THE VALUATION OF COMMODITY-LINKED BONDS

by

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Abstract

Commodity-linked bonds provide a potential vehicle for developing countries to raise money on the international capital markets with lower default risk than standard forms of financing. In this dissertation we apply the techniques of option pricing theory to obtain a model to value a commodity-linked bond. Assuming that the price of the reference commodity, the interest rate, the convenience yield and the value of the firm issuing the bonds follow Wiener diffusion processes, we can apply Ito's lemma and standard arbitrage arguments to obtain a partial differential equation closely related to Schwartz (1982) for pricing the security. Solving this partial differential equation is a non-trivial exercise. However, by imposing different restrictions on the model along the lines of Black and Scholes (1973), Merton (1973), Geske (1979) and Schwartz (1982) special closed form solutions can be obtained. Numerical solutions are obtained using finite difference procedures in cases where closed form solutions are unavailable.

A contribution of the thesis is to estimate a joint diffusion process followed by gold prices and interest rates over the period in the 1980's. This forms the basis for valuing hypothetical gold-linked bonds that could have been issued. The values of these hypothetical bonds were analyzed under four different pay-off scenarios. Consistent with our economic intuition, the estimates of the par coupon rates show that bonds with a call feature attracted the smallest coupon rate. The par coupon rate of the fully indexed bonds was found to be greater than those with a call feature but less than that of conventional bonds. Bonds with a put feature were found to pay the highest par coupon rates.
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DEDICATION

To my late father, Nana Atta-Mensah I, who substituted my childhood toys with books; my mother, Madam Mercy Ama Akuffo Newman, whose wish is to understand what I am studying; and my brothers, Edward, Edmund and Robert for always being able to count on them.
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<tr>
<td>$\alpha_i$</td>
<td>Instantaneous expected rate of change of variable $i$.</td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>Instantaneous standard deviation of variable $i$.</td>
</tr>
<tr>
<td>$dz_i$</td>
<td>Stochastic variable with mean zero and standard deviation of $dt$ (per unit change in time, $dt$).</td>
</tr>
<tr>
<td>$\rho_{ij}$</td>
<td>Instantaneous correlation coefficient between $dz_i$ and $dz_j$ for $i \neq j$.</td>
</tr>
<tr>
<td>$P$</td>
<td>Price of the commodity to which the bonds are indexed.</td>
</tr>
<tr>
<td>$V$</td>
<td>Value of the firm issuing the commodity linked bonds.</td>
</tr>
<tr>
<td>$D$</td>
<td>Rate of dividends paid out to shareholders of the firm issuing the commodity linked bond.</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Convenience yield.</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Time left to maturity.</td>
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<tr>
<td>$r$</td>
<td>Interest rate.</td>
</tr>
<tr>
<td>$G(\cdot)$</td>
<td>Price of a default free pure discount bond of $1.00$ maturity value.</td>
</tr>
<tr>
<td>$Q(\cdot)$</td>
<td>Same definition as $G(\cdot)$.</td>
</tr>
<tr>
<td>$A(\cdot)$</td>
<td>Price of any other security whose value depends only on $r$ and $\tau$.</td>
</tr>
<tr>
<td>$J$</td>
<td>Value of a portfolio containing proportions of $G(\cdot)$ and $A(\cdot)$.</td>
</tr>
<tr>
<td>$I(\cdot)$</td>
<td>The price of a security whose value depends on $\delta$ and $\tau$.</td>
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<tr>
<td>$\lambda_1$</td>
<td>The market price of interest rate risk.</td>
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\( \lambda_2 \)  
The market price of convenience yield risk.

\( B(\cdot) \)  
The price of the commodity linked bond whose value depends on \( V, P, r, \delta, \tau \).

\( H \)  
Value of a portfolio that contains various proportions of the commodity linked bond, the commodity referenced to the commodity linked bond, the value of the firm, default free government discount bond and security \( I(\cdot) \).

\( K \)  
Exercise price of the commodity referenced to the commodity linked bond.

\( F \)  
Face value of the commodity linked bond.

\( L(\cdot) \)  
Value of a European call option on \( P \) with an exercise price of \( F \).

\( \hat{L}(\cdot) \)  
Value of a European call option on \( P \) with known dividends \( \delta P \) and an exercise price of \( F \).

\( U \)  
Futures price of the commodity referenced by the commodity linked bonds.

\( X \)  
Forward price of the commodity referenced by the commodity linked bonds.

\( \bar{L}(\cdot) \)  
Value of a European call option on \( U \) with an exercise price of \( F \).

\( \bar{P}(\cdot) \)  
Value of a European put option on \( P \) with an exercise price of \( F \).

\( E \)  
Exercise price of a commodity linked bond.
\( \text{CB}(\cdot) \) Value of a European call option on the commodity linked bond with exercise price \( E \)

\( \text{PB}(\cdot) \) Value of a European put option on the commodity linked bond with exercise price \( E \).

\( \hat{B}(\cdot) \) Value of a two commodities linked bond.

\( M \) Value of a zero investment portfolio which comprises proportions of two commodities linked bond, the two commodities indexed to the bond and a pure discount bond.

\( \hat{C}(\cdot) \) Value of a European compound call option.

\( \hat{P}(\cdot) \) Value of a European compound put option.
CHAPTER ONE

INTRODUCTION

In an economic environment of high interest rates and uncertainty, borrowers in the financial markets, especially corporations, are on the lookout for sources of capital which have more favorable terms than those that already exist on the money market. Investment banks have responded by designing new financial instruments which satisfy both their corporate clients and also the bearer of such investment instruments. Derivative securities, or contingent claims, are among those which have become popular in the market and have also drawn much academic research. One such derivative security is a bond whose payoff is linked to the current market price of a commodity.

Commodity linked bonds are different from conventional bonds in terms of the payoffs to the holder. The bearer of the conventional bond receives fixed money coupon (interest) payments during the life of the bond and face value (principal) at maturity. However the principal of a commodity linked bond is paid in either the physical units of a reference commodity or its equivalent monetary value. Similarly the coupon payments may or may not be in units of the commodity to which it is indexed. The structural difference, therefore, between the two bonds is that the nominal return of the conventional bond held to maturity is known with certainty, although the real return is unknown due to inflation uncertainty, while both the nominal and real return of the commodity linked bond are not known.

In both the conventional and the commodity linked bonds the payments referred to are promised (or contractual) payments. If the issuer is
unable or unwilling to make the contractual payments, default occurs, and the bearer receives a smaller or zero payment. In the event of default, substantial bankruptcy, legal, and renegotiating costs may be incurred, and new uncertainties introduced (especially in international borrowing). These are deadweight losses (as opposed to simple wealth transfer) to the parties that are involved with the contract. Derivative securities may serve to minimize these deadweight losses in the sense that the state contingent payments may be tailored to the risk preferences of either borrower or lender. This avoids transaction costs of using other markets to accomplish this and also minimizes the probability of default.

Commodity indexed bonds are of two types\(^1\): the forward type and the option type. With the forward type the coupon and/or principal payment to the bearer of the bond are linearly related to the price of a stated amount of the reference commodity. With the option type the coupon payments are similar to that of a conventional bond, but at maturity the bearer receives the face value plus an option to buy or sell a pre-determined quantity of the commodity at a specified price. Alternatively, to minimize the default risk, the borrower may be given the option to pay the minimum of the face value and the value of the reference amount of commodity at the maturity date. The use of commodity indexed bonds, as mentioned in O'Harra (1984), dates as far back as 1863 when the Confederate States of America (CSA) issued bonds payable in bales of cotton. In recent years several commodity indexed bonds have been issued on the financial markets.

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\(^1\)Technically, the forward type is known as the commodity indexed bond; and the option type as the commodity linked bond. However in the thesis, unless otherwise stated, we have used commodity indexed bond and commodity linked bond interchangeably.
The purpose of this thesis is to apply the theory of option pricing to value commodity linked bonds. Assuming that the price of the reference commodity, the interest rate, the convenience yield and the value of the firm issuing the bonds follow diffusion processes, we can apply Ito's lemma and standard arbitrage arguments to obtain a partial differential equation closely related to Schwartz (1982) for pricing the security. Solving this partial differential equation is a non-trivial exercise. However by imposing different restrictions on the model along the lines of Black and Scholes (1973), Merton (1973), Geske (1979) and Schwartz (1982) special closed form solutions can be obtained. Numerical solutions are obtained using finite difference procedures in cases where closed form solutions are unavailable.

The thesis is organized as follows. Chapter One is the introductory chapter. A brief discussion of the experiences with the issue of various indexed securities are presented in Chapter Two. Chapter Three derives the valuation equation for pricing a commodity indexed security. Chapter Four presents some properties of commodity linked bonds. The estimation of the joint diffusion processes for Libor rate, gold price and the convenience yield of gold is provided in Chapter Five. Numerical estimates of equilibrium par coupon rates are also calculated in Chapter Five. Chapter Six of the thesis applies the model to problems in international finance. The research there looks at the benefits to developing countries, in terms of minimizing their default risk on international loans and also in terms of their ability to raise funds on the international money markets through the issue of commodity linked bonds. Summary and conclusions are provided in Chapter Seven.
CHAPTER TWO

A BRIEF BACKGROUND TO THE ISSUE OF COMMODITY-LINKED BOND

In this chapter we present a brief review of previous experiences with commodity-linked bonds, which appear in Fall (1986) and Prio volos and Duncan (1989).

1. GOLD LINKED BONDS

The most popular form of commodity indexed bond is of the one referenced to specified units of gold. A well known example of gold bonds is the one issued in 1973 by the French government and has been accepted in the financial markets as the "Giscard". The "Giscard" carries a 7% nominal coupon rate and a redemption value indexed to the price of 1 Kg. bar of gold. The bearers of the "Giscard" were protected by a safe guard clause which stated that interest and the face value payments would be indexed to a Kg. bar of gold should the French franc lose its parity with gold and other currencies. In 1977, to the disappointment of the French government, the French franc was forced by other European currencies to float. Furthermore, in 1978 the International Monetary Fund (IMF) abolished the linkage of currencies to gold. The consequence of these two economic events made the safe guard clause operative and therefore in 1980 the
government of France paid 393 francs as interest payments for every single bond issued instead of 70 francs as was originally planned for. Moreover each of the issued bonds, which was traded at par in 1977, matured in January 1988 with a redemption value of 8910 francs. Thus there was an increase of about 700 percent in value over 10 years.

Other types of gold linked securities have come into existence since the issue of the "Giscard." Unlike the "Giscard" which had only the redemption value indexed to a specified amount of gold, the others have the principal and/or the interest payments indexed to gold. An example is the gold bonds issued in 1981 by the Refinement International Company. The company issued 3.29% gold linked bonds with an aggregate principal of 100 000 ounces of gold. The maturity date for the bonds is February 1996. Interest payments are made annually. The bearer of these bonds have the option to receive both interest and principal in either the monetary value of the specified amount of gold indexed to the bond or the physical quantity of gold referenced. Claims for the units of gold may be made in London or Zurich.

Another type of gold indexed security is the gold warrants issued by Echo Bay Mines Ltd of Canada in 1981. 1 550 000 preferred voting shares were issued by the company. Holders of these shares were entitled to an annual dividend of US$ 3.00 and 4 gold warrants per share. Each warrant when exercised guaranteed the holder 0.0706 ounces of gold from the Mines at a price of US$ 595 per ounce. These 4 warrants had to be exercised on different dates. 31 January 1986, 31 January 1987, 31 January 1988 and 31 January 1989 were the exercise dates for the first, second, third and fourth warrants respectively. The warrants were allowed to be traded to a
It must be mentioned that the exercise of the warrants was dependent on the completion of the Lupin Gold project.

2. SILVER LINKED BONDS

In 1980 the Sunshine Mining Company, a large silver mine in the United States, issued US$ 25 million silver indexed bonds in order to hedge against the fluctuations in the price of silver. Each US$ 1 000 bond is indexed to 50 ounces of silver, pays a coupon rate of 8.5% and has a maturity of 15 years. At the maturity the bearer of the bond receives the maximum of the face value of US$ 1 000 or the market value of 50 ounces of silver. The bond is only redeemable on or after 15 April 1995 if the average silver price for 30 consecutive days is above US$ 40 per ounce.

A second issue of silver indexed bonds was also issued by the Sunshine Mining company in April 1985. Each of the US$ 1000 bond was now referenced to 58 ounces of silver and the coupon rate was also increased to 9.75%. On the maturity date of April 2004, the holder of the bond had the option of choosing the face value of US$ 1000 or the market value of 58 ounces of silver.

Unlike the gold bonds there are not very many silver linked securities. An economic reason is that the market price of silver has not fluctuated very much. Hence silver producers do not have the incentive to issue silver bonds for the sole purpose of hedging against changes in silver prices.
3. OIL LINKED BONDS

Oil backed bonds appeared in the financial market during the late 1970s. The government of Mexico is believed to be the first to issue such bonds. These bonds, known in the financial markets as Petrobonds, were issued on behalf of the government by the National Financiere S.A. (NAFINSA) which is a development bank owned by the Mexican government. Each 1000 peso bond was linked to 1.95354 barrels of oil. The coupon rate was 12.65823% per annum and matured at the end of three years. On the maturity date the Petrobonds were redeemed at a value equal to the maximum of the face value or the market value of the referenced units of oil plus all coupons received during the life of the bond. With this issue the government was not only raising new money at low nominal cost but was also hedging part of its oil production against fluctuation in oil prices. On the other hand bearers of the Petrobonds were hoping to benefit from an upswing in the price of crude oil.

In 1981 Petro-Lewis Corporation of Denver issued US$ 20 million worth of oil indexed notes. Each note had a life time of 5 years and paid an annual coupon rate of 9%. As explained in Fall (1986), each note is expected on the maturity date to pay the face value (principal), the accrued interest and a contingent interest. The contingent interest which had a feature of a cap was defined as the increase over US$ 668.96 of (a) the average crude oil price of 18.5 barrels of crude oil for the 3 months ending February 28, 1986 or (b) if greater, the highest average price of 18.5% barrels of crude oil, up to a maximum of US$ 1258 or US$ 68 per barrel for any calendar quarter through the quarter ending December 31,
1985. This feature enabled an investor to make at most an additional US$ 589 per bond. The oil notes of Petro-Lewis differed from the Petro bonds in the sense that the repayment of the face value included a call option on oil prices and therefore offered a protection to the bearers from a fall in oil prices. However in the case of Petro bonds the payment of the principal was fully indexed to specified units of oil.

4. OTHER FORMS OF COMMODITY INDEXED SECURITIES

The financial market has seen the issuance of other bonds indexed to the other types of precious metals.

As reported in Privolos and Duncan (1989) Inco, which is one of the world’s largest producer of nickel, copper, silver, cobalt and platinum, in 1984 raised Cdn$ 90 million on the financial market through the issuance of bonds linked to the the price of nickel or copper. The bonds which matured in 1991 pays coupon rate of 10% per annum. Holders of the bonds have the option to receive at the maturity date the face value or the monetary value of a specified amount of nickel or copper. With this issue Inco was able to get out of its financial difficulties in 1984.

Cominco Ltd of Canada also raised US$54 million in 1987 by issuing preferred share and commodity indexed warrants (CIS). Holders of the CIS had the right to exchange each each warrant on or before August 1992 for a number of common shares of the corporation based on the average market price of zinc or copper and on the market value of common stocks on the exercise date.

The largest producer of copper in the U.S., Magna, is also noted to
have issued copper indexed notes in 1988. The notes which mature in 1998 link the interest payments to the price of copper. Thus the interest rates on the notes ranged between 21% per annum at average copper prices of US$2 per pound and above, and 12% per annum at average copper prices of US$0.80 per pound and below. The indexation of the interest payments to copper prices enabled Magna to reorganize its liabilities and therefore control the risk and the net worth of the company.

Commodity indexed bonds are also reported to be in use in the developing countries to finance development projects. The government of Malaysia is noted to have accepted a loan from the Citibank which was indexed to prices of palm oil. Similarly the investment in the copper belt of Papua New Guinea by Metallgesellschaft was done with copper indexed financing.

5. THE ECONOMIC REASON FOR COMMODITY BACKED FINANCING

An economic rational for the issue of commodity indexed bonds is that it allows governments and corporations in need of investment funds to share the appreciation of the market value of underlying commodity with the bond holders in return for a lower coupon rate. Furthermore, as argued in Budd (1983), it offers an opportunity for commodity-producing issuers and international commodity organizations to borrow at below-market interest rates. The less developed countries (LDCs) could, through the issue of such bonds, gain access to the U.S. and Eurobond markets. Myers and Thompson (1989) buttress this point by suggesting that the LDCs could, through the issue of bonds linked to their main exports, hedge against the
fluctuations in their export earnings. The main causes of the debt crises of these countries could be linked to the decline in the earnings from commodity exports, together with the simultaneous increase in the debt service obligation. We therefore concur with Myers and Thompson (1989) that if the debt had been issued in the form of commodity-linked bonds, debt service obligation would have fallen along with commodity prices, thus easing the adjustment burden.

In a world of inflation, and the general uncertainties in the markets, the availability of the commodity, indexed to the bonds, greatly reduces the default risk of the bonds. Hence, issuers of the bonds must maintain a threshold level of inventory similar to what banks hold as the reserve requirements. Moreover, issuers without the commodity must back the bonds with a long position in the forward or futures contracts whose maturity is timed with the redemption date of the bonds.
CHAPTER THREE

A THEORETICAL MODEL TO VALUE COMMODITY-LINKED BOND

In this chapter we shall present a theoretical model for the pricing of the commodity linked bond. It is assumed in our analysis that this bond is of the option type. Besides the final payment, commodity linked bond holders also receive coupon payment at a constant instantaneous coupon rate of c.

1. CONVENIENCE YIELD

Some commodities, such as the precious metals, crude oil, etc., are commercially strategic. They serve to provide to its owners both quantitative and qualitative services. These services, known in the literature as convenience yield, which stem from the ability of the owner to exploit supply shortages, to avoid transportation costs, to control consumption, etc., do not accrue to the owners of commodity futures contracts. A brief discussion on the convenience yield will now be put forward before the presentation of the valuation model for the pricing of the commodity indexed bonds.

