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AN INVESTIGATION OF
THE PROBLEMS OF LEARNING FRACTIONS
AND THEIR MEANINGS

by

ROGER VICTOR MARTIN SANDFORD
Bachelor of Education (Secondary)
University of British Columbia, 1970

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS (EDUCATION)
in the Faculty
of
Education

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AN INVESTIGATION OF THE PROBLEMS OF LEARNING FRACTIONS AND THEIR MEANINGS.

Author:

Roger Victor Martin Sanford

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18th September, 1974

(date)
It was established that there is widespread agreement that children in school have difficulty mastering fractions. Since there is little research in this area it was decided to examine the teacher training literature to find out why the learning of fractions presents such a problem.

Firstly, it became evident that, although there are many physical applications of rational number structure and symbolism, the fact that they all have a common operational structure is not fully utilized in school. In practice rational numbers are often taught with the meaning confined to one particular application, namely fractional parts of a whole object.

Secondly, it was established that the teacher training literature indicated mathematical confusion between the many different applications of the rational number symbolism. On the other hand, the mathematical literature read did not treat fractions extensively enough for the mathematical questions requiring answers to be dealt with.

It was decided to attempt to correct this defect in the literature and a mathematical treatment of number using
Peano's axioms was developed so that answers to some of the questions were obtained.

One suggestion for improving the learning of rational number symbolism and operational structure is that they be taught prior to, and free of, the many applications commonly met with in school so that when each application is eventually studied the structure will not be psychologically tied to some limited meaning. It was considered necessary to test this suggestion with children in order to get some indication as to whether it was based on real considerations for effective learning.

A small field study was conducted with two groups of grade V students. One group was taught fractions as equivalence classes of ordered pairs and their operations, free from reference to any physical applications. The other group was taught the same material using more standard approaches and several physical applications.

Questions purporting to test understanding were extracted from established standardized tests and a teacher-constructed test for understanding was produced. An established standardized test for fraction computation skills was also used. Both tests were given before and after the teaching period and the data analysed using appropriate statistical techniques to test the following two null hypotheses:
I. There will be no significant difference between the means of the test scores of the group taught using the traditional approach to fractions and those taught the same material using ordered pair symbolism and no reference to physical applications, with respect to understanding.

II. There will be no significant difference between the means of the test scores of the group taught using the traditional approach to fractions and those taught the same material using ordered pair symbolism and no reference to physical applications, with respect to computational skills with fractions.

The results of the statistical analysis for both hypotheses indicated no significant difference between the two approaches.

From these results we conclude that an introduction to fractions using ordered pair symbolism and no reference to physical applications is a proposal that merits more exact research techniques at a future date.
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And, finally, the Board of the School District of Cowichan whose co-operation made possible the experiment.
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CHAPTER I

THE NATURE AND THE PURPOSE OF THE STUDY

Introduction

This study is concerned with those ordered pairs having an equivalence relation as follows:

\[(a, b) = (c, d) \text{ if and only if } a \cdot d = b \cdot c\]

\[a, b, c, d \in \mathbb{I} \quad b, d \neq 0\]

For the purposes of this study such pairs will be called 'fraction ordered pairs'. Since such fraction ordered, pairs are traditionally written \(\frac{a}{b}, \frac{c}{d}\) etc., the above equivalence becomes:

\[\frac{a}{b} = \frac{c}{d} \text{ if and only if } a \cdot d = b \cdot c\]

It is this symbol form that is usually presented to children of third grade and upwards and called 'fractions'. In this study such 'fractions' will be called 'fraction symbols' and it will be established that students seem to have difficulty coping with them.

Very little research seems to have been done on the reasons for this learning difficulty and because of this the teacher training literature was examined as well. As will be seen in this study several reasons were given but there
was a wide variation of opinion, and even conflict, among the authorities. Some admitted to mathematical confusions while fraction symbolism in the mathematical literature was not treated extensively. Consequently it was decided to develop an axiomatic approach from one of the mathematical treatments so that some of the important mathematical questions could be answered.

When this is done one can appreciate that a fraction symbol has many meanings, although as introduced by many teachers, it often becomes associated with only one meaning. Furthermore if, as the young student examines each application, there is a tendency to treat each one in isolation the pedagogical value of the shared structures is missed.

It was decided that a curriculum change might avoid the two above problems. Fraction symbols could initially be introduced free from any application. Subsequently, when applications are dealt with, the previously learned pure mathematical structure would more likely be emphasised and play an integrating role.

Since none of the teacher training texts mentions the possibility of this kind of introduction and subsequent approach, it was considered necessary, though not sufficient, to do a small field study. Necessary, to give indications that the suggestion was probably firmly based on real considerations for effective learning by children; not sufficient because it is quickly apparent that, to test
thoroughly such an introduction and subsequent approach, a much larger field study would be essential with time and facilities available to enable greater equalization of the variables emphasised. For these reasons, the field study became a necessary but not sufficient follow-up to the theoretical development and because of its limited scope could only be an indication of a need for further research.

2. Theoretical Development

The equivalence relation for the fraction symbol is defined as follows:

\[ \frac{a}{b} = \frac{c}{d} \text{ if and only if there exists } e \frac{f}{g} \]

such that \( \frac{a}{b} = ne \) and \( \frac{c}{d} = me \) where all \( \frac{b}{nf} \) and \( \frac{d}{mf} \) letters represent integers and neither \( b \) nor \( d \) is equal to zero.

From this definition one may readily derive the following theorems that:

(i) \( \frac{a}{b} = \frac{c}{d} \) if and only if \( a \cdot d = b \cdot c \)

and (ii) \( \frac{a}{b} = n \cdot a \) which is the equivalence relation \( \frac{b}{n \cdot b} \)

commonly used in grade V.

The four operations \( \text{Op1}, \text{Op2}, \text{Op3}, \text{Op4} \) are defined as follows:

\[ \text{Op1: } \frac{a}{b} \text{ Op1 } \frac{c}{d} = \frac{ad+bc}{bd} \]
\[ \text{Op2: } \frac{a}{b} \text{ Op2 } \frac{d}{d} = \frac{ad-bc}{bd} \]
The student working through grades K-12 will usually meet fraction symbols exhibiting the above structure at several stages for different applications.

From each physical situation is abstracted two integers each of which can be given a physical meaning and, since the above structure is probably, in itself, not very difficult to understand, it is possible that the ambiguities of the fraction symbol inherent in some of the physical settings can create confusion, especially if the student psychologically tends to think of the fraction symbol in one particular setting.

Normally the student first meets ordered pairs exhibiting the above structure when he is introduced to division of whole numbers. Any division requires two numbers (or terms) and if we define natural number quotients, for the purpose of this study, as those divisions producing a remainder of zero and give them the symbol \( \frac{a}{b} \), we will see that they have the same structure as shown above.

For example

\[
\frac{4}{2} = \text{remainder 0} \quad \text{and} \quad \frac{16}{8} = \text{remainder 0}
\]

so it would be true to say

\[
\frac{4}{2} = \frac{16}{8} \quad \text{and} \quad 4 \cdot \frac{8}{8} = 16 \cdot \frac{2}{2} \quad \text{which accords}
\]
with theorem (i) above.

Again \( \frac{4}{2} = 2 \) remainder 0 and \( \frac{8}{4} = 2 \) remainder 0 and since 2 remainder 0 plus 2 remainder 0 equals 4 remainder 0 we can write

\[
\frac{4 + 8}{2} = \frac{4 \cdot 4 + 8 \cdot 2}{2 \cdot 4}
\]

since \( \frac{4 \cdot 4 + 8 \cdot 2}{2 \cdot 4} = \frac{32}{8} = 4 \) remainder 0.

In a similar way we see that

\[
\frac{4 - 2}{2} = \frac{4 \cdot 1 - 2 \cdot 2}{2 \cdot 1}
\]

\[
\frac{8 \times 4}{4} = \frac{8 \cdot 4}{2 \cdot 4}
\]

and \( \frac{8 \div 4}{4} = \frac{8 \cdot 2}{4 \cdot 4} \)

By the similarity in the patterning with the structure of the fraction symbol equivalence relation and its operations we see that the two systems are isomorphic and \( \text{Op1} \) corresponds to addition, \( \text{Op2} \) corresponds to subtraction, \( \text{Op3} \) corresponds to multiplication and \( \text{Op4} \) corresponds to division.

The abstraction expressed as above is usually in school taken from two different physical settings. \( \frac{8}{2} \) is either obtained from eight objects shared equally into two sets or as eight objects distributed into sets each containing two objects, the answer to the division operation being either the number of objects in the two sets or the
number of sets of two objects that can be obtained (4 in this example).

The symbol $4:2$ is first introduced in grade 2\(^1\) yet the alternate bar symbol $\frac{4}{2}$ is not used until grade 6\(^2\) when all basic work on the fraction symbol as a part of a whole object has been completed. No mention has been discovered, either in teacher training texts or school texts, of natural quotients following the structure of the fraction symbol. It is certainly possible to study this use of the fraction symbol before studying fractional parts of whole objects and it may be better pedagogically. For example the three following factors could make it easy for students to handle them:

(a) Operations identical to the natural number operations are used.

(b) A rapid conversion of any symbol to the corresponding natural number can be made.

(c) Physical ambiguities are kept to a minimum, thereby reducing confusion.

As shown in Appendix A the historical and logical development of fraction symbols is the extension of the natural

\(^{1}\)Hartung, M. L. et al., *Seeing Through Arithmetic. 2.*, W. J. Gage Ltd. 1966, p. 143.

division symbols to the "indicated division" symbols, those divisions having a non zero remainder and written \( \frac{2}{3}, \frac{16}{9} \) etc.

This concept of indicated operation is important to the extension of the numbers capable of being symbolized for when placed in a very similar but slightly different axiomatic basis to the natural divisions they produce, the set of rational numbers.

A complementary approach would be to consider \( \frac{a}{b} \) as a compound operator where \( a \) of \( c = \frac{a \times c}{b} \). Again, as can be seen more fully in Appendix A, this compound operator shows some structure 'isomorphism', identical to that of our original fraction symbols with the four operations.

What has happened traditionally is that the students in grades 5 and 6 have been introduced to fraction symbols without reference to previous work done on division. Fractional parts of objects and groups of objects are the physical source for abstraction. In such a case a student can easily see the ordered pair symbol as a static description of a limited physical situation. The numerator represents the number of equipartitions while the denominator represents the number of such equipartitions required to make up the whole object or group of objects being considered. Initially the operand, in physical form, is all important but quickly it ceases to be mentioned as
we move from; for instance,

\[
\frac{1}{2} \text{ of a cake} + \frac{1}{3} \text{ of the same cake} = \frac{5}{6} \text{ of the cake}
\]

\[
to \quad \frac{1}{2} + \frac{1}{3} = \frac{5}{6}
\]

The \( \frac{1}{2}, \frac{1}{3}, \text{ and } \frac{5}{6} \) in the sentence \( \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \) are all numbers unassociated with a physical meaning since the operand has ceased being "a cake" or some other physical object or group of objects and has become an abstract "oneness". Thus \( \frac{1}{2} \)

l is a fraction acting on 1 producing the symbol \( \frac{1}{2} \) which is indistinguishable from a member of the set of rational numbers, more fully described in Appendix A. Skemp prefers to call this number symbol a "fractional number" because the physical model from which it is derived is different from the ratio model that rational numbers derive from. He writes:

Fractional numbers are properties of equivalence classes of fractions, which are abstracted from actions of collecting and sharing. Rational numbers are properties of equivalence classes of ratios, which are abstracted from a certain kind of correspondence between sets.3

Whether a treatment of fractional numbers separately from that of rational numbers (but forming isomorphic systems) is preferable to treating them as synonyms is a

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question unanswered in this study.

Even if the dynamic view of a fraction operating on an object or group of objects is maintained, a multiplicity of physical ambiguities are met. \( \frac{2}{3} \) of \( \frac{3}{3} \) using a rectangular object can refer to either of the following situations:

\[
\begin{align*}
\text{2 of } \frac{1}{3} & \quad \text{of 3} \\
\text{1 of } \frac{2}{3} & \quad \text{of 3}
\end{align*}
\]

and the situation becomes even more complicated in its ambiguities when groups of objects are equipartitioned. This will be seen later in the study.

In Grade 6, students are introduced to comparison pairs where each of two numbers is associated with some designated characteristic of two sets and again is symbolized \( a \). A particular subset of the comparison pairs \( \frac{3}{3} \) are the rate pairs which are comparison pairs following the equivalence relation defined above.

Thus if the first term represents distance and the second represents time then physical meaning can be given to

\[
\frac{2}{3} + \frac{8}{4} = \frac{2 \cdot 4 + 3 \cdot 8}{3 \cdot 4}
\]

since we can visualize a man on a moving staircase where his speed is 2 steps per 3 seconds and the staircase's speed is...
8 steps every 4 seconds and the $\frac{2.4 + 3.8}{3.4}$ represents the combined speed relative to a stationary object. Using similar models one can readily demonstrate that rate pairs produce symbols having isomorphic operations to fractions, quotients, rational numbers and compound operators and that from sets of equivalent classes of such rate pairs are derived the symbols for the rational numbers.

3. Background and Review of the Literature

School children experience difficulty with the ideas associated with fractions. As Spitzer says:

The literature of arithmetic, both ancient and modern abounds with evidence in support of the thesis that fractions are difficult.

