f \) which leaves \( \bar{a} \) fixed. This is an isomorphism of two finite substructures of \( G \) and can thus be extended to an automorphism of \( G \). This automorphism fixes \( \bar{a} \) and therefore the restriction of it to \( H \) is an automorphism of \( H \) which extends \( f \). This completes the proof.

For the remainder of this section we assume that

\[ T \models \forall \forall \forall (v_0 \exists \exists \exists (v_0 \neq v_1)). \]

Let \( A \models T \) and \( a_0, a_1, a_2 \) be three members of \( A \). We say that \( \{a_0, a_1, a_2\} \) is a 3-cycle in \( A \) just in case \( A \models Ra_0 a_1 \land Ra_1 a_2 \land Ra_2 a_0 \). We say that \( \{a_0, a_1, a_2\} \) is a transitive triple just in case \( A \models \forall \{a_0, a_1, a_2\} \exists \forall \forall \forall (v_0 \land \forall v_1 \land \forall v_2 \land (Rv_0 v_1 \land Rv_1 v_2 \to Rv_0 v_2)). \) The statement "\( A \) is a 3-cycle" has the obvious meaning.

We may now state the first of the two major results of this section.

**Theorem 4.2** If \( T \) has a finite model then the only possibility for \( T \) is the theory of a 3-cycle.

Proof. Let \( G \models T \) and let \( a \in G \). By Lemma 4.4 we see that

\( C_a = \{ b : G \models R \text{ab} \} \) and \( I_a = \{ b : G \models R \text{ba} \} \) determine substructures of \( G \) which either have one member or are finite tournaments whose theories admit elimination of quantifiers. Since the cardinalities
of \( O_a \) and \( I_a \) are independent of the choice of \( a \) and since
\[
\sum_{a \in G} |O_a| = \sum_{a \in G} |I_a|,
\]
we see that \( |O_a| = |I_a| \). But then \( |G| = 2m+1 \)
where \( m = 1 \) or \( m \) is the cardinality of a finite tournament whose
theory admits elimination of quantifiers. But then the cardinality
of \( G \) must be \((2^k - 1)\) for some \( k \geq 2 \), and for all \( k' \) where
\( 2 \leq k' \leq k \) there must be a tournament of power \((2^{k'} - 1)\) whose theory
admits elimination of quantifiers. It will thus suffice to show that
there is no such tournament of power seven.

For contradiction assume that \( G \) is a tournament of power seven
whose theory admits elimination of quantifiers. Let the universe of
\( G \) be \( \{a_i : i < 7\} \) where \( O_{a_0} = \{a_1, a_2, a_3\} \) and \( I_{a_0} = \{a_4, a_5, a_6\} \).
By the above there is no loss in assuming that \( G \models Ra_{a_2} \land Ra_{a_3} \land
Ra_{a_1} \) and that \( G \models Ra_{a_5} \land Ra_{a_6} \land Ra_{a_4} \). Now there is no loss
of generality in assuming that \( G \models Ra_{a_4} \land Ra_{a_5} \) since by inspection
there must be two elements in \( I \cap O \). Then \( G \models Ra_{a_6} \). Now
consider the edge \((a_0, a_1)\). To each \( j \) with \( 2 \leq j \leq 6 \) we may associate
an ordered pair \( p_j = (k, l) \) from \{0, 1\} determined by
\[
G \models (Ra_{a_0} a_j)^k \land (Ra_{a_1} a_j)^l.
\]
Then these pairs are \((0,0), (0,1), (1,0), (1,0), \) and \((1,1)\) in
order. Now it is obvious that any automorphism of \( G \) which fixes
\((a_0, a_1)\) must also fix \( a_2, a_3 \) and \( a_6 \) since they are determined
by the pairs \( p_2, p_3 \) and \( p_6 \) respectively. Also there must be an
automorphism which fixes \((a_0, a_1)\) and carries \( a_4 \) to \( a_5 \). This
must interchange them since it fixes \( a_0, a_1, a_2, a_3 \) and \( a_6 \).
This completes the proof of Theorem 4.2. We are grateful to A.H. Lachlan for helping us simplify the above proof.

It is natural to ask what examples exist which have infinite models. There are, of course, DO and the model generic for the class of all finite tournaments. There is also another countable model in which the successors of a point are linearly ordered by $R$. The countable model $C$ may be described in the following way. First choose a countable dense subset $|C|$ of the unit circle in the plane with the property that if $\alpha \in |C|$ then the opposite end of the diameter through $\alpha$ does not belong to $|C|$. Now let

$$R_C = \{ (\alpha, \beta) \in C \times C : \text{signed angle subtended by the arc } \overset{\rightarrow}{\beta\alpha} \text{ lies in } (0, \pi) \},$$

that is $\beta$ lies on the semicircle clockwise from $\alpha$ and is not $\alpha$ itself. Let $C = <|C|, R_C>$. We claim that $Th(C)$ admits elimination of quantifiers and that the set of successors of a point is linearly ordered by $R_C$.

The second part of the claim is easy to see for if $(\alpha, \beta_1), (\alpha, \beta_2), (\alpha, \beta_3) \in R_C$ then whether $(\beta_i, \beta_j) \in R_C$ depends on the magnitude of $\overset{\rightarrow}{\beta_i\alpha}$ and $\overset{\rightarrow}{\beta_j\alpha}$. To see that $Th(C)$ admits elimination of quantifiers we shall apply the results of Chapter 3. By Lemma 3.4 it suffices to show that $C$ is $\Sigma$-homogeneous where $\Sigma$ is the class of finite structures which can be embedded in $C$. By Lemma 3.1 to establish this it is sufficient to show that if

$\bar{\alpha} = <\alpha_0, \ldots, \alpha_{n-1}>$ and $\bar{\beta} = <\beta_0, \ldots, \beta_{n-1}>$ are two sequences from $C$ such that the mapping $g$ taking $\alpha_i$ to $\beta_i$ for $i \leq n$ is an
isomorphism, and if $\alpha \in C$ then there is $\beta \in C$ such that the extension to $\bar{a} \cap \langle \alpha \rangle$ of $g$ which takes $\alpha$ to $\beta$ is an isomorphism. If $n = 0$ or $\alpha \in \{\alpha_0, \ldots, \alpha_{n-1}\}$ this is trivial. Otherwise the atomic formulae satisfied by $\langle \alpha_i, \alpha \rangle$ permit one to translate the problem into a question about an intersection of open intervals in the rationals, when it can be seen to be the case that $\beta$ must exist as above.

It turns out that the rationals and $C$ are the only two infinite models in which the successors of a member are linearly ordered.

Theorem 4.3 If $T$ satisfies the following three conditions then $C$ is a model of $T$.

(i) $T$ has an infinite model.

(ii) There is a three cycle in $T$.

(iii) The successors of a point are linearly ordered.

Proof. Let $G$ be countable and let $G \models T$. We shall show that $G \models C$. First however we establish the following two claims.

Claim 1. The predecessors of a point are linearly ordered, that is

$$T \models \left[ \bigwedge_{1 \leq i \leq 3} Rv_i v_0 \rightarrow Rv_i v_2 \rightarrow Rv_2 v_3 \rightarrow Rv_1 v_3 \right].$$

For a contradiction assume that

$$G \models Rb_0 b_1 \land Rb_1 b_2 \land Rb_2 b_0 \land \bigwedge_{i \leq 3} Rb_i a.$$ 

Now since $T$ admits elimination of quantifiers there is $c \in G$ such that $G \models Rac \land Rcb_0$. It is easy to see that $c$ is distinct from $a, b_0, b_1, b_2$ since $G \models Rac$. Consider $\{b_2, c\}$. Since $G \models Rb_2 c$ would mean that the three cycle $\{a, c, b_0\}$ was in the set of successors
of \( b_2 \) we must have that \( G \models \neg Rcb_2 \). Also, since \( G \models \neg Rb_1c \) would mean that the three cycle \( \{b_2, a, c\} \) lies in the outset of \( b_1 \) we have \( G \models \neg Rcb_1 \). But then \( b_0, b_1, b_2 \) lie in the outset of \( c \), contradiction.

**Claim 2.** Let \( a \in G \) and let \( B_0 = \{ b : G \models Rab \} \) and let \( B_1 = \{ b : G \models Rba \} \). Then \( \langle B_1, R_G \upharpoonright B_1 \rangle \models DO \) for \( i = 0, 1 \).

This claim is an easy consequence of elimination of quantifiers. To give the idea we show that \( B_i \) is infinite for \( i = 0, 1 \). Clearly one of \( B_0, B_1 \) is infinite, and each is linearly ordered by \( R_G \). Suppose \( B_0 \) is infinite. Then there are \( b_0, b_1, b_2 \) such that \( G \models \bigwedge_{i < 3} Rab_i \land Rb_0 \lor b_1 \land Rb_1 \lor b_2 \land Rb_2 \). Now \( a, b_2 \) realize the same 1-type and it easily follows that \( R_G \) is a dense order on \( B_0 \) and \( B_1 \) and they are infinite.

It is clear that \( G \) is the generic structure for the class of finite structures which can be embedded in \( G \). By Lemma 3.3 to show \( G \cong C \) it will suffice to show that if \( H \) is a finite structure then \( H \) can be embedded in \( G \) just in case it can be embedded in \( C \).

We show the condition holds by induction on the cardinality \( n \) of \( H \). When \( n \leq 3 \) the condition holds trivially. Let \( H \) be given and assume that whenever \( |H| \geq 4 \) and whenever \( |H_0| < |H| \) then \( H_0 \) can be embedded in \( G \) just in case it can be embedded in \( C \).

Case 1. \( H \) is a linear order. Since \( B_0, B_1 \) are infinite this case is trivial.
Case 2. Otherwise there is a 3-cycle \( \{ \bar{h}_0, \bar{h}_1, \bar{h}_2 \} \) in \( H \), where
\[
H \models \Gamma_{\bar{h}_0} \land \Gamma_{\bar{h}_1} \land \Gamma_{\bar{h}_2} \land \Gamma_{\bar{h}_0}.
\]
We shall assume that \( H \subseteq G \) and show that \( H \) can be embedded in \( C \). The other implication is proved similarly. There is no loss of generality in assuming that \( h \in H \setminus \{ \bar{h}_0, \bar{h}_1, \bar{h}_2 \} \) and \( H \models (\Gamma_{\bar{h}_0} \leftrightarrow \Gamma_{\bar{h}_1}) \). Let \( H_0 \) be the substructure determined by \( H \setminus \{ h \} \) and let \( H_1 \) be that determined by \( H \setminus \{ h_2 \} \). By the induction hypothesis \( H_0 \) and \( H_1 \) can be embedded in \( C \). Since \( C \) is generic for the class of structures which can be embedded in it we may assume the embeddings agree on \( H_0 \cap H_1 \).

Let these embeddings be \( e_0 \) and \( e_1 \) respectively. Now
\[
C \models (\Gamma e_0(h)e_0(h) \leftrightarrow \Gamma e_0(h)e_0(h_1)).
\]
Thus \( e_0(h) \neq e_1(h_2) \). But also it entails that \( G \models \Gamma_{h_2} \) iff \( C \models \Gamma e_0(h)e_1(h_2) \). But then \( e_0 \cup e_1 \) is an embedding of \( H \) into \( C \).

This establishes Theorem 4.3. An open question (Q7) is just what other countable tournaments have theories which admit elimination of quantifiers.

4. Undirected graphs

In this section we turn to another case in which combinatorial arguments have characterized those finite structures whose theories admit elimination of quantifiers—symmetric irreflexive graphs. In this section we assume that the following formula is a theorem of the theory \( T \)
\[
\forall v \forall v_1 (R v v_1 \leftrightarrow R v_1 v_0).
\]
The models of such theories are just undirected graphs.

