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MODE STRUCTURE IN ACOUSTIC CAVITIES

by

RICHARD CHARLES HUGHES

B. Sc. (Hons.), University of Manitoba, 1970

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OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
in the Department

of

Physics

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Mode Structure in Acoustic Cavities

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A variety of techniques was used to study "modes" of coupled acoustic cavities under both active (with acoustic gain) and passive conditions. One member, and sometimes the only member, of each composite cavity was an oriented single crystal of cadmium sulfide, polished flat and parallel (phonon maser). The other members (buffers) of the various composite cavities were polished samples of fused silica, CdS and sapphire. As CdS is a piezoelectric semiconductor, the coupling between the electron system and the phonon system allows acoustic amplification under appropriate conditions of conductivity and electrical bias. If acoustic gain is sufficient to overcome all losses, the systems break into spontaneous oscillation.

Detailed calculations of the electrical impedance of single and coupled cavities are given. Measurements of the electrical impedance of cavities allowed the determination of the acoustic loss (or gain) and of the condition of acoustic resonance (normal modes of the cavity). Light diffraction and optical heterodyning studies were made of coupled cavities under conditions of oscillation. These measurements allowed determination of phase velocities, frequencies, wavelengths, polarizations, directions of propagation and spatial extent of acoustic fields.
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a

t ratio of the amplitudes of the forward acoustic wave to the return wave parameter, see Eq. 2.55a, b

\((1/a_2 + 1/a_3)\)

a_t

real and imaginary parts respectively of \(a_t\)

A, A

amplitude

\(A_m\)

amplitude of the \(m^{th}\) order

A

area of platelet

\(\alpha_a\)

attenuation constant, \(\text{Im}\{-k_a\}\)

\(\alpha_0\)

zero order attenuation constant, \(\text{Im}\{-k_0\}\)

\(\alpha_L\)

lattice attenuation constant

\(\alpha_{A'}, \alpha_B\)

attenuation constants for Cavities A and B respectively

b, a

reference amplitude for heterodyne detection

B_v

variable susceptance

B_{act}

active susceptance of a resonator

\(B_{ij}, B_{i}\)

relative dielectric impermeability tensor and matrix respectively

\(B_{i}'\)

relative dielectric impermeability matrix under strain

\(\Delta B_{i}\)

\(B_{i}' - B_{i}\)

\(\beta\)

piezoelectric constant, \(\beta_{11}\), or single transit phase change, \(q_{Bd_B}\)
\( \beta \), \( \beta_2 \), \( \beta_3 \)

- single transit phase change, \( q_2 \) and \\
- \( q_3 \) respectively

\( \beta_{ijkl} \)

- piezoelectric tensor

\( c, c_A, c_B \)

- elastic constant, \( c_{ijkl} \)

\( c_{ijkl} \)

- elastic tensor

\( C_0, C_A \)

- static capacitance, \( \varepsilon A/(4\pi d) \)

\( C_V \)

- variable capacitance

\( x \)

- piezoelectric coupling constant, \\
- \( 4\pi \beta^2/(cc) \)

\( d, d_A, d_B \)

- thickness of cavity

\( d_D \)

- effective indium diffusion depth

\( D \)

- electric displacement, \( D_1 \)

\( D_{i,j} \)

- electric displacement vector

\( D_a \)

- electric displacement for modes of propagation

\( D_{I, II, I, m, II, m} \)

- electric displacement along axes of index ellipse

\( D_{Y,m} \)

- \( m \)th order diffracted light passing \\
- through an exit polarizer at angle \( \gamma \) to \\
- \( x \)-axis

\( D_{II} \)

- diffusion constant, \( D_{II} \)

\( D_{ij} \)

- diffusion tensor for electrons

\( \delta \)

- velocity dispersion parameter, see \\
- Eq. 2.75b

\( \delta_{i,j} \)

- Kronecker delta function

\( e \)

- magnitude of electronic charge

\( E \)

- electric field, \( E_i \)

\( E_i \)

- electric field vector
electric field for modes of propagation
d.c. electric field
electric field for threshold oscillation
ionization energy of i^{th} trap level
dielectric constant, \( \epsilon_{11} \)
dielectric permittivity tensor
fraction of space charge electrons in conduction band
limit of \( f \) as \( \omega \to 0 \)
the real and imaginary parts of \( f \) respectively
parameter, see Eq. 3.32b
variable conductance
active conductance of a resonator
forward acoustic wave velocity parameter, \( 1 - v_0 / s_0 \)
backward acoustic wave velocity parameter, \( 1 + v_0 / s_0 \)
single transit acoustic losses; \( \alpha_d, \alpha_A, \alpha_B \) respectively
round trip cavity loss
Planck's constant divided by \( 2\pi \)
parameter, see Eq. 2.60d, or detected optical signal
electric current
intensity of m^{th} order signal
electric current density, \( J_1 \)
electric current density vector
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<td>$J_a$</td>
<td>electric current density for modes of propagation</td>
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<td>$J_m(a)$</td>
<td>$m^{th}$ order Bessel function</td>
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<td>$k$</td>
<td>propagation constant</td>
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<td>$k_a$</td>
<td>propagation constant for modes of propagation</td>
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<tr>
<td>$k_0, k_0^a$</td>
<td>uncoupled or zero order ($\beta = 0$) propagation constants</td>
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<td>$k_A, k_B$</td>
<td>propagation constants for Cavities A and B</td>
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<td>$\vec{k}$</td>
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<td>$k_B$</td>
<td>Boltzmann's constant</td>
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<td>$L$</td>
<td>thickness of acoustic beam</td>
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<tr>
<td>$\lambda$</td>
<td>wavelength of light in vacuo</td>
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<td>$\Lambda$</td>
<td>acoustic wavelength</td>
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<td>$m_e^*$</td>
<td>effective mass of the electron</td>
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<td>$\mu$</td>
<td>electron mobility</td>
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<td>$n$</td>
<td>excess electron concentration over equilibrium value or index of refraction</td>
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<tr>
<td>$n'$</td>
<td>index of refraction under acoustic strain</td>
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<td>$n_0$</td>
<td>equilibrium electron concentration or undisturbed index of refraction</td>
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<td>$n, n', n_2$</td>
<td>principal indices of refraction</td>
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<td>$n_0, n_e$</td>
<td>ordinary and extraordinary indices of refraction</td>
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<td>$n_c$</td>
<td>electron concentration in conduction band</td>
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<tr>
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equilibrium electron concentrations
time dependent electron concentrations
lengths of semi-axes of unstrained index ellipse
length of semi-axes of strained index ellipse