This difficulty faced by students is the central problem of this study though only one aspect of the pedagogical problem is investigated. None of the many teacher training texts or articles listed in the bibliography tackles in depth, the reasons for this difficulty. Spitzer mentions two possibilities, namely:

First there is the matter of the dual concept versus a single concept. Consider, for example, the two expressions "three fourths mile" and "three miles". In the former expression the "three" tells "how many" but the "fourths" do not. The fourths refers to the size of the

---

parts. On the other hand, the "three" of the latter expression has only the "how many" meaning. A second factor which makes fractions more difficult to learn than whole numbers is that with whole numbers we use ordered series over and over, acquiring strong basic notions of number. With fractions the ordered series is more complex and is used less frequently. Our basic ideas of halves, thirds and fourths come from fewer experiences and from relating the whole number ideas to the denominator aspect of fractions.5

Thus, Spitzer's explanation why fractions are found to be difficult is twofold. Firstly, there are two concepts to manipulate and understand while secondly, the student is relatively inexperienced with their manipulation. Swanson also noticed that the literature about fractions tends to avoid a thorough treatment and explanation of what fractions mean. As she says, using meaning in a plural sense:

Too much time and energy have been spent struggling with poorly understood and misunderstood mechanical operations with fractions. If half that amount of time were spent helping learners understand the fractions themselves, the effects might be as revolutionary as the orbiting of weather satellites [on weather forecasting]. Whether the fraction is an expression of a number or a number relationship, the important consideration is that emphasis should rest on the meaning represented by the symbol.6

5Ibid., p. 173.

In roughly the same area of difficulty Rappaport suggests that:

The difficulty that children and adults have had with fractions in the past has often resulted from the fact that they were not aware that there are several meanings to the symbol. 7

Thus Rappaport underlines that it is perhaps unwise and confusing to introduce the symbol \( \frac{a}{b} \) without explaining the several meanings. It is possible that students become confused because so many meanings and models are used at the outset. Perhaps it is right to suggest that initially the structure and operations of the fraction symbols should be learned by the student followed by the many applications of this structure to various models.

Since it is rare to see listed the several meanings alluded to by Rappaport above, it would be germane to consider what meanings can be given to the symbol \( \frac{a}{b} \) by listing them under headings and then giving a brief explanation of each.

The symbol \( \frac{a}{b} \) can mean, if \( a, b \) are real or complex and \( b \neq 0 \):

(a) One or more parts of

(i) an object

(ii) a group of objects.

(b) A compound arithmetic operator.
(c) A comparison of two numbers.
(d) A division.
(e) A number symbol for the answer to a division.
(f) A rational number.
(g) A complex fraction.
(h) A solution to an irrational equation.
(i) A solution to a complex number equation.
(j) An exponent.

Let us consider each of these in turn:

(a) This "fractional part" definition of \( \frac{a}{b} \) is usually the first approach to the student's studies of fraction symbols. It seems obvious that \( a \), the "numerator", is the number of pieces while \( b \), the "denominator", is the "size" of the pieces and all that is needed to consolidate the idea is some paper pie plates to be cut up with some scissors, and an apple or two. It would seem that only a very poor teacher would fail to get the simple idea across. As has been indicated in the introduction this is not however always the case. One possible explanation is that the model changes drastically as one considers the fractional part of an object and then the fractional part of a group of objects. Again the physical visualization is quite ambiguous and
is rarely spelled out as to which particular model is being studied.

\[ \frac{2}{3} \text{ of a pie can mean either} \]

(a) \[ \begin{array}{c}
\includegraphics[width=1cm]{pie1.png} \\
\includegraphics[width=1cm]{pie2.png}
\end{array} \]

\[ \text{two pieces} \]

or (b) \[ \begin{array}{c}
\includegraphics[width=1cm]{pie3.png}
\end{array} \]

\[ \text{one piece} \]

Again \( \frac{2}{3} \) of a tray of six pies can mean

(a) \[ \begin{array}{cccccc}
\includegraphics[width=0.2cm]{pie4.png} & \includegraphics[width=0.2cm]{pie5.png} & \includegraphics[width=0.2cm]{pie6.png} & \includegraphics[width=0.2cm]{pie7.png} & \includegraphics[width=0.2cm]{pie8.png} & \includegraphics[width=0.2cm]{pie9.png}
\end{array} \]

\[ \text{four pies} \]

(b) \[ \begin{array}{cccccccccccc}
\includegraphics[width=0.2cm]{pie10.png} & \includegraphics[width=0.2cm]{pie11.png} & \includegraphics[width=0.2cm]{pie12.png} & \includegraphics[width=0.2cm]{pie13.png} & \includegraphics[width=0.2cm]{pie14.png} & \includegraphics[width=0.2cm]{pie15.png} & \includegraphics[width=0.2cm]{pie16.png} & \includegraphics[width=0.2cm]{pie17.png} & \includegraphics[width=0.2cm]{pie18.png} & \includegraphics[width=0.2cm]{pie19.png} & \includegraphics[width=0.2cm]{pie20.png} & \includegraphics[width=0.2cm]{pie21.png}
\end{array} \]

\[ \text{twelve pieces} \]

(c) \[ \begin{array}{cccccccc}
\includegraphics[width=0.2cm]{pie22.png} & \includegraphics[width=0.2cm]{pie23.png} & \includegraphics[width=0.2cm]{pie24.png} & \includegraphics[width=0.2cm]{pie25.png} & \includegraphics[width=0.2cm]{pie26.png} & \includegraphics[width=0.2cm]{pie27.png} & \includegraphics[width=0.2cm]{pie28.png}
\end{array} \]

\[ \text{six pieces} \]

When scissors are being used it is difficult to avoid these physical differences but they are not usually pointed out in the texts.

(b) As a compound operator \( \frac{a}{b} \) can be considered in two ways:

either (i) Multiply the operand by \( a \) and then divide the product quantity by \( b \).

or (ii) Divide the operand by \( b \) and then multiply the quotient quantity by \( a \).

So \( \frac{2}{3} \) of 6 pies is \( \frac{2 \times 6}{3} \) pies using either method but again any use of scissors would result in three
(c) When \( a \) is considered as a comparison we are faced with two possibilities. Either:

(i) Certain number characteristics of two sets are compared as in

\[
\text{number of girls} = 5 \\
\text{number of boys} = 4
\]

or (ii) We have certain number characteristics of two sets compared but where for each comparison an infinite set of equivalent comparisons may be obtained using the equivalence relation.

\[
a \cdot c \iff a \cdot d = b \cdot c \text{ where } a, b, n \in \mathbb{R} \quad n, b \neq 0
\]

We call the second type of comparison a 'rate pair' and if the measure unit used on each set is the same the rate pair is called a ratio. For example a rate pair would be a speed so that \( 60 \text{ miles} = \frac{120 \text{ miles}}{1 \text{ hour}} \) and a ratio would be derived from similar triangles when

\[
x \text{ inches} = y \text{ inches} \text{ in the diagram:}
\]

\[
a \text{ inches} \quad b \text{ inches}
\]
(d) As a division $a$ is usually taught as an alternate symbolism for $a \div b$. The equivalence $a \div b = (n \times a) \div (n \times b)$, and hence $a = \frac{na}{b}$ is usually mentioned.

But when one considers how important it is in cancelling in arithmetic calculation, and in algebra, one would suspect that it should be emphasised in elementary arithmetic teaching.

(e) The division $\frac{8}{2}$ need not be associated with the number 4.

It can be left as $\frac{8}{2}$. It is of course usual to write the quotient number as 4 but advantages can result from the alternate symbolism. However the indicated division $\frac{8}{3}$ does not have a natural number answer but if we use the symbol $\frac{8}{3}$ as an indicated quotient answer to the division then the union of the indicated quotients and the integer answer quotients gives a number field isomorphic to fraction symbols. It is possible that quotients are the least complicated application of equivalent classes of ordered pairs for students to understand since:

$$\frac{12}{3} + \frac{10}{2} = \frac{2 \times 12 + 3 \times 10}{3 \times 2} = \frac{24 + 30}{6} = \frac{54}{6} = \frac{9}{1} = 9$$

$$4 + 5 = 9$$

(f) The symbol $\frac{a}{b}$ can represent a rational number defined as
a set of ordered integers \((a, b)\) such that \(b \neq 0\) and exhibiting the equivalence relation \((a, b), (c, d)\) or \(\frac{a}{b} = \frac{c}{d}\) where two operations are defined producing a field.

(g) A use of the symbol \(a\) can be as a "complex fraction" as in the example:

\[
\begin{align*}
1 + \frac{9}{16} & = \frac{1}{1 - \frac{3}{4}} \\
\end{align*}
\]

(h) \(a\) can represent the solution to an irrational equation such as \(2x = \pi\) and is again an indicated quotient with an irrational member:

\[
x = \frac{\pi}{2}
\]

(i) Similar to (h) above we can have \(a\) the solution to a complex number equation such as \((1-i)x = -2+3i\) and can then be symbolized by the indicated quotient having complex members:

\[
x = \frac{-2 + 3i}{1 - 1}
\]


\(^{10}\) Ibid., p. 460.

\(^{11}\) Ibid., p. 409
(j) The symbol \( a^b \) can be used as an exponent where \( a \) can mean the power to which the base is to be raised while \( b \) can represent the root to which the result is to be taken such as \( 4^{3/2} = 8 \).

It is recognized that the last four meanings are only likely to be met in the later grades but one would expect to see the first six headings described in teacher training literature. Lovell, Shipp, and Spitzer hardly mention explicitly more than one meaning. Bruckner and Rappaport list four meanings while Copeland, Dutton, and Corle quote five


14 Spitzer, op. cit.


16 Rappaport, op. cit. p. 117.


meanings. Swanson comprehensively describes eight meanings. Swanson\textsuperscript{20} comprehensively describes eight but there is every indication of uneven treatment of the fraction symbol. For convenience an analysis table is included below.

TABLE I

ANALYSIS OF THE DEFINITION OF THE SYMBOL $\frac{A}{B}$ IN SELECTED TEACHER TRAINING TEXTS

<table>
<thead>
<tr>
<th>AUTHOR</th>
<th>Bruckner</th>
<th>Rappaport</th>
<th>Copeland</th>
<th>Dutton</th>
<th>Corle</th>
<th>Swanson</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>Definition:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fractional part of whole</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Fractional part of group</td>
<td>X</td>
<td>O</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Compound operator</td>
<td>O</td>
<td>X</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Comparison</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Rate pair</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Ratio</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
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</tr>
<tr>
<td>Division</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Quotient</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Rational number</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

Y, X, mentioned. N, O, not mentioned.

The teacher training texts manifest widely differing emphasis. Mueller misses the point of the isomorphism.

\textsuperscript{20} Swanson, op. cit., pp. 303-312.
between rational and fractional numbers when he says:

Most ratio problems can be solved by expressing ratio as a fraction and employing the usual arithmetic of fractions.²¹

and later hints at philosophical problems in his comment:

Today, many would find it difficult to accept Smith's "artificial number" interpretation (of fractions) because accepting the fraction as a number can become logically awkward. The ground is much more firm if we classify the fraction as a numeral even though we must fumble linguistically over its number referent.²²

Doubt and hesitancy appear in other texts, for example in Developing Arithmetic Concepts and Skills we read:

Traditional texts define ratio as implied division. The ratio of the number 12 to the number 4 is 12:4 or \( \frac{12}{4} \). Also, we write 12:4 and this \( \frac{12}{4} \) is read "the ratio of 12 to 4". This definition is very nearly the same as the traditional definition of common fractions. Some texts note that fractions "express" a ratio and treat fractions and ratios as equivalent. One might wonder why they should be treated separately if they are really the same thing.²³

In fact the overall impression from the texts is that the problem of "fractions" is not treated as deeply as


²² Ibid., p. 205.

²³ Shipp and Adams, op. cit., p. 297.
it merits. Titchmarsh and Skemp clearly explain that the four usual operations with natural numbers are different from the operations with rational numbers even though they traditionally share the same symbols of $\times$ and $\div$. Only one teacher training text studied, namely Trends in Elementary Education, makes the point that many applications share the same structure when it says, albeit indirectly:

This expression $\frac{a}{b}$ also develops
the whole numbers and it may be
seen that finding sums and products
of elements of the whole numbers is
equivalent to finding sums and
products of corresponding members
of the rational numbers. Extending
the observation a bit, if the
structures of the two systems
(axiom systems) have models so much alike that one may be regarded as a
mere relabelling of the other, they
are said to be isomorphic.

On the other hand, all the texts are unanimous in stating that fractions should be introduced using concrete objects.

4. Purpose of the Study

Cheney surveyed elementary teachers arithmetic

---

understanding and found that they had difficulty with fraction symbols and since little research has been done on reasons for such difficulties among students or teachers it was decided to study the teacher training literature. Each treatment therein indicated differing approaches and emphasis but most left many questions unanswered. Some authors admitted to confusion while only one book contained the important linking idea of isomorphism between the various applications of the fraction symbol. Some gave long lists of different meanings emphasising the pedagogical importance of establishing the variation in meaning in the student's mind, while others hardly mention explicitly more than one meaning. None indicated that the fraction symbol operations of addition, subtraction, multiplication and division are in any way different from their counterparts in the natural number system. Several questions listed below were left unanswered after reading the teacher training literature.

A. Mathematical Questions

1. Are the operations between fraction symbols identical to the corresponding operations in the natural number system? If they are different are they very different?

2. Is it valid at any time to multiply a whole number symbol by a fraction symbol?
3. Is a 'fraction' - as learned in Grade 6 - a rational number?

4. Is a rational number identical to a fractional number?

5. Is a fraction a ratio?

B. Pedagogical Questions

1. Does it improve student understanding and skill with fraction symbols if a student is aware of their many meanings?

2. Does the introduction of fraction symbols as fractional parts of whole objects psychologically impede future learning of other meanings?

3. Does the introduction of the fraction symbol in a physical setting create confusion because of multiple physical ambiguities?

4. Does it matter whether the fraction symbol is taught as a symbol for a numeral or as 'one number over another'?

5. Does the traditional fraction symbol present problems to the student by its physical configuration since the usual left to right eye movement is interrupted?

