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CALCULUS STUDENTS' UNDERSTANDING OF DYNAMIC SITUATIONS

by

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THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Faculty of

Education

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SIMON FRASER UNIVERSITY

July 1992

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CALCULUS STUDENTS' UNDERSTANDING OF DYNAMIC SITUATIONS

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ABSTRACT

The objective of this research project was to study calculus students' understanding of dynamic situations. The questions I wished to investigate were:

- What do calculus students understand about the concepts of change, and rate of change?
- How does this contribute to their mastery of formal course material?

These questions were investigated by conducting four longitudinal single-subject case studies. Each case study consisted of a set of eight biweekly interviews. Throughout the interviews I used a variety of non-standard tasks to promote unpractised responses from the subjects. The tasks addressed their understanding of dynamic situations in descriptive, graphical, numerical, and algebraic settings. These settings were used to describe situations involving the relative motion of two vehicles, the height of water flowing into an assortment of containers, the populations of two microbe cultures, the weights of two groups of children, and various situations represented only by mathematical functions.

The subjects were chosen from volunteers solicited from introductory calculus courses at Vancouver Community College, Langara Campus. Their understanding was analyzed by considering the proximity between their informal concept
images and the concept fields for the notions of change and rate of change. From the data, I drew detailed portraits of three of the subjects' understanding of dynamic situations.

The results of the study indicated that all the subjects had a good intuitive sense of dynamic situations but that connections between this intuitive sense and the formal constructs to study such situations were, generally, quite varied. Each of the three portraits showed that a different aspect of the notion of rate played a dominant role in the students' understanding of this concept. All subjects showed some difficulty distinguishing between the behaviour of a function and the values of the function. This seemed to be intimately connected to developing facility with the notion of negative rate. The subjects' conceptual command seemed to be quite independent of their computational command.

Many of the activities in the interviews seemed to promote conceptual command, and the connections between subjects' intuition and formal mathematical notions inherent in studying dynamic situations.
This work is dedicated to my father, Wesley Lidstone.

(March 25, 1910 - March 20, 1992)

He held standards high. I have tried to follow.
As sure as the sunrise, as sure as the sea,
    As sure as the wind in the trees,
We rise again in the faces of our children.

    Leon Dubinsky,

    Englishtown, NS
ACKNOWLEDGEMENTS

I extend a hearty thank-you to Harvey Gerber, Tom O'Shea, and David Pimm, the members of my Supervisory Committee. They each contributed a great deal, in very different ways, to this thesis.

There are many people at Langara to whom I am indebted for their help with this project. I thank the members of the Educational Leave Adjudication Committee for half-time leave from my instructional duties throughout the last year of these graduate studies. My thanks also go to the members of the Langara Research Committee for a grant of $1500 to pay for tape transcription. I am grateful to those departmental colleagues who granted me permission to solicit volunteers from their classes. I am most grateful to Murray Besler. In his role as the Langara Mathematics and Statistics Department Chair throughout my pursuit of these graduate studies, he granted me scheduling considerations that made an otherwise impossible situation manageable.

I also thank very much the students who were involved in this project. Their commitment of time and energy has been invaluable.

My warmest thanks go to my family members, Barb, Austin and Ryan, for their support throughout this pursuit. There have been too many weekends and evenings without their company. I look forward to the changes that completing this project will bring to us.
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CHAPTER I
THE NATURE AND PURPOSE OF THE STUDY

Calculus is the study of how systems change, or of *dynamic situations*. James Gleick (1987) applauds the discipline as being "one of the most ingenious creations of humans trying to model the changeable world around them" (p. 67). The concepts and methods that are studied in calculus have been created by the mathematical world to describe the dynamics of a system undergoing change. These concepts and methods constitute a symbolic formality or symbolic technology for quantitatively representing dynamic situations, and for providing a means to describe the nature of how such situations change.

In this technology, dynamic situations are represented with variables, functions and graphs. The technical language of studying the nature of change addresses the amount, and, most importantly, the rate of change. Amount of change in a quantity is often formally represented by differences in function values, and the algebraic sign of this difference describes whether the quantity has increased or has decreased. Rate of change is formalized as the ratio of change in one quantity to change in a second quantity. In addition to such symbolic formalities, these issues also have corresponding graphical images as vertical line segments (intervals), orientation, and slope.
Rationale for the Study

Since the mid-1980s the Mathematical Association of America (MAA) has been addressing what they describe as "a state of disarray and near crisis [in calculus instruction] at most American colleges and universities, the principal evidence being a failure rate of about 50% (MAA, 1986, p. 116)." They have called for detailed attention to the issue.

We need to know more about what students learn in mathematics classes. A close look at students' work (by means of interviews, videotapes of students working on problems, etc.) is often a disturbing, but valuable source of information. More detailed research on student's learning would be helpful, both to tell us about current difficulties in instruction and to suggest ways that might help us to improve. (p. xx)

The mathematical education community has also been active on this issue.

One of the reasons for such crises [as the state of calculus instruction] is precisely that in college mathematics teaching there is usually no consideration of cognitive processes but only of mathematical content. (Dreyfus, Artigue, Eisenberg, Tall, & Wheeler, 1990, p. 115)

In a review of mathematics education literature in Chapter II, I will describe research that has addressed how students think about a variety of concepts that are fundamental to the study of calculus. For the most part, this research does not address how calculus students think about quantitative change, the very object of study in calculus.

In my own instructional practice I endeavour to address dynamic situations for which students should have an immediate intuitive sense. For example, early in the term I
pose a problem about the relative motion of a car and a truck along a street near our campus. This situation is addressed again and again throughout the term as our study becomes more and more technical. I am surprised at the number of students who seem to feel that the mathematical situation is not at all connected with the actual situation. They do not seem to see that the information from the mathematical model—the truck quickly overtakes the car before the car has gained enough speed to pass the truck—is in accord with what should be their intuitive sense of how the positions of the vehicles could change. This prompted me to consider students' informal sense of how a dynamic situation might unfold.

The formal notions for studying dynamic situations are such an important part of studying calculus, that I felt they too must be considered in any study of calculus students' understanding. A common error I have observed throughout my teaching experience is that students act as if quantities are not changing or as if a situation is not dynamic when, in fact, they are or it is. For example, in Figure 1 is a problem that I address early in the term as part of setting the stage for further study.

A baseball diamond has the shape of a square with sides 90 feet long (see diagram). If a player runs from first base to second base, determine the distance between the player and home as a function of the distance the player has run.

Figure 1. A problem about baseball that I use in the early part of my calculus courses
A common reaction to this problem is to put the runner at second base despite evidence in the problem statement that suggests the player's position is changing. The technical phrase "determine the distance between the player and home as a function of the distance the player has run" is a request to represent this dynamic situation symbolically with variables. Implicit in this phrase is the statement that the distance the player has run and the distance between the player and home are both varying. The task is to determine an algebraic relationship between these distances that describes how the variation of one is connected to the variation of the other.

Such use of variables and functions to represent changing quantities is part of the symbolic technology of calculus. In putting the player at second place, students are focusing on the states of being at the bases and overlooking the variation that takes one between these states. Unfortunately, my experience in the classroom suggested that many calculus students do not readily see variables as varying. For this reason, I also wanted to explore what they understand about formal representations of dynamic situations.
Purpose of the Study

The objective of this research project was to study freshman calculus students' understanding how the formal constructs of mathematics represent dynamic situations, and to study what students could understand about change and rate of change without focusing on this symbolic formality. The questions I wished to investigate were:

- What intuition do students have about the concepts of change and rate of change?
- How does this contribute to their mastery of formal course material?

At the outset of the study, my assumption was that the concepts of change and rate of change were difficult for students to see and comprehend. My suspicion was that intuition about dynamic situations was not strong in calculus students, and that such lack of intuition contributed to their difficulties with the symbolic formalism of calculus.

Overview of the Study

Throughout the months from January to April of 1992, I conducted four longitudinal single-subject case studies. Each case study consisted of a set of eight biweekly interviews. The subjects were chosen from a collection of volunteers solicited from introductory calculus courses at Vancouver Community College, Langara Campus (Langara).

During the interviews, I used a variety of non-standard
tasks to promote informal or unpractised responses. The tasks addressed their understanding of change in descriptive, graphical, numerical, and symbolic settings. These settings were used to describe dynamic situations involving the relative motion of two vehicles, the height of water flowing into a variety of containers, the populations of two microbe cultures, the weights of two groups of children, and situations represented only by mathematical functions.

The interviews took place in an audio studio at Langara, and were recorded on audio-cassettes. The subjects were asked to vocalize their thoughts as they addressed the tasks. I also offered tutorial help with questions they had from their course work, but only after all my chosen tasks had been addressed. Transcriptions of the interviews were made from the tapes and served as the main source of my data. In addition, I took field notes and collected whatever written work the subjects produced for the tasks. A research grant of $1500 from the Langara Research Committee covered the cost of the tape transcription.

As a result of this study, I feel that a dynamic sense is quite accessible, although it probably does not easily occur spontaneously. Rather, it can be nurtured by activities similar to those I addressed in the study. More importantly though, I now feel that the connection between such a dynamic sense and the mathematical formalities used to study dynamic situations quantitatively warrants more
attention than we, as calculus instructors, often afford it.

Organization of the Thesis

I start Chapter II by presenting the framework I used to analyze students' understanding of dynamic situations. The next three sections of this chapter present a review of literature that addresses students' understanding of concepts that are fundamental to a quantitative study of dynamic situations. Each of these sections also includes some description of what the mathematical community at large considers, or has considered, about these concepts. The fifth section addresses literature on calculus students.

A detailed description of methodology for the study is presented in Chapter III. It starts with a section on clinical interview methodology. The next section describes some details about the pilot stage and how this stage shaped the main study. The third and fourth sections of Chapter II present the details of the methodology for the main study. The second of these is a thorough description of the tasks used throughout the interviews to promote informal or unpractised responses from the subjects. This chapter closes with a description of how the data were analyzed and reported.

Within the framework that I outline in Chapter II, the data that I collected were used to draw portraits of three of the subject's understanding. These portraits constitute the
primary goal of the data collection and are presented in Chapter IV. They are preceded by an overview of the subjects' performance on the interview tasks.

Sections of the last chapter constitute a discussion about these portraits and my conclusions from the study. In this chapter I also address limitations of the study and suggestions for further study.
CHAPTER II
LITERATURE REVIEW

In this chapter, I review literature that is relevant to my study. The first section addresses work pertaining to the framework that I use to analyze calculus students' understanding of dynamic situations. The next three sections review literature on students' understanding of variables, functions and graphs, and rates. These are concepts that I see as fundamental to the study of dynamic situations. These sections also include a brief look at what the mathematical community at large considers, or has considered, about these concepts. The fifth section reviews what calculus students understand about the notion of limits.

A Framework for Analyzing Students' Understanding of Dynamic Situations

The framework I use to analyze my subject's understanding of dynamic situations is a variation on a scheme promoted by Tall and Vinner (1981). I introduce the terms informal concept image and concept field which are modifications of terms that have been introduced by these authors. In later sections these two notions will be used as I review students' understanding of concepts such as variable, function, graph, and rate, that are inherent in mathematical studies of dynamic situations.
Concept Image and Concept Definition

Tall and Vinner use the terms concept definition and concept image to distinguish between "mathematical concepts as [they are] formally defined and the cognitive processes by which they are conceived" (1981, p. 151). They do so because "the human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics" (p. 150). Cornu (1981) shares this view of non-logical aspects of mathematical thought, and argues that is not the formal definition of a concept that guides mathematicians' activities, rather it is their mental image of the concept.

La représentation mentale que le mathématicien a d'une notion donne a cette notion son aspect dynamique, vivant. Elle permet au mathématicien de faire fonctionner la notion. Au contraire, la définition mathématique formelle est bien souvent figée; elle reste bien sûr le recours et le garant permanent, et elle permet l'écriture et la communication. Mais à elle seule, elle ne suffit pas à déclencher l'activité mathématique. [The mental image that a mathematician has of a concept gives the concept its power, its life. It allows the mathematician to work with the concept. On the other hand, the formal mathematical definition is often rigid; it remains, certainly, the recourse and the ultimate guarantee, and it allows for writing and communication. But on its own, it is not sufficient to initiate mathematical activity.] (p. 323)

A concept definition is described as "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). The adjectives formal and personal are used to draw a

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1 Throughout this literature review any translations are done by me, and are added within square brackets for the benefit of the reader.
distinction between a concept definition accepted by the mathematical community at large and one used by an individual. So, a formal concept definition is a form of words used by the mathematical community at large to specify a concept. A personal concept definition is a form of words used by an individual to specify a concept. For example, a formal concept definition for the notion of continuity of a function at the point $c$ is "the limit as $x$ approaches $c$ of the function is equal to the value of the function at $c"$ (symbolically written as $\lim_{x \to c} f(x) = f(c)$). A student's personal concept definition might be "it is all in one piece" (p. 167).

Concept image is defined to be "the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes" (p. 152). The amalgam of such mental representations will be unique to an individual. A personal concept definition will be part of an individual's concept image, whereas a formal concept definition may or may not be. Some examples of mental pictures that students associate with continuity come from responses to a questionnaire that Tall and Vinner administered to 41 students with an A or B grade in A-level mathematics in England. The students offered reasons for functions not being continuous such as "it is not in one piece", "it is not given by a single formula", or "there is a
sudden change in gradient" (p.167).

A learner's concept image will include a variety of aspects which may or may not be consistent with each other or with the formal concept definition. These inconsistent aspects are described as potential conflict factors (p. 153). With continuity at a point, students might associate the notion of being able to draw the graph without raising the pencil. This is a global phenomenon, whereas, continuity at a point is a local phenomenon. Hence, this is a potential conflict factor in the image of continuity. Analyzing a student's understanding of a concept would seem to involve considering the nature of the potential conflict factors in their concept image and the potential for resolution of these conflicts.

The importance of the role played by concept image in guiding mathematical activity, suggests further elaboration of this notion as it pertains to students of mathematics. Although he does not use the terminology of concept image, Cornu (1981) introduces notions that provide such elaboration. He defines the term modèles spontanés [spontaneous models] for those aspects of a learner's concept image that exist prior to instruction—what many now seem to refer to as preconceptions. For example, prior to instruction on the concept of continuity, the word might suggest certain mental images in a student, such as being all in one piece or continuing forever. These images will form
part of the concept image for the notion. Cornu argues that these preconceptions interact with formal instruction on a concept to form what he refers to as *modèles propres* [personal models], and that this interaction is of significant influence.

Leurs modèles propres sont extrêmement marqués par la conception initiale. [Their personal models are influenced very much by their initial conceptions.] (p. 326).

Indeed, these preconceptions can be seen to provide potential conflict factors.

[La définition mathématique] va entrer en conflit avec les modèles spontanés. Il va se produire des mélanges, des adaptions, pour finalement aboutir chez l'élève à des modèles engendrés à la fois par les modèles spontanés et la définition mathématique. [The formal definition will conflict with preconceptions. It will produce interactions and changes to finally bring students to their own models generated by both preconceptions and the formal definition.] (p. 323)

Cornu argues that students' errors should not be seen as careless. Rather, they should be considered as logical consequences of students' personal models. Hence it is by studying student errors that one can identify potential conflict factors in a concept image.

**Informal and Formal Concept Image**

It seems that Cornu's use of the term *spontaneous* has been expanded to include post-instructional conceptions. An example of such usage comes from work by Williams (1991) with second-semester calculus students who presumably had studied the notion of limit during their first semester. With them, Williams studied "informal models of limit ...[or] what Cornu
(1981) called students' spontaneous models of limits" (p. 219). The term seems to have grown to apply to any conceptions and responses that are unprompted or unrehearsed, and perhaps occur even in spite of instruction. For example, in a study of children's spontaneous representations of changes in population and in speed, Tierney and Nemirovsky (1991) worked with children who "had been introduced to the making of bar graphs in school prior to this experiment and some had experience with line graphs" (p. 183). In working with bar graphs, a zero value is denoted by an empty cell. However, they report a tendency for their subjects not to do this—rather, their subjects spontaneously represented situations involving zero in a manner very different from how other values were represented (p. 184). Davis and Vinner (1986) are quite explicit about the importance they place on unprompted responses.

The instructions for the written test were deliberately vague, in order to avoid test language that might trigger the retrieval of correct ideas....It was not the possession of correct ideas that was in question, so much as the possible presence of incorrect naive ideas. (p. 294)

Cornu suggests that students can quite adequately handle standard exercises with inappropriate personal models.

Les exercices qu'un étudiant rencontre lors de sa scolarité sont assez semblables les unes aux autres, et on voit souvent des étudiants réussir à leurs études sans trop de difficultés avec des modèles propres mathématiquement faux, car ces modèles propres ont été suffisants pour le champ couvert par les exercises rencontrés. [The exercises that students encounter throughout their schooling are very similar to each
other, and one can often see students having little difficulty completing their studies with personal models that are mathematically incorrect, because these models were sufficient for the ground covered by the exercises encountered.] (1981, p. 324)

Because of these differences in the nature of well-rehearsed responses and spontaneous ones, I feel it is useful to identify two categories of aspects within a concept image. I do so in a spirit similar to Tall and Vinner's distinction that results in the two categories of formal and personal concept definition. Since concept images are all personal, the use of the same two terms would be inappropriate. Because of this, I use the adjective formal to describe those aspects of a concept image that can be directly attributed to instruction, and might occur as well-rehearsed responses to tasks. In some sense, this use of the word formal will be similar to its use in the expression formal concept definition, because one would hope that what can be directly attributed to instruction is acceptable to the mathematical community at large. The adjective informal will be used for those aspects of a concept image that are not necessarily seen as a direct response to instruction but instead are unrehearsed, or occur spontaneously. To be sure, the dividing line between these two categories will be fuzzy. Indeed, one might argue that formal aspects become informal as understanding improves.

Like Davis and Vinner, I feel well-rehearsed responses to standard exercises do not seem to be particularly
revealing of concept image. As such, I feel that non-practised or informal responses provide a better view of students' understanding than formal responses would provide. I agree with Cornu that standard exercises are often quite easily done correctly without appropriate understanding, mainly because their similarity to one another provides a stage for well-rehearsed responses. Hence, my study of student understanding will use non-standard exercises to focus on the students' informal concept images.

There is a widespread belief that students' use of language is of paramount utility for analyzing their informal concept image. Herscovics (1989) offers the view that "it is often by a chance remark that [students'] lack of understanding surfaces" (p. 82). Bell and Janvier (1981) assert "a language approach is in essence indirect and natural" (p. 40, my emphasis). So, such an approach might be seen to promote spontaneous responses. Studying students' informal concept image will then involve addressing non-standard tasks and including a focus on the language used while addressing such tasks.

The Notion of Concept Field

Although concept image lies within an individual, there seems to be utility in addressing what is accepted by the mathematical community at large. However, to this end I find Tall and Vinner's notion of formal concept definition to be too restrictive. Certainly, the words used to specify a
concept are of considerable importance, but, what such words attempt to embrace and what we will be able to deduce from them seem to also be of particular importance. I would like to be able to address that which surrounds a concept and is accepted by the mathematical community at large. For this reason, I introduce the term concept field to refer to that which the experienced mathematical community acknowledges as being involved in, embraced by, and/or immediately deducible from a formal concept definition.

For example, with the concept of continuity of a function, a formal concept definition would be $f$ is continuous at the point $c$ if and only if $\lim_{x \to c} f(x) = f(c)$.

This formal concept definition involves the notion of evaluating a function at a point and the notion of the limit of a function. The definition attempts to embrace the notion that the function values stay close together. It is equivalent to the statement for all $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. These aspects that surround the concept definition will be part of the concept field.

A concept field is by no means fixed over time, nor is

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2 I wish to emphasize that this is a "Lidstonian" term. In particular, I do not want it to be confused with the term champ conceptuel evidently used often by Brousseau and Vergnaud. This term is apparently usually translated as conceptual field. Please notice the difference.
it independent of context. Indeed, as a concept field acquires more definition in a photographic sense, we can often see the emergence of new aspects, contexts, and even concepts. For example, in the latter part of the nineteenth century Karl Weierstrass saw the importance of particular points in the notion of continuity and hence distinguished the notions of pointwise continuity and uniform continuity. Despite this pliable nature of what would constitute a concept field, I feel it will be useful to be able not only to refer to a concept's conventional definition, but also to that which conventionally surrounds the concept.

I choose the term field not only because of the image of a field surrounding something, but also because there is an underlying photographic sense that applies. Image has a photographic sense of including everything that surrounds the subject of the photograph. A definition by Stroebel and Hollis (1974) of a photographic sense for field includes "the entire subject area imaged within the circle of good definition of a lens" (p. 72, my emphasis). For a concept field, the role of the lens would be played by the mathematical community at large.

**Summary**

In my analysis of student understanding I will work within these modifications of the Tall and Vinner framework. My focus will be on studying the proximity between the concept field and students' informal concept image of
mathematical notions inherent in dynamic situations. In what follows, I will use this framework as I review literature on what I see as fundamental notions in calculus—variables, functions and graphs, and rates. I will also review literature on calculus students that I have encountered which, for the most part, addresses the notions of functions and limits. The literature on the fundamental issues of change and rate of change that I have encountered has, for the most part, involved students prior to their study of calculus. My work will draw these two categories together.

**Concept Images and Field for Variables**

Throughout my experience with teaching freshman calculus courses, the most prevalent difficulties I noticed were with related rates problems—those in which students are asked, from given information, to relate the rates at which two or more quantities change. Often the textbook formulation of such problems includes mention of a particular instant and a common error by students is to use that given instant from the outset of their analysis, thus overlooking the dynamic nature of the problems.

I obtained a hint of the origin of this error through tutorial work with precalculus students. A typical problem from one of our precalculus texts is presented in Figure 2.
In the figure, \( s \) is the length of the shadow cast by a 6-foot person standing \( x \) feet from a light source that is 24 feet above level ground. Express \( s \) as a function of \( x \). (Sobel & Lerner, 1987, p. 159)

Figure 2. The shadow problem, a typical problem from a precalculus text.

Despite my heroic efforts at walking students through the intricacies of lights casting shadows, or similar triangles, or functional relationships, a common comment at the end of my efforts was "But what is \( x \)?" Retorts such as "The distance between the person and the lamp-post." or "The length of this line segment." would not help—"But what is \( x \)?" was invariably posed again.

It took me some time to recognize that what was expected for \( x \) was a particular number, that is, a solution to an equation. Such encounters opened my eyes to the various uses of letters in mathematics. In an overview of students' command of algebraic issues, Küchemann (1981) offers the following comment on this situation.

The blanket use of the term "variable" in generalized arithmetic is a common practice which has served to obscure both the meaning of the term itself and the very real differences in meaning that can be given to letters. (p. 110)

In this section, I review literature that addresses concept images of variables. I introduce what I see as three distinct perspectives that are adopted in the use of variables. These are called a particular view, a general view, and a dynamic view. I show how these three
perspectives have manifested themselves throughout the
history of mathematics, and that, although all three have
their place in the concept field of variable, it is the
general view of variable that receives the most formal
emphasis.

Use of Letters

Küchemann (1981) summarizes the results of the algebra
test from the Concepts in Secondary School Mathematics and
Science [CSMS] research programme based at Chelsea College,
London, in the middle and late 1970s. The study identified
six different ways in which students interpreted and used
letters (p. 104).

- **Letter evaluated:** ... The letter is assigned a
  numerical value from the outset [so that it need not be
  operated upon].
- **Letter not used:** Here the children ignored the letter,
or [worked] without giving it a meaning.
- **Letter used as an object:** The letter is regarded as
  shorthand for an object or an object in its own right,
  [rather than a quantity associated with an object. For
  example, $s$ refers to the side of a square, rather than
  the length of such a side].
- **Letter used as a specific unknown:** Children regard a
  letter as a specific but unknown number, and can operate
  upon it directly.
- **Letter used as a generalized number:** The letter is seen
  as representing, or at least being able to take, several
  values rather than just one.
- **Letter used as a variable:** The letter is seen as
  representing a range of unspecified values, and a
  systematic relationship exists between two such sets of
  values.

The first three categories seem to have the effect of
avoiding the algebraic use of letters. The last three
categories are more fully distinguished as follows.
With the letters regarded as specific unknowns, the relationship $5b+6r=90$ is seen simply as a statement which happens to be true for a particular, albeit unknown, pair of numbers. This statement is essentially static, it involves no idea of change. Alternatively, when the letters are used as generalized numbers, $5b+6r=90$ becomes a statement that is satisfied by several, but still essentially isolated, pairs of numbers, namely some or all of $(6,10), (12,5), (0,15), (18,0)$. This view involves the idea that the values of $b$ and $r$ can change, but does not in itself indicate how they change, for which it is necessary to compare the values with one another in some way. (p. 110)

Using the letter as a variable is seen as involving the ability to carry out such a comparison. They present the operational definition "letters are used as variables when a second (or higher)-order relationship is established between them" (p. 111). The statement that "very few children reached the high degree of understanding required to interpret a letter as a variable" (p. 105) suggests that this use of letters is very challenging for students.

The CSMS work was incorporated by Sutherland (1991) in a study of the use of variables in a computer setting by 8 students of age 11 to 14. The study compared the students' understanding of algebra related ideas in a Logo context (computer environment) and their understanding of these issues in an algebraic context (paper & pencil environment). Sutherland used the following five categories as a framework (p. 42).

- Understanding that a variable name represents a range of numbers.
- Understanding that different variable names can represent the same value.
- Understanding the "lack of closure" in a variable-
dependent expression.
• Understanding the nature of the second order relationship between two variable dependent expressions.
• Ability to use variables to express a general method.

Sutherland argues that students' mastery of these issues is mostly affected by the environment in which they work rather than by any developmental stage at which they might be working.

The most important result from this study was that the links which pupils made between variable in Logo and variable in algebra depended very much on the nature and extent of their Logo experiences. So, for example, it turned out that those pupils who could correctly answer the question about whether or not different variable names can represent the same value had used this idea whilst programming in Logo. (p. 42)

The data presented seem to suggest that for each of the five categories described above, there was considerable progress resulting from experience with Logo. Sutherland mentions that there was a "need for relatively 'heavy' teacher intervention in order to provoke pupils to use the idea of variable in Logo" (p. 42). This phenomenon is also reported by Kieran, Booker, Filloy, Vergnaud and Wheeler (1990) with the statement that "the concept of variable does not occur spontaneously in young students working in a Logo environment" (p. 110).

Informal Images of Change

Work addressing children's spontaneous representations of dynamic situations offers some interesting insight about their command of formal representations such as variables. In a study of representations of the continually changing
speed of a car by children in Grade 4, Tierney and Nemirovsky (1991) report that their subjects established discrete categories such as "park", "fast", "slow", "moderate", and "stop" (p. 185). These children are not alone with their difficulty of seeing continuous change. With six and seven year-old students, Bednarz and Dufour-Janvier (1991) also observed a focus on particular states.

These procedures originate from the children's excessive centration [sic, focus] on the states, and reveal a static conception of the relations underlying these situations. Everything occurs as if the thinking process of the child acted only on states without being able to reconstruct the changes. (p. 141)

Ponte's work with 16-year olds suggested his subjects "tended to think of continuous variables in terms of discrete states" (cited in Dreyfus et al., 1990, p. 121). As I mentioned in Chapter I, when addressing the baseball problem in Figure 1, many of my students seem to focus on the state of being at second base. In some sense the letter used as a specific unknown category identified by Küchemann can also be seen as a consideration of a state--the particular numerical solution of a state represented by an equation. All of these examples involve something particular, and provide examples for the first view of variable I wish to identify.

Three Views of Variables

A particular view would see a variable as denoting particular states or solutions. Such a view would include the possibility of a letter representing more than one value
but not a continuous range of values. I do not see this view as applying to the use of the letters π, e or φ as particular numbers because these symbols, their meanings and values are predetermined. As an example of how such a view would be used, consider the model of distance covered in free fall that is given by the equation \( s = \frac{1}{2}gt^2 + v_0 t \). The use of \( g \) might involve a particular view because its value will be 32 or 9.8 depending on the particular choice of units. So too would the use of \( v_0 \) involve a particular view because, although unspecified, it is intended to represent the initial state. Answering a question such as "how long does it take a ball to fall 50m?" would involve a particular view.

Such a view need not embrace variation, and letters used in this way would more properly be called unknowns or parameters (although not in the sense of the continually varying parameter involved in the study of parametric equations). While elaborating on the generalized number category, Küchemann (1981) says "a distinction can be made between the idea of a letter taking on several values in turn and a letter representing a set of values simultaneously" (p. 109). I make this distinction by including the former in the particular view of variable, and the latter as an aspect of a general view of variable.

A general view would see a letter as representing a set of values simultaneously. The set of values that is
represented would in some sense be extraordinarily large--usually an interval but I do not wish to rule out such sets as "all positive integers", or "all rational numbers between e and π". I see this view as involving the notion of any or all values from such a set being freely substituted, however not necessarily with any requirement of order. The five categories that Sutherland describes would be embraced by this view. In particular, understanding that a variable name represents a range of numbers would be essential to adopting this view of variable. Below, in the sections on functions and graphs, and on rates, I will review literature addressing the difficulties students experience with reading intervals in a graphical setting. I see this as a challenge of working with sets of values, and hence as part of the challenge involved in adopting a general view of variable.

With respect to the shadow problem in Figure 2, I would describe the students with whom I was working as operating with a particular view when they needed to consider a general view. A successful solution of the baseball problem in Figure 1 would require generalizing the distance run by the player and the distance between the player and home with letters. As applied to the free fall model, a general view would see the equation between s and t as holding at any time throughout the flight of the object, but need not embrace the notion of time flowing.

As with the particular view, I see the general view of
variable as being static. Küchemann's elaboration of "generalized number" that I quoted on page 22 above, seems static because there is no sense of flowing between the different pairs. The free-fall equation, or the equations describing the length of the shadow in the shadow problem, or the distance between the player and home in the baseball problem, would give "numerical snapshots" but need not offer a "numerical movie" of the situations. Of course, a movie is composed of snapshots, or still-frames, viewed in an ordered succession that is sufficiently rapid, or dense, to illustrate the transition between states. A similar condition will be required for a dynamic view of variable.

