VALUATION OF MORTGAGE-BACKED SECURITIES

by

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ABSTRACT

This paper attempts to provide a method for the valuation of mortgage-backed securities (MBS).

Specifically, Modified Goldman Sachs model is selected to describe mortgagors’ prepayment behaviour, which takes account of mortgage’s refinancing incentive, aging effect, month effect and burnout effect. In this study, I will use above model to price a pass-through MBS and a plain vanilla MBS respectively.

Furthermore, I use scenarios testing to discuss how MBS’s price and standard deviation (risk) change if mortgage property and model parameters change.

Keywords: Mortgage-backed Securities; Prepayment Model; Monte Carlo Simulation
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1 INTRODUCTION

Mortgage-backed securities (MBS) are investment instruments that represent ownership of a group of mortgages. Usually, financial institutions treat MBS as a method to fund the capital pool brought by lending mortgages.

Up to 2006, MBS market is one of the fastest growing, as well as one of the largest financial markets in the United States. There are now three major agencies in U.S. issuing MBS – Ginnie Mae (the Government National Mortgage Association, or GNMA), Fannie Mae (the Federal National Mortgage Association, or FNMA), and Freddie Mac (the Federal Home Loan Mortgage Corporation, or FHLMC). These agencies support the secondary market by issuing an MBS in exchange for pools of mortgages from lenders. This ensures that lenders have funds to make additional loans. In general, MBS issued through these three government-supported agencies are considered to have no credit or negligible credit risk.

The paper will be structured in this way. In section 2, we will look at risks behind mortgage-backed securities and factors causing those risks. Mortgage pricing models developed in the recent past will be presented in section 3. In section 4, we will discuss some prepayment models and Modified Goldman Sachs model is chosen to price MBS. In section 5, we will choose the interest rate process. Then in section 6, we will introduce the specific CMO structure we will price later – plain vanilla. In the next section, section 7, MBS cash flow streams are discussed. In section 8, the computation process is detailed, and we will present the numerical results. Section 9 will be the last section where the conclusion is presented.
2 RISK OF MORTGAGE-BACKED SECURITIES

The biggest uncertainty of mortgage-backed securities is the prepayment behaviour. In U.S., mortgagors could prepay their loans at any time with no or only little penalty. The above earlier payments lead to unstable future cash flow, thus uncertainty of MBS fair price today. To some extent, mortgages can be treated as a callable bond and people could exercise their option any time they like. So before we begin to price MBS, What we want to know is at what time those borrowers will exercise the call option and what the value of that embedded option is. Roughly, there are three reasons causing the prepayment behaviours.

2.1 Refinancing

When interest rates decline, mortgagors originally signed the higher rate contract would like to refinance through entering a new mortgage contract, thus to take advantage of the prevailing low interest rates.

2.2 House Turnover or House Sales

Many mortgage contracts specify the settlement of remaining loan before selling out the house. So people would like to prepay before mortgage maturity date if they want to sell the house.

2.3 Involuntary Prepayment

Prepayment could occur unexpectedly. Such early termination events include marriage, divorce, death and other extraordinary events.
Lots of studies then report that several factors affect those behaviours – interest rate as mentioned above, mortgage seasoning, borrowers' heteroscedasticity, overall economy, burnout effect, even geographic diversity and schooling time.... It seems to be a tough work if we want to construct a model with these multitudinous factors, but still, several models have been built through chasing one or several above variables.
3 MORTGAGE MODELLING OVERVIEW

Many literatures on mortgage valuation have been published in the past 20 years.

3.1 Structural Model

Kenneth Dunn was the first to introduce an MBS model and then a series of the papers with McConnell in 1981. They treat prepayment as a call option and their model assumes optimal refinancing decisions. Although these models consistently link valuation and prepayment, their prepayment predictions assume an instant prepayment action by all the mortgagors when interest rates hit a threshold. Timmis, Dunn and Spatt, and Johnston and Van Drunen improved this model by incorporating transaction cost or other frictions to explain refinancing decisions. Noticing that some people may not exercise their options when interest rate is low, in the following years, this option-based method has been improved by adding so called option probability model or other determined variables. Downing, Stanton and Wallance established a 2 factor model use both interest rate and house price as a joint threshold and option probability model by a hazard rate function. Other researchers, like Kariya, Ushiyama, Pliska construct a 3 factor model using treasury yield, mortgage rate and house price.

The general idea of the structural model begins with the “public believed” movement of determined variables, usually interest rate and house price. Then Ito’s lemma is used to get a “pure” PDE equation and prepayment noises are added to compute the “real” PDE equation. Finally we solve this PDE equation with boundary condition.