The purpose of this study is to answer some of these questions and to suggest a possible improvement to the usual introduction to the fraction symbol incorporating a small field test to check whether further research should follow.
The mathematical texts consulted did not answer all
the mathematical questions above so it was decided to
develop Bell's\textsuperscript{28} treatment using Peano's axioms. In this
way the fraction symbol is examined closely and some
answers to the mathematical questions are attempted in
Appendix A.

To establish answers to the five pedagogical
questions would require considerable research and technique.
However, arising from the second question is the possibility
of introducing the fraction symbol free from any single
application so that one could hope for something like the
following curriculum sequence in school:

(1) **Grades I - IV.**
Manipulation of whole numbers.

(2) **Grade V**
A description of the structure of fraction symbols and
the four operations without reference to any
application.

(3) **Grade V - XII**
Illustrations of the learned structure utilizing:

(i) natural divisions

(ii) indicated divisions

(iii) compound operators

(iv) fractional measure

\textsuperscript{28}Bell, A. W., *Algebraic Structures.* John Wiley and
(v) fractional parts of a single object
(vi) fractional parts of a group of objects
(vii) rate pairs
(viii) rational numbers
(ix) complex fractions
(x) rational algebraic expressions
(xi) solutions to irrational equations
(xii) solutions to complex number equations
(xiii) fractional exponents

Step 2 above in the theoretical sequence is counter to a strong tradition supporting the present introduction of fraction symbols using fractional parts of objects together with much emphasis on physical models. Because of this it was considered necessary to test step 2 above to find whether it is based on a realistic consideration for effective learning.

Trivett has suggested an alternative treatment of fraction symbols using fraction ordered pairs, a discovery game and a pure mathematical approach. It was realized that, since Grade 5 students in Duncan have already met the fraction symbol in the two previous years, Trivett's schema could usefully be modified (henceforth called the "modified Trivett schema") to introduce the fraction symbol.

structure free of application connotations to students familiar with fractional parts of whole objects. It was thought that the fraction ordered pairs looked so little like the equivalent fraction symbols that the structure could be introduced without even implied applications.

It was recognized that the modified Trivett schema introduced the two fresh variables of game versus non-game and fraction ordered pair symbol versus fraction symbol. Nevertheless, it was decided that the advantages outweighed the disadvantages.

To do this, two similar classes of Grade V students were selected with the advice of the district Superintendent of Schools and the principal of the school involved. One class was taught fraction symbolism and the four operations of addition, subtraction, multiplication and division of fractions; using the previous experience of the students with fractions, conventional symbolism and many physical models to illustrate the ideas. The other class was taught precisely the same material but using fraction ordered pairs, no reference to experience with fractions or concrete models to illustrate the ideas. Each class was administered tests of understanding and skills with fractions before and after the teaching period and a fuller description of these tests will be found in Chapter III of this study.

The data obtained from these tests was analysed
using appropriate techniques in order to test specifically the following null hypotheses:

1. There will be no significant difference between the scores of the group of students taught fractions using ordered pairs, with no previous experience or physical models and those taught fractions using traditional symbolism, with previous experience and physical models with respect to understanding fractions.

2. There will be no significant difference between the scores of the group of students taught fractions using ordered pairs, no previous experience or physical models and those taught fractions using traditional symbolism, previous experience and physical models with respect to manipulative skills with fractions.

5. Limitations of the Study

This study is not a continuation of previous research and, as a beginning, draws very little from other established experimental work.

Fraction symbolism is not treated extensively or in depth in either the teacher training literature or in the mathematical texts. Thus because there is little authoritative work in this area it happens that more questions are raised than are answered and even the suggested improvement to the teaching of the fraction symbol contained in this study is not only not found in the teacher
training literature but is in opposition to their unanimous opinion that applications are essential.

The theoretical work in Appendix A is to be found in no single book. Parts are quite original and some questions will not be answered. The following three questions specifically are not answered:

1. Is a fractional number the same as a rational number, though they have different derivations?
2. Is a number independent of its symbol?
3. How is it that a subset of the division numbers has identical operations to the natural numbers while its complement does not?

The small field study was designed solely to indicate the practicality of teaching fraction pair structure without reference to applications and to see whether more exact research was required in the future. It was not designed to test whether the theoretical curriculum schema was better than the traditional sequence of instruction. To do this would probably require work lasting at least several years, involve many teachers and an experimental design which would limit the variables to one.

Although the numbers of children in the field study groups were quite small it was felt that the limited goal of the study to achieve an indication of practicality was appropriate.

Because of time limitations and the need to equalize teaching time between the groups, justice was not done to
all the possible physical models of fractional parts of wholes and groups.

Since one might expect to introduce fraction symbol structure using fraction symbols and not fraction ordered pair symbols, an unwanted variable was introduced into the investigation.

The pronounced game structure of the modified Trivett schema introduced another variable into the investigation though an attempt was made to minimize this by making each class equally enjoyable.

Since only one teacher was used for both classes, similar teaching attitude was to be expected but an observer was asked to check for any differences in teacher attitude during the progress of the investigation.

Due to the above-mentioned restrictions the results of this field study are subject to further investigation and can only be taken as evidence for suggesting more research.

6. Organization of the Remainder of the Thesis

The thesis is subdivided into four additional chapters:

a. Chapter II will describe in general the lessons and procedure followed in the two classes being studied.

b. Chapter III will outline the design of the small field study describing the tests used and their derivation.
c. Chapter IV will describe the results obtained from the tests and the statistical analysis applied to them.

d. Chapter V will give the conclusions, the delimitations, generalizations and areas suggested for further investigation.
CHAPTER II

A DESCRIPTION OF THE TWO APPROACHES TO THE INTRODUCTION OF FRACTIONS USED IN THE STUDY

1. Introduction

In British Columbia, as in most other places, the fraction is introduced to school children by using card and paper pies, cakes, rectangles, hexagons, circles and sometimes even scissors to cut them up. Addition and subtraction are illustrated by using equal sized slices. For example, in *Seeing Through Arithmetic*, we see \( \frac{1}{2} + \frac{1}{3} \) as \( \frac{5}{6} \).

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Division and multiplication are usually not taught using concrete illustrations since the Programme of Studies for the Intermediate Grades counsels against it on the basis that:

It is almost impossible to illustrate the process for multiplication of fractions in a concrete way because the basis of the rule is mathematical.\(^{31}\)

This lack of using physical settings for multiplication and division is common in North America and Dwight summarizes the situation when he says:

Usually, physical settings are provided in an attempt, with varied degrees of success, to get pupils to recognize the character of the operations on the new numbers. Then pupils, generally, are willing to accept the fundamental laws of the operations on the enlarged set of numbers with little or no activities with concrete settings to justify doing so.\(^{32}\)


Dwight succinctly exposes the subject of investigation of this study when he includes, as an aside, the words "with varied degrees of success". It is possible that the concrete illustrations and models do not facilitate understanding and skill with fractions.

Since equivalent classes of ordered pairs, though seldom recognized as such, are met with as quotients in Grades 3 and 4, as fractions in Grades 4, 5 and 6, as rational numbers in Grades 7 and 8 and in algebra as quotients, ratios, fractions, exponents and rational expressions in Grades 9, 11 and 12, it would seem reasonable to encourage the students to learn the shared structure at the outset free from any particular application. In this way the known structure could be applied year by year to each application as it is met, thus consolidating the known structure as it is used and avoiding the necessity of learning new material each time by establishing isomorphisms and differences.

The question remains whether the typical structure of equivalent classes of ordered pairs can be taught effectively without recourse to the physical models of the traditional approach. Trivett has suggested a schema for doing this. In it students are invited to play a combined game of "guess the rule" fitting number pairs into families which later become understood as equivalent classes. When the pupils have discovered the rules for "naming a family"
(finding the ordered pair reduced to its lowest term) and the rules for placing any given ordered pair into its family, then possible operations between the pairs are invented using made-up names and symbols. The class is directed to the four usual operations and games are played to test the versatility and skill of the students with these four. The whole number manipulation exercise is considered as a game and since the ordered pairs have little similarity to the conventional fraction symbolism no support or interference is met from prior experience with fraction concepts.

These two different approaches were used in the investigation, one with each group, and every effort was made to teach both methods enthusiastically and skillfully. With the object of reporting on what he saw and heard, the principal of the school was asked to frequently observe each of the classes. A letter from him will be found in Appendix B, giving his opinion that both classes were taught with equal enthusiasm and that the diaries in the Appendix B were a fair description of what happened in each class.

2. The Lessons Taught Traditionally

Approximately eighteen hours were spent with fifteen students over eighteen periods between the twelfth of February and the eighth of March, 1973. The lessons were at nine o'clock, eleven o'clock, one o'clock and two o'clock on a cyclical basis of four days. To reduce "new teacher"
anxieties that the students may experience, small candies were offered as rewards for enthusiastic and/or correct work, whether oral or written, throughout the duration of the experiment. A day by day description of the lesson plans will be found in Appendix B of the thesis.

3. The Lessons Taught Using A Modified Trivett Schema

Approximately eighteen hours were spent with fifteen students over eighteen periods between the fifteenth of January and the seventh of February, 1973. The lessons followed the same time pattern as above and as before, small candies were offered as rewards for enthusiastic and/or correct work. A day by day description of the lessons will be found in Appendix B of the thesis.
CHAPTER III

THE DESIGN OF THE FIELD STUDY

1. Quasi-experimental Design

The research design for this field study can be called "Pretest - Post-test Control Group Design". It is depicted schematically by the following figure:

Group 1: $R \ M_D \ + \ T_1 \ + \ M_a$

Group 2: $R \ M_D \ + \ T_2 \ + \ M_a$

The $R$ in the model indicates that of the two groups of Grade 5 students the selection into Group 1 or Group 2 was considered random though no attempt at random sampling was undertaken. $M_D$ corresponds to measurement before the experiment (pretest), $T_1$ and $T_2$ refer to two different approaches to the introduction of fraction concepts and $M_a$ signifies measurement after the teaching session (post-test).

In this field study the two groups were taught at separate times for approximately one hour each day for eighteen lessons. The lessons were not at the same period each day but were on a four day cycle beginning at 9.00 a.m., 11.00 a.m., 1.00 p.m. and 2.00 p.m. respectively. The class that comprised twenty students was taught fractions using the modified Trivett schema from the 15th January, 1973 to the 8th February, 1973. The class that comprised
fifteen students was taught fractions using traditional methods from the 12th February, 1973 to the 8th March, 1973.

Two tests were administered, one to indicate understanding and the other to indicate skill at manipulating fractions. The same tests were given to both groups as pretests and as post-tests. The post-tests for the group learning equivalent classes of ordered pairs were identical in content to the other tests but the symbolism necessarily had to be altered. All tests are included in this report as part of the Appendix B.

The data gathered was analysed using analysis of covariance to test the null hypotheses given on page 27 of the thesis.

The Pilot Project

In order for the investigator to gain personal experience, a pilot project involving two Grade XI general mathematics classes was run in October and November, 1972.

Problems were able to be anticipated and the quality of instruction in the actual study improved beforehand. As a result of this project, the instruction time was set and some details were altered in both approaches to avoid difficulties and save time. Finally, the tests were administered for the investigator to gain insight into adequate procedures for the main study.
2. Sample

The thirty-five students utilized for the main study were children living in Somenos, a rural area to the north of Duncan on Vancouver Island. They were in two classes and no apparent reason for their selection was evident or able to be offered by the principal of the school or by the experimenter. As a result the classes were considered randomly selected in effect though strictly all that can be said is that there appeared no special weighting with respect to sex, ability or attainment.

3. Description of Tests

Test A - Test for Understanding

Much controversy surrounds testing for understanding but as the Report of the Cambridge Conference says:

When we take so much care to develop understanding and creativity in the student, it would be a pity to test his achievement only in terms of the mechanical skills and rote responses he has learned.33

However there is little work published on testing for understanding and as Brownell wrote as late as 1967:

The reason why related research (on understanding) is not mentioned and reviewed is that there is none. In research studies that might have been cited paper and pencil testing has been the technique used and the

---

criterion of effectiveness has been proficiency; for example skill and accuracy in computation and problem solving. 34

Nevertheless several commercially-produced tests were discovered which purported to test for understanding of fractions. They were usually only small parts of conventional manipulative skill tests. It was decided to compose an understanding test which was a combination of questions from three of these very well-known tests:–

(a) Canadian Tests of Basic Skills 35
(b) Stanford Diagnostic Test 36
(c) Science Research Associates Achievement Series Arithmetic Concepts. 37

The following table indicates the composition of the test for understanding with references to the three source tests as listed above as (a), (b) or (c).


All questions from the three understanding sections alluded to above were selected solely under the criterion that the question tested a concept to be learned during the investigation period and can be seen in Appendix B.

The three tests receive favourable reviews in the Mental Measurements Yearbooks though for reasons mentioned previously the testing for understanding is treated cautiously. The Stanford Diagnostic test is reviewed by

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<table>
<thead>
<tr>
<th>Question on Test</th>
<th>Reference Test</th>
<th>Question Number on Reference Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>b</td>
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</tr>
<tr>
<td>15</td>
<td>c</td>
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</table>
Rogers in the *Seventh Mental Measurements Yearbook*. The Canadian Tests of Basic Skills by Birch in the same yearbook and the S.R.A. test by North in the *Fifth Mental Measurements Yearbook*. Each review confirmed that the tests were appropriate to the ability and age level of the students taught in the experiment.

Test B - Test for Manipulative Skills with Fractions

The test for manipulative skills was test 3(b) of the *Stanford Diagnostic Arithmetic Test, Level II, Form W.* In consultation with professors of Simon Fraser University it was decided that this test for computation skill would be suitable. Rogers reports that the split half reliability coefficients are in the acceptable range of .82 and .95 and that high correlations are obtained when the validity was tested against other well-established tests so he concludes:

> It would appear, on the basis of these data that the S.D.A.T. is suitable for achievement testing at these grade levels (4.5 - 8.5)

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41 Rogers, op. cit., p. 923.
Two versions of the post-tests were produced, one with ordered pairs written \((a,b)\) while the other used the usual \(a\) for fractions. This was necessary so that a post-test could be applied to those students who had studied equivalent classes of ordered pairs using the symbolism \((a,b)\). However the questions and layout were otherwise identical.