A dynamic view of variable would include a general view, but would also involve an orderly succession of values throughout a set. The challenge of adopting such a view is addressed by Bednarz and Dufour-Janvier (1991) with the claim that "situations, involving the mental reconstruction of a dynamic process, are often perceived as statics [sic] by children" (p. 140). As I review more literature in the sections below I will offer further evidence of the challenges of adopting such a view. I see a dynamic view as being necessary for Küchemann's letter used as a variable category in order to "indicate how they change" (p. 110). A dynamic view applied to the shadow problem might see the length $s$ decreasing as $x$ decreases. Applied to the baseball problem, such a view might involve imagining a point moving
along the segment between first base and second base and seeing the distance between this point and home plate increase. With such a view of the free fall model, $s$ might be seen to continually increase, and do so more rapidly as time progresses. It is a dynamic view of variable that would provide appropriate motivation for questions of differential calculus. As students progress into calculus and are asked to consider the rate at which quantities are changing, it will be helpful for them to embrace such a view.

**Historical Considerations**

These three views of variables have their place in the history of mathematics, and at times their coexistence has presented a challenge. The early stages of algebra emphasized the particular view with a focus on determining particular unknown quantities.

Classical algebra was introduced about the year 830 A.D. ... [and] was presented as a list of rules and procedures needed to solve specific linear and quadratic equations. (Herscovics, 1989, p. 60).

Kieran (1990) gives a broader view of the development of algebra and offers some comment on the emergence of a general view.

The second stage [in the development of algebra], syncopated algebra, extended from Diophantus [middle of the third century], who introduced the use of abbreviations for unknown quantities, to the end of 16th century. Harper has pointed out that the concern of algebraists during these centuries was exclusively that of discovering the identity of the letter or letters as opposed to an attempt to express the general. The third stage, symbolic algebra, was initiated by Vieta's use of letters to stand for given quantities. At this point it
became possible to express general solutions. (p. 97)

So, in the development of algebraic practice we see a transition between two very different perspectives that fit the first two views of variable I have introduced.

There is also a history of challenges in adopting a dynamic view. In a study of the purpose of Zeno's arguments against motion, Cajori (1915) comments that Aristotle saw the arguments as being fallacious and offered a rationale for his view by stating "this is false, for the reason that time is not composed of individual, indivisible Now's, as also no other quantity is so composed" (p. 8). This seems to speak to the inappropriateness of conceiving of something that flows, like time, in terms of particular instants. Cajori presents the case that some historians see exactly this point as being the purpose of Zeno's arguments.

Tannery has advanced the view that Zeno directed his arguments against the notion that space is the sum of points, and time the sum of instants. In others words, Zeno did not deny motion, but wanted to show that motion was impossible under the conception of space as the sum of points. (p. 15)

This seems to speak to the incompatibility of a particular view and a dynamic view. To conclude his paper, Cajori states "Tannery's hypothesis will probably stand as the most acute and scholarly interpretation of Zeno's arguments" (p. 20).

A dynamic view seems to have its place in the origins of calculus, although Grabiner (1983) presents a case against
dynamic issues being central in the development of calculus.

Medieval classifications of variation helped to lead Galileo in 1638, without the benefit of calculus, to his successful treatment of uniformly accelerated motion....Were such studies the origin and purpose of calculus? The answer is no. However plausible this suggestion may sound...physical questions were in fact neither the immediate motivation nor the first application of the calculus....The first problems to be solved, as well as the first applications, occurred in mathematics, especially geometry. (p. 198, my emphasis)

The geometric problems to which she refers were ones of determining tangents or areas. The general solution of such problems became issues during the seventeenth century because of the rise of symbolic algebra and analytic geometry. Kleiner (1989) suggests that the rise of these two branches of mathematics contributed to a shift towards a dynamic view.

Both of these developments suggested a dynamic, continuous view of the functional relationship as against the static, discrete view held by the ancients. (p. 283)

Moreover, in an overview of Newton's mathematical works, Pepper (1988) provides a very clear statement of the role played by a dynamic view in the development of calculus.

Newton's patterns of thought were more physical. His variable quantities varied with time--they were flowing quantities (fluents) with a velocity, or a rate of change (a fluxion)--as opposed to Leibniz' more formal and static infinitesimals. (p. 67)

It seems then that a dynamic view, as adopted by Newton, did enrich the development of calculus.

By the nineteenth century, the notions of calculus had gained sufficiently wide utility and attention to generate more rigorous consideration. Cauchy provided a logical
structure that started with the concept of limit. He gave the following definition.

> When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up differing from it by as little as one could wish, this last value is called the limit of all the others. (Fauvel & Gray, 1987, p. 566)

The mention of "successively" suggests to me a dynamic view because it implies an ordered passage through values. Certainly, Cauchy's use of variable was more involved than a particular view would entail. "When Cauchy needed a limit property in a proof, he used the algebraic inequality-characterization of limit" (Grabiner, 1983, p. 204). Indeed, Cauchy apparently overlooked the role of the particular in that "given an ε, he chose a δ which he assumed would work for any x" (Grabiner, p. 205, my emphasis).

This oversight was brought to the surface largely by Weierstrass with the distinction of pointwise convergence from uniform convergence (that is, convergence at a particular point from convergence over a set of points). As a teacher, Weierstrass paid considerable attention the role of variables in analysis. A course he taught in the summer of 1861 started with a definition of variable and quantities that vary continuously (Dugac, 1973, p.56). Boyer (1949) suggests that, rather than considering the movement through values, Weierstrass stressed the assumption of particular values.
In retrospect, it is pertinent to remark that whereas the idea of variability had been banned from Greek mathematics because it led to Zeno's paradoxes, it was precisely this concept which, revived in the latter Middle Ages and represented geometrically, led in the seventeenth century to the calculus. Nevertheless, as the culmination of almost two centuries of discussion as to the basis of the new analysis, the very aspect which had led to its rise in a sense was again excluded from mathematics with the *static* theory of the variable that Weierstrass had developed. The variable does not represent a progressive passage through all values of an interval, but the disjunctive assumption of any one of the values in the interval. (p. 288, my emphasis)

Evidently, the acceptance of a dynamic view of changing quantities has fluctuated throughout the history of mathematics.

I do not intend this historical aside to be a thorough account of the development of the notion of variable. I make this aside only to point out there have been conceptions of variables, and variable quantities, within the mathematical community that have involved three distinct views. I see these three views as *particular*, *general* and *dynamic*. They are increasingly complex, and their coexistence has not been free from controversy.

**Concept Field for Variables**

In reference to the six categories from the CSMS study, Küchemann says "it is often convenient to switch from one use to another in the course of solving a problem" (1981, p. 110). This is exactly what calculus students encounter with related rates problems--they need to consider such problems dynamically in order to incorporate the derivative as a rate
of change in their solution, but then consider a particular instant in order to address what is usually posed. Such considerations may not always be part of their concept image of variables. Indeed, I suggest that a dynamic view is not very dominant in the concept field for variables.

In the context of a calculus course, the notion of a variable does not seem to warrant a formal definition. For example Larson and Hostetler (1982), the textbook for the calculus courses at Langara, distinguishes dependent and independent variables but does not define the term variable (p. 39). Of the five other elementary calculus texts in my book case, only one mentions the term in its own right. The volume on analysis in the series Fundamentals of Mathematics (Behnke, Bachmann, Fladt & Süss, 1962/1986) is quite encyclopedic in that it starts with very fundamental principles. Yet, the term variable is addressed only in the context of separable differential equations.

Sobel and Lerner (1987), one of the precalculus texts at Langara, says only "letters, called variables, are used to represent real numbers" (p. 8). The first volume in the Fundamentals of Mathematics series has the following description of the term.

Variables are letters that do not refer to any definite entity but rather to a definite range of entities, whose names can be substituted for these variables. (Vol. I, p. 11)

A dictionary of mathematics compiled by Millington and
Millington (1966) includes the following definition.

Any symbol for any member of a set of numbers, points, values, etc. An element of the set is then called a value of the variable, and the whole set its range. (p. 251)

These all seem to embrace mainly a general view, and do not readily suggest the representation of a dynamic object.

A dynamic sense might be somewhat nebulous and hence not easily captured in a formal concept definition. However, it does seem to be useful and does entail something beyond a general view. Indeed, Küchemann (1981) states the following about this.

The concept of a variable clearly implies some kind of understanding of an unknown as its value changes, and if this is to go beyond the ideas already present in seeing letter as a specific unknown and a generalized number, it would seem reasonable to argue that the concept implies, in particular, some understanding of how the values of an unknown change, though precisely what this might mean is hard to pin down. (p. 110)

I have tried to pin this down by describing the notion of a dynamic view of variable. This is an aspect of the concept field of variable that seems to be largely ignored, however, I feel it is very useful because it provides appropriate motivation for the concepts of calculus.

Summary

In this section, I have reviewed the literature addressing students' understanding of the use of letters in mathematics. I have argued that the use of variables in algebra and calculus involves three different views that I have called particular, general, and dynamic.
These views, and the challenges of simultaneously adopting them or of switching from one to another, have manifested themselves throughout the history of the study of dynamic situations. It is the general view that is dominant in the concept field of variable.

Adopting a general view seems to be a common challenge for students of algebra. Adopting a dynamic view seems to be most appropriate for students of calculus. Moreover, the ability to move among all three views seems particularly useful in a mathematical study of dynamic situations. Although a dynamic view of variable has been (in Boyer's words) "banished", I believe it remains crucial to an appreciation of the technology of calculus by providing the appropriate image for problems in calculus.

Variables are the most basic way of representing changing quantities--other representations are functions and graphs. In subsequent sections, I will show how the particular, general and dynamic perspectives emerge in students' views of these other formal constructs that are inherent in a mathematical study of dynamic situations.

**Concept Images and Field for Functions and Graphs**

Like variables, functions and graphs are important tools for mathematically representing dynamic situations. Unfortunately, often students are not able to use or interpret them as such. For example, Ponte's 16-year olds
"represented the variation of continuous phenomena by a set of unconnected points on a graph and used dot-by-dot strategies to represent continuous variation" (cited in Dreyfus et al., 1990, p. 121). Bednarz and Dufour-Janvier (1991) comment that "graphic codes and conventions which are in use with the intention of recreating change are mostly interpreted in a static way by children" (p. 141). In this section I will review literature on students' understanding of functions and graphs that speaks to the challenges students face in adopting a dynamic view of these representations.

**Personal Concept Definitions**

With a survey of 271 college students and 36 junior high school teachers, Vinner and Dreyfus (1989) studied personal concept definitions for functions and explored the subjects' concept images of functions. Within their sample they identified five groups based on mathematical experience: low, intermediate, high, mathematics majors, and teachers. The relationships between these groups and definition categories was also studied. They were able to identify six categories of definitions as follows: *correspondence* (the Dirichlet-Bourbaki definition of a function), *dependence relation* (y depends on x), *rule* (something that is regular rather than the arbitrariness that is possible with "correspondence"), *operation* (something that is done), *formula* (or equation) and *representation* (the function is identified with a
representation such as a graph). Their data showed that more of the groups with intermediate and low mathematical experience use the latter categories.

A similar categorization of students' understanding of the definition of function was carried out by Gooya (1988) in a study of twelve second semester calculus students. She identified four categories: some elements of a formal concept definition, a relation between variables, an equation, and what she diplomatically refers to as "idiosyncratic responses" (p. 96). Here too we see that student definitions of a function are quite varied.

**Informal Images of Functions**

Vinner and Dreyfus (1989) also observed a tendency to emphasize the operational aspect of a function. "One has to do something to x in order to obtain the corresponding y" (p. 364). Work by Anna Sfard on 16- and 18-year olds' understanding of functions showed "the majority of the students who were tested conceived of functions as a process rather than as a static construct" (Kieran et al., 1990, p. 109). Sfard (1991) shows this procedural emphasis to be prevalent in the history of the development of the concept of functions.

By examining the roles played by the aspects of one-valuedness, of discontinuity, of a split domain and of an exceptional point, Vinner and Dreyfus obtained further insight into their subjects' concept images of functions.
For example, many students rejected a discontinuous graph as representing a function, and many rejected a numerical correspondence with one exceptional point as representing a function. The respondents seemed to expect a certain amount of regularity from a function.

The idea that a graph of a function has to have a stable character throughout its entire domain plays a crucial role for these respondents....The idea that a function should be given by a single formula plays a major role. (1989, p. 362)

Perhaps the most striking observation from the data collected by Vinner and Dreyfus was that, except for the students of mathematics, most subjects who gave the Dirichlet-Bourbaki definition for a function did not use this definition while addressing other tasks on the questionnaire. This provides a clear example of the power of students' informal concept image in guiding their mathematical activity.

Reading Graphs

Bell and Janvier (1981) studied a variety of aspects of the command of graph reading that was displayed by 17 secondary school students in Britain. These aspects included "the recognition of global features by a progression from point reading to interval and gradient reading" and "measuring intervals or gradients and comparing intervals or gradients" (p. 37). By global features they mean such features as "the general shape of the graph, intervals of rise or fall, or of maximum increase" (p. 34).

They attempted to develop students' skills at
translating between situations and graphs, and between equations and graphs. This was done by emphasizing the nature of variation with the expectation that "using this notion of variation together with sketching a graph allows the pupil to go directly from a situation to its graph" (p. 36). Such an emphasis is included in the South Nottinghamshire Project where it is described as follows.

The main objectives of this topic therefore are to identify and describe the way the numbers will vary in different situations and relate this to a sketch graph. (p. 35)

Bell and Janvier identified aspects of the students' difficulties with making sense of the general shape of graphical representations.

[Among these are] various *distractors*, in particular pictorial distractors, when the shape of the graph is confused with that of a hill being climbed or the racetrack being traversed, and *situational* distractors, when experience of the situation interferes with attention to the meanings of the abstract features of the graph. (p. 37)

Later in the paper they describe a comparative teaching experiment between groups using graphs and using numerical tables. The first group was encouraged to speak meaningfully about the graphs representing data they had collected in a number of experiments. The second group did the same with data they had recorded in numerical tables.

The use of tables proved a powerful tool to study "how variables change". The results conclusively show that the table approach certainly spelled out many ideas to the extent of making possible transfers from tables to graphs, [and] suggest that the use of tables should be included in our graph teaching scheme. (p. 41)
This numerical emphasis seems to offer a means to overcome the difficulties of the "mental reconstruction of a dynamic process" (Bednarz & Dufour-Janvier, 1991, p. 140) and foster a dynamic view of variables and other representations of dynamic situations.

**Beyond Point Reading**

Bell and Janvier also identified "the reading of intervals and gradients as compared with points" to be a "key aspect of difficulty" (p. 42) for students. For example, when given graphs representing two populations and asked "when is population B greater than population A?", subjects tended to give a point response rather than an interval response. Often times this was despite considerable prompting (p. 37). Such a task requires working with a set of values simultaneously and hence I see it as involving a general view of whatever quantity is being represented, whereas, point reading would only require a particular view.

The challenge of reading intervals has also been addressed in work by Monk (1990). In a discussion of students working with two graphs representing in-flow and out-flow of water in a reservoir he states the following.

We never saw a student spontaneously use the vertical distance between the graphs as any kind of indicator of a quantity of water in the reservoir. Moreover, questions that I would now describe as quite leading, never elicited this notion from these students. (p. 134)

The Bell and Janvier question involves an interval of abscissa (pre-image) values, and the Monk question involves
an interval of ordinate (image) values. Monk classifies the nature of such questions respectively as backward and forward (p. 138). He uses four other categories, of which the most interesting for my purposes is across-time questioning.

*Across-time* questioning seems to require a *dynamic view* by asking for patterns in the variation of a quantity as the independent variable progresses. As an example of such questioning, given a distance versus time graph Monk asks "tell whether or not the distance covered in 5-minute intervals gets bigger or smaller in the period from 20 to 40 minutes" (p. 138). In the setting of area under a graph he poses "suppose the value of x increases from x=4.8 to x=6.0. Tell whether the function A(x) increases or decreases" (p. 139). In the context of a sliding secant line on a graph the across-time question posed is "as [the point Q moves towards the point P], does the slope of the line S increase, decrease or stay the same?" (p. 140). In the context of a speed versus time graph he poses "tell whether or not the cars are coming closer together in the time period t=½ to t=1 hour" (p. 140). Monk offers ample evidence of students being quite comfortable with forward questioning, but having difficulty with across-time questioning. He concludes that "they find Across-Time Questions much more difficult than questions that can be answered by looking at one point at a time" (p. 146).

I feel that across-time questioning involves much more
than "looking at one point at a time", which would only require a particular view. It also seems to require going beyond a general view of considering intervals, and requires a dynamic view in order to see the nature of whatever change is taking place.

**Concept Field**

The Dirichlet-Bourbaki definition referred to above is one that sees a function as a "one-valued correspondence between two sets" (Dreyfus et al., p. 119). The formal definition given by Bourbaki (1968) is as follows.

A graph \( F \) is said to be a functional graph if for each \( x \) there is at most one object which corresponds to \( x \) under \( F \) (Chapter I, §5, no. 3). A correspondence \( f=(F,A,B) \) is said to be a function if its graph \( F \) is a functional graph and if its source \( A \) is equal to its domain \( Pr,F \).

In other words, a correspondence \( f=(F,A,B) \) is a function if for every \( x \) belonging to the source \( A \) of the relation \((x,y) \in F\) is functional in \( y \) (Chapter I, §5, no. 3); the unique object which corresponds to \( x \) under \( f \) is called the value of \( f \) at the element \( x \) of \( A \), and is denoted by \( f(x) \) (or \( f_x \), \( F(x) \), or \( F_x \)). (p. 81)

I include this formal definition merely for the sake of completeness, I encourage the reader to keep in the mind only the quotation from Dreyfus et al. which paraphrases the definition. This is the sense of function presented in the textbook for the calculus courses at Langara (see Larson & Hostetler, 1982, p. 39).

Although this is a common approach in today's textbooks, it is not the only one. In talking about the intention of a graphical focus in the South Nottinghamshire Project, Bell
and Janvier (1981) offer the following.

Let us emphasise the fact that, here, functions are more than a subset of ordered pairs but a "definite connection between two things which change, also called variables". (p. 35)

Here are some who wish to view the notion of a function in a dynamic sense. However, in tracing the rise of the Bourbaki definition, Sfard (1991) points out that this dynamic view was not seen to be very workable.

The main problem with the early definitions of function was that they leaned heavily on the concept of variable, which by itself was rather fuzzy and escaped every attempt at reification. (p. 14)

Summary

In this section I have reviewed literature that addresses concept images of functions and of graphs, two mathematical constructs for representing dynamic situations. The work on functions provides some clear examples of the distinction between informal and formal concept images. The work on graphs describes student difficulties that I see as being tied to the differences between adopting the three views of particular, general, and dynamic as they apply to graphs. A dynamic view has seemingly been purged from the concept field of functions, however there are some who will still support such a perspective.
Concept Images and Field for Rates

Above, I have reviewed literature addressing students' understanding of variables, functions, and graphs. These mathematical constructs symbolically represent quantitative change. The nature of such change is quantitatively studied by the notion of rate of change. The larger the rate of change, the more a quantity will change. A negative rate of change describes a quantity that is decreasing. Formally, a quantity would be represented by a function, and the nature of the change in the quantity is described by the behaviour of the function. The behaviour of a function is related to the values of the function's derivative. In this section, I review literature that addresses the notion of rate of change. Much of this literature has involved subjects prior to their study of calculus.

Rate/Amount Distinction

A common difficulty for students seems to be the ability to distinguish between rate and amount. Work with high school students by Rubin and Nemirovsky (1991b) suggests they tend to confuse the slope of a line and the height of the line. They offer the example of students proposing Car A as the answer to the question in Figure 3 (p. 3).

Which car is going faster at t=2?

Position

Figure 3. A problem on comparing the speeds of two cars
Similarly, Bell and Janvier (1981) report that when students tried to determine which of two populations was growing faster over a given time interval, the "responses show fully the confusion among rate of increase (gradient), amount of increase, and greatest value" (p. 38).

In his study of calculus students' understanding of functions represented by graphs, Monk (1990) reports:

In effect, they had changed the meaning of the vertical axis from the amount that has flowed into the reservoir over some time period to the rate of flow into the reservoir at a given time....Something prompts these students to begin to act as if the graphs were of rates not amounts. (p. 135)

The notion of rate is important not only in the study of mathematics. Indeed, this notion is fundamental to most scientific study, and students' difficulties with it are also evident in other fields. For example, in describing the results of student interviews about basic concepts involved in the study of chemical equilibrium, Bergquist and Heikkinen (1990) identify difficulties of distinguishing between amount and rate, or quantity and relative quantity.

Students fail to distinguish between how fast a reaction proceeds (rate) and how far the reaction has gone (extent)....[they display] a general inability to distinguish between mass and concentration....[and a] confusion between amounts (moles) and concentration (molarities). (p. 1000)

Bergquist and Heikkinen observe that "a thread linking some of these misconceptions appears to be that many students are unable to grasp the proportional aspects of [these] concepts" (p. 1002). The proportional aspects of relative amount can
be seen as a problem of rates relative to the total amount. Such widespread confusion of rate and amount suggests that a more thorough look at this situation is warranted.

**Derived Quantities**

Monk distinguishes between variables or quantities that are displayed by a graph (that is, the labels on the axes) and other quantities which could be determined from what is displayed. He calls the latter "derived quantities" (p. 137). For example, for a graph of volume of water plotted against time, *derived quantities* might include such quantities as change in volume over a specified time interval, volume per one hour time intervals, or rate of change in volume with respect to time. He offers the following comment about the effectiveness with which *derived quantities* were used by beginning college students in a calculus course for social science and business majors.

What we found in these studies is that students tend to use graphs as if they were tables in which one could read the amount of water (In or Out) for each time and do arithmetic on these amounts...there was little or no use of the various derived quantities that we might see and use in this situation--such as Net Increase or Net Change in the amount of water over various time intervals. (p. 135)

The work of Bell and Janvier (1981) includes the following statement about this matter.

The greatest increase, it seemed, had to be connected with the greatest value. This "increase vs. value" distraction reappeared constantly in this and other tasks. The second observation was that increases, when needed, were generally obtained by referring back to the axes, reading off the values and subtracting them. More
sophisticated methods, such as reading the difference directly from the scale on the axis, without subtraction, or reading it even more directly from the grid in the body of the graph, were rarely used. (p. 37)

The first observation in this quotation seems to suggest that not only rate, but also other derived quantities tend to be confused with amount. The second observation seems to speak, as does the quotation from Monk, to the difficulty of working with anything other than what is immediately present. The second part of Monk's paper describes work with students from grades nine and ten on the notion of speed.

None of these students referred directly to rate of change of distance—or any single-number measure of speed. [One of them] used these distance/time pairs, such as "she went 10 miles in 5 minutes", with greater or lesser effectiveness. This tends to confirm my view that, regardless of how we would like students to think about ratio, correspondences among pairs is the way they most readily use this concept. (p. 147)

I would like to suggest that because working with derived quantities involves considering more than one value, it requires moving beyond a particular view and adopting a general view. This seems similar to what I reviewed above pertaining to students' difficulty with anything other than point reading from a graph. The propensity to focus on only what is immediately present, and to overlook what could be derived from this, may be a significant factor in the confusion students experience between rates and amounts. Moving beyond such a focus may involve a view of variables and functions which is more sophisticated than many students possess.
Features Shared by a Function and its Derivative

Even when students have made progress towards distinguishing rate from amount, there seems to be a tendency to assume the two concepts share features. In my own teaching practice, I have tried to assess the ability to distinguish between properties of a function and those of its derivative with problems such as in Figure 4.

Determine if the following statement is true or false, and give a BRIEF reason for your answer.

If \( f' \) is increasing on the interval \((a,b)\) then \( f \) is increasing on \((a,b)\).

Figure 4. A problem on features of \( f \) and \( f' \).

Of 28 responses to this problem the last time I used it, only 3 were correct, and one of these provided no rationale.

The work by Rubin and Nemirovsky (1991b) with precalculus students addresses the concept of rate of change in three technical environments: a motion detector coupled to a computer which displays graphs of the motion (distance or velocity) of a Lego car, a pump and an air bag coupled to a computer which displays graphs of air flow or volume of air in the bag, and a spreadsheet environment allowing subjects to define functions in terms of first and second differences and initial values. Subjects were given a velocity (or air flow, or first difference) graph, asked to predict the graph of position (or volume or function), and then asked to produce such a graph on the computer. They found that their subjects tended to "assume resemblances between the behavior or appearance of a function and its derivative" (p. 7).
assert "we believe this propensity is not a simple matter of confusing a function with its derivative" (p. 32).

For example, we know from everyday life that if we close a sink faucet slowly, the volume of water accumulated in a bucket will continue to increase. For us it is not an explanation to say "Dan doesn't know about that relationship", rather we try to understand what elements of the situation...may help or hinder the emergence and articulation of that knowledge to illuminate the problems he is dealing with. (p. 23)

They identify two classes of guiding ideas for such tendencies: "isomorphic variation and geometrical patterns" (p. 9). The first involves learners overgeneralizing experiential situations where a function and its derivative do behave the same. For example, the faster a car goes the more distance it gains--more velocity is more distance. This seems to be unduly generalized to draw the conclusion that less velocity cannot be more distance. The second class involves learners reproducing an observable pattern. These tendencies to assume common features among rate and amount were addressed in my own study.

Mathematical Conventions

In a second paper that considers the differences among their three technological environments, Rubin and Nemirovsky (1991a) describe difficulties with mathematical conventions. The car environment involved the complication of motion being relative to an arbitrary fixed reference point. One subject was particularly articulate about her difficulty with negative rate.
I still don't quite get the concept of negative velocity....If you're going you're going, what difference does it make if you're going forward or backwards? (p. 171)

This difficulty with negative rate of change was even more pronounced when addressing acceleration in the motion setting because "students' language was based on the absolute value of the velocity (i.e. speed)" (p. 172). Language used in the motion environment made no distinction between positive and negative. However, in the air bag environment, such a distinction was readily made. This environment offered a very natural reference point of zero volume and a much more natural meaning for negative rates. In this environment, their subjects overcame the difficulties of reference points specified by arbitrary convention and of negative rates.

Oh, now I see, all right....Then I pull it down and it [the flow rate] came back down to negative....The volume can never go down to be less than zero. The flow rate can. (p. 171)

Difficulties with conventional use of the negative sign might continue well into students' mathematical experience.

Monk's (1990) college students experienced difficulty with the notion of zero rate of change and how it should be represented graphically.

They would then begin to conclude that no water had come in over this time. But this conflicted, for many, with their strong belief that, "If no water had come in over this time, then the graph would have to be down on the x-axis which it is not." (p. 135)

Another challenge students seem to face with rate is
seeing it as a local property. For example, in describing his work with students from grades nine and ten, Monk (1990) makes the following comment.

It is perhaps significant that when asked questions about the speed [of Sally's trip] at one time or another, all of these students tended to go back to the beginning of Sally's trip, as if speed is the result of an overall pattern and not a "local" property. (p. 147)

This might very well be the consequence of the difficulties of moving to a general view in order to address derived quantities and then to a particular view in order to address the mathematical convention of instantaneous speed.

Concept Field for Rates

The notion of rate involves the consideration of two quantities, and hence is, in Monk's terms, a derived quantity. Although it has this extra layer of complexity, people are exposed to this notion daily in contexts such as interest rates at their bank or price per kilogram at their butcher. Such rates, however, are not as dynamic as the formal notion of rate is. Moreover, the formal notion of rate appears in many guises.

Precalculus instruction usually involves some emphasis on rate/time/amount problems that is restricted to situations in which the derived quantity of rate is constant. With the technology of function notation, students are introduced to the notion of average rate of change over an interval \([a,b]\) where it is defined to be the difference quotient \(\frac{f(b) - f(a)}{b - a}\).
The much briefer notation $\frac{\Delta y}{\Delta x}$ is used by Larson and Hostetler (1982) for this notion (p. 107).

In a graphical setting this ratio gives the rate of change of height with respect to horizontal distance, or the notion of slope. Students initially encounter this concept in the context of a straight line where slope is constant. In the setting of non-linear functions, students need to expand this prior knowledge to consider slopes that vary.

The notion of average rate of change is preliminary to the study of the derivative, and through the technology of limits, students move from this idea to the notion of instantaneous rate of change. The formal concept giving instantaneous rate of change is the derivative. The values of a function's derivative give information about the behaviour of the function. This connection between values of $f'$ and the behaviour of $f$ is summarized in the following Test for Increasing or Decreasing Functions (Larson & Hostetler, p. 166).

Let $f$ be a function that is differentiable on the interval $(a,b)$.

i. If $f'(x)>0$ for all $x$ in $(a,b)$, then $f$ is increasing on $(a,b)$.

ii. If $f'(x)<0$ for all $x$ in $(a,b)$, then $f$ is decreasing on $(a,b)$.

iii. If $f'(x)=0$ for all $x$ in $(a,b)$, then $f$ is constant on $(a,b)$.

The Derivative

Gooya's (1988) subjects displayed concept images of the derivative that seemed to preclude change. "It was hard for
them to conceptualize that the function [for the] derivative remained the same while its numerical values were changed by changing the point of tangency" (p. 98). They did not see that the derivative of a function would itself be a function.

On the other hand, their concept image of derivative did include computational mastery. "None of the students had difficulty in doing straightforward computation of the derivative" (p. 99). The straightforward computation she refers to is the derivative of \[1 + 1/(7-5x)^{1/2}\]\textsuperscript{50}! I have often witnessed this same computational mastery without conceptual mastery in my own teaching practice. For example, one of my most regular customers for tutorial help showed remarkable competence with derivative computations but was quite stumped with the most basic problems involving rate of change. When I posed the question "what is a derivative?", his response was "that's a good question, 'what's a derivative?' I've been doing them for days but I don't know what you mean."

The operational aspect mentioned above in connection to functions, seems to be at work in this context also. However, perhaps such an operational focus would not be so dominant if there were a strong dynamic view of functions as representing continually changing quantities.