A. Gardiner [4; Th. 7, Th. 12] proved the following:

Theorem 4.4 Let $G \not\models T$ and assume that $G$ is finite. Then either $G$ or $\bar{G}$ is isomorphic to one of the following:

(i) $K^n_k$ - a model of $E^n_k$ for some $n, k<\omega$,

(ii) $P$, the pentagon,

or (iii) $L(K_3,3)$.

Here the pentagon $P$ is the structure with universe \{i:i < 5\} and with $R_p = \{(i,j) : i = j+1 \lor j = i+1 \lor (i = 4 \land j = 0) \lor (i = 0 \land j = 4)\};$ and $L(K_3,3)$ is the line graph on $K_3,3 = K^2_3$; the vertices of $L(K_3,3)$ are the unordered pairs \{a,b\} such that $(a,b) \in R_{K_3,3}$, and the edges of $L(K_3,3)$ are the pairs $(e,f)$ such that $e \cap f \neq \emptyset$. $L(K_3,3)$ is easily seen to be isomorphic to the product of two copies of the complete graph on three vertices. Note that $P$ and $L(K_3,3)$ are both self-dual.

Following Gardiner we call a graph $G$ ultrahomogeneous just in case it is $\Sigma$-homogeneous where $\Sigma$ is the class of finite structures which may be embedded in $G$.

In the remainder of this section we assume that $T$ has an infinite model. An interesting family of ultrahomogeneous graphs is provided by the methods of Chapter 3. Let $K_n$ be the complete graph with $n$ vertices. For $2 \leq n$ let $\Sigma_n$ be the class of undirected graphs which omit $K_n$. Then $\Sigma_n$ has arbitrarily large members, for example $K_m$ for $m \in \omega$. We may see that $\Sigma_n$ has the amalgamation property as follows.
If \( G, H_0, H_1 \in \Sigma_n \) and \( |H_0| \cap |H_1| = |G| \) then let \( H \) be the structure with \( |H| = |H_0| \cup |H_1| \) and \( R_H = R_{H_0} \cup R_{H_1} \). Then \( H \in \Sigma_n \) trivially. Now let \( G_n \) be a \( \Sigma_n \)-generic structure by Lemma 3.2. \( G_n \) is ultrahomogeneous.

The main result of this section is the following theorem which lends some further support to \( C^7 \).

Theorem 4.5 There are only four countable ultrahomogeneous undirected graphs which omit the triangle, \( K_3 \). The proof of the theorem is broken down into a sequence of lemmas. For the remainder of this section graph will mean undirected graph. Also fix \( G \) a countable ultrahomogeneous graph which omits \( K_3 \). Let \( \Sigma \) be the class of finite graphs which can be embedded in \( G \). Let \( S_2 \) be the graph \( \langle \{a_0, a_1, a_2\} \rangle \), where \( |S_2| = \{a_0, a_1, a_2\} \) and \( R_2 = \{(a_i, a_j) : i \neq j \wedge i \cdot j = 0\} \). Let \( M^* \) be the graph \( \langle b_0, b_1, b_2 \rangle \), i.e. \( \langle b_0, b_1, b_2 \rangle \), \( \langle b_1, b_2 \rangle \), \( \langle b_2, b_1 \rangle \).

Let \( \mathbb{K}_k^n \) be a model of \( \mathbb{E}_k^n \), that is an equivalence relation with \( n \) classes each containing \( k \) members. Recall the abbreviations given in §1.

Lemma 4.5 If \( M^* \notin \Sigma \) or \( S_2 \notin \Sigma \) then \( G \) is isomorphic to one of the following:

(i) \( K_1^\omega \)

(ii) \( K_2^\omega \)

(iii) \( K_2^2 \)
Proof. If $M^* \notin \Sigma$ then $\Delta_0 v_0 v_1(G)$ is an equivalence relation and the equivalence classes have the same cardinality. Since $K_3 \notin \Sigma$, there are at most two classes for if $g_1, g_2, g_3$ all belong to different classes we must have $(g_1, g_j) \notin R(G)$ for $i \neq j$ which is impossible. Thus $G$ is isomorphic to $K_1^\omega$ or $K_2^\omega$.

If $S_2 \notin \Sigma$ then $\Delta_0 v_0 v_1(G)$ is an equivalence relation all of whose equivalence class have the same cardinality. The fact that $K_3 \notin \Sigma$ ensures that each class contains at most two elements. Thus $G$ is either isomorphic to $K_1^\omega$ or $K_2^\omega$, and the proof of Lemma 4.5 is completed.

For the remainder of this section we assume that $M^*$ and $S_2$ belong to $\Sigma$. From Lemma 3.5 $\Sigma$ has the amalgamation property and we rely very heavily on this fact below. We shall show that $\Sigma = \Sigma_3$, whence $G$ will be isomorphic to $G_3$. To establish that $\Sigma = \Sigma_3$ some definitions are required.

For $n \geq 2$ define the $n$-star $S_n = \langle \{a_i : i \leq n\}, R_n \rangle$ where

$$R_n = \{(a_i, a_j) : i \neq j \land i \cdot j = 0\},$$

i.e. $S_n = \begin{array}{c}
  a_1
  \\
  a_0
  \\
  a_n
\end{array}$.

Let $I = K_1^1$, i.e. $I = \cdots$ and let $M = K_2^1$, i.e. $M = \begin{array}{c}
  a_1
  \\
  a_0
  \\
  a_n
\end{array}$.

Let $\{H_\alpha : \alpha \in \Lambda\}$ be a nonempty family of graphs where $H_\alpha = \langle |H_\alpha|, R_\alpha \rangle$. Define the disjoint union $<H, E> = \bigcup\{H_\alpha : \alpha \in \Lambda\}$ as follows:

$$|H| = \{(h, \alpha) : h \in H_\alpha \land \alpha \in \Lambda\}$$

and

$$E = \{((h, \alpha), (g, \beta)) : \alpha = \beta \land (h, g) \in R_\alpha\}.$$
If $\kappa$ is a cardinal and $H$ is a graph ($\kappa \cdot H$) is the disjoint union of $\kappa$ copies of $H$. There is no loss in assuming that $\cup$ is commutative and associative. If $H$ and $J$ are graphs then $(0 \cdot H) \cup J$ is just $J$. Note that $M^* \cong I \cup M$.

We shall complete the proof of Theorem 4.5 by proving the following four lemmas:

**Lemma 4.6** For all $n$ $(n \cdot I) \cup S_2 \in \Sigma$

**Lemma 4.7** $(n \cdot S_2) \in \Sigma$ for all $n \geq 1$

**Lemma 4.8** $(n \cdot S_m) \in \Sigma$ for all $n \geq 1$ and $m \geq 2$

**Lemma 4.9** $\Sigma = \Sigma_3$

Of course the theorem is immediate from Lemma 4.9 for which the other lemmas are a necessary preliminary.

**Proof of Lemma 4.6** We prove by induction that $(n \cdot I) \cup S_2 \in \Sigma$ and $(n+1) \cdot I) \cup M \in \Sigma$. With $n=0$ this is simply that $S_2$ and $M^*$ belong to $\Sigma$. The induction step follows from Propositions 1, 2, and 3 below.

**Proposition 1.** If $(n \cdot I) \cup S_2 \in \Sigma$ then $((n+1) \cdot I) \cup S_2 \in \Sigma$ or $(n \cdot I) \cup S_3 \in \Sigma$.

**Proof.** For an application of AP we may assume that $(n \cdot I) \cup S_2 \cong B_0$ where $|B_0| = N \cup \{a_0, a_1, a_2\}$ and $R(B_0) = \{(a_i, a_j) : i \neq j \& i \cdot j = 0\}$

Let $A$ be the substructure of $B_0$ generated by $N \cup \{a_1, a_2\}$. Let $b$ be new and define $B_1$ such that $|B_1| = N \cup \{a_1, a_2, b\}$, and $B_1 \cong (n+3) \cdot I$.

Now $A, B_0 \in \Sigma$. Also $B_1 \in \Sigma$ since from Ramsey's theorem we easily get that $\omega \cdot I$ can be embedded in $G$. 


Thus let \( C \in \Sigma \) and let \( f_0 : B_0 \rightarrow C \) and \( f_1 : B_1 \rightarrow C \) be embeddings such that \( f_0 \upharpoonright A = f_1 \upharpoonright A \). Then \( f_0(a_0) \neq f_1(b) \) since \( (a_0, a_1) \in R(B_0) \) while \( (b, a_1) \notin R(B_1) \). The disjunction depends on whether \( (f_0(a_0), f_1(b)) \in R(C) \).

The reader may find the proofs less difficult to follow from consideration of the figures which we shall present. The following figure corresponds to Proposition 1.

Figure 1.

Proposition 2. If \( (n \cdot I) \cup S_3 \in \Sigma \) and \( (((n+1) \cdot I) \cup M) \in \Sigma \) then \( ((n+1) \cdot I) \cup S_2 \in \Sigma \).

Proof. The figure for this proposition is figure 2.

Figure 2.
For an application of AP we may choose \( B_0 \cong (n \cdot I) \cup S_3 \) where
\[
|B_0| = N \cup \{a_0, a_1, a_2, a_3\} \quad \text{and} \quad R(B_0) = \{(a_i, a_j) : i \neq j \text{ and } i, j = 0\}.
\]
Let \( A = N \cup \{a_2, a_3\} \) and let \( b \) be new and let \( B_1 \cong ((n+1) \cdot I) \cup M \) where \( |B_1| = A \cup \{b\} \) and \( R(B_1) = \{(a_2, b), (b, a_2)\} \).

Now let \( C \in \Sigma \) and let \( f_0 : B_0 \to C \) and \( f_1 : B_1 \to C \) be embeddings such that \( f_0 \upharpoonright A = f_1 \upharpoonright A \). Then \( f_1(b) \neq f_0(a_0) \) since \((a_0, a_3) \in R(B_0)\) while \((b, a_3) \not\in R(B_1)\) and similarly \( f_1(b) \neq f_0(a_2) \) because of \( a_2 \). There is no loss of generality in assuming that \( f_0, f_1 \) are inclusions. Now \((a_0, b) \not\in R(C)\) since \( K_3 \not\in \Sigma \). There are now two cases:

Case 1. \((a_1, b) \not\in R(C)\)

and Case 2. \((a_1, b) \in R(C)\).

In Case 1 consider \( N \cup \{a_0, a_1, a_3, b\} \). This subgraph is isomorphic to \( ((n+1) \cdot I) \cup S_2 \).

In Case 2 consider \( N \cup \{b, a_1, a_2, a_3\} \) which is isomorphic to \( ((n+1) \cdot I) \cup S_2 \).

Proposition 3. If \(((n+1) \cdot I) \cup S_2 \in \Sigma \) then \(((n+2) \cdot I) \cup M \in \Sigma \).

Proof. For this proposition consider figures 3 and 4.

Figure 3.
For a first application of AP let $B_0 \cong ((n+1) \cdot I) \cup S_2$ where

$$|B_0| = N \cup \{b, a_0, a_1, a_2\} \quad \text{and} \quad R(B_0) = \{(a_i, a_j) : i \neq j \text{ and } i \cdot j = 0\}.$$  

Let $A = N \cup \{b, a_1, a_2\}$ and letting $c$ be new define $B_1 \cong B_0$ by

$$|B_1| = A \cup \{c\} \quad \text{and} \quad R(B_1) = \{(c, b), (b, c), (c, a_1), (a_1, c)\}.$$  

Now let

$$c \in \Sigma \quad \text{and} \quad f_0 : B_0 \to C \quad \text{and} \quad f_1 : B_1 \to C$$

be embeddings such that

$$f_0 \upharpoonright A = f_1 \upharpoonright A.$$  

Then $f_0(a_0) \neq f_1(c)$ and without loss of generality $f_0$, $f_1$ are inclusions. Since $K_3 \not\subseteq \Sigma$ we have $(a_0, c) \not\in R(C)$.