Scholarly work on the convenience yield of a commodity was first introduced in the economic literature by Kaldor (1939). Kaldor defines the convenience yield as that return which goods obtain when measured in terms of themselves and therefore excludes any return due to appreciation of its
nominal value. With this definition, Kaldor offered an explanation on the relationship between the spot and futures prices of a commodity. He argues that the spread between a futures and a spot price, in equilibrium, must equate the marginal expenditure on the rent for storage space for the commodity, interest cost, and handling charges minus the marginal convenience yield. Furthermore, since marginal convenience yield is inversely related to the stock or inventory, the marginal convenience yield may exceed the marginal expenditure on physical storage when stocks are relatively small, and therefore result in the futures price lying below the spot price. The reverse will hold when there is a large stock of inventory. Later work on convenience yield may be found in Working (1948), Working (1949), and Brennan (1958). Recent research on this subject appear in Brennan (1986), Fama and French (1987), and Gibson and Schwartz (1990). All these studies concur with the Kaldor hypothesis that generally there is a negative relationship between the level of aggregate inventories and the convenience yield of commodities.

The conclusions reached in the academic papers cited above strongly imply that the valuation equation for derivative securities that are indexed to a commodity must take account of the convenience yield of the commodity linked to it. We therefore assume a stochastic convenience yield in our valuation model. This differs from the usual assumption made by Brennan and Schwartz and others that the convenience yield may be postulated as a function of the spot price of the commodity. Our choice of a stochastic convenience yield follows from the recent work by Gibson and Schwartz (1990) who noted that an assumption of constant convenience yield leads to an inaccurate pricing model. We now present the model which is
closely related to Schwartz (1982).

2. ASSUMPTIONS

The following assumptions are postulated:

(i) Assets are traded in a frictionless or perfect market where there are no taxes, transaction costs, or short sale restrictions, and all assets are perfectly divisible;

(ii) trading of assets is done continuously; and

(iii) the value of the firm that issues the bond, the price of the referenced commodity, the interest rate and the convenience yield follow a continuous time diffusion process.

These stochastic processes are:

\[
\frac{dP}{P} = (\alpha_p - \delta)dt + \sigma_P dz_P
\]

\[
\frac{dV}{V} = \left(\alpha_V - \frac{D}{V}\right)dt + \sigma_V dz_V
\]

\[
dr = \alpha_r(r)dt + \sigma_r dz_r
\]

\[
d\delta = \alpha_\delta(\delta)dt + \sigma_\delta dz_\delta
\]

P is the price of the commodity to which the bonds are indexed, r is the instantaneous default free rate of interest, V is the value of the firm and \(\delta\) is the convenience yield of the commodity. \(D\) is the rate of total
pay outs to all the security holders of the firm (dividends, interests, etc). Hence \((\alpha_v - D/V)\) is the expected rate of appreciation in the value of the firm. In equations (3.1) - (3.4), the terms preceding the \(dt\)'s are the expected rates of change of the state variables, \(P, V, r\) and \(\delta\). The respective instantaneous standard deviations, \(\sigma_p, \sigma_v, \sigma_r\) and \(\sigma_\delta\) are assumed, for now, to be constant\(^2\); \(dz_p, dz_v, dz_r\) and \(dz_\delta\) follow a joint Gauss-Wiener process with correlations:

\[
E[dz_idz_j] = \rho_{ij} dt \quad \text{for } i \neq j \text{ and } i, j = P, V, r, \delta.
\]

The correlation coefficients, \(\rho_{ij}\)'s are assumed to be constant. In Chapter Five the functional forms of \(\alpha_r(r)\) and \(\alpha_\delta(\delta)\) are defined. However, throughout the thesis we assume \(\alpha_p\) and \(\alpha_v\) to be constants.

3. THE GENERAL MODEL

Let \(G(r, \tau)\) be the price of a default free pure discount bond of $1 maturity value with time \(\tau\) left to maturity. \(G(\cdot)\) is assumed to be twice continuously differentiable in \(r\), and continuously differentiable in \(\tau\). Furthermore we assume that all default free bond prices depend on this same single factor, \(r\). Hence by the application of Ito's lemma we have:

\[
dG = G_r dr + \frac{1}{2} G_{rr} (dr)^2 - G_t dt
\]

\(^2\)In the empirical analysis in chapter five, we relax this assumption for \(r\) and define \(\sigma(r) = \sigma_r^{1/2}\).
Let \( A(r, \tau) \) be the price of any other security whose value depends only on the spot rate of interest, \( r \) and the time to maturity \( \tau \). Applying Ito's lemma again we have:

\[
\frac{dA}{A} = \alpha_A \, dt + \sigma_A \, dz_r
\]  

(3.7)

where

\[
\alpha_A \equiv [ \alpha_r A_r + \frac{1}{2} \sigma_r^2 A_{rr} - A_\tau ] / A
\]

and

\[
\sigma_A \equiv \frac{\sigma_r A_r}{A}
\]
Now let us construct an investment portfolio, of arbitrary non-zero value $J$, in which the investor puts proportion $\omega$ of his wealth in $G$ and the remaining $1-\omega$ in $A$. The return on the portfolio will be:

$$\frac{dJ}{J} = \omega \frac{dG}{G} + (1 - \omega) \frac{dA}{A}$$

$$= [\omega \alpha_G + (1 - \omega)\alpha_A]dt + [\omega \sigma_G + (1 - \omega)\sigma_A]dz_r$$  \hspace{1cm} (3.8)

This return could be made riskless if $\omega$ is chosen to satisfy:

$$\omega\sigma_G + (1 - \omega)\sigma_A = 0$$

This will be the case if the investor continuously maintains a short position of $\frac{\sigma_G}{\sigma_A}$ of asset $A$ for each unit of $G$ held. For there to be no arbitrage profits made, this riskless portfolio must return the instantaneous riskless rate of interest. Equating this to the drift term of (3.8) we have:

$$\alpha_G - \frac{\sigma_G}{\sigma_A} \alpha_A = r \left[ 1 - \frac{\sigma_G}{\sigma_A} \right]$$

$$\frac{\alpha_G - r}{\sigma_G} = \frac{\alpha_A - r}{\sigma_A}$$  \hspace{1cm} (3.9)

Condition (3.9) will hold for all assets whose sole underlying state
variable is the spot rate of interest. Hence equation (3.9) may be defined as the risk-premium prevailing in a market where these assets are traded. Let us define:

\[
\frac{\alpha_G - r}{\sigma_G} = \lambda_1(r)
\]  

(3.10)

where \(\lambda_1(r)\) is the market price of interest rate risk.

The risk premium for convenience yield may be obtained by similar steps outlined above. Let us assume there exists a security whose price \(I(\delta, \tau)\) depends solely on stochastic convenience yield. More will be said about such a security later on in Chapter Five.

By applying standard Ito's lemma we obtain:

\[
\frac{dI}{I} = \alpha_I dt + \sigma_I dz_I
\]  

(3.11)

where,

\[
\alpha_I = \frac{\alpha_\delta I_\delta + \frac{1}{2} \sigma^2_\delta I_\delta - I_\tau}{I}
\]

and

\[
\sigma_I = \frac{\sigma_\delta I_\delta}{I}
\]

By the arbitrage argument we followed to determine the market price of interest rate risk, we can find the risk premium for convenience yield as:
where \( \lambda_2(\cdot) \) is the market price of convenience yield risk.

Given our assumptions, the total market value of the commodity linked bond issued by the firm is expressed as \( B(P, V, r, \delta, \tau) \). The drift and the diffusion of this bond is determined by the application of Itô's lemma. The application of the lemma gives:

\[
dB = B_p dp + B_v dv + B_r dr + B_\delta d\delta - B_t dt + \frac{1}{2} B_{pp}(dp)^2
\]

\[
+ \frac{1}{2} B_{vv}(dv)^2 + \frac{1}{2} B_{rr}(dr)^2 + \frac{1}{2} B_{\delta \delta}(d\delta)^2
\]

\[
+ B_{pv}(dp)(dv) + B_{pr}(dp)(dr) + B_{p\delta}(dpd\delta) + B_{vr}(dv)(dr)
\]

\[
+ B_{v\delta}(dv)(d\delta) + B_{r\delta}(dr)(d\delta)
\]

Making the substitution from equations (3.1) - (3.4) gives:

\[
dB = [(\alpha_p - \delta)PB_p + (\alpha_v V - D)B_v + \alpha_r B_r + \alpha_\delta B_\delta + \frac{1}{2} P^2\sigma^2 B_{pp}]
\]

\[
+ \frac{1}{2} \sigma_v^2 B_{vv} + \frac{1}{2} \sigma_r^2 B_{rr} + \frac{1}{2} \sigma_\delta^2 B_{\delta \delta} + \rho_{pv} \sigma_p \sigma_v PB_{pv}
\]

\[
+ \rho_{pr} \sigma_p \sigma_r PB_{pr} + \rho_{p\delta} \sigma_p \sigma_\delta PB_{p\delta} + \rho_{vr} \sigma_v \sigma_r VB_{vr} + \rho_{v\delta} \sigma_v \sigma_\delta VB_{v\delta}
\]
\[
+ \rho_{r}\delta_{r} \delta_{r} B_{r} - B_t \] \ dt + \sigma_{P B} dz + \sigma_{V B} dz_v + \sigma_{r B} dz_r \\
+ \sigma_{B r} dz_r \\
\text{(3.14)}
\]

Equation (3.14) can be expressed in a compact form as:

\[
\frac{d B}{B} = (\alpha - \frac{c}{B}) dt + \eta_p dz_p + \eta_v dz_v + \eta_r dz_r + \eta_\delta dz_\delta \\
\text{(3.15)}
\]

where \(c\) is the instantaneous coupon rate and \(\alpha\):

\[
\alpha = [(\alpha_p - \delta)PB_p + (\alpha_v - V - D)B_v + \alpha_{B r} + \alpha_{\delta r}] + \\
+ \frac{1}{2} \sigma_{P P}^2 + \frac{1}{2} \sigma_{V V}^2 + \frac{1}{2} \sigma_{r r}^2 \\
+ \frac{1}{2} \sigma_{B B}^2 + \sigma_{P V} P V B + \sigma_{P r} P r B + \sigma_{\delta B} \delta B \\
+ \sigma_{V B} V B + \sigma_{V \delta} V \delta B + \sigma_{r \delta} r \delta B - B_t + c] / B \\
\text{(3.16)}
\]

also,

\[
\eta_p = \frac{\sigma_{p B}}{B}, \quad \eta_v = \frac{\sigma_{v B}}{B}, \quad \eta_r = \frac{\sigma_{r B}}{B}, \\
\eta_\delta = \frac{\sigma_{\delta B}}{B}, \quad \sigma_{p v} = \rho_{p v} \sigma_{p v}, \quad \sigma_{p r} = \rho_{p r} \sigma_{p r} \\
\sigma_{p \delta} = \rho_{p \delta} \sigma_{p \delta}, \quad \sigma_{v r} = \rho_{v r} \sigma_{v r}, \quad \sigma_{v \delta} = \rho_{v \delta} \sigma_{v \delta}
\]
Now consider forming a portfolio, whose value is $H$, by investing
amounts of wealth in the following manner:

- $\omega_1$ in the underlying commodity
- $\omega_2$ in the firm
- $\omega_3$ in the default free pure discount bond, $G(\cdot)$,
- $\omega_4$ in asset $I(\delta, \tau)$ or an asset which is maximally correlated
  with the convenience yield and,
- $\omega_5$ in the commodity linked bond.

The instantaneous return on such a portfolio, $dH$, will be:

$$dH = \omega_1 \frac{dP + \delta dt}{P} + \omega_2 \left( \frac{dV + Ddt}{V} \right) + \omega_3 \frac{dG}{G} + \omega_4 \frac{dI}{I} + dB + cdt + \frac{\omega_5}{B}$$

and its value will be $H = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5$. Substituting equations (3.1), (3.2), (3.6), (3.11) and (3.15) into (3.17) we have:

$$-dH = (\omega_1 \alpha_P + \omega_2 \alpha_V + \omega_3 \alpha_G + \omega_4 \alpha_I + \omega_5 \alpha_B) dt$$

$$+ (\omega_1 \sigma_P + \omega_5 \eta_P) dz_P + (\omega_2 \sigma_V + \omega_5 \eta_V) dz_V$$

$$+ (\omega_3 \sigma_G + \omega_5 \eta_R) dz_R + (\omega_4 \sigma_I + \omega_5 \eta_\delta) dz_\delta$$

(3.18)
The risk of the portfolio can be hedged away if \( \omega_1, \omega_2, \omega_3, \omega_4, \) and \( \omega_5 \) are selected to satisfy:

\[
\begin{align*}
\omega_1 \sigma_p + \omega_5 \eta_p &= 0 \\
\omega_2 \sigma_v + \omega_5 \eta_v &= 0 \\
\omega_3 \sigma_G + \omega_5 \eta_r &= 0 \\
\omega_4 \sigma_I + \omega_5 \eta_\delta &= 0
\end{align*}
\]  

Since there are only four equations for the five unknowns, let us arbitrarily let \( \omega_5 \) equal to one. Then the solution of equation (3.19) is given as:

\[
\begin{align*}
\omega_1 &= -\eta_p/\sigma_p, \quad \omega_2 = -\eta_v/\sigma_v, \quad \omega_3 = -\eta_r/\sigma_G, \quad \omega_4 = -\eta_\delta/\sigma_I, \quad \omega_5 = 1
\end{align*}
\]

An investment position of the above amounts would make the investment riskless. Hence for there to be no arbitrage profits, the instantaneous return on the portfolio must be equal to that of a risk-free asset, \( r \).

Thus:

\[
dH = r \left( 1 - \frac{\eta_p}{\sigma_p} - \frac{\eta_v}{\sigma_v} - \frac{\eta_r}{\sigma_G} - \frac{\eta_\delta}{\sigma_\delta} \right) dt \quad (3.20)
\]

Equating (3.20) to the drift term of (3.18) with the substitution of the
\( \omega's \) we have:

\[
\begin{aligned}
&- \frac{\eta_p}{\sigma_p} \alpha_p - \frac{\eta_v}{\sigma_v} \alpha_v - \frac{\eta_r}{\sigma_r} \alpha_r - \frac{\eta_\delta}{\sigma_\delta} \alpha_\delta + \alpha_B = \\
&\quad 1 - \frac{\eta_p}{\sigma_p} - \frac{\eta_v}{\sigma_v} - \frac{\eta_r}{\sigma_r} - \frac{\eta_\delta}{\sigma_\delta}
\end{aligned}
\]

Rearranging, we have:

\[
\begin{aligned}
&\left( \alpha_B - r \right) - \frac{\eta_p}{\sigma_p} \left( \alpha_p - r \right) - \frac{\eta_v}{\sigma_v} \left( \alpha_v - r \right) - \frac{\eta_r}{\sigma_r} \left( \alpha_r - r \right) - \frac{\eta_\delta}{\sigma_\delta} \left( \alpha_\delta - r \right) = 0
\end{aligned}
\]

Substituting expressions (3.10), (3.12) and (3.16) into (3.21), and rearranging, gives the partial differential equation for valuing the commodity linked bond:

\[
\begin{aligned}
&\frac{1}{2} \sigma_p^2 B_{pp} + \frac{1}{2} \sigma_v^2 B_{vv} + \frac{1}{2} \sigma_r^2 B_{rr} + \frac{1}{2} \sigma_\delta^2 \delta \delta + \sigma_{p} P B_{p} + \sigma_{v} V B_{v} + \sigma_{r} V B_{r} + \sigma_{p} P B_{p} + \sigma_{r} r B_{r} + (r - \delta) B_{p} + (rV - D) B_{v} + (\alpha_r - \sigma_r \lambda_1) B_{r}
\end{aligned}
\]
\[ + (a_\delta - \sigma_\delta \lambda_2) B_\delta - B_t - rB + c = 0 \quad (3.22) \]

It must be pointed out that this differential equation, and hence the value of the bond, is independent of the expected returns on the commodity and on the firm. It does depend, however, on the levels of volatilities and covariances of all the state variables, the expected rates of change in interest rates and convenience yield, and the market risk premiums for interest rate and convenience yield risk.

4. THE BOUNDARY CONDITION

Let us assume that on the maturity date the bearers of the commodity linked bond receive the maximum of the face value \( F \) and the monetary value of a prespecified units of the referenced commodity. By specifying the final payment in this manner we have assumed that the indexed bond has a call option feature attached to it.\(^3\)

If we assume that the units of commodity indexed to the bond is \( F/K \), then the final payment of the indexed bond is \( \text{Max}[F, (F/K)\cdot P] \). This is equivalent to holding a straight bond with face value of \( F \) plus an option to buy \( F/K \) units of the reference commodity at an exercise price of \( K \) per unit of the commodity; a price agreed upon by the issuer and the bearer on the date of issue of the indexed bond. The promised payments can be made by

\(^3\) Other forms of the boundary conditions will be specified later on in the text.
the issuer if the value of the firm at the maturity date is greater than that amount. If the legal arrangements of the firm are such that in the case of default bond holders can costlessly take over the firm, then the boundary condition for the solution of equation (3.22) on the maturity date would be:

$$B(P, V, r, \delta, 0) = \min [V, \max \{F, (F/K) \cdot P\}]$$

$$B(P, V, r, \delta, 0) = \min [V, F + (F/K) \cdot \max \{0, P - K\}]$$  \hspace{1cm} (3.23)

The solution of equation (3.22) subject to (3.23) is not a trivial task. Hence an attempted solution could be made by considering special cases where some stronger assumptions are made.