**Summary**

Although we might have plenty of material world experience with the notion of rate of change, as an object of formal study it is quite complex because it involves
considering something beyond what is immediately present. The concept field for rate has many facets. In a numerical setting, rate involves the derived quantity of a ratio of two quantities, and in a geometric setting it appears as slope. In an algebraic setting, the average rate of change is considered, and in a calculus setting instantaneous rate is an interpretation of the derivative.

There is literature suggesting that students tend to confuse rate and amount, and tend to make assumptions that a function and its derivative will behave the same. This may have to do with developing enough sophistication to work with derived quantities, and may also be connected with an ability to work with a view of variables, functions and graphs that looks beyond the particular.

In my study, I set out to explore what sense students make of rate, and how this sense would contribute to their understanding of dynamic situations. The tasks I set for my subjects addressed their ability to work with the derived quantity of rate in graphical, numerical and algebraic settings.
Concept Images for Limits

In the sections above I have reviewed literature pertaining to students' understanding of variables, functions and graphs, and rates. These concepts are inherent in a mathematical study of dynamic situations. The symbolic technology for addressing dynamic situations comprises the field of study of calculus. In this section, I will review literature that addresses some of the challenges faced by calculus students. For the most part, any such literature I encountered focused on their command of the concept of limits, and so that will be the focus of this section. Although this concept is not germane to my study, some of the student conceptions that are reported are relevant to my study. Moreover, this section offers further examples of the exploration of informal concept images, and the role played by such images in mastering a concept. However, because the concept of limits is not germane to my study, I do not consider its concept field.

Language Concerning Limits

In an effort to identify preconceptions and personal models, as described in the first section of this review, Cornu (1981) studied how students use the language of the notion of limit. He asked subjects to use the word limite [limit] and the expression tends vers [tends to], in sentences. His subjects consisted of a group of students in classe de seconde (Grade 10 in France) who had not yet
studied the notion, and other groups at higher levels of study of mathematics. At the outset tends vers was not part of his subjects' vocabulary in a non-mathematical context. Eventually, it came to be used in the spirit of the word resembles—"Ce bleu tends vers le violet" [this blue resembles purple] (p. 324). He comments "elle peut ne pas contenir d'idée de variation effective" [it is possible for this use to not contain any actual sense of variation] (p. 324). In a mathematical context, he identified four models for the phrase:

-approaching something distant (for example, as a number goes from 1 to 3 it tends vers 10),
-approaches until it reaches a final state at which it stops (for example, as x goes from 1 to 3, x+1 tends vers 4 but then ça ne tends plus (p. 325)),
-approaches but never reaches, (for example, 1/x tends vers 0 as x tends to infinity),
-is close to, or is like (2.8 tends vers 3).

The first three involve a dynamic sense of variation, and were, in fact, only used when there was variation. "Une fonction constante ne peut pas tendre vers quelques chose" [A constant function could not tends vers something] (p. 325).

Preinstructional use of the word limite was more evident in the students' vocabulary than the use of tends vers. Limite was generally used to address something static and
non-mathematical, such as a geographic border. As with tend vers he identifies four models in a mathematical context:

- an unpassable boundary,
- an upper bound or lower bound (Apparently a number around which a sequence or function oscillates would not be seen as a limit in this model.)
- a number that is attained,
- a number that is unattainable.

Evidently, his subjects did not embrace a dynamic view of limit.

On notera que pour beaucoup, la notion de limite ne contient aucune idée de variation, de mouvement, de rapprochement de cette limite. [We should note that for many, the notion of limit contained no hint of variation, of movement, or of getting closer to the limit.] (p. 325)

Cornu also remarks that the two phrases were generally not used in the same context. Limite was used in precise contexts, whereas tend vers was used in vague contexts, so that unbounded sequences had no limit but could tend vers infinity. The sequence .9, .99, .999, .9999, and so on, has a limite of 1 but tend vers .999....

This particular distinction of usage was not evident in a study by Monaghan (1991) with 16-year-olds in their first year of A-level studies in England. He explored the use of the words tend to, approaches, converges, and limit, as they pertain to the above sequence, and to six graphs. This was done by asking the subjects whether or not they agreed to the
use of the phrases such as "the sequence tends to 1". With respect to the sequence, his subjects displayed comparable agreement to the use of the phrases tends to .9 (74%) and tends to 1 (66%) (p. 21). Similarly, they agreed with use of both of the phrases approaches .9 and approaches 1. The subjects were equally divided on whether or not the sequence converges to .9 and were marginally in favour of the limit is .9. However, most did not feel it converges to 1 and most did not agree that the limit is 1. This seems to indicate similar use of the dynamic phrases approaches and tends to, and similar use of the technical phrases converges to and limit is. However, it suggests the usage of the dynamic phrases is different from that of the technical phrases. Such a trend can also be seen in the use of these phrases as they pertain to the six graphs. Indeed, in the context of sequences and graphs of functions Monaghan concludes the following.

Tends to and approaches were very often seen as the same. They represent movement towards a terminus without ever getting there....Converges is generally seen as different to the other two verbs. (p. 23)

Monaghan also studied the use of these words in context free sentences. Everyday use of limit was for the most part as a boundary. Such use of approaches was mostly one of "drawing nearer" (p. 21), and of tends to was almost always one of "personal inclination" (p. 22). Everyday use of the word converges involved "two continuous objects coming
together and touching" (p. 23) and this led to some difficulties with its mathematical usage. One of the subjects states that "you'd have to have two sequences coming in on each other. I don't think you can have one sequence converging" (p. 23). This is a powerful illustration of how informal models can be influenced by spontaneous conceptions, and of the complexities that can emerge in a concept image.

Reaching the Limit

Tall and Vinner (1981) surveyed incoming university students in an attempt to identify concept images for limits of sequences. They conclude that a significant potential conflict factor in students' concept image of $s_n$ converging to $s$ is that the limit is never reached.

When a small group of four such students [who held this belief that a sequence cannot reach its limit] were shown the example $s_n = \begin{cases} 0 & (n \text{ odd}) \\ 1/2n & (n \text{ even}) \end{cases}$ they insisted that it was not one sequence but two. The even terms tended to one [sic, zero] and the odd terms were equal to one [sic]. So this was not a genuine example of a sequence some of whose terms equalled the limit! (p. 160)

Perhaps, these students were focusing on the different formulas defining this sequence and overlooking the sequence being the infinite ordered list of numbers $\left\{ \frac{1}{2}, 0, \frac{1}{4}, 0, \ldots \right\}$.

One wonders how they might have reacted to a sequence given by one formula, such as $s_n = \left(\frac{1}{n}\right) \sin\left(\frac{\pi}{2}\right)$, which displays the same type of behaviour.

In a later study involving 15 twelfth graders studying calculus in an academically selective high school, the notion
of limit was introduced "in such a way which was supposed to prevent concept images like the above being formed" (Vinner, 1986, p. 3). Such was not the case because, among other difficulties, these students also demonstrated an understanding of the unreachability of the limit. "Indeed these [students'] misconceptions were in general the same old familiar ones that seem to be revealed in all studies of beginning calculus students" (Davis & Vinner, 1986, p. 300).

**Dynamic Sense of Limits**

Unlike the students in Cornu's work, the subjects in the Tall and Vinner (1981) study did seem to embrace a dynamic view of the limit of a function. They were working with 70 students each with an A or B in A-level mathematics. Of the 49 of these students who gave a definition for \( \lim_{x \to a} f(x) = c \), 31 incorporated some dynamic sense by using words such as approaches, gets close to, or tends to. In response to a request for an explanation of \( \lim_{x \to 1} \frac{x^3-1}{x-1} \), "54 out of 70 used a dynamic approach in the example" (p. 163). They state the following about the role that a dynamic view plays in the concept image of limits of a function.

We would conjecture that the intuitive approach prior to the definition (if the latter is given at all) is often so strong that the feeling of the students is a dynamic one---as \( x \) approaches \( a \), so \( f(x) \) approaches \( c \)—with a definite feeling of motion. (p. 161)

The distinction that Monaghan's subjects showed between the use of the dynamic phrases and the use of the technical
phrases may suggest that the above conjecture is open to challenge.

Williams (1991) summarizes the research by Cornu, Vinner and others with the following statement.

Conceptions of limit are often confounded by issues of whether a function can reach its limit, whether the limit is actually a bound, whether limits are dynamic processes or static objects, and whether limits are inherently tied to motion concepts. (p. 219)

He administered a questionnaire, whose design was based on this research, to 341 second semester calculus students. The results of the questionnaire indicate that the aspects of a dynamic sense for a limit and of the unreachability of a limit were most prominent, and were strong even with those who had some sense of a formal concept definition.

Among those who recognized a paraphrased delta-epsilon definition (Statement 3) as true, roughly the same proportions also recognized Statements 1 and 4 [dynamic and unreachable, respectively] as true as for the whole group. (p. 225)

Fifty students from this population volunteered for further study, and from this group, 10 were chosen for a series of interviews to explore their conceptions of limit in depth. These interviews included giving the subjects two opposing statements attributed to hypothetical calculus students. The subjects were asked to choose the statement they found most agreeable, and to describe what they found disagreeable about the other statement. There were three such pairs of statements addressing the reachability of a limit, a dynamic sense for a limit, and the sense of a limit as a boundary.
The dynamic sense was widespread and was robust.

All 10 students in the study expressed at some point a view of the limit as dynamic....No real change occurred in the 10 students' dynamic view of limit. (p. 229)

This view involved the use of words such as approaching or getting close, with two apparent interpretations.

As describing the physical process of evaluating a function at different numbers, which are over time to be closer and closer to the value s, or as describing the mental process of imagining the points on a graph moving closer to the limit point. (p. 228)

The second interpretation clearly involves motion, hence I would see it as embracing a dynamic view of variable.

Because of the phrase "over time to be closer and closer" I also see the first interpretation as embracing such a view of variable. As such, this seems to offer some support for the Tall and Vinner conjecture mentioned above. Indeed, one of Williams' subjects had a model that went well beyond a dynamic process and included the notion of rate of change.

If the function was like this [draws a monotone increasing graph approaching a horizontal asymptote from below], then when it's very close and if you were considering x-values as a kind of speed, when this was moving very fast, the y-values are moving very slowly....That's they way I'd like, that I think I understand best. For this type of function. Now if we were doing a line both their speeds would just be proportionate to the slope. (p. 231)

Procedural Emphasis

While addressing reachability, six of the subjects seemed to focus more on the utility of the process of taking a limit--if a function could reach its limit then you would not need to find its limit.
You never would really want to find a limit, you know, where there is a point, a continuous function. So it's, you know, kind of a moot point, that you can even use it as one....If you don't do the limit or anything on it, you can plug in that 1 and you're going to get the value. You're going to get the exact value. But that would be a totally different separate problem, I mean nothing to do with limits. (p. 226)

Both the rationale for using limits offered in this quotation and the mention of "the process of evaluating" in the quotation about interpretations of approaching and getting close seem to speak to an operational emphasis within the concept images of these students. Indeed, Davis and Vinner (1986) state that "the algorithm by means of which a numerical concept is calculated becomes the definition in the minds on many students" (p. 298).

The models of limits that Williams' subjects adopted were very much influenced by their schooling experience.

Students tended to favor the view of limit that was expedient....In many cases the types of functions presented to them [in the interviews] were not the kinds they were used to seeing in the classroom or on tests....[A student said] "But for those in our textbook, they usually showed simple ones that don't create problems." (1991, p. 233)

This is an example of what Cornu states about usual classroom practice being quite manageable with incomplete concept images (see page 14 above).

Summary

This review of literature addressing calculus students understanding of limits offers some insight into informal concept images of limits. We see language reserved for
separate contexts, even though such language might be synonymous when used in a mathematical sense. The issue of whether a limit can be reached seemed to be a major concern. The procedure of computing a limit was seen as having little use if the limit could be reached. A dynamic sense was quite evident in this context.

The fact that these concept images are so common across geographical locations (such as France, England, and various regions of the U.S.A.), and across time (over a decade), speaks to the challenging nature of these aspects involved in understanding the notion of limits. Indeed, Cornu states the following.

L'étude de l'histoire de la notion de limite permet de voir que la plupart des modèles que nous avons rencontrés chez des élèves ont existé et ont joué un rôle dans l'évolution de la notion de limite. [A study of the history of the concept of limit allows us to see that most of the models we encountered in the students have existed and played a role in the evolution of the concept of limit.] (p. 326)

Chapter Summary

Much of the literature reviewed in this chapter involved students prior to their study of calculus. The material that did involve calculus students, for the most part, addressed concepts not essential to my research questions. Nonetheless, drawing these two categories together is essential for my study. The research I have reviewed above provided the framework in which I will analyze students' understanding of dynamic situations, and offered many
examples of how this framework supports analysis of other concepts.

The framework by Tall and Vinner (1981) for analyzing students' conceptual understanding distinguishes the way in which a concept is conceived from the way in which it is defined mathematically because the human brain is too complex to operate within the logical restrictions of mathematics. I use a variation on this framework that analyses a student's understanding of a concept by considering the student's informal concept image in relation to the concept field. The latter includes everything that the mathematical community at large sees as surrounding a concept. The former consists of everything within an individual's conceptual structure that is spontaneously associated to a concept. In other parts of the chapter, this framework was considered as it applies to concepts that are fundamental in a mathematical study of dynamic situations.

As a result of reviewing literature concerning students' understanding of the concept of variable, and of considering a historical perspective of developments in mathematics, I feel it is useful to identify three increasingly complex views of variables and changeable quantities. These views see a variable as representing a particular object, a general object, or a dynamic object. Much of the literature on learning algebra suggested that students face a challenge in moving from a particular view to a general view. There has
been a history of challenges in adopting a dynamic view within the mathematical community at large. This view seems to be largely ignored by this community.

The work I reviewed on students' understanding of functions showed how varied their concept images can be. It also offered an example of how formal aspects of a concept image are not put to use with the same power as informal aspects. The material that addressed students' abilities with graphs suggested that any task involving a perspective beyond a particular view was challenging for students.

Some important questions about students' command of rate came out of this literature review. There seems to be a tendency to confuse rates and amounts, some difficulty working with derived quantities, a tendency to attribute the same properties to a function and its derivative, and some difficulties with the mathematical conventions of negative and zero rates.

The section on limits offered examples of how language tasks are used to study informal concept images, and further evidence of the differences between informal and formal concept images. It is worth repeating that students in these studies maintained separate contexts for words and phrases that would be seen as mathematically synonymous, and that these students did bring a dynamic sense to the study of limits.

The object of my study is what calculus students
understand about dynamic situations. This literature offered some sense of what to consider as I study students' images of the basic concepts such as variables, functions, graphs and rate, that are involved a mathematical description of dynamic situations. I am interested in whether a dynamic view, which apparently is put to work in studying limits, is applied to an understanding of dynamic situations. Will subjects in my study be challenged by working with intervals or other derived quantities, as were students in the work of Bell and Janvier, and Monk? Will the student difficulties with rates that are suggested by the literature be evident in my subjects?

The methodology for my study was, for the most part, shaped by the literature reviewed in this chapter. Many of the tasks that I set to explore students' understanding of various aspects of a mathematical study of dynamic situations were borrowed or adapted from this material. The next chapter will include a detailed description of these tasks.
CHAPTER III
METHODOLOGY

In this chapter, I present the methodology I used to collect, analyze and report the data for my study. The first section addresses general interview strategies I employed and why I used them. The second section describes how the methodology for the pilot study affected the main study. Methodology for the main study is presented in the third and fourth sections—the latter is a detailed account of the tasks I set for my subjects. The chapter closes with a description of how the data were analyzed, and how the analysis has been presented.

Clinical Interview Methodology

Research into mathematical thinking has three basic aims: the discovery of cognitive processes; the identification of cognitive processes; and the evaluation of competence. Theoretical analysis shows that the clinical interview is the most appropriate method for accomplishing these aims, although it is far from foolproof and although other procedures, chiefly naturalistic observation and standard tests, may also have appropriate uses. (Ginsburg, 1981, p. 10)

The three aims identified by Ginsburg were paramount in my study of calculus students' understanding of dynamic situations. Naturalistic observation (insofar as classroom and tutorial interactions are natural) and standard testing (inasmuch as usual assignments, tests and exams for a calculus course are standard) played a role in formulating
the questions I wished to investigate. However, investigating these questions required something very different from the types of interactions that contributed to formulating the questions. Ginsburg states some "investigators believe that the clinical method is superior to standard testing and makes possible a degree of insight that is virtually impossible to obtain through standard tests" (p. 5).

Posner and Gertzog (1982) comment on the breadth of purposes to which a clinical interview can be put.

[Interviewing] is clearly a common phenomenon with a wide range of purposes. Probably the most useful way to consider this range is as a continuum: At one end is the situation of interviewer seeking information from the interviewee, and at the other a troubled interviewee seeking help from the interviewer. (p. 195)

My use of the interview methodology spanned both ends of this continuum. I sought information from the interviewees by having them address a number of tasks of my design. However, I also offered the interviewees tutorial help during each interview after such tasks had been addressed.

These tutorials were recorded and data from them was also taken into consideration for the study. I felt that the subjects' investment of time and attention in my project warranted some reward, and that such tutorial help was all that I could offer them as a reward. I do not feel this jeopardized the study in any way because the questions I wished to address with the study did not require that I consider only how their classroom activities contributed to
their understanding. On the contrary, this tutorial service enriched the study by offering unique scenarios during which the subjects wrestled hard with problems of calculus. In this sense, the interviews were what Posner and Gertzog (1982) describe as elite because each interviewee was given some "special non-standardized treatment" (p. 198).

Ginsburg (1981) acknowledges that "in any given study the distinctions among [the three aims above] may be blurred, and more than one aim, may be involved" (p. 5). Nonetheless, he addresses the aims separately and describes interview techniques appropriate for each of the three aims.

When the discovery function is stressed, the clinical interview procedure begins with (a) a task, which is (b) open ended. The examiner then asks further questions in (c) a contingent manner, and requests a good deal of reflection on the part of the subject. (p. 6)

In my study, even those tasks for which there was a correct response were addressed in an open-ended spirit to the extent that I emphasized to my subjects that my main concern was how they thought about the tasks, rather than their ability to produce a correct response. Although the tasks I used were predetermined, my interview questioning was contingent in the sense that each subject addressed the tasks in his or her own way and I responded to each particular method. Moreover, when the subjects were puzzled I offered tutorial help usually by posing questions about their method.

With respect to the aim of identification, Ginsburg asserts that "the clinical interview is intended to
facilitate rich verbalization which may shed light on underlying process" (p. 7). To this end, the subjects were continually reminded to vocalize their thoughts for the tape recorder. In addition to being asked to describe what they were doing or thinking, they were also asked to describe why they were doing it or thinking it.

As for the issue of competence, Ginsburg states that "often researchers want to know what the child 'really knows' about division even though his performance in class or on an achievement test may be poor" (p.8). Conversely, as I stated in Chapter II, researchers such as Cornu (1981) have argued that good performance on usual test materials does not necessarily imply good understanding of the concepts these materials test. Van den Brink (1990) suggests that one should do "justice to the object of study by examining it from different aspects" (p. 35). To this end I tried to study what calculus students really understand about dynamic situations by having them consider a variety of tasks that involved various aspects of understanding such situations.

Posner and Gertzog (1982) suggest four types of tasks appropriate for investigating conceptual change (p. 207). One of these types that I used was a general open-ended task, such as constructing a focused list, commenting on a quotation, or commenting on proposed solutions to a problem. Most of the tasks I chose, though, were of a second type mentioned by Posner and Gertzog--the problem-solving type.
Students were asked to think aloud as they addressed problems that I set. Addressing these problems, at times, involved a third type of task proposed by Posner and Gertzog that they call the garden-path type. "These tasks are designed to lead the students 'down the garden path' to conclusions that are counterintuitive" (p. 207). A detailed account of the tasks that I chose and the rationale of these choices is presented in the section below entitled Interview Tasks.

In summary, I chose the interview methodology because of the richness of the data it affords. I designed the interview tasks, and practised interview techniques, in an effort to meet the goals of discovery and identification of cognitive processes, and evaluation of competence. I felt that well practised pencil and paper responses to traditional tasks can offer misleading information about students' understanding of the tasks at hand. I was interested in informal or non-practised responses to non-standard tasks because I saw these as providing a better view of students' understanding than well rehearsed responses would provide.

The verbal exchange and the intimacy of the interview environment provided very deep insight into the processes the students experienced. If there were a scale that could measure an instructor's access to what a student understands, I would suggest that the readings on the scale would increase as the environment changed from classroom interaction, to assessing submitted work, to office hour interaction, to non-
Methodology for the Pilot Study

The methodology for my study was piloted throughout the summer of 1991 with two students over a sequence of four interviews. Before starting this stage of my study, appropriate consent was requested, and received, from the chair of the Department Mathematics and Statistics at Langara (see Appendix A).

I polled those departmental colleagues teaching introductory calculus courses for permission to solicit volunteers from their classes. In the classes where such permission was granted, I described the study and distributed a form for students to return to me if they were interested in volunteering (see Appendix B). This method of soliciting volunteers was also used for the main study.

Two subjects for the pilot stage of the study were chosen, from eight who volunteered, based on their written responses on the form. From a very practical perspective, this turned out to be a poor criterion on which to base a choice of interview subjects. Although the written response was quite articulate, the speech patterns from one of the subjects were so difficult to comprehend that the tapes of the interviews ended up being very expensive to transcribe. As such, I resolved to interview all volunteers for the main
part of the study, and to choose my subjects after they had been interviewed once.

At the outset of the interviews, the subjects were informed of their prerogative to leave the study, or any part of the study, at any time. A sample of the consent form they signed is included in Appendix C. This form was also used for the main study.

The pilot stage was also used to sort out the technological details of recording student interviews. For this stage, the interviews were done in my office at Langara and were recorded with two portable tape recorders. The office setting proved to be too noisy, and so for the main study I arranged to use an audio studio in the Instructional Media Centre at Langara.

The use of two machines to record the interviews turned out to be an invaluable strategy. Two machines generally run at different speeds. As such, there is no data lost when a tape ends. Also, there were occasions on which one of the two machines malfunctioned but, again, no data was lost. Although such technicalities are minor issues, learning these lessons contributed to a main study that ran smoothly.

The most important way in which the pilot study shaped the main study was in the nature of the interview tasks, and in the very questions that I ended up studying. For example, some of the open-ended tasks, such as describing a dynamic situation, did not generate as much data about the subjects'
intuitive sense of change as I would have liked. My lack of experience with interview technique was also an obstacle to generating such data. I used relatively fewer of this open ended type of task in the main study.

Some of the problems addressed in the pilot stage were of a fairly open ended computational nature as in Figure 5.

A truck is travelling north along Main Street at a rate of 15 m/s. Before the truck comes to the light at 49th, the light turns green and a car waiting there accelerates at a rate of 2 m/s² also travelling north. Under these conditions the distance (measured in meters north of 49th) travelled by the truck and car respectively in t seconds after the truck goes through the intersection will be given by T(t)=15t and C(t)=t²+12t+18. Determine the values of t for which the truck will be ahead of the car. What can you say about the distance between the vehicles at the time of t=1 second? And at t=2 seconds?"

*Figure 5. A sample problem from the pilot stage.*

The subjects in the pilot study made so little progress with such problems that I felt some other type of activity was necessary in order to promote responses. In the section below entitled Interview Tasks, I describe how this problem was modified for the main study in an effort to generate better data. Other tasks chosen for the main study were far less open-ended in nature so that data might be more closely related to the students' understanding of the concepts of change and rate of change.

At the outset of the pilot study, I was interested only in the subjects' intuition about the dynamic aspects of change—that is, what they thought about quantities that continuously change. In trying to study this, I recognized
that isolating one aspect, such as intuition, does not do justice to the complexity of learning calculus. This is a subject that is filled with formal representations. My study would only be worthwhile to me as an instructor if I also considered the connections that students make between their intuitive sense and the formal representations.

The interviews during the pilot stage were scheduled to be between 60 and 90 minutes duration--this I found too long. Also, during this stage, I found that tutorial work with the subjects was every bit as informative as the tasks I had chosen. Hence, the interview tasks for the main study were designed to take at most about 45 minutes so that there would be some time left for tutorial work before an hour had passed.

Data collected during the pilot study was not used in the main study, but was used for a paper written as part of my course work. This gave me practice in analyzing tape transcripts. Even though the data from the pilot study was used only in this way, the pilot stage was invaluable. It contributed to shaping the main study as described above, and it afforded me an opportunity to practice such skills as interview techniques and transcript analysis.
Methodology for the Main Study

As with the pilot study, subjects for the main study were solicited by visiting introductory calculus courses. At Langara we have four such courses: one for economics and business majors, one for science majors, and a stream that splits the science course into two courses. This stream starts with Math 153, which introduces all differential calculus concepts but considers only algebraic functions. This is followed by Math 253 which revisits the issues of calculus as they pertain to transcendental functions. Three of the subjects for the study were from Math 153, and the fourth subject was from Math 253.

Seven students volunteered for the study, and I made arrangements to interview all of them. Only five of these students kept their appointment with me. These five all seemed very cooperative and articulate, so a second interview was arranged with each of them. One of the five missed the second appointment, and I did not try to arrange any further appointments with that student. The other four subjects stayed with me for the duration of the project. I gave these subjects the pseudonyms Amy, Cam, Margo and Ralph.

All of the subjects returned to college studies after some absence from previous schooling. As such, they were representative of a significant part, although not a full cross-section, of Langara's student population. They all had considerably more life experience than most high school
graduates.

Each subject took part in a sequence of eight interviews. Each interview lasted at most one hour, with about two weeks between successive meetings. During the interviews, subjects addressed tasks of my design, and then were encouraged to ask me questions about matters from their course work that puzzled them. They almost always had some issue for this tutorial work.

The interviews were audio-taped, and then were transcribed. The transcription work was contracted out to a professional with such skills. This person had no connection to the subjects, and had little experience with mathematics education. The tapes were transcribed verbatim with little use of mathematical notation.

The design of interview tasks was based largely on the literature I reviewed. Hence, for the most part, their construction was not influenced by subjects' responses to tasks from previous interviews. One exception to this was the design of the sixth interview which was based on responses from the third interview. Although most tasks were preplanned in this sense, they were prepared as the interviews progressed. A detailed description of each task is provided in the next section.

During the interviews I was mainly interested in informal responses, and hence the tasks were generally not what would be seen as standard calculus exercises. However,
as part of the study I was curious about how their informal understanding contributed to their mastery of the formal course work. Hence, in addition to what I was able to infer about their formal progress from tutorial interaction, I considered their responses to some of the problems on their final examination. The problems I considered were those that addressed rates, and the relationship between a function and its derivative. I had no part in making up any of these examination problems.

From the Math 153 examination, I considered two problems pertaining to average and instantaneous rates of change, and a related rates problem. These are presented in Figure 6.

Answer true or false with a brief explanation for your answer.
If $f(x)$ and $f'(x)$ are defined over the interval $[a,b]$, then the average rate of change of $f(x)$ over $[a,b]$ is $\frac{f'(a)+f'(b)}{2}$.

The volume of water in a leaking tank is given by $V(t)=(1-t^2)^2$, where $t$ is measured in hours, $0\leq t\leq 1$.

a) Find the average rate of change of the volume between $t=0$ and $t=\frac{1}{2}$.
b) Find the instantaneous rate of change at $t=\frac{1}{2}$.

Wheat is poured through a chute at a rate of 10 m$^3$/min, and falls in a conical pile whose bottom radius is always half the height of the pile. How fast will the circumference of the base be increasing at the instant the pile is 8m high? (Volume of a cone is $V=\frac{1}{3}\pi r^2h$)

Figure 6. Problems from the Math 153 examination.

The Math 253 examination had more of a computational emphasis, so for example, there was nothing explicitly requesting students to distinguish between the concepts of average rate of change and instantaneous rate of change. It included two graphing problems and two related rates problems whose solutions I considered in my data analysis. They are presented in Figure 7.
Sketch a good graph of \( y = x - \cos(x) \) over \([0,2\pi)\) on the grid below. Include an analysis of \( y' \) and \( y'' \) in your work, and indicate intervals where the graph is increasing or decreasing, and concave up or concave down. Don't bother finding x-intercepts.

Sketch a good graph of \( f(x) = \frac{\ln(x)}{x^2} \) over \((0,4)\). Show all intercepts, asymptotes, relative and absolute extrema, and inflection points on your graph.

At a certain instant, each edge of a cube is 5cm long, and the volume is increasing at a rate of 2cm³/sec. How fast is the total surface area of the cube expanding at that same instant?

A revolving light is located on a stationary ship which is 4km from a straight shoreline. If the light makes one revolution every 10 seconds, how fast is the beam of light moving along the shoreline, when it makes an angle of 45° with the shore?

**Figure 7. Problems from the Math 253 examination.**

As I stated above, these questions were used only to gather some data about how the subjects' informal understanding contributed to their mastery of the formal course material. The bulk of the data I analyzed came from tasks I set in order to study their informal understanding. The next section describes these tasks in detail.

**Interview Tasks**

**Interview #1**

The first task addressed in this session was a *focused list* on the word variable. The subjects were asked to write down and comment on anything that came into their mind about the word variable. The name and design of this activity were borrowed from the work of Cross and Angelo (1988, p. 22). However, they suggest its use as a review activity. My purpose in using it was to promote context-free informal associations with the focus—in the spirit of Cornu's
spontaneous models.

We subsequently addressed the problem in Figure 8, which I will refer to as the informal car/truck problem.

Imagine a car sitting at the traffic light on Main Street and a truck approaching the light heading in the same direction as the car. The light turns green before the truck reaches it and the car starts away from the light. Describe the possible relative motion of the two vehicles.

Figure 8. The informal car/truck problem addressed during the first interview.

Subjects were asked to picture the scene in their mind and then describe what they saw. They were then given a toy truck and a toy car, and asked to recreate what they had described. These stages of imaging, describing, modelling and then redescribing seemed to promote much deeper discussion, and hence generate more data, regarding the car/truck scene than what had transpired in the pilot stage of the project where only a description was requested.

Interview #2

The first activity for this session was a focused list on change. As described above, the subjects were asked to write down and comment on anything that came into their mind about the word change.