4. Administration of Achievement and Understanding Tests

Some students who were very slow were given ample time to attempt each question. The students were instructed that they were not to worry if any question was too difficult, but simply miss it out and move on to the next one.

5. Course Content

Each class was taught the following concepts, not necessarily in order:

1. A fraction is a symbol with two members.
2. Fractions form equivalent classes.
3. The operations of addition, subtraction, multiplication and division on (a) like fractions and (b) unlike fractions.
4. Reciprocal.
5. Factors, common factors and the greatest common factor.
The class studying fractions learned about lowest common multiples while the class studying ordered pairs studied dual naming of the terms in an ordered pair binary operation. The actual lessons are described in more detail in Chapter II of this study and in Appendix B.
CHAPTER IV

THE RESULTS OF THE INVESTIGATION

1. Introduction

This chapter includes the results of the data analysis of tests given on certain aspects of rational numbers as equivalent classes of ordered pairs when taught with the usual fraction introduction and when taught using a modified Trivett schema:

a. with respect to understanding
b. with respect to computation skills

2. Statistical Analysis

The two classes of children were pretested and the results of these tests used as the predictor (or controlled) variable (x). The post-test results (y) were adjusted to obtain a statistical partialling out of the some of the variation of the two groups of students being tested. The statistical adjustment was obtained by using the appropriate procedures for the analysis of covariance as outlined in Roscoe's *Fundamental Research Statistics*,42 and the results were corroborated using the appropriate computer program at Simon Fraser University. The raw scores and the analysis of their

covariance is given in the Appendix and only summary figures and tables will be given in this section.

Test of Understanding

Null Hypothesis I.

The null hypothesis to be tested using the analysis of covariance was: The means of the scores of the test for understanding are equal when one group is taught using the modified Trivett schema and the other is taught using traditional methods, the effects of any initial disparity being partially controlled statistically.

At the .05 level of significance with 1 and 31 degrees of freedom, Null Hypothesis I would be rejected if

\[ F \geq 4.16. \quad 43 \]

At the .01 level of significance with 1 and 31 degrees of freedom, Null Hypothesis I would be rejected if

\[ F \geq 7.53. \quad 44 \]

---

43 Ibid., p. 322.
44 Ibid., p. 323.
TABLE III

SUMMARY FOR THE ANALYSIS OF COVARIANCE OF THE SCORES
OF THE TWO INVESTIGATED CLASSES ON UNDERSTANDING

<table>
<thead>
<tr>
<th>Source</th>
<th>Number of degrees of freedom</th>
<th>Sum of squares of the pretest scores</th>
<th>Sum of the products of the pre- and post-test scores</th>
<th>Sum of the squares of the post-test scores</th>
<th>Number of degrees of freedom for the adjusted post-test scores</th>
<th>Sum of squares of the adjusted post-test scores</th>
<th>Adjusted mean square of the post-test scores</th>
<th>Adjusted mean square of modified Trivett schema</th>
<th>Adjusted mean square of the traditional schema</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>1</td>
<td>70.44</td>
<td>24.16</td>
<td>8.29</td>
<td>1</td>
<td>3.92</td>
<td>3.92</td>
<td>XXX</td>
<td>XXX</td>
</tr>
<tr>
<td>Within</td>
<td>32</td>
<td>163.73</td>
<td>102.27</td>
<td>334.28</td>
<td>31</td>
<td>270.39</td>
<td>8.72</td>
<td>XXX</td>
<td>XXX</td>
</tr>
<tr>
<td>Total</td>
<td>33</td>
<td>234.17</td>
<td>126.43</td>
<td>342.57</td>
<td>32</td>
<td>274.31</td>
<td>XXX</td>
<td>909</td>
<td>989</td>
</tr>
</tbody>
</table>

\[ F - \text{ratio} = \frac{\text{Adjusted Mean Square Between}}{\text{Adjusted Mean Square Within}} = \frac{3.92}{8.72} = 0.45 \]

Hence Null Hypothesis I is not rejected and there is no significant difference between the mean scores.
Test for Computational Skill

Null Hypothesis II.

The null hypothesis to be tested using the analysis of covariance was: The means of the scores of the test for computational skills are equal when one group is taught using the modified Trivett schema and the other group is taught using traditional methods, the effects of any initial disparity being partially controlled statistically.

At the .05 level of significance with 1 and 31 degrees of freedom, Null Hypothesis II would be rejected if

\[ F \geq 4.16. \]

At the .01 level of significance with 1 and 31 degrees of freedom, Null Hypothesis II would be rejected if

\[ F \geq 7.53. \]

\[ ^{45} \text{Ibid., p. 322.} \]

\[ ^{46} \text{Ibid., p. 323.} \]
TABLE IV

SUMMARY FOR THE ANALYSIS OF COVARIANCE OF THE SCORES OF THE TWO INVESTIGATED GROUPS ON COMPUTATIONAL SKILLS

<table>
<thead>
<tr>
<th></th>
<th>Number of degrees of freedom</th>
<th>Sum of squares of the pretest scores</th>
<th>Sum of the products of the pretest scores and the post-test scores</th>
<th>Sum of the squares of the post-test scores</th>
<th>Number of degrees of freedom for the adjusted scores</th>
<th>Sum of squares of the adjusted post-test scores</th>
<th>Adjusted Mean Square of the adjusted post-test scores</th>
<th>Adjusted Mean of the post-test scores of Trivett schema</th>
<th>Adjusted Mean of the post-test scores of the traditional schema</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>1</td>
<td>2.76</td>
<td>7.04</td>
<td>18.02</td>
<td>1</td>
<td>9.14</td>
<td>9.14</td>
<td>XXX</td>
<td>XXX</td>
</tr>
<tr>
<td>Within</td>
<td>32</td>
<td>69.13</td>
<td>48.50</td>
<td>1700.15</td>
<td>31</td>
<td>1666.12</td>
<td>53.75</td>
<td>XXX</td>
<td>XXX</td>
</tr>
<tr>
<td>Total</td>
<td>33</td>
<td>71.89</td>
<td>55.54</td>
<td>1718.17</td>
<td>32</td>
<td>1675.26</td>
<td>10.32</td>
<td>10.05</td>
<td></td>
</tr>
</tbody>
</table>

\[ F - \text{ratio} = \frac{\text{Adjusted Mean Square Between}}{\text{Adjusted Mean Square Within}} = \frac{9.14}{53.75} = 0.17. \]

Hence Null Hypothesis II is not rejected and there is no significant difference between the mean scores.

Summary of Results

Both of the hypotheses under investigation were not rejected when the pretest and post-test scores were subjected to statistical analysis of covariance.
CONCLUSIONS, DELIMITATIONS AND AREAS OF FURTHER INVESTIGATION

1. Summary of Results

In the section entitled 'Purpose of the Study' five mathematical questions were raised and a mathematical study was undertaken in an attempt at answering them.

It became clear in the mathematical treatment that the operations of addition, subtraction, multiplication and division between fraction symbols seem to have a subtly different axiomatic structure from the corresponding operations in the natural numbers.

The question whether it is valid to multiply a fraction symbol by a whole number symbol requires a two-part answer. Firstly the 'multiply' is a symbol for the compound operation if the fraction symbol is the multiplier or for repeated addition if the whole number is the multiplier. Secondly it is most important which meaning is attached to the fraction symbol. As a compound operator there would be no problem in carrying out the 'multiplication' but as a rational number it would have little meaning since multiplication between rational numbers is defined between pairs.
The traditional treatment of the 'fraction' in grade six is not as a rational number but as a fractional number since neither ratio nor the operation of division usually enters the definition of a fractional part of a whole. It is possible that the fractional number when used as an operator could be better treated as a compound operator in school.

No decision is made as to whether a fractional number and a rational number are identical or not. There is no doubt that they derive from widely differing areas and that the terms have widely differing meanings. However, even if the terms have different connotations, the symbols are apparently identical and philosophical questions are consequently raised: Is a number independent of its symbol? Is a number independent of its derivation?

The question posed in the literature as to whether a fraction and a ratio are the same can be clearly answered. Both are applications of ordered pairs but the connotations of the terms are quite different so although similar structures do exist pronounced differences also exist.

Another result of the mathematical study was that clear isomorphism was shown between natural divisions, indicated divisions, rational numbers and rate pairs. When the fraction number is used as an operator there are subtle changes in the operational structure though familiar patterns are still noticed.
The fact that natural divisions are isomorphic to a subset of the rational numbers is a most important finding since this is neglected in school despite the fact that they are probably more easily understood than fractional parts.

The suggestion that the fraction symbol be introduced in grade five on a pure mathematical basis gives every indication of being worthy of proper research investigation since there was no significant difference between the scores of the classes taught experimentally or traditionally.

2. Delimitations

As was explained earlier some extraneous variables were automatically introduced with the modified Trivett schema since it was desired that no hint of fraction applications would enter that part of the study. There are six possible reasons for an absence of significant difference in the test results of the investigation.

1. The time for manipulation of physical models was severely limited by the time available for the investigation if the instructional time was to be equal for both classes.

2. The smallness of the groups could undermine the statistical validity of the analysis of the test results.

3. The tests contained no reference to physical models and
hence there was no opportunity given for special skills and insights with physical models to be reflected in the test results. It was necessary to exclude such fraction diagrams since those using the experimental schema were not even aware that they were studying fractions and therefore would have been at a great disadvantage.

4. Because of the many applications of fractions and the variation of physical models, confusion could result.

5. Since the symbols $\frac{a}{b}$ and $(a,b)$ are different from the point of view of spacial configuration, it is possible that numerical manipulation is easier with one than the other. It was noted that fraction ordered pairs encouraged a left to right flow while the traditional notation tended to require a vertical flow.

6. The use of operation symbols from the natural numbers used for non-identical operations in fraction symbolism yet identical in form could cause confusion when fractions are studied traditionally while in the modified Trivett schema this problem is avoided.

3. Conclusions and Areas of Further Investigation

The results of this study indicate that the theoretical schema for introducing fraction symbolism may be realistic and that the suggestion that fractions be introduced using ordered pair symbolism with no reference to physical applications merits more exact research techniques.
This result is indicated statistically but is subjectively confirmed by the investigator. The atmosphere in the class using the modified Trivett schema was more carefree than in the other class. To some extent the students had invented "the game" and if they forgot the rules then it was no grave matter. In contrast, in the class taught traditionally there was a slight indication of emotion at 'not understanding' the physical situations from which the symbols derived. It should be emphasised that the field study was not a proper research experiment and no firm conclusions may be taken from the results.

The most rewarding aspect of the study on the author's part was that he was able to reach a far greater understanding of the widely misunderstood area of fractions.

As was indicated in the section entitled "Summary of Results" many questions remain unanswered. It was in the intention of the small field study to indicate the realism of one part of a whole curriculum schema with a view to possible future more exact research, so we are still left with the questions as to the weight of the following three factors:

1. Relaxation accompanying a game situation versus the usual lesson format.

2. Number manipulation without physical applications versus physical situations leading to number manipulation.

3. Different nomenclature and symbolism justified and discovered by the students versus traditional nomen-
clature and vertical symbolism derived from tradition.

Other questions posed for further investigation are:

(4) Does it improve student understanding and skill with fraction symbols if a student is aware of their many meanings?

(5) Does the introduction of fraction symbols as fractional parts of whole objects psychologically impede future learning of other meanings?

(6) Does the introduction of the fraction symbol in a physical setting create confusion because of multiple physical ambiguities?

(7) Does it matter whether the fraction symbol is taught as a symbol for a number or as 'one number over another'?

(8) Are natural divisions easier for students to handle than fractional parts of wholes?

It is believed that fraction symbols are not inherently difficult so that the results of this thesis are an indication that their teaching has become subject to a tradition which is possibly not the best way of handling them. Whether the suggested theoretical schema of utilizing a shared structure as each application is met with is justified is left for further investigation. That a better approach to fractions is needed in our teacher training literature and schools is confirmed by Skemp in the quotation:

Some people, indeed, go through life without ever really understanding
fractional numbers; and small blame to them. Their teacher probably never understood them either.\textsuperscript{47}

and by Copeland when he concludes:

\begin{quote}
Fractions are today the nemesis of many a fifth-grader and many an adult as well.\textsuperscript{48}
\end{quote}

\textsuperscript{47} Skemp, op. cit., p. 45.

\textsuperscript{48} Copeland, op. cit., p. 116.
# APPENDIX A

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I. INTRODUCTION

The essential concept of an indicated operation will be used to show the development of the integers from 'indicated subtractions' of natural numbers and of the rational numbers from 'indicated divisions' of integers.

The concept of an indicated operation is the idea that a symbol for a number may be written in the form of a binary operation between two elements. For example, the symbol for the number six may be the standard symbol '6', or the non-standard (8 - 2), (2 + 4), (2 x 3), (12 ÷ 2). It will be shown that the natural and integer quotients, for example \( \frac{12}{2}, \frac{18}{3} \), exhibit precisely the same operational structure as the rational numbers and in fact are isomorphic to a subset of the rational numbers.

It will be seen that a fractional part is merely the application of a rational number on some object, group of objects, measure or number but that the operations between such operators is similar but no isomorphic to the earlier structures of integer quotients and rational numbers.