5. SPECIAL CLOSED FORM SOLUTIONS

In this section we find special closed form solutions to equation (3.22) under the boundary condition (3.23). This will be done by considering special cases where we impose certain restrictions on the model. For simplicity and without loss of generality we shall assume throughout our analysis, unless otherwise stated, that the exercise price $K$ is set equal to the face value, $F$, of the bond.
CASE 1: Uncertain Commodity Price

In this section we consider a situation in which we have no default risk, no convenience yield and interest rates and the coupon payment are constant. Hence the only source of risk is the commodity price. Under these conditions the partial differential equation for valuing the commodity linked bond and the boundary condition reduce to:

\[
\frac{1}{2} \sigma^2 P B P + rPB B_t - rB + c = 0 \quad (3.24)
\]

and

\[
B(P, 0) = F + \text{Max}[0, P - F] \quad (3.25)
\]

The solution of equation (3.24) subject to equation (3.25) is given in Schwartz (1982) as:

\[
B(P, t) = \frac{c}{r} (1 - e^{-rt}) + F e^{-rt} + L(P, F, \tau) \quad (3.26)
\]

where \( L(P, F, \tau) \) is the value of a European call option on \( P \) with exercise price \( F \). Using the Black and Scholes formula, \( L(P, F, \tau) \) can be expressed as:

\[
L(P, F, \tau) = P N_1(d_1) - Fe^{-rt} N_1(d_2) \quad (3.27)
\]

where,
\[
    d_1 = \frac{\ln(P/F) + (r + \frac{1}{2} \sigma_p^2) \tau}{\sigma_p \sqrt{\tau}}
\]

\[
    d_2 = d_1 - \sigma \sqrt{\tau}
\]

and \(N_1(\cdot)\) is the cumulative standard normal distribution function.

---

**CASE 2 : Uncertain Commodity Price and deterministic convenience yield**

Here we maintain the assumptions in case 1. However we add another assumption that the convenience yield is deterministic and proportional to the price of the commodity indexed to the bond. Hence the convenience yield is \(\delta P dt\), where \(\delta\) is a proportionality constant. With this assumption the instantaneous return of the commodity is \((\alpha_p - \delta) dt\). Incorporating the proposed assumptions, the partial differential equation for valuing the commodity linked bond reduces to:

\[
    \frac{1}{2} \sigma_p^2 P^2 \frac{\partial^2 B}{\partial P^2} + (r - \delta) P \frac{\partial B}{\partial P} - \frac{\partial B}{\partial t} - rB + c = 0
\]

(3.27)

and

\[
    B(P, 0) = F + \text{Max}[0, P - F]
\]

(3.28)

Treating the convenience yield as a dividend to the bearer of the commodity, we can apply Merton (1973) or Geske (1978) to obtain the a closed form solution:
where \( \hat{L}(P, F, \tau) \) is the value of a European call option on \( P \) with known dividend \( \delta P \) and exercise price \( F \). The value of \( \hat{L}(\cdot) \) is:

\[
\hat{L}(P, F, \tau) = e^{-\delta \tau} P N_1(\hat{d}_1) - F e^{-\tau \tau} N_1(\hat{d}_2)
\]

(3.30)

where,

\[
\hat{d}_1 = \frac{\ln(e^{-\delta \tau} P/F) + (r + 1/2 \sigma_p^2) \tau}{\sigma_p \sqrt{\tau}}
\]

\[
\hat{d}_2 = \hat{d}_1 - \sigma_p \sqrt{\tau}
\]

and \( N_1(\cdot) \) is the cumulative normal distribution function.

CASE 3: Uncertain commodity price with stochastic convenience yield

The assumptions of constant interest rate and no default risk are maintained. However, suppose that both the commodity price and the convenience yield are stochastic. This brings up the problem of finding the right financial instrument to hedge the bond against the convenience yield risk. Moreover, even if we find the appropriate hedge instrument, we
shall have to deal with the problem of estimating the market price of convenience yield. One solution to this problem is suggested by Ingersoll (1982). He recommends using commodity forward or futures contracts rather than the physical commodity to hedge the commodity linked bond. However the delivery date of the forward or futures contract must coincide with that of the maturity date of the commodity linked bond for this to work.

Following the suggestion of Ingersoll we shall use a futures price dynamic in place of that of the spot to construct our hedge portfolio needed to price the bond. Let \( U(t) \) denote the futures prevailing at time \( t \) for one unit of the commodity to be delivered at time \( T \), the maturity date of the bond. Assume that the path followed by the futures price is expressed by the stochastic differential equation:

\[
\frac{dU}{U} = \alpha_u dt + \sigma_u dz_u \quad (3.31)
\]

where \( \sigma_u^2 \) and \( \alpha_u \) are, respectively, the variance and the mean of the rate of change of the futures price. It is assumed that futures and spot prices converge by the delivery date; i.e., \( U(T) = P(T) \).

Using equation (3.31) in place of equation (3.1), and also noting that there is no cost involved in entering into a futures contract, we have the partial differential equation for valuing the bond as:

---

\( \text{Equation (3.32) is obtained by constructing an arbitrage portfolio by borrowing } \frac{B}{u} \text{ at the instantaneous riskless rate. Use the borrowed funds to purchase one unit of the commodity linked bond and simultaneously short } \frac{B}{u} \text{ units of the forward contract. Since no initial wealth is involved} \)
and

\[ B(U, F, 0) = F + \text{Max}[0, U - F] \quad (3.33) \]

Note that the differential equation is similar to equation (3.24) but with one term missing. This term drops out because of the fact that the value of a futures contract is zero.

Following Black (1976), the value of the bond under this condition is:

\[ B(U, t) = \frac{C}{r}(1 - e^{-rt}) + F e^{-rt} + \bar{L}(U, F, \tau) \quad (3.34) \]

\( \bar{L}(U, F, \tau) \) is the value of a European call option on \( U \), which is given by the expression:\(^5\)

\[ \frac{1}{2} \sigma_u^2 U B_{uu} - B_t - rB + c = 0 \quad (3.32) \]

the terminal value of this riskless portfolio must be zero. In other words the instantaneous return of the portfolio is:

\[ \frac{1}{2} \sigma_u^2 U B_{uu} + \alpha U B - B_t + cB/r - (B/B)U \alpha = 0 \]

Rearranging the above expression yields equation (3.32).

\(^5\) The validity of (3.35) requires \( \sigma_u \) to be constant. Technically, this is inconsistent with \( \sigma_p \) being constant and convenience yield being stochastic.

I.e., if there are really two factors pushing around \( P \) and \( U \), then options on one or the other must require a two factor model. Therefore, there is a subtle shift in the specification of the model between cases 2 and 3. We make this point because it makes (3.31) (constant \( \sigma_u \) is assumed) incompatible with (3.1) and (3.4) where \( \sigma_p \) and \( \sigma_\delta \) are also assumed to be constants.
where,
\[
    \bar{d}_1 = \frac{\ln(e^{-r\tau}U/F) + (r + 1/2 \sigma_u^2)\tau}{\sigma_u \sqrt{\tau}}
\]
\[
    \bar{d}_2 = \bar{d}_1 - \sigma_u \sqrt{\tau}
\]

and \(N_1(\cdot)\) is the cumulative normal distribution function. It must be noted that equation (3.35) is similar to the formula for valuing a European call option on a stock that makes continuous dividend payments at a rate equal to the stock price times the interest rate.

CASE 4 : Uncertain commodity price and default risk

The assumptions of this case are that the interest rate is constant, the convenience yield is stochastic, no dividend payments, the commodity price is uncertain and there is a positive probability of the issuing firm defaulting. Furthermore we assume that the commodity linked bond is of the discount type, which means there are no coupon payments to bearers of the bond. As suggested in Case 3, under a stochastic convenience yield regime it is appropriate to use the futures price dynamics rather than the spot price dynamics. The partial differential equation for the pricing of the
bond under these assumptions is:

\[
\frac{1}{2} \sigma_u^2 B_{uu} + \frac{1}{2} \sigma_v^2 B_{vv} + \sigma_{uv} B_{uv} + rB_{v} - B_t - rB = 0 \tag{3.36}
\]

where \( \sigma_{uv} = \rho_{uv} \sigma_u \sigma_v \). The boundary condition at maturity is:

\[
B(U, V, 0) = \text{Min}[V, F + \text{Max}[0, U - F]] \tag{3.37}
\]

In equation (3.36) we have the term \( rUB_u \) missing. Again, this is due to the fact that no cash is required from parties entering into a futures contract. The solution to equation (3.36) subject to equation (3.37) is given in Carr (1987) as:

\[
B(U, V, t) = V \left[ 1 - N_2 \left( h_1 \left( \frac{V}{Fe^{-rt}}, \sigma_v \right), h_1 \left( \frac{V}{Ue^{-rt}}, \sigma_{v/u} \right); \rho_{v, v/u} \right) \right]
\]

\[
+ Ue^{-rt} N_2 \left( h_3 \left( \frac{V}{Fe^{-rt}}, \sigma_v, \sigma_v \right), h_2 \left( \frac{V}{Ue^{-rt}}, \sigma_{v/u} \right); \rho_{v, v/u} \right) \right]
\]

\[
+ \left[ Fe^{-rt} N_2 \left( h_2 \left( \frac{V}{Fe^{-rt}}, \sigma_v \right), - h_2 \left( \frac{Ue^{-rt}}{Fe^{-rt}}, \sigma_u \right); \rho_{uv} \right) \right]
\]

\[
- Ue^{-rt} N_2 \left( h_3 \left( \frac{V}{Fe^{-rt}}, \sigma_{vu}, \sigma_v \right), - h_1 \left( \frac{Ue^{-rt}}{Fe^{-rt}}, \sigma_u \right); \rho_{uv} \right) \right]
\]

(3.38)
where, following, Carr's notation, \( N_2(a, b; \rho) \) is the standard bivariate normal distribution function evaluated at \( a \) and \( b \) with correlation coefficient \( \rho \). Also

\[
\begin{align*}
    h_1(y, \sigma) &= \frac{\ln(y) + I(\sigma)/2}{\sqrt{I(\sigma)}} \\
    h_2(y, \sigma) &= \frac{\ln(y) - I(\sigma)/2}{\sqrt{I(\sigma)}} \\
    h_3(y, \sigma_1, \sigma_2) &= \frac{\ln y - 1/2I(\sigma_1)}{\sqrt{I(\sigma_2)}}
\end{align*}
\]

\[
I(\sigma) = \int_0^\tau \sigma^2 \, ds
\]

\[
\sigma^2_{v/u} = \sigma_v^2 + \sigma_u^2 - 2 \sigma_{vu}
\]

\[
\rho_{v,v/u} = (\sigma_v^2 - \sigma_{vu})/\sigma_v \sigma_{v/u}
\]

and

\[
\hat{\sigma}_{vu} = \sigma_v^2 - 2 \sigma_{vu}
\]
CASE 5: Uncertain commodity price, interest rate risk and no default risk

The assumption of no default risk is once again maintained. However the convenience yield, interest rate and the commodity price are assumed to be stochastic. Furthermore, it is assumed that the commodity linked bond makes no coupon payments. A forward contract that matures on the same date as the commodity linked bond would be used to hedge the bond against the commodity price risk and the convenience yield risk. The reason for this hedge is similar to that given in Case 4. We use the forward instead of the futures contract for the theoretical reason that they are not the same when interest rates are stochastic, and it is easier to obtain a closed form solution with the forward contract as a hedge. Let $X(t)$ be the price payable now for one unit of the commodity to be delivered at time $T$. The price, $X$, of this forward claim is assumed to be lognormally distributed with dynamics:

$$\frac{dx}{X} = \alpha_x dt + \sigma_x dz_x$$

(3.39)

where $\alpha_x$ is the instantaneous expected return of the forward contract, $\sigma^2_x$ is the instantaneous variance and $dz_x$ is a Gauss-Wiener process with correlation between $dz_x$ and another Gauss-Wiener process $dz_t$, being

---

6 This is not to be confused with the conventional forward price $\tilde{F}(t)$ prevailing at $t$, where both payment and delivery take place at time $T$. However the two are related by $X(t) = \tilde{F}(t)Q(t)$, where $Q(t)$ is defined later in the text as the price of a discount bond. Furthermore, a purchase of one unit of $X$ can be equivalently achieved by taking a long forward contract position for one unit of the good at price $\tilde{F}$ and investing $\tilde{F}Q$ dollars in discount bonds maturing at $T$. 

33
The assumption of stochastic interest rate makes it difficult to find closed form solutions due to the market price of interest rate risk. Numerical algorithms provide one way of solving the valuation partial differential equation in the presence of the market price of interest rate risk. Merton (1973) suggests a different approach. The approach, used by Schwartz (1982) and Carr (1987), is to hedge interest rate risk using a default free pure discount bond paying $1.00 at maturity date. This discount bond whose value is denoted by $Q(\tau)$, is assumed to follow the lognormal process:

\[
\frac{dQ}{Q} = \alpha_d t + \sigma_d dz
\]  \hspace{1cm} (3.40)

where $\alpha_d$ and $\sigma_d^2$ are, respectively, the instantaneous expected return and the instantaneous variance of the discount bond price. Under these assumptions the value of the commodity linked bond can be represented as $B(X, Q, \tau)$. Given the distributional assumptions on $X$ and $Q$, the price of the commodity linked bond can be shown by the application of Ito's lemma to follow:

\[
\frac{dB}{B} = \alpha_B dt + \gamma dz_x + \psi dz_q
\]  \hspace{1cm} (3.41)

where $\alpha_B$, $\gamma$ and $\psi$ are defined to be:

\[
\alpha_B = \left[ \frac{1}{2} \sigma_x^2 X^2 + \frac{1}{2} \sigma_q^2 Q^2 - \sigma_{XQ} XQ + \alpha_X B + \alpha_Q B - B \right] / B
\]
In order to value the commodity-linked bond we shall follow Merton (1973) by creating a standard self financing hedge portfolio. This would require investing \( \omega_1 \) in the commodity-linked bond and \( \omega_2 \) in the forward claim matures at the same time as the commodity linked bond. This investment is financed by short selling \( \omega_3 \) (\( =-(\omega_1 + \omega_2) \)) amount of the discount bonds, with their maturity dates the same as the commodity linked bond. The instantaneous return on this hedge portfolio would be:

\[
dY = \frac{\omega_1 dB}{B} + \frac{\omega_2 dX}{X} - (\omega_1 + \omega_2)\frac{dQ}{Q}
\]

Substituting equations (3.39), (3.40) and (3.41) into (3.42) we have:

\[
dY = \omega_1 (\alpha - \alpha_q)dt + \omega_2 (\alpha - \alpha_q)dt + [\omega_1 \gamma + \omega_2 \sigma] dz
\]

\[+ [\omega_1 \psi - (\omega_1 + \omega_2)\sigma_q] dz \quad (3.43)\]

If the portfolio is fully hedged against the risk attributed to the forward price and the price of the discount bond, and the net investment for creating the portfolio is zero, then the aggregate return on the portfolio must also be zero to prevent arbitrage profits. This is embodied in the following three conditions:

\[
\omega_1 \gamma + \omega_2 \sigma_x = 0 \quad (3.44)
\]
A non trivial solution (i.e. $\omega_1$ and $\omega_2$ are different from zero) to equations (3.44) - (3.46) can only exist if:

\[ \frac{\sigma_x}{\gamma} = \frac{-\sigma_q}{\psi - \sigma_q} = \frac{\alpha - \alpha_q}{\alpha_B - \alpha_q} \]  \hspace{1cm} (3.47)

Substituting equations (3.39) - (3.41) into (3.47) we have:

\[ \frac{\sigma_x}{\gamma} = \frac{-\sigma_q}{\psi - \sigma_q} \]

which implies,

\[ \frac{\sigma_x}{\sigma_{xB}} = \frac{-\sigma_q}{\sigma_{QB}/B - \sigma_q} \]

or,

\[ B = xB + QB \]  \hspace{1cm} (3.48)

The next condition requires:

\[ \frac{\sigma_x}{\gamma} = \frac{\alpha - \alpha_q}{\alpha_B - \alpha_q} \]
Substituting for the expressions we get:

\[
\sigma \left( \frac{1/2 \sigma^2 \frac{\sigma^2 B_{xx}}{x^2} + 1/2 \sigma^2 \frac{\sigma^2 B_{qq}}{q^2} + \sigma^2 \frac{\sigma XQB_{xx}}{x^2} + \sigma^2 \frac{\sigma XQB_{qq}}{q^2} + \alpha \frac{\sigma XQB_{xq}}{x^2} + \alpha \frac{\sigma XQB_{qq}}{q^2} - B_t}{B - \alpha} \right)
\]

\[
\frac{\sigma^2 \frac{\sigma^2 XB_{xx}}{x^2}}{B} = \frac{(\alpha - \alpha_q)}{B}.
\]

Rearranging gives,

\[
\frac{1/2 \sigma^2 \frac{\sigma^2 B_{xx}}{x^2} + 1/2 \sigma^2 \frac{\sigma^2 B_{qq}}{q^2} + \sigma^2 \frac{\sigma XQB_{xx}}{x^2} + \sigma^2 \frac{\sigma XQB_{qq}}{q^2} + \alpha \frac{\sigma XQB_{xq}}{x^2} + \alpha \frac{\sigma XQB_{qq}}{q^2} - B_t - \alpha B}{B} = \frac{\alpha \frac{\sigma XQB_{xq}}{x^2} - \alpha \frac{\sigma XQB_{qq}}{q^2}}{B}
\]

Simplifying by combining equations (3.48) and (3.49) gives the following partial differential equation for valuing the bond:

\[
\frac{1/2 \sigma^2 \frac{\sigma^2 B_{xx}}{x^2} + 1/2 \sigma^2 \frac{\sigma^2 B_{qq}}{q^2} + \sigma^2 \frac{\sigma XQB_{xx}}{x^2} + \sigma^2 \frac{\sigma XQB_{qq}}{q^2} - B_t}{B} = 0
\]

The boundary condition that must be satisfied at maturity is:

\[
B(Q, X, 0) = F + \text{Max}[0, X - F]
\]

Appealing to Carr (1987), the solution to equation (3.50) subject to
equation (3.51) is:

\[ B(Q, X, t) = FQ + XQN_1 \left( h_1 \left( \frac{X}{F}, \sigma_{x/q} \right) \right) - FQN_1 \left( h_2 \left( \frac{X}{F}, \sigma_{x/q} \right) \right) \tag{3.52} \]

where \( N_1(\cdot) \) is the univariate normal distribution function, \( h_1, h_2 \) and \( \sigma_{x/q} \) are as defined in case 4.