The subjects were then given the task shown in Figure 9, which I will call the formal car/truck problem. They were asked to comment on each of the proposed solutions by stating whether or not they agreed with it, and by describing what they liked or disliked about it.
A truck is travelling north along Main Street. Shortly before the truck comes to the light at 49th, the light turns green and a car waiting at the stop-line there accelerates away from the intersection also travelling north. We can study the distances travelled by the truck and by the car as functions of the time elapsed after the truck crosses the stop-line. By measuring the distance as metres north of the stop-line the formula for how far the truck has travelled is given as $T=15t$ and that for the car is given by $C=t^2+6t+18$. When is the truck ahead of the car?

**STUDENT A**

By comparing the equation for the car, which is $t^2+6t+18$, with the one for the truck, which is $15t$, I see the number 18 is bigger than the number 15. So the car starts ahead of the truck and proceeds onward staying ahead of the truck.

**STUDENT B**

I would try $15t>t^2+6t+18$. So that $0>t^2-9t+18$. I can factor $t^2-9t+18$ as $(t-3)(t-6)$ and so I would think about the polarity.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>t-2</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>t-7</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$t^2-9t+18$</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

So the truck passes the car after 2 seconds and then the car passes the truck after another 5 seconds.

**STUDENT C**

By trying $t=1$ we would get $C=25$ and $T=15$. Let me make a table...

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>25</td>
<td>34</td>
<td>45</td>
<td>58</td>
</tr>
<tr>
<td>T</td>
<td>15</td>
<td>30</td>
<td>45</td>
<td>60</td>
</tr>
</tbody>
</table>

So the truck is ahead for 4 seconds.

**STUDENT D**

I think we should solve $15t=t^2+6t+18$. Let's see, collecting like terms gives $t^2-9t+18=0$. Factoring gives $(t-3)(t-6)=0$ so $t$ is either 3 or 6. 6 is bigger, so I think the truck is ahead of the car for six seconds.

Figure 9. The formal car/truck problem addressed during the second interview.
This task was suggested by the work of Williams (1991), and was chosen because, as mentioned in the section above on the pilot study, the subjects in pilot stage were not able to make very much progress with the algebraic problem in Figure 5. The offering of alternatives for subjects to choose from and comment on seemed like an effective means for getting around the need for subjects to be inventive.

I wrote the proposed solutions based on student conceptions of variables as described in the literature, or observed by me in the pilot stage or in my teaching practice. All proposed solutions contain errors— it was the nature of the errors and the view of variable that differed among the proposals.

The first of the proposals ignored the presence of the time variable altogether and drew conclusions from the nature of the coefficients that were present. I see this as being similar to the first three categories of use of letters identified by Küchemann (1981).

The second solution contained algebraic errors in the factoring of a quadratic expression but was based on a dynamic view of variable that looked for a range over which time varied and used a polarity chart to analyze the solution of a quadratic inequality. The polarity chart is a piece of symbolic technology that helps in the analysis of the algebraic sign of an expression. At Langara, we include this topic as part of our instructional emphasis. Of the four
proposed solutions, this is the one I saw as being the most correct conceptually.

The third proposed solution involved evaluating the expressions at particular instants and drawing a conclusion that confused a particular instant of time with a duration of time. This proposal was based on a mixture of a particular view and a general view of variable.

The fourth proposal was based on a particular view of variable. It involved setting the two expressions for distance to be equal and then solving the resulting quadratic equation. The algebra was correct but the conclusion drawn from the two answers overlooked the relevance of the two answers as the boundaries of the required solution.

Interview #3

The first activity for this session was a focused list on rate. The subjects were then asked to sketch graphs representing what transpired in a dynamic situation that I set up for them. Water, that was stored in a burette, was allowed to flow into a variety of containers (a 50 ml beaker, a 50 ml graduated cylinder, a 50 ml Erlenmeyer flask, a stoppered funnel that we called "the Martini glass", and a volumetric flask). The subjects were asked to sketch graphs representing the height of the water in each container as time progressed. They were also asked to comment on why they sketched what they did. This activity was inspired by the work of Bednarz and Dufour-Janvier (1991) on spontaneous
representations, and of Bell and Janvier (1981) on the interpretations of graphs. I call this the flowing water task. A theoretical analysis, based on Torricelli's law, of the height of the water in each container is given in Appendix D.

**Interview #4**

The converse of the flowing water task was addressed as part of Interview #4. It asked the subjects to identify the situation from which a given graph arises. The task, called bottles, was borrowed from research by Saw Hoon Teah as described by Rosalinde Scott-Hodgetts (1988, p. 271). I learned about this reference after having addressed the flowing water task in the third interview. The instructions to the subjects are reproduced in Figure 10.

Here are pictures of six bottles and nine graphs. Identify which graph goes with each bottle. For those graphs left over, sketch the shape that the bottle should have.

![Diagram of bottles and graphs](image)

**Figure 10.** Bottles, a task addressed during the fourth interview.
The main activity for this session explored the subjects' command of interpreting a graph representing a dynamic situation. The task, called microbes, was borrowed from the work of Bell and Janvier (1981, p. 37), and is presented in Figure 11.

The graphs below indicate two populations of microbes in a laboratory culture throughout a twenty-four hour period. The populations are measured in units of millions of microbes per cubic centimetre--let's denote this mill/cc. Please respond to each of the questions below and vocalize your thoughts as you address each question.

**POPULATION (in millions per cubic centimetre)**

1) What is the population of culture A at 2 am?
2) What is the population of culture B at 4 pm?
3) When is population B larger than 2.1 mill/cc?
4) When is population A smaller than .9 mill/cc?
5) When is population B greater than population A?
6) How much does population A grow between 2 pm and 6 pm?
7) How much does population B grow between 4 am and 6 am?
8) Between 2 am and 6 am which is the faster growing population? Between 1 pm and 4 pm?
9) Does population A grow faster between 4 am and 6 am or between 3 pm and 4 pm?
10) When is population B growing most rapidly?

**Figure 11.** Microbes, a task addressed during the fourth interview.
The questions were developed from the works of Bell and Janvier (1981), and of Monk (1991). Questions 1 and 2 addressed the subjects' ability to properly interpret what the graphs represent—Bell and Janvier call this point reading. The next five questions involved the subjects' ability to read intervals determined in various ways. I see these types of questions as involving a general view of variable because of the need to work with more than a particular value of the variable. Questions 3 and 4 addressed their ability to identify a range of values of the independent variable from a value of the dependent variable—what Monk calls backwards questioning (p. 138). Question 5 addressed the same ability as above but this time given a criterion that is not particular. Questions 6 and 7 explored the ability to read a range of values for the dependent variable—what Monk calls forward questioning (p. 138). Question 8 used this skill to read a derived quantity in order to judge a choice between the two graphs. Question 9 addressed the same issues as they apply to two parts of the same graph. Question 10 addressed a graphical view of instantaneous rate.

**Interview #5**

The first activity for this session asked subjects to work with a numerical table representing a dynamic situation. The setting of comparing weights of two groups and the idea of working with a numerical table both came from Bell and
Janvier (1981). This activity is called *weights*. The table and questions are presented in Figure 12. These questions were chosen to parallel those in *microbes*, however, there were some small changes. Questions 3 and 4 from *microbes* were collapsed to one (question c), and Questions 6 and 7 were collapsed to question e.

The table below gives the average weight in kilograms of a population of girls and a population of boys throughout the first eighteen years of their lives. With reference to this table, please answer the questions below and vocalize your thoughts as you address each question.

<table>
<thead>
<tr>
<th>Age (in years)</th>
<th>Girls average weight</th>
<th>Boys average weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.3</td>
<td>6.2</td>
</tr>
<tr>
<td>2</td>
<td>9.5</td>
<td>9.5</td>
</tr>
<tr>
<td>3</td>
<td>12.2</td>
<td>12.3</td>
</tr>
<tr>
<td>4</td>
<td>13.8</td>
<td>14.0</td>
</tr>
<tr>
<td>5</td>
<td>14.3</td>
<td>15.2</td>
</tr>
<tr>
<td>6</td>
<td>16.1</td>
<td>17.3</td>
</tr>
<tr>
<td>7</td>
<td>17.5</td>
<td>19.2</td>
</tr>
<tr>
<td>8</td>
<td>19.2</td>
<td>22.1</td>
</tr>
<tr>
<td>9</td>
<td>22.3</td>
<td>25.8</td>
</tr>
<tr>
<td>10</td>
<td>25.8</td>
<td>28.2</td>
</tr>
<tr>
<td>11</td>
<td>30.1</td>
<td>30.4</td>
</tr>
<tr>
<td>12</td>
<td>35.4</td>
<td>32.3</td>
</tr>
<tr>
<td>13</td>
<td>40.6</td>
<td>35.8</td>
</tr>
<tr>
<td>14</td>
<td>48.7</td>
<td>44.6</td>
</tr>
<tr>
<td>15</td>
<td>50.5</td>
<td>50.1</td>
</tr>
<tr>
<td>16</td>
<td>52.7</td>
<td>58.9</td>
</tr>
<tr>
<td>17</td>
<td>55.2</td>
<td>65.6</td>
</tr>
<tr>
<td>18</td>
<td>56.1</td>
<td>68.4</td>
</tr>
</tbody>
</table>

a) What is the average weight of the boys at age nine?
b) What is the average weight of the girls at age four and a half?
c) When do the boys weigh less than 24 kg?
d) When do the girls weigh more than the boys?
e) What is the increase in the girls average weight between ages three and eight?
f) Does the boys weight increase faster between the ages two and five, or between the ages nine and eleven?
g) From ages nine to ten which group grows faster?
h) When do the boys grow most rapidly?

*Figure 12*. Weights, a task addressed during the fifth interview.
The second activity for this session had the subjects address the same types of questions but work with an algebraic representation of a dynamic situation that was without context. They were given two functions and asked virtually the same questions in this setting as they addressed in the numerical setting. The numbers were chosen so that correct algebra would lead to numerical results that were not too complicated. I call this activity functions, it is reproduced in Figure 13. Solutions are presented in Appendix E.

Consider the functions $f(x)=1+3.2x-x^2$ and $g(x)=.2x^3$. Please answer the questions below and vocalize your thoughts as you address each question.

a) What is the value of $f$ at $x=.4$?
b) When is the value of $g$ larger than 3.9?
c) When is $f(x)$ smaller than 2.92?
d) When is $f(x)$ larger than $g(x)$?
e) By how much does $g$ change between $x=2.3$ and $x=2.6$?
f) Does $f$ grow faster for $x$ between .7 and .9 or for $x$ between .6 and 1.2?
g) Which function grows faster for $x$ between .7 and .9?
h) Which function grows faster at $x=1.55$?

Figure 13. Functions, a task addressed during the fifth interview.

**Interview #6**

This session consisted of four activities that addressed translating words into graphical representations and sketching the graph of $f$ given the graph of $f'$.

The first of these involved reviewing some of the interaction from the flowing water task in interview #3. With each subject, I reviewed the words they used to describe the height of the water in each of two containers and the
graphs they drew for these containers. The dialogue excerpts were chosen on the basis of how well the subjects described the rate of change of height as they drew a sketch for the height. This was done in an effort to establish continuity between their own use of language and the tasks that were to follow. After this review, the subjects were asked to sketch a graph of h' based on the words they used and the pictures they drew for h.

For the second activity in this session the subjects were asked to draw graphs involving dynamic situations whose context might be familiar to them. The contexts were the growth of interest on a monetary deposit, and the growth of the student population at our college. They were then asked to address the same issues in a context free setting.

The descriptions of the dynamics of these situations are presented in Figure 14.

Consider the interest earned on a deposit of money in a savings account (assume simple interest). What happens to it as time progresses? Sketch a graph displaying how you think the interest might grow if the interest rate were constant. Sketch a graph of how the interest rate would look. Lately, we have seen interest rates that are quite low, however, I heard yesterday that they went up. If interest rate were increasing, in your opinion what would the graph of the interest earned on a deposit in a savings account look like? How about the graph of the interest rate?

Sketch a graph of the number of students at Langara, if the growth of the student population is decreasing.

Can you sketch a graph of function f that decreases at a constant rate? What would the graph of f' look like?

Sketch what you think will be the graph of a function that decreases, but does so more slowly.

Figure 14. Dynamic situations whose graphs were drawn during the sixth interview.
I chose these tasks in an effort to observe how the use and understanding of language pertaining to a dynamic situation guided a graphical representation of the situation, and to gain insight into the subjects' informal understanding of how a function and its derivative are related. The second and third activities are called graphs of dynamic situations with and without context, respectively.

The fourth activity for this session asked the subjects to sketch the graph of \( f \) given the graph of \( f' \) (rather than words describing the dynamics of the situation). This was a variation on work by Rubin and Nemirovsky (1991b) who had their subjects do this with the help of technological environments. The task was worded as in Figure 15.

The graph on the top row denotes the derivative, \( f' \), of some function, \( f \). Describe each of these graphs for the tape recorder. On the axes below each, sketch what you think will be the graph of the function, \( f \).

![Figure 15](image)

*Figure 15. Drawing the graph of \( f \) from the graph of \( f' \).*
Interview #7

The subjects were presented with the set of exercises in Figure 16, that I call the paper and pencil test.

1) Determine if the statement below is true or is false and describe your reasoning for your conclusion.
   If \( f' \) is increasing on an interval \((a,b)\) then \( f \) is increasing on the interval \((a,b)\).

2) If the information below is all that you know about a function's derivative, \( f' \), what can you say about the function, \( f \)? ("nd" indicates "not defined")

<table>
<thead>
<tr>
<th>( f' )</th>
<th>+</th>
<th>-</th>
<th>+</th>
<th>+</th>
<th>nd</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3) From the graph of a function, \( f \), given below, determine all intervals over which the derivative, \( f' \), of the function is negative.

4) Below is the graph of the derivative of function \( f \), that is the graph of \( y=f'(x) \).
   a) State the intervals on which \( f \) is increasing and the intervals on which \( f \) is decreasing.
   b) Where do the local extrema of \( f \) occur, and for each one say whether it is a local maximum or a local minimum.

5) Determine the intervals on which the function \( f(x)=2x^3+3x^2-12x \) increases and the intervals on which it decreases, and identify all relative extremes.

Figure 16. The paper and pencil test addressed during the seventh interview.
This was most like a traditional paper and pencil test, hence the choice of name. I did my best to relieve any anxiety that might usually accompany the writing of a test by offering comment for clarity, encouragement and tutorial help. The problems were taken from my own classroom practice, except for #4 which is from material distributed by the Harvard Calculus Consortium. The first four are somewhat non-standard problems promoting unrehearsed responses. The last is a standard computational exercise similar to many in the subjects' textbook. This was to be the major measure of their command of the formal technology of calculus.

**Interview #8**

Our last session was a structured interview to wrap up the project. Firstly, we addressed focused lists for change and for rate again. This repetition of tasks addressed in interviews #1 and #3 was in an effort to observe any changes in the responses. As before, I asked the subjects to write down and comment on anything that came into their mind about the word change and about the word rate. I also asked "What words would you use to describe the way in which things change?"

I then presented the subjects with the quotation "[calculus] stands as one of the most ingenious creations of humans trying to model the changeable world around them" (Gleick, 1987, p. 67). I asked them to react to and comment on the statement. I suggested that they might want to bear
in mind what they learned in class or throughout our discussions. As with the focused lists, this was in an attempt to study their informal concept images of calculus issues by promoting discussion and considering the language they used.

The third activity involved reviewing what transpired throughout their courses and our interviews. The subjects were asked to identify an aspect of their course that they found most difficult, and an aspect of their course that they found easiest. They were then asked to do the same for what transpired throughout our interviews. Also, I asked them to identify anything we did in the interviews that they found helpful in their course. These requests were in an effort to solicit their comment on the challenges they faced, and to measure their insight about the connection between interview tasks and classroom tasks.

These interview tasks constituted a broad collection of instruments with which to gather data about the subjects' understanding of dynamic situations. The manner in which the data from these tasks was analyzed and the purpose to which the analysis was put are described in the next section.
Methodology for Analyzing and Reporting Data

As mentioned above, I made fields notes about each interview after its completion. These notes were expanded when written responses from all the interviews were reviewed after all data was collected. The results of this review are presented as an overview of the responses to the interview tasks in the first section of the next chapter. I found it useful to keep in mind this global view of the responses as I considered the similarities among and differences between the subjects in the study. The overview suggested some features among the subjects' responses that were sufficiently consistent for further elaboration. However, the unit of analysis for this study was the individual student. The intent of the analysis was to draw a portrait of each student's understanding of dynamic situations.

To this end, in chronological order, I reviewed the tapes and transcripts of the sequence of interviews for Amy, Cam, and Margo. With the images from the chronological review, I constructed a portrait of each these subject's understanding of dynamic situations. This portrait considered the subject's intuition about dynamic aspects of change, and his or her images of concepts fundamental to a mathematical study of dynamic situations. I focused on aspects that seemed consistent through the variety of tasks. Moreover, I was most interested in informal aspects of their concept images. So, for example, I did not place as much
emphasis on a one-time response such as "the derivative is rate of change" as I did on how such a statement was put to use throughout the interviews.

Three features for the portraits were suggested by the overview of the subjects' responses. These were the subjects' dynamic view, their images of rate, and the means by which they drew a distinction between the value and the behaviour of a function. I paid particular attention to exchanges that illuminated these features, and was most interested in what appeared to be significant learning sequences where the subjects moved from being quite puzzled to displaying some command.

Of the four subjects, these three were chosen merely in alphabetical order. My intention at the outset of the study was to consider all four subjects in detail. However, a portrait for the fourth subject, Ralph, was ultimately omitted because of time constraints. These constraints were due, in part, to personal commitments beyond my life as a graduate student. They were also due, in part, to the nature of the study--I will expand on this in the next chapter when I address limitations of the study.

Ralph's work with the interview tasks was considered for, and is mentioned in, the overview of performance on the interview tasks. This work was not particularly different from that of the other subjects--he too displayed a dynamic sense, had images of rates that were put to work, and
wrestled with distinguishing between the value and the behaviour of a function. So, had I been able to produce a portrait of Ralph's understanding, its general features would be the same as those for the other subjects. However, his portrait would, obviously, include details that are different from those of the other subjects. The results of the study are limited by not having these details to consider.

As stated above, the intent of the analysis was to draw a detailed description of these students' understanding of dynamic situations, it was not to carry out a detailed discourse analysis. The interview data served only as the palette with which I drew this image. Consequently, in reporting my analysis, I decided to include edited excerpts from the transcripts, even though I worked with complete verbatim transcripts.

I made this decision because I feel the spoken word does not translate well to the written page. Many unstated meanings or references can be missed, and many utterances may be included that have little bearing on the train of thought being pursued in the dialogue. These editorial changes to the verbatim transcripts were made only in an effort to present the dialogue clearly. I have been particularly conscientious about faithfully reproducing what transpired.

For example, the verbatim transcription of part of a tutorial on rates with one of the subjects, whose pseudonym is Margo, is as follows.
M: I guess ... so the rate would be the ... it should be one thing compared to another ... so it would be ... I don't know ... it would be, what, the pressure compared to the altitude again. That's all I see there.
I: I think that would certainly be a very reasonable interpretation of these words. In which case ...
M: ... there's no problem.
I: ... there's not a problem ... well how would you find that, what would you do ... 
M: I would just stick the 10,000 feet into the function given for pressure and I'd get the pressure at 10,000 feet.
I: That would be the pressure; that wouldn't be the rate of ...
M: Right.
I: ... the rate has to be one thing compared to another. Rather than the one thing. Although I think that you're certainly very close with your ... What you proposed to do was...
M: I don't know what it wants.
I: Okay. But you made the decision that, or you made the assumption that it wants the rate of change of pressure with respect to altitude. Okay, so under that assumption of the rate of change ...
M: Oh! So we've put the 10,000 into the derivative.

What I wish to illustrate with this scene is that Margo had determined she was to compute the rate of change of pressure with respect to altitude. However, her initial reaction was to compute the pressure--she confused the rate with the amount. With some tutorial intervention she was able to make the appropriate distinction. This scene is reported as follows.

M: So the rate would be the pressure compared to the altitude again. That's all I see there.
I: Well how would you find that, what would you do?
M: I would just stick the 10,000 feet into the function given for pressure and I'd get the pressure at 10,000 feet.
I: That would be the pressure, that wouldn't be rate. The rate has to be one thing compared to another rather than one thing.
M: I don't know what it wants.
I: But you made the decision that it wants the rate of change of pressure with respect to altitude.
M: Oh! So I put the 10,000 into the derivative.
The changes I made included adding descriptive comments, or unstated meanings and references. Such additions are made in square brackets. Also, I have omitted mumbling or disjointed phrases that were not essential to the point being addressed. In order to give the reader some sense of the raw data with which I worked, I have included one of the verbatim transcripts as Appendix F.

At times, these excerpts are punctuated with sketches drawn by the subjects. These sketches were photo-reduced to fit the text, and at times I have traced over their work in order to darken the image. However, these are the only ways in which the subject's hand drawn sketches were altered.

Chapter Summary

In this chapter, I described the methodology I used to collect, analyze and report the data for my study. I chose to interview students because of the richness of data that such an interaction affords. My interviews focused around tasks of my choice and design, but included requests for tutorial attention from the subjects. From these interactions I drew images of how the subjects were thinking about issues fundamental to a quantitative study of dynamic situations. The intent of the analysis was to draw these images together to create portraits of the subjects' understanding of dynamic situations. These portraits are presented in the next chapter.
CHAPTER IV
DATA FROM THE STUDY

In this chapter, I present the data collected throughout the study. The first section is an overview of the performance on the tasks from the interviews. This is useful in order to get some global sense of how the subjects responded to the tasks. The bulk of the chapter, though, consists of three sections each of which presents a portrait of a student's understanding of dynamic situations. These are what the analysis of the data was intended to produce.

Summary of Performance on the Interview Tasks

This section describes the performance of all four subjects on the tasks addressed throughout the interviews. I found it useful to keep a global view of this performance in mind as I considered an individual's responses. Moreover, even though this study was not designed in an effort to replicate results elsewhere, I was curious about how the responses from my subjects compared with those from the subjects of the studies I reviewed. For the purposes of this report I have grouped the tasks that shared a theme.

Focused Lists for Variable and for Change

These tasks were pursued in an effort to record informal images of the words variable and change, and the mathematical concepts associated with these words. I was particularly
interested in any images that suggested a dynamic sense.

The images conjured up in the subjects by these two words were quite distinct. The context of variable was seen to be mathematical, whereas the context of change was much more general and seldom included mathematics. I found it interesting that three of the four lists for variable included the word change but that these subjects' lists for change did not include the word variable.

Only Cam was very clear about a dynamic view of variable. Amy and Margo did speak to a dynamic view, but did not seem to operate with one during later interactions. Ralph mentioned "distance versus time" which suggested some sense of continuous change.

The focused lists for change generally included states, such as "your mind" (from Ralph), "mood" (from Cam), "Tai Chi [postures]" (from Margo), and "from one thing to another" (from Amy). Margo and Amy spoke about the transition between such states, and Ralph included "over time" in his list. So these three seemed to attach a dynamic sense to the word change.

During the last interview, when we again addressed a focused list for change there seemed to be little difference from what had been offered during the first interviews.

Given that images conjured up by the two words were very different, the subjects seemed not to see the words as being synonymous. All but Cam attached a dynamic sense to the word
change, and Cam attached a dynamic sense to the word variable, so such a sense did seem to accompany one or the other of the words.

**Car/Truck Problems**

The goal of these exercises was also to explore the subject's dynamic sense. The informal part of the problem was exploring their intuitive sense of how a dynamic situation might unfold, and the formal part of the problem was exploring how they viewed an algebraic representation of the dynamic situation.

With the informal part, all subjects described a scene where the relative positions of the vehicles were quite variable, however they seemed to use different initial conditions. For Cam, the car started ahead of the truck, then the truck overtook the car, and then the truck was, in turn, overtaken by the car. Amy had the car start ahead of the truck and stay that way, but the distance between them changed as the speeds changed. Margo had the vehicles start at the same position but because the truck was moving it immediately moved ahead of the car, and then the distance between them decreased until the speeds were the same.

When acting out their descriptions with the toy truck and car, the subjects were consistent in reconstructing their initial description, however, some elaborated on or extended it. Coupled with the focused lists for variable and change, this task suggested to me that these subjects had a good
informal dynamic sense.

Analysis of the proposed solutions for the formal car/truck problem generated considerable discussion that supported the subjects' views of variable as articulated in the focused list. Only Cam initially welcomed the solution from Student B, the initial reaction from all others was to dismiss the polarity chart. All subjects quite quickly dismissed the proposal from Student A, they knew that the time variable would play a role in the solution. For the most part the discussion centred around the proposed solutions from Student C and Student D. All the subjects recognized the confusion between a duration of time and an instant of time in these proposed solutions. They seemed to find this a novel exercise, and expressed surprise at a mathematics problem having a variety of solutions.

Focused List for Rates

All subjects included an interpretation of rate as ranking. I was surprised by this because I had focused so much on the use of rate as a noun that I overlooked its use as a verb. Only Cam mentioned "interest rate".

Their lists all included a variety of mathematically technical descriptions--some more workable than others. For example, Amy and Ralph considered rates with respect to time, whereas the others included a view that went beyond only time rates. It was interesting to observe how this variety of technical descriptions was put to use throughout the
interviews. I will expand on this in the subjects' portraits.

When this task was repeated during the last interview, the list had grown to include more technical phrases such as "related rates" and "rate of change".

**Flowing Water Task and Bottles**

I have grouped these two tasks in this summary because they are converses of each other. The first had subjects draw a graph when presented with a situation, and the second had them determine a situation when presented with a graph.

The performance on these graphing tasks was much better than I had imagined it would be. All four subjects drew fairly accurate graphs representing how the height increased. Their language addressed the rate at which the height increased, and they all seemed to have a good intuitive sense of how the rate at which a quantity changes governs the shape of a graph of the quantity.

**Microbes, Weights, and Functions**

These three tasks are grouped together because they involved the same types of considerations but in different settings. All involved the reading of points, the reading of intervals, and the comparison of rates.

None of the subjects had any difficulty with the problems addressing point reading—-even when interpolation between numerical values or the use of function notation was involved.
Reading intervals was very well done in the graphical and the numerical settings. However, the algebraic setting caused some difficulty, and the extent of progress varied considerably. Ralph seemed most comfortable with the requisite algebraic skills for the backwards interval problems (see p. 87 above), and had the most success with them. Margo and Cam described the problems with the appropriate inequality signs but had difficulty with carrying out the manipulations correctly. For example, Margo used the strategy that if \(x(a-x)<b\) then \(x<b\) or \((a-x)<b\). Amy displayed a remarkably accurate inversion of the operations of \(g(x)\) in question b of functions (see Figure 13), but then a terrible command of the syntax of the inequality sign.

A: Um, okay, I go backwards. 3.9 minus 3 equals \([.9]\), divided by \(.2\) equals \([4.5]\). When \(x\) is 4.5, it's equal to 3.9. At 5 it's 1 plus 3 is 4, so it's greater. When \(x\) is greater than 4.5. [writes \(x<4.5\)]

Similarly, after having correctly solved the quadratic inequality \(1+3.2x-x^2<2.92\), she wrote \(0.8>x>2.4\). It was remarkable to observe the subjects do this type of problem so well in the graphical and numerical setting, yet experience so many difficulties in an algebraic setting.

Although Ralph had been the most efficient at working with the inequalities, he seemed to have the most difficulty with the forward interval problem in functions. The others easily computed the difference between the two appropriate values of \(g(x)\) during their interviews. However, Ralph's
interview was scheduled some days after the other interviews had taken place, and during the time that elapsed his class had studied differentials. The following exchange transpired.

I: By how much does \( g \) change between \( x=2.3 \) and \( x=2.6 \)?
R: We just started something like this in differentials. But I don't [pause] we just took it last class. We just took an hour of it. And I haven't done any homework on it. So somehow, offhand, by how much does \( g \) change? 0.06, I guess. I'm taking a stab at that one. I don't know if it's correct or not. We've done something technically like this, taking delta \( x \) itself. This is the way I would do it. And for me it seems right. That's how much it would change from this point to that point. That would be the answer.

His answer and intuition were correct, but he proceeded towards his answer with utmost caution. This seemed to be a very powerful example of how classroom instruction can affect what a student brings to a problem.

There were mixed results on problems that addressed a comparison of rates. None of these students were thrown by the distractor of higher value as had been the case with subjects in the studies reviewed in Chapter II (Bell & Janvier, 1981; Rubin & Nemirovsky, 1991b). For example, in problem #8 of microbes (see Figure 11) between 1pm and 4pm, B is the larger population but its rate of growth is smaller than that of population A.

Although none of the subjects were caught by this distractor, Amy was the only subject to consider the derived quantity of rate in order to address problem #9 of microbes, and part f of weights. The other three considered the change
rather than the rate of change on these problems and on part f of functions (Amy had run out of time before getting to part f).

Despite this oversight, these three did consider the derivative when they addressed instantaneous rate in part n of functions. They were initially puzzled by the mention of one instant. Cam quickly jumped to the derivative at that instant, whereas, Ralph and Margo did so very cautiously.

In the graphical setting, all subjects initially considered a valley to peak comparison when asked for the instant of most rapid growth. Only Cam moved away from that initial perspective and considered the slope of the curve at points between a valley and a peak. Cam also considered how he would do these problems using the derivative.

**Graphs of Dynamic Situations**

The two types of activities here amounted to converse problems. The first involved sketching a graph of a function given words that described the behaviour of the function, and then sketching the function's derivative. The second involved sketching the graph of a function given the graph of the function's derivative.

All the subjects were quite good at sketching the graph of a function from words describing its behaviour. The first two such exercises involved situations with a material world context and last two were without material world context. There was no difference between the performance on these two
types. The subjects appeared to have a very good intuitive sense of how the rate of change of a function affected the shape of a graph for the function.

However, sketching the derivative of these functions proved to be very challenging. The most obvious challenge that all subjects met was working with negative rates. All the subjects initially overlooked when the derivative would be negative, and only changed their representation after tutorial intervention from me. This situation will be more thoroughly described in the portraits below.