But then the substructure $C_0$, determined by $N \cup \{a_2, a_0, a_1, c\}$ belongs to $\Sigma$. For a second application of AP let $B = N \cup \{a_2, a_1, c\}$.

Let $d$ be a new individual and let $C_1 \cong ((n+1) \cdot I) \cup S_2$ be such that

$$|C_1| = B \cup \{d\} \quad \text{and} \quad R(C_1) = \{(a_1, c), (c, a_1), (a_1, d), (d, a_1)\}.$$  

Now let

$$D \in \Sigma \quad \text{and} \quad g_0 : C_0 \to D \quad \text{and} \quad g_1 : C_1 \to D$$

be embeddings with

$$g_0 \upharpoonright B = g_1 \upharpoonright B.$$  

Now $g_0(a_0) \neq g_1(d)$ since $(a_0, a_2) \in R(C_0)$ while $(d, a_2) \not\in R(C_1)$. Thus we may assume that $g_0, g_1$ are inclusions.

Since $K_3 \not\subseteq \Sigma, (a_0, d) \not\in R(D)$. Now the substructure determined by

$$N \cup \{c, d, a_0, a_2\}$$

is isomorphic to $((n+2) \cdot I) \cup M$.

This completes the proof of Lemma 4.6.

We now turn to the proof of Lemma 4.7: If $(n \cdot I) \cup S_2 \in \Sigma$ for all $n$ then for all $n \geq 1$ $(n \cdot S_2) \in \Sigma$. 

---

Figure 4.
Proof. We prove by induction on $n \geq 1$ that $((m \cdot I) \cup (n \cdot S')) \in \Sigma$ for all $m$. With $n=1$ this is simply the conclusion of Lemma 4.6.

The induction step involves three applications of AP and is greatly simplified by consideration of figures 5, 6, and 7.

Figure 5.

Figure 6.

Figure 7.
In the first two amalgamations a label is attached to an outer vertex of \( S_2 \) and with this a 3-star is created. Then the 3-star is used to create two 2-stars. For the reader who can follow the diagrams this should suffice. However to avoid any possible obscurity let \( m \geq 1 \) be fixed and fix \( K \cong ((m \cdot I) \cup ((n-1) \cdot S_2)) \).

Claim. \( K \cup (2 \cdot I) \cup S_3 \in \Sigma \).

Fix new entities \( a_0, a_1, a_2, b_0, b_1 \), and \( b \). For an application of AP let \( B_0 \) be determined by \( |B_0| = |K| \cup \{a_0, a_1, a_2, b_0, b_1\} \) and

\[
R(B_0) = R(K) \cup \{(a_i, a_j) : i \neq j \text{ and } i \cdot j = 0\}
\]

and let

\[
A = K \cup \{a_1, a_2, b_0, b_1\} \quad B_0 \cong ((m+2) \cdot I) \cup (n \cdot S_2) \quad \text{and so belongs to } \Sigma. \quad \text{Let } B_1 \text{ be such that } |B_1| = |A| \cup \{b\} \quad \text{and}
\]

\[
R(B_1) = R(K) \cup \{(a_2, b), (b, a_2)\}. \quad \text{Now } B_1 \in \Sigma.
\]

Let \( C \in \Sigma \) and let \( e_0 : B_0 \to C \) and \( e_1 : B_1 \to C \) be embeddings with \( e_0 \uparrow A = e_1 \uparrow A \). Now \( e_0(a_0) \neq e_1(b) \) since \( (a_0, a_1) \in R(B_0) \) while \( (b, a_1) \not\in R(B_1) \). Thus without loss \( e_0, e_1 \) are inclusions. \( (a_0, b) \not\in R(C) \) since \( K_3 \not\in \Sigma \).

For another application of AP let \( C_0 \) be the subgraph determined by \( K \cup \{a_0, a_1, a_2, b_0, b_1, b\} \). Let \( B = K \cup \{a_0, a_1, b, b_0, b_1\} \).

Let \( a_3 \) be new and let \( C_1 \) be such that \( |C_1| = B \cup \{a_3\} \) and

\[
R(C_1) = R(K) \cup \{(a_3, a_0), (a_0, a_3), (a_0, a_1), (a_1, a_0)\}. \quad \text{Then}
\]

\[
C_1 \cong ((m+3) \cdot I) \cup (n \cdot S_2) \in \Sigma.
\]

Now let \( D \in \Sigma \) and let \( f_0 : C_0 \to D \) and \( f_1 : C_1 \to D \) be embeddings with \( f_0 \uparrow B = f_1 \uparrow B \). By inspection \( f_0(a_2) \neq f_1(a_3) \) and without loss \( f_0, f_1 \) are inclusions. \( (a_3, a_2) \not\in R(D) \) since \( K_3 \not\in \Sigma \). But then the subgraph determined by \( K \cup \{a_0, a_1, a_2, a_3, b_0, b_1\} \) is isomorphic to \( K \cup (2 \cdot I) \cup S_3 \in K \), establishing the claim.
With another application of AP we may see that $K \cup (2 \cdot S_2) \in \Sigma$.

Let $D_0$ be the subgraph determined by $K \cup \{a_0, a_1, a_2, a_3, b_0, b_1\}$ in the above. Let $e$ be new and let $D_1$ be determined by

$$|D_1| = K \cup \{a_1, a_2, a_3, b_0, b_1, e\} \quad \text{and} \quad R(D_1) = R(K) \cup \{(a_3, e); (e, a_3), (b_0, e), (e, b_0), (b_1, e), (e, b_1)\}.$$

Let $E = D_0 \cap D_1$. Then $D_0, D_1 \in \Sigma$.

Let $F \in \Sigma$ and let $g_0 : D_0 \to F$ and $g_1 : D_1 \to F$ be embeddings with $g_0 \upharpoonright E = g_1 \upharpoonright E$. Then $g_0(a_0) \neq g_1(e)$ and we may take $g_0, g_1$ to be inclusions. The substructure of $F$ determined by $K \cup \{a_0, a_1, a_2, e, b_0, b_1\}$ is isomorphic to $(m+1) \cdot I$ $\cup$ $((n+1) \cdot S_2)$. This completes the proof of Lemma 4.7.

Now we turn to the proof of Lemma 4.8, that $(n \cdot S_m) \in \Sigma$ for $n \geq 1$ and $m \geq 2$.

Proof. The proof of this lemma is very similar to the first part of the proof of Lemma 4.7. Using the amalgamation property we tag one of the outer vertices of an $m$-star and then by another amalgamation produce an $(m+1)$-star. This is shown in figures 8 and 9.

Figure 8.
Formally we prove by induction on $m \geq 2$ that for all $n \geq 1$ $(n \cdot S_m) \in \Sigma$.

With $m = 2$ this is just the conclusion of Lemma 4.7. Assume that $(k \cdot S_m) \in \Sigma$ for all $k \geq 1$. Now we prove by induction on $n \geq 0$ that for all $k \geq 1$ $((k \cdot S_m) \cup (n \cdot S_{m+1})) \in \Sigma$. Assume that $((k \cdot S_m) \cup (n \cdot S_{m+1})) \in \Sigma$ for all $k \geq 1$. Let $K \equiv ((k \cdot S_m) \cup (n \cdot S_{m+1}))$ and let $a_0, \ldots, a_m$ be distinct new individuals. Define $B_0 \equiv (K \cup S_m)$ by $|B_0| = |K| \cup \{a_i : i \leq m\}$ and $R(B_0) = R(K) \cup \{(a_i, a_j) : i \neq j \text{ and } i \cdot j = 0\}$.

Let $A = K \cup \{a_1, \ldots, a_m\}$. Let $b$ be new and define $B_1$ by $|B_1| = A \cup \{b\}$ and $R(B_1) = R(K) \cup \{(b, a_i), (a_i, b)\}$. Then $B_1 \in \Sigma$.

By AP let $C_0 \in \Sigma$ and let $f_0 : B_0 \to C_0$ and $f_1 : B_1 \to C_0$ be embeddings for which $f_0 \upharpoonright A = f_1 \upharpoonright A$. Then $f_0(a_0) \neq f_1(b)$ since $(a_0, a_1) \in R(B_0)$ while $(b, a_1) \notin R(B_1)$. Thus we may assume that $f_0$ and $f_1$ are inclusions and then $(a_0, b) \notin R(C_0)$ since $K_3 \notin \Sigma$.

For another application of AP let $B = K \cup \{a_0, \ldots, a_{m-1}, b\}$ and let $a_{m+1}$ be new and let $C_1$ be such that $|C_1| = B \cup \{a_{m+1}\}$ and $R(C_1) = R(B) \cup \{(a_{m+1}, a_0), (a_0, a_{m+1})\}$. Then $C_1 \in \Sigma$. 
Let \( D \in \Sigma \) and let \( g_0, g_1 \) be embeddings of \( C_0 \) and \( C_1 \) into \( D \) respectively such that \( g_0^{-1}B = g_1^{-1}B \). Now \( g_0(a_m) \neq g_1(a_{m+1}) \) since \( (a_m, b) \in R(C_0) \) while \( (a_{m+1}, b) \notin R(C_1) \). Without loss \( g_0, g_1 \) are inclusions. Now \( (a_m, a_{m+1}) \notin R(D) \) and \( K \cup \{a_0, \ldots, a_m\} \) is isomorphic to \( (k \cdot S) \cup ((n+1) \cdot S + 1) \).

This completes proof of Lemma 4.8.

We now turn to the proof of Lemma 4.9, that is \( \Sigma = \Sigma_3 \).

Proof. It is clear that \( \Sigma \subseteq \Sigma_3 \). We prove by induction on the lexicographic order on pairs \( (|B|, |R(B)|) \) that if \( B \in \Sigma_3 \) then \( B \in \Sigma \).

Thus let \( H \in \Sigma_3 \) and assume that if \( B \in \Sigma_3 \) and \( B \) has fewer vertices than \( H \) or if \( |B| = |H| \) but \( B \) has fewer edges than \( H \) then \( B \in \Sigma \). We now consider three cases.

Case 1. There are elements \( h_0, h_1, h_2 \in H \) such that \( (h_0, h_1), (h_0, h_2) \in R(H) \) and such that the mapping \( f : (H \setminus \{h_1\}) \rightarrow (H \setminus \{h_2\}) \) which has \( f(h_2) = h_1 \), and \( f(h) = h \) for \( h \in H \setminus \{h_1, h_2\} \) is not an isomorphism.

By the induction hypothesis \( H \setminus \{h_1\} \) and \( H \setminus \{h_2\} \in \Sigma \) and we may choose \( K \in \Sigma \) and embeddings \( f_0 : (H \setminus \{h_1\}) \rightarrow K \) and \( f_1 : (H \setminus \{h_2\}) \rightarrow K \) such that \( f_0 \) and \( f_1 \) agree on \( H \setminus \{h_1, h_2\} \). Then \( f_0(h_2) \neq f_1(h_1) \) since the mapping \( f \) is not an isomorphism. Thus there is no loss in taking \( f_0, f_1 \) to be inclusions. We have \( (h_1, h_2) \notin R(K) \) since \( K \notin \Sigma \). But then \( H \subseteq K \subseteq G \) and \( H \in \Sigma \).
Case 2. There are \( h_0, h_1, h_2 \in H \) such that \((h_0, h_1), (h_0, h_2) \in R(H)\) and such that the mapping \( f : H \setminus \{h_1\} \to H \setminus \{h_2\} \) defined in Case 1 is an isomorphism but such that \(|R(H \setminus \{h_1\})| + 2 < |R(H)|\).