For example, in 2 factor model, where we assume house price and interest move as follows.
\[ dH = (u_H - s)Hdt + \sigma_HHdz_H \]

Where \( s \) represents risk adjusted drift when we applied to the risk neutral world.

\[ dr = \gamma(\theta - r)dt + \sigma_r\sqrt{r}dz_r, \quad \text{CIR (1985)} \]

We could get PDE for mortgage \( X(r,H,t) \) (That is the core idea of the structural model)

\[
\frac{1}{2} H^2 \sigma^2 \frac{\partial^2 X}{\partial H^2} + \rho H \sigma \frac{\partial X}{\partial H} \frac{\partial}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 X}{\partial r^2} + \gamma(\theta - r) \frac{\partial X}{\partial r} + (r - s)H \frac{\partial X}{\partial H} + \frac{\partial X}{\partial t} - rX = 0
\]

Combining this pure equation with boundary condition and option exercise probability model, we could get the final numerical answer. Today, different finite backward methods are used to solve this problem.

It seems to be a compromise of reality, since option-base technique is a “perfect, classical, non-arbitrage” thought that investors act optimally, while option exercise probability model is generally a result of irrational exercise behaviours.

### 3.2 Reduced Form Model

Since above theoretical process asks for very complex calculation, practitioners in industry refer to find other comparatively easy-to-understand solution. Reduced form model assumes option exercise is built in a prepayment model and evaluates mortgage price by discounting future cash flow to its present value. It is easy to modify and needs less computation. No doubt when those academic geniuses were hired by those big investing companies or financial institutions they all choose to use this method.

The basic idea of this method is that the price of MBS today should be the present value of total cash inflow which could be earned in the future.
Here we assume a 30-year mortgage, $v_1, v_2, \ldots$ are explainable variables, $z_1, z_2, \ldots$ are implied yield.

Usually, cash flows are determined by a prepayment model and Monte Carlo simulation is used to simulate interest process and cash flows. In the paper we prefer this method and will use it to compute MBS pool, but first, let us look at some different prepayment models.

\[
P_M = \frac{CF_1(v_1, v_2, \ldots)}{(1 + z_1)} + \frac{CF_2(v_1, v_2, \ldots)}{(1 + z_2)^2} + \ldots + \frac{CF_{360}(v_1, v_2, \ldots)}{(1 + z_{180})^{360}}
\]
4 PREPAYMENT MODEL

4.1 12-year Life Method

According to the material from Federal Housing Administration (FHA), a 30-year mortgage loan will be fully paid in the 12th year, so GNMA obtain MBS price by assuming that the 30-year mortgage will have normal payment in the first 143rd months then repay the all the balance remaining in the 144th month. This coarse method is quickly substituted by other more complex method since there is evidence that the prepayment model is determined by a number of different factors.

4.2 FHA Experience Method

FHA built a yearly survivorship table according to its historical 30-year mortgage data. Prepayment rate is lower during initial loan period, then increases, and reaches its crest between the 5th and 8th year, afterwards the number drops. The advantage of this method is prepayment could vary every year. The disadvantage is that those numbers only reflect the prepayment rate in a specified period and do not take interest rate into account.

4.3 CPR (Conditional Prepayment Rate) Method

There has been a proliferation of CPR models in financial literatures. These models include those by PSA (Public Security Association), Asay, Guillaume and Mattu (1987), Chinloy (1989), Davidson, Herskovitz and Van Drunen (1989), Giliberto and Thibodeau (1989), Richard and Roll (1989), Schwartz and Torous (1989). Among those models, PSA method is an “easy to implement” one and a learning objective in recent CFA curriculum. The last two models are more popular and referenced time after time in the recent papers, so we will discuss them in detailed.
4.3.1 **PSA (Public Security Association) Standard Method**

Public Security Association supposes the prepayment rate in the first year is 0.2% and this rate grows by 0.2% every month until the 30th month. It could be formulated by:

\[ CPR = 6\% \times \frac{t}{30} \quad \text{when } t<30 \]

\[ CPR = 6\% \quad \text{when } t\geq30 \]

PSA treats above formula as its 100% mortgage prepayment benchmark. For different PSA mortgage, for example, 150% PSA, you just need to multiply 1.5 on the above formula.

4.3.2 **Schwartz and Torous's Proportional Hazard Model (PHM)**

They use the following proportional hazard model to compute CPR.

\[ CPR = \pi(t; \nu, \beta) = \pi_0(t; \Gamma, \rho) \times \exp(\beta \nu) \]

\( \nu \) is the vector of determined factors; \( \beta \) is the vector of coefficients.

Particularly, they use the following four factors.

\[ \nu_1(t) = c - l(t-s), s \geq 0 \]

Here \( \nu_1 \) represents the difference between mortgage rate and current long term treasury rate. \( c \) is mortgage rate; \( l \) is long term treasury rate at time \( t-s \), where \( s \) is the lag factor. \( s=3 \) is used in this model.

\[ \nu_2(t) = [c - l(t-s)]^3, s \geq 0 \]

Here \( \nu_2 \) is just cubed equation of \( \nu_1 \), representing the accelerate effect when treasury rate is sufficiently lower than the mortgage rate.
\[ v_3(t) = \ln(BAL(t) / BAL^*(t)) \]

\( v_3 \) represents the burnout effect where \( BAL(t) \) is the mortgage pool outstanding at time \( t \) and \( BAL^*(t) \) is pool outstanding in the absence of prepayment.