Lastly it will be shown that comparisons have certain similarities but also definite differences from earlier meanings of the fraction symbol.
It should be emphasised that no text was discovered which contained all the material in this Appendix and in fact parts were taken from no book and, as far as is known, are original.
II. AXIOMS OF EQUALITY

The following axioms will be accepted throughout this chapter:

1. Reflexive axiom of equality (REF):
   For all numbers, \( a \), \( a = a \)

2. Symmetric axiom of equality (SYM):
   For all numbers \( a, b \), if \( a = b \) then \( b = a \)

3. Substitution axiom of equality (SUB):
   If \( \text{op} \) stands for any binary operation then for all numbers \( a, b, c, d \) if \( a = b \)
   and \( b \ \text{op} \ c = d \) then \( a \ \text{op} \ c = d \).
   If \( a = b \) and \( c \ \text{op} \ b = d \) then \( c \ \text{op} \ a = d \)

4. Transitive axiom of equality (TRAN)
   For all \( a, b, c \), if \( a = b \) and \( b = c \)
   then \( a = c \).
III. NATURAL NUMBERS

The axioms for the natural number sequence \((N)\) will follow Peano's axioms\(^{49}\) and are as follows:

A natural number sequence is a set, \(N\), of elements together with a successor function \((^+)\) satisfying the following axioms:

Axiom N.1 For each element \(a\) of \(N\), there is a unique element \(a^+\) in \(N\), called the "successor" of \(a\).

Axiom N.2 No two elements \(a, b\) have the same successor.

Axiom N.3 \(N\) contains a unique element \(1\) which is not the successor of any element of \(N\).

Axiom N.4 Every set \(S\) of elements of \(N\) such that

(a) \(1\) is in \(S\), and (b) if \(a\) is in \(S\), then its successor \(a^+\) is in \(S\) for any element \(a\), then \(S\) consists of the whole set \(N\).

A segment \(<ln>\) of a natural number sequence \(N\) will be defined as a subset \(S\) of \(N\) such that

A set will be said to have number, n, (or to contain n elements) if its elements can be placed in one-to-one correspondence with the members of the segment $\langle ln \rangle$. The number of two natural number sequences will be said to be equal if the segments $\langle lm \rangle$ and $\langle ln \rangle$ may be put in one-to-one correspondence with each other. The sum, $a+b$, of two natural numbers will be defined by the following:

$$a + l = a^+ \quad (S.N.1)$$
$$a + b^+ = (a + b)^+ \quad (S.N.2)$$

and the operation is called addition.

**Theorem 1.**

Natural numbers obey an associative law for addition. For all natural numbers $a, b, c$ \( (a + b)^+ + c = a + (b + c) \)

**Theorem 2.**

Natural numbers obey a commutative law for addition. For all natural numbers $a, b$ \( a + b = b + a \)

**Theorem 3.**

Natural numbers obey a cancellation law of addition. For all natural numbers $a, b, c$ if $a + c = b + c$ then $a = b$.

If for any two natural numbers $a, b$, there is a natural number $c$ such that $b + c = a$, we say that $a$ is greater than
b (a > b) or we say b is less than a (b < a). Furthermore, we call c the "difference" between a and b and write 
c = a − b where, for the purpose of this study only, the operation sign between a and b is called natural subtraction since the difference between a and b is a natural number. If the difference were not a natural number we would use the usual subtraction sign, so if a < b then we would write a − b as an 'indicated' subtraction.

Theorem 4.
There are three associativity relations for natural subtractions.
For all natural numbers a,b,c provided that each difference is a member of the natural numbers
(a) a − (b − c) = (a − b) + c
(b) a − (b + c) = (a − b) − c
(c) a + (b − c) = (a + b) − c

Theorem 5.
There is an unlimited number of natural subtractions equal to any given natural number. If a,b,c,n are all natural numbers and

c = b − a then c = (b + n) − (a + n)

The multiplication of any two elements a,b of the natural number sequence, symbolized by either a·b or a x b will be defined as follows:
\[ 1 \cdot b = b \quad \text{(N.N.1)} \]

and \[ a^* \cdot b = a \cdot b + b \quad \text{(N.N.2)} \]

Theorem 6.
Natural numbers obey distributive laws of multiplication over addition. For all natural numbers \( a, b, c \),
(1) \( a \cdot (b + c) = a \cdot b + a \cdot c \)
(2) \( (b + c) \cdot a = b \cdot a + c \cdot a \)

Theorem 7.
Natural numbers obey a commutative law for multiplication. For all natural numbers \( a, b \)
\[ a \cdot b = b \cdot a \]

Theorem 8.
Natural numbers obey an associative law for multiplication. For all natural numbers \( a, b, c \)
\[ (a \cdot b) \cdot c = a \cdot (b \cdot c) \]

Theorem 9.
Natural numbers obey a cancellation law for multiplication. For all natural numbers \( a, b, c \)
If \( a \cdot c = b \cdot c \) then \( a = b \)

If, for any two natural numbers \( a, b \) there is a natural number \( c \) such that
We will call \( c \), the "natural quotient" of "a by b" and write \( c = \frac{a}{b} \) calling b, the "divisor" of a.

It should be noted that \( a \) is also conventionally written \( a \div b \) or \( a \) but the wavy line symbol is invented for the purpose of this study to differentiate a natural from an indicated quotient. We use the symbol \( \frac{a}{b} \) where \( a, b \) are natural numbers and \( a \neq 0 \pmod{b} \), to denote an indicated quotient. To make this symbolism clear \( 8 \div 4 \) can be written \( 8 \) but \( 4 \div 8 \) would have to be written \( 4 \). Referring back to natural subtraction and indicated subtraction, four can be written \( 8 - 4 \) but \( 4 - 8 \) may not be written with the wavy line, according to our convention.

Theorem 10.

For all natural numbers \( a, b, c, p, q \)

if \( \frac{b}{c} = p \) and \( \frac{a}{b} = q \) then there exists \( \frac{a}{c} \)

Theorem 11.

Division Theorem

If \( a \) and \( b \) are natural numbers and \( b < a \) then either there exists a natural number \( p \) such that
or there exists natural numbers $q$ and $r$ with $r < b$ such that

$$a = bq + r$$

We will define the number zero, 0, as a number such that

1. $a + 0 = a$ where $a \in \mathbb{N}$

and 2. the set consisting of the natural number sequence and 0 has all the previously proved properties, and will be denoted $\mathbb{N}^0$.

Theorem 12.

Zero is a unique element of $\mathbb{N}^0$.

Theorem 13.

For all natural numbers $a$,

$$a \cdot 0 = 0 \cdot a = 0$$
IV. THE INTEGERS

We have seen that the indicated subtraction $a - b$, where $a, b$ are natural numbers and $a < b$, has no corresponding natural number but in some physical situations can acquire meaning and be utilized temporarily to produce consistent results. For example, a deposit in a bank account of three dollars followed by a withdrawal of four dollars can be considered with a subsequent deposit of ten dollars as:

$$(3 - 4) + 10 = 9.$$  

The indicated subtraction $(3 - 4)$ has no meaning within the natural number system but its very usefulness in the market place, as a debit, has led mankind to use indicated subtractions. This motivated mathematicians to postulate the existence of numbers than can be symbolized by these indicated subtractions.

Theorem 14.

If $a, b$ are natural numbers $a < b$ and a number $y$ exists such that $a - b = y$ then there is an unlimited number of indicated

Subtractions equal to $y$.

$$y = (a + n) - (b + n) \text{ where } n \in \mathbb{N}$$

Since $(a - b)$ and $(a + 1) - (b + 1)$ are both equal to $y$, we say that they are equivalent symbols for $y$ and all such equivalent symbols form a set called an equivalence class of differences which can be denoted $[a - b]$ where $(a - b)$ is a member of the equivalence class. Natural subtractions also form equivalence classes of differences where a unique natural number is the common difference for each member of a particular equivalence class of difference (Theorem 5).

Since this new symbolism covers the numbers already studied in the set $\mathbb{N}$ and new numbers derived from the indicated subtractions, the whole system of numbers defined by the new symbolism is called the integers and new axioms are introduced with the intention of maintaining an isomorphism between the natural subtractions and the natural numbers and producing consistency when the indicated subtractions are used.

The "opposite", $x'$ of an integer $x = (a - b)$ is $x' = (b - a)$. Thus each indicated subtraction is the opposite of some natural number since if $a < b$ then $b > a$. This fact usefully leads to another symbolism for the integers since it is the union of two sets of natural number sequences where one is the opposite of the other and a set containing zero.
Now let us consider an axiom set for the integers. The set \( I \) of integers consists of an element zero, together with the elements of two natural number sequences, one designated "positive" \((+a)\) and the other "negative" \((-a)\) satisfying the following:

**Axiom I 1.** Addition, \(+\), is defined for two integers \( a, b \) by the following
\[
a + 1 = a^+ \quad (S.I.1.)
\]
\[
a + b^+ = (a + b)^+ \quad (S.I.2.)
\]
while multiplication, \( \cdot \) or \( \times \), is defined for two integers \( a, b \) by the following
\[
1 \cdot a = a \quad (M.I.1.)
\]
\[
a^+ \cdot b = a \cdot b + b \quad (M.I.2.)
\]
It should be noted that these definitions have identical structure to the addition and multiplication of natural numbers, \( S.N.I., S.N.2., M.N.I. \) and \( M.N.2. \) though the elements are integers in these cases.

Both defined operations obey the associative, commutative, distributive and cancellation laws as follows:

If \( a, b, c \) are integers:

**Associative Laws**
\[
I (a) \quad a + (b + c) = (a + b) + c
\]
\[
I (b) \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c
\]
Commutative Laws
I (c)  \( a + b = b + a \)
I (d)  \( a \cdot b = b \cdot a \)

Distributive Law
I (e)  \( a(b + c) = a \cdot b + a \cdot c \)

Cancellation Laws
I (f)  \( a + b = c + b \Rightarrow a = c \)
I (g)  \( a \cdot b = c \cdot b \Rightarrow a = c \) provided that \( b \neq 0 \)

Axiom I 2.

Subtraction and the limited form of division
where the quotient represents an integer
(in this study designated 'integer division'), are defined from addition and
multiplication as follows:
If \( a,b,x,y \) are integers then
2(a)  \( (a - b) \) is a number \( x \) such that
\[ b + x = a \] (if such a number exists)
2(b)  If \( b \neq 0 \) then \( a \) is a number \( y \) such
\[ \frac{1}{b} \] that \( b \cdot y = a \) (if such a number
exists).

Axiom I 3.

The positive and zero numbers combine in
the same way as the natural numbers and
zero to which they correspond.

If \( a,b \in \mathbb{N} \) then
\[ +a + +b = + (a + b) \quad -a + +b = + (a \cdot b) \]

Axiom I 4.

If \(a \in \mathbb{N}\) then
\[ +a + -a = 0 \]

Theorem 15.

If \(a, b\) are members of the set of natural numbers then

(a) \(-a + -b = -(a + b)\)
(b) \(+a + -b = -b + +a\)
(c) if \(b < a\) then \(+a + -b = +(a - b)\)
(d) if \(a < b\) then \(+a + -b = -(b - a)\)

Theorem 16.

If \(a, b\) are members of the set of integers then for all \(a, b\)
\[ a - b = a + b' \]

Theorem 17.

The zero element, 0, is unique in the set of integers

Theorem 18(a)

For all integers \(a, b\) \[ a \cdot b' = (ab)' \]

Theorem 18(b)

For all natural numbers \(a, b\)
\[ +a \cdot +b = -a \cdot -b = +a \cdot b \]
\[ +a \cdot -b = -a \cdot +b = -a \cdot b \]

Thus we now have the capacity to symbolize both positive and negative whole numbers but the integer symbolism fails to symbolize non-whole numbers.
V. INTEGER AND NATURAL QUOTIENTS

An alternative symbolization of the numbers symbolized by the integers is the ordered pair \( \frac{a}{b} \), where \( a, b \in \mathbb{I} \) and \( a \equiv 0 \pmod{b} \).

As has already been explained we will call these quotients "integer quotients" if the number pair comprises integers or "natural quotients" if the components are natural numbers. In a similar way to the ordered pair symbolism for an integer an infinite number of symbols are equal to any given number which can also be symbolized by an integer. We have equivalence classes of such symbols, each symbolizing the same number viz:

Theorem 19.

If \( \frac{r}{p} = q \) then \( \frac{n \cdot r}{n \cdot p} = q \) when \( n, p, q, r \in \mathbb{I} \) and \( n, p \neq 0 \).

As before, square brackets can symbolize the equivalence class and \( \left[ \frac{a}{b} \right] \) will represent any particular member of the equivalence class where \( a, b \) are mutually prime, thus:

\[
\left[ \frac{a}{b} \right] = \left\{ \frac{a}{b}, \frac{2a}{2b}, \frac{3a}{3b}, \ldots \right\}
\]

Theorem 20.

If \( p \in \mathbb{I} \), then \( \frac{1}{p} = \frac{1}{71} \).
Theorem 21.

If $p \in \mathbb{I}$, $p \neq 0$, then $p = \frac{1}{p}$

Theorem 22.

If $n, p, r \in \mathbb{I}$, $p \neq 0$, then $n \cdot \frac{r}{p} = \frac{n \cdot r}{p}$

Theorem 23.

If $n, p, r \in \mathbb{I}$, $n, p \neq 0$, then $\frac{r}{p} = \frac{r}{n \cdot p}$

Theorem 24.

If $a, b, d \in \mathbb{I}$ and $d \neq 0$, then

$$\frac{a + b}{d} = \frac{a + b}{d}$$

Theorem 25.

If $a, b, c, d \in \mathbb{I}$ and $b, d \neq 0$, then

$$\frac{a + c}{b} = \frac{d \cdot a + b \cdot c}{d \cdot b}$$

Theorem 26.

If $a, b, c, d \in \mathbb{I}$, $b, d \neq 0$, then

$$\frac{a \cdot c}{b} = \frac{a \cdot c}{b \cdot d}$$

Theorem 27.