Then let \( k \) be new and let \( K_2 \) be the graph for which
\[ |K_2| = (H \setminus \{h_1\}) \cup \{k\} \text{ and } R(K_2) = R(H \setminus \{h_1\}) \cup \{(h_2, k), (k, h_2)\}. \]
Now \( K_2 \in \Sigma_3 \) and by the induction hypothesis \( K_2 \in \Sigma \) since
\[ |R(K_2)| = |R(H \setminus \{h_1\})| + 2. \]

Let \( K_1 \) be the graph such that \( |K_1| = (H \setminus \{h_2\}) \cup \{k\} \) and \( R(K_1) = R(H \setminus \{h_2\}) \). \( K_1 \in \Sigma \) and \( K_1 \cap K_2 = (H \setminus \{h_1, h_2\}) \cup \{k\} \).

By the amalgamation property let \( K \in \Sigma \) and let \( f_1 : K_1 \to K \) \( f_2 : K_2 \to K \) be embeddings such that \( f_1 \upharpoonright (K_1 \cap K_2) = f_2 \upharpoonright (K_1 \cap K_2) \).

Then \( f_1(h_1) \neq f_2(h_2) \) since \((h_1, k) \not\in R(K_1)\) while \((h_2, k) \in R(K_2)\). Thus without loss \( f_1, f_2 \) are inclusions. Note that \((h_1, h_2) \not\in R(K)\). But then \( H \subseteq K \) and so \( H \in \Sigma \).

Case 3. Otherwise for all \( h_0, h_1, h_2 \in \Sigma \) if \((h_0, h_2), (h_0, h_1) \in R(H)\) then \((h, h_1) \in R(H)\) iff \((h, h_2) \in R(H)\) for all \( h \in H \) since Case 1 fails; and indeed if \((h, h_1) \in R(H)\) then \( h = h_0 \) since Case 2 fails.

But then it is easy to see that there are \( m, n \) such that \( H \) is isomorphic to a subgraph of \( n \cdot S_m \) and so \( H \in \Sigma \).

This completes the proof of Lemma 4.9 and hence the proof of Theorem 4.5.

We conjecture

(C8) For each \( n \) there are a finite number of countable undirected graphs whose theories admit elimination of quantifiers and which omit \( K_n \).
The extension of the techniques employed in this paper is not obvious even for the case $n=4$.

5. Finite ultrahomogeneous graphs

In this section we examine briefly the problem of describing all finite graphs which model $\lnot R_{0}v_{0}$ and whose theories admit elimination of quantifiers. This project is far from complete, but continues from the work of Theorem 4.2 and the work of Gardiner [4]. In this section $G$ is assumed to be a finite ultrahomogeneous graph which models $(\lnot R_{0}v_{0})$.

Lemma 4.4 holds out the hope of obtaining some inductive characterization of such $G$. If $a \in G$ then each of the substructures determined by \{b : $G \models \Gamma ba\}$, \{b : $G \models \Gamma ab\}$, \{b : $G \models \Delta_{0}ab \land a\not\sim b\} \text{ and } \{b : G \models \Delta_{1}ab \land a\not\sim b\}$ is a graph whose theory admits elimination of quantifiers.

If $G \models \Delta_{0}v_{0}v_{1}$ or if $G \models \Delta_{1}v_{0}v_{1}$ then $G$ is $K_{n}$ or $\overline{K}_{n}$ for some $n$. If $G \models \Delta_{2}v_{0}v_{1}$ then $G$ is $K_{1}$ or a 3-cycle. The work of Gardiner describes those $G$ in which $(\Delta_{0}v_{0}v_{1} \lor \Delta_{1}v_{0}v_{1})$ is valid.

In this section we give some results on those $G$ which model $(\Delta_{0}v_{0}v_{1} \lor \Delta_{2}v_{0}v_{1})$. For the remainder of this section assume that $G \models \Delta_{1}v_{0}v_{1} \lor \Delta_{2}v_{0}v_{1}$.

Say that $G$ is connected just in case there are no non-empty $G_{0}, G_{1} \subseteq G$ with $G_{0} \cap G_{1} = \emptyset$, $G_{0} \cup G_{1} = G$ and $(R_{G} \uparrow G_{0}) \cup (R_{G} \uparrow G_{1}) = R_{G}$.

Component has the usual meaning.
The first easy observation is the following:

Lemma 4.10. If $G$ is not connected then $G$ is isomorphic to $\overline{K}_n$ for some $n$ or $G$ is a disjoint union of 3-cycles.

Proof. Since $G$ is finite the equivalence relation "$a$ lies in the same component as $b$" is a definable equivalence relation. From this it follows that any automorphism of $G$ must permute its components. The subgraph determined by a component $C$ is then itself ultrahomogeneous since the restriction to $C$ of the appropriate automorphism of $G$ will furnish an automorphism of $C$ extending an isomorphism of subgraphs of $C$.

If $a,b$ lie in different components of $G$ then $G \models \Delta_1 ab$. Hence each component models $\Delta_2 v_0 v_1$ by elimination of quantifiers. Therefore each component is either a 3-cycle or each component is a singleton.

In the sequel we assume that $G$ is connected. Then we conjecture (C9) $G$ has a Hamiltonian circuit.

This is supported by

Lemma 4.11 $G$ has a spanning circuit, that is there is some sequence $<a_n : n \leq m>$ such that $G = \{a_n : n \leq m\}$, $a_0 = a_m$ and for all $n < m$

$G \models Ra_n a_{n+1}$.

Proof. If $|G| = 1$ there is nothing to prove. Thus assume that $G \models \exists v_0 \exists v_1 (\Gamma_0 v_{01})$. Now since $G$ is finite the relation $Sab$, defined on $G$ by "there is a sequence $<a_n : n \leq m>$ such that $a_0 = a$, $a_m = b$ and $m > 0$ and $G \models Ra_n a_{n+1}$ for all $n < m"$, is definable.
Note that $S$ is transitive. Since $\text{Th}(G)$ admits elimination of quantifiers $|\{b : \text{Sab holds}\}|$ is independent of the choice of $a$.

Fix $a \in G$ and let $a_0$ be such that $G \models Ra_0a$. Since $S a_0 a$ holds we have $\{b : \text{Sab holds}\} \subseteq \{b : S a_0 b \text{ holds}\}$ whence equality follows.

Clearly $S a a$ holds, i.e. $S$ is reflexive. If $\text{Sab}$ holds, then $\text{Sba}$ holds since $S a a$ holds and $\{c : S a c \text{ holds}\} = \{c : S b c \text{ holds}\}$.

But then $S$ is a definable equivalence relation on $G$. If $G \models \text{Rab}$ then $a, b$ are in the same equivalence class under $S$.

Since $G$ is connected any $a, b \in G$ are $S$-equivalent. The desired conclusion now follows easily.
Chapter 5

The Theory $T_0$

In this chapter we construct an example of a complete theory $T_0$ in a small language which has a nonprincipal 1-type $p$ containing infinitely many 2-types $q$ and such that $T_0$ has a finite number of isomorphism types of countable models. The idea is to generalize the construction used by Peretyiatkin [8]. We produce a dense tree with infinite branching and indexing of subtrees above a node by the rationals.

Let $L$ be the language with one unary function symbol $U$, a binary predicate symbol $<$, a binary function symbol $\wedge$, a binary function symbol $I$ and a constant symbol $c$.

Let $T$ be the theory with language $L$ and nonlogical axioms:

**Group 1.** $\langle U, \wedge, E U \rangle$ is a tree, otherwise $\wedge$ is trivial.

Thus letting $x \wedge y \equiv_{DF} Ux \wedge Uy \wedge (x \wedge y = x) \wedge x \neq y$ the axioms are

$$
\begin{align*}
    v_0 \wedge v_1 &= v_1 \wedge v_0, \\
    (v_0 \wedge v_1) \wedge v_2 &= v_0 \wedge (v_1 \wedge v_2), \\
    v_0 \wedge v_0 &= v_0, \\
    v_0 \wedge v_2 &= v_1 \wedge v_2 ; \\
    v_0 \wedge v_0 \wedge v_1 &= v_1 \wedge v_0 \wedge (v_0 = v_1), \\
    \neg U v_0 &\rightarrow (v_0 \wedge v_1 = c), \neg U c
\end{align*}
$$

**Group 2.** $\langle \neg U \setminus \{c\}, < \rangle$ is a linear order, otherwise $<$ is trivial:

$$
\begin{align*}
    v_0 < v_1 &\rightarrow \neg U v_0 \wedge \neg U v_1 \wedge v_0 \neq c \wedge v_1 \neq c, \\
    (v_0 < v_1) \wedge (v_1 < v_2) &\rightarrow v_0 < v_2, \neg v_0 < v_0, \\
    \neg U v_0 \wedge \neg U v_1 \wedge v_0 \neq c \wedge v_1 \neq c \wedge v_0 \neq v_1 &\rightarrow v_0 < v_1 \vee v_1 < v_0
\end{align*}
$$

\[73.\]
Group 3. \( f \) indexes the subtrees above a node by points of the linear order:

\[
\neg v_0 \alpha v_1 \land I(v_0,v_1) = c \\
v_0 \alpha v_1 \land v_1 \alpha v_2 \rightarrow I(v_0,v_2) \\
v_0 \alpha v_1 \rightarrow \neg U(I(v_0,v_1)) \land (I(v_0,v_1) \neq c).
\]

Let \( \Sigma \) be the class of finite models of \( T \).

Lemma 5.1 \( \Sigma \) has the amalgamation property.

Proof. It will suffice to show that if \( A,B_0,B_1 \in \Sigma \), and \( A \subset B_0 \), \( A \subset B_1 \) and \( |B_0| \cap |B_1| = |A| \) then there is \( C \in \Sigma \) with \( B_i \subset C \) for \( i = 0,1 \). Let \( A,B_0,B_1 \) be as above. We prove by induction that the operations of \( B_0,B_1 \) may be extended to \( |B_0| \cup |B_1| = |C| \) to construct the required \( C \).

Let \( U_C = U_{B_0} \cup U_{B_1} \) and let \( c_C = c_{B_0} \).

It is not difficult to extend \( \prec_{B_0} \cup \prec_{B_1} \) to a linear order \( \prec_C \) on \( |C| \setminus (U_C \cup \{c_A\}) \). First define \( \prec \) by \( c \prec d \) iff there is a sequence \( <z_i : i \leq n> \) where \( n > 0 \) and \( c = z_0 \), \( d = z_n \) and for all \( i < n \) \( (z_i, z_{i+1}) \in \prec_{B_0} \cup \prec_{B_1} \). An induction on the minimum length of the sequence \( <z_i> \) shows that if \( c \prec d \) and \( c,d \in B_i \) then \( (c,d) \in \prec_{B_i} \). Indeed \( c = z_0 \) and \( d = z_{n+1} \) are both in \( B_i \).

If \( z_n \in B_i \) we are done for \( (z_0,z_n) \in \prec_{B_i} \). Otherwise \( z_n \in B_{1-i} \setminus A \)
whence both \( z_{n-1} \) and \( z_{n+1} \in B_{1-i} \) assuming \( n \geq 1 \). Now \( z_{n-1} <_{B_{1-i}} z_{n+1} \) so the sequence \( <z_i : i \leq n+1> \) can be shortened.
Thus in each case the shortest sequence has length 2 if \( c, d \in B_i \).

That \( \prec \) can be extended to a linear order is clear.