\[ v_4(t) = \begin{cases} 1 & \text{when } t = \text{May through August} \\ 0 & \text{when } t = \text{September through April} \end{cases} \]

\( v_4 \) represents the seasoning factor on prepayment decision.

\[ \pi_0(t; \Gamma, p) \] is baseline function which explain the CPR when factor vector \( v=0 \), and they use log-logistic function to describe \( \pi_0 \)

\[ \pi_0(t; \Gamma, p) = \frac{\Gamma p \Gamma t^{p-1}}{1 + \Gamma t^p} \]

This model is intuitive because it divides the mortgage prepayment risk into two parts. The first part, \( \pi_0 \) (also called aging effect) tells us that mortgage will suffer a baseline risk at any time of mortgage period. The second part, \( \exp(\beta v) \) represents all the mortgage specified risk.

### 4.3.3 Richard and Roll's Modified Goldman Sachs Model

The model is first constructed when Richard and Roll were hired by Goldman Sachs as financial advisors and then developed by OTS (Office of thrift supervision). They point out that CPR is determined by the following four factors and measure CPR as the product of those four factors:

\[ \text{CPR} = (\text{Refinancing Incentive}) \times (\text{Seasoning Factor}) \times (\text{Month Factor}) \times (\text{Pool Burnout Factor}) \]
Where CPR is the conditional annual prepayment rate.

Refinancing incentive in their model could be measured by the ratio or the spread of weighted average mortgage coupon rate and mortgage refinancing rate.

\[ RI = a + b \arctan(c + d(WAC - R)) \]

or

\[ RI = a + b \arctan(c + d(WAC / R)) \]

Seasoning reflects the observation that newer loans tend to prepay slower than older or “seasoned” loans. This factor follows the rationale behind the PSA standard prepayment model. This industry convention adopted by the Public Securities Association models mortgage prepayment rates as increasing linearly from 0.2% CPR at issue to 6% CPR at thirty months and then remaining constant. In the paper, the formula for Seasoning is:

\[ \text{Age}(t) = \min(1, \frac{t}{30}) \]

Month factor measures different prepayment behaviours during 1 year. It is believed that prepayments peak in the summer and decrease in the winter. One of the sources of prepayments due to seasonality is housing turnover. This could be due to weather and school schedules. In this paper, the monthly parameters for ith month were taken from the figure in Richard and Roll’s paper (1989).
Burnout takes into account the tendency for prepayment to diminish over time, even when refinancing incentives are favourable. Richard and Roll try to quantify burnout by measuring how much the option to prepay has been deep in-the-money since the pool was issued. They suggest the more the prepayment option has been deep in-the-money, the more burned out the pool is, and the smaller prepayments are, all other things being equal. The Burnout factor is calculated as follows:

\[ BM(t) = 0.3 + 0.7 \frac{B(t)}{B(0)} \]

Where \( B(t) \) is the mortgage balance in the beginning of month \( t \), \( B(0) \) is the initial mortgage balance.
5 INTEREST RATE MODEL

Interest rate is notoriously difficult to predict. Recent interest model could be categorized into two groups – equilibrium models and non-arbitrage model. The main difference of the two lies in the different description of interest rate drift. In equilibrium model, drift is not set as a function of time while non-arbitrage model takes it into consideration. In the paper we will introduce non-arbitrage models which are more commonly used in MBS evaluation.

5.1 Vasicek Model

Vasicek (1977) proposes a kind of interest rate model

\[ dr = a(b - r)dt + \sigma dz \]  

(Vasicek Model)

Where \( a \) is instantaneous drift speed, \( b \) is long term equilibrium interest rate, \( \sigma \) is the instantaneous standard deviation and \( dz \) is a Wiener process.

The Advantage of this model is the built-in mean-reversion idea in the equation. When short-term interest rate deviates from long time equilibrium level \( b \), it will then be reversed to \( b \) at the speed of \( a \).

5.2 CIR Model

Cox, Ingersoll and Ross (1985) improve Vasicek’s model by modifying instantaneous standard deviation from \( \sigma \) to \( \sigma \sqrt{r} \)

\[ dr = a(b - r)dt + \sigma \sqrt{r}dz \]  

(CIR Model)
This model follows the idea of mean-reversion phenomenon of interest rate. Besides, it will not generate negative interest rate. So I pick this model to build our interest rate path. Parameters in our interest rate model are described in the "Mortgage Pool Pricing" section.
6 PLAIN VANILLA MBS

Mortgage pool is not expected to be sold in one pile. Usually, those agencies will issue securities by redirecting the cash flows of mortgages in order to attract different investors. They redirect cash flows in such way that one classes of bond will be only exposed to one form of prepayment risk. When the cash flows of mortgage-backed products are redistributed to different bond classes, the resulting securities are called collateralized mortgage obligations (CMOs). Plain Vanilla CMOs, also called Sequential Pay CMOs, is the most basic class within CMOs structure. This class reallocates collateral principal payments sequentially to a series of bonds, namely A, B, C, D, .... Bond A will receive principal payments, including prepayments first. When it is fully retired; then principal payments are redirected to the next sequential class, bond B. This process continues until the last sequential class is fully retired. While one class is receiving principal payments, the other existing classes only receive monthly interest payments at their coupon rate on their principal.
7 MBS CASH FLOWS

As mentioned above, mortgage is just like an ordinary coupon bond (without final principal payment) if there is no prepayment. The interest part and principal part in every schedule payment under no prepayment assumption will follow the pattern in figure 2.