When trying to sketch a graph for a function given the graph of its derivative, all subjects showed a tendency to represent the behaviour of the function as being the same as the behaviour of its derivative. This is consistent with the data from Rubin and Nemirovsky (1991b). Here too, after tutorial intervention they were able to make significant progress towards distinguishing the behaviour of a function and its derivative. It appeared to be addressing the significance of the algebraic sign of the derivative that contributed to this progress. This situation will also be more thoroughly addressed in the portraits below.

**Paper and Pencil Test**

Amy and Cam were very quick to offer correct responses and rationale for problem #1 (see Figure 16), whereas Ralph and Margo answered the first problem incorrectly. However, their confusion seemed to lie in the universality of the
statement.

R: See the thing is that you don't have a situation to work from. That's what I always have problems with. It's a rule that they want to prove and it's kind of like, well, it could be true for this equation but it's not true for this equation.

All four subjects seemed to find the next three exercises quite accessible, although they again paused to wrestle with distinguishing between $f'$ and $f$. Each of them seemed to have a point when the penny dropped. The speed and accuracy with which they proceeded from that point was remarkable.

Margo and Ralph addressed problem #5 by differentiating from the outset, whereas Amy and Cam both initially worked with the function rather than with its derivative. All of them eventually came to correct solutions. It was interesting to notice that the two students who demonstrated immediate command of problem #1, a conceptual problem, did not display immediate command of problem #5, a computational problem, and vice versa.

Review of the Course and the Interviews

There was considerable variation with what was identified as the easiest and as the hardest parts of the course. Cam confessed to having found anything that required algebraic manipulation difficult and commented that the topic of related rates easiest. Amy felt that limits were the most challenging part of the course and that computing derivatives was the easiest. Ralph also identified this as the easiest
part of the course, and he found optimization problems the most difficult part. Margo identified the logarithm functions as the most challenging topic in her course, and never really pointed to an aspect of the course that was easy.

There was considerably less variation in opinions about the interview tasks. All subjects identified the flowing water task as either the most helpful or the easiest part of the interviews. Ralph and Amy both identified the problems of drawing the graph of a function given the graph of its derivative as being the most difficult part of the interviews. However, Amy also commented that this was the most helpful activity we addressed. Margo mentioned that a tutorial session we had on the role of the independent variable in the derivative was the most helpful of our activities. Cam did not like the focused lists. Margo found analyzing the proposed solutions in the formal car/truck problem most difficult.

Summary

This overview gives some idea of how the responses of the subjects compared to each other, and to what was reported in the literature reviewed in Chapter II. However, my purpose in gathering this data was not to compare and contrast these subjects, or to replicate other studies. Rather it was to try to better understand how calculus students think about dynamic situations by considering a
detailed study of individuals.

These responses to the tasks suggested to me that these individuals did have a sense of dynamic situations that manifested itself in different ways throughout the various tasks. The overview also suggested that the subjects did have images of rates that they brought to bear on situations in different ways and with different degrees of effectiveness. In a variety of places, all subjects displayed some difficulty with distinguishing the values and the behaviour of functions. In the next three sections I will present portraits of three of these students' understanding of dynamic situations that provides detail to these features of dynamic sense, images of rates, and value/behaviour distinction.
Portrait of Amy's Understanding of Dynamic Situations

Amy was registered in Math 153, our introduction to differential calculus. She graduated two years prior to this study from a highly respected secondary school on Vancouver's west side. Her most recent experience with mathematics was Algebra 12 at that time. The portrait I sketch shows that she tended to work with a particular view of variables, yet had a keen sense of dynamic situations. She was very astute in her ability to add detail to how quantities were changing. Despite such insight, she seemed to have some difficulties with making connections between symbolic formalities and what they modelled. As the interviews progressed she met such challenges, and displayed that she was an extremely capable student. The calculations of calculus presented little difficulty for her--she scored 90% on the final exam. Details of her portrait are presented below.

Dynamic View

Data from the focused list for variable suggested that Amy's concept image of variable involved mainly a mathematical context. She considered a non-mathematical setting only after prompting from me, and in such a setting, variables, for her, were factors or states affecting something.

I: [Are there] any other ideas that come to mind, anything beyond mathematics?
A: No, most of the time, for me, I think of math. [contemplative pause] I think of it for flying also. I'm a pilot so I have to take in the variables when I'm going
flying, such as the weather. That's a main one. And the runway conditions. That's different than math but
[pause]. Yeah, it's different options that I have to consider when I'm making a decision for flying.

In a mathematical context, Amy seemed to speak to all three views (particular, general, and dynamic) of variable.

A: Changing, constantly changing, and the first thing I see is $x$ because that's always used as a variable. Numbers, when I think of variable I often think of numbers. It's used in different equations. It's an unknown.

Although she spoke to general and dynamic views of variable, she seemed to operate mostly with a particular view. For example, when she discussed the formal car/truck problem she found the solution by Student C, which considered particular values for $t$, to be the most agreeable.

A: Student C, well their way makes the most sense. Yeah, to me, putting it in a table and doing that.

She admitted to not being spontaneously disposed to consider the algebraic representations. However, after further reflection she seemed ready to admit to some utility in such a consideration.

A: And I don't know why they say $15t = t^2 + 6t + 18$. The same with Student B, I'm not sure why they give that, [or] would say $15t \geq t^2 + 6t + 18$. I wouldn't do that. I'm not sure. Because they're not [contemplative pause] oh, I see. I guess after [thinking about it a bit] what they're saying is "after how many seconds will they have travelled the same distance?" And they'll travel the same distance after either 3 seconds or 6 seconds.

She recognized the confusion of an instant of time with a duration of time that is present in the proposed solution from Student C and she distinguished these uses with the words "after" and "for".
A: The truck's not ahead for 4 seconds, as I said, it's after 3 seconds.

These observations on her part suggested to me that she had an emergent general view of variable.

At the end of this interview, during a tutorial on the slope of a curve, she made a good deal of progress with moving from a particular view to a general view of variables.

I: [This gives] the slope of the tangent line at this point [whose coordinates are \((x,f(x))\)]. Now the thing is that this formula [for slope] holds regardless of where we are on the curve.

A: I don't understand how, I mean, because we could be on a curve that changes, how can it be the same for the curve?

I: Let's just try to explore that. [Sketches the curve in question \((y=\sqrt{x-4})\), locates a few particular points along the curve, and explains the representation of a general point on the curve as \((x,\sqrt{x-4})\).]

What does this [formula] have to do with that [general point]? Well, \(1/2\sqrt{x-4}\) is the slope of this line [pointing to a sketch of a tangent line at the point \((x,\sqrt{x-4})\)]. This line has slope \(1/2\sqrt{x-4}\). Now that [statement] does include a whole lot of generalities. Let's see what it means, say here, at \(x=5\). If \(x\) is 5, the point of tangency is \((5,1)\), the slope of the tangent is ... [Amy says "1/2"]. As you move along, say, go up to 8, the point would be \((8,2)\) [interrupted by]

A: [interjects with] 1/4

I: [continues with] and the slope would be 1/4.

A: Oh, so the formula, this formula [pointing to \(1/2\sqrt{x-4}\)], stays the same as well everywhere on that curve. So does that work? Is that always going to be true even if your curve has different, like when you've given an equation where there's different curves, like this one [pointing to the graph of the function \(f(x)=x^2/x^2-9\), which has two vertical asymptotes and hence is in three pieces]. If I find out \(f\) prime of \(x\) for this equation, \(x^2/x^2-9\), then it doesn't matter what point I take, the slope of the tangent to the curve at that point is always going to be whatever \(f\) prime of \(x\) came out to.

I: That's right.

A: Oh, okay! Oh that makes sense now. Then it is useful. You see I didn't think it was very useful trying to figure this out if it was only true for one point. Right? I couldn't [see the generality of the computation]. [I thought] "well, I suppose it's useful for that one point
but then you have to go through this whole thing over again. But of course not, because you're using the same symbol, you're not using any numbers.

Although Amy made this shift towards working with a general view, subsequent interactions suggested the progress was fragile. After we had finished the paper and pencil test in interview #7, we returned to what had been omitted from functions in interview #5. In response to question #4 of this task, she suggested graphing as a solution, but did not consider the utility of solving an inequality. Indeed, her only other approach was to consider particular points.

I: Can you think about a way in which to address that problem without using a picture?

A: I don't know if I can. \( g(x) \) is a straight line, and \( f(x) \) is a parabola. [contemplative pause] I don't know how I would do that without figuring out what they are at different points. Like at 0, \( g(x) \) is greater than \( f(x) \) [pause]. When is \( f(x) \) smaller than \( g(x) \)? Oh, okay, at 1 they're the same. At 2, \( f(x) \) is larger than \( g(x) \) but I don't know where it changes between that. Although I could plug in numbers and find out, but I don't know how to do it besides plugging in numbers.

Even very heavy prompting on my part did not bring out the consideration of solving an inequality.

I: In the answers to your previous questions you've got things like \( x<4.5 \) and \( x<.8 \). Does that give you any idea as to how you could go about doing this without pictures?

A: Well it's gonna be [pause]. Well, I could do it in that notation but I still don't know how to do it without plugging numbers in.

I was hoping that "that notation" would have prompted her to consider solving an inequality. Unfortunately, she focused on particular values for the variable and hence did not use the power of a general view of a variable.
None of the dialogue around the focused list for variable seemed to suggest a dynamic view of variable—even the mention of "constant change" did not necessarily imply any order or progression to the change. However, discussion about the focused list for change suggested that she had a very strong dynamic view of change. She was quite emphatic about considering the transition between states rather than just the states.

I: You've written addition, subtraction, from one thing to another, mutation, and a difference. And you've then commented that the most important one is a difference.

A: It's a difference between two things. There's a change. There's a difference from what you had to what you go to. When something changes, it's a mutation. But, it goes from one to another, from what it was to what it becomes.

I: Now, when you think about that, what do you think about most readily? Do you think about the one and then the other? [What] we might call the one state and [then] another state?

A: No, the change.

I: The change. So the transition from one to another. You focus on that more.

A: Yeah, it's a process, from one thing to another.

The dialogue around many of the tasks suggested to me that she had a remarkable intuitive sense of dynamic situations. Her language often specified quantities that were changing and described the manner in which such quantities were changing. For example, her discussion of the informal car/truck problem included a mention of how velocities might be changing.

A: So the truck would be coming up but it would be slowing down. The light would turn green and the car would take off and truck would start to accelerate and they would go constant but the car would still be going faster [than the truck] so it would be moving away and the light would turn
red. Another light would turn red and [the car] would slow down but it would slow down faster whereas the truck would be back here and start to slow down halfway instead of a quarter way up the block, say.

While addressing weights, she expressed a tension between the yearly snapshots depicted by the figures in the table and the continuous nature of aging. This seemed to suggest a dynamic view of time and growth.

A: The growth isn't constant and all the numbers I've given you have been constant numbers for one year. It's not constant over the year. It changes over the year.

So, although she did not seem to work with a dynamic view of variable, she certainly did embrace a dynamic view as it pertained to situations involving change.

Images of Rate

Her focused list for rate presented me with a couple of surprises. The first was the omission of the notion of interest rate. The second was the inclusion of a sense of ranking. I had been so focused on the use of rate as a noun that I had overlooked its use as a verb--the subjects did not share this oversight.

I: Could you describe "judgement" a bit more?
A: When you rate something, you judge it. You rate it on a scale of 1 to 5 or something.

Her list also included "per time" (which she wrote as "/time"), speed, and acceleration. She elaborated on these as follows.

A: Per time: when you have to give a rate of something you have to give it over time and usually you have to give it over a unit of [whatever] per one unit of time you're using. And it can be [contemplative pause] I guess the
unit of time can vary but more often the whatever you're comparing it to varies. Which is why I don't have anything there [in the numerator]. Speed is a rate, miles per hour, minutes per second, and I don't know [contemplative pause] if acceleration would be a rate, because acceleration is metres per second squared. So I don't know if that is a rate.

I: That's a pretty strange unit, isn't it, metres per second squared.
A: Yeah. I'm not sure about that one. Rate per time.
I: It's a rate per time. What rate per time?
A: Speed. [continuing the list started before the tangential thought about acceleration] Rate of growth, or rate of erosion, or rate of climb, it's all something per time.

By leaving the numerator blank in "/time" she has provided a non-standard representation of something unspecified or general--this may be related to the difficulties she experienced in working with a general view of variables. That aside, this seemed to be an exceptionally accurate description of the technicalities of the notion of rate. When other tasks in the interviews were addressed, she demonstrated the ability to work with this informal representation of rate. Indeed, she experienced no difficulty working with the derived quantity of rate during the microbes and weights tasks. Throughout these interviews, the notion of speed played a very dominant role in her concept image of rates.

Her language surrounding the various containers in the flowing water task provided accurate details of how the height increased by addressing the rate at which it increased.

A: Well the first time that I watched [the water flowing into the beaker], to me it looked like it went much faster in
the middle part of it. That was with this small beaker. Visually. But I'm not actually sure if it's doing that. [Watches the water flow again] Well, I think it goes slower at the end. But I don't know that [pause]. It [the height] starts at zero, zero time. So the height would increase first. It's [the rate is] fairly constant although [pause]. Yeah, it's fairly constant. It [the rate] becomes less.

In studying the graduated cylinder she was very observant and articulate about her observations.

A: Okay. I've marked here the same amount of time [as it took the water to flow into the beaker]. But it goes up much faster, much, much faster and of course much greater distance. I think it's slower at the top, too. But I'm not sure if that's an optical illusion or not.

With all the other containers it was the rate at which the height changed that she focused on.

A: That one [the Erlenmeyer flask] goes up constant until at the very end it's a little bit slower. That one [the height in the martini glass] very definitely decreases. That one [the volumetric flask] increased, well, it increased all the way but it increased most quickly in that part [pointing to the stem].

This facility with distinguishing the dynamics of a situation was also evident during discussion around the bottles task. All the conversation was around the rate at which the height was increasing.

A: Bottle #2 is going to start off slowly and speed up and it's going to be relatively constant so I think it's D. [Bottle] #1 is going to be constant and then it's going to increase. F and G both do that but I think it's F because it goes higher. [For bottle] #4, the slope will decrease constantly. So it will be A. Bottle #5, it will start off, it will increase and then it will decrease, the slope will decrease again.

Similarly, during the sixth interview, when we addressed graphs of dynamic situations with and without context, she
very easily translated the words about rate into graphical representations which, except for some trouble with algebraic sign, were very accurate.

Despite her clear intuitive sense of rate and how it governs a graphical representation of height, she did not readily see the connections to formalities that had been studied in the classroom. The following exchange from the third interview suggested that her focus on speed seemed to contribute to some difficulty that she had with seeing slope as rate.

I: You say "quite that slow". How is it in your graph that you are measuring slowness?
A: This height [pointing to the graph for the martini glass] only took that distance and here [pointing to the graph for the volumetric flask] it took about twice that distance to go the same height.
I: Can you think of any words from your math class to describe that?
A: Describe the difference in [puzzled pause].
I: The difference that you just described.
A: [Stumped pause]
I: Slope doesn't ring a bell?
A: Slope? Oh, I guess so. It would be, I changed the slope of the graph, yes. So if you took the tangent to one of these graphs you'd get--I don't know--I was going to try to figure out what you would get. You get velocity if you do a distance, distance over time, maybe it's still velocity. It's still velocity? I didn't think of it as slope at the time.

In terms laid out in Chapter II, it would appear that the notion of slope was a formal aspect of her concept image of rate that had not yet become informal.

In the next interview, as we addressed microbes, her responses indicated that this rate/slope connection had become quite strong. Problem #8 asked for a comparison of
rates for the two graphs. She promptly responded as follows.

A: Population A.
I: And why do you say that?
A: Because the slope of the graph is greater.

Similarly, in question #9, she quite readily worked with the appropriate derived quantities to compare rates of change over two different time intervals on one graph.

A: So between 4am and 6am it went up from 2 to 2.5 which is 0.5 in two units of time. And from 3pm to 4pm it went from 1 to 1.3 which is 0.3 units in one unit of time. If you divide 0.5 by two it's 0.25. So it went up .25 units in one unit or it went up .3 units in one of time. So .3 millions of microbes in one unit of time is steeper, greater.

She was also very competent with this slope/rate connection while addressing the problems in weights that used the same skills but in a numerical setting.

Both microbes and weights used particular points with these computations of derived quantities, whereas functions was designed to address this skill in an algebraic setting. Unfortunately, Amy ran out of time before getting to the part of the functions task that addressed average rates using difference quotients. Moreover, as I will show below, she experienced some difficulty using this aspect of rate, and connecting the notion of the derivative with rate. I suspect that had we been able to address all of functions, she might have been more comfortable with the role that the difference quotient, and perhaps hence the derivative, plays in the concept field for rates.
While reviewing the flowing water task during the sixth interview we went through a wonderful exchange that seemed to contribute somewhat to the process of connecting the derivative with slope/speed/rate. In reference to the graph below which she had drawn during the third interview, we were reviewing what she had said about the rate at which the height of water was changing.

She had been referring to this rate as "it". When questioned about what the word "it" was referring to, the conversation proceeded as follows.

A: The slope of the graph.
I: Ah, good. The slope of the graph is playing the role of "it". Could you now sketch a picture for the slope of the graph?
A: Sketch a picture for the slope? How? Like, this is the slope, right? If this was the slope [interrupted by]
I: [Interjecting with] No, no. This is a picture of the [height]. So now I want us to move away from that picture. You say, "It's fairly constant. It becomes less."
A: Oh, so you want the [contemplative pause].
I: You say, "the slope is fairly constant, the slope becomes less."
A: Alright.
I: Could you picture that graphically?
A: Yes. Um [contemplative pause] velocity, time and velocity? [pause] I think that it [pause] although it's constant [pause] because it slows at the end it's going to continuously slow. It's not going to be constant and then all of a sudden start to slow. That makes a difference. It's probably more decreasing like that. [draws the graph below]
I: This picture is height, and you thought about the slope of height.
A: Which would be speed, right? That's a distance over time. Height is a distance.
I: Good, good. The speed of the height, certainly. Good.
Now can you think of another symbol to represent the slope of $h$? A formal symbol that you may have seen in your class?
A: [Contemplative pause] Is it [uncertain pause] the derivative?
I: Indeed it is.
A: I don't think of it that way.
I: You don't think of it that way.
A: Not at all. Because maybe calculus is new to me but when I think about it that way [contemplative pause while she reconsiders]. So this could be $h'$, the slope of [the graph]. Oh! Neat! Alright!

Here again, we see an example of a formal aspect of her concept image of rate (slope is derivative) that was becoming informal. The progress made in this exercise is made clear by the quickness and perspicacity with which she addressed the same issue with the volumetric flask. Moreover, as mentioned above, she was also very capable with the next tasks of drawing graphs of dynamic situations with and without context.

Although in a graphical setting her perception of the connection between rate and the derivative improved, she continued to have some difficulty with the algebraic setting for average rate. This was despite somewhat of a
breakthrough during the fourth interview.

A: In one of the questions further back in the book, it asked for the rate of [something]. It was asking for rates and I couldn't figure out [what rate]. It didn't ask for velocity when they were talking about distance, right? It asked for the rate of something and I didn't know how you would take the rate of it. Until recently when I thought, they're just asking for the derivative. I mean, I did the derivative because that's what it said to do [in the solution manual] but I didn't connect that taking the derivative meant that you were taking it as a rate.

I: The thing is, when we talk about average rate of change, there does have to be a "per" in it--something per something else. [Here we have] the change in f per change in x.

A: Oh! Okay! That makes sense because that's the slope of the line. And f is y on the x and y axis. [The values of f are plotted along the y-axis of the coordinate system.]

In making the connection that rate is derivative, she has again expanded her informal concept image of rate.

Despite this breakthrough, the algebraic setting for average rate caused her problems on the final exam. Of the five true/false problems on the exam, the only one she answered incorrectly was the one on average rate. She had used the correct formula of \[ \frac{f(b)-f(a)}{b-a} \] with particular points throughout the microbe and weights tasks, however on the exam, instead of using this formula she used the average of the instantaneous rates. On the problem that addressed the distinction between average and instantaneous rate she displayed the same misconception. She answered the part on instantaneous rate correctly but for average rate she again used the average of instantaneous rates. Moreover, although she correctly answered the related rates problem on the exam,
she worked with differentials rather than derivatives. As such she was not explicit about what the rates were with respect to, and she did not make explicit use of the chain rule. I would have included such expectations as part of a solution to this type of problem, and so would not have been as generous with marks for her solution as her instructor was. I do not record these observations in order to criticize her performance on the exam--there is little to criticize, these oversights aside her performance was exceptionally good. I record these observations in order to emphasize that her concept image of rate did not include the notion of average rate as defined in their textbook. For her, average rate seemed to be the average of two rates. This is despite many other very strong images in her understanding of rate. In particular, per time and speed seemed so strong as to preclude "per something else".

During the last interview we again addressed focused lists for change and for rate. There was virtually no difference in the list for change. The list for rate now seemed to distinguish between speed and rate of change, allowing only the former to be constant.

A: [Pause] Speed, [pause], I first think of speed and then I think of rate of change of something. How fast something is changing or how slow it is changing.

I: Can you distinguish between, or describe the difference between your first thought and the second, the speed and the rate of change?

A: A speed can be constant. Rate of change means that it changes. If I was talking about speed it would be acceleration. It would be the rate at which the speed is
changing. Rate of change is [pause] yeah, curved.
The words slope and derivative seemed to be conspicuously
absent from the list.

Amy quite readily recognized rate as the theme of our
interviews.

I: Can you identify a theme that would apply to all the
activities that we have done?
A: I've sort of been trying to do that since the beginning.
A theme that would apply to all of them. It would all
have to do with rates, the ones that I'm thinking of. The
car and the truck, and the boys and the girls, and the
water--the rates were changing.

Value/Behaviour Distinction

As I addressed above, Amy displayed a good intuitive
sense of rate of change. Nonetheless, there were some
instances where she experienced confusion between a quantity
and its rate of change. In fact, much of the dialogue
reported above used the pronoun it, and so there is not
always a clear sense of what was being referred to. The
following excerpt includes two examples of confusing quantity
and rate.

A: So, it starts out constant, and then decreases. It
decreases all the way to 0. It would decrease to 0 when
velocity eventually stops because there's no more water,
right? Velocity, speed of the--I guess it's not velocity
because it's not constant--it's the speed at which the
water is coming out of the burette. So, it's coming out
at the speed of gravity but that's acceleration but I
didn't [think] that it would be acceleration.

Notice that the velocity did not stop, rather, it was the
height of the water that stopped increasing. Also, we have a
confusion between speed and acceleration.
The confusion of amount and rate was most evident during the graphs of dynamic situations task, where she was to draw a graph for the derivative of a function from a graph of the function, and vice versa. This task involved recognizing that the value of $f'$ provides information about the behaviour of $f$. Amy generally seemed to focus on and describe the behaviour of $f'$, and then attributed similar behaviour to $f$. It seemed difficult for her to distinguish the value of $f'$ from the behaviour of $f'$. Drawing such a distinction was a very enlightening learning sequence, so I will describe it in some detail.

She was quite competent with drawing $f$ from the words describing the behaviour of $f$. For example, in response to "sketch a graph of a function that decreases at a constant rate" she drew the graph below.

I: Good. Beside that, could you sketch $f'$ and describe what you are sketching.
A: Alright. If $f(x)$ is decreasing, the slope of $f(x)$ is decreasing at a constant rate. No. $f(x)$ is decreasing at a constant rate. $f'(x)$ is constant, $f'(x)$ being the slope of $f(x)$. So the graph of $f'(x)$ to $x$ would be straight line [draws the graph below]. So long as the function $f(x)$ decreases at a constant rate.
Her graph showed the correct behaviour for \( f' \)--it is constant. She overlooked only the fact that the values of \( f' \) should be negative. This oversight of the significance of negative rate continued for some time.

I: What do you think would be the graph of a function that decreases but does so more slowly--the rate slows down?

A: Alright, so it's [contemplative pause] like that. [draws the graph below]

I: Now beside that function can you sketch what you think \( f' \) will be. [She draws the following graph]

Here too, she has overlooked the role of algebraic sign in rates, but otherwise she was very reasonable with her graph,
after all f is becoming flatter. The oversight of the value of $f'$ being negative results in the graph of $f'$ resembling that of $f$.

Although she did quite well with going from the graph of $f$ to the graph of $f'$, the next exercise required her to go in the opposite direction. Here the difficulty with drawing a distinction between the ordinate on a graph (value of a function) and the behaviour in a neighborhood of the ordinate (value of the function's derivative) became quite evident. Much of this difficulty also seemed closely connected to providing a meaning for a negative value of slope.

Her initial response to the first of the graphs is pictured in Figure 17 beside the graph for $f'$, and her rationale is as follows.

A: The slope of the graph decreases which means as it decreases it becomes more horizontal as opposed to more vertical.

![Graphs](image)

**Figure 17.** Amy's initial response to the first part of drawing the graph of $f$ from the graph of $f'$

I was not certain which of the graphs she was referring to when she spoke about slope. If she was referring to the slope of $f'$ then she was incorrect because its slope is
constant. Amy did not make many perceptual errors so I interpreted the graph whose slope was decreasing to be the graph of $f$. Under this interpretation, she overlooked the fact that the values of $f'$ are positive (and hence $f$ increases). However, she did draw $f$ becoming horizontal which is consistent with the values of $f'$ decreasing to 0.

The following excerpt describes her response for the second of the graphs for $f'$ (see the graphs below).

A: It increases at a constant rate.
I: What's "it"?
A: The slope. The slope increases at a constant rate. The slope of the graph of $f$ increases. Yes, the slope of the graph increases.

Despite her words, it does not seem clear as to whether "the slope increases at a constant rate" referred to the slope of $f$ (value of $f'$) which does increase at a constant rate, or to the slope of $f'$ which does not increase (it is constant). The graph she drew (on the right above) is consistent with the first interpretation and the literal meaning of the words she used. However, it is not consistent with the values of $f'$ being negative. Moreover, as she addressed the third graph in this exercise (see the graphs below), she said "in the third one, the slope of $f'$
increases". The behaviour of $f'$ in this graph is exactly the same as the behaviour in the second graph—only the values are different.

The literal meaning of the phrase she used here ("the slope of $f'$ increases") is incorrect because the slope of $f'$ is constant. However, the values of $f'$ increase, so she seems to have focused on behaviour (of $f'$). As above, there seemed to be a confusion between the values of $f'$ and the slope of $f'$. She drew a graph for $f$ whose slope is increasing which is consistent with the behaviour of $f'$. Unfortunately, it is not consistent with the values of $f'$. I drew her attention to the values of $f'$ rather than the behaviour of $f'$, and the following sequence unfolded.

I: What's the difference here [pointing to the graph on the left above] between being above the x-axis and below the x-axis?
A: I'm not certain. [pause] The number that you get out is a negative number.
I: Uh-huh. What would it mean for the number that comes out, in other words, $f'$, to be negative? What does it mean for the slope to be negative?
A: Well [pause] is the slope negative? If that equals, if you put in $x$ and you get out a negative number then the slope is negative which means that it's decreasing.
I: Do you want a new sheet?
A: Yes, please. [Pause] So if you get, alright, alright, if the number is negative then the slope is decreasing. And if the number is positive then the slope [interrupted by] I: [interrupting with] Just a sec.
A: If $f'$ is a negative number then the slope of $f$ is decreasing. That doesn't seem right.
I: This [pointing to the graph of $f'$] is the slope of $f$.
A: No, that's the slope of $f'$.
I: No, no. This is a picture of $f'$. The slope of $f'$ is whatever the slope of that line is [pointing to the picture of $f'$]. But that [the picture of $f'$] represents the slope of $f$.
A: So here [pointing to the graph of $f'$ in Figure 17], this shows the slope of $f$ as being positive.
I: Does that agree with what you've drawn here?
A: Not at all. Being positive [and] I'm reaching 0 and it's decreasing.
I: What's "it"?
A: The slope of $f$ is decreasing.
I: Good.
A: It would be more like that [draws the logarithmic shape that is pictured in the second row of Figure 18]. Oh dear me.
I: Why do you say "oh dear me"?
A: Because now I've done these wrong. [she quickly returns to the other graphs and quite readily corrects all errors, see Figure 18]. So, for the second graph, $f'$ is negative and reaching zero. Which is this way [draws the second graph in the second row of Figure 18]. So in this one it's decreasing and then it reaches zero and then its increasing. So it's a parabola, okay, decreasing and increasing?

![Graphs](image)

Figure 18. Amy's second responses to the task of drawing the graph of $f$ from the graph of $f'$
Despite being able to correct these errors, it took some time for her to focus on the main issue here—the polarity of f'—however, when she did so focus her reaction was quite remarkable.

I: If f' is six, [then] whether it's increasing or decreasing
interrupted by]
A: [interjecting with] at that particular point in time, f is increasing. f is positive. No, it doesn't have to be positive because it can be below the x-axis. It's increasing.
I: Similarly, what would you be able to conclude about f if, for example, f' were -3?
A: It's decreasing.
I: It's decreasing.
A: f
I: And it seems to me that the aspect that you overlook is that positive [interrupted by]
A: [interjecting with] f' must be negative if f is going to be decreasing. Oh! Oh, oh, it doesn't matter what the slope of f' is doing, it is what f' is equal to!

She has distinguished behaviour and value with her own words, "doing" and "equal to". Again, her informal concept image seems to have expanded. Although this progress was exciting, it was also quite fragile.

I: So this was to represent a graph of a function that decreases at a constant rate. Would you like to modify f' at all?
A: Yeah, because, f [puzzled pause] no, wait a second. It decreases at a constant rate. It's decreasing, it's not negative. It's decreasing, f' is negative. I'm still having trouble with that. That right? It is, the slope is decreasing so f' is negative, but it is constant.
I: The slope is constant.
A: The slope of f(x) is decreasing at a constant rate so
interrupted by]
I: [interjecting with] No.
A: No?
I: f is decreasing at a constant rate.
A: f is decreasing at a constant [rate]. The slope of f(x) is constant. So f'(x) has a constant slope decreasing [reconsidering] so f'(x) is negative and constant.
By the seventh interview, which took place during the last week of classes, she seemed to still be somewhat tentative about the relationship between $f$ and $f'$ but had basically come to grips with it. She responded to problem #1 on the paper and pencil test as follows.