**CASE 6: Uncertain commodity price, interest rate risk and default risk**

Here we add to the assumptions already made in case 5 the assumption that the firm could default on the bond payments. The firm is assumed to make no dividend payments to shareholders.

Under these assumptions, and following the standard arbitrage arguments we went through in case 5, it can be shown that the partial differential equation for valuing the commodity linked bond is:

\[
1/2 \sigma^2 Q^2_B \frac{\partial^2}{\partial x^2} + 1/2 \sigma^2 Q^2_B \frac{\partial^2}{\partial q^2} + 1/2 \sigma^2 V^2 \frac{\partial^2}{\partial v^2} + \sigma QxB \frac{\partial}{\partial q} + \sigma QV \frac{\partial}{\partial x} + \sigma \frac{\partial}{\partial v} + \sigma \frac{\partial QV}{\partial v} - B_t = 0 \tag{3.53}
\]

with the boundary condition being:
The solution to equation (3.53) subject to equation (3.54), which is similar to that in Carr (1987), is given as:

\[ B(X, V, Q, t) = V \left[ 1 - N_2 \left( h_1 \left( \frac{V}{FQ}, \sigma_{v/q}, \sigma_{v/x} ; \rho_{v/q,v/x} \right) \right) + XQN_2 \left( h_3 \left( \frac{V}{FQ}, \sigma_{v/q}, \sigma_{v/q} ; \rho_{v/q,v/q} \right), h_2 \left( \frac{V}{XQ}, \sigma_{v/x} ; \rho_{v/q,v/x} \right) \right) \right. 

\left. + \left[ FQN_2 \left( h_2 \left( \frac{V}{FQ}, \sigma_{v/q} ; \rho_{v/q,v/q} \right), - h_2 \left( \frac{X}{F}, \sigma_{x/q} ; \rho_{v/q,q/x} \right) \right) - XQN_2 \left( h_3 \left( \frac{V}{FQ}, \sigma_{v/q}, \sigma_{v/q} ; \rho_{v/q,v/q} \right), - h_1 \left( \frac{X}{F}, \sigma_{x/q} ; \rho_{v/q,q/x} \right) \right) \right] \]  

(3.55)

where,

\[ \rho_{v/q,v/x} = \frac{\sigma_{v}^2 - \sigma_{v} \cdot \sigma_{v} - \sigma_{q} \cdot \sigma_{v} + \sigma_{q} \cdot \sigma_{q}}{\sigma_{v/q} \cdot \sigma_{q/x}} \]

\[ \rho_{v/q,v/x} = \frac{\sigma_{v} \cdot \sigma_{v} - \sigma_{v}^2 - \sigma_{q}^2 + \sigma_{q} \cdot \sigma_{q}}{\sigma_{v/q} \cdot \sigma_{q/x}} \]
Following Carr (1987), we can provide some intuition behind equation (3.55) by dividing the expression into three parts. The first term represents the value of the firm which goes to bondholders unless the firm is not bankrupt and the forward price of the commodity indexed to the bond is less than the value of the firm at the maturity date. The second term is the value of the commodity linked bond if the firm is solvent and the bond holders always get X (the forward price) at the maturity date. The last two terms adjust for the fact that the bond holders receive F when at the maturity date of the bond X is below F.
CHAPTER FOUR

PROPERTIES AND EXTENSIONS TO THE PRICING OF THE COMMODITY-LINKED BOND

1. PROPERTIES OF THE COMMODITY - LINKED BONDS

In this section we shall present comparative statics results for the option type of commodity-linked bond. We shall also show that the knowledge of the value of the commodity linked bond can be used to price other options. For simplicity our analysis will be carried out assuming a constant interest rate, no convenience yield, no default risk and that the commodity bond pays no coupons. Like the section on the special closed form solutions in Chapter Three, we shall assume that the exercise price, K, is set equal to the face value F. Under these assumptions the only source of risk is the price changes of the commodity linked to the bond.

In Chapter Three, Section Five, Case 1, it was shown that the value of the commodity linked bond under these conditions is:

\[ B(P, \tau) = F e^{-r\tau} + PN_1(d_1) - Fe^{-r\tau}N_1(d_2) \]

\[ = F e^{-r\tau}(1 - N_1(d_2) + PN_1(d_1)) \quad (4.1) \]

where \( N_1(\cdot) \), \( d_1 \) and \( d_2 \) are as defined in Case 1 of Section Five.

As in Chapter Three B(P, F, \( \tau \)), P, F and \( \tau \) stand respectively for the value of the commodity linked bond, the price of the commodity to which the
bond is indexed, the face value of the bond and the time left to maturity of the bond.

**PROPOSITION 1:**

The value of a European call option, $C(P, F, \tau)$, to purchase one unit of the commodity at an exercise price $F$, with time to expiry equal to that left for to maturity of the commodity linked bond, is:

$$C(P, F, \tau) = B(P, F, t) - F e^{-r\tau}$$

**PROOF:**

Construct two portfolios, A and B. In portfolio A place a call option to purchase one unit of the commodity at an exercise price of $F$. In portfolio B purchase a commodity linked bond with face value $F$ and simultaneously short a Treasury bill which matures at the same time as the call with face value of $F$. At maturity if the $\text{Max}[P, F] = F$ then portfolio A is worthless. The value of portfolio B is $F - F = 0$. However, if the $\text{Max}[P, F] = P$ the value of portfolio A is $P - F$. Portfolio B would be valued at $P - F$. This completes the proof.

**PROPOSITION 2:**

The value of a commodity linked bond which makes no coupon payments can be replicated with long positions in the commodity to which the bond is linked and a European put option on the commodity with exercise price of $F$ expiry date equal to the maturity date of the bond. Notationally this statement implies that:
\[ B(P, F, \tau) = P + \tilde{P}(P, F, \tau) \]  \hspace{1cm} (4.3)

where \( \tilde{P}(\cdot) \) is the value of a European put option on the commodity indexed to the bond with an exercise price of \( F \).

**PROOF:**

Let portfolio A contain the commodity linked bond which pays no coupons. For portfolio B purchase one unit of the commodity and an European put option to sell the commodity at the price \( F \) and the time to mature equals the time left for the maturity of the commodity linked bond. If on the maturity date the \( \text{Max}[P, F] = F \), then the value of portfolio A is \( F \). The value of portfolio B in this case is \( P + F - P = F \). On the other hand if \( \text{Max}[P, F] = P \) then the value of portfolio A is \( P \). Portfolio B is worth in this state \( P + 0 = P \). Hence the result.

**REMARK:**

Propositions 1 and 2 together establish the classic put-call parity theorem known in the options literature. Using equations (4.1) to (4.3) the value of a European put option on the commodity indexed to the commodity linked bond with an exercise price of \( F \) is:

\[
\tilde{P}(P, F, \tau) = B(P, F, \tau) - P = F e^{-r\tau} \left( 1 - N(d_2) + P(N(d_1) - 1) \right) \hspace{1cm} (4.4)
\]
where the notation has already been defined.

**PROPOSITION 3:**

Let \( CB(B, E, \tau) \) be the value of a European call option on the commodity-linked bond, \( B(P, F, \tau) \), with an exercise price \( E \). The pay off of this option at the maturity date is given as:

\[
CB(B, K, 0) = \max(0, B(\cdot) - E)
\]

\[
= \max(0, \max(P, F) - E)
\]  \hspace{1cm} (4.5)

(It can easily be shown that the partial differential equation governing the option process is similar to that of the commodity-linked bond. Numerical solution of the option value subject to (4.5) can be obtained.) Under the assumptions postulated at the beginning of this chapter, and for \( E \) greater than the face value \( F \), of the commodity linked bond, the value of this option on the bond is given as:

\[
CB(B, K, \tau) = B(P, E, \tau) - Ee^{-\gamma\tau}
\]  \hspace{1cm} if \( E > F \)  \hspace{1cm} (4.6)

where \( B(P, E, \tau) \) is the value of a commodity linked bond whose maturity payment is \( \max(P, E) \). The value of the option for \( E \) equal to \( F \) is:

\[
CB(B, E, \tau) = B(P, F, \tau) - Fe^{-\gamma\tau}
\]  \hspace{1cm} if \( E = F \)  \hspace{1cm} (4.7)

Lastly if \( E \) is set less than \( F \) then the price of the option is:
PROOF:

Let us consider the case in which the underwriters of the European call option on the commodity linked bond set the exercise price $E$ greater than $F$. In this case construct two portfolios $A$ and $B$. In portfolio $A$ purchase one call option on the commodity linked bond with exercise price $E$. In portfolio $B$ purchase one unit of commodity-linked bond which pays $\max(P, E)$ on the maturity date and simultaneously short Treasury bills of value $Ee^{-rT}$. If at the maturity date $\max(P, F) = F$, then the value of portfolio $A$ is worthless. The value of portfolio $B$ is $E - E = 0$. On the other hand if $\max(P, F) = P \leq E$, then the value of portfolio $A$ is again worthless. Portfolio $B$ is also worth $E - E = 0$. If on the maturity date we have $\max(P, F) = P > E$, then portfolio $A$ is worth $P - E$. The value of portfolio $B$ is worth $P - E$ under this circumstance. Hence the proof of equation (4.6).

The proof of equation (4.7) is similar to that of equation (4.6). In portfolio $A$ place one unit of the call on the commodity linked bond. However in portfolio $B$ hold one unit of commodity linked bond with face value $F$ and simultaneously short $Fe^{-rT}$ of $T$-bills. The value of portfolios $A$ and $B$ are the same in all possible states of the world.

Next consider $E < F$. Again construct portfolios $A$ and $B$. In portfolio $A$ purchase one unit of the European call option on the commodity linked bond. For portfolio $B$ hold one unit of the commodity linked bond with face value $F$ and simultaneously short Treasury bills of value $Ee^{-rT}$.
If on the maturity date $\text{Max}(P, F) = F$, then the value of portfolio A is $F - E$. The worth of portfolio B is also $F - E$. On the other hand if $\text{Max}(P, F) = P$, portfolio A would be worth $P - E$. Portfolio B would also be valued at $P - E$.

This completes the proofs. Hence under various values of $E$ the value of the European call option on the commodity linked bond must be priced by equations (4.6) - (4.8).

**PROPOSITION 4:**

Let $PB(B, E, \tau)$ be the value of a European put option on the commodity linked bond with an exercise price of $E$. The payoff of this option at the maturity date is given as:

$$PB(B, E, 0) = \text{Max}[0, E - B(\cdot)]$$

$$= \text{Max}[0, E - \text{Max}(P, F)] \quad (4.9)$$

Under the assumptions postulated earlier in the chapter and if the exercise price, $E$, is greater than the face value of the commodity linked bond then the present value of the European put option is given as:

$$PB(B, E, 0) = B(P, E, \tau) - B(P, F, \tau) \quad \text{if } E > F \quad (4.10)$$

However the European put option, $PB(\cdot)$, is worthless if $E$ is set less than or equal to $F$. 

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PROOF:

Let us consider when $E > F$. Form portfolios A and B. Hold one unit of the European put option on the commodity linked bond, with exercise price $E$, in portfolio A. For portfolio B purchase one unit of commodity linked bond which pays $\max(P, E)$ on the maturity date. Also for portfolio B short one unit of commodity linked bond whose final payment is $\max(P, F)$. On the maturity date if $\max(P, F) = F$ then the value of portfolio A is $E - F$. Portfolio B would be worth $E - F$. On the other hand if $\max(P, F) = P < E$, then portfolio A is $E - P$. The worth of portfolio B is also $E - P$. Lastly if $\max(P, F) = P > E$, then portfolio A is worthless. Portfolio B would have a value of $P - P = 0$. Hence equation (4.10).

It is obvious that the European put option is worthless when $E \leq F$.

PROPOSITION 5:

The value of the commodity linked bond increases monotonically as the price of the commodity indexed to the bond increases.

PROOF:

By appealing to equation (4.1) and the Black-Scholes formula, differentiate $B(P, F, \tau)$ with respect to $P$. Thus:

$$\frac{\partial B(\cdot)}{\partial P} = -\frac{F e^{-\tau \tau}}{P \sigma \sqrt{\tau}} \left\{ \frac{N'(d_2) + N_1(d_1) + P(N'(d_1))}{P \sigma \sqrt{\tau}} \right\}$$
\[ N'(\cdot) \text{ is the derivative of the cumulative normal distribution which is defined as:} \]

\[ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} \quad (4.12) \]

Using equation (4.12) in (4.11) we have:

\[ \frac{\partial B(\cdot)}{\partial P} = N_1(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} -(1/2) \frac{d_1^2}{\sigma} e^{-\frac{-r \tau}{P\sigma \sqrt{2\pi \tau}}} - (1/2) d_2^2 \]

\[ = N_1(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} \left[ -(1/2) d_1^2 e^{-\ln(P/F) - r \tau - (1/2)d_2^2} \right] \]

\[ = N_1(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} \left[ -(1/2) d_1^2 e^{-d_1^2 e^{\ln(P/F) - r \tau - (1/2)d_2^2}} \right] \]

\[ = N_1(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} \left[ -(1/2) d_1^2 - e^{\ln(P/F) - r \tau - (1/2)d_2^2} \right] \]

\[ = N_1(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} \left[ -(1/2) d_1^2 - e^{\ln(P/F) - r \tau - (1/2)d_2^2} \right] \]

\[ = N_1(d_1) \geq 0 \quad (4.13) \]
Hence the proof of proposition 5.

**PROPOSITION 6:**

The value of the commodity linked bond increases monotonically as the face value, \( F \), of the bond increases.

**PROOF:**

Using a similar algebraic manipulation employed in the proof of proposition 5 it can be shown that:

\[
\frac{\partial B(\cdot)}{\partial F} = e^{-r\tau}(1 - N_1(d_2)) 
\]

(4.14)

However, since \( 0 \leq N_1(\cdot) \leq 1 \) then:

\[
\frac{\partial B(\cdot)}{\partial F} \geq 0 
\]

QED

**PROPOSITION 7:**

The change in the value of the commodity linked bond is indeterminate when the time to maturity increases.

**PROOF:**

By differentiating \( B(\cdot) \) with respect to \( \tau \) and simplifying we get:

\[
\frac{\partial B(\cdot)}{\partial \tau} = -rF e^{-r\tau}(1 - N_1(d_2)) + \frac{\sigma P}{2\sqrt{\tau}} N'(d_1) 
\]

(4.15)
It is obvious that the sign of (4.15) is indeterminate.

**PROPOSITION 8:**

The value of the commodity linked bond increases as the price of the commodity indexed to the bond becomes more volatile.

**PROOF:**

Differentiating \( B(\cdot) \) with respect to \( \sigma_p \) and upon simplification yields:

\[
\frac{\partial B(\cdot)}{\partial \sigma_p} = \sqrt{\tau} p N_1(d_1) > 0
\] 

(4.16)

QED.

**PROPOSITION 9:**

If the interest rate is a constant then the value of the commodity linked bond decreases monotonically as the interest rate rises.

**PROOF:**

Taking the differential of \( B(\cdot) \) with respect to \( r \) and simplifying would give the following result:

\[
\frac{\partial B(\cdot)}{\partial r} = -\tau e^{-rT} (1 - N_1(d_2)) < 0
\]

(4.17)

Equation (4.17) is negative since \( 0 \leq N_1(d_2) \leq 1 \). QED.
Let us consider the intuition behind Propositions 5 to 9. The commodity linked bond is equivalent to a portfolio consisting of a discount bond with face value $F$ and a European call option on the commodity indexed to the bond with an exercise price of $F$. The maturity dates of the discount bond, the call option and the commodity linked bond are assumed to coincide. We examine this portfolio rather than the commodity linked bond.

The explanation for Proposition 5 is that as the commodity price increases the probability of the call contained in the portfolio ending up in the money increases. The portfolio, therefore, appreciates in value. Consequently, there is an increase in the value of the commodity linked bond.

Proposition 6 is explained by the fact that the increase in the face value of the commodity linked bond leads to an increase in the value of the discount bond and a decrease in the value of the call contained in the portfolio. The call falls in value since the increase in $F$ increases the probability that the call finishes out of the money. However the net value of the portfolio is increased since the value of the discount bond dominates the call.

The reason for the indeterminateness of the direction of the change of the value of the commodity linked bond in Proposition 7 may be due to the fact that an increase in the time to maturity decreases the value of the discount bond and increases the value of the call due to the fall in the present value of the exercise price, $F$. The magnitude of these changes cannot be ascertained. Hence the ambiguity in the change in the value of the commodity linked bond.
In Proposition 8 the change in the volatility of the price of the commodity indexed to the bond would affect only the call in the portfolio. The increase in the volatility of the commodity price \( \sigma_p \) increases, increases the value of the call. This is because a call option has no downside risk (that is, the value of the call is zero irrespective of how far it finishes out of the money). An increase in \( \sigma_p \), therefore, goes to increase the chances that the call option will expire in the money. Hence Proposition 8.

Lastly our intuition into Proposition 9 is that the increase in the interest rate reduces the present value of the discount bond and that of the exercise price of the call. However the resulting increase in the value of the call is offset by the fall in the value of the discount bond. Hence the net fall in the worth of the portfolio and consequently the commodity linked bond.

**PROPOSITION 10:**

The value of a commodity linked bond, with final pay-off equal to \( \max(P, F) \), would dominate a bond with face value \( F \) and maturity time the same as that of a commodity linked bond.

**PROOF:**

Set up portfolios A and B. In portfolio A purchase one commodity linked bond. In B purchase the bond with face value \( F \). At maturity date if \( \max(P, F) = F \) then the values of A and B are the same. On the other hand if \( \max(P, F) = P \) then portfolio A is worth \( P \) and B is valued at \( F \). Hence the proposition.
PROPOSITION 11:

The value of the commodity linked bond would dominate a portfolio of one unit of the commodity indexed to the bond.

PROOF:

Construct a similar argument proposed for the proof of proposition 10.

PROPOSITION 12:

The value of the commodity linked bond is convex in \( F \).