Figure 2  Cash Flow Pattern for Regular Bonds

Source: www.riskglossary.com

But when prepayments happen, the interest part and the principal part will be uncertain. It may follow the pattern below.
In Fabozzi’s paper, the cash flows can be calculated using the following formula:

\[ MP(t) = B(t) \times \frac{WAC/12 \times \frac{1}{1 - \left(1 + \frac{WAC}{12}\right)^{-WAM \times t}}}{12} \]

Where \( MP(t) \) is the scheduled mortgage payment for period \( t \), \( B(t) \) is the mortgage balance remaining at \( t \), \( WAC \) is weighted average coupon rate and \( WAM \) is the weighted average maturity of mortgage.

\[ IP(t) = B(t) \times \frac{WAC}{12} \]

Where \( IP(t) \) is the interest payment at \( t \).

\[ SP(t) = MP(t) - IP(t) \]

Where \( SP(t) \) is the scheduled principal payment for period \( t \);

\[ PP(t) = SMM(t) \times (B(t) - SP(t)) \]

Where \( PP(t) \) is the principal prepayment for period \( t \) and \( SMM \) is single month mortality and computed as
\[ SMM(t) = 1 - \sqrt[2]{1 - CPR(t)} \]

We could calculate cash flow using CPR(t), the conditional prepayment rate, which was derived from our prepayment model. Once CPR(t) is known, everything else can be calculated and total cash flow at time t CF(t) will equal:

\[ CF(t) = SP(t) + PP(t) + IP(t) \]

Then we could compute the price of mortgage pool by discounting monthly cash flow.

\[ P_M = \frac{CF(1)}{(1 + S_1 + OAS)} + \frac{CF(2)}{(1 + S_2 + OAS)^2} + ... + \frac{CF(360)}{(1 + S_{360} + OAS)^{360}} \]

Where S is the spot yield curve, computed by our simulated interest rate process.

\[ S_n = \sqrt[2]{(1 + r_1)(1 + r_2)...(1 + r_n)} - 1 \]

OAS is the implied spread from historic MBS data that makes the computed price of mortgage pool equal to its market price. In this paper OAS is set to zero, since we only want to compare pure outcomes between different classes.

The only difference between an entire mortgage pool and a Plain Vanilla MBS is that all the principal payments and prepayments will be used to retire first sequential bonds, say Bond A. The principal of Bond B will begin to decline until Bond A is fully amortized, so the calculation will be as follows if we assume there are two classes, Bond A and Bond B.

\[ P_{BondA} = \frac{CF(1)}{(1 + S_1 + OAS)} + \frac{CF(2)}{(1 + S_2 + OAS)^2} + ... + \frac{CF(t)}{(1 + S_t + OAS)^t} \]

Where t1 is the time Bond A is fully retired and cash flow CF(t) will be
Where $IPA(t)$ is the interest payment according to the remaining balance of bond A at time $t$, $SPT(t)$ and $PPT(t)$ is the schedule principal payment and prepayment computed from the total mortgage pool.

$$CF(t) = SPT(t) + PPT(t) + IPA(t)$$

$P_{BondB} = \frac{IP}{(1 + S_1 + OAS)} + \frac{IP}{(1 + S_1 + OAS)^2} + \ldots + \frac{CF(t_1)}{(1 + S_{t_1} + OAS)^{t_1}} + \ldots + \frac{CF(t)}{(1 + S_t + OAS)^t}$

where $IP$ is fixed interest payment of bond $B$ when bond $A$ is in the process of retiring.

$CF(t_1) = SPT(t_1) + PPT(t_1) + IPB(t_1)$ is the first cash flow of bond $B$ that contains prepayment and schedule principal payment.
8 MBS PRICING

After every computation processes are introduced, we could then price MBS through the following flow chart.