We now show by induction on \(|U_C|\) that there exist functions
\( \wedge_C, I_C \) on \(|C|\) extending \( \wedge_{B_i}, I_{B_i} \) respectively for \( i = 0,1 \)
such that \( \langle |C|, U_C, \prec, \wedge_C, I_C, \prec \rangle \) is a model of \( T \). Let \( \wedge_i, I_i \)
denote \( \wedge_{B_i}, I_{B_i} \) respectively. Let \( \alpha_i \) be the partial order
corresponding to \( \wedge_i \) as in the definition in Group 1.

There are now two cases.

Case 1. \( \exists i < 2 \exists x \in B_i \setminus A \forall z \in B_i (\neg x \alpha_i z) \).

Choose such an \( i \) and \( x \). By the induction hypothesis there
are extensions \( \wedge', I' \) of \( \wedge_{1-i}, I_{1-i} \upharpoonright (B_i \setminus \{x\}) \) and \( \wedge_{1-i}, I_{1-i} \upharpoonright (B_i \setminus \{x\}) \)
to \( C \setminus \{x\} \) such that the resulting structure is a model of \( T \).

Now let \( x_0 \) be the maximum of \( \{y : y \alpha_i x\} \) and for \( z \in C \) define
\[ z \wedge x = x \wedge z = \begin{cases} x & \text{if } z = x \\ z \wedge' x_0 & \text{otherwise} \end{cases} \]
and
\[ I(x,z) = c_A \]

\[ I(z,x) = \begin{cases} I'(z,x_0) & \text{if } z \notin \{x_0,x\} \\ I_i(x_0,x) & \text{if } z = x_0 \end{cases} \]

Case 2. For every \( x \in B_i \) there is \( a \in A \) such that
\( x = a \lor x \alpha_i a \).

Choose \( i \) from 0,1 and \( x \in B_i \setminus A \) such that
\[ \forall y \forall z (x \alpha_i y \rightarrow y \in A \wedge x \alpha_i y \wedge y \alpha_{1-i} z \rightarrow z \in A) \]
Let \( A_0 = \{a : x \alpha_i a\} \).
As in Case 1 let $\land'$, $I'$ extend $\land_{i-1}'$, $i \uparrow (B_i \setminus \{x\})$ and
$I_{i-1}'$, $I_i \uparrow (B_i \setminus \{x\})$ respectively to $C \setminus \{x\}$ such that one obtains
a model of $T$.

Extend $\land'$ to $C$ by letting $x_0$ be the least element of $A$
which is greater than $x$ and for all $z \in C$
$$z \land x = x \land z = \begin{cases} 
  x & \text{if } z \in A_0 \cup \{x\} \\
  x_0 \land' z & \text{otherwise}
\end{cases}$$

Let
$$I(x, z) = \begin{cases} 
  I_i(x, z) & \text{if } z \in A_0 \\
  c_A & \text{otherwise}
\end{cases}$$

and
$$I(z, x) = \{I'(z, x_0) \} \text{ if } z \neq x.$$  
It is an easy calculation to check that $C$ is a model of $T$. This
completes the proof of Lemma 5.1.

$\Sigma$ clearly has a member which can be embedded in each other member
- a structure with one element which is a model of $T$. Also $\Sigma$ has
arbitrarily large finite members and is closed under substructure.

Finally it is easy to see that there is a bounding function for the
 cardinality of generated substructures and so we may apply Lemmas 3.2
and 3.4. Let $M$ be a $\Sigma$-generic structure. Then Th($M$) admits elimination
of quantifiers.

Let $c_n : n \in \omega$ and $d_n : n \in \omega$ be new constant symbols and let $L_0$
the language $L \cup \{c_n : n \in \omega\} \cup \{d_n : n \in \omega\}$. Let $M_0$
be an expansion of $M$ to the language $L_0$ to model:
Let $T_0 = \text{Th}(M_0)$. We shall show that $T_0$ has 9 countable models and that there is a nonprincipal 1-type which contains infinitely many 2-types.

First note that $T_0$ admits elimination of quantifiers since $\text{Th}(M)$ does.

The 9 models are characterized by the upper bounds for $\{c_n : n \in \omega\}$ and $\{d_n : n \in \omega\}$. Let the models $H, K$ agree on the following four infinitary sentences:

$$
\exists x \left( \bigwedge_{i < \omega} c_i < x \right)
$$

$$
\exists x \left( \bigwedge_{i < \omega} d_i \alpha x \right)
$$

$$
\exists x \forall y \left( \bigwedge_{i < \omega} c_i < x \ & \left( \bigwedge_{i < \omega} c_i < y \right) \rightarrow x \leq y \right)
$$

$$
\exists x \forall y \left( \bigwedge_{i < \omega} d_i \alpha x \ & \left( \bigwedge_{i < \omega} d_i \alpha y \right) \rightarrow x \alpha y \right)
$$

The first two sentences assert that $\{c_i\}, \{d_i\}$ respectively have upper bounds, the third and fourth sentences that $\{c_i\}, \{d_i\}$ respectively have least upper bounds. When these least upper bounds exist we shall denote them by $\sup\{c_i\}, \sup\{d_i\}$ respectively.

We shall show that $H \cong K$.

Write $\neg U(A)$ for $(\neg U v_0)(A)$. 

We may apply the argument for the Ehrenfeucht structures discussed in [12, §6] to obtain an isomorphism

\[ \sigma : [ \bigcup \Gamma (H)]_H \rightarrow [ \bigcup \Gamma (K)]_K \]

where \([S]_A\) denotes the substructure of \(A\) generated by \(S\).

We now apply a back and forth argument on \(U^H\) and \(U^K\) to extend \(\sigma\) to an isomorphism of \(H\) and \(K\).

Let \(\{e^n : n<\omega\}\) enumerate \(U^H\) and let \(\{f^n : n<\omega\}\) enumerate \(U^K\). We construct enumerations \(\{h^n : n<\omega\}\) of \(U^H\) and \(\{k^n : n<\omega\}\) of \(U^K\) such that for each \(n\) there is an isomorphism

\[ \rho^n : [ \bigcup \Gamma (H) \cup \{h^n_i : i<n\}]_H \rightarrow [ \bigcup \Gamma (K) \cup \{k^n_i : i<n\}]_K \]

and such that the following conditions are satisfied for all \(n\)

(i) \(\rho^n\) extends \(\sigma\)

(ii) \(\rho^n (h^n_i) = k^n_i\) for \(i<n\)

(iii) \(h^0_0 = (\sup \{d^n_i\})_H\) and \(k^0_0 = (\sup \{d^n_i\})_K\) if these exist

(iv) \(h^n_n\) is an upperbound of \(\{d^n_i\}\) in \(H\) iff \(k^n_n\) is an upper bound of \(\{d^n_i\}\) in \(K\).

There is no difficulty in choosing \(h^0_0\) and \(k^0_0\) to be \(\sup \{d^n_i\}\) in \(H,K\) respectively when these exist since for any upper bound \(h\) of \(\{d^n_i\}\) in \(H\) and any upper bound \(k\) of \(\{d^n_i\}\) in \(K\)

\[ H \models I(d^m_n, h) = c^0_0 \quad \text{and} \quad K \models I(d^m_n, k) = c^0_0 \]

for all \(m\).

Thus we may assume that \(\{h^n : n<N\}\) and \(\{k^n : n<N\}\) have been chosen to satisfy the conditions and \(N>0\) if \(\sup \{d^n_i\}\) exists. We present the argument for even \(N\), with \(N\) odd we extend the range of \(\rho^n\) in a symmetric way.
Let $n$ be minimal such that $e_n \notin \{h_n : n < N\}$ and set $h_N = e_n$.

Let $H' = H \upharpoonright L$ and $K' = K \upharpoonright L$. There are now two cases:

Case 1. $H \models d_m \alpha h_N$ for all $m$.

Let $A$ be the finite substructure of $H'$ generated by $\{h_n : n < N+1\}$.

Let $a$ be the least of the upper bounds of $\{d_i\}$ which belongs to $A$.

If $a \in \{h_n : n < N\}_H$, then let $B$ be the substructure of $H'$ generated by $\{h_n : n < N\} \cup (A \cap \Gamma(U(H)))$. Now $\rho_N \upharpoonright B$ is an embedding and $K'$ is $\Sigma$-universal and $\Sigma$-homogeneous. Thus $\rho_N \upharpoonright B$ can be extended to an embedding of $A$ in $K'$. Let $k_N$ be the image of $h_N$ in this embedding. This determines $\rho_{N+1}$. We do not verify all the details that $\rho_{N+1}$ is an $L_0$-isomorphism but check the relevant clause to see that $\rho_{N+1}$ respects $I$. If $k \in \rho_{N+1} (\{h_1, \ldots, h_N\} \cup \Gamma(U(H)))$, and $k \alpha k_N$, then either $k \alpha \rho_N(a)$ in which case $I_K(k, k_N) = I_K(k, \rho_N(a))$, or $\rho_N(a) \neq k$ in which case the extension of $\rho_N \upharpoonright B$ to $A$ ensures that $\rho_{N+1}$ is an isomorphism.

Now suppose that $a \notin \{h_n : n < N\}_H$. Then $\{d_i\}_H$ has no least upper bound, but there are upper bounds. If there are no upper bounds for $\{d_i\}_H$ which belong to $\{h_n : n < N\}_H$, then we may choose for $k_N$ any upper bound of $\{d_i\}_K$. Otherwise let $b$ be the minimum of those upper bounds for $\{d_i\}_H$ which belong to $\{h_n : n < N\}_H$. Choose an upper bound $k$ for $\{d_i\}_K$ such that $k \alpha \rho_N(b)$. Let $j \in \Gamma(U(H))$ be such that $\sigma(j) = I_K(k, \rho_N(b))$. The idea from here is to see that we can use $k$ in the way we used $\rho_N(a)$ in the first part of this case.

We construct a member of $\Sigma$ in order to use the $\Sigma$-generic nature of $H'$ to find a suitable preimage for $k$, the analogue of $a$ in the first part.
Let $S$ be the substructure of $H'$ generated by $\{h_i : i \leq N\} \cup \{j\}$ and let $h$ be a new individual. Consider the structure $C$ for $L$ whose universe is $|S| \cup \{h\}$ and which satisfies in addition to the condition that $C \models |S| = S$ the conditions

(i) $h \in U_C$

(ii) $x \wedge_C h = h = h \wedge_C x$ for $x \in S$ satisfying $a_S x$ or $x = h$

(iii) $x \wedge_C h = h \wedge_C x = \begin{cases} c_H & \text{if } x \notin U_C \\ x \wedge_S a & \text{if } x \in U_S \text{ but } a \notin_S x. \end{cases}$

(iv) $I_C(h, x) = \begin{cases} j & \text{if } x \in S \text{ and } a_S x \\ c_H & \text{otherwise} \end{cases}$

(v) $I_C(x, h) = I_S(x, a)$ for $x \alpha_C h$.

Then $C \in \Sigma$ and since $H'$ is $\Sigma$-generic there is no loss of generality in assuming that $h \in H$.

Now observe that the substructure of $H'$ $\left[\{h_i : i \leq N\} \cup \{h\} \cup (\neg \cup (C))\right]_{H'}$, and the substructure of $K'$ $\left[\{k_i : i \leq N\} \cup \{k\} \cup (\sigma(\neg \cup (C)))\right]_{K'}$, are isomorphic by the unique mapping $\sigma_0$ compatible with $\rho_N$ which takes $h$ to $k$. Using the genericness of $K'$ we can find $k_N$ so that $\sigma_0$ extends to an isomorphism $\sigma_1 : \left[\{h_i : i \leq N+1\} \cup \{h\} \cup (\neg \cup (C))\right]_{H'} \to \left[\{k_i : i \leq N+1\} \cup \{k\} \cup (\sigma(\neg \cup (C)))\right]_{K'}$, where $\sigma_1(h) = k_N$.