PROOF:

Take the second partial derivative of \( B(\cdot) \), as expressed by equation (4.1), with respect to \( F \) and simplify. The result is:

\[
\frac{\partial^2 B(\cdot)}{\partial F^2} = \frac{e^{-r \tau}}{F \sigma \sqrt{\tau}} N'(d_2) \geq 0
\]  (4.18)

QED.

PROPOSITION 13:

The value of the commodity linked bond is convex in the price of the commodity indexed to the bond.

PROOF:

Take the second partial derivative of \( B(\cdot) \) with respect to \( P \). This
PROPOSITION 14:

The value of a perpetual \((t = \infty)\) commodity linked bond is equal to the value of one unit of the commodity indexed to the bond.

**PROOF:**

Set \(\tau\) in equation (4.1) to \(\infty\). The result is:

\[
B(P, F, \infty) = PN_1(\infty)
\]

\[
= P
\]  \hspace{1cm} (4.20)

Note that \(N_1(\infty) = 1\). QED.
CASE 1: Issuer of the bond given the option to determine final payment

Until this section we have carried out our analysis on the assumption that the issuer of the commodity linked bond pays the bearer, on the maturity date, the maximum of the face value, F and the monetary value of prespecified units of a commodity indexed to the bond. In this case the option lies with the bearer to determine the value of the final payment. Such an arrangement could increase the chances of the issuer defaulting on the final payment. This tendency could occur when the commodity linked bonds are issued by developing countries to finance their domestic projects and these countries are not able to meet their obligations due to serious balance of payments problems caused by low prices for their exports.

At a higher coupon rate the countries in question could minimize the probability of defaulting on the final payment by paying the bearers of the commodity linked bond the minimum of the face value, F, and the monetary value of a pre-specified units of a commodity to which the bond is indexed to. The reduction in the probability of default occurs due to the fact that contractual debt payments are reduced in precisely those circumstances when balance of payments problems occur - namely, low export prices. Furthermore, under this arrangement the maximum the issuer would pay on the maturity date is F. However, under the other arrangement where the bearer has the option to choose the final payment, the minimum the issuer would pay on the maturity date to the bearer is F and an unbounded maximum.

The partial differential equation for finding the value of the
commodity linked bond when the issuer determines the final payment would be as that of equation 3.22. However, the boundary conditions would be:

\[ B(P, V, r, \delta, 0) = \min(V, \min((F/K)P, F)) \quad (4.24) \]

If we abstract from the problem of default and also set \( F = K \) then:

\[ B(P, V, r, \delta, 0) = \min(P, F) \]

\[ = F - \max(0, F - P) \quad (4.25) \]

Equation (4.25) suggests that a commodity linked bond which pays the minimum of the face value, \( F \), and the monetary value of prespecified units of a commodity, can be replicated by a portfolio of a discount bond with face value \( F \) and a short position in a commodity put option with exercise price \( F \). Postulating certain assumptions we can obtain closed form solutions, similar to those obtained in Section 5 of Chapter 3, for the commodity linked bond with boundary conditions given by equations (4.24) and (4.25).

CASE 2 A model for a bond indexed to two commodities

Our analysis carried up to this stage has centered on the valuation and properties of a bond indexed to one commodity. We discussed in Chapter 2 that a reason for a firm or a country issuing these type of bonds is to
attract lenders. However, a country or a firm can increase the attraction of lenders by issuing a bond which is linked to two commodities. The issuing firms do not have to produce the commodities since they can take long positions on the commodities futures market. Furthermore, the issuer of this two commodity indexed bond could exchange the appreciation in the commodities prices for a lower interest payment on the bond if the bearer of the bond is given the option on the final payment. On the other hand if the issuer is given the option of the final payments then the interest cost to the issuer would be higher than on conventional debt.

The purpose of this section is to apply arbitrage argument to construct a valuation model for the pricing of such types of bonds. In so doing we shall study the two types of options (i.e., when the option on the final payment lies with bearer and when the issuer is given the option).

Before we find the value of the bond under the two types of options we would first find the partial differential equation governing the pricing formula.

The assumptions we make, in addition to the usual frictionless market, are that the commodities indexed to the bond have zero convenience yields, the interest rates are stochastic and the prices of the commodities, \( P_1 \) and \( P_2 \) are lognormally distributed. Furthermore, since our pricing method would be based on the technique used by Merton (1973) we also assume that the stochastic interest rate can be hedged using a default free pure discount bond which pays $1.00 at the maturity date. This discount bond whose value would be denoted by \( Q(\tau) \) is also assumed to be lognormally distributed. Lastly we assume that the firm or the country issuing the bond is very solvent and therefore there is no question of default risk. Under these
assumptions we postulate that the discount bond price and the prices of the commodities indexed to the bond follow a continuous time diffusion process of the following form:

\[
\begin{align*}
\frac{dp_1}{p_1} &= \alpha_{p_1} dt + \sigma_{p_1} dz \\
\frac{dp_2}{p_2} &= \alpha_{p_2} dt + \sigma_{p_2} dz \\
\frac{dQ}{Q} &= \alpha_{Q} dt + \sigma_{Q} dz
\end{align*}
\]  

(4.26) (4.27) (4.28)

where the \( \alpha \)'s and the \( \sigma \)'s are respectively drift and diffusion for the dynamics. Under these assumptions the value of the two commodities indexed bond would be represented as \( \hat{B}(P_1, P_2, Q, \tau) \). Applying Ito's lemma the rate of change of the value of the bond is given by:

\[
\frac{d\hat{B}}{\hat{B}} = (\alpha_{\hat{B}} - \frac{c}{\hat{B}}) dt + \psi_{\hat{B}} dz + \psi_{P_{1}} dz + \psi_{P_{2}} dz + \psi_{Q_{1}} dz + \psi_{Q_{2}} dz
\]

(4.29)

where \( c \) is the instantaneous coupon rate and \( \alpha_{\hat{B}} \) is expressed as:

\[
\alpha_{\hat{B}} = \left[ \alpha_{P_{1} \hat{B}} + \alpha_{P_{2} \hat{B}} + \alpha_{Q \hat{B}} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{i j}^{2} \right]_{P_{1} P_{1} P_{2} P_{2} Q_{1} Q_{2}}
\]

\[
+ \frac{1}{2} \sigma_{P_{1} P_{1} P_{2} P_{2}}^{2} + \frac{1}{2} \sigma_{Q_{1} Q_{2}}^{2} + \sigma_{P_{2} \hat{B}} P_{2} P_{1}
\]

58
\[ + \sigma_{P - 1 Q^B} P_{1 - 1} + \sigma_{P - 2 Q^B} P_{2 - 2} - \hat{B}_t + c \right)_B \] (4.30)

also,

\[ \psi_1 \equiv \frac{\sigma_{P - 1 Q^B}}{\hat{B}} \quad \psi_2 \equiv \frac{\sigma_{P - 2 Q^B}}{\hat{B}} \quad \psi_3 \equiv \frac{\sigma_{Q^B}}{\hat{B}} \]

\[ \sigma_{P - 1 P_2} = \rho_{P - 1} \sigma_{P - 1 P_2} \quad \sigma_{P - 1 P_2} = \rho_{P - 1} \sigma_{P - 1 P_2} \quad \sigma_{P - 1 Q^B} = \rho_{P - 1} \sigma_{P - 1 Q^B} \]

A portfolio is formed containing the two commodities linked bond, the two commodities and pure discount bonds with maturity date the same as that of the two commodities linked bond. Our purpose is to create a riskless zero-investment portfolio, which, to avoid arbitrage profits, will have a zero rate of return. This zero investment portfolio, \( M \), will contain the two commodities, the pure discount bonds and the commodities linked bond with weights, \( \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \) respectively, such that:

\[ M = \omega_{P - 1} + \omega_{P - 2} + \omega_{Q - 3} + \omega_{Q^B - 4} = 0 \] (4.31)

Note that \( \omega_4 = - (\omega_1 + \omega_2 + \omega_3) \). The rate of return of this portfolio is:

\[ dM = \omega_{P - 1} \frac{dP_1}{P_1} + \omega_{P - 2} \frac{dP_2}{P_2} + \omega_{Q} \frac{dQ}{Q} + \omega_{Q^B} \left( \frac{dB + cdt}{B} \right) \] (4.32)

Substituting equations (4.29) - (4.31) into (4.32) we get:
The portfolio can be nonstochastic if the choice of the weights obey the following conditions:

$$
\frac{\omega_p B}{B} \hat{p}_1^{B} \\
\omega_1 = - \frac{\hat{p}_1^{B}}{B} \tag{4.34}
$$

$$
\frac{\omega_p B}{B} \hat{p}_2^{B} \\
\omega_2 = - \frac{\hat{p}_2^{B}}{B} \tag{4.35}
$$

$$
\frac{\omega q B}{B} \hat{q}_q^{B} \\
\omega_3 = - \frac{\hat{q}_q^{B}}{B} \tag{4.36}
$$

To prevent arbitrage profits the return on this nonstochastic, zero investment portfolio must also be zero. Hence combining (4.34) - (4.36) with (4.33) and also substituting the expression for $\omega_p$ and simplifying, results in the partial differential equation for pricing the two commodities indexed bond. This partial differential equation is:

$$
\frac{1}{2} \sigma^2 p_1^{2B} p_1 p_1 + \frac{1}{2} \sigma^2 p_2^{2B} p_2 p_2 + \frac{1}{2} \sigma^2 q^{2B} q q + \sigma \hat{p}_1^{2B} p_1 p_1 + \sigma \hat{p}_2^{2B} p_2 p_2 \\
+ \sigma \hat{q}_q^{2B} q q - \hat{B}_t + c = 0 \tag{4.37}
$$
The boundary conditions for the solution of equation (4.37) depends on how the final payment is structured. If the issuer gives the bearer the option on the form of the final payment, then on the maturity date the bearer would receive the maximum of the face value of the bond, $F$, the monetary values of prespecified units of commodities one and two. For simplicity and without the loss of generality, we shall assume that the bond is referenced to one units of commodities one and two. Hence the boundary condition under this arrangement and with no default risk is:

$$\hat{B}(P_1, P_2, Q, 0) = \text{Max}(F, P_1, P_2)$$

$$= F + \text{Max}(0, P_1 - F, P_2 - F) \quad (4.38)$$

The solution of equation (4.37) subject to (4.38) could be obtained by using numerical methods. However, closed form solutions, although difficult, can be obtained. Equation (4.38) suggests that the two commodities linked bond could be replicated by a portfolio containing a regular bond (which pays $F$ on the maturity date) and a European compounded call option on the two commodities, which pays on the expiry date $\text{Max}(0, P_1 - F, P_2 - F)$. Equating the present value of the two commodities bond to that of this hypothetical portfolio, then the value of the bond, if we assume no coupon payments, is:

$$\hat{B}(P_1, P_2, Q, \tau) = FQ + \hat{C}(P_1, P_2, F, \tau) \quad (4.39)$$

$\hat{C}(\cdot)$ is the value of a European compound call option. Cheng (1987)
appealing to tedious mathematical manipulations, has shown that the value of \( C(\cdot) \) is given as:

\[
\hat{C}(P_1, P_2, F, \tau) = P_{1/2} N\left(\frac{a}{\sigma_a}, \frac{a-b}{\tau}, \frac{\sigma^2 - \sigma_{ab}}{\sigma_a}\right)
\]

\[
+ P_{1/2} N\left(\frac{b}{\sigma_b}, \frac{b-a}{\tau}, \frac{\sigma^2 - \sigma_{ab}}{\sigma_b}\right)
\]

\[
+ FQ(1 - N_2\left(-\frac{a}{\sigma_a}, \frac{\sigma^2}{2}\right), -\frac{b}{\sigma_b}, \frac{\sigma^2}{2}, \frac{\sigma_{ab}}{\sigma_a\sigma_b}\right)) \quad (4.40)
\]

where, following Cheng's notation, and noting that \( N_2(a, b; \rho) \) is a bivariate normal distribution evaluated at \( a \) and \( b \) with correlation coefficient \( \rho \). We define:

\[
P_1 = \ln\left(\frac{a}{\sigma_a}\right) \quad (4.41)
\]

\[
P_2 = \ln\left(\frac{b}{\sigma_b}\right) \quad (4.42)
\]

\[
\sigma^2_a = \int_0^\tau \sigma_a^2 \, ds \quad (4.43)
\]

\[
\sigma^2_b = \int_0^\tau \sigma_b^2 \, ds \quad (4.44)
\]
Hence the solution of (4.37) under the boundary condition of equation (4.38) (and setting \( c \) to zero) is given by equations (4.39) and (4.40).

If the issuer of the two commodities linked bond has the option on the final payment then the bearer would receive on the expiration of the bond the minimum of the face value of the bond, \( F \), and the monetary values of pre-specified units of commodities one and two. The boundary condition, assuming no default risk, would then be specified as:

\[
\hat{B}(P_1', P_2', Q, 0) = \min(F, P_1', P_2')
\]

\[
= F - \max(0, F - P_1', F - P_2')
\]  \hfill (4.47)

If the two commodities indexed bonds pay no coupons, then the solution of equation (4.37) under the boundary condition of equation (4.47) is given as:

\[
\hat{B}(P_1', P_2', Q, \tau) = FQ - \hat{P}(P_1', P_2', F, \tau)
\]  \hfill (4.48)

where \( \hat{P}(\cdot) \) is a European compound put option on the two commodities with maturity payment equal to \( \max(0, F - P_1', F - P_2') \). Again applying Cheng (1987) the value of \( \hat{P}(\cdot) \) is expressed as:
\[
\hat{P}(P_1, P_2, F, \tau) = FQ(1 - N_2 \left( -\frac{a}{\epsilon_a} + \frac{\epsilon_a}{2}, -\frac{b}{\epsilon_b} + \frac{\epsilon_b}{2}, \frac{\epsilon_{ab}}{\epsilon_a} \right))
\]

\[
- P_{12} N \left( -\frac{a}{\epsilon_a} - \frac{\epsilon_{a}}{2}, \frac{a-b}{\epsilon} - \frac{\epsilon^2 - \epsilon_{ab}}{\epsilon_a} \right)
\]

\[
- P_{22} N \left( -\frac{b}{\epsilon_b} - \frac{\epsilon_b}{2}, \frac{a-b}{\epsilon} - \frac{\epsilon^2 - \epsilon_{ab}}{\epsilon_b} \right)
\]  \hspace{1cm} (4.49)

with definitions (4.41) - (4.46).
In Chapter three we mentioned solving equation (3.22) (the partial differential equation for valuing commodity-bonds) subject to the boundary condition given by equation (3.23) (the final pay-off of the bond) is a non-trivial exercise. The use of numerical methods to solve this equation would require a five dimensional grid - each dimension representing a level for the five state variables; i.e., the diffusion processes followed by the value of the firm issuing the bond (V), the stochastic convenience yield (δ), the interest rate (r), the price of the indexed commodity (P) and time (t). It would also require a great deal of computing time. Such resources, if available, do not come cheap. As a result, most researchers restrict their models to two state variables and then employ numerical methods to obtain the solution. Examples are the Brennan and Schwartz models, Gibson and Schwartz (1990) and Jones and Jacobs (1986). We shall, therefore, follow these researchers and solve equation (3.22) for only two state variables at a time. We consider three cases.

The first case will be to consider the valuation of the commodity-linked bonds in a regime where the interest rate and the price of the referenced commodity are stochastic. Case one also assumes zero convenience yield. The second case will involve the same assumptions as
case one but extended to the case where the convenience yield is a deterministic function of the spot price of referenced commodity. In the third and last case we assume a deterministic interest rate process with the commodity price and convenience yield following stochastic processes. Note that in all the three cases we have implicitly assumed the probability of default by the issuer of the commodity-linked bond to be zero.

However, before we can embark on the numerical procedure we have to obtain estimates for the parameters, \( a(r) \), \( a(\delta) \), \( \sigma_p \), \( \sigma_r \), \( \sigma_\delta \), \( \rho_{pr} \), \( \rho_{p\delta} \), \( \rho_{r\delta} \), \( \lambda_1(r) \) and \( \lambda_2(\delta) \). These parameters were defined in Chapter Three. (See equations 3.1 - 3.4, 3.10 and 3.12). We shall now present our method of estimating these parameters and the numerical procedure for valuing a hypothetical bond. The numerical exercise involves simulating the prices of a hypothetical gold-linked bond.

1. ESTIMATION PROCEDURE

IMPLIED SPOT CONVENIENCE YIELD

**PROPOSITION 1:** Given \( P_t \) as the spot price of a commodity at time \( t \), then the price, \( U_t(P, T) \), of a futures contract expiring at time \( T \) is obtained as:

\[
U_t(P, T) = P_t e^{(r - \delta)(T - t)}
\]  

(5.1)

where \( r \) is the continuously compounded interest rate and \( \delta \) is the compounded net convenience yield.

**PROOF:** Recall equation (3.1) of Chapter Three and let the diffusion process followed by spot price of commodity, \( P_r \) be:
\[
\frac{dP}{P} = (\alpha_p - \delta)dt + \sigma_p dz_p
\] (5.2)

The above parameters have already been defined in Chapter Three.

Applying Ito's lemma it follows that the diffusion process of the future price, \( U_t(P, t) \) is given as:

\[
\frac{dU}{U} = \alpha_u dt + \sigma_u dz_p
\] (5.3)

where

\[
\alpha_u = \frac{1}{2\sigma_p^2} \frac{P_p^2 U_{pp}}{U} + (\alpha_p - \delta)P_U + U_t \bigg/U
\] (5.4)

\[
\sigma_u = \frac{\sigma_p U}{U}
\]

Now construct a riskless arbitrage portfolio by borrowing \( P \) at the instantaneous riskless interest rate, \( r \). Use the borrowed funds to purchase one unit of the commodity at the price \( P \) and simultaneously short \( U/U_p \) of the futures contract. Since this portfolio must be riskless and an initial wealth of zero is used, then the instantaneous return of this portfolio must also be zero. In other words the instantaneous return of the portfolio is:

\[
\alpha_P \frac{U}{U_p} - \frac{1}{2\sigma_p^2} \frac{P_p^2 U_{pp}}{U} + (\alpha_p - \delta)P_U + U_t \bigg/U - rP = 0
\]

Rearranging we get:

\[
1/2\sigma_p^2 P_p^2 U_{pp} + (r - \delta)P_U + U_t = 0
\] (5.5)
The solution of (5.5) is subject to the following boundary condition:

\[ U_t(P, T) = P_T \]  

(5.6)

Since \( r \) and \( \delta \) are constants the solution of (5.5) subject to (5.6) yields:

\[ U_t(P, T) = P_t e^{(r - \delta)(T - t)} \]

QED.