If $a, b, c, d \in \mathbb{I}$, $b, c, d \neq 0$, then

(a) $$\frac{a}{b} = \frac{a \cdot d}{b \cdot c} = \frac{a \cdot d}{b \cdot c}$$

(b) $$\frac{a}{b} = \frac{a \cdot c}{b} = \frac{a \cdot c}{b}$$
Thus we have a symbol system which does not extend the numbers capable of being symbolized by the integers. However, theorems concerning the operations met with in the integer section are shown and it has been seen that the four operations can be carried out in certain patterned ways using integer quotients. Natural quotients show precisely the same properties and, to save tedious repetition and space, theorems particular to natural quotients will not be quoted.
VI. THE RATIONAL NUMBERS.

If we wish to extend the number of numbers beyond those that the integer quotients will symbolize we must consider the "indicated quotients". The "indicated quotient" is a quotient $\frac{a}{b}$ where $a \not\equiv 0 \pmod{b}$ and $a,b \in \mathbb{N}$.

If we postulate the existence of a number $\frac{a}{b}$ then by the definition of division

$$ a = \frac{a \cdot b}{b} $$

for example $2 = \frac{2 \cdot 3}{3}$

The union of the integer quotients and these indicated quotients produces a symbol system which extends the number of numbers capable of being symbolized. This symbol system is called the rational numbers and is a set, $\mathbb{Q}$, of integer ordered pairs having the symbol $\frac{a}{b}$, read $a$ over $b$, where $b \neq 0$, and satisfying the following axioms:

Axiom Q 1

An equivalence relation similar to that for integer quotients in theorem 19 exists. Addition, $\cdot$ is defined for two rational numbers, $a,b$ by the
following:

If \( n \in \mathbb{I} \),
\[
\begin{align*}
a \oplus \frac{n}{n} &= a^+ \quad \text{(SQ1)} \\
a \oplus b^+ &= (a + b)^+ \quad \text{(SQ2)}
\end{align*}
\]

Multiplication \( \otimes \) or \( \odot \) is defined for two rational numbers \( a, b \) by the following. If \( n \in \mathbb{I} \),
\[
\begin{align*}
n \odot a &= a \\ n \odot \frac{n}{n} &= a^+ \quad \text{(MQ1)} \\
a^+ \odot b &= a \odot b + b \quad \text{(MQ2)}
\end{align*}
\]

It should be noted that these two operations are not identical with either of the natural number operations or of the integer operations. These two operations obey the associative, commutative, distributive and cancellation laws. If \( a, b, c \in \mathbb{Q} \)

**Associative Laws**

1(a) \( a \oplus (b \oplus c) = (a \oplus b) \oplus c \)
1(b) \( a \odot (b \odot c) = (a \odot b) \odot c \)

**Commutative Laws**

1(c) \( a \oplus b = b \oplus a \)
1(d) \( a \odot b = b \odot a \)

**Distributive Law**

1(e) \( a \odot (b \oplus d) = a \odot b + a \odot d \)

**Cancellation Laws**

1(f) \( a \oplus c = b \oplus c \Rightarrow a = b \)
1(g) \( a \odot c = b \odot c \Rightarrow a = b \)
Axiom Q2.

Subtraction and division are defined from addition and multiplication as follows. If \( a, b, x, y \in \mathbb{Q} \)

- \( a \ominus b \) is a number \( x \) such that \( b \oplus x = a \) (if such a number \( x \) exists).
- \( a \odot b \) is a number \( y \) such that \( a = b \circ y \) (if such a number \( x \) exists).

Axiom Q3(a).

The rational numbers \( \frac{p}{q} \), where \( p \equiv 0 \pmod{q} \)

combine in the same way as the integer quotients in theorems 18-27.

Axiom Q3(b).

The rationals, with the second member \( 1 \frac{p}{l} \)

combine in the same way as the integers \( p \).

Axiom Q4(a).

If \( q \in \mathbb{I}, q \neq 0 \)

\[
\frac{q}{l} \odot \frac{1}{q} = 1
\]

Axiom Q4(b).

If \( p, q \in \mathbb{I}, q \neq 0 \)

\[
\frac{p}{q} = \frac{p \odot 1}{q}
\]

Theorem 28

If \( k, p, q \in \mathbb{I}, q \neq 0 \), then \( \frac{kp}{kq} = \frac{p}{q} \)
The reciprocal of a rational number \( a = \frac{p}{q} \) is symbolized \( a^{-1} \)
and \( a^{-1} = \frac{q}{p} \) where \( p, q \in \mathbb{Z} \), \( p, q \neq 0 \)

Theorem 29
If \( a, b \in \mathbb{Q} \) and \( b^{-1} \) exists then \( a = \frac{a \circ b^{-1}}{b} \)

Theorem 30
If \( p, q, r, s \in \mathbb{Z} \), \( q, s \neq 0 \) then
(a) \( \frac{p}{s} \circ \frac{q}{s} = \frac{p + q}{s} \)
(b) \( \frac{p}{q} \circ \frac{r}{s} = \frac{ps + qr}{qs} \)

Theorem 31
If \( p, q, r, s \in \mathbb{Z} \), \( q, s \neq 0 \) then
(a) \( \frac{p}{q} \circ s = \frac{p}{s} \)
(b) \( \frac{p}{q} \circ \frac{r}{s} = \frac{p \cdot r}{q \cdot s} \)

Theorem 32
If \( a, b, c, d \in \mathbb{Q} \) and \( a = c \) and \( a = d \) then \( c \) and \( d \) belong to the same equivalence class of rational numbers.

Theorem 33
If \( a, b, c \in \mathbb{Z} \), \( c \neq 0 \) then by axiom Q3(a)
(a) \( a \circ b = \frac{a \circ b}{c} = \frac{ab}{c} \)
(b) \( b \circ a = \frac{b \circ a}{c} = \frac{ab}{c} \)
(c) \( a \circ b = \frac{a \circ b}{c} = \frac{ac + b}{c} \)
(d) \( \frac{b}{c} + a = b \otimes a = \frac{b \div ac}{c} \)

(e) \( \frac{b}{c} \div a = b \otimes a = \frac{b}{ac} \) \( a, c \neq 0 \)

(f) \( \frac{a}{c} \div b = a \otimes b = \frac{ac}{\frac{b}{c}} \) \( b \neq 0 \)

The number that \( a \) symbolizes is precisely the same number that \( a \) symbolizes so they are equivalent symbols and may replace each other.
VII. FRACTIONAL PARTS

Skemp considers a fraction involves "sharing and combining" as an "action" on a "standard object", characterized by a "double operation". He would consider that any object's magnitude, \( y \), can be "shared and combined" in two ways. If the "sharing" is into \( b \) congruent parts and the "combining" producing \( a \) of the congruent parts then

\[
\frac{a}{b} \text{ of } y \text{ means } y \cdot \frac{a}{b} \text{ or } y \div b \cdot a.
\]

We shall call the first the "initial multiply" definition of the fraction "action" or operation while the second will be called the "initial divide" definition of the fraction operation.

By the first definition

\[
\frac{a}{b} \text{ of } y = (y \times z) \div b \quad \text{if } b \neq 0
\]

\[
y \times \frac{a}{b} = \begin{array}{c}
\frac{a}{b} \\
1
\end{array}
\quad \text{Axiom Q3(a)}
\]

\[
y \times \frac{a}{b} \circ \frac{1}{b} = \frac{a}{b} \quad \text{Axiom Q3(b)}
\]

\[
y \times \frac{1}{b} \circ \frac{a}{b} = \frac{a}{b} \quad \text{Axiom Q1(d)}
\]

\[
y \div b \circ \frac{a}{b} = \frac{a}{b} \quad \text{Axiom Q3(b)}
\]

\[
y \div b \times a \quad \text{Axiom Q3(a)}
\]

\footnote{Skemp, op. cit., p. 189.}

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which is the definition of the fraction operation using the second definition. Hence the magnitude of the result of the fraction operation is the same whether the "initial divide" definition is used or the "initial multiply" definition is used. Thus:

Theorem 34

\[
\frac{a}{b} \text{ of } y = (a \times y) \div b = (a \div b) \times y = \frac{a \cdot y}{b}
\]

By theorem 33, \( \frac{a \times y}{b} = \frac{a}{b} \cdot y \), so if \( y = 1 \) then the fraction operations would result in the rational numbers. However, the multiplication of a magnitude by a rational number has precisely the same effect as the operation of a fraction. Thus an alternate definition of a fraction is a rational number acting as a multiplication operator.

Theorem 35

If \( a, b, c, n \in \mathbb{N} \), \( n, b, \neq 0 \)

then \( \frac{a}{b} \text{ of } c = \frac{na}{nb} \text{ of } c \)

so

"\( \frac{a}{b} \text{ of } c \)" and "\( \frac{na}{nb} \text{ of } c \)" are equivalent operations.

Theorem 36

If \( a, b, c, d, e \in \mathbb{N} \), \( b, d \neq 0 \)

then \( \frac{a}{b} \text{ of } (\frac{c}{d} \text{ of } e) = \frac{a \cdot c}{b \cdot d} \text{ of } e \)

so

"\( \frac{a}{b} \text{ of } c \)" of is equivalent to "\( \frac{a \cdot c}{b \cdot d} \text{ of } e \)"
The fraction operator is often used as a fraction of a measure and if \( K \in Q \) and \( \mu_1 \) and \( \mu_2 \) are verbal units of measure then

Theorem 37

If \( a, b, c, d \in I, \quad b, d \neq 0 \),

then \( \left( \frac{a}{b} \right) \cdot \left( \frac{d}{e} \right) = \left( \frac{a + d}{b + e} \right) \) of \( c \)

Theorem 38

There always exists units of measure \( \mu_1', \mu_2' \) such that

\[ \frac{a}{b} \text{ of } K\mu_1 = aK\mu_2, \quad a, b \in I, \quad b \neq 0 \]

Theorem 39

If \( a, b, n \in I, \quad b, n \neq 0 \),

then \( \frac{a}{b} \text{ of } K\mu_1 = \frac{na}{nb} \text{ of } K\mu_1 \)

Theorem 40

If \( a, b, c, d \in I, \quad b, d \neq 0 \),

then \( \frac{a}{b} \text{ of } \frac{c}{d} \text{ of } K\mu_1 = \frac{a \cdot c}{b \cdot d} \text{ of } K\mu_1 \)

Theorem 41

If \( a, b, c, d \in I, \quad b, d \neq 0 \),

then \( \frac{a}{b} \text{ of } K\mu_1 + \frac{c}{d} \text{ of } K\mu_1 = \left( \frac{a}{b} \oplus \frac{c}{d} \right) \text{ of } K\mu_1 \)
A comparison is a number pair where each number is associated with designated characteristics of two sets and is symbolized a:b or \( \frac{a}{b} \) where \( a, b \in \mathbb{I} \).

If more than one pair of numbers measure and are associated with the same two designated characteristics of two sets, they represent the same "rate" if they are in the following relation; if \( a, b, c, d \in \mathbb{I} \) and \( a:b, c:d \) are the two comparison pairs then \( a \cdot d = b \cdot c \)

or \( \frac{a}{b} = \frac{c}{d} \) \( b, d \neq 0 \)

Such rate pairs exhibiting this equivalence relation are said to be "in the same proportion" and a set of such rate pairs is called a "proportion". The symbol used for such an equivalence relation is \( a:b - c:d \) meaning the rate pair \( a \) to \( b \) is equivalent to the rate pair \( c \) to \( d \) so \( ad = bc \).

Theorem 42

If \( a, b, n \in \mathbb{I}, b, n \neq 0 \)

then \( a : b - na : nb \)

We define the addition of rate pairs

\[ a : b + c : d = (ad + bc) : bd \]

if \( a, b, c, d \in \mathbb{I} \) and \( a \) and \( c \) refer to the same characteristic quality and \( b \) and \( d \) refer to the same
characteristic quality. The multiplication of these rate pairs is defined as

\[ a : b \otimes c : d = a \cdot c : b \cdot d \]

if \( a, b, c, d \in \mathbb{I} \)

Theorem 43

Addition of the elements of a proportion is commutative.

If \( a, b, c, d \in \mathbb{I} \) and \( a : b \doteq c : d \)
then \( a : b \oplus c : d = c : d \oplus a : b \)

Theorem 44

Addition of the elements of a proportion is associative.

If \( a, b, c, d, e, f \in \mathbb{I} \) and \( a : b \doteq c : d \doteq e : f \)
then \( a : b \oplus (c : d \oplus e : f) = (a : b \oplus c : d) \oplus e : f \)

Theorem 45

Multiplication of the elements of a proportion is commutative

If \( a, b, c, d \in \mathbb{I} \) and \( a : b \doteq c : d \)
then \( a : b \otimes c : d = c : d \otimes a : b \)

Theorem 46

Multiplication of the elements of a proportion is associative

If \( a, b, c, d, e, f \in \mathbb{I} \) and \( a : b \doteq c : d \doteq e : f \)
then \( a : b \otimes (c : d \otimes e : f) = (a : b \otimes c : d) \otimes e : f \)
Theorem 47

Multiplication of the elements of a proportion is distributive

If $a, b, c, d, e, f \in I$ and $a : b c : d - e : f$
then $a : b \times (c : d + e : f) = a : b \times c : d + a : b \times e : f$

In recent times rate pairs have been written as a fraction symbol where $a : b$ corresponds to $\frac{a}{b}$. The fraction symbol notation makes it abundantly clear that the structure of the rate pair operation symbol system illustrated in theorems 42-47 is isomorphic to the rational number symbol system illustrated in theorems 28-33.