Since $k \alpha k_N$ $k_N$ is an upper bound of $\{d_i\}_K$ and for $\rho_{N+1}$ we can take a restriction of the unique common extension of $\sigma_1$ and $\rho_N$. It is easy to see that $\rho_{N+1}$ is an $L_0$-isomorphism. This completes the first case.
Case 2. Otherwise, i.e. \( \exists m \ [(d_m)_H \not\in h_N] \).

Let \( m \) be minimal such that \( (d_m)_H \not\in h_N \). Let \( A \) be the substructure of \( H' \) generated by \( \{h_i : i < N+1\} \cup \{d_i : i < m+1\} \). Let \( B \) be the substructure of \( A \) generated by \( \{h_i : i < N\} \cup \{d_i : i < m+1\} \cup \sigma(\tau(A)) \).

By hypothesis \( \rho_N \upharpoonright B \) is an \( L \)-isomorphism from \( B \) to the substructure of \( K' \) generated by \( \{k_i : i < N\} \cup \{d_i : i < m+1\} \cup \sigma(\tau(A)) \). Since \( K' \) is \( \aleph \)-generic we may find \( k_N \) a suitable image in \( K \) for \( h_N \) and an \( L \)-embedding \( \sigma_1 \) of \( A \) in \( K' \) which extends \( \rho_N \upharpoonright B \) such that \( \sigma_1(h_N) = k_N \). As in the first case we let \( \rho_{N+1} \) be the unique common extension of \( \rho_N \) and \( \sigma_1 \). Again it is easy to see that \( \rho_{N+1} \) is an \( L_0 \)-isomorphism.

Thus \( H \cong K \) as was to be proved.

We have shown that the isomorphism type of a countable model of \( T_0 \) is determined by the truth values of four infinitary sentences. These sentences are not independent because existence of a least upper bound implies existence of an upper bound. The reader will easily see that there are in fact 9 different possibilities for the quadruple of truth-values. From this it is easy to see that \( T_0 \) has 9 countable models.

Let \( p \) be the type determined by \( \{ \forall x \wedge d_i \alpha x : i < \omega \} \). \( p \) is nonprincipal. Also there are \( \omega \) 2-types \( q \) in \( pxp \). There are at most \( \omega \) since \( I(T_0, \omega) < \omega \) and \( \{q_n : n \in \omega \} \) where \( q_n \) is the 2-type in \( pxp \) determined by \( \{ x \alpha y \wedge I(x,y) = c_n \} \) is a family of distinct 2-types in \( pxp \).

We summarize these results in the following theorem.
Theorem 5.1 There exists a complete theory with a small language which has 9 countable models and a nonprincipal 1-type which contains infinitely many 2-types.

In the next chapter we shall see how to construct a complete theory $T$ with a small language with countably many unary predicate symbols such that $T$ has 4 countable models and a nonprincipal 1-type containing infinitely many 2-types. The following is open.

Q9: If $T$ has finitely many nonlogical symbols except for constant symbols, if $T$ admits elimination of quantifiers and $I(T,\omega) < 9$ can $T$ have a nonprincipal 1-type containing infinitely many 2-types?

Another question is the following:

Q10: If $I(T,\omega) = 3$ and $T$ has a small language can $T$ have a nonprincipal 1-type containing infinitely many 2-types?
Chapter 6

C1, C4 and the Theory $T_1$

1. C1 and C4

In this chapter we shall consider further complete theories with a small language which have a finite number of countable models. In this section we describe some properties of theories which satisfy C1 or C4. In the next section we shall construct the theory $T_1$ which satisfies C1 but for which C4 fails.

In this section we assume that $T$ is a complete theory with a countable language and that $T$ has a finite number of countable models and more than one. Let $M$ be a countable saturated model of $T$, and let $p$ be a nonprincipal 1-type of $T$.

In the next lemma we explore the consequence of assuming that $T$ satisfy a strengthening of C1.

Lemma 6.1 Assume that the relation $R$ on $p(M) = \{ a \in M : a$ realizes $p \}$ defined by

$$aRb \iff b \text{ is prime over } a$$

is a linear order. Then, $T$ satisfies the following conditions:

(i) there are a finite number of 2-types in $p$

(ii) $R$ is definable (in the sense that there is a formula whose restriction to $p$ defines $R$)

(iii) $R$ is a dense order.
Proof. Let $\{\theta_n : n < \omega\}$ be a family of formulas in $(L)$ such that if $a, b$ realize $p$ in $M$ and $aRb$ then for some $n \theta_n$ generates $tp(a,b)$ over $p$, and for $n \neq m$ $p \cup \{\exists \nu \theta_n \land \theta_m\}$ is contradictory.

To see (i) assume for contradiction that there are infinitely many 2-types in $p$. Then $\alpha = \omega$. But then by the compactness theorem there is a 2-type $q$ in $p \times p$ containing

$$\{\forall \theta : n \in \omega\} \cup \{\exists \nu \theta_\nu : n \in \omega\} \cup \{v_0 = v_1\}.$$  

But then if $a', b'$ realize $q$ in $M$ we have neither $a'Rb'$ nor $b'Ra'$ a contradiction. Thus (i) holds.

We now easily obtain (ii) from (i), for $\alpha < \omega$ and $\forall \theta$ will define $R$ on $p(M)$.

Since $p$ is nonprincipal it is clear that $p(M)$ is infinite and that $R$ is without endpoints. Now let $n < \alpha$ and observe that for $b$ realizing $p$ in $M$ $\{a \in M : M \models \theta_n (a,b) \land a \text{ realizes } p\}$ must be infinite since $R$ is linear. For the type $q_n$ generated by $\theta_n$ over $p$ is not principal in its second coordinate, i.e. $T_{1,1}(q_n)$ is not principal over $p$. But now if $a_1, a_2, b$ realize $p$ in $M$ and $M \models \theta_n(a_1,b) \land \theta_n(a_2,b) \land a_1 \neq b$ we must have that $a_1Ra_2$ holds or $a_2Ra_1$ holds. Thus $\exists \nu_2 (\nu \models (v_0, v_2) \land \nu \models (v_2, v_1) \land \nu \models (v_2, v_1))$ belongs to $q_n$. Since this is true for each $n < \alpha$ it easily follows that $R$ is dense on $p(M)$. This completes the proof of the lemma.

In the next lemma we present a strengthening of a result of Benda [1, Thm. 2]. He showed that if every inessential extension of $T$ has
a finite number of countable models then $T$ has a universal model which is not saturated.

**Lemma 6.2** If every inessential extension of $T$ has a finite number of countable models then $T$ has a universal model prime over a finite set.

**Proof.** Recall first that if a theory has a finite number of countable models then the theory has a powerful $n$-type for some $n$. An $n$-type is powerful just in case every model of the theory realizing the type realizes every $m$-type in the theory [1, p. 111].

Assume that every inessential extension of $T$ has a finite number of countable models. Let \{$a_n : n \in \omega$\} be an enumeration of $M$. Let $\bar{a}_n$ be the $n$-tuple \{$a_j : j < n$\}. Now $\text{Th}(M, \bar{a}_n)$ has a finite number of countable models. Let $\Pi_n$ be a powerful $k_n$-type in $\text{Th}(M, \bar{a}_n)$ and let $\bar{b}_n$ realize $\Pi_n$ in $(M, \bar{a}_n)$. Let $\Sigma_n$ be the $(n+k_n)$-type of $\bar{a}_n \cap \bar{b}_n$. Note that $\Sigma_0$ is a powerful $k_0$-type in $T$.

We shall show that the model $N$ prime over $\bar{b}_0'$ realizing $\Sigma_0'$ is universal.

We choose by recursion sequences \{$a'_n : n \in \omega$\} and \{$b'_n : n \in \omega$\} in $N$ such that $\bar{a}'_n \cap \bar{b}'_n$ realizes $\Sigma_n$ where $\bar{a}'_n$ is the sequence \{$a'_j : j < n$\}. Then $M$ is isomorphic to an elementary substructure of $N$ so that $N$ is universal.

Assume that $a'_{m-1}$ and $b'_{m}$ have been chosen for $m < n$ such that $\bar{a}'_m \cap \bar{b}'_m$ realizes $\Sigma_m$ for $m < n$, and $n > 0$. Now $(N, \bar{a}'_{n-1})$ is a model of $\text{Th}(M, \bar{a}_{n-1})$ and $\bar{b}'_{n-1}$ realizes $\Pi_{n-1}$, a powerful type in $\text{Th}(M, \bar{a}_{n-1})$. But then it is easy to see that we may choose
a'_{n-1}$ and $b'_{n}$ in $N$ such that $a'_{n} \cap b'_{n}$ realize $\Sigma_{n}$ since the type of $<a'_{n} \cap b'_{n}$ in $\text{Th}(M, a'_{n-1})$ must be realized in $(N, a'_{n-1})$.

This completes the proof of Lemma 6.2.

2. The Theory $T_{1}$

In this section we construct a theory $T_{1}$ which has 4 countable models but which has an inessential extension $T_{2}$ having infinitely many countable models. $T_{1}$ is in a small language with countably many unary predicate symbols. $T_{1}$ satisfies $C_{1}$, indeed it has a nonprincipal $1$-type $p$ on which the relation "$a$ is prime over $b$" is linear.

The construction of $T_{1}$ uses the tools of Chapter 3 and resembles the development in Chapter 5. First an $\omega_{0}$-categorical theory is constructed, the models of which are linear orders with two sorts of elements: "rationals", and "irrationals" indexed by increasing triples of rationals. This theory is obtained by amalgamation techniques.

Consider the language $L$ with nonlogical symbols: a unary predicate symbol $U$, a binary relation symbol $<$, a 3-ary function symbol $I$, and a constant symbol $c$. Let $T$ be the theory with language $L$ and the following axioms.

Group 1. $<$ is a linear order with least elements $c$:

$$(x < y \land y < z \rightarrow x < z), \neg x < x,$$

$$(x < y \lor x = y \lor y < x), \quad c \leq x.$$  

Group 2. $I$ is a mapping from increasing triples of $U$ to $\neg U \setminus \{c\}'$ and is otherwise trivial:
Group 3. Distinct increasing tuples of $U$ index distinct irrationals:

$$I_{v_0,v_1,v_2} \neq c \quad \Rightarrow \quad v_0 < v_1 \wedge v_1 < v_2 \wedge Uv_0 \wedge Uv_1 \wedge Uv_2,$$

$$I_{v_0,v_1,v_2} \leq v_0 \wedge \neg U(I_{v_0,v_1,v_2}).$$

Let $\Sigma$ be the class of finite models of $T$.

Lemma 6.3 $\Sigma$ has the amalgamation property.

Proof. To see this it suffices to consider $A_0, A_1, A_2 \in \Sigma$ with $A_0 \subseteq A_1$, $A_0 \subseteq A_2$ and $|A_0| = |A_1| \cap |A_2|$. There is no loss of generality in assuming that $U_{A_1} \cup U_{A_2} \cup \{c_{A_0}\}$ contains no triple and

$$(\neg Uv_0 \wedge v_0 \neq c)(A_i) = \{(u,v,w) : A_i \models u < v \wedge v < w \wedge Uu \wedge Uv \wedge Uw\}$$

for $i = 0, 1, 2$.

First let $<'$ be an extension of $<_{A_1} \cup <_{A_2}$ to a linear order of $|A_1| \cup |A_2|$. In the proof of Lemma 5.1 we have already seen how to amalgamate two linear orders.