From equation (5.1) the implied spot net convenience yield is given as:

\[ \delta_T = r_T - \frac{1}{(T - t)} \log \left( \frac{U_t(P, T)}{P} \right) \]  

(5.7)

where \( \delta_T \) is the spot convenience yield of a \((T - t)\) futures contract at time \( t \) and \( r_T \) also denotes the \((T - t)\) period spot interest rate.

In view of our numerical exercise we computed the implied spot convenience yield using the weekly spot price of gold, the price of futures on gold maturing in three months and the three month Libor rate (adjusted

\[ U_{pp} = 0, \quad U_P = e^{(r - \delta)(T - t)} \quad \text{and} \quad U_t = -(r - \delta) e^{(r - \delta)(T - t)} \]

Substituting above into (5.5) shows that (5.1) is the solution to (5.5) subject to (5.6).
to annual value by multiplying by 365/360). The period of observation was from January 1982 to May 1987. The number of weekly observations was 277. The implied spot convenience yield was found to exhibit a mean reverting drift. The conclusion of mean reversion was drawn after regressing the first difference \( \Delta \delta_t = \delta_t - \delta_{t-1} \) on the lagged value \( \delta_{t-1} \) of the convenience yield. The coefficient on \( \delta_{t-1} \) was found to be negative and statistically significant.\(^8\)

With our empirical evidence supporting mean reversion, we specified the diffusion process of the convenience yield as:

\[
\dot{\delta} = \kappa_1 (\delta - \delta) dt + \sigma_\delta dz_\delta
\]

(5.8)

The formulation of equation (5.8) specifies that the spot convenience yield is being pulled towards an average of \( \delta \). A linear discretized approximation of equation (5.8) is given as:

\[
\delta_t = \kappa_1 \delta + (1 + \kappa_1) \delta_{t-1} + \epsilon_t
\]

(5.9)

where \( \epsilon_t \) is distributed as \( \text{N}(0, \sigma_\delta^2) \). An OLS regression of equation (5.9) would imply the intercept would be \( \kappa_1 \delta \) and the coefficient of \( \delta_{t-1} \) would be \( (1 + \kappa_1) \). The estimate of \( \sigma_\delta \) would be the standard deviation of \( \epsilon_t \).

\(^8\) Using the implied 3 month spot convenience yield, our regression analysis estimated the coefficient on \( \delta_{t-1} \) to be -0.30549. The t-statistic was computed to be -7.1747.
INTEREST RATE PROCESS

Cox, Ingersoll and Ross (1985) suggests that the process governing the path followed by the interest to be of the form:

\[ \text{dr} = \kappa_2 (\theta - r) \text{dt} + \sigma_r \sqrt{r} \text{dz}_r \] (5.10)

in which \( \sigma_r \), \( \kappa_2 \) and \( \theta \) are constants. Equation (5.10), referred to in the literature as the square root process, implies that interest rates are pulled towards a long run average value, \( \theta \).

A linear discretization of equation (5.10) would be of the form:

\[ r_t = \kappa_2 \theta + (1 - \kappa_2) r_{t-1} + \eta_t \] (5.11)

Comparison of equations (5.10) and (5.11) reveals that the variance of the error term, \( \eta_t \), is \( \sigma_r^2 r_t \). Hence applying data to equation (5.11) and performing an OLS regression would introduce the problem of heteroscedasticity. A transformation of the interest rate process is needed to remove the problem heteroscedasticity.

However Schuss (1980), and also reported in Lo (1988), has shown that if a state variable \( X(t) \) follows a diffusion process of the form:

\[ dX = \alpha(X, t) dt + \beta(X, t) dz \] (5.12)

then there exists a transformed process \( Y(t) \) of \( X(t) \) for which the coefficient functions are independent of \( Y(t) \) if \( \alpha(\cdot) \) and \( \beta(\cdot) \) satisfy the following reducibility condition:
This implies that for some suitable change of variable \( F(X(t)) = Y(t) \), the application of Ito's lemma will yield:

\[
\frac{\partial}{\partial X} \left[ \frac{1}{\beta} \frac{\partial \beta}{\partial t} - \frac{\partial}{\partial X} \left( \frac{\alpha}{\beta} \right) + \frac{1}{2} \frac{\partial^2 \beta}{\partial X^2} \right] = 0 \tag{5.13}
\]

where \( \phi \) is a vector of parameters. Lo (1988) further demonstrates that the transition density function, \( \rho_k(Y, t) \), for the transformed data is given as:

\[
\rho_k(Y, t) = \left[ 2\pi \int_{t_{k-1}}^t q^2 d\tau \right]^{-1/2} \exp \left[ \frac{(Y - Y_{k-1} - \int_{t_{k-1}}^t v d\tau)^2}{2 \int_{t_{k-1}}^t q^2 d\tau} \right] \tag{5.15}
\]

Following Lo (1988) and Schuss (1980) we conducted a reducibility test on equation (5.10) before looking for a transformation for the interest rate, \( r \). An application of equation (5.13) to equation (5.10) implies that we check whether:

\[
\frac{\partial}{\partial r} \left[ \frac{1}{\sigma_r^{1/2}} \frac{\partial \sigma_r^{1/2}}{\partial t} - \frac{\partial}{\partial r} \left( \frac{\kappa_r}{\sigma_r^{1/2}} \right) + \frac{1}{2} \frac{\partial^2 \sigma_r^{1/2}}{\partial r^2} \right] = 0 \tag{5.16}
\]

is equal to zero. However, the evaluations of equation (5.16) yields:
Clearly equation (5.17) is different from zero. Hence equation (5.10) fails the reducibility test. Hence any transformation of \( r \) would have one or all the coefficients of the transformed process to be dependent on the transformed variable.

Nevertheless we transformed \( r \) by defining:

\[
R = 2r^{1/2}
\]  

(5.18)

The application of Ito's lemma yields:

\[
dR = \left(2(\kappa_2 \theta - \frac{1}{4} \sigma_r^2)R^{-1} - \frac{1}{2} \kappa_2 R\right)dt + \sigma_r dz_r
\]  

(5.19)

Equation (5.19) clearly supports Lo (1988) and Schuss (1980) that equation (5.10) did not satisfy the reducibility condition since the process, \( dR \), has its drift depending on \( R \). Discretization of equation (5.19) gives:

\[
R_t = 2(\kappa_2 \theta - \frac{1}{4} \sigma_r^2)/R_{t-1} + (1 - \frac{1}{2} \kappa_2)R_{t-1} + \xi_t
\]  

(5.20)

Note that the error term \( \xi_t \) is homoscedastic.

We have tried to minimize the bias introduced by the fact that the drift depends on \( R \) by using weekly, as opposed to less frequent, observations on interest rates. Jacobs and Jones (1986) found the discretization bias to be negligible at this sampling frequency.
GOLD PRICE

As specified in equation (3.1) (see Chapter Three) the gold price process was assumed to be of the lognormal form. Thus:

\[ dP = (\alpha_p - \delta)Pdt + \sigma_P Pdz \]

A transformation of \( S = \log(P) \) was conjectured after finding that (5.21) satisfied the reducibility condition of Schuss. Hence, applying Ito's lemma, we obtain the diffusion process for \( S \) as:

\[ dS = \left( (\alpha_p - \delta) - \frac{1}{2} \sigma^2_p \right) dt + \sigma_p dz \]

Discretization of equation (5.22) leads to:

\[ S_t = \left( (\alpha_p - \delta) - \frac{1}{2} \sigma^2_p \right) + S_{t-1} + \nu_t \]

The error term \( \nu_t \) is homoscedastic and it's distribution is normal with mean equal to zero and variance equal to \( \sigma^2_p \).

THE MARKET PRICE OF INTEREST RATE RISK

The implementation of the Valuation model of equation (3.22) requires an estimate of the market price of interest rate risk, \( \lambda_1 \) \(^9\). Knowledge of the preference functions of investors would be the best way of measuring the true value of the risk premium to be paid to these investors who have

\[^9\text{For our numerical exercise } \lambda_1(\cdot) \text{ and } \lambda_2(\cdot) \text{ were estimated as constants.}\]
claims to financial assets which are solely influenced by the interest rate. However, it would not be an easy exercise to find the utility functions of bearers of financial contracts which are dependent on a stochastic interest rate. An alternative, which we use in this thesis, is to compute the implied premiums using observed market and theoretical prices of returns on assets solely dependent on the interest rate.

In Chapter Three (see section 3.3) we showed that if $G(r, \tau)$ is a default free pure discount bond and $\lambda$ is the interest rate risk premium then the partial differential equation for valuing such a bond is:

$$\frac{1}{2} \sigma^2 G_{rr} + (\alpha - \lambda \sigma_r) G_r - G_\tau - rG = 0 \quad (5.24)$$

where all the variables have been defined already in Chapter 3. Assuming a mean reversion interest rate process of the Cox, Ingersoll and Ross type (see equation 5.10) equation (5.24) becomes:

$$\frac{1}{2} \sigma^2 G_{rr} + (\kappa_2 (\theta - r) - \lambda \sigma_r^{1/2}) G_r - G_\tau - rG = 0 \quad (5.25)$$

Specifying an appropriate boundary condition to (5.25), the Crank-Nicholson's algorithm is used to numerically value the discount bond.

This is done by converting the partial differential equations of (5.25) into a set of difference equations which are then solved iteratively. For a good exposition on the topic of numerical methods for valuing derivative securities see Chapter 9 of Hull (1989), Jones and Jacob (1986) and Brennan and Schwartz (1979).

Specifying the boundary condition to equation (5.25) as $G(r, 0)$ we use
numerical methods to obtain theoretical prices of a bond of specified maturity. This was done by substituting the estimates of the underlying parameters and an initial guess of $\lambda_1$ into equation (5.22). Interpolation methods were then used to compute the corresponding prices of the bond using the $N$ observations of the one month libor rate as the spot rate of interest. Let $G^*(r, \lambda_1, \tau)$ represent the theoretical price at time $t$. Then the theoretical yield to maturity $r^*_t(\lambda_1, \tau)$ for the length of maturity of the bond is given as:

$$r^*_t(\lambda_1, \tau) = \frac{1}{\tau} \log \left( \frac{G(r, 0)}{G^*(r, \lambda_1, \tau)} \right)$$  \hspace{1cm} (5.26)$$

where $\tau (= T - t)$ is the time to maturity of Bond and $G(r, 0)$ is the face value of the discount bond.

Our final step involved finding the $\lambda_1$ that minimized the mean square errors between the actual observed spot market rate, $r^*_t$ with the same time frame as the theoretical one. Thus our objective function was:

$$\min_{\lambda_1} \frac{1}{N} \sum_{t=1}^{N} (r^*_t - r^*_t(\lambda_1, \tau))^2$$  \hspace{1cm} (5.27)$$

This objective was achieved by using numerical iteration to search for $\lambda_1$ that gave the minimum mean square error expressed by (5.27). In this thesis we set $G(r, 0)$ to $\$100.00$, $\tau$ to be 3 months and, therefore, $r^*_t$, was the 3 month Libor rate.
THE PRICE OF CONVENIENCE YIELD RISK

The estimate of the market price of convenience yield risk is computed in a similar fashion as that for the interest rate risk.

However, to embark on this exercise we had to construct a hypothetical security which was solely influenced by the convenience yield. Following Brennan (1986) we define a "convenience claim as a claim to net (of storage costs) flow of services yielded by a unit of inventory over a specified time period." As in chapter 3, let \( I(\delta, \tau) \) be a convenience claim of maturity \( \tau \). This asset is exactly replicated by a portfolio made of a unit of the commodity referenced to the commodity-linked bond and a short position in a commodity futures contract of maturity \( \tau \). If \( P(t) \) is the current price of a unit of the commodity, \( U_t(P, \tau) \) is the futures prices and \( r(\tau) \) is the riskless rate of interest on holding a T-bill with maturity \( \tau \), then:

\[
I(\delta, \tau) = P(t) - U_t(P, \tau)e^{-r(\tau)t}
\]

Since by construction, the convenience claim has no value on the maturity date, then the boundary condition will be:

\[
I(\delta, 0) = 0
\]

From equation (3.12) the partial differential equation governing the

---

10 The formulation of the convenience claim, as expressed by equation (5.28), is correct in a regime of constant interest rates. However interest rates only matter because of the discounting of the convenience yield, and since we use futures contract of less than one year to maturity then the discounting effect is, for a first approximation, negligible.
valuation of this convenience claim would be given as:

\[
\frac{1}{2}\sigma^2 I_\delta \delta I_\delta + (\alpha_\delta - \lambda_2 \sigma_\delta) I_\delta - I_{\tau} - rI = 0 \quad (5.30)
\]

However, assuming that the stochastic convenience yield process follows a mean reversion process, we have equation (5.30) becoming:

\[
\frac{1}{2}\sigma^2 I_\delta \delta I_\delta + (\kappa_\delta (\delta - \delta) - \lambda_2 \sigma_\delta) I_\delta - I_{\tau} - rI + \delta = 0 \quad (5.31)
\]

Note that \( \delta \), which acts like a coupon rate, is average per unit flow of convenience yield. Equation (5.31) was solved subject to equation (5.29) by numerical methods. Letting \( I^A(\cdot) \) be the observed price of convenience claim with maturity \( \tau \), (where \( I^A = P(t) - Ue^{-r\tau} \)), and \( I^*(\cdot) \) be the theoretical price obtained numerically then our pricing error is \( I^A - I^*(\lambda_2, \tau) \).

The objective was then to find \( \lambda_2 \) that minimized the mean pricing error. Thus our objective function was:

\[
\text{Min} \quad \frac{1}{N} \sum_{i=1}^{N} (I^A - I^*(\lambda_2, \tau))^2 \quad (5.32)
\]

This objective was achieved by finding estimates for the parameters in equation (5.31) and using numerical iteration to search for \( \lambda_2 \) that gave the minimum mean pricing error. In our construction of \( I(\cdot) \) we used weekly observations of the price of future on gold maturing in three months. The results are stated in the next section.
2. NUMERICAL VALUATION OF THE BOND

As mentioned in the introduction to this chapter, the valuation of the commodity-linked bond was valued numerically for three cases. We proceeded in this way since we cannot compute the bond value numerically when we have more than two underlying state variables.

The numerical exercise was carried out by constructing a variety of hypothetical gold-linked bonds. Numerical values were obtained under four different pay-off scenarios. Under the first scenario the bearer of the gold-linked security receives a final payment of the \( \text{Max}[1000, (1000/K)*P] \), where \( P \) is the price of an ounce of gold at maturity and \( K \) is the exercise price of the option feature attached to the bond. In the second scenario bearers receive \( \text{Min}[1000, (1000/K)*P] \) as the final payment. Under the third scenario, the bearers were paid $1,000.00 on the maturity date of the bond. This is the conventional nominal bond. Under the fourth scenario, the final payment to bearers is the \( (1000/K)*P \). This is the fully indexed bond. In the rest of the text we shall refer to these bonds as bond(1), bond(2), bond(3) and bond(4) respectively.

CASE 1

Here the hypothetical gold bond is evaluated under the assumption that only interest rates and gold prices are stochastic. Convenience yield is assumed to be zero. The partial differential equation for the valuing the gold-linked bond reduces to:
In equation (5.33) we have assumed that the interest rate process is of the form expressed by equation (5.10). Equation (5.33) is solved for the four different final payoff scenarios.

Equation (5.33) requires the estimates of \( \sigma_p, \sigma_r, \rho_{pr}, \kappa_2, \theta \) and \( \lambda_1 \). With the exception of the market price of interest rate risk, the other parameters were estimated jointly. With the aid of the Shazam Computer package we used a seemingly unrelated regression model to estimate these parameters. The estimation procedure involved non-linear maximum likelihood methods. We ran the regression model using the discretized approximation of equation (5.20) in conjunction with equation (5.23) with \( \delta \) set to zero. The data used was obtained from the Data Resources Incorporated. Weekly observations were used from January 1982 to May 1987. For the interest rate process, we used the 3 month Libor rate annualized by multiplying each observation by \( 365/360 \). The annualized estimated parameters are summarized below:

<table>
<thead>
<tr>
<th>( \sigma_p )</th>
<th>( \sigma_r )</th>
<th>( \rho_{pr} )</th>
<th>( \kappa_2 )</th>
<th>( \theta_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.23645</td>
<td>0.07502</td>
<td>-0.2521</td>
<td>0.74568</td>
<td>0.0801</td>
</tr>
</tbody>
</table>

Before inserting these parameter estimates into equation (5.33) we estimated \( \lambda_1 \). The method of estimating \( \lambda_1 \) has been outlined earlier. By minimizing the function given by equation (5.27) we estimated \( \lambda_1 \) as
With the estimated value of $\lambda_1$ and of the coefficients of the joint stochastic process we computed the prices of the hypothetical bond numerically using the Alternating Direction Implicit method (ADI) described in McKee and Mitchell (1970).

The results obtained were similar to those obtained in case 2. Hence we shall combine the results of cases 1 and 2. These results will be discussed under case 2.

**CASE 2**

This case is similar to that in case 1. However, we add the assumption of the convenience yield being proportional to the spot price of gold. Hence the partial differential equation for pricing the gold linked bond becomes:

\[
\frac{1}{2} \sigma_B^2 \frac{\partial^2 B}{\partial p^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 B}{\partial r^2} + \rho_{pr} \sigma_p \sigma_r \frac{\partial B}{\partial p} \frac{\partial B}{\partial r} + (r - \delta)PB_p + (\kappa_2(\theta - R) - \lambda_1 \sigma_r)B_r - B_T - rB + c = 0 \quad (5.34)
\]

In this section we repeated the estimation procedure carried out in case 1 using equations (5.20) and (5.23). The annualized estimated parameters are:

<table>
<thead>
<tr>
<th>$\sigma_p$</th>
<th>$\sigma_r$</th>
<th>$\rho_{pr}$</th>
<th>$\kappa_2$</th>
<th>$\theta_r$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.23645</td>
<td>0.07502</td>
<td>-0.2521</td>
<td>0.74568</td>
<td>0.0801</td>
<td>0.00143</td>
</tr>
</tbody>
</table>
With the estimated value of $\delta$ and of the estimates of the other parameters we carried out the same numerical procedure embarked on in case 1 to obtain the price of the hypothetical gold bond under different final pay-off scenarios.