Figure 4  Computation Process

8.1 Pricing a Mortgage Pool

In this paper, I assume a mortgage-backed security with collateral value equal to 1000000. Its principal values, coupon rates, issue date, maturity are described as follows (assuming Gross coupon rate = Net coupon rate, Settle date = Issue date).
First, I use CIR model to generate interest rate process. Particularly, I use the discretization form of CIR model, an AR(1) model to generate interest rate path.

\[ r(t+1) - b = \phi(r(t) - b) + \sigma_d \sqrt{r(t)} \epsilon_i \]

Above AR(1) model is said to have the same mean and stationary variance if

\[ \phi = e^{-a} \text{ and } \sigma_d^2 = \sigma^2 \frac{1 - \phi^2}{2a} \]

The long-run average, \( b \), was set to 7.75%. \( a \) was set to 0.1 and the volatility term, \( \sigma \), was set to 0.0225. These parameters are picked from Buser and Hendershott's article (1984). The starting interest rate was set to 10%. Figure 5 plots one path of the interest rate process used in my simulation.
Here I choose Modified Goldman Sachs prepayment model to valuate MBS. Particularly, refinancing incentive is computed using following formula and parameters are picked from Akesson and Lehoczky's simulation paper (2000).

\[ RI = a + b \cdot \arctan(c + d(WAC - R)) \]

\[ a = 0.28 \]

\[ b = 0.14 \]

\[ c = -8.571 \]

\[ d = 430 \]

WAC is weight average coupon rate
R(t,T) is calculated using following method.

According to Hull (2000), if interest rate follows CIR movement,

\[ dr = a(b - r)dt + \sigma \sqrt{r} dz \]

then bond price \( P(t,T) \) will equal to

\[ P(t,T) = e^{-R(t,T)(T-t)} = A(t,T)e^{-B(t,T)\gamma r} \]

where \( B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{\gamma + a(e^{\gamma(T-t)} - 1) + 2\gamma} \)

\[ A(t,T) = \left[ \frac{2\gamma e^{(\gamma + a)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma} \right]^{2ab/\sigma^2} \]

with \( \gamma = \sqrt{a^2 + 2\sigma^2} \)

so \( R(t,T) = -\frac{1}{T-t} \ln A(t,T) + \frac{1}{T-t} * B(t,T) * r(t) \)

Figure 6 plots one path of the R process used in my simulation.
Recall that $RI = a + b \cdot \text{arctan}(c + d(WAC - R))$ and $\text{CPR} = (\text{Refinancing Incentive}) \times (\text{Seasoning Factor}) \times (\text{Month Factor}) \times (\text{Pool Burnout Factor})$

Figure 7 then plots one path of the CPR process used in my simulation.
I ran 1000 simulations to get the MBS price. Making it simple, I set the market price of the collateral and the bonds to their respective present values, assuming OAS to equal 0 and observe what will happen if we change the parameters of the model.
8.1.1 Change Coupon Rate

Table 2 Estimated Results for Mortgage Pool (Different Coupon Rate)

<table>
<thead>
<tr>
<th>Coupon rate</th>
<th>4%</th>
<th>6%</th>
<th>8%</th>
<th>10%</th>
<th>12%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pool Price</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>804,809</td>
<td>927,047</td>
<td>1,054,560</td>
<td>1,188,868</td>
<td>1,325,394</td>
</tr>
<tr>
<td>r(0)=8%</td>
<td>784,392</td>
<td>904,781</td>
<td>1,030,721</td>
<td>1,164,248</td>
<td>1,298,989</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>773,483</td>
<td>890,820</td>
<td>1,016,203</td>
<td>1,147,316</td>
<td>1,281,814</td>
</tr>
<tr>
<td>Pool Std</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>26,203</td>
<td>27,241</td>
<td>27,707</td>
<td>27,684</td>
<td>27,633</td>
</tr>
<tr>
<td>r(0)=8%</td>
<td>25,597</td>
<td>25,969</td>
<td>28,395</td>
<td>27,883</td>
<td>28,724</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>25,459</td>
<td>27,957</td>
<td>27,903</td>
<td>28,832</td>
<td>30,499</td>
</tr>
</tbody>
</table>

Table 2 above presents the price and standard deviation of the mortgage pool. Here I set initial interest rate to 5%, 8% and 10%, one higher than, one lower than and one close to the long-term rate and pick different coupon rate (4%, 6%, 8%, 10% and 12%). As coupon rate increases, prepayment becomes more probable, thus the price of mortgage pool increase. Compared to initial interest rate 10%, pool prices at 5% interest rate level are generally higher, reflecting a more frequent prepayment behaviour in the beginning period when interest rates have not yet converged to the long term rate, 7.75%. Figure 8 below presents the relationship between pool price and coupon rate at two initial interest level (5% and 10%).
8.1.2 Change Mortgage Term

Table 3  Estimated Results for Mortgage Pool (Different Mortgage Term)

<table>
<thead>
<tr>
<th>Mortgage Term</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pool Price</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>1,038,228</td>
<td>1,044,449</td>
<td>1,048,359</td>
<td>1,050,507</td>
<td>1,054,560</td>
</tr>
<tr>
<td>r(0)=8%</td>
<td>1,016,757</td>
<td>1,022,209</td>
<td>1,026,044</td>
<td>1,029,940</td>
<td>1,032,925</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>1,002,264</td>
<td>1,008,403</td>
<td>1,011,768</td>
<td>1,013,658</td>
<td>1,016,203</td>
</tr>
<tr>
<td>Pool Std</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>17,701</td>
<td>21,466</td>
<td>22,594</td>
<td>24,728</td>
<td>25,885</td>
</tr>
<tr>
<td>r(0)=8%</td>
<td>20,291</td>
<td>22,474</td>
<td>24,478</td>
<td>25,702</td>
<td>27,306</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>20,444</td>
<td>22,909</td>
<td>25,003</td>
<td>26,313</td>
<td>27,903</td>
</tr>
</tbody>
</table>

Table 3 above presents the price and standard deviation of the mortgage pool in different mortgage term. We could see from the table that both price and standard deviation accrue when we increase the mortgage term. It can be indicated that uncertainty of prepayment increases when the mortgage term is much longer.