Now define $A_3$ as follows:

$$U_{A_3} = U_{A_1} \cup U_{A_2},$$

$$|A_3| = |A_1| \cup |A_2| \cup \{c_{A_0}\} \cup \{(u,v,w) : u,v,w \in U_{A_3}, \quad u <' v, v <' w\},$$

$$c_{A_3} = c_{A_0},$$

$$I_{A_3}(u,v,w) = \begin{cases} (u,v,w) \text{ if } u,v,w \in U_{A_3} \text{ and } u <' v, v <' w, \\ c_{A_0} \text{ otherwise}, \end{cases}$$
$\prec_{A_3} \{ (u,v,w) : (u,v,w) \in [A_3] \setminus (|A_1| \cup |A_2|) \}$ is the lexicographic order induced by $\prec$.

$\prec_{A_3} (u,v,w) < x$ for $x \in (|A_1| \cup |A_2|) \setminus c_{A_1}$ and $(u,v,w) \in [A_3] \setminus (|A_1| \cup |A_2|)$ and $\prec_{A_3}$ is a linear order which extends $\prec$ to $|A_3|$.

Then clearly $A_1 \subseteq A_3$, $A_2 \subseteq A_3$ and $A_3 \in \Sigma$.

This completes the proof that $\Sigma$ has AP and Lemma 6.3 is established.

Clearly there is a one element structure in $\Sigma$ which can be embedded in every structure in $\Sigma$, $\Sigma$ has arbitrarily large finite members, and $\Sigma$ is closed under substructure. Also there is a bounding function for the cardinality of generated substructure so we may apply Lemmas 3.2 and 3.4.

Let $M$ be a countable $\Sigma$-generic structure. $\text{Th}(M)$ admits elimination of quantifiers. It is easy to see that $\prec |M|$, $\prec_M$ is a dense order with first element in which $U_M$ and $(|M| \setminus U_M)$ are dense subsets. Also $M \models \forall Uv_0 \exists v_1 \exists v_2 \exists v_3 ( Iv_1 v_2 v_3 = v_0 )$.

Now let $L_1$ be the first order language obtained from $L$ by adding countably many unary predicate symbols $\{ U_i : i < \omega \}$. Let $M$ be expanded to a structure $M_1$ for $L_1$ to model for each $i < \omega$:

\begin{align*}
U_i v_0 & \land \forall v \prec v_1 + U_i v_1, \\
U_{i+1} v_0 & \rightarrow U_i v_0, \\
U_i v_0 & \rightarrow \exists v_1 ( v_0 < v_1 \land U_i v_1 ), \\
\neg U_i v_0 & \rightarrow \exists v_1 ( v_0 < v_1 \land \neg U_i v_1 ), \\
\neg U_i c & , \\
\text{and} & \exists v_0 ( \neg U_{i+1} v_0 \land U_i v_0 ) .
\end{align*}
That is we expand \( M \) by distinguishing an increasing sequence of irrational cuts in \( <_M \).

Now let \( T_1 = \text{Th}(M_1) \). We shall show that \( T_1 \) has four countable models. Consider the following four conditions on a model \( A \) of \( T_1 \):

**Condition I.** \( \bigcap_{i<\omega} U_i(A) = \emptyset \).

**Condition II.** There is \( a \in U_A \) such that \( a \in \bigcap_{i<\omega} U_i(A) \) and for all \( b < a \), \( b \not\in \bigcap_{i<\omega} U_i(A) \).

**Condition III.** There is \( a \in A \setminus U_A \) such that \( a \in \bigcap_{i<\omega} U_i(A) \) and for all \( b < a \), \( b \not\in \bigcap_{i<\omega} U_i(A) \).

**Condition IV.** None of Conditions I, II, III, that is for all \( a \in A \) there is \( b \in A \) such that \( a \in \bigcap_{i<\omega} U_i(A) \) \( \Rightarrow (b < a \land b \in \bigcap_{i<\omega} U_i(A)) \).

If Condition II or III holds we say \( a \) is \( \text{sup}\{U_i\} \). It is clear that the conditions are exhaustive and mutually exclusive. We shall show that if \( A \) and \( B \) are two countable models of \( T_1 \) which satisfy the same condition then \( A \cong B \). To do this we first require some further properties of \( M_1 \).

For \( j < \omega \) let \( L(j) \) be \( L \cup \{U_i : i < j\} \) and let \( \Sigma(j) \) be the class of finite structures for \( L(j) \) which are expansions of members of \( \Sigma \) which model for each \( i < j \):

\[
U_{i+1}v_0 \land v_0 < v_1 \rightarrow U_i v_1 \\
U_i v_0 \rightarrow U_i v_0 \text{ when } i+1 < j \\
\neg U_i c
\]
We then have $C(0) = Z$ and $M_1 \models L(0) = M$ and the following lemma.

**Lemma 6.4.** $M_1 \models L(j)$ is $\Sigma(j)$-generic for each $j < \omega$.

**Proof.** The proof is by induction on $j$. With $j = 0$ this is just the choice of $M$. Assume that $M_1 \models L(j)$ is $\Sigma(j)$-homogeneous and $\Sigma(j)$-universal. We shall let a symbol refer to the interpretation in $M_1$ unless otherwise specified.

We first show that $M_1 \models L(j+1)$ is $\Sigma(j+1)$-homogeneous. Assume that $H, K$ are finite substructures of $M_1 \models L(j+1)$ and that $\sigma$ is an isomorphism of $H$ onto $K$. In order to see that we may extend $\sigma$ to an automorphism of $M_1 \models L(j+1)$ by a back and forth argument it will suffice to see that if $H_0$ is a finite substructure of $M_1 \models L(j+1)$ with $H \subseteq H_0$ then $\sigma$ may be extended to $H_0$.

The idea is to reduce the problem to $\Sigma(j)$-genericness by introducing "brackets" for $U_j$. Given a structure $N \in \Sigma(j+1)$ define the canonical extension $N' \in \Sigma(j)$ of $N$ as follows.

Let $\lambda^N, \gamma^N$ be new individuals and let $N'$ satisfy

1) $|N'| = |N| \cup \{\lambda^N, \gamma^N\} \cup D$ and $N \models L(j) \subset N'$

   $U^{N'}_i = U^N \cup \{\lambda^N, \gamma^N\}$ for $i < j$;

2) $\prec^{N'}$ is a linear order of $|N'|$ such that $\lambda^N < \gamma^N$ and

   (a) if $a \in |N| \setminus (U_j)$ then $a <^{N'} \lambda^N$

   (b) if $a \in (U_j)$ then $\gamma^N <^{N'} a$.

3) $D = \{(u, v, w) : \text{at least one of } u, v, w \text{ is } \lambda^N \text{ or } \gamma^N \text{ and } u, v, w \in U^N, \text{ and } N' \models (u < v \lor v < w)\}$. 

4) $<_N'^1 \uparrow D$ is the lexicographic order induced by 

$<_N' \uparrow U_{N'}$, 

5) $N' \models \alpha \in d \land d < b$ for $d \in D$ and $b \in |N'| \setminus (D \cup \{c_N\})$, 

6) $N' \models \neg \forall i \in d$ for $d \in D$ and $i < j$, 

and 7) $N' \models I_{uvw} = (u,v,w)$ for $(u,v,w) \in D$.

We say that $l_{N'}' r_{N'}$ bracket $(U_j)'_N$. 

We first show that we may take the canonical extension of $K$ to be a substructure of $M_1 \uparrow L(j)$ with $l_K \notin U_j$ and $r_K \in U_j$. We first choose first approximations $l_0, r_0$, that is we choose $l_0 \in U$ such that

(a) $l_0 \notin U_i$ for $i < j$ 
(b) $l_0 \notin U_j$ 

and (c) $k \notin U_j \rightarrow k < l_0$ for $k \in K$; and choose $r_0 \in U$ such that

(a) $r_0 \notin U_i$ for $i \leq j$ 
(b) $k \in U_j \rightarrow r_0 < k$ for $k \in K$.

Let $K^1$ be the substructure of $M_1 \uparrow L(j)$ generated by $|K| \cup \{l_0, r_0\}$. We may assume that $l_K, r_K$ are new. We now construct a common extension $K^2$ of $K^1$ and $K'$ such that $K^2 \in \Sigma(j)$. 

Let $K^2$ satisfy clauses 1), 3), 4), 5), 6) and 7) of the definition of canonical extension with $K^1$ in place of $N$, $K^2$ in place of $N'$, $l_0$ for $l_N'$, and $r_0$ for $r_N'$; and let $K^2$ satisfy $2'$) $<_2^K$ is a linear order of $|K^2|$ such that $l_0$ is the immediate successor of $l_K$ and $r_0$ is the immediate predecessor of $r_K$. 


By the induction hypothesis there is no loss in assuming that
\[ K^2 \subseteq M_1 \upharpoonright L(j) \]. But then \( K' \subseteq M_1 \upharpoonright L(j) \) and since \( \ell_0 \notin U_j \), \( \ell_K \notin U_j \) and since \( r_0 \in U_j \) we have \( r_K \in U_j \).

The next step is to introduce brackets for \( H_0 \). It is clear that \( H'_0 \) can be taken to be a substructure of \( M \upharpoonright L(j) \) and that the substructure of \( H' \) generated by \( \langle H \rangle \cup \{ \ell_{H_0}, r_{H_0} \} \) is isomorphic to \( H' \) and so we may identify it and \( H' \). Now \( \sigma \) may be immediately extended to an isomorphism \( \sigma_1 \) of \( H' \) and \( K' \) with \( \sigma_1(\ell_{H_0}) = \ell_K \) and \( \sigma_1(r_{H_0}) = r_K \). But then since \( M_1 \upharpoonright L(j) \) is \( \Sigma(j) \)-generic we must have that \( \sigma_1 \) can be extended to an isomorphism \( \sigma_2 \) of \( H' \) and a substructure of \( M_1 \upharpoonright L(j) \). Let \( K_0 \) be the substructure of \( M_1 \upharpoonright L(j+1) \) generated by the image of \( \langle H \rangle \) under \( \sigma_2 \) and let \( \rho \) be the restriction of \( \sigma_2 \) to \( \langle H \rangle \). Then \( \rho \) is an isomorphism of \( H_0 \) and \( K_0 \) since \( \ell_K \) and \( r_K \) bracket \( U_j \), for
\[
h \in (U_j)_{H_0} \rightarrow r_{H_0} < h < r_K < \rho(h) \rightarrow \rho(h) \in (U_j)_{K_0}.
\]
and
\[
h \notin (U_j)_{H_0} \rightarrow h < \ell_{H_0} < \rho(h) < \ell_K < \rho(h) \notin (U_j)_{K_0}.
\]
This completes the proof that \( M_1 \upharpoonright L(j+1) \) is \( \Sigma(j+1) \)-universal.

Next we show that \( M_1 \upharpoonright L(j+1) \) is \( \Sigma(j+1) \)-homogeneous. Let \( H \in \Sigma(j+1) \). Consider \( H' \). Without loss of generality we may assume that \( \ell_H, r_H \in M_1 \) and \( \ell_i \in \cap U_{i<j} \) \( i \neq j \) and \( r_H \in U_j \). Now the substructure of \( H' \) generated by \( \{ \ell_H, r_H \} \) is a substructure of \( M_1 \upharpoonright L(j) \) and so by the \( \Sigma(j) \)-genericness of \( M_1 \upharpoonright L(j) \) we may assume that \( H' \subseteq M_1 \upharpoonright L(j) \). But then it is easy to see that
H ⊂ M₁ ⊨ L(j+1) since \( h \notin U_j(H) \rightarrow h < \lambda_H \) and \( h \in U_j(H) \rightarrow r_H < h \).