For all our simulations, we take the exercise price of the bond to be 700, the time to maturity of the bond to be five years and the coupon payments to be made semi-annually at an annual rate of 4% of the final pay-off. Our results are found in tables 1 to 4 in the appendix to this chapter.

In Table 1, which is the simulated prices for bond(1), we find that the value of the bond falls with the rise in interest rates and rises when the spot price of gold goes up. We also find that the interest rate sensitivity to value of the bond falls as the spot price of gold rises.

Table 2 reports the simulated prices of bond(2). In agreement with our economic intuition we find that the value of bond(2) also rises with the rise in the spot price of gold and falls with the rise in the interest rates. We also find that the rate of increase in the value of bond(2) falls with the rise in the spot price of gold.

Table 3 gives the prices of bond(3) at different spot prices of gold and interest rates. Since bond(3) is a conventional bond it is not surprising that we find in table 3 that that the value of bond(3) is insensitive to changes in the spot price of gold. However, as expected, we find that the value of bond(3) is fairly sensitive to changes in the interest rates. That all bond values are below par indicates that 4% is below the par coupon rate for 5 year bonds, even for very low levels of short term rates.

The prices of bond(4) at various interest rates and spot prices of
gold are contained in table 4. In accord with our economic intuition we find that the value of bond(4) is insensitive to changes in the interest rates. The value however rises in direct proportion to the spot price of gold. Notice that at $P$ equal to $700$, where the current value of the gold delivered at maturity is exactly $1000$, the bond is worth more than par ($1192$). This reflects the fact that the bond pays a 4% per year coupon, whereas gold does not, making the bond strictly preferable to holding gold.

Comparing tables 1 to 4 we find that the value of bond(1) dominates the values of the other bonds at all levels of the spot price of gold and interest rates. Bond(2) is also found to be the least valuable of all the bonds. However bond(4) dominates bond(3) for prices of gold of 500 and beyond. For spot prices of gold of 300 and below bond(3) is more valuable than bond(4). We also find that the value of bond(2) converges to that of bond(3) as the spot price of gold rises. Similarly, the value of bond(4) approaches bond(1) as the spot price of gold rises. Although not tested, we believe that the value of bond(1) will converge to that of bond(3) at prices of gold below 100.

**CASE 3**

In this section we evaluated the hypothetical gold linked bond under the assumption of interest rates being deterministic. However, the price of a unit of gold and convenience yield are postulated to be stochastic, and follow a Brownian motion. Under these assumptions the pricing partial differential equation for the gold bond becomes:

\[
\frac{1}{2} \sigma_p^2 B_{\delta \delta} + \frac{1}{2} \sigma_{\delta}^2 B_{\delta \delta} + \rho_p \sigma_p \sigma_{\delta} B_{\delta \delta} + (r - \delta) B_{\delta} \]

82
Note that in equation (5.35) we have assumed that the convenience yield process follows the mean reversion process expressed by equation (5.8). As in case 1, we would have to find the estimates of \( \sigma_p, \sigma_\delta, \rho_{p\delta}, k_1, \delta \), and \( \lambda_2 \). With the exception of \( \lambda_2 \) we estimated the other variables jointly. Like the estimation in case 1 we ran a seemingly unrelated regression using a non-linear maximum likelihood methods. We ran the regression model using the discretized approximation of equation (5.9) in conjunction with equation (5.23). The annualized estimated parameters are summarized below:

<table>
<thead>
<tr>
<th>( \sigma_p )</th>
<th>( \sigma_\delta )</th>
<th>( \rho_{p\delta} )</th>
<th>( k_1 )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.23645</td>
<td>0.09843</td>
<td>-0.0321</td>
<td>14.83872</td>
<td>0.0051864</td>
</tr>
</tbody>
</table>

Next we estimated \( \lambda_2 \) by the method described earlier in the section on estimation procedure. Our estimate of \( \lambda_2 \) is -0.3895.

Substituting the estimates of the parameters into (5.35) the prices of the hypothetical gold bond were obtained under different final pay-off scenarios.

Similar to cases 1 and 2, for all our simulations in case 3, we take the exercise price of the bond to be 700, the time to maturity of the bond

---

Note that the large value of \( k_1 \) implies any divergence of \( \delta \) from \( \bar{\delta} \) is short lived.
to be five years and the coupon payments were made semi-annually at annual rate of 4% of the final pay-off. The instantaneous interest rate was set at 11.40% p.a. Our results are found in tables 5 to 8 in the appendix to this chapter.

Table 5 contains the price of bond(1) at various price of gold and convenience yield. The results there show that the value of the bond falls with the rise in the spot convenience yield, though the effect is not great. Our intuition suggests that a rise in the convenience yield reduces the growth rate of the price of gold and as a result the value of the bond falls. Furthermore as expected the value of the bond rises with the rise in the price of gold.

In table 6 we find that the value of bond(2) also falls with the rise in the convenience yield and rises as the spot price of gold goes up. However the sensitivity of the convenience yield weakens with the rise in the spot price of gold.

The value of bond(3) at different levels of gold price and convenience yield can be seen in table 7. In agreement with our economic intuition we find that the value of bond(3) is insensitive to changes in both the convenience yield and the spot price of gold. The price of the bond is calculated as 713.6689 for all levels of the price of gold and convenience yield. Not to our surprise, the figure of 713.6689 is equal to the present value of a conventional bond with a face value of 1000, matures in five years, makes coupon payments semi-annually at the rate of 4% of the face value and discounted continuously at a constant interest rate of 11.40% p.a.

In table 8 we find that the value of bond(4) is sensitive to changes in gold prices and convenience yield. It falls with the rise in the
convenience yield and rises with the rise in gold prices.

Similar to cases 1 and 2 we find that when we compare tables 5 to 8 we also find that the value of bond(1) dominates the values of the other bonds at all levels of the spot price of gold and convenience yield. Bond(2) is also found to be the least valuable of all the bonds. However bond(4) dominates bond(3) for prices of gold of 700 and beyond. For spot prices of gold of 500 and below bond(3) is more valuable than bond(4). We also find that the value of bond(2) converges to that of bond(3) as the spot price of gold rises. Similarly, the value of bond(4) approaches bond(1) as the spot price of gold rises. Although not tested, we also believe that the value of bond(1) will converge to that of bond(3) at prices of gold below 100.

**EQUILIBRIUM PAR COUPON RATES**

An interesting question we attempted to answer with our model is that if investors are to pay 1000 for each of the bonds, then what coupon rate must these investors be offered on each bond.

The results of our simulation exercise are contained in tables 9 to 11. In the simulation exercise we set the current price of gold at 400. The coupon payments were made semi-annually. However the interest rates, the times to maturity, exercise prices and convenience yields were allowed to vary. We shall refer to the par coupon rates of bond(1), bond(2), bond(3) and bond(4) respectively as \( c_1, c_2, c_3 \) and \( c_4 \). Also figures reported in tables 9 to 11 are to be read as percentages per year.

In table 9 we report the par coupon rates under the assumption that the spot price of gold and interest rates are stochastic with no convenience yield. However, since the results are similar to that of table 10 and the model used in table 10 takes account of a deterministic
convenience yield (which is equal to 0.001433*P) we shall, therefore, discuss table 10.

It can be seen in table 10 that, at all levels of arbitrarily chosen interest rates and times to maturity, the coupon rates for \( c^1 \) and \( c^4 \) are negative for exercise prices of 400 and below. Our economic intuition suggests that at these exercise prices, the bonds are so valuable that to induce issuers of the bond to trade the bonds at a price of 1000 then investors who bear the bonds would rather have to pay the issuer at those respective coupon rates. With the exception of \( c^4 \), all the coupon rates rise as interest rates rise for given time to maturity. The result is due to the fact that a rise in the interest rates reduces the present value of the bonds. The reason it does not for \( c^4 \) (the pure commodity bond) is that as interest rates rise, so does the expected rate of appreciation in equilibrium gold prices. The coupon rates would, therefore, have to go up in order to keep the current price of the bonds at 1000. Consistent with our economic intuition, at exercise prices of 400 and below it was found out that \( c^4 \) and \( c^1 \) are almost the same rates. At times to maturity of 5 and 10 \( c^1 \) attracted the smallest coupon rates. The rates for \( c^4 \) was computed to be less than that of the conventional bond, \( c^3 \). Not surprising \( c^2 \) offered the highest coupon rates. Also for exercise prices of 400 and below \( c^2 \) converges to \( c^3 \). As the time to maturity rose, \( c^1 \) and \( c^4 \) were seen to fall. This may be due to the fact that the value of the bonds increased with the time to maturity. As expected \( c^3 \) rose with the increase in the times to maturity since the value of a conventional bond falls with the rise in the time to maturity. \( c^2 \), however, for exercise prices of 700 and above, was seen to fall with the rise in the times to maturity. For exercise prices of 400 and below \( c^2 \) was observed to rise with the increase
in the times to maturity. Note also from Table 10 the normal term premium present in the conventional yield curve. That is at \( r \) equal to 8% (approximate value of \( \theta \)) we get 10.52%, 12.34% and 12.79% for 1, 5, and 10 year bonds respectively.

Our last simulation exercise is contained in table 11. There we computed the par coupon rates under the assumption of spot gold prices and convenience yields being stochastic and deterministic interest rates.

We found out that \( c^3 \) was insensitive to changes in the convenience yield. However, the rates \( c^1, c^2 \) and \( c^4 \) were found not to change very much with changes in the convenience yield. Also for exercise prices of 400 and below \( c^1 \) and \( c^4 \) are almost equal. Furthermore, \( c^1 \) was again observed to attract the smallest coupon rates. \( c^2 \) was computed to be the highest coupon rates. Also \( c^4 \) was seen to be greater than \( c^3 \) for exercise prices of 700 and above. However at exercise prices of 400 and below, \( c^4 \) was observed to be smaller than \( c^3 \). As our intuition predicted it was observed that \( c^3 \) was close to the deterministic interest rate. The effect of changes in the times to maturity is similar to that observed in tables 9 and 10\(^{12}\).

We conclude this chapter by making the following observations. It was found that allowing the convenience yield to be stochastic did not have a large quantitative impact on theoretical bond values compared to a constant convenience yield assumption. Therefore, this might be safely ignored for other empirical work. Our other observation was that the additional coupon

\(^{12}\)Note throughout that none of the figures presented alter 'cost of capital' to the borrower. Also the \( c^3 \) rates are for default free debt. The rates would be higher in equilibrium if the probability of default was positive.
rate required to compensate lenders for the put option feature of the commodity linked bond (i.e., $c^2 - c^3$) is very small when the option is initially just at or "out of the money" (i.e., $K \leq 400$). This suggests viability of this financing vehicle over the straight debt. The small premium for the put feature, in this case, probably occurs because of the rather low convenience yield of gold. Hence high expected appreciation rate. For commodities with higher convenience yields this premium would undoubtedly be larger.
APPENDIX TO CHAPTER FIVE

RESULTS OF THE SIMULATION EXERCISES
TABLE 1

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to $\text{Max}[1000, (1000/K) \times P]$.*

<table>
<thead>
<tr>
<th>P \ r</th>
<th>0.01</th>
<th>0.04</th>
<th>0.07</th>
<th>0.10</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>769.7225</td>
<td>741.6148</td>
<td>714.5397</td>
<td>688.4593</td>
<td>663.3373</td>
</tr>
<tr>
<td>300</td>
<td>804.9933</td>
<td>781.4941</td>
<td>759.3712</td>
<td>738.5819</td>
<td>719.0858</td>
</tr>
<tr>
<td>500</td>
<td>973.5059</td>
<td>959.3125</td>
<td>946.4711</td>
<td>934.8873</td>
<td>924.4734</td>
</tr>
<tr>
<td>700</td>
<td>1238.2732</td>
<td>1231.2058</td>
<td>1225.0770</td>
<td>1219.7816</td>
<td>1215.2265</td>
</tr>
<tr>
<td>900</td>
<td>1550.4633</td>
<td>1547.2351</td>
<td>1544.5394</td>
<td>1542.2970</td>
<td>1540.4401</td>
</tr>
<tr>
<td>1100</td>
<td>1880.6571</td>
<td>1879.1643</td>
<td>1877.9536</td>
<td>1876.9750</td>
<td>1876.1875</td>
</tr>
<tr>
<td>1300</td>
<td>2217.1467</td>
<td>2216.4308</td>
<td>2215.8637</td>
<td>2215.4160</td>
<td>2215.0638</td>
</tr>
<tr>
<td>1500</td>
<td>2556.0057</td>
<td>2555.6493</td>
<td>2555.3726</td>
<td>2555.1585</td>
<td>2554.9934</td>
</tr>
<tr>
<td>1700</td>
<td>2895.8242</td>
<td>2895.6405</td>
<td>2895.5004</td>
<td>2895.3939</td>
<td>2895.3131</td>
</tr>
<tr>
<td>1900</td>
<td>3236.0553</td>
<td>3235.9576</td>
<td>3235.8842</td>
<td>3235.8293</td>
<td>3235.7883</td>
</tr>
</tbody>
</table>

* This model assumes that the price of gold, $P$, and the interest rate, $r$, are stochastic with a constant convenience yield which is set to equal 0.00143*P.

The bond has a maturity of 5 years, an exercise price, $K$, of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
TABLE 2

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to Min[1000, (1000/K)*P].

<table>
<thead>
<tr>
<th>P \ r</th>
<th>0.01</th>
<th>0.04</th>
<th>0.07</th>
<th>0.10</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>165.2146</td>
<td>165.3820</td>
<td>165.5582</td>
<td>165.7421</td>
<td>165.9325</td>
</tr>
<tr>
<td>300</td>
<td>470.5412</td>
<td>466.1002</td>
<td>461.3241</td>
<td>456.2169</td>
<td>450.7813</td>
</tr>
<tr>
<td>500</td>
<td>642.6260</td>
<td>628.8792</td>
<td>614.8216</td>
<td>600.5090</td>
<td>585.9912</td>
</tr>
<tr>
<td>700</td>
<td>718.4562</td>
<td>697.5833</td>
<td>676.8132</td>
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<td>635.8355</td>
</tr>
<tr>
<td>900</td>
<td>746.8635</td>
<td>722.1515</td>
<td>697.9482</td>
<td>674.2942</td>
<td>651.2193</td>
</tr>
<tr>
<td>1100</td>
<td>757.2671</td>
<td>730.8197</td>
<td>705.1314</td>
<td>680.2135</td>
<td>656.0693</td>
</tr>
<tr>
<td>1300</td>
<td>761.3749</td>
<td>734.1506</td>
<td>707.8187</td>
<td>682.3700</td>
<td>657.7904</td>
</tr>
<tr>
<td>1500</td>
<td>763.1133</td>
<td>735.5295</td>
<td>708.9072</td>
<td>683.2249</td>
<td>658.4583</td>
</tr>
<tr>
<td>1700</td>
<td>763.8923</td>
<td>736.1357</td>
<td>709.3768</td>
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<td>658.7360</td>
</tr>
<tr>
<td>1900</td>
<td>764.2586</td>
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<td>658.8582</td>
</tr>
</tbody>
</table>

* This model assumes that the price of gold, P, and the interest rate, r, are stochastic with a constant convenience yield which is set to equal 0.00143*P.

The bond has a maturity of 5 years, an exercise price, K, of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
TABLE 3

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to 1000.*

<table>
<thead>
<tr>
<th>P \ r</th>
<th>0.01</th>
<th>0.04</th>
<th>0.07</th>
<th>0.10</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>764.6384</td>
<td>736.6982</td>
<td>709.7992</td>
<td>683.9027</td>
<td>658.9710</td>
</tr>
<tr>
<td>300</td>
<td>764.6384</td>
<td>736.6982</td>
<td>709.7992</td>
<td>683.9027</td>
<td>658.9710</td>
</tr>
<tr>
<td>500</td>
<td>764.6384</td>
<td>736.6982</td>
<td>709.7992</td>
<td>683.9027</td>
<td>658.9710</td>
</tr>
<tr>
<td>700</td>
<td>764.6384</td>
<td>736.6982</td>
<td>709.7992</td>
<td>683.9027</td>
<td>658.9710</td>
</tr>
<tr>
<td>900</td>
<td>764.6384</td>
<td>736.6982</td>
<td>709.7992</td>
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<td>658.9710</td>
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<td>736.6982</td>
<td>709.7992</td>
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<td>658.9710</td>
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<td>709.7992</td>
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<tr>
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</tr>
<tr>
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<td>736.6982</td>
<td>709.7992</td>
<td>683.9027</td>
<td>658.9710</td>
</tr>
</tbody>
</table>

* This model assumes that the price of gold, P, and the interest rate, r, are stochastic with a constant convenience yield which is set to equal 0.00143*P.

The bond has a maturity of 5 years and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
TABLE 4

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to (1000/K)*P.*

<table>
<thead>
<tr>
<th>P \ r</th>
<th>0.01</th>
<th>0.04</th>
<th>0.07</th>
<th>0.10</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
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<td>170.2987</td>
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</table>

* This model assumes that the price of gold, P, and the interest rate, r, are stochastic with a constant convenience yield which is set to equal 0.00143*P.

The bond has a maturity of 5 years, an exercise price, K, of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
### TABLE 5

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to \(\text{Max}[1000, (1000/K)\times P]\).*

<table>
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<th>(P \times \delta)</th>
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<th>0.07</th>
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* This model assumes that the price of gold, \(P\), and the convenience yield, \(\delta\), are stochastic with a constant annual interest rate set at 11.40%.