8.1.3 Change CIR Parameters

Interest rate played an important role when we valuate the mortgage-backed securities. Here I present the MBS results corresponding to different mean reverse speed ($a=0.04, a=0.07, a=0.1, a=0.13, a=0.16$) and different interest rate volatility ($\sigma=0.0025, \sigma=0.0125, \sigma=0.0225, \sigma=0.0325$). The results are shown in table 4 and table 5 respectively.
Table 4  Estimated Results for Mortgage Pool (Different Mean Reverse Speed)

<table>
<thead>
<tr>
<th>Mean Reverse Speed</th>
<th>0.04</th>
<th>0.07</th>
<th>0.10</th>
<th>0.13</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pool Price</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>1,059,097</td>
<td>1,056,553</td>
<td>1,054,560</td>
<td>1,050,852</td>
<td>1,049,826</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>988,296</td>
<td>1,007,704</td>
<td>1,016,203</td>
<td>1,020,210</td>
<td>1,023,411</td>
</tr>
<tr>
<td>Pool Std</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>41,170</td>
<td>31,736</td>
<td>27,707</td>
<td>21,885</td>
<td>18,602</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>50,562</td>
<td>34,147</td>
<td>27,903</td>
<td>22,183</td>
<td>18,239</td>
</tr>
</tbody>
</table>

Table 5  Estimated Results for Mortgage Pool (Different Interest Volatility)

<table>
<thead>
<tr>
<th>Interest Volatility</th>
<th>0.0025</th>
<th>0.0125</th>
<th>0.0225</th>
<th>0.0325</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pool Price</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>1,056,620</td>
<td>1,055,040</td>
<td>1,054,560</td>
<td>1,051,589</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>1,017,040</td>
<td>1,016,982</td>
<td>1,016,203</td>
<td>1,012,153</td>
</tr>
<tr>
<td>Pool Std</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td>3,107</td>
<td>15,625</td>
<td>27,707</td>
<td>35,949</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td>3,181</td>
<td>15,953</td>
<td>27,903</td>
<td>38,282</td>
</tr>
</tbody>
</table>

It seems CIR parameters have a huge effect on the pool standard deviation. Pool standard deviation soars if we decrease mean reverse speed and increase interest volatility. What is worth mentioning is that, when initial rate is 10%, pool prices increase with the increase of mean reverse speed; It changes in opposite direction when initial rate is set to 5%. It could be explained by the intuition that, at a lower initial rate, more prepayment behaviours are expected at lower mean reverse speed than at higher reverse speed, it will increase the cash flows in the first several years, thus increase the pool price. On the other hand, at a higher rate, more prepayment behaviours are expected under higher mean reverse speed, so the pool price will be higher.
8.2 Pricing a Plain Vanilla MBS

After we analyze the mortgage pool, let us look at a simple CMO structure. Table 6 below presents a four tranche Plain Vanilla MBS. Coupon rates in every tranche are picked to make the sum of tranche coupon payments equal to pool coupon payments.

Table 6 CMO Structure

<table>
<thead>
<tr>
<th>Principal</th>
<th>Issue Date</th>
<th>Mortgage Term</th>
<th>Coupon Rate</th>
<th>Coupon Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000000</td>
<td>Jan 01, 2000</td>
<td>30 year</td>
<td>8%</td>
<td>80000</td>
</tr>
<tr>
<td>Tranche A</td>
<td>350000</td>
<td></td>
<td>8.2%</td>
<td>28700</td>
</tr>
<tr>
<td>Tranche B</td>
<td>200000</td>
<td></td>
<td>9%</td>
<td>18000</td>
</tr>
<tr>
<td>Tranche C</td>
<td>300000</td>
<td></td>
<td>7.1%</td>
<td>21300</td>
</tr>
<tr>
<td>Tranche D</td>
<td>150000</td>
<td></td>
<td>8%</td>
<td>12000</td>
</tr>
<tr>
<td>Total Coupon Payment</td>
<td></td>
<td></td>
<td></td>
<td>80000</td>
</tr>
</tbody>
</table>

The main difference is we now have interest payment for every tranche and monthly principle payments are paid in sequence. Let $IP(t)_i$ denote the interest payments for tranche i in month t.

$$IP(t)_i = B_i(t) \times \frac{WAC_i}{12}$$

I ran 500 simulations to get tranche prices and I will repeat the similar process (set initial rate to 5% and 10%) to analyze the prices and risks under different situations.
8.2.1 Change Mortgage Term