This completes the proof that \( M_1 \models L(j+1) \) is \( \Sigma(j+1) \)-generic and the lemma follows.

With lemma 6.4 we can prove the following theorem.

**Theorem 6.1** \( T_1 \) has four countable models.

**Proof.** Assume that \( A \) and \( B \) are two countable models of \( T_1 \) which satisfy the same one of Conditions I, II, III or IV. We shall prove that \( A \) and \( B \) are isomorphic by a back and forth argument.

Let \( \{ e_n : n \in \omega \} \) and \( \{ f_n : n \in \omega \} \) be enumerations of \( U_A \) and \( U_B \) respectively. We construct enumerations \( \{ a_n : n \in \omega \} \) and \( \{ b_n : n \in \omega \} \) of \( U_A \) and \( U_B \) such that for each \( n \):

1) if \( A \) satisfies Condition II then \( a_0 = \sup \{ U_i \} \) in \( A \)
and \( b_0 = \sup \{ U_i \} \) in \( B \).

2) if \( A \) satisfies Condition III then \( I(a_0, a_1, a_2) \) is
\( \sup \{ U_i \} \) in \( A \) and \( I(b_0, b_1, b_2) \) is \( \sup \{ U_i \} \) in \( B \).

3) There is an isomorphism \( \rho_n \) of \( [\{ a_j : j < \omega \}]_A \) and
\( [\{ b_j : j < \omega \}]_B \) which sends \( a_j \) to \( b_j \) for \( j < \omega \).

Here \( [S]_N \) is the substructure of \( N \) generated by \( S \). When \( S \) is
finite and \( N \models T_1 \) then \( [S]_N \) is finite also. It is easy to satisfy
1). Also 2) is straightforward for if \( a_0, a_1, a_2 \in A \) and
\( b_0, b_1, b_2 \in B \) are such that
\( A \models a_0 < a_1 \land a_1 < a_2 \land U_i Ia_0 a_1 a_2 \) for \( i < \omega \)
and \( B \models b_0 < b_1 \land b_1 < b_2 \land U_i Ib_0 b_1 b_2 \) for \( i < \omega \)
then \( [\{ a_0, a_1, a_2 \}]_A \) and \( [\{ b_0, b_1, b_2 \}]_B \) are isomorphic since there
are no other increasing tuples in \( U \). We may assume \( n > 0 \) in case Condition II holds and \( n > 2 \) in case Condition III holds.

We present only the argument for even \( n \). When \( n \) is odd we extend \( \{b_i : i < n\} \) symmetrically.

Let \( m \) be minimal such that \( e_m \notin \{a_i : i < n\} \). Let \( a_n = e_m \).

Let \( A_n = \{a_i : i < n\} \), \( B_n = \{b_i : i < n\} \), and \( A_{n+1} = \{a_i : i < n+1\} \).

Case 1. \( a' = \min\{a \in A_n : a \in \bigcap_{i < \omega} U_i(A)\} = \min\{a \in A_{n+1} : a \in \bigcap_{i < \omega} U_i(A)\} \)

or \( \{a \in A_{n+1} : a \in \bigcap_{i < \omega} U_i(A)\} \) is empty.

In this case let \( j > 0 \) be sufficiently large that for \( a \in A_{n+1} \)

if \( a \in U_{j-1}(A) \) then \( a \in \bigcap_{i < \omega} U_i(A) \). Now \( B \upharpoonright L(j) \) is \( \Sigma(j) \)-generic from Lemma 6.4 and \( A_n \) and \( B_n \) are isomorphic under \( \rho_n \). Also \( A_{n+1} \upharpoonright L(j) \in \Sigma(j) \). Thus there is an extension \( \sigma \) of \( \rho_n \) which embeds \( A_{n+1} \upharpoonright L(j) \) in \( B \upharpoonright L(j) \). Let \( b_n = \rho(a) \). Then it is not difficult to see that we may take \( \rho_{n+1} = \sigma \) to obtain an \( L_1 \)-isomorphism of \( A_{n+1} \) and \( \{b_i : i < n+1\} \).

The only detail to check is that

\( a \in U_k(A) \leftrightarrow \rho_{n+1}(a) \in U_k(B) \), for \( a \in A_{n+1} \) and \( k \geq j \).

If \( k \geq j \) and \( a \in U_k(A_{n+1}) \) then \( a \in \bigcap_{i < \omega} U_i(A) \) and so

\( \rho_n(a') \leq \rho_{n+1}(a) \) so \( \rho_{n+1}(a) \in \bigcap_{i < \omega} U_i(B) \). Otherwise \( a \notin U_i(A) \) for some \( i < j \) and the result follows.

Case 2. Otherwise, that is there is \( a \in A_{n+1} \) with \( a \in \bigcap_{i < \omega} U_i(A) \)

and for all \( a' \in A_n \) with \( a' \in \bigcap_{i < \omega} U_i(A) \) \( a \leq a' \).

In this case \( A \) and \( B \) must satisfy Condition IV. Before applying the techniques of Case 1 we must first establish an upper bound in \( B \)
to play the role of $p_n(a')$, suited to the introduction of a preimage to play the role of $a'$.

Let $b'' \in \cap_{i<\omega} U_i(B)$ be such that $b'' \in U(B)$ and $b'' < b$ for each $b \in \cap_{i<\omega} U_i(B)$. Since $B \upharpoonright L(1)$ is $\Sigma(1)$-generic we may choose $b' \in U(B)$ such that

1) $b'' \leq b'$

2) if $b \in \cap_{i<\omega} U_i(B)$ then $b' < b$

3) $B \models \neg \bigcup_0 (I(u,v,w))$ whenever $u,v,w \in \cap_{i<\omega} U_i(B) \cup \{b'\}$ and at least one is $b'$

4) $B \models I(u,v,w) < b$ when $b \in B \setminus \{c_B\}$, $u,v,w \in \cap_{i<\omega} U_i(B) \cup \{b'\}$ and at least one of $u,v,w$ is $b'$

5) the ordering on $\{I(u,v,w) : u,v,w \in U(B) \cup \{b'\}\}$, one of $u,v,w$ is $b'$, and $B \models u < v \wedge v < w$ is the lexicographic order.

We leave the explicit application of the techniques of the proof of Lemma 6.4 to the reader.

Now choose $j > 0$ such that for all $a \in A_{n+1}$ if $a \in U_{j-1}(A)$ then $a \in \cap_{i<\omega} U_i(A)$.

From Lemma 6.4 $A \upharpoonright L(j)$ is $\Sigma(j)$-generic. Thus we can choose $a' \in U(A)$ such that

1) $a' \in U_{j-1}(A)$

2) if $a \in A_{n+1}$ and $a \in U_{j-1}$ then $a' < a$

3) $A \models \neg \bigcup_0 (I(u,v,w))$ whenever at least one of $u,v,w$ is $a'$ and $u,v,w \in \cap_{i<\omega} U \cup \{a'\}$.
4) $A \models I(u,v,w) < a$ when $a \in A_{n+1} \setminus \{c_i\}$ at least one of $u,v,w$ is $a'$ and $u,v,w \in A_{n+1} \cup \{a'\}$.

5) the ordering on $\{I(u,v,w) :$ one of $u,v,w$ is $a'$, $u,v,w \in U(A_{n+1}) \cup \{a'\}$, and $A \models u < v \land v < w \}$ is the lexicographic order.

Now $[|A_n| \cup \{a'\}]_A \uparrow L(j)$ and $[|B_n| \cup \{b'\}]_B \uparrow L(j)$ are isomorphic by the extension of $\rho_n$ which takes $a'$ to $b'$. Since $B$ is $\Sigma(j)$-generic this may be extended to an embedding $\rho$ of $[|A_{n+1}| \cup \{a'\}]_A \uparrow L(j)$ into $B \uparrow L(j)$. Let $b_n = \rho(a_n)$ and let $\rho_{n+1} = \rho \uparrow A_{n+1}^{+\omega}$. The choice of $a'$, $b'$ and $\rho$ is sufficient to show that $\rho_{n+1}$ is an isomorphism by the argument from case 1.

Thus $A \cong B$, and $T_1$ has at most 4 countable models. Since $T_1$ is easily seen to have at least 4 countable models the proof of the theorem is complete.

Let $p$ be the $l$-type of a member of $\bigcap U_i$, i.e. the $l$-type generated by $\{U_i \land \neg v_0 : i < \omega\}$. Then there are only a finite number of 2-types in $p \times p$ - each principal over at least one coordinate: the types $q_0, q_1, q_2$ where $q_0 \supset \{v_0 = v_1\}$, $q_1 \supset \{v_0 < v_1\}$ and $q_2 \supset \{v_1 < v_0\}$. Indeed one may show that if $A$ is the countable saturated model of $T_1$ then given $a,b$ in $A$ $a$ is prime over $b$ or $b$ is prime over $a$. One need only modify the first steps in the proof of the theorem to construct the desired automorphism of $A$.

There are, however, infinitely many 3-types in $p$. For each $j > 0$ there is the 3-type $r_j$ in $p$ such that $r_j \supset \{v_0 < v_1 \land v_1 < v_2 \land \bigcup_{j-1} (I v_0 v_1 v_2) \land \neg U_j (I v_0 v_1 v_2)\}$. 


Clearly \( r_j \neq r_k \) if \( j \neq k \). There are also the 3-types \( r_0 \) and \( r_\omega \) in \( p \) where

\[
 r_0 \supset \{ v_0 < v_1 \land v_1 < v_2 \land \neg U_0 (I v_0 v_1 v_2) \}
\]
and

\[
 r_\omega \supset \{ v_0 < v_1 \land v_1 < v_2 \} \cup \{ U_i (I v_0 v_1 v_2) : i < \omega \}.
\]

\( T_1 \) has an inessential extension \( T_2 \) which has infinitely many countable models. Let \( c_0, c_1 \) be new constant symbols and let \( T_2 \) be a completion of \( T \cup \{ U_j c_0 : j < \omega \} \cup \{ c_0 < c_1 \} \). There are now infinitely many models satisfying Condition II. For each \( j < \omega \) there is a model \( M_j \) with an element \( a = \sup \{ U_i \} \) such that \( (a, (c_0)_{M_j}, (c_1)_{M_j}) \) realizes \( r_j \). Thus the \( M_j \) are distinct countable models of \( T_2 \).

The construction of \( T_1 \) uses an infinite number of unary predicate symbols. We have not succeeded in finding a construction that gives an example in a small language that is finite except for constant symbols. Also \( T_1 \) has 4 countable models. This raises the following questions:

**Q11:** If \( T \) has a finite number of countable models and if \( T \) is in a small language which has only a finite number of symbols other than constant symbols, does every inessential extension of \( T \) have a finite number of countable models? , and

**Q12:** If \( T \) has three countable models does every inessential extension of \( T \) have a finite number of countable models?
Conclusion

It is to be hoped that the techniques employed in this work can be used to develop more examples of theories with a finite number of countable models. The reader will have noticed that all of the examples considered have an underlying dense, albeit partial, order and thus satisfy C2. Some further examples which cast light on this conjecture would be welcome.

The properties of theories with a small language are of independent interest as shown by Q11 and C6. The class of such theories with function symbols appears to be much richer than the class of theories in a small language which do not have function symbols.

However, the results of Chapter 4 indicate that considerable complexity is possible even for those theories with one binary relation symbol. The problem of classifying all finite ultrahomogeneous graphs is an illustration.

Finally it appears that theories with three countable models may have very special properties since they have middle models. A separate investigation of such theories would be worthwhile. An open question is whether the existence of a middle model is equivalent to the property of having three countable models.
Bibliography