The bond has a maturity of 5 years, an exercise price, \(K\), of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to Min[1000, (1000/K)*P].

<table>
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<th>P</th>
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<th>0.07</th>
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* This model assumes that the price of gold, P, and the convenience yield, δ, are stochastic with a constant annual interest rate set at 11.40%.

The bond has a maturity of 5 years, an exercise price, K, of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
TABLE 7

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to 1000*

<table>
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<th>P \ δ</th>
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</tr>
<tr>
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<td>713.6689</td>
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<tr>
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<td>713.6689</td>
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* This model assumes that the price of gold, P, and the convenience yield, δ, are stochastic with a constant annual interest rate set at 11.40%.

The bond has a maturity of 5 years and makes coupon payments at an annual rate of 4% of the final pay-off.
**TABLE 8**

The Value of the 'hypothetical' gold-linked bond with a final pay-off equal to \(1000/K \times P\). *

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* This model assumes that the price of gold, \(P\), and the convenience yield, \(δ\), are stochastic with a constant annual interest rate set at 11.40%.

The bond has a maturity of 5 years, an exercise price, \(K\), of 700.00 and makes a semi-annual coupon payments at an annual rate of 4% of the final pay-off.
Notations used in tables 9 to 11

\( \tau \) = Time left for the maturity of the bond.

\( K \) = The exercise price of the bond.

\( r \) = The instantaneous spot rate of interest.

\( \delta \) = The instantaneous convenience yield.

\( c_1 \) = The coupon rate that an investor must be paid for holding the gold-linked bond which trades currently for $1000.00 and offers to pay bearers a final payment of \( \text{Max}[1000, (1000/K) * P] \).

\( c_2 \) = The coupon rate that an investor must be paid for holding the gold-linked bond which trades currently for $1000.00 and offers to pay bearers a final payment of \( \text{Min}[1000, (1000/K) * P] \).

\( c_3 \) = The coupon rate that an investor must be paid for holding the gold-linked bond which trades currently for $1000.00 and offers to pay bearers a final payment of 1000.

\( c_4 \) = The coupon rate that an investor must be paid for holding the gold-linked bond which trades currently for $1000.00 and offers to pay bearers a final payment of \( (1000/K) * P \).
TABLE 9

The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.*

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<td>20.19</td>
<td>7.60</td>
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TABLE 9 (CONT'D)
The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.

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The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.*

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* This model assumes that the spot price of gold, $P$, and the interest rates, $r$ are stochastic with no convenience yield. Also the coupon payments are made semi-annually.
TABLE 10
The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario. *

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TABLE 10 (CONT'D)

The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.

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TABLE 10 (CONT'D)
The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.*

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* This model assumes that the price of gold, \( P \), and the interest rate, \( r \), are stochastic with a constant convenience yield which is set to equal \( 0.00143 \times P \). Also the coupon payments are made semi-annually.
TABLE 11

The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.*

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TABLE 11 (CONT'D)

The equilibrium par coupon rates for gold-linked bonds under different pay-off scenario.*

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<th>$\tau$</th>
<th>$\delta$</th>
<th>K</th>
<th>$C^1$</th>
<th>$C^2$</th>
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</tr>
<tr>
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* This model assumes that the price of gold, $P$, and the convenience yield, $\delta$, are stochastic with a constant annual interest rate. The interest rates were set at 0.083 for $\tau = 1$, 0.1140 for $\tau = 5$ and 0.1217 for $\tau = 10$. Also the coupon payments are made semi-annually.
CHAPTER SIX

COMMODITY - LINKED BOND AS AN INSTRUMENT FOR LDCS TO RAISE FOREIGN CAPITAL

1. BACKGROUND

The less developed countries (LDCs) have for years been faced with a colossal foreign debt. The retirement and/or servicing of these debts have been a major problem for these countries and their creditors due to the volatility of the prices of export commodities and hence export revenue of the LDC's. The crisis created by these debt "overhangs" has drawn academicians and practitioners to research into the ways and means for creditors to receive the interest payments on the debt (if not the principal) and also allow the LDCs to continue to contract new loans.

The difficulty faced by the LDCs in meeting their debt obligations could be minimized if they could embark on measures that would protect them from export commodity price volatilities. One measure suggested in the literature is the adoption of hedging strategies by these debt laden developing countries. Rolfo (1980) suggests the use of futures markets by these countries while Lessard (1977) calls for developing countries shifting commodity price risk to the financial markets. A few of the means available to the LDC for hedging, as explained in Fall (1986), are (i) international commodity agreements (ICA's), (ii) futures markets, and (iii) counter-trade.
The ICA’s which have involved commodities like cocoa, coffee, natural rubber, olive oil, sugar and tin have been around for a while. Through these agreements the LDCs and consumer countries sign a pact which seeks to stabilize the world prices of the commodities. The stabilization scheme of price is carried out in a way to attract importers and to satisfy the interest of producing countries. Producing countries, as mentioned in Fall (1986), prefer price supporting systems which are achieved through export quotas or buffer stocks. The ICA’s allow prices of commodities to fluctuate freely with an agreed upon range. Whenever prices fall through the floor, export quotas are applied or the buffer-stock manager will enter the market and purchase sufficient amounts of the commodity. Either action would raise the price of the commodity to fall within the predetermined range. On the other hand, should the prices go through the ceiling, export quotas are relaxed or the buffer-stock manager sells the commodity in the spot market in order to drive the price down in the range. The ICA’s have been fraught with problems. One problem is the asymmetry in the incentives of the importers and the producing LDC’s in entering into the agreement. The consumers (importers) are mainly concerned with higher prices reducing their purchasing power of imports; while the producers are concerned with low prices. Another problem is with the buffer stock. The manager is faced with limited funds to purchase the commodity whenever the price falls though the floor. The problem with the export quotas has been with enforcement of the quota by all the signatories to the agreement.

The LDCs can also use the futures market to hedge against fluctuations in commodity prices. By entering into the futures market LDCs can lock in the price at which the commodity will be sold in the future. However,
futures contracts have their limitations. Firstly, their term to maturity is about two years. Secondly, regulations at the exchanges where they trade restrict investors (and therefore the LDCs) from taking huge positions in the markets for fear of having them corner the market or manipulate prices. These problems suggest that the LDCs may not be in a position to hedge all their exports through the futures market.

Counter trade, which is defined as a financing scheme in which settlements are made in the form of physical goods instead of money, has also been another hedging strategy that is employed by the LDCs. This strategy comes in three forms. One form is the good old barter system. In this case LDCs can have bilateral or multi-lateral arrangements with developed economies in which they could exchange their export commodities for other goods produced by the developed countries. The transactions could take place instantaneously or within a year. The weakness in the barter system is the inability to match the interests of participating parties. This problem is known in the literature as the "double coincidence of wants." The second form of the counter-trade scheme is the "buybacks arrangements" where the LDCs import production facilities and agree to deliver a specified amount of output at some future date. These arrangements most often involve the financing of processing plants in the LDCs. Under this scheme the developing countries are able to lock in the present the future earnings of output. Although the scheme does not insulate the producer countries from the risk of the volatility of commodity prices, it is however project specific. The third form of the counter-trade is what is known as "counter purchase agreement." As presented in Fall (1986), "the arrangement typically has one party importing certain goods or commodities and committing itself to export at
an agreed date, a specified amount of a commodity." By this arrangement we have the LDCs protected against price risk. Furthermore the transaction made under this arrangement is like the importing LDC entering into a combination of spot and forward contracts with the developed economies. Hence under this scheme the LDCs enjoy similar advantages offered by forward contracts.

Despite the availability of the above schemes an enormous debt continues to "overhang" the LDCs. This has brought about a call for the reorganization of the LDC's debt. The United States which has been a major creditor of the LDCs, has come out with two plans to help relieve and solve the debt crisis. The first one, which has been accepted as the "Baker plan," was proposed by the then U.S. Secretary of the Treasury, Mr. James Baker at the October 1985 annual meeting of the IMF and World Bank in Seoul, South Korea. The Baker plan, which comes in three parts, was aimed at solving the debt problem through a program of sustained growth of the economies of the LDCs. The three parts of the plan, as reported in Kenen (1990), are, firstly, the international financial institutions encouraging debtor countries to embark on comprehensive macroeconomic and structural policies which would enhance growth, balance of payment adjustment and reduction in the inflation rate. Secondly, under the supervision of the IMF, the multilateral development banks continue to lend to LDCs with structural adjustment policies. Thirdly, the private banks increase their lending in support of comprehensive economic adjustment programs. It was the aim of Secretary Baker that the use of austere economic measures by the LDCs would help curb inflation and produce trade surpluses needed to service their foreign debt. Furthermore, the structural adjustment and new
foreign lending would ensure economic growth for the LDCs and consequently reduce their debt load. However, the Baker plan was not able to achieve its purpose. The failure may be attributed to the private and the multilateral banks not increasing their lending and the LDCs, for political reasons, were not able to implement the structural adjustment policies. Hence in March 1985 the U.S. changed its strategy on the debt relief program of the LDCs with a scheme known as the "Brady Plan."

The plan which was announced by Mr. Nicholas Brady, U.S. Secretary of the Treasury, calls for the forgiveness of part of the debt of the LDCs. It also proposes that the IMF and the World Bank go to the aid of debt-reduced nations in the form of lending which could be used to collateralize debt-for-bond exchanges at discounts, cash buybacks of debt and also be used to ameliorate the interest payments on new or modified debt contracts. As explained in Kenen (1990) "the IMF and the World Bank adopted guidelines to implement the Brady Plan and the IMF extended new credits to Mexico, Costa Rica and the Philippines in accordance with those guidelines."

These two plans of the United States have led to academic research being conducted on the debt relief of LDCs. Advocates of debt relief, such as Krugman (1989), suggests that reducing the debt of an LDC with a debt overhang could increase that country's economic efficiency and consequently its real income which in turn leads to a reduction in the default risk. Kenen (1990) supports the position of Krugman (1989) and Sach (1988) by arguing that a country with a large debt overhang suffers from a fall in economic efficiency in two ways. Firstly, he (Kenen) maintains that "high debt service payments require high tax rates that discourage capital
formation and the repatriation of flight capital." Secondly, since governments of heavily indebted LDCs are responsible for making the debt-service payments which appear in its budget then it might not institute a devaluation policy that may be required to improve its foreign reserve position and in turn ameliorate the debt crises.

The reasons for the government action, as given in Dornbush (1988), may be due to the fact that devaluation increases the domestic-currency cost of servicing foreign-currency debt, raising the budget deficit, increasing the growth rate of the money supply and consequently a rise in the rate of inflation. These reasons suggest that the governments of the LDCs may resort to the use of inefficient economic methods to produce the trade surpluses needed to service its foreign debt.

Other economists, like Krugman, have used the concept of debt Laffer-curve to argue when forgiveness of debt would be beneficial to LDCs. They propose that if the LDC is on the correct (inclining) side of the debt Laffer curve then debt forgiveness will lead to a reduction in the market value of outstanding debt and, therefore, will be detrimental to creditors. The reverse holds when the debtor country is on the wrong (declining) side of the Laffer Curve. This calls for the determination of the position of a debtor country on the Laffer Curve before a decision of forgiveness might be made.

Froot, Scharfstein and Stein (1989) have pointed out the moral-hazard effect of forgiveness. They argue that the amount of relief required to induce investment in the LDCs may depend on a variety of factors, some of which may be known only by the borrowing country. A borrowing country would know the level of austere economic measures it can impose on its
citizens without causing serious disruptions. Hence, in negotiating for
debt relief, this country might conceal part of the private information it
has on its citizens in order to receive more relief. They (Froot et al.)
believe that these problems can be resolved if the forgiven countries would
index their future debt-service payment to commodity prices. We,
therefore, propose that the LDCs should consider raising capital on the
financial markets through the issue of commodity linked bonds.

2. THE MERITS OF ISSUING COMMODITY - LINKED BONDS

In the introductory chapters we pointed out the economic rational for
the issuing of commodity linked bonds. There we mentioned that the LDCs
would place themselves in an advantageous position by being linked to the
international financial markets through the issue of commodity-linked
bonds. Furthermore, Myers and Thompson (1989) have argued that "by issuing
bonds linked to the prices of commodities which they export, developing
countries can hedge against unexpected deterioration in export earnings."
It is the view of Myers and Thompson (1989) that the debt crisis faced by
the LDCs is primarily due to a fall in exports revenue and a simultaneous
rise in world interest rates and debt-service payments. We therefore agree
with Myers and Thompson (1989) that if debt had been issued in the form of
commodity-linked bonds, debt-service payments would have declined along
with exports prices (or export revenues), thus lightening their debt load.
Opponents against the strategy of LDCs issuing commodity-linked bonds to
hedge against fluctuations in export prices would suggest that LDCs use the
futures market to control for commodity price risk.

Regulators of the futures markets have limits to the movement of the futures price in a single day. Hence, as put by Fall, (1986) futures prices cannot move quickly to accommodate new information. Such limits are not in place for commodity options and, therefore, commodity linked bonds, which are a combination of straight bonds and commodity options, would react to the arrival of new information to form the equilibrium price.

Another advantage commodity linked bonds have over futures contract is that futures contracts have a maturity of less than a year and exist for a limited number of commodities. By the issuance of commodity-linked bonds, the LDCs can have longer term maturity and also index the bonds to any commodity of their choice.

It must be mentioned that the issuance of commodity-linked bonds also minimizes the default risk faced by financiers of LDC loans. However, there is still the need to find a way of addressing the collateral arrangements that must be reached between the LDCs and the developed nations who would be major holders of the bond. A way suggested by Lessard (1977), with which we agree, is that a legal contract be reached between the LDCs and investing nations such that holders of commodity linked bond be empowered to seize any proceeds from the LDCs exports in any of the signatory countries in the case of default. The drawback of such a proposal is the ability to enforce such a contract and the enormous transaction cost that would have to be incurred to settle a dispute between LDC and a bearer of the bond.

The use of commodity linked bonds for external financing would also minimize the enormous transactions cost that would be incurred if the LDCs
were to dynamically hedge their export revenue with futures contract.
CHAPTER SEVEN

SUMMARY AND CONCLUSION

The reason for this research was put across in chapter one as an application of the theory of option pricing theory to value commodity-linked bonds. We also provided the definition of the different types of commodity-linked bonds.

Previous experiences with commodity-linked bonds were provided in chapter two. It was also argued in chapter two that the economic rational for the issue of these type of bonds is that it allows governments and corporations in need of investment funds to share the appreciation of the market value of underlying commodity with the bond holders in return for a lower coupon rate. Alternatively, to minimize the default risk, the borrower may be given the option to pay the minimum of the face value and the value of the reference amount of the commodity at the maturity date.

Chapter three derives the valuation equation for pricing of a commodity-linked bond. The valuation model was obtained by assuming that the value of the commodity-linked bonds is influenced by the price of the reference commodity, the interest rate, the convenience yield and the value of the firm issuing the bond. Under a further assumption that these variables that affect the bond's value follow Wiener diffusion processes, we applied Ito's lemma and standard arbitrage methods to derive a partial differential equation for pricing the bond. The solution to the partial differential equation was purported to be a non-trivial exercise. However by imposing different restrictions on the model we obtained special closed
Properties and extensions to valuing the commodity-linked bonds are contained in chapter four.

By constructing a hypothetical gold-linked bond under different pay-off scenarios we used numerical methods to obtain prices of the gold-linked bonds. Our simulation exercise supported our economic intuition that bonds with a call option feature were found to be the most valuable and therefore attracted the smallest equilibrium par coupon rates. The par coupon rate of the fully indexed bonds were found to be greater than those with a call feature but less than that of conventional bonds. This is due to the fact that the value of the fully indexed bonds is greater than that of a conventional bond but less than bonds with a call feature attached to them. Bonds with the put feature were found to pay the highest par coupon rates since they are least valuable. We also observed that allowing the convenience yield to be stochastic did not have a large quantitative impact on theoretical bond values compared to a constant convenience yield assumption. This suggests that idea of stochastic convenience yield might be safely ignored for other empirical work. Also the additional coupon rate required to compensate lenders for the commodity linked bond with the put option feature (i.e., \( c^2 - c^3 \)) is very small when the option is initially just at or "out of the money". This suggests viability of this financing vehicle over straight debt. The small premium for the put feature, in this case, probably occurs because of the rather low convenience yield of gold (and, therefore, high expected appreciation rate). However, for commodities with higher convenience yields this premium would undoubtedly be larger.

In chapter six we discussed the application of our model to problems
in international finance. The research there argued that the issue of commodity-linked bonds will provide an opportunity for commodity-producing developing countries to borrow at below market interest rates on the international markets. We also suggested that the LDCs could, through the issue of bonds linked to their main exports hedge, against the fluctuations in their export earnings.

We conclude by emphasizing that our pricing model has a wide variety of application. It may be used by resource based industries to raise financial capital at a lower cost of capital on the money markets. It could be applied also to all other financial securities whose pay-offs are contingent on the value of traded commodities.

Like most economic models there are limitations to our model. The viability of a commodity-linked bonds market cannot be guaranteed by simply letting risk prone speculators issue these bonds to risk averse hedgers. Hence the commodity-linked bonds market must be commercially guided and participants must be the major market makers like corporations and governments. In a world of inflation, and the general uncertainties in the markets, the availability of the commodity, indexed to the bonds, greatly reduces the default risk of the bonds. Hence, issuers of the bonds must maintain a threshold level of inventory similar to what banks hold as the reserve requirements. Furthermore, issuers without the commodity must back the bonds with a long position in the forward or futures contracts whose maturity is timed with the redemption date of the bonds.

In our future research we hope to look at the pricing of the commodity-linked bond in an environment of positive transaction costs, taxes and default risk. We also hope to research into econometric techniques like the generalized methods of moments and the ARCH processes.
to estimate the diffusion processes assumed for the underlying state variables of the bond.
REFERENCES


Donbusch, Rudiger. "Our LDC Debts," in Feldstein, Martin, editor,


Ingersoll, J., "The pricing of commodity - linked bonds, Discussion."
Merton, R. C., "Theory of rational option pricing." Bell Journal of