A: It's false. It's false because if $f'$ is negative and increasing, $f$ is still negative. Although [contemplative pause] this is what I was getting confused with before, right? If $f'$ is negative, then the slope is decreasing? [puzzled pause] although it could be approaching 0 [contemplative pause] yeah. No, that's wrong. That's false. And it's false because if $f'$ is negative and increasing, $f$ is still decreasing although it's decreasing at a slower rate. It's getting closer to 0 [that is] the angle is getting closer to 0. [Draws the graph below as a counter-example]

\[\text{\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Counter-example graph}
\end{figure}}\]

Indeed, she handled problems #3 and #4 quite easily and hence seemed to have drawn the connection between value of $f'$ and behaviour of $f$.

A: $f'$ is negative between 0 and 2, and $f'$ is negative between 3 and 4. So the slope, therefore, the slope of $f$ is negative which means that $f$ is decreasing. And then $f$ is increasing from 2 until 3.

However, she still seemed to have difficulty transferring such progress to algebraic representations. For problem #5 she did not spontaneously consider the polarity of $f'$, rather, she devoted a great deal of attention to parts of a graphing algorithm that were irrelevant to the task at hand.
Amy's Perspicacity

There are a few more scenarios that warrant mention because they offer evidence of Amy's intellectual character. Despite some difficulties in making connections between what was done in her class and in our interviews, she was able to make a connection between what was done in her math class and what was done in the ground school of her flying lessons.

A: [It's] one of the things that I'm learning in my flight ground school, because we're doing graphs also. She [the ground school teacher] drew [a graph for the coefficient of lift over drag] the other day and she said "this goes off to infinity", so it goes up. A certain type of drag is induced entirely because of lift, and there's a point at which you get no more lift, when your angle has increased. So it's all drag, and it goes off [to infinity]. That's calculus, right? That's directly calculus.

She also expressed an appreciation of mathematical demonstration, something that I have very rarely heard from any of the nearly two thousand students that I have had in introductory calculus courses.

A: And I do understand it, with the proofs that he [the mathematics teacher] gives us. I find proofs very helpful actually. I don't remember them but if I have to I can go back and look at them and then I understand. Like, I understand why it works then.

Lastly, I found some of her comments about the quotation by James Gleick that we discussed in the last interview (see p. 93 above) to be most insightful.

A: I've never thought of calculus as a model or in terms of looking at the world as a model. Ever. Oh, well, models help to predict or give an insight into something that is going to happen which, now that I say it out loud, is what calculus does. But I never have thought of it in those terms at all. Yeah.
Summary

I found Amy to be a very discerning student, and, in particular, to have a keen sense of dynamic situations. However, she seemed to have not yet made strong connections between symbolic formalities and what they model. For example, despite her dynamic intuition about change, she seemed to operate mainly with a particular view of the concept of variable. She quite readily worked with the derived quantity of rate during the microbes and weight tasks, but did not work with the difference quotient for the average rate of change of a function. She was particularly astute in her ability to draw graphs representing dynamic situations, however she had some difficulty giving meaning to algebraic sign in the context of slope. Distinguishing rate from amount was easy for her in numerical and graphical settings with context, but more challenging in algebraic settings or in graphical settings without context. In a graphical setting, she had some difficulty distinguishing the behaviour of a function from the values of a function. However, she made great progress towards drawing such a distinction and it seemed to be confronting the issue of negative rate that brought about this progress. Although there were a few issues that Amy found challenging, she was an exceptionally good calculus student, and throughout this study she demonstrated mastery of the computations of calculus.
Portrayal of Cam's Understanding of Dynamic Situations

Math 153 was the only course Cam was taking, as he maintained a full-time job. Prior to this course, his most recent study of mathematics took place 11 years ago when he graduated from Grade 12 in Alberta. The portrait I draw shows he had strong intuition about changing quantities, and a good conceptual command of the formal representations of changing quantities. His understanding included a dynamic view of variables, an ability to represent dynamic situations graphically, and strong connections between slope, rate of change, and the derivative.

Despite the overall conceptual command, there were some evident computational difficulties. Exercises that involved comparing rates were readily addressed in a graphical setting, but presented much more of a challenge in numeric or algebraic settings. There was some conceptual difficulty with distinguishing values of a function from the behaviour of the function, however, considerable progress was made with this issue.

Dynamic View

Cam articulated a very rich set of images in the focused list for variable.

C: *Fluctuate.* Want me to explain why I think of that? When I think of variable I think of something that changes, fluctuate comes to mind. Well obviously I think of math because that's where the concept was first introduced to me. I also think of research because I was involved in a fair amount of research and we had to use independent and dependent variables. I think of control because of
research also. We needed to control one variable and the other one was independent. Okay, variable. Equation comes to mind, I don't know why.

I: Can you describe any differences between what comes into your mind about variables in a math context and variables in the context of the research that you had experience with.

C: Okay, yeah. A variable in math would just be, represents a number to me. Whereas variables in research represent concepts, people, actions, changes.

For the most part he seemed to attach a mathematical context to the word, however, when prompted he also included the context of research. He was quite clear in distinguishing between the two contexts. In the mathematical context, words such as "fluctuate" and "changes" suggested to me that he had a fairly strong dynamic view of variables. Indeed, he seemed to try to block a particular view with the phrase "equation comes to mind, I don't know why".

This dynamic view was reinforced throughout the interview tasks. His description of the informal car/truck problem involved the truck overtaking the car initially, but then in turn being overtaken by the car once "the car's acceleration, momentum, kicks in". During the formal car/truck problem he was immediately drawn to the response from Student B.

C: Well, I think the first one doesn't involve any real math. It's just comparing the constant with the 15t. So I don't think I would agree with [Student] A. The second one is perhaps a better idea. But, now, with this [polarity chart] I guess you could graph each of the [equations]. Right, and with that, you could see exactly when they cross. So, I like Student B's idea, if he were to put it onto a graph which you could do with these two equations.
He indicated a dynamic interpretation of the numbers that came from the algebraic representation of the situation, and he identified the confusion of an instant of time with a duration of time.

C: According to this [table of numbers] the car takes off, darts ahead of the truck. Then it takes 3 seconds for the truck to catch up to the car and on the fourth second he's passed him and he's gone 60 metres to the car's 58.

I: How does that fit with the conclusion that the student has made, that the truck is ahead for 4 seconds.

C: Well, that would be false. He's ahead after 4 seconds.

During the sixth interview he remarked "assuming this is time, time is not going backwards". This seemed to suggest that he recognized order as being important with a dynamic use of variables. This evidence suggested to me that Cam had a good understanding of variables as dynamic objects.

Such a dynamic view was not apparent in the focused list for change, where he seemed to focus on items, situations or states that can be changed.

C: Clothes, relationship, attitude, diet, mood, government, decision, stations, return, discussion. So, I started off with thinking of things that you can change.

There was no mathematical context for these initial responses, and even when prompted for a mathematical context he was hard pressed to provide one.

C: You can [contemplative pause] mathematical [pause] change. Well, you can change your signs as opposed to positivity and negativity. I don't know. I'm drawing a blank.

It is interesting to notice that he mentioned change when asked about variable, but did not reciprocate by mentioning variable when asked about change.
Images of rate

The focused list for rate was also rich.

C: Okay, well, first thing is interest, as in interest rates, which is relevant today. Rate of change, we were just doing that in [class] as far as derivatives are concerned. Rate, ten. I just seen a girl in the waiting area and I thought she rated a ten. Don't see that very often. Rate of change would be how quickly something is climbing or falling with respect to another variable.

The informal images included the notion of ranking. The inclusion of, and the elaboration on, the phrase "rate of change" seemed quite formal, but he displayed a very good command of this notion throughout the subsequent tasks.

The notion of slope eventually played an strong role in Cam's concept image for rate, although the connection was not spontaneous. This may have been because the graphical setting, which involves the notion of slope, was not easily seen as being appropriate for representing a dynamic situation. During the initial stage of the flowing water task we had the following exchange.

C: What is it exactly that you want me to do? Okay, you want me to draw what's going on. Now what's going on is moving and what I'll be drawing obviously will be static. One picture. Or do you want me to draw a series of pictures? Do you mean like in a table or a graph or do you mean a picture of a beaker with water going in it?
I: What would you rather do?
C: Well, a graph or flow chart, something that could show motion. But one picture couldn't.
I: Okay, let's think about a graph, okay? Think about representing the height of the water in the container graphically. Do you want to see it again?
C: No, I pretty well get the idea. The water goes in and then it goes up. In a graph, I could sketch it in a somewhat linear fashion by saying that this is a height of water and this is an amount entering the beaker. So as the amount increases so does the height. Height of water
in millimetres, and amount in millilitres.

I: How about if rather than height versus volume, we think of it as height versus time.

C: As time passes, the height increases as well to a certain point, until the water stops emptying into it. I mean, this may not be totally linear. Until it stops. I imagine it would be the same thing except that eventually it would go like this and then stay the same because it stops entering. [draws the graph below]

Once Cam saw how to represent the situation graphically, he proceeded very quickly and accurately with the subsequent containers. For example, with the graduated cylinder we had the following description.

C: Here, the slope would be greater so it would be more like this because [it] takes a shorter time [for the water level] to go up higher in this case because the graduated flask is narrower than the beaker. It would reach a greater height obviously than the beaker because it is taller so in the same amount of time it would reach a higher height. [draws the graph below]

Although he was quite accurate with depicting the behaviour graphically, he restricted his references to the graph and did not include mention of the behaviour of the height that was being represented. Notice that the language he used
described the slope, and that up until this stage he had not
yet used language pertaining to rate. It was not until we
addressed what he had drawn for the Erlenmeyer flask that the
language of rates emerged.

C: No. That's fine. Okay, well this one would look more
like this. It would start off and increase something like
that. This one gets narrower so the height would have
increased a little faster than this one. And that
continues because it continues to get narrower. So it
would increase, the rate of change for height--hey, the
rate of change, where's that piece of paper? [referring to
the focused list from the first part of the interview]--
the rate of change for height would increase for this one.

Once the language of rates had been introduced, it was used
with the remaining containers. For example, here is an
excerpt from the exchanges around the martini glass and the
volumetric flask.

C: It's essentially the opposite of the last one. The base
is really narrow so the height is going to increase really
quickly. As it starts to widen out the height will, the
rate of change will decrease which is happening here.
Okay, what happens [in the volumetric flask] is that it
starts off slow and it slowly increases up to a certain
point and then it no longer [increases], the rate of
change stays the same.

By this stage, "it" referred to the rate of change of the
height rather than to the height. When asked to elaborate on
the change in the nature of his language, Cam was very
articulate about what had transpired concerning the phrase
"rate of change".

C: What happened was, when I wrote down rate of change, I
[had] remembered the word from the textbook but I couldn't
think of any real applications for it. What I gave you is
really abstract and I probably couldn't have elaborated on
it. But when I saw this [water flowing], it sort of sunk
in. What I was doing with rate of change in the
exercises, I was doing in class. This is how I understand
rate of change now. Actually, I was fishing for a word to describe this [how the water level was increasing] and rate of change came to my mind and it all sort of sunk in at once. Serendipitous, you might say.

He seemed to have made a strong connection between slope and rate, because in subsequent interviews he used the notions interchangeably. For example, when asked to compare rates in #8 of microbes (see Figure 11), he rationalized his responses in terms of slope.

C: Well, the slope of the curve [A] looks like it's greater [than that for B] so I would say it's growing faster. Between 1pm and 4pm, again it looks like A. A's slope looks a little steeper.

When he considered the graphs in bottles (see Figure 10), his responses were very accurate and his rationale was all in terms of rate.

C: I'll use the term rate of change again because the amount of water going in, the height, is changing with time. So as the water goes in, the rate of change decreases because the width of the glass increases. And it looks rather gradual. Um [pause] this looks like #1. Yeah, it looks like 1. [Here], it's constant up to a certain point here and when it gets to about here it increases rapidly. Um [pause] this looks like 2.

Although he developed a strong geometric image of rate, there was evidence to suggest that his numeric and algebraic sense of the notion was somewhat fragile. For example, while considering a comparison of rates during weights (part f in Figure 12), he worked with the change in weight rather than the rate of change.

C: Does the boys' weight increase faster between the ages 2 and 5 or between the ages 9 and 11? [Computational pause] 9.5 to 15.2, or 9 and 11, 25.8 to 34. Here we have 5.7 lb and there we have, say 5. It's more between ages 2 to 5.
Similarly, while considering part f in functions (see Figure 13) he worked with the change in the function rather than the rate of change. By the time of the exam, the situation seemed to have improved somewhat. He recognized that the true/false question about average rate of change was incorrect. However, he did not express the correct formula with appropriate symbolism—he wrote \( \frac{f(x)-f(ab)}{f(a)-f(b)} \). He used the appropriate derived quantity for average rate of change of volume over a duration of time, but made an arithmetic error.

Cam seemed to have a strong connection between the derivative, instantaneous rates, and slope. For example, we dealt with microbes (see Figure 11) quite early in the term, yet he considered questions #9 and #10 from that task in terms of values of the derivative.

C: Well, I could probably check the slope with the formula. I think I would need to know the equation.
I: If you had the equation, what would you do with it?
C: Well, if I had the equation I would find out the slope by taking the derivative of the population at this point.

He also displayed that he saw a strong connection between rate, slope and the derivative when he addressed Question #10 on instantaneous rate.

C: When is population growing most rapidly? Again, if I had the equation of the graph, what I could do is find out when the derivative is the greatest.

When he considered the problem about comparing rates in functions he spontaneously addressed derivatives, and in so doing showed remarkable command of the rate of linear growth.
C: Which function grows faster at 1.55? Now I think this calls for a derivative. I'm looking for a slope here. [He computes \( f'(1.55) \), and writes \( m=.1 \).] Oh actually, it's \( [g(x) \text{ is}] \) a straight line so it's going to grow at the same rate and that's at .2 so this one [would grow faster].

In the sixth interview, when we reviewed the graphs from the flowing water task, he again quite naturally asserted the connection between rate of change and the derivative.

I: Can you think of something, some other way, in which to denote this phrase, rate of change of \( h \)?

C: Yeah, this would be the derivative of that.

On the final exam, except for a computational error in the instantaneous rate problem and a translational error in the related rates problem, his command of the connection between rate and derivative was evident.

During the last interview he identified rate of change as the theme of our interviews, related rates as the easiest topic in the course, and the flowing water task as his favourite activity during the interviews.

C: Well you seem to have keyed on rate of change throughout what I can remember. The focused lists, the cars, the beakers, everything seemed to be directed towards my understanding of rate of change and how it applies to the physical world and how I viewed it with and without math involved. Any concept that we dealt with in class I could get a handle on my own unless it involved detailed algebra. But that aside the related rates seem to be the easiest. What I enjoyed the most was the beakers [flowing water task], finding the rate of change, and [it] also gave me a bit of understanding into rate of change.

By the end of the sequence of interviews, his concept image of rate had grown to include a good deal of technical detail. The focused list at this stage was as follows.
C: First one is rate of change. The words that I used before come back to my mind. I remember using the ten, as in rating someone a ten. I'll see if I can come up with something new this time. Okay, well, there's interest rate, there's related rates. A rate occurs in relation to another variable. It can exist as an average or instantaneously.

Value/Behaviour Distinction

Although Cam drew the connections between the derivative, slope and rate of change, these connections were challenged from time to time, generally when confronted with an anomalous situation. During the fourth interview, we had a tutorial session in which he expressed concern about the non-differentiability of \( f(x) = 2x - 3x^{2/3} \) at \( x=0 \).

I: So \( f \) is defined when \( x \) equals zero, the function's derivative is not.
C: Right. Oh, I see. Okay. Right. So this is \( f(x) \), not \( f'(x) \).
I: We're drawing the graph of \( f \), \( f' \) gives us information about the graph of \( f \). In particular this fact, that \( f' \) is not defined gives us the information that as we come into \( (0,0) \) [the graph will] be sharp.
C: Oh, okay. Right. Ahhh, yes. Thank you.

The anomaly of this situation, a cusp at the origin, brought to the surface the distinction between values of \( f(x) \) and values of \( f'(x) \). He overlooked the fact that the graph depicts values of \( f(x) \) but the behaviour of the graph is depicted by values of \( f'(x) \).

Similarly, the issue of the algebraic sign of the slope only presented difficulty in the case when it was negative. He displayed his command of positive slope early in the interviews when we addressed microbes.
I: You're restricting yourself to this part of the curve. Why did you decide to do that?
C: Well, I'm taking the points where the slope's positive, because that's the only time it's growing.

However, when presented with tasks involving negative slope, he floundered considerably. For example, during the sixth interview we had the following exchange as we reviewed his words describing how the martini glass was filling.

C: Actually, that's what's confusing me is the words I'm using. This [pointing to the graph of height in Figure 19] is going to increase and the height with respect to time will decrease.

![Figure 19. Cam's graph of the height of water in the martini glass.]

I: You said the height with respect to time is going to decrease. [In] what you've drawn here, with respect to time, the height goes up.

C: Right.

Here, I used (and heard) the phrase "height with respect to time" to mean height as a function of time. However, Cam seemed to use the phrase to mean the rate of change of height with respect to time. I tried to clarify the situation.

I: So now we're asked to asked to step away from the height and represent not the height but the rate of change of the height.

C: Right, okay. See, I'm picturing it going negative but for some reason that doesn't seem right to me. That's where I'm confusing myself because intuitively it seems like if \( h' \), if the rate of change is decreasing, [pause] oh, hold it, hold it. The slope here is 1, then here it's 1/2. It would be decreasing. Here it's 1/4. Here it's 0. So it
would be decreasing. So, how to represent that?
[tentatively draws the graph below]

\[h\]

\[\frac{dh}{dt}\]

I: Well, what you've drawn here, as we go out this way, goes down. In other words, it decreases.
C: Right. But why doesn't that seem right to me? I mean, if this was an exam question, I would really be stuck to put that down as an answer. So if we called this 1, this 1/2 and so on, [pause]. Okay, yeah, I guess that would be right. I don't know why it doesn't seem right.
I: Can you think about why you feel it doesn't seem right?
C: Yeah. The reason it doesn't feel right is because, for some reason it seems like if you've got a graph with a positive slope here [pointing to the graph in Figure 19], the slope of its derivative should be positive also. But that is not necessarily the case. As I see here. It's a concept that I actually had trouble grasping when we were doing the problems as I recall. Picture this as being negative, just because the slope is decreasing. So \(h'\) would be decreasing. But yet looking at it that's a positive slope, right?

The distinction between the value of the slope (positive) and the behaviour of the slope (decreasing) is a challenge.

This confusion of value and behaviour continued when sketching a graph representing a student population of Langara that is declining but the decline is slowing down. He quite easily drew a correct sketch for the population.
C: So here is a population. [draws the graph below]
When asked to consider a graph for the derivative of population he stated the following.

C: Now its derivative would look like this. It would look more like that because the slope of this curve is decreasing. No, I mean the slope of the curve is actually increasing because this would be a slope of -1, and here it would be like \(-\frac{1}{2}\) and here it would be 0. So, if you're representing the slope it would increase over time so I'll call this \(p'\) prime and \(t\). [draws the graph below]

\[ f' \]

\[ t \]

Notice that he has correctly identified that values of the slope are going from -1 to \(-\frac{1}{2}\) to 0, and hence are increasing. As such, he has drawn a graph depicting this behaviour, but he has not represented that these values are negative. It took some intervention to have him consider how to represent the value of the slope being negative.

C: It would look more like that, then. Because the slope would eventually reach 0. [draws the graph below]

\[ f' \]

\[ t \]

A little more intervention moved him from this picture to the following graph which is correct.
This difficulty with negative rate continued with the exercises on drawing $f'$ from $f$ without context, and seemed intimately connected to Cam's tendency to confuse the behaviour of $f'$ with the behaviour of $f$. However, with the second of these exercises he seemed to cross a threshold. When asked to draw a graph of a function $f$ such that "$f$ decreases and does so faster", he quickly drew the appropriate graph (pictured below).

When he initially considered the graph for $f'$, he again missed the negative values, rather he saw that $f$ was getting steeper and drew $f'$ as going up.

C: Okay. Assuming it starts off at 0 here, this would be 0 at 0. Then it would, the slope here, would be increasing so the slope would increase like so. Until it's almost vertical. Okay. So it would look something like that [draws the graph below].
When asked to reconsider, he then acknowledged that the values of the derivative should be negative but his initial attempt at representing the negative values put the curve on the left of the y-axis. He checked this inclination and eventually came up with the graph below.

C: Okay. Let me run this through my head again. Slope increases so, if you've got 0 here, -1/2, -1, -2. Let's try that one there [draws the graph below]. Now I think we're catching on. So here it is, for the record. As time increases, the rate of change of the slope increases negatively.

Notice that he was still struggling with words to distinguish the value and behaviour of the slope. He could have reworded "the rate of change of slope is increasing negatively" by stating "the slope is negative and is decreasing".

Language difficulties aside, he evidently had drawn the distinction because he attacked the next task of drawing a graph of \( f \) from the graph of \( f' \) with utmost efficiency and accuracy. Also, he responded to problem #1 on the paper and pencil test (see Figure 16) correctly, and used an argument involving concavity. Similarly, he clearly articulated the significance of a negative value for slope in his response to #3 on this task.
C: When the slope is going from left to right it would be negative. When it's going from right to left it's positive. So from -4, which looks like it would be about 0, to -1, the slope is going from left to right which would be negative. And then again from 1 to 2 it's doing the same thing. And from 3 to infinity.

I find it interesting that his rationale is all in terms of slope, and that there is no mention of the behaviour of $f$, that is, no mention of whether the function decreases or increases. Indeed, when asked to determine where a function is increasing and where it is decreasing (problem #4, Figure 16), there was again some uncertainty about what to consider.

C: And here $f$ prime is decreasing so $f$ would be [long pause]. Okay, I've got to run this over in my head again. I'm getting confused between $f$, $f$ prime and $f$ double prime. Okay, now when $f$ is increasing, $f$ prime is positive and, okay so $f$ prime is negative then $f$ must be decreasing. $f$ prime is still negative but it's increasing. Now if $f$ prime is increasing to 0 what would $f$ be doing?

The answer to the question "what would $f$ be doing" lies only in the value of $f'$ not in the behaviour of $f'$. Cam made considerable progress towards making with this distinction in a graphical setting, but the progress was not robust. In an algebraic setting the significance of the statement $f'(x)>0$ still presented quite a challenge. It took him some 15 minutes of very tentative work to come up with a correct solution to #5 on the paper and pencil test.

**Summary**

I feel Cam had an excellent intuitive grasp of the nature of dynamic situations, and of the interpretation of variables and functions used to represent such situations.
Unfortunately, he did not work with variables very well, and his algebraic skills sometimes interfered with making sense of problems about such situations. The notion of slope played a strong role in his understanding of dynamic situations. He developed strong connections between slope and rate, and he became quite competent with using slope to describe dynamic situations graphically. Cam also developed a strong sense of the connection between the derivative and the notion of rate. He did experience some difficulties in distinguishing the value of a function from the behaviour of a function. This was particularly evident in his view of how the graph of \( f \) and that of \( f' \) relate to each other. Most of this difficulty seemed to be connected to the issue of the rate being negative. However, I feel he made admirable progress towards drawing this distinction between the behaviour and the values of a function, and towards the connection of the behaviour of a function with the values of the function's derivative. Coming to grips with the issue of negative rates seemed to be the principal factor in making this progress.
Portrait of Margo's Understanding of Dynamic Situations

Throughout the study, Margo was taking Math 253, the second half of our introductory differential calculus course. As such, she had one term of experience with calculus. She was in her late thirties and returned to school to train for a career as a mathematics teacher. Margo seemed to work primarily with a particular view of variables. She had a good intuitive sense of rate of change that provided appropriate guidance with her graphical representations of dynamic situations. Her images of rate included a sense of comparing two quantities, however, this sense was seemingly not put to work when she addressed problems of rate in more formal contexts. This is one example of the challenges that Margo seemed to face in connecting her intuitive sense with the formalities of mathematics, even though she generally displayed good computational command with these formalities.

Dynamic View

The focused list for variable that Margo offered included three distinct uses, one of which suggested a dynamic view.

M: Something that changes, something that stands for something, or holds the place of something [pause] that's about it. The variable, at one time it could be one thing or at one time it could be something else. Like $x=1$ in one place and $x=3$ somewhere else. So it's something that changes.

I found it interesting that she should distinguish the other two uses, and I found it difficult to understand the nature
of such a distinction. I asked her to expand on these uses.

M: We'll, two and three, they're two separate things. See, I could say a variable is something that stands for something else, as being a symbol that means something else or I could say that the variable holds the place for something. Like if you have a big long calculation like a physics problem and you've got your variables holding the places in the calculation. They're pretty much the same [pause] but a little bit different.

I: Can we think about a specific example? You had mentioned an equation in physics and one that comes most readily to my mind is the distance travelled [by an object with constant acceleration]. The formula would be \( s=\frac{1}{2}at^2+v_0t \).

M: This is a good way of explaining it because you've got three variables. You've got \( s \), you've \( v_0 \) and you've got \( t \). On, four [variables], and \( a \). So they're all holding places in the formula. If you were to mix them and put the \( s \) here or [make it] \( at+\frac{1}{2}v_0t^2 \) it's not the same thing. So they're holding places in the formula.

She seemed to draw a distinction between representing a physical concept, such as velocity, and representing the role that a numerical value of that concept would play.

Although she mentioned that variables could change, there was not necessarily a sense of flowing from "\( x=1 \) in one place and \( x=3 \) somewhere else". This suggested to me that a dynamic view of variables was not very prominent. During the formal car/truck problem, she did not initially work with variables as dynamic objects. Indeed, she initially ruled out the solution by Student B, and displayed much more of a particular view of variables as she considered this problem.

M: I'm confused because they all sound like reasonable ways of going about it, except for this polarity one [the solution from Student B].

I: That's not reasonable?

M: Well, it certainly wouldn't have come into my mind to do it that way.

I: Student A has noticed the 18 for the car and the 15 for the truck and drawn the conclusion [that the car is always
ahead of the truck].

M: I sort of did that myself before I got to reading the solution there. I just stuck 1 in there for $t$ and thought the truck is never going to pass the car. I'd put a 1 in and try to get a rough idea of what's going on before I would try and solve it exactly. Try and get a ballpark figure on what's happening. So I would sort of agree with Student A because that's what they've done. The first thing they had done was just take a look and get some idea of what's going on. And the same with Student C, except [pause while she considers the response from Student C], except that I wouldn't have made a table. I would try a number like Student C. I'd try the number 1. [After further consideration] Well, I think Student A is just making it too simple. He's not looking at the whole thing. I don't think it's quite that simple.

I: Not looking at the whole thing, what has been overlooked?

M: [Pause] But then he's right. $t^2+18$ is always going to be bigger than $15t$. Unless it's negative and it can't be negative. So maybe that is enough.

She may have considered that these algebraic expressions could change, however, she did not display much dynamic sense in asserting that $t^2+18$ will always be larger than $15t$.

As she further considered the problem she recognized the error in this statement by considering another value for $t$.

She went on to make considerable progress towards working with a dynamic view of the situation.

I: Can you describe what's going through your mind?

M: Well I said before that this $t^2+18$ has always got to be bigger than $15t$ but if you put 4 in, it isn't, is it? Well, at four seconds, the truck's ahead of the car. So A is wrong. [Now] I'm reading [the answer from] Student B and I can see that it's very reasonable. At first I thought, I dismissed it as "what's this guy doing?" but now I see that [it might make sense].

I: Why have you come to now see that it could be reasonable.

M: I figured that my first impression that the car's ahead of the truck all the time was wrong by looking at [the solution from] Student C and seeing the numbers. So then I looked at Student D to see just where does it change. Ah-hah, change. And I see he's gotten the points where they're in line with each other, so I thought oh well, a polarity chart would be the way to see which is ahead and
which is behind. So that's when I looked at Student B to see how he's done it. So I can see that that's the correct one now. That's funny because that's the one I dismissed right at the beginning.

It is interesting to notice that she realized the word change has a place in this algebraic setting. Below, I will address her focused list for change and show that such a setting was not readily associated with change.

As we finished the formal car/truck problem she presented an interpretation of the algebraic representation that seemed to embrace a dynamic view.

I: As you think about this problem, can you think about what we talked about last time, and do you see any agreement with what we chatted about last time?
M: I had the truck passing the car and then the car accelerating to meet the speed of the truck. I never had the car ahead of the truck. I was thinking about it a little bit differently. In this problem the car starts to move before the truck actually catches up to the car. I had them caught up, the truck still going and then the car started to move.

This description suggested to me that she had good intuition about translating an algebraic representation of a dynamic situation.

Her focused list for change offered some sense of a dynamic view, however it seemed to be restricted to transition between states that need not include numerical states.

I: You've written Tai Chi, life, everything.
M: Everything is about change. I practice Tai Chi and it's about change. You change from one movement to the next to the next to the next to the next and all the movements change, sort of from the inside. The inside stays the same and the outside moves and changes and changes and changes. That's the way my teacher always
talks about it. Change. So that's sort of preoccupies my mind. And life is about changes. And that's about the only thing guaranteed in life is that things are going to change. Everything changes. Nothing ever stays the same.

I: You mentioned that in Tai Chi you change from one posture to another. I'm curious about that. In Tai Chi, is it the states that you hold, the postures that you hold, that become the focus of your attention or is it the transition of one to another that becomes the focus of your attention?

M: Well, if you get really good at it the focus of your attention is your centre that doesn't change at all but yeah, it's how one gets from one posture to the next that's sort of the important part.

This focus on transition spawned a mathematical context for change that similarly embraced transition. However, rather than from one posture to another, the transition in a mathematical context was from one algebraic form to another.