Now the rate pairs, when members of a proportion, can be seen on the one hand as picturing a variation in terms but on the other hand a constancy if the terms are divided. As the numbers change in magnitude the quotients of the members of each rate pair in the proportion remains constant. Thus a rational number, which may be considered a quotient (Axiom Q4(b)), is a common characteristic of the rate pairs in a proportion. Thus given any rate pair $a : b$ there is a unique quotient (or rational number) that corresponds to it, namely $\frac{a}{b}$. Thus a rate pair could be considered an alternative symbol for a rational number and since an alternative name for a rate pair is a ratio, indirectly it gives its name to the number.
# APPENDIX B

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I. TEST FOR UNDERSTANDING OF
   SOME FRACTION CONCEPTS

Change as indicated:

1. \( \frac{3}{4} = \frac{8}{8} \)

2. \( \frac{8}{3} = \frac{3}{3} \)

3. \( \frac{5}{6} = \frac{24}{24} \)

4. \( \frac{31}{7} = \frac{7}{7} \)

Reduce to simplest terms:

5. \( \frac{25}{35} = \)

6. \( \frac{16}{24} = \)

7. \( \frac{64}{18} = \)

8. \( \frac{30}{16} = \)

9. \( \frac{54}{81} = \)

10. In which pair are the fractions equivalent?

    A. \( \frac{5}{6}, \frac{5}{12} \)

    B. \( \frac{2}{3}, \frac{8}{12} \)

    C. \( \frac{3}{7}, \frac{4}{7} \)

    D. \( \frac{3}{5}, \frac{9}{10} \)
11. Which of these fractions is NOT equal to the other three?

A. \( \frac{2}{3} \)

B. \( \frac{8}{12} \)

C. \( \frac{3}{4} \)

D. \( \frac{4}{6} \)

12. \( \frac{2 \times 3}{2 \times 4} \) is the same as (a) \( \frac{1}{2} \) (b) \( \frac{3}{4} \) (c) \( \frac{4}{3} \) (d) \( \frac{3}{2} \)

13. Which of the following are true?

A. \( \frac{3}{4} = \frac{3 + 2}{4 + 2} \)

B. \( \frac{3}{4} = \frac{3 \times 3}{4 \times 3} \)

C. \( \frac{3}{4} = \frac{3 - 2}{4 - 2} \)

D. \( \frac{3}{4} = \frac{3 \times 3}{4 \times 4} \)

14. Which of these is a correct way of finding a fraction equivalent to \( \frac{9}{12} \)?

A. \( \frac{9}{12} = \frac{9 + 3}{12 + 3} = \frac{12}{15} \)

B. \( \frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4} \)

C. \( \frac{9}{12} = \frac{9 - 3}{12 - 3} = \frac{6}{9} \)

D. \( \frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4} \)
15. Which of the following sets is composed of like fractions?

A. \( \frac{1}{5}, \frac{2}{5}, \frac{1}{3}, \frac{2}{3} \)

B. \( \frac{1}{2}, \frac{1}{6}, \frac{5}{12}, \frac{5}{6} \)

C. \( \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{3}{9} \)

D. \( \frac{1}{7}, \frac{3}{7}, \frac{6}{7}, \frac{8}{7} \)
II. TEST FOR MANIPULATIVE SKILLS WITH BASIC FRACTION OPERATIONS

Write all answers as improper or proper fractions in simplest form.

ADD:

1. \( \frac{3}{5} + \frac{1}{5} = \)

2. \( \frac{1}{2} + \frac{1}{4} = \)

3. \( \frac{5}{8} + \frac{1}{2} = \)

4. \( \frac{2}{3} + \frac{5}{6} = \)

5. \( \frac{1}{4} + \frac{1}{3} = \)

6. \( \frac{4}{5} + \frac{2}{3} = \)

7. \( \frac{3}{10} + \frac{3}{8} = \)

8. \( \frac{11}{6} + \frac{7}{10} = \)

SUBTRACT:

9. \( \frac{4}{5} - \frac{1}{5} = \)

10. \( \frac{15}{4} - \frac{11}{8} = \)

11. \( \frac{11}{3} - \frac{2}{3} = \)

12. \( \frac{4}{1} - \frac{7}{3} = \)

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13. \( \frac{12}{5} - \frac{4}{5} = \)

14. \( \frac{10}{3} - \frac{5}{3} = \)

15. \( \frac{17}{4} - \frac{8}{3} = \)

16. \( \frac{13}{4} - \frac{13}{5} = \)

**MULTIPLY:**

17. \( \frac{1}{4} \times \frac{12}{1} = \)

18. \( \frac{2}{3} \times \frac{18}{1} = \)

19. \( \frac{4}{1} \times \frac{19}{8} = \)

20. \( \frac{3}{4} \times \frac{2}{3} = \)

21. \( \frac{13}{5} \times \frac{5}{6} = \)

22. \( \frac{27}{8} \times \frac{11}{3} = \)

**DIVIDE:**

23. \( \frac{3}{4} \div \frac{2}{1} = \)

24. \( \frac{4}{1} \div \frac{2}{3} = \)

25. \( \frac{4}{5} \div \frac{2}{3} = \)

26. \( \frac{3}{5} \div \frac{1}{4} = \)

27. \( \frac{8}{3} \div \frac{2}{3} = \)

28. \( \frac{28}{5} \div \frac{8}{5} = \)
### Table V

**Raw Scores Obtained in the Study**

| SUBJECT | Understanding Test | Skills Test | |
|---------|--------------------|-------------|
|         | Group Traditionally Taught | Group Experimentally Taught | |
|         | Before | After | Before | After | Before | After | Before | After |
| 1       | 1      | 7     | 4      | 5     | 0      | 1     | 0      | 4     |
| 2       | 2      | 11    | 8      | 15    | 4      | 15    | 0      | 12    |
| 3       | 2      | 7     | 5      | 14    | 0      | 13    | 0      | 17    |
| 4       | 1      | 14    | 8      | 14    | 3      | 21    | 3      | 0     |
| 5       | 1      | 8     | 8      | 14    | 0      | 17    | 0      | 20    |
| 6       | 0      | 6     | 6      | 12    | 0      | 4     | 0      | 15    |
| 7       | 3      | 13    | 7      | 13    | 1      | 5     | 1      | 20    |
| 8       | 9      | 11    | 4      | 4     | 0      | 21    | 2      | 1     |
| 9       | 5      | 8     | 6      | 10    | 5      | 19    | 4      | 13    |
| 10      | 2      | 6     | 7      | 11    | 1      | 4     | 0      | 16    |
| 11      | 0      | 7     | 4      | 3     | 0      | 9     | 0      | 5     |
| 12      | 0      | 8     | 3      | 8     | 0      | 0     | 1      | 8     |
| 13      | 8      | 11    | 6      | 8     | 4      | 19    | 0      | 11    |
| 14      | 2      | 8     | 7      | 7     | 1      | 10    | 0      | 6     |
| 15      | 5      | 8     | 3      | 9     | 0      | 16    | 1      | 9     |
| 16      |        |       | 4      | 8     | 1      | 2     |        |       |
| 17      |        |       | 4      | 12    | 0      | 10    |        |       |
| 18      |        |       | 7      | 8     | 1      | 4     |        |       |
| 19      |        |       | 5      | 8     | 0      | 5     |        |       |
| 20      |        |       | 6      | 14    | 0      | 25    |        |       |
Lesson I. 12th February, 1973

The class was asked what they thought a fraction was. It was agreed after some discussion that it is a "top" number or "counting" number of the slices, separated from a "bottom" number which told one the number of slices needed to make a whole pie. The separation was a short horizontal line between the "top" number and the "bottom" number.

After the three drawings for \( \frac{3}{4} \) were put on the board

\[
\begin{array}{ccc}
\:\:\:\:\:\:\:\:\:\:\:
\end{array}
\]

three pieces

it was agreed that the top number must be a counting number for the equal sized pieces.

The symbols \( \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5} \) etc. were used to illustrate wholeness since in words we read them as two halves, three thirds, four quarters, etc. In this way the concept of equivalence was introduced and children were encouraged to go to the board and draw one half in many different assemblies of equal-sized slices. For example:

\[
\begin{array}{ccc}
\:\:\:\:\:\:\:\:\:\:\:
\end{array}
\]

are all half a pie.
Lesson 2. 13th February, 1973

Using cuisenaire rods the class as a group working on the floor of the classroom reviewed the basic ideas of fractions discussed in lesson 1. Many examples of equivalent fractions were put on the board and a rule that new equivalent fractions can be obtained by multiplying the numerator and denominator by the same number was discovered without using cuisenaire rods.

Lesson 3. 14th February, 1973

The class was very confident that \( \frac{3}{4} \) is equivalent to \( \frac{9}{12} \) and one of the more confident students explained to the class that this was so because:

\[
\frac{3}{4} \times 3 = \frac{9}{12}
\]

Several oral questions were put to the class members and it transpired that the rule for equivalence was known but that the multiplication tables were not. A multiplication grid was constructed on the board by the class and then each member of the class in turn was asked for an equivalent fraction to \( \frac{2}{3} \). It was orally noted by pupils in the class that an unlimited number of equivalent fractions could be generated. Generation of equal fractions was practiced by dividing the top and the bottom of given fractions by the
same number. It was found that there always came a time when the fraction could be reduced no more. For homework the class was asked to reduce $\frac{128}{384}$ to its simplest form.

Lesson 4. 15th February, 1973

Some students reduced $\frac{128}{384}$ by dividing the top and bottom by the greatest common factor while others obtained the same answer by repeated division by 2. The class played the game of "teacher". Each student in turn went to the board and wrote a pair of equivalent fractions leaving out a term and asked anyone in the class to fill in the blank. A good-humoured fifteen minutes ensued as the teacher acted the part of a student who had difficulty answering any questions but asked many. A duplicated multiplication table sheet was issued and fifty questions were also distributed, typical questions being:

(6) $\frac{19}{19} = \frac{38}{38}$  (25) $\frac{16}{3} = \frac{16}{3}$  (34) $\frac{5 \times 3}{4} = \frac{15}{4}$

Lesson 5. 16th February, 1973

Because a student asked about division by zero, considerable time was spent discussing division and the alternate bar symbolism for division. $\frac{1}{0}$ would not fit into our fraction framework since the symbol would be interpreted as one slice having such a size that none of them would make a whole cake. As a division it was seen to be a meaningless
question and quotients were shown to have a similar

\textit{equivalence relation to fractions}. The concept of \textit{like fractions} was introduced and the word reciprocal explained. It was agreed that a group of like fractions have bottom numbers the same and that a reciprocal was a fraction turned upside down. The students were asked to reduce $\frac{24}{64}$ to its simplest terms.

\textbf{Lesson 6. 19th February, 1973}

A game was played to review the equivalence relation in the form of generating equivalent fractions and in the form of reducing a fraction to its simplest form. Since concepts of odd, even and prime numbers were inherent in the way the game was played, a useful introduction to these ideas was obtained. It was interesting to see that not all the students were able to discern even numbers. As the game progressed the concept of \textit{lowest common multiple} was introduced. The lesson finished by practising factoring by primes on the blackboard.

\textbf{Lesson 7. 20th February, 1973}

The ease of \textit{adding like fractions} was established by utilizing the denominator as a naming device. Since 6 hammers are readily added to 7 hammers to give 13 hammers, and 3 hammers added to 2 hamsters leaves one confused as to a name for the five objects, it was orally discovered by the
class that twenty quarters added to five quarters give a total of twenty five quarters. Yet three quarters added to one third was hard to cope with.

It was discovered that the class was not yet happy with obtaining the lowest common multiple so the class did a little work with them on the blackboard and for homework the students were asked to find the smallest number that 5 and 4 would divide into. Two more like questions using the numbers 8 and 4 and 12 and 30 were given at the same time.

Lesson 8. February 21st, 1973

As was expected, several members of the class were unhappy about finding the lowest common multiple of two numbers. The homework was done for the students using a class activity game:

1. Each child in the class was given a large card with a prime number drawn on the front.
2. Each of the numbers was prime factored. Some used the simple algorithm taught the previous day; some did it by observation.
3. Children were called out to represent the numbers like this:

```
  12
[] []
 30
[] []
```
4. The children above representing the numbers were asked to join each other, those with like numbers hiding one behind the other but never more than two together. After some discussion we ended up with:

\[ 2 \times 2 \times 3 \times 5 = 60 \]

Each homework question was done in this way and five other questions besides. Students then were asked to find lowest common multiples on the board and then to find the lowest common multiples with six pairs of numbers.

Lesson 9. 22nd February, 1973

The addition of fractions was introduced using the fact that adding objects with the same name is easy but when the names are different difficulties arise. With cuisenaire rods if we considered a red rod against an orange rod, the red rod would be \( \frac{1}{5} \) and 3 red rods added to 6 red rods is obviously 9 red rods or 9. But just 1 red rod added to a yellow rod seemed difficult to name using a single name. However by replacing the red and yellow rods with equivalent lengths of white rods it was readily seen that we had seven white rods or \( \frac{7}{10} \) since 10 white rods are needed to make one orange rod. The 10 is the lowest common multiple of the
numbers of the \(\frac{1}{5}\) and the \(\frac{1}{2}\). Several examples were given and as we added the rods we did the typical calculations on the blackboard.

\[
\begin{align*}
\frac{1}{5} + \frac{1}{2} &= \frac{2}{10} + \frac{5}{10} \\
&= \frac{7}{10}
\end{align*}
\]

Some subtractions were done and it was seen that similar procedures were necessary if the fractions were unlike.

Lesson 10, 26th February, 1973

The class reviewed writing equivalent fractions, lowest common multiples and the addition and subtraction of fractions using rectangles drawn on the board.