Table 7 Tranche Price (Different Mortgage Term)

<table>
<thead>
<tr>
<th>Mortgage Term</th>
<th>Tranche A</th>
<th>Tranche B</th>
<th>Tranche C</th>
<th>Tranche D</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>359,087</td>
<td>213,692</td>
<td>300,735</td>
<td>158,501</td>
<td>1,032,015</td>
</tr>
<tr>
<td>15</td>
<td>360,544</td>
<td>216,450</td>
<td>299,524</td>
<td>160,278</td>
<td>1,036,796</td>
</tr>
<tr>
<td>20</td>
<td>361,634</td>
<td>218,640</td>
<td>297,814</td>
<td>161,276</td>
<td>1,039,364</td>
</tr>
<tr>
<td>25</td>
<td>362,935</td>
<td>220,795</td>
<td>297,235</td>
<td>161,994</td>
<td>1,042,958</td>
</tr>
<tr>
<td>30</td>
<td>363,449</td>
<td>221,831</td>
<td>296,145</td>
<td>162,407</td>
<td>1,043,832</td>
</tr>
</tbody>
</table>

Table 7 above presents the tranche prices in different mortgage term. As expected, tranche A has smaller standard deviation if we take principal magnitude into account. It can be
explain by the fact that senior tranches are retired at earlier period when interest rate is still away from the long term equilibrium rate, so people are pretty sure whether they should prepay or not. Surprisingly, all tranche prices are growing with the increase of mortgage term except tranche C. Besides, tranche C has the largest standard deviation, almost twice as large as tranche A. These findings may reveal the fact that tranche C’s payments are in the most unstable period and many elements impact the price of that tranche.
### 8.2.2 Change CIR Parameters

#### Table 8 Tranche Price (Different Mean Reverse Speed)

<table>
<thead>
<tr>
<th>Mean Reverse Speed</th>
<th>0.04</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tranche Price</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>363,011</td>
<td>363,577</td>
<td>363,449</td>
<td>362,738</td>
<td>361,508</td>
</tr>
<tr>
<td>Tranche B</td>
<td>218,311</td>
<td>220,967</td>
<td>221,831</td>
<td>221,998</td>
<td>221,071</td>
</tr>
<tr>
<td>Tranche C</td>
<td>304,673</td>
<td>299,594</td>
<td>296,145</td>
<td>295,061</td>
<td>293,784</td>
</tr>
<tr>
<td>Tranche D</td>
<td>166,310</td>
<td>163,878</td>
<td>162,407</td>
<td>161,892</td>
<td>161,248</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>1,052,304</td>
<td>1,048,015</td>
<td>1,043,832</td>
<td>1,041,689</td>
<td>1,037,611</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>343,877</td>
<td>348,868</td>
<td>351,102</td>
<td>352,266</td>
<td>353,093</td>
</tr>
<tr>
<td>Tranche B</td>
<td>204,394</td>
<td>210,913</td>
<td>213,627</td>
<td>215,038</td>
<td>215,570</td>
</tr>
<tr>
<td>Tranche C</td>
<td>279,161</td>
<td>284,634</td>
<td>284,698</td>
<td>285,615</td>
<td>286,080</td>
</tr>
<tr>
<td>Tranche D</td>
<td>153,513</td>
<td>156,106</td>
<td>156,239</td>
<td>156,776</td>
<td>157,035</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>980,945</td>
<td>1,000,521</td>
<td>1,005,666</td>
<td>1,009,697</td>
<td>1,011,778</td>
</tr>
<tr>
<td><strong>Tranche Std</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>4,482</td>
<td>4,713</td>
<td>5,064</td>
<td>4,754</td>
<td>4,285</td>
</tr>
<tr>
<td>Tranche B</td>
<td>5,629</td>
<td>5,214</td>
<td>5,543</td>
<td>4,695</td>
<td>4,307</td>
</tr>
<tr>
<td>Tranche C</td>
<td>20,208</td>
<td>13,490</td>
<td>10,933</td>
<td>8,278</td>
<td>7,565</td>
</tr>
<tr>
<td>Tranche D</td>
<td>13,067</td>
<td>7,857</td>
<td>6,262</td>
<td>4,655</td>
<td>4,430</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>40,865</td>
<td>28,674</td>
<td>25,978</td>
<td>20,962</td>
<td>19,366</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>10,736</td>
<td>7,709</td>
<td>5,961</td>
<td>4,835</td>
<td>4,281</td>
</tr>
<tr>
<td>Tranche B</td>
<td>9,979</td>
<td>7,142</td>
<td>5,601</td>
<td>4,702</td>
<td>4,190</td>
</tr>
<tr>
<td>Tranche C</td>
<td>22,699</td>
<td>15,466</td>
<td>10,395</td>
<td>8,537</td>
<td>7,080</td>
</tr>
<tr>
<td>Tranche D</td>
<td>13,267</td>
<td>8,880</td>
<td>6,027</td>
<td>4,952</td>
<td>3,997</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>53,384</td>
<td>36,844</td>
<td>26,041</td>
<td>21,570</td>
<td>18,453</td>
</tr>
</tbody>
</table>
### Table 9  Tranche Price (Different Interest Volatility)