M: In fact I was telling my [Tai Chi] teacher that math is like Tai Chi because you just change it from one form to the next to the next to the next to the next until you finally have it in a form that you want for your particular application. When I mentioned that to him, I was doing Math 152. We were doing [pause] I don't even remember all that stuff, but you have to get the equations for the parabolas in the right form so that you can read off the particular things when you have it in the right form. So you do it by changing. Well the same thing with simplifying those derivatives. You know, you do the derivative you get this big long thing and then you just keep simplifying it, you just keep changing it down to a form that you like. I say that because I've been having problems with that. I either don't simplify enough or too far or I come up with different answers than the problem book does every time.

I: Any other mathematical things with change?

M: Not that comes to mind right away.

There was not an association with the word variable, even though her focused list for variable had included the notion of changing from one number to another.

During the last interview when we addressed the focused list again, there was still not a quantitative association
with change, despite prompting on my part.

I: Can I focus on, say, quantities changing. Can you think about what words you might use to describe quantities that are changing, or how quantities are changing.

M: Describe how quantities are changing. I don't know what to say.

She offered a very insightful comment during this interview that suggested to me she had made considerable progress towards adopting a dynamic view in the course of studying calculus, but remained somewhat tentative about seeing variables in this way.

M: I guess up until the point of learning calculus all math has been just numbers, sort of. Very static sort of a thing. Fairly static in that this equals that. But to be able to apply it to something else, to something that's changed, to some different point in time or at some different quantity of x, is [pause] I still don't really understand it.

Images of Rates

Margo displayed strong intuition about the technical aspects of rate, in particular the presence of two entities in the concept.

M: Well, the word rate, the first thing that comes to mind is related rate problems which I still don't know how to do, I don't know why. But rate is one thing compared to another. What else would you say a rate is? [pause] That pretty well sums it up. It's a relationship between two things, usually time but not necessarily. Your pay rate is how many dollars you're going to get paid for the amount of time you put in, or you could get paid piece work where you go so many dollars per amount of work done. So, it's a relationship between two things.

When pressed for a non-mathematical context she offered the interpretation of ranking.

M: Rate, you could think of it as ratings, giving something a value, you could talk about TV ratings.
During the flowing water task she demonstrated a very good intuitive sense of rate and used this intuition well as a guide in sketching the graphs of the height of the water in the containers. For the first two containers, the beaker and the graduated cylinder, she was able to see the rate decrease and was easily able to contrast the behaviour of the height of the water in the two containers. She accounted for the changing rate, and she drew a connection between the rate and the slope. Throughout the discussion her language focused on rate.

M: At first I thought it [the height of water in the beaker] was going to go straight up but it actually slows down a little bit as it fills up. Oh, it's much easier to see how it slows down in this one [the graduated cylinder]. It's slowing down I think because of the way it comes out of the burette, it has nothing to do with what's catching it. But it's just more visible to see that it slows down as it fills up in the graduated cylinder. So I made it [the graph] a little flatter.

Similarly, the nature of how the height of the water increased in the other containers was very easily discerned, and the height was accurately represented.

M: Well, [for the Erlenmeyer flask] the height rises at a steady rate. I'm surprised because I thought it was going to go on an exponential graph like that [indicating the general shape of an exponential curve with her hand], judging by the shape of the container. But, I guess the shape of the container cancels out the slowing down of the burette and you come up with a straight line. Well, [in the martini glass] you have the level rising real quickly at first but then levelling off to an almost imperceptible change in height. [The volumetric flask] actually filled fairly steadily until it got to the neck and then it went quickly.
This graphical intuition was also evident when she addressed bottles, the inverse of this task, and when she addressed drawing the graph of a function from words describing how the function behaves (graphs of dynamic situations in the sixth interview).

Such intuition contributed to correct answers to the problems of comparing rates in microbes during the fourth interview.

M: Between 1pm and 3pm, A is the faster growing. Because I'm looking at the slope, the sharpness. I have my words all mixed up. Well, the steepness of the curve, the steepness, that's the word I'm looking for, so the steepness of A. A is more steep than B.

I: Does population A grow faster between 4am and 6am or between 3 pm and 4 pm?

M: Can't tell from just looking. I'm going to have to figure this out by the slope.

However, during the fifth interview when this type of problem was addressed numerically in weights and algebraically in functions, she considered only the change in quantity rather than rate of change. This inconsistency seems even more remarkable given the attention she devoted to "one thing compared to another" in her focused list. Unfortunately, in the numerical and algebraic setting she seemed to have overlooked what plays the role of "the other". This oversight also surfaced when we addressed her connection between rate and the derivative.

Probably because of her experience with Math 153, Margo readily saw the connection between the flowing water task and the formalities of rate of change.
M: It's a good preliminary to related rate problems. I don't know why I can't grasp them because I understand the concepts of changing rates of things. But I just can't do the problems.

In spite of having seen this connection, and in spite of her strong intuition about rate, Margo confessed to having difficulty with related rate problems. We used this very opportune moment for a tutorial session on this topic.

I: The problem reads "the atmospheric pressure at an altitude $h$ feet above sea level is given by $P = 15e^{-0.0004h}$ lb/in$^2$. A jetliner is at 10,000 feet and climbing at a rate of 1,000 feet per minute. Find the rate of external pressure at that instant". What kind of sense do you make of what you're asked?

M: Well, you've got the atmospheric pressure, $P$. So the change in pressure is going to be the derivative of this function $P$. Right? So $P'$ is going to be the change in pressure. Right?

She misinterpreted the derivative as the change in the function, rather than as the rate of change of the function.

Further discussion showed that she was overlooking the role played by the independent variable in a derivative. As such, she did not readily have available another quantity for comparison.

I: Can you try to explain that [phrase], "if $P$ is pressure, $P'$ is change in pressure" to me. What you mean by that?

M: Well, I understand that derivatives are the rate of change of something so if $P$ is what's given, then the rate of change of $P$ is going to be the derivative of the function.

I: Do you see any difference between the phrase rate of change and the word change?

M: Well, rate of change is [pause] saying there's got to be something else in there that it's changing against. Which I suppose is the key. So it's the change in pressure with respect to what, I don't know [pause] altitude, I guess.

I: I think it's essential for you to be very specific about that. You've mentioned "one thing compared to another". I want to suggest that this is where Leibniz notation becomes absolutely invaluable.
M: That's what the teachers keep saying except that I can never figure out which letters you put where. So then it becomes confusing.

I: [uses her own words "one thing compared to another" to explain what Leibniz notation depicts]

M: Okay, what I want is the rate of external pressure at that instant. I imagine that instant is the 10,000 feet. So the rate would be the [pause] it should be one thing compared to another. I don't know what, the pressure compared to the altitude again. That's all I see there.

I: I suspect there is another possible interpretation.

M: Time. Because you put this, that 1,000 feet per minute. Oh, so they want to know the rate of change of pressure per time.

We addressed a second problem in which this issue did not arise, and then a third problem where she demonstrated a better command of the role played by the independent variable in the derivative.

I: According to a logistic model based on the assumption that the earth can support no more than 40 billion people, the world's population, in billions of people, \( t \) years after 1960 will be approximated by the function
\[
p(t) = \frac{40}{1+12e^{-0.08t}}.
\]

The first questions asks, if this model is correct at what rate will the world's population be increasing in 1985? Before you think about what you do, can I suggest you think about what you want.

M: What we want, the rate of increase of the world's population in 1985. The rate of increase, so that's the number of people per some time. Year, I guess, because they're talking \( t \) years.

I: So that's what we're after in your language. Now, let's try to express the same idea in this language, the language of calculus. Your ideas are the same, just the way in which we're writing the ideas is a little bit different.

M: Well, if the function is \( p(t) \) then the derivative of this function \( p \) is going to be the rate of change per year, of population, the number of people per year. So that would be \( dp/dt \).

In the sixth interview we had a tutorial on related rates problems with trigonometric functions during which there was evidence that the role of the independent variable
was quite clearly understood. Also, the solutions to the related rates problems on the final exam included clear and accurate use of Leibniz notation to indicate the appropriate rates. During the last interview she acknowledged that our work on improving the connection between the notion of rate and the derivative had been worthwhile.

M: This is what I've gotten from you, from our interviews here. As soon as I see rate of increase, ah they mean the derivative of something.

Value/Behaviour Distinction

We encountered confusion between value and rate when we addressed the problem about atmospheric pressure during the tutorial on related rates (see p. 162 above).

M: So the rate would be the pressure compared to the altitude again. That's all I see there.
I: Well how would you find that, what would you do?
M: I would just stick the 10,000 feet into the function given for pressure and I'd get the pressure at 10,000 feet.
I: That would be the pressure, that wouldn't be rate. The rate has to be one thing compared to another rather than one thing.
M: I don't know what it wants.
I: But you made the decision that it wants the rate of change of pressure with respect to altitude.
M: Oh! So I put the 10,000 into the derivative.

When we addressed functions during the fifth interview, the issue arose again, however, on this occasion she sorted out the distinction by herself.

M: Well, I'll just plug this 1.55 in and see what we come up with. Well that just gives me the value of f(x) at 1.55. That just gives me the value, it doesn't tell me which one's growing faster. For that you need the derivative. Ah. Okay. So I'll take f' which is 3.2-x and g prime which .2 and put in the x.
I: Were there any words in here that cued that response to consider the derivative?
"Which one is growing faster" is the slope, so which one's going to be steeper, so the value had nothing to do with it.

Although Margo made progress in distinguishing the value from the rate in an algebraic setting, in a graphical setting the confusion again arose. Here, it appeared in the form of having difficulty representing a negative rate and in the focusing on the behaviour of $f'$ rather than on the values of $f'$. When asked to draw a graph of a function that decreases at a constant rate, she quite easily drew an appropriate graph. However, she then had some difficulty drawing a graph for the derivative of such a function.

M: Well, it [the graph of $f'$] would be a curve just going to a horizontal asymptote because you couldn't have it going into the negative.
I: Why is that?
M: Hmmm. I don't know.

Some prompting from me brought out the representation of constant rate as a horizontal line for $f'$, but she had some trouble determining the appropriate place to locate such a line.

M: Oh, well it's constant. It's a straight horizontal line.
I: Let me ask you this. What would you estimate the slope of this line [for $f$] to be?
M: -1.
I: And what [have you drawn for] $f'$?
M: 0.
I: But what you're asked to think about is the rate of this function [$f$].
M: Oh, okay, so right, -1 and it's never changing.
When she addressed a function that decreases and does so more slowly, she again drew an appropriate graph for $f$ but wrestled with drawing $f'$ (see below). She missed the negative value for slope, focused on the behaviour of $f$, and represented $f'$ going toward zero in order to bring about this behaviour.

M: Let's see. Well, the changes are getting slower. So it should be a straight line going down towards the right.

![Graphs of f and f']

This difficulty continued with her first attempts at drawing $f$ from the graph of $f'$.

M: Well, the first one, $f'$ is a straight line decreasing towards the right so that means the rate of change is decreasing. So that would be a curve getting flatter and flatter and flatter as it goes towards the right [see what she has labelled "first try" in Figure 20].

![Graphs of f and f' with labels]

**Figure 20.** Margo's sketches for the first part in the task of drawing the graph of $f$ from the graph of $f'$. 

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Her logic was impeccable except that she overlooked the fact that the derivative would not have negative values. So although \( f \) would be getting flatter it would not be decreasing, rather it would be increasing.

She seemed to overlook this with her initial responses to the next two graphs also. Some prompting about the significance of the derivative being zero (crossing the axis), and the interpretation of negative slope brought about a change in her responses.

I: If you keep in mind the interpretation of \( f' \) as slope, what would this say about the slope?

M: Oh, it would say that's a negative slope and that's [pointing to the right of the x-intercept] a positive slope so it should probably look more like a parabola, like that, with a minimum [see Figure 21, second try].

Figure 21. Margo's sketches for the third part in the task of drawing the graph of \( f \) from the graph of \( f' \).

After this she quickly went back and corrected the other two graphs.

M: I'm looking at the second one now. Yeah. Okay, if that's negative, oops I have a feeling it should be decreasing. But [pause] it's getting less and less negative until it hits zero so if it's getting less and less negative then it will be flattening out so it would be a curve [see Figure 22, second try]. And the first one ... yeah, and the first one would be that way. It's flattening out but it's flattening out going up [see Figure 20, second try].
She made these changes to her initial responses quite quickly, and so seemed to have readily drawn the distinction between the behaviour of a function and the behaviour of the function's derivative, and drawn the connection between the behaviour of a function and the values of the function's derivative. For the most part, this progress was reinforced with the work on the paper and pencil test in the seventh interview. For example, as she addressed the third problem (see Figure 16) she did so in terms of the behaviour of $f$.

M: That would be where the function's decreasing, so that would be from here to here.

She similarly displayed the connection between the value of $f'$ and the behaviour of $f$ when she attempted the fourth problem which was the converse of the above problem.

M: Okay, I'm just trying to picture this as a polarity number line like I usually use because I don't usually see $f'$ on a graph.

Although, she had wrestled with this connection throughout the interviews, she used the connection well on the graphing problems on the final exam.
Connecting Intuitive Sense with Formal Study

In the sections above, I commented that the level of Margo's intuition about dynamic situations was not always matched by her understanding of the formal representations. In particular, although she demonstrated computational skills with the formal operations, at times she expressed concern about her understanding of these operations. This was most evident in the last interview as we reviewed the term.

M: Actually I think that's where I'm finding problems. I don't really understand, so it's more difficult to memorize a bunch of rules and just apply them in the right places if you're kind of fuzzy about what's going on. Well, [Math] 253 as opposed to [Math] 153, I find is quite different. [In Math] 153 you're learning all the concepts and in 253 you're just learning a whole bunch more rules to apply them to different functions.
I: Do you find that easier or more difficult, do you favour one over the other?
M: Well, I mean, you have to have both, you have to have a little bit of both. Like I say, I'm finding now with the how-to approach, every time I get a different problem I'm kind of stuck. What do I do with it? I go look through my notes for one that's the same so I can follow the step by step procedure because I'm having trouble reasoning it out and seeing where it's coming from, to be able to figure out the step by step [procedure] by myself.

During the second interview she had expressed a willingness to suspend meaning for some time and a desire to focus on manipulative skills.

M: I don't understand why that [property of logarithms] works, but for now I'll take the teacher's word for it that these are the rules you follow and just do it.
I: What is it that you don't understand.
M: This whole log business. What it is? But as I say, it's okay, I'll just learn the rules and apply them for now and then I suppose later on I'll understand it. I find that's sort of the best way to approach this is to just do it for awhile first instead of trying to understand it. It will come later.
Unfortunately, on this occasion it seemed that understanding did not come later because during the last interview she commented that the most difficult part of the course had been the study of the logarithm functions.

M: Well, the most difficult part I guess is the ln functions for me because I'm pretty fuzzy on what this logarithm business is and where they come from, where they get these functions in the first place.

Her willingness to suspend understanding seemed to be connected to her experience as a student ("the best way to approach this"). She evidently had the intuition to make better sense of these formalities—consider the progress she made in understanding the role of the independent variable in the derivative.

**Summary**

In a variety of situations Margo seemed to display good intuitive sense but experienced difficulty connecting this intuition to mathematical formalities. However, she generally made good progress towards making such connections throughout the course of our interviews.

Early in the term she seemed to operate more with a particular view of variables than with a general view or a dynamic view. As the term progressed, she did make a shift, albeit tentative, towards operating with a dynamic view of variables.

The image of comparing two things played a very strong role in her understanding of rate. Yet she had difficulty bringing this image to bear on the problems involving a
comparison of rates in a numeric and in an algebraic setting. She also seemed to overlook this image as it applied to the derivative and hence had difficulty with understanding the significance of the role played by the independent variable in the derivative. As the interviews progressed, she developed a much improved understanding of this issue.

Margo was able to readily connect her intuition about rate to graphical representations of dynamic situations. During the last interview, she commented that the flowing water task had been most helpful in this regard.

M: Well, I liked the water going into the different containers. That made the exponential and log functions, it made them real.

I interpret the reference to exponential and log functions as pertaining to the shape of the graphs. Her performance on all of the problems that involved depicting a function from a description of its behaviour indicated that she had an excellent sense of how rate affects the shape of a graph.

Despite this skill of representing shape, she displayed some troubles with distinguishing the behaviour of a function from that of its derivative, as did the other subjects in the study. This situation also improved as the interviews progressed. As with the other subjects, it seemed to be addressing the significance of negative values for the derivative that acted as a conduit through this obstacle.
At the outset of this study, my hypothesis was that the concepts of change, and rate of change, were difficult for students to see and/or comprehend. My suspicion was that intuition about dynamic situations was not strong in calculus students, and I felt that such lack of intuition contributed to their difficulties with understanding the symbolic formalism of calculus.

As a result of this study, I feel that a dynamic sense is quite accessible, although it probably does not easily occur spontaneously. More importantly though, I feel the connections between a dynamic sense and the mathematical formalities used for a quantitative study of dynamic situations warrant more attention than we, as calculus instructors, often afford them. Such connections could be nurtured by activities similar to those employed in this study. I observed that distinguishing slope and value is a robust problem for students, and is intimately connected to understanding the significance of negative rates. Lastly, this study contributed to my current belief that computational command and conceptual command are quite independent. In this chapter, I will discuss these conclusions.
Performance on the Interview Tasks

The subjects in my study displayed performance that seemed comparable with what was reported in the literature I reviewed in Chapter II. Although the intent and details of this study were very different from those works, I find it useful to consider this comparison in order to get a sense of how widespread these issues might be.

As with the students involved in the works on the language concerning limits (Cornu, 1981; Monaghan, 1991), the subjects in my study seemed to maintain different contexts for words that might be seen as mathematically synonymous. In the case of my study these words were variable and change. I see this as suggesting some differences between their informal concept image and the concept field.

The subjects in my study displayed somewhat mixed results concerning the command of formal representations of change. All of them were very good at representing dynamic situations graphically. Their work with intervals in an graphical and numerical setting seemed to be consistent the results reviewed in Chapter II (Bell & Janvier, 1981; Monk, 1990). They were quite competent at working with intervals, like the college students in Monk's work, and unlike the high school students in Bell and Janvier's work. However, in an algebraic setting, their performance on problems of determining intervals did not match the command they demonstrated in the other two settings. I see this as
evidence of gaps that need to be bridged among formal constructs.

**Dynamic View**

Working with a *dynamic view* of variables would seem to make the process of connecting variables to dynamic situations much easier. The work of my subjects suggested that some of the challenges identified by Küchemann (1981) that younger students face in working with variables seem to persist even into this stage of students' mathematical development. Working with either a *general* view or a *dynamic* view of variables at times seemed to present a challenge. For example, Margo's analysis of the proposed solutions for the formal car/truck problem very much involved a *particular* view of variables. She did shift towards a dynamic view as the analysis, indeed as the term, progressed but such movement seemed tentative.

During the second interview Amy wrestled with the issue of the formula for slope holding regardless of where one is on the curve. As mentioned in the literature review, this issue was also confronted by Gooya's (1988) subjects. Although there may be many ways to account for this phenomenon, I conceive the difficulty of not seeing the generality of the formula as one of working with a *particular* view. It might be surprising to discover the prevalence of this view at such an academic level.
Only Cam seemed to have a dynamic view of variables from the outset, but his algebraic skills were such that he was not predisposed to work with letters. For example, he initially agreed with the proposed solution from Student B on the formal car/truck problem but mentioned a preference for seeing graphs for the algebraic expressions. Although the focused lists and responses to proposed solutions in the formal car/truck problem seemed to mainly involve a particular view of variables, the subjects were quite able to work with interval responses and hence did seem able to move beyond such a view.

A dynamic sense was much more evident from those activities in the study that did not emphasize the role of a variable. Both of the women attached a dynamic sense to the word change. During the flowing water task, all of the subjects in the study were quite adept at seeing how the height of the water in the containers was changing and at representing this change with a graph. Throughout other interview tasks the subjects demonstrated good command of using graphs to display the way in which quantities changed. The graphs they sketched from words that described how a function changes were all very good. I conclude that a dynamic view was used by all subjects.

Most of the work on calculus students' understanding of limits that I reviewed in Chapter II (Monaghan, 1991; Tall & Vinner, 1981; Williams, 1991) mentioned students' tendency to
apply a dynamic sense to limits. The work of the subjects in my study suggested to me that they all had a good intuitive sense of dynamic situations. As such, I set aside my initial hypothesis that the concepts of change, and rate of change, were difficult for students to see and/or comprehend. My suspicion that calculus students' intuition about dynamic situations was not strong has proven to be unfounded. However, connecting this intuition to the formal symbols of calculus so that, for example, they can conceive of variables as flowing quantities like Newton did, seems to require some attention. I maintain that a dynamic view of variables, graphs and functions promotes meaning for the manipulations of calculus, and hence warrants closer consideration in the calculus classroom.

Images of Rate

As I outlined in Chapter II, the concept field for rate is multi-faceted. It includes global notions of average rate of change, slope between two points and the difference quotient, as well as the instantaneous notions of rate of change, slope, and derivative. Working with the global notions involves working with something beyond what is immediately present and hence these notions are what Monk (1990) identified as derived quantities. The subjects in this study easily worked with some derived quantities by readily giving interval responses. Also, they did not fall
prey to such distractors as higher ordinate values (see p. 106). However, working with the derived quantity of average rate of change presented some difficulties. Like the subjects in the work of Bell and Janvier (1981), my subjects generally considered the amount of change rather than the rate of change.

The notions of rate of change, slope, and derivative are, in a formal sense, synonymous. Seeing that these notions are synonymous, however, involves three connections for students (slope/rate, slope/derivative, rate/derivative). I was surprised to observe that these connections were not made equally, nor in the same manner. Indeed, each subject seemed to have a different connection play a dominant role in his or her concept image of rates. I was also surprised to observe very different informal images of rates for each subject, and to see how these images were put to varying degrees of use.

For Amy, the notions of "per time" or "speed" were very powerful in her informal image of rate, so much so that they seemed to inhibit making connections to slope and to the derivative. It was wonderful to observe her as she made these connections. Although her informal images may have contributed to the difficulties of making such connections, they were put to appropriate use in problems of comparing rates. She was the only subject to use the derived quantity of rate to address the comparison problems in the weights
Cam's images of rates seemed much more formal, yet he put them to good use. He seemed more comfortable than Amy was with considering rates with respect to some variable other than time. The graphical image of slope seemed to be dominant in his conception of rate and he used this image well. The derived quantity of average rate was not used well in a numerical setting or in an algebraic setting, where he focused on amount of change rather than rate of change.

In these settings, Margo also focused on change rather than rate of change, yet she had a powerful image of rate as "one thing compared to another". This image seemed somewhat more general than Amy's "per time" because it had no restriction on the independent variable. We put this image to use in a tutorial on the significance of the independent variable in the derivative. This emphasis on connecting her own words to the formalities of the derivative seemed to be a very effective tutorial strategy.

The concept of rate is complex and yet it is fundamental to understanding the technology of calculus. My subjects offered a variety of dominant images for this notion. I was surprised to learn that their images could be so rich. I was also surprised to see the difficulties they had in making connections between the various aspects of rate. Emphasizing this multiplicity of aspects of rate and connecting the students' dominant image with the other aspects seemed to
enrich the subjects' understanding of rate. This provides an example of how the connections between a student's intuition and mathematical formalities can be strengthened.

Value/Behaviour Distinction

As with the work reviewed in Chapter II (Bell & Janvier, 1981; Bergquist & Heikkinen, 1990; Monk, 1990) the subjects in this study did, from time to time, have difficulty distinguishing value from rates. Drawing a distinction between amount and rate seems to involve being able to work with something beyond what is immediately presented. The graphical equivalent seems to be drawing a distinction between values on a graph and the behaviour of the graph. It was this distinction that the subjects of this study found most challenging. However, this challenge was overcome by all subjects through attention to the possibility that rate could be negative.

Rubin and Nemirovsky (1991b) state "we believe this propensity [to assume resemblances between a function and its derivative] is not a simple matter of confusing a function with its derivative" (p. 32). Certainly it is not a simple matter, but I believe the tendency is intimately tied to the confusion between the behaviour of \( f \) (value of \( f' \)) and the behaviour of \( f' \). For example, for the drawing \( f \) from \( f' \) task in the sixth interview, all the original graphs that Amy drew for \( f \) resemble the graphs of \( f' \). This is consistent with the
data from Rubin and Nemirovsky, however, these were all a result of not distinguishing the value of $f'$ from the behaviour of $f'$. Indeed, Amy's own words were "oh, it doesn't matter what the slope of $f'$ is doing, it is what $f'$ is equal to!" (see p. 133). Once this distinction was made, all the graphs for $f$ were easily corrected.

There are, no doubt, a number of ways to attempt to account for this confusion of value and behaviour. Value of a function is a pointwise phenomenon whereas behaviour is a global phenomenon. A particular view is all that is required in order to read points, whereas a dynamic view seems appropriate for observing global phenomena. The confusion of value and behaviour may be one of using a particular view when a dynamic view would be more appropriate.

Rubin and Nemirovsky (1991b) try to account for this propensity by identifying the guiding principles of *isomorphic variation* and *geometrical patterns* (see p. 49). The graph that Cam initially drew for the derivative of a function that decreases and does so faster (see p. 150) clearly did not result in the same shape but it did reflect the same behaviour of getting steeper. Hence, I cannot imagine that he was using *geometrical patterns* as a guiding principle.

Rather, Cam was apparently guided by *isomorphic variation*, however in a manner different from what Rubin and
Nemirovsky call an "overgeneralization of experiential situations" (p. 10). Margo's rationale for her graphs similarly resulted in the behaviour of $f$ being the same as the behaviour of $f'$. However, what each of the subjects in my study overlooked was the algebraic sign of the rate, slope, or derivative. Their rationale was in terms of what the absolute value of the slope was doing, and did not consider the orientation of the slope. It was recognizing the significance of a negative rate that seemed to provide a conduit for negotiating this obstacle of distinguishing the value of $f'$ from the behaviour of $f'$, and hence connecting the value of $f'$ to the behaviour of $f$. This slope/value confusion seems to be more robust than we might imagine.

**Promoting Connections Between Intuition and Formalities**

In providing rationale for trying to attribute the difficulty with the value/behaviour distinction to something other than merely confusing a function with its derivative, Rubin and Nemirovsky state the following.

We know from everyday life that if we close a sink faucet slowly, the volume of water accumulated in a bucket will continue to increase. For us it is not an explanation to say "Dan doesn't know about that relationship". (p. 23)

They seem to be asserting that students' have the intuition that a function and its derivative are different. My subjects were able to display such intuition in any graphing exercise that asked them to represent a function from given
information about how the function changed.

Nonetheless, they displayed the same tendencies to show resemblances between \( f \) and \( f' \) as did the subjects in Rubin and Nemirovsky's study. Throughout my study I observed students having difficulty connecting their intuition with the formalities of mathematics. As such, I would like to suggest that although "Dan" might know that the bucket keeps filling as the flow is decreased, he might have difficulty connecting flow to slope. For Rubin and Nemirovsky's subjects, the experience with the air-bag environment seemed to promote the connection to the algebraic sign of rates. Sutherland asserts that the connections her students made with variables depended very much on how they had used variables in a LOGO setting (see p. 23). My study included a variety of activities that seemed to promote drawing connections between the subjects' intuition and their formal study.

Experiences such as the flowing water task and sketching graphs of quantities from descriptions of how the quantities change seemed to promote a dynamic view of the formal representations of change--variables, functions and graphs. All the subjects commented on how the flowing water task helped improve their sense of rate of change. The scales involved in this activity seemed particularly appropriate for such contribution. The situation happens quickly enough that it can be observed from start to finish, unlike many
situations such as, say, the growth of a plant. The water flows slowly enough that the continuously changing height is very noticeable, unlike say the motion of a baseball thrown by a professional pitcher. The apparatus is small enough to bring the dynamic situation into in a room, unlike perhaps the population of a city. Yet, the apparatus is large enough for the phenomenon to be easily observed, unlike say a population of bacterial culture.

All the subjects wrestled with the significance of negative rate, despite the Test for Increasing and Decreasing Functions (see p. 52) being a topic of study in the course. This difficulty with the negative sign was also reported in Rubin and Nemirovksy's (1991a) work. Difficulties with zero rate were reported in Monk's (1990) work. Both zero and the negative sign are formal mathematical conventions that do not seem to be easily connected to intuitive sense. The subjects in Rubin and Nemirovsky's study were able to make these connections by working in a technological environment that offered a natural reference point for zero, and a natural interpretation for the rate being negative. The rationale used by the subjects in my study for drawing the graph of $f$ from the graph of $f'$ seemed to only overlook algebraic sign. They were not so fortunate as to have technological resources at their disposal, however they were able to make progress with the significance of negative rate by confronting it intellectually while sketching the graph of a function given
As I said above, the slope/value distinction was challenging, and distinguishing the two was not something that these students spontaneously mastered. However, they all did make such progress. I found that using their own descriptions of how a function behaved, and focusing on what they were referring to when they used the word it, were very useful in promoting this value/behaviour distinction.

The task of drawing $f$ from a picture of $f'$ also seemed very useful in this regard. Indeed, during the last interview, Amy identified this task as being the most helpful thing that we did throughout the interviews.

A: Yeah, that whole [pause] the graphs, the same thing that I had backwards. I don't know whether I had it backwards because I wasn't thinking it through. But to actually be aware of it made a difference to me.

Such activities seemed to promote connections between these students' intuitive sense of dynamic situations and the mathematical formalities of describing behaviour.

**Conceptual Command and Computational Command**

At the outset of my study I was concerned about the role that student's intuition about dynamic situations played in their command of the formalities of calculus. Throughout my study, the subjects were quite competent at identifying and describing change without the symbolic formalism of calculus. They demonstrated fairly good understanding of change and
rates of change when presented with non-standard graphical or numerical tasks. Their comments suggested that these activities helped with their conceptual mastery of what was being studied in their course. I am led to believe that the fundamental concepts of calculus are quite accessible. It is some of the formalities, such as the role of negative rates, that present obstacles. As such, I feel that we should incorporate more conceptual activities in our most elementary courses, and that the connection between the object of study and the formalities of the study deserves increased emphasis in the classroom.