The students were then put in selected pairs, the confident with the worried, and a worksheet issued, the first fifteen questions being in this form:

\[
\begin{align*}
(13) \quad \frac{7}{8} + \frac{2}{3} &= \quad (14) \quad \frac{8}{9} - \frac{7}{8} = \\
\end{align*}
\]

while the next twelve were laid out as shown below:

\[
\begin{align*}
(17) \quad \frac{11}{12} - \frac{3}{8} = \quad (27) \quad \frac{7}{18} + \frac{4}{45} \\
\end{align*}
\]

Questions were answered individually as they arose and the worksheet was completed for homework.
Lesson 11. 27th February, 1973

The worksheet was corrected and any incorrectly done questions corrected by the students on the blackboard.

Examples of addition and subtraction of fractions were put on the board and the class played the game of teacher while the teacher acted the dullard. The students were asked to attempt the problem:

\[
\frac{7}{45} + \frac{1}{18} + \frac{1}{9} =
\]

Lesson 12. 28th February, 1973

The homework was done on the blackboard and then using cuisenaire rods the pattern for \( \frac{a}{b} \) of \( \frac{c}{d} = \frac{ac}{bd} \) discovered by the class. The class called it "of-ing" and it was explained to them that "of-ing" has the multiply symbol and is called multiplication. It was discovered that a product of a fraction and its reciprocal is always 1. The sets of division sums below were put on the board and the students were encouraged to finish each problem before the teacher had completed the right-hand side and call out the answer.

\[
8 \div 4 = 2 \quad 9 \div 1 = 9 \\
4 \div 2 = 2 \quad 199 \div 1 = 199 \\
16 \div 8 = 2 \quad 300 \div 1 = 300 \\
32 \div 16 = 2 \quad 4 \times 5 \div 1 = 4 \times 5
\]

This was to point out that if both terms in a division are
multiplied or divided by the same number the answer remains unchanged. Secondly we see that the easiest division of all is the one where the second term is 1.

The division of two fractions was shown by converting the second term of the division into 1 by multiplying both terms by the reciprocal of the second term. Because too much material had been compressed into this one lesson no homework was given.

Lesson 13. 1st March, 1973

A review of the fraction work was done and a sheet of miscellaneous fraction computation questions was issued, to be done in pairs, the teacher was available for individual help.

Lesson 14. 2nd March, 1973

Math baseball was played using cards; the set of cards having questions on many aspects of the work covered.

Lesson 15. 5th March, 1973

Five question tests were given. Marking was done by selected students each doing one of the questions on the board. Five such tests were completed in the lesson, two of which are given below as examples:

Test 1. a. Give the reciprocal of \( \frac{2}{3} \)
   b. Spell "reciprocal"
   c. Spell "equivalent"
d. $\frac{6}{7} = \frac{6 + 2}{7 + 2}$ true or false?

e. $\frac{6}{7} = \frac{6 - 2}{7 - 2}$ true or false?

Test 3. 

a. Give LCM of 4, 12.

b. $\frac{3 + 5}{4} = \frac{12}{12}$

b. $\frac{3 + 5}{4} = \frac{12}{12}$

d. $\frac{3}{4} \times \frac{5}{12} = \frac{12}{12}$

e. $\frac{3}{4} \div \frac{5}{12} = \frac{12}{12}$

Lesson 16. 6th March, 1973

A worksheet of thirty-five miscellaneous questions was worked on for the whole period.

Lesson 17. 7th March, 1973

The whole lesson was devoted to five-question tests and students working at the blackboard in preparation for the test.

On the 8th March, 1973, a post-test was given which was precisely the same test as the pretest which had been given on the 12th February, 1973. These tests are described more fully in Appendix A.
Lesson 1. 15th January, 1973

The class was introduced to the "what's my rule" game where the students give numbers to the teacher and for each number given a corresponding number is supplied by the teacher according to some private rule. As the evidence increases the class is able to discover the rule. In this way the teacher was able to gauge the ability of the class and win some confidence from the students.

Lesson 2. 16th January, 1973

By question and answer the reasonableness and convenience of the ordered pair symbolism \((a,b)\) was established. Each student was asked to call out an ordered pair and the teacher put them, as they were called out, into columns of equivalence classes called families. The class was challenged to find the teacher's rule for putting the pairs into families. As each new pair was created the class was asked to say whether a new family column should be started or whether it could be put into an established column. After each of the students had offered two ordered pairs most of the students seemed to know the game. The
students were asked to write the families into their notebooks and under (15,20) was placed ( , ) and only when they had finished were they to fill in the boxes. For homework the class was asked to pretend that (3,4) was called Orphan Annie and that she was lost. Could they discover her family for her?

Lesson 2. 17th January 1973

Sheets of the columns of families of ordered pairs were duplicated since few of the students were able to write numerals small enough to get the information conveniently on one sheet. The class was eager to put "Orphan Annie" (3,4) into her family. The game of the previous day was repeated but this time each pair was such that each member was factored so that the reduced form was evident in the factors. Any child who claimed to know the rule was appointed teacher to take the next number pair and put it into the appropriate family. The first four families were:

<table>
<thead>
<tr>
<th>FAMILY NAME</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6x1, 6x2)</td>
</tr>
<tr>
<td>(30x1, 30x2)</td>
</tr>
<tr>
<td>(10x1, 10x2)</td>
</tr>
<tr>
<td>(9x1, 9x2)</td>
</tr>
<tr>
<td>(14x1, 14x2)</td>
</tr>
<tr>
<td>(7x1, 7x2)</td>
</tr>
</tbody>
</table>
Lesson 4. 18th January, 1973

Sheets showing the families in factored form were duplicated so that the boardwork of the previous day could be recalled. The following were drawn on the board:

| (20x1,20x2) | (6x1, 6x1) |
| (16x1,16x2) | (42x1,42x1) |
| (21x1,21x1) |
| (11x1,11x1) |

The class was asked what the three pictures had in common and one student said that they were all bears. The following was drawn on the board:

Fred Bear
Partly Bear
Completely Bear

The class was asked what these three had in common. The class laughingly replied that they all had big ears and long noses. The class was then asked to look at the first column on their duplicated sheets and find out what all the members of that family had in common. The class had no difficulty in seeing that they all had a one in the first member and a two in the second member and that each was multiplied by the
same number. The class was agreeable to name this family the (1,2) family and using this idea they were quickly able to fill in at the top of each column the names of each of the families listed on the duplicated sheet. The students were asked to add another number pair to each family so that it belonged to that family. The class discussed how to find the family name of the ordered pair (24,9) and afterwards several such ordered pairs were given and three quarters of the class seemed to have grasped the idea.

For homework the class was told that (24,56) was a little girl who was lost and they were asked to find out her 'family' name.

Lesson 5. 19th January, 1973

The concepts of factor, common factor and greatest common factor were discussed and oral practice followed on finding the greatest common factor of the terms of various ordered pairs using lists of factors of each of the numbers and choosing the largest one that appeared in each group. The concept of reciprocal was explained and practice with this idea followed.

Lesson 6. 22nd January, 1973

The work covered to date was reviewed with emphasis on reducing any given ordered pair to its simplest terms. The class was split up into fours and each person asked to set the rest of the group an ordered pair and check whether
they could name the family. Two students were found to be unable to cope and on checking this with their regular teacher, he confirmed that both students had had great difficulty with all their schoolwork.

For homework the students were asked to find the family names of several number pairs.

**Lesson 7. 23rd January, 1973**

The class practised finding the standard names of the family of any given ordered pairs and then writing equivalent ordered pairs to any given ordered pair.

**Lesson 8. 24th January, 1973**

The problem of reducing a number pair (24,84) was discussed and the method of dividing both terms by the greatest common factor was chosen by some students but others preferred to repeatedly divide each term by a common factor until the terms were mutually prime. A duplicated sheet of questions on writing equivalent ordered pairs was issued and the first twenty of the forty-one questions were completed. Typical questions were as follows:

(9) \((6,12) = (3, \ )\)

(18) \((2,20) = (10) = (4, \ )\)

(36) \((7,6) = (8\times7, \ )\)

The rest of the questions were offered as homework if the students so wished.
Lesson 9. 25th January, 1973

The rules for finding the standard name of a family and for creating new members of a family were verbalized by the class and written on the board.

The class played the "What's my rule" game but this time they gave the teacher two ordered pairs. For example the first three such problems given were:

\[(2,3) \times (4,7) = (8,21)\]
\[(4,3) \times (3,7) = (12,21) = (4,7)\]
\[(5,8) \times (2,7) = (10,56) = (5,28)\]

Most of the class had discovered the rule by the fifth example and were able to complete the right-hand side of the equation before the teacher completed it. Three students were unable to see the rule so the first members of each of the ordered pairs were coloured in red chalk and the second members in yellow chalk and this enabled two more to understand the rule and complete the right-hand side of the equation. One student could not see the rule though the rest of the class had no difficulty with ten further examples.

The class decided to call the operation of multiplying first terms to get the first term of the answer and multiplying the second terms to obtain the second term of the answer "gunking".

For homework the children were asked to discover a rule for the four examples using a different rule from
"gunking".

\[(2,3) \triangle (4,2) = (4,12) \quad (3,5) \triangle (7,3) = (9,35)\]
\[(2,1) \triangle (5,8) = (16,5) \quad (4,5) \triangle (7,2) = ( , )\]

Lesson 10. 26th January, 1973

"Gunking" was reviewed by examining the gunking of any ordered pair with its reciprocal and the gunking of any ordered pair with any member of the \((1,1)\) family. Using four students at the board each one standing under a number like this:

\[(12 \quad 10) \triangle (4 \quad 2)\]

Each number position was given two names. Just as each student had two possible names, his first name and his surname, so each number in an operation had two names. Any number in the "12" position could be called a "first" or an "outer", any number in the "10" position could be called a "second" or "inner", any number in the "4" position could be called a "first" or an "inner" while any number in the "2" position could be called either a "second" or "outer". For brevity the word term was excluded from each position name. The class practised using the names on different gunking operations. The rule for gunking was verbalized using the new descriptions of positions and the class decided that "to
gunk, you multiply the firsts to get the first in the answer, and then multiply the seconds to get the second in the answer.

The homework was discussed and the rule verbalized into "to get the first in the answer multiply the outers and to get the second in the answer multiply the inners" and the new operation given the name "junking".

Lesson 11. 29th January, 1973

On a "what's my rule" basis the students were introduced to the operation of "twiddling". Help was given by noting against each problem the inner products and outer products as in, for example:

\[(3,4) \oplus (1,2) = (10,8)\]  
inner product 4  
outer product 6

Eight examples were given and nearly all the class had discovered the rule.

For homework the students were challenged to discover the rule for "twaddling":

\[(12,10) \ominus (2,3) = (16,30)\]  
inner product 20  
outer product 36

Lesson 12. 30th January, 1973

Only four students had discovered the rule for "twaddling" so it was decided to review the rules for "gunking", "junking" and "twiddling". The rules were obtained from the class and written on the board and a quiz game played as an aid to review.
The two examples:

\[ (2,3) \oplus (2,3) = (12,9) = (4,3) \]

\[ (5,3) \oplus (2,3) = (21,9) = (7,3) \]

were put on the board and the students were asked to find a quick way of arriving at the answer not using the rule for "twiddling". It was suggested that they work out several examples where the second terms were the same and then look for the pattern.

Lesson 13. 31st January, 1973

Only two people had discovered the rule for "twiddling" like ordered pairs so on a discovery basis the class tackled the rule for "twaddling" and nearly all the class had discovered this rule after five examples had been put on the board. The four rules were given by the class and written on the board for reference.

The students were put to work in pairs on a worksheet on the four operations. Individual help was given but was only needed when operation signs were confused. The first seventeen questions were completed and marked by the whole class. Examples of questions from the sheets were as follows:

(6) \( (4,3) \odot (3,5) = \)

(13) \( (7,3) \odot (2,3) = \)

(27) \( (2,5) \odot (4,5) = \)
Lesson 14. 1st February, 1973

Since six students were absent it was decided to use the lesson for review. The class was split into a team of boys and a team of girls. The first person from one team asked any opponent one of the questions from the homework sheet. The opponent went to the board and explained how the question was done. If the answer was correct he won a point and if he was wrong the questioner could gain a point by answering the question himself.

Each person took a turn at being a questioner and the involvement of the students was considerable.

Lesson 15. 2nd February, 1973

Since the boys had won the previous day's game and five students were absent, it was decided to have a rematch. The questions were invented by the questioners. Very few errors were made and the children seemed to be happy with the work.

Lesson 16. 5th February, 1973

Although six students were absent it was decided to examine "gunking", "junking", "twiddling" and "twaddling" ordered pairs which have like second terms called 'like ordered pairs'. The class readily established the rules for "twiddling" and "twaddling", and with a little more trouble "junking". It was agreed that we could not find a shortcut rule for "gunking" like ordered pairs.
Lesson 17. 6th February, 1973

A review lesson was given playing mathematics baseball.

Lesson 18. 7th February, 1973

A general review sheet was issued and the children worked in pairs completing them.

On the 8th February, 1973, a post-test was given which, though written in ordered pair nomenclature, was precisely the same test as the pretest which had been given on the 15th January, 1973. These tests are described more fully in Appendix A.
Roger Sandford,  
R.R.1 Tzouhalem Road,  
Duncan.  

Dear Roger,  

This letter is my response to your request that I listen to the matter, manner and enthusiasm of your lessons to the two grade V classes at my school in January, February and March, 1973.

I have no hesitation in saying that you tackled each teaching assignment as enthusiastically as you were able. Both classes seemed to enjoy your lessons and although such things are difficult to state with complete certainty I do feel that both classes received roughly the same treatment from the aspect of teacher attitude and student enjoyment.

I noted that the class that I usually teach were taught material which was not apparently fractions but was a game of ordered pairs with various rules and you used no reference to fractions or physical models in that class. On the other hand, I noted that you taught the other class fractions in the usual manner using physical models and usual symbolism.

Yours sincerely,  

Henry Spencer, B.Ed. M.A.
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C. UNPUBLISHED SOURCES