<table>
<thead>
<tr>
<th>Interest Volatility</th>
<th>0.0025</th>
<th>0.0125</th>
<th>0.0225</th>
<th>0.0325</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tranche Price</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r(0)=5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>364,005</td>
<td>363,813</td>
<td>363,449</td>
<td>361,758</td>
</tr>
<tr>
<td>Tranche B</td>
<td>222,955</td>
<td>222,762</td>
<td>221,831</td>
<td>219,340</td>
</tr>
<tr>
<td>Tranche C</td>
<td>295,591</td>
<td>295,899</td>
<td>296,145</td>
<td>297,417</td>
</tr>
<tr>
<td>Tranche D</td>
<td>162,255</td>
<td>162,360</td>
<td>162,407</td>
<td>162,745</td>
</tr>
<tr>
<td>Total</td>
<td>1,044,806</td>
<td>1,044,834</td>
<td>1,043,832</td>
<td>1,041,260</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>351,434</td>
<td>351,310</td>
<td>351,102</td>
<td>350,454</td>
</tr>
<tr>
<td>Tranche B</td>
<td>214,749</td>
<td>214,458</td>
<td>213,627</td>
<td>212,184</td>
</tr>
<tr>
<td>Tranche C</td>
<td>284,384</td>
<td>284,339</td>
<td>284,698</td>
<td>286,324</td>
</tr>
<tr>
<td>Tranche D</td>
<td>156,120</td>
<td>156,034</td>
<td>156,239</td>
<td>156,474</td>
</tr>
<tr>
<td>Total</td>
<td>1,006,688</td>
<td>1,006,141</td>
<td>1,005,666</td>
<td>1,005,435</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Tranche Std</strong></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>r(0)=5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>600</td>
<td>2,994</td>
<td>5,064</td>
<td>7,069</td>
</tr>
<tr>
<td>Tranche B</td>
<td>654</td>
<td>3,274</td>
<td>5,543</td>
<td>6,840</td>
</tr>
<tr>
<td>Tranche C</td>
<td>1,221</td>
<td>6,109</td>
<td>10,933</td>
<td>14,530</td>
</tr>
<tr>
<td>Tranche D</td>
<td>716</td>
<td>3,492</td>
<td>6,262</td>
<td>8,292</td>
</tr>
<tr>
<td>Total</td>
<td>2,975</td>
<td>14,846</td>
<td>25,978</td>
<td>33,612</td>
</tr>
<tr>
<td>r(0)=10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tranche A</td>
<td>719</td>
<td>3,505</td>
<td>5,961</td>
<td>8,261</td>
</tr>
<tr>
<td>Tranche B</td>
<td>707</td>
<td>3,404</td>
<td>5,601</td>
<td>7,335</td>
</tr>
<tr>
<td>Tranche C</td>
<td>1,214</td>
<td>5,926</td>
<td>10,395</td>
<td>14,939</td>
</tr>
<tr>
<td>Tranche D</td>
<td>694</td>
<td>3,361</td>
<td>6,027</td>
<td>8,603</td>
</tr>
<tr>
<td>Total</td>
<td>3,138</td>
<td>15,208</td>
<td>26,041</td>
<td>36,211</td>
</tr>
</tbody>
</table>

Table 8 and 9 examine the CIR parameter effect on the tranche price and standard deviation. Compared to junior tranches, senior tranches are less influenced by the change of interest volatility and mean reverse speed. Their standard deviations change slower than the junior ones which accord with the thought that senior tranches have comparatively stable cash flow streams.
9 CONCLUSION

Our paper provides a big picture of mortgage-backed securities and a general idea of how to price mortgage-backed securities. In the paper, we use reduced form method to pricing MBS and CMOs. Interest rates were simulated by using CIR model. A set of cash flow was generated by the prepayment model and final price is computed by discounting all the cash flows. The numerical results presented help us determine how risk is allocated in mortgage pool and each tranches.

The results reveal that the price of mortgage pool is unconditionally positive related with mortgage term and coupon rate and negative related with interest volatility. When we change the interest reverse speed, the pool price gives us two different tendencies. It goes up with the increase in the reverse speed, if we set initial rate higher than mortgage coupon rate and falls if we set initial rate lower than mortgage coupon rate. Mortgage’s standard deviation goes up with the increase of mortgage term, coupon rate and interest volatility. It rocket even we change volatility by 1 percent. Its risk declines if we increase the interest reverses speed, which could be explained by the more predictable cash flows.

Things become more complex when we deal with a plain vanilla structure MBS. The numbers still realize some tendencies of the change of the tranche price and tranche standard deviation. However, those tables reveal that risks have been redistributed in tranches in some forms and could be studied by changing the coefficients. In this paper, I only assume a plain vanilla structure; however, my simulations can be easily implemented to any other type of CMO and more complicated CMO structures could be studied by this method.
REFERENCE LIST