Despite their apparent improvement in conceptual command, my subjects at times displayed difficulty with the computational aspects of these formalities. Throughout the interviews I often observed good understanding without manipulative skills and vice versa. For example, Cam displayed the most robust connections between rate, slope, and the derivative, yet had the most computational difficulties. On the other hand, Amy was extremely competent with computations, yet seemed to be the one who wrestled most with the connections between rate, slope and the derivative. The clearest example of this situation is the performance on the first and last questions on the paper and pencil test (see Figure 16). Margo and Ralph did the computational problem (#5) most efficiently but struggled with the conceptual problem (#1), whereas the performance on the two
problems was exactly opposite for Amy and Cam.

This leads me to suggest that conceptual mastery and computational mastery are relatively independent. Tradition might suggest that the latter is a prerequisite for the former, but I now feel that such tradition is open to challenge. Others are also challenging such tradition. In summarising how his subjects worked with limits, Williams (1991) states "their procedural knowledge...is largely separate from their conceptual knowledge" (p. 233). Heid (1988) reports on a project in which microcomputers were used to carry out computational algorithms while instruction emphasized concepts. Only during the last three weeks of the semester were the routine computational skills taught. She concludes "this concepts-first curriculum challenged popularly held beliefs that students could not adequately understand concepts without prior, or at least simultaneous, mastery of basic skills" (p. 22). My work suggests that such technology is not necessary to promote conceptual mastery, there are other activities that will do. However, the common thread here is that conceptual command and computational command seem quite independent.

Current calls for revitalization in the instruction of calculus include an increased emphasis on understanding and are sometimes met with cries of "that will make the courses harder". As a result of this study, I believe this not to be the case. Conceptual understanding is not necessarily
harder, but it certainly is different. Anna Sfard (1991) has suggested there are two sides to every mathematical concept—the object side and the process side. I see the first as involving conceptual command and the second as involving computational command. It seems to me that we can and should distinguish between these two aspects within a course ("okay next week we will learn how to compute, for now we will focus on what the results of the computations tell us"). Perhaps, like Heid we could offer courses where machines do the computation but still call them mathematics courses.

I feel that maintaining only an expectation of computational mastery does a disservice to a healthy view of mathematics. For example, Margo had developed the willingness to suspend understanding and just "take the teacher's word for it that these are the rules you follow". This seemed to contribute to her not really getting a handle on the role that the independent variable plays in the derivative until quite late in her calculus experience. I believe that the conceptual side deserves more attention than traditional practice has afforded it, that there are very accessible activities to promote such attention, and that such attention will make the study of calculus more accessible to a wider student population.
Limitations of the Study

The intent of this study was to draw detailed portraits of students' understanding of dynamic situations. As such, it is most unfortunate that I was unable to draw such a portrait of Ralph's understanding. This was because of time constraints generated, in part, by the complexity of dynamic situations and of an individual's conceptions.

I may have been unduly ambitious in trying to consider such a broad view of students' understanding of dynamic situations. If I were to pursue further exploration of this issue, I would maintain a much more narrow focus such as the understanding of negative rates, or the use of language to describe dynamic situations, or a comparison of skills in graphical, numerical and algebraic settings.

The subjects involved in this study all had considerable life experience that may have enriched their intuition about dynamic situations. The youngest of them, Amy, was a pilot. This is hardly a pursuit of the average high school graduate. The others were all older than most freshman calculus students. As such, it is conceivable that a wider population of freshman calculus students would not display such strong intuitive sense of dynamic situations.

I conceived of three perspectives with which formal constructs describing change can be viewed and considered students' understanding of such constructs in terms of these perspectives. There are, no doubt, very different
perspectives that are useful in considering students' understanding of these concepts. I mentioned Sfard's (1991) work on conceiving of mathematical objects operationally or structurally. If a function is seen as an operation that must be done to variables, then it might be challenging for it to be seen as an object representing changing quantities, and hence difficult to consider such notions as rate of change. Similarly, if a derivative is seen as an operation it might be difficult to conceive of it as being something of the same type as what the operation started with (and hence having a graph or an equation as at the start). This might account for students' difficulties with distinguishing the values of a function from the behaviour of the function, or seeing the generality of the formula for slope. Although in the literature there was some hint of the students having an operational emphasis in their conceptions of functions and limits, exploring such an emphasis was not considered in this study.

The value/behaviour distinction proved to be widespread and robust--however, I did not set out to study this phenomenon. Had I anticipated its importance, there probably would be tasks better suited for its examination than those in this study. In retrospect, I feel a more thorough study of this phenomenon might include informal images of slope and of negative numbers, and perhaps more exploration of local versus global issues, such as height versus shape. Despite
such limitations concerning the study of the value/behaviour distinction, it is clear that confronting negative rates provided a conduit for this obstacle.

Suggestions for Further Research

In the section above, I suggested some questions for further consideration that came out the focus of my study. However, there was a great deal of data collected that was not germane to the focus of my study (wide as that focus was).

The most suggestive of this data was what I observed about the subjects' algebraic skills. I have alluded to Cam having little command of algebraic manipulations, yet he displayed good conceptual mastery of issues in calculus. This was part of the evidence that led to my conclusion about the separation of computational and conceptual mastery within calculus. There was also evidence to suggest there is little connection between the computations of algebra and those of calculus.

Amy also displayed some difficulty with algebraic skills and she admitted concern about the situation.

A: I've discovered that is my biggest fault when I'm doing tests. It's my algebra, not in my concepts. That bothers me.

Despite her difficulties with symbolic manipulations from algebra, she mastered the symbolic manipulations of calculus very well. The most obvious evidence for this statement is
her score of 90% on the final exam, but there were also a number of scenarios from the interviews in which this mastery was displayed. So, she seems to provide an example of a student whose algebra computational skills are not robust but whose calculus computational skills are robust. This situation seems to warrant considerably more attention than I was able to grant it.

Another interesting issue that arose was the students' understanding of the universal nature of mathematical definitions and statements. This was most clearly an issue for Ralph when he addressed the first problem on the paper and pencil test (see Figure 16). He felt the statement could be true sometimes and false other times. The subjects in Williams' (1991) study seemed to experience the same difficulty and hence examples, non-examples or counter-examples that were presented in an effort to change their concept image did not produce such change.

I thought about all the definitions we dealt with and I think they are all right--they're all correct in a way and they're all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions. (p. 232)

Ralph's response in our interview, sparked my memory of a comment from a second semester student some time ago who, while considering some true-false questions, asked "If it's correct do we need to do anything?" Students' understanding of the nature of mathematical statements was by no means a topic for this study, yet such evidence surfaced. This is
another topic for further research.

**Personal Benefits**

The intimate view of students' understanding that this study afforded me will be invaluable in my future instructional practice. I must confess to having, in the past, held the expectation that performance and understanding were highly correlated. This expectation has been abandoned, I now see conceptual mastery and computational mastery to be relatively independent. As such, I will devote much more attention to non-standard evaluation instruments in an effort to assess understanding that might otherwise be missed.

I also witnessed the variety of aspects that can play dominant roles in students' understanding of fundamental concepts of calculus. This will lead me to be much more accepting of different conceptions. I will try to keep this in mind especially when working in a tutorial setting with students, where I will make every effort to identify and to work with whatever informal images are particularly strong in the individual.

The flowing water task seemed to be most useful in connecting intuition to symbolic formality. I was so impressed with what it showed about their dynamic sense and their ability to display this sense graphically that I plan to use the task as a central laboratory activity the next time I present a calculus course. I now have three
laboratory activities planned around this task. The first will be essentially what was done in this study--sketching graphical representations for the various containers. The second will be using data from the first lab to deduce Torricelli's law from a least squares fit of a line to a log/log plot of the data from the first lab--this is an optimization problem. The third lab will be to come up with some of the explicit formulae for the height of the water in the containers as I did in Appendix D.

This study also provided the last step in changing my philosophical stance about what we do with mathematics, and this too I find beneficial. I have abandoned the ubiquitous Platonist position that asserts the objectivity of truth and our need to discover such truth. Such a position would assert that mathematics is inherent in situations and our task (as students) is to see it there. This has been replaced with a much more constructivist view that looks at mathematics as not being inherent in a situation, rather as being projected onto a situation. I find this a much more democratic stance for it allows for different projections.
Summary

I have drawn three major conclusions from this study. The first is that these students do have an intuitive sense about dynamic situations. Their informal images of change are rather varied, but can be put to good use in a mathematical study of dynamic situations. There are some activities, such as the flowing water task, that can nurture this intuition.

The second conclusion is that student difficulties with the formalities of calculus appear to be in making connections between their intuition and these formalities. Activities practised in this study seem to promote making such connections. In particular, the formal issue of connecting the value of the derivative with the behaviour of the function was difficult for these students, but was accessible by attention to the significance of negative rates.

The last conclusion that I draw is that conceptual command and computational command seem to be very different issues.

The study has been invaluable to my professional development. It has offered me a very thorough view of students' abilities and difficulties with the concepts and technical aspects of calculus. Such a view will shape my expectations and practices in the classroom.
LIST OF REFERENCES


Cajori, F. (1915). The purpose of Zeno's arguments on motion. Isis, 3, 7-20


This letter is to acknowledge that I am aware of research on calculus students' understanding of change or dynamics being conducted in my department by Dave Lidstone, a student in the Masters Degree Program in Mathematics Education at SFU.

I understand the research methodology to consist of three longitudinal single-subject case studies. Each study will consist of weekly teaching-interviews of sixty to ninety minutes duration to take place in Mr Lidstone's office at VCC, Langara. The interviews will be audio-taped and these tapes will provide the primary source of data to be studied. Mr Lidstone will visit first semester calculus courses in our department to describe his study and will call for student volunteers with a written announcement. The three subjects will be chosen from those students who volunteer. At the outset, the subjects will be informed of their prerogative to leave the study, or any part of the study, at any time.

I see these activities as being well within the educational context of VCC, Langara. The results of this study will prove most beneficial to our department.

M. J. A. Besler
Chairman of the Department of Mathematics and Statistics, VCC Langara
Call for Volunteers to Participate in a Study of Issues in Calculus

I am soliciting volunteers for interviews that I will be conducting to explore students' understanding of issues in calculus. This research is an effort to better understand the challenges that students face in learning calculus. It is part of my graduate work at Simon Fraser University.

Each volunteer will be subjected to about ten interviews throughout the term. Each interview will be about forty minutes long and will address problems related to learning calculus. Throughout the interviews I may act in a tutorial capacity. The interviews will be audio-taped for analysis and the tapes will be destroyed after the study is completed. Your identity will be protected when the data are reported and you will have access to such reports. These interviews will be scheduled according to your convenience and will take place in my office here at Langara (A382c). I cannot offer any financial reward but will offer you free drop-in tutorial service after your part in the study is complete. This activity will not be for marks in your calculus course. It will involve no homework or preparation.

You may choose to leave the study, or any part of it, at any time. Moreover, should you become dissatisfied with or have complaints about the research procedures, you will be able to express your concerns to appropriate parties.

If you are interested in participating in this activity, please fill in the questionnaire on the back of this sheet and return it to my office (A382c). Thank you very much for your attention.

D. Lidstone
Questionnaire for Volunteers Participating in a Study of Issues in Calculus

Name: ______________________________________________________________________

Date of birth: ______________________________________________________________________

Current math course and section number: ______________________________________________________________________

Why are you taking this course? ______________________________________________________________________

____________________________________________________________________

What other courses are you taking? ______________________________________________________________________

____________________________________________________________________

What math course did you take prior to this one? ______________________________________________________________________

____________________________________________________________________

Where did you take this last course? ______________________________________________________________________

____________________________________________________________________

When was this last course taken? ______________________________________________________________________

Do you have a job? If yes, how many hours a week do you work? ______________________________________________________________________

What times would you find most convenient to be interviewed? (Circle your preferences and indicate their order with numbers.)

MON. 830 930 1030 1130 1230 1330 1430 1530 1630 1730

TUE. 830 930 1030 1130 1230 1330 1430 1530 1630 1730

WED. 830 930 1030 1130 1230 1330 1430 1530 1630 1730

THU. 830 930 1030 1130 1230 1330 1430 1530 1630 1730

FRI. 830 930 1030 1130 1230 1330 1430 1530 1630 1730

Please indicate a telephone number or mailing address at which you can be reached for further communication. Thank you very much for your participation.
I, ____________________________, have volunteered to be a subject for Dave Lidstone's study of calculus students' understanding of change or dynamics through teaching-interviews. I understand this research will not be for marks in my calculus course but is being conducted in order to better understand students' challenges in learning calculus. I also understand that my identity will be protected when the data are reported and that I will have access to such report. I am aware that I may choose to leave the study, or any part of the study, at any time. I understand that should I become dissatisfied with or have complaints about the research procedures, I may express my feelings to the Chair of the Department of Mathematics and Statistics at VCC, Langara or to Mr. Lidstone's advisors: Dr. H. Gerber at 291-3377 or Dr. T. O'Shea at 291-4453.

Signature: ____________________________

Date: ____________________________
APPENDIX D
A THEORETICAL ANALYSIS OF
THE HEIGHT FUNCTIONS FROM THE FLOWING WATER TASK.

Volume of Water in the Containers

Torricelli's law asserts that the rate of change of volume with respect to time of water draining from a container under the influence of gravity is proportional to the square root of the water's depth in the container. This implies that the rate is largest initially and decreases as the height decreases—which is in accord with intuition about the nature of water draining. In the case where water drains from a burette, Torricelli's law reduces to the rate of change of depth being proportional to the square root of depth. This is because a burette is a cylinder and the volume of water in a cylinder will be the cross-sectional area of the cylinder (which is constant) times the depth of the water. So if \( h \) represents the depth of the water in the burette, Torricelli's law amounts to the differential equation \( \frac{dh}{dt} = k\sqrt{h} \). After separating the variables and writing this equation using the notation of differentials we have \( h^{-\frac{1}{2}}dh = kdt \). Integrating both sides of this equation gives us that \( 2h^{\frac{1}{2}} = kt + c \). This shows that the height of water in the burette, and hence also the volume of this water, is quadratic in time.

Since we always started with about 55ml of water in the burette and the volume of water in each container is 55 minus what is in the burette, the volume of water in the containers will also be quadratic in time. Let \( V \) be the volume of water in the containers, so as a function of time it will have a formula in the form \( V = at^2 + bt \) where \( a \) and \( b \) are determined by considering two conditions. Two conditions that will be sufficient to determine \( a \) and \( b \) are the time it took for all the water to drain from the burette, which was about 14 seconds, and the total volume of water, which was 55ml. These conditions give us that, as a function of time, the volume of water in any of the containers is given by the formula \( V = 7.86t - .28t^2 \). Since this holds for all the containers, as a function of time the height of the water in each container will depend only on the shape of the container.

The 50ml Beaker

This is a cylinder as pictured at the right. As such, the volume of water at any time is the cross sectional area of the cylinder multiplied by the...
height of the water. Hence the height of the water as a function of time will be given by \( h = (l/A)V = (l/A)(7.86t - .28t^2) \). The diameter of the flask is 4 cm, so the quadratic giving the height of the water in the beaker at any time is the function \( h = .625t - .022t^2 \). Graphs for this function and its derivative are given below.

Graduated Cylinder

As with the beaker, this is a cylinder as pictured at the right. As above, the volume of water at any time is the cross sectional area of the cylinder multiplied by the height of the water. This time the dimensions of the cylinder imply the function for the height of the water has the formula \( h = 1.48t - .053t^2 \). Graphs for this function and its derivative are given below.

Martini Glass

The shape here is a cone and hence the volume \( (V = \frac{1}{3}\pi r^2 h) \) is jointly proportional to the square of the radius of the surface of the water and the height of the water. By considering the side view of the cone, we can use similar triangles to
show that the radius of the surface of the water is proportional to the height of the water. As such, we have the volume of the water being proportional to the cube of the height. Inverting this, and considering the dimensions of the Martini glass would give us the formula for the height of the water as function of time to be $h=(7.76t-.276t^2)^{1/3}$. Graphs for this function and its derivative are given below.

![Graph](image)

**Other Containers**

The shapes of the Erlenmeyer flask and the volumetric flask are such that an algebraic analysis becomes too involved for the purposes of using this exercise in a calculus class. As such, I will omit them here.

Comparing the results of this theoretical analysis with the sketches drawn by the subjects of my study, gives a sense of the accuracy of their intuition about rate of change.
APPENDIX E
SOLUTIONS TO THE INTERVIEW TASK FUNCTIONS

Consider the functions $f(x)=1+3.2x-x^2$ and $g(x)=.2x+3$. Please answer the questions below and vocalize your thoughts as you address each question.

a) What is the value of $f$ at $x=.4$?
We replace $x$ by the number .4 in the expression for $f(x)$ to get the value of $f$ at $x=.4$ to be $f(.4)=1+3.2(.4)-(.4)^2=2.12$

b) When is the value of $g$ larger than 3.9?
We consider the inequality $.2x+3>3.9$. This is equivalent to $.2x>.9$, which implies that $x>4.5$.

c) When is $f(x)$ smaller than 2.92?
We consider the inequality $1+3.2x-x^2<2.92$. This is equivalent to $x^2-3.2x+1.92>0$. The quadratic formula can be used to find the roots of the quadratic to be .8 and 2.4. Hence the inequality becomes $(x-.8)(x-2.4)>0$. Considering the polarity of each of the factors and their product might be the easiest way to see that the solution to the inequality is $(-\infty,.8)\cup(2.4,\infty)$.

d) When is $f(x)$ larger than $g(x)$?
We consider the inequality $1+3.2x-x^2>.2x+3$. This is equivalent to $x^2-3x+2<0$. The quadratic expression factors quite easily, so that the inequality becomes $(x-1)(x-2)<0$. Considering the polarity of each of the factors and their product might be the easiest way to see that the solution to the inequality is the interval $(1,2)$.

e) By how much does $g$ change between $x=2.3$ and $x=2.6$?
We consider $g(2.6)-g(2.3)$, which is equal to .06.

f) Does $f$ grow faster for $x$ between .7 and .9 or for $x$ between .6 and 1.2?
We consider the derived quantities $\frac{f(.9)-f(.7)}{.9-.7}$ and $\frac{f(1.2)-f(.6)}{1.2-.6}$.
The first of these is 1.6 and the second is 1.4, so $f$ grows faster between .7 and .9.

g) Which function grows faster for $x$ between .7 and .9?
Since $g$ is linear, the rate of growth for $g$ is constant and given by its slope, which is .2. Hence from the work in part f) we see that $f$ grows faster for $x$ between .7 and .9.

h) Which function grows faster at $x=1.55$?
As above we see the rate of growth for $g$ is .2. Consider $f'(x)$ which is given by 3.2-2x. Hence $f'(1.55)=-.1$. So $g$ grows faster at $x=1.55$. 

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I: The table here gives average weight in kilograms of a population of girls and a population of boys throughout the first 18 years of their lives. With reference to this table, please answer the questions below and vocalize your thoughts as you address each question.

C: These are the same people at age 1, age 2, age 3, age 4 ...

I: That's right. The populations are the same. The girls and the boys aren't the same [laughing] but the people within this group are constant throughout and the people within this group are constant throughout. Okay?

C: Average weight of the boys at age 9 would be ... do you have a straight ... 25.8.

I: Okay. Here. This can act as a straight edge. That pen's not working?

C: It's died. Hold it. We've got it going. What is the average weight of the girls at age 4 1/2.

I: You might need some scrap paper and we've got lots.

C: Okay, well, it's going from 13.8 to 14.3 so [mumbling] ... 14. Age 4 1/2. Oops. I pressed the wrong numbers. Let's try it again. 13.8, right? 4 1/2. 14.3. 14.05.

I: Mmm-hmm.

C: When do the boys weigh less than 24 kg? Average?

I: Yeah, all of the weights are average.

C: They weigh less than 24 before age 9. So ... do you want specifically, like years and months?

I: Whatever you want to give.

C: Well, less than age 9 they do.

I: Mmm-hmm.

C: When do the girls way more than the boys? Well, they weigh more than the boys at ... 1, 12, 13, 14, 15. What is the increase in the girls' average weight between ages 3 and 8? 3 is 12.2 and 8 is 19.2. 7. Do boys, does the boys' weight increase faster between the ages 2 and 5 or between the ages 9 and 11? Umm ... 9.5 to -- 2 and 5? -- to 15.2 or 9 and 11, 25.8 to 30.4. Here we have 5.7 lb and there we have say 5. It's more between ages 2 to 5. [Mumbling] From ages 9 to 10, which group grows faster. [Mumbling] So they grow ... um ... 2.7, 3.5 and they are, the girls do. When do the boys grow most rapidly? Well, um, here they grow ... let's just take a look quickly. [Pause, mumbling]

I: Can I interrupt?

C: Yes.

I: As you're pointing at these, what is it that you're considering as you ...?

C: The difference between the two numbers.

I: Do you want to record the difference?

C: All of them? Alright. Sure. Why not. [Mumbling while
recording numbers][Long pause] Hold it a second here. I think ... [mumbling] ... okay, it's right here. I wasn't ... just wait ... one more .... It's right here between ages 15 and 16. I could have saved myself a lot of trouble if I'd looked at them first.

I: Can you describe what happened there?
C: I hadn't actually looked at these ages down here and I was thinking that they all increase around 2 to 3 from 1 to 18. I didn't take into account that if I would have looked down here I would have noticed that some of these intervals are a lot larger than these ones up here. Which of course makes sense because teenage boys grow quite a bit between those ages. I could have intuitively thought that one out. But yeah, that's what happened.

I: Is there anything more you want to say about these exercises? Okay let's go on to exercise #2. Here it is. As you can see, it reads, consider the functions f of x = 1 + 3.2x - x^2 and g of x = .2x + 3. This is a period, a full stop, rather than a decimal point. Please answer the questions below and vocalize your thoughts as you address each question, the first one being, what is the value of f at x=.4.

C: Okay. Well, that would be 1 + 3.2 times 4 - 4^2 ...

[mumbling] which would be ...

I: If I could interrupt ... this is a decimal point here.
C: Oh, that's point 4. Oh, I'm sorry. That changes everything.

I: Everything? All you have to do is add a point.
C: Really?

I: All you had to do here was put the point in front of the 4.
C: Oh right ... [mumbling] 2.2. When is the value of g larger than 3.9. Okay ... well, [mumbling] ... that's .2x, right? + 3 ... .2x ... [mumbling] so x = 4.5. So when ... g is larger than 3.9 when x is 4.5, when x is greater than 4.5.

I: The third question. When is f of x smaller than 2.92?
C: Okay ... x - x^2 ... okay ... [mumbling] now, what happens if we take ... okay, take ... I can't divide it by ... well, okay, this is an area where I have trouble.

I: Mmm-hmmm. What kind of equation are you seeing here?
C: Well, that's a trinomial, right?

I: There are three terms so certainly we could say that that's a trinomial. We usually might choose other language to describe the nature of that type of equation. Language that indicates presence of the x^2.

C: A quadratic.

I: It's a quadratic equation, that's right.
C: Right. So ... okay, well I could do this, then. But I'm trying to find when f of x is smaller. Now if I made it equal to 2.92 and then made the whole thing equal to zero, would that give me my answer? I'm not too sure if it
would or not. Ummmm... okay, if I make...

I: When you said make the whole thing equal to zero, what whole thing were you making reference to?

C: Well, if I make this initial equation equal to 2.92 to find out what the value of x would be, to make this equal to 2.92, now if I bring the 2.92 into the equation and make that equation equal to zero, would that... achieve my goal? That's what I'm asking myself. Yeah, actually, it should. It should in fact do that. So let's try that. So... -3.2 plus or minus square root of 3.2 squared minus ...[mumbling] ...3.2 squared... that would be 10.24 minus...

I: So... you're getting for 3.2 squared? Yeah, okay, sorry.

C: [Mumbling continues ...] 17.92... 3.2 over [mumbling] 4.48 [mumbling] ... 2.12. Okay, assuming I did everything right it would equal zero at 1.6 plus 2.12 and 1.6 minus 2.12. So answering the question, when is f of x smaller than 2.92, well, now, I can go and plug it back into the formula and find out because one of them might not be right. So [mumbling] [pause]... okay, well it's not the first one. [Mumbling][long pause] .39. Well that one's not right either.

I: Mmm-hmm. How did you get this 1.92?

C: That would be c which would be this, 2.92-1.

I: I see. Now, you say that would be c, so I guess you're meaning in the quadratic form of this, ax^2 + bx + c = 0. Okay. Notice that this has all the terms on one side of the equation and zero on the other. So in order to apply this pattern here, you have to get zero here. That would be -.192 + 3.2x - x^2 so you're doing a good job here except that would be -c or c would be minus that.

C: Right.

I: How about we go through it and change that arithmetic and let me also coach you a bit on calculators and in particular the use of the memory.

C: Mmm-hmmm.

I: Okay? Actually, no, I'll have to check that... I realize we can do this without using memory but also without making any record of our partial computations. We want 3.2 and we want it squared. Now the calculator will keep track of operations so this is a minus. We do want to subtract 4 times 1.92. So I go minus 4 times 1.92 and the calculator knows to multiply first and then subtract so now I press the equal sign and 3.2^2 minus 4 times -1 times 1.92 is 2.56 so what I now need to do is compute the square root of that which is 1.6. I can now put that in memory. I can think about 3.2... oops, I need to change the sign on that, but no, we've got another sign down here so the minus signs here will cancel out so we can think about 3.2 plus recall the memory, and divide... whoops, I just made an error, I shouldn't have divided yet, but then I divided by 1 and that didn't affect things, so, the 3.2
plus the 1.6 is 4.8. Now I want to divide by 2 to get 2.4. So ... minus 3.2 plus or minus the square root of 2.56 over -2. One of them will be 2.4 and we can get the other one by looking at 3.2 minus what's in memory, equals, divide by 2 to get .8. Okay?

C: Okay.
I: All I've done here is picked out this little slip. Now that we have these numbers, can we get back to the question that we're addressing?
C: Well, I can put those back in. 1 + 3.2 times 2.4 minus 2.4² = .84. Okay, so that wouldn't be the right one according to this equation. And ... the other one is .8 + 3.2 times .8 - .8². 2.92. Okay, so it would be when x is less than .8. That would be the answer. x less than .8.
I: How did you settle on the less than?
C: Well, because 2.4 is greater and the answer was smaller.
I: I see. Okay.
C: Of course, well, that could be wrong. What I might want to do is do a polarity chart which I no longer have room for. I can use the back of this paper.
I: We've got other paper that you can ...
C: Okay. Have you got another pen?
I: Let's try this.
C: Okay. Polarity chart.
I: And don't, don't feel you have to pursue this. If you want to move on to another one, that's fine.
C: Yeah, I'm ready. I will stop there.
I: It's not a matter of ... it's not a test.
C: Okay, the next one. When is f of x larger than g of x. According to x, right?
I: I'm not sure what you mean.
C: With respect to x. Is that what they're talking about? Yes ... I guess so. Okay, well, .2x + 3 [mumbling] ... hold it ... let's do this ... so ... [mumbling][pause] ... is 2.
I: Good.
C: Okay. So now -x and -2 [mumbling] ...
I: Maybe for the tape we can just describe what you've done. You have said .2x + 3 is less than 1 + 3.2x - x². What was the rationale there?
C: When is f of x larger than g of x. So I make f of x larger than g of x, greater than, and I'm solving for x.
I: Okay, good, so then you're doing some algebra here. And it looks good. Good for you.
C: -x² + 3x -2. Now, what adds to, or what multiplies to 2 and adds to 3 [mumbling] ... so at 2 and 1, x would be larger, f of x would be larger than g of x. Okay, how much does g change between 2.3 and 2.6? Well, I can just plug it in there. .2 x 2.3 ... [mumbling] ... I think it's supposed to be .46 unless I did something wrong here. Yeah. It changes roughly .6. Does f grow faster for x between .7 and 9 or for x between .6 and 1.2? You know, I
have a funny feeling there's a lot easier way of doing this than the way I'm doing it.

I: Oh? Why do you say that?

C: Because I think that if I would have graphed these at the beginning, I would have had all the answers in front of me, rather than having to go back to the equation each time. But ... seeing as I'm almost done. Okay, .7 and 9 ... [pause][mumbling] ...

I: Could I interrupt? What's the 2.56 that you recorded?

C: 2.57. That's f at .7.

I: Okay, good. Thanks.

C: [Mumbling] Okay, I think that's ... [mumbling] ... okay, it grows faster between .6 and 1.2.

I: Okay.

C: For x between 7 and 9 here. .52 and for this one [pause][mumbling] ... it grows faster for f of x ... the function grows faster at x = 1.55.

I: Umm...wait a sec. In 7, why did you say that it's f of x. Could you just describe your rationale?

C: The difference between the points is less than, is less for g of x than it is for f of x.

I: Okay. Good.

C: Which function grows faster at 1.55. Now I think this calls for a derivative. I'm looking for a slope here. So the derivative of this would be 3.2 minus 2x and 1.55 ... would be .1 there and .2 ... well, the derivative of that equation is .2. Now I'm not sure what I'd do [pause]. Oh actually, it's a straight line so it's going to grow at the same rate and that's at .2 so this one would grow faster.

I: Now you've concluded that g grows faster. Can you for the tape describe why?

C: Well, g is a straight line and it's growing at .2 always. This is the slope at that point, at 1.55, of f is .1. So, no, hold it. Yeah, .1 would be greater than .2. No it wouldn't. That's right. So the slope of .1 would be greater than .2. Holy cow.

I: Try again?

C: The slope of .2 would be greater than the slope of .1 so g of x would be greater.

I: Okay, good. Great.

C: Great. That's it.

I: Do you have any comments that you want to say about these exercises?

C: Well, um ... yeah, I did. I forgot actually. There was something here.

I: Can you try to focus on which one it was. Was it in #3 here where you were ...

C: Oh yeah, I know. Graphing this equation, this one would be simple, this is just a straight line, but graphing this equation ... would ... you know, when I do a bunch of
problems, something like this will come easy, but when I
don't work on something like that for awhile, now we're
doing related rates and we're not really doing graphing,
what I'm confused about is if this was factored, right ...
Let's make this simple. How would we graph this function?
[tape ends]