\( n_0, n_i, n_t \)
\( n_{c1}, n_{i1}, n_{t1} \)
\( n_I, n_{II} \)
\( n_I', n_{II}' \)
\( \Delta n, \Delta n_{I}, \Delta n_{II} \)
\( N \)
\( N_c \)
\( N_i, N_t \)
\( N_c' \)
\( \nu, \nu_m \)
\( \nu_N, \bar{\nu} \)
\( \Delta \nu, \overline{\Delta \nu} \)
\( \delta \nu_N \)
\( \omega, \omega_0, \Omega \)

normal mode (acoustic resonance) number
effective density of states in conduction band
concentration of traps
frequency
frequency of maximum gain, \( \frac{1}{2\pi \sqrt{\nu D}} \)
frequency of \( N^{th} \) normal mode
frequency relative to \( N^{th} \) normal mode, \( \nu - \nu_N \)
"unity" coupling frequency of \( N^{th} \) normal mode
deviation from "unity" coupling, \( \nu_N - \nu_N' \)
angular frequency
angular frequency of undiffracted light
angular frequency of acoustic waves or ohms
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<td>unstiffened acoustic wave vector, $\omega/s_0$</td>
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xx
electric admittance

acoustic impedance; \( \frac{c_A}{s_A}, \frac{c_B}{s_B} \)
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CHAPTER 1

INTRODUCTION

1.1 Historical Overview

Crystals which exhibit both electrical conductivity and piezoelectricity are of interest since the piezoelectric properties serve to couple the electron system and the acoustic waves. Normally this interaction, or coupling, produces an attenuation of the acoustic waves. However, acoustic gain is possible through a transfer of energy from the electron system to the acoustic waves if the drift velocity of the electrons (in the presence of a d.c. electric field) exceeds the acoustic velocity.

At very high conductivities ($\sigma >> \frac{\varepsilon \nu}{\sigma}$, where $\sigma$, $\varepsilon$ and $\nu$ are the conductivity, dielectric constant and operating frequency respectively, expressed in c.g.s. units) the piezoelectric fields are effectively shorted out and all piezoelectric effects disappear. For this reason piezoelectric semiconductors were not realized until 1960, when Hutson (1960) managed to compensate samples of ZnO and CdS sufficiently to be able to measure their piezoelectric properties. The large (as compared to quartz) piezoelectric coefficients measured for both materials provide an explanation for the earlier observations of an anomalous thermo-
electric response in ZnO (Hutson 1959) and of a photosensitive acoustic attenuation in CdS (Nine 1960).

Working with photoconducting CdS, Hutson, McFee and White in 1961 reported the first observation of acoustic amplification in a piezoelectric semiconductor using an applied d.c. electric field. Previously Weinreich (1956) had attempted to amplify acoustic waves by the application of a d.c. electric field to a non-piezoelectric semiconductor, where the deformation potential was to have provided the necessary coupling between the electron system and the acoustic waves. However, the deformation potential typically produces effects 5 orders of magnitude smaller than piezoelectricity (McFee 1966) and acoustic gain was not experimentally observed.

Building on earlier theoretical work by Kyama (1949, 1954), Hutson and White (1962) developed a linear, one-dimensional theory which considers the piezoelectric interaction between the electron system and the acoustic waves in the absence of an applied electric field. An extension of this theory to the case of an applied d.c. electric field (White 1962) predicted an acoustic amplification which was in good agreement with experiment (White, Handelman and Hanlon 1965).

Various extensions of the original theory by White (1962) have been presented. Uchida et al. (1964) included
the possibility of electron traps with an arbitrary relaxation time (i.e., a complex trapping factor). Electron trapping effects become important at low conductivities. An extension of White's work was developed by Blotekjaer and Quate (1964) who included the effect of perturbed carrier density waves as well as perturbed acoustic waves. The small signal expressions for acoustic gain and dispersion in an infinite medium are equivalent to those derived by White (1962), but the inclusion of carrier density waves can produce important effects when physical boundaries are considered.

As noted in the original papers (Hutson, McFee and White 1961, White 1962), the attenuation suffered by an acoustic wave travelling against the electron drift can be less than the gain achieved by an acoustic wave travelling in the drift direction. If the excess gain (forward gain - return loss) is larger than the reflection losses at the ends of the sample, spontaneous oscillation will occur. Continuous coherent oscillation in a thin platelet of photoconducting CdS was first reported in 1965 (White, Handelman and Hanlon 1965) with a more detailed study following (White and Wang 1966).

A considerable number of papers have been written on acousto-electric oscillations in thin platelets of CdS and ZnO. Many of these are listed at the end of the biblio-
graphy (page 266). A few of the major developments will be commented on. Gurevich and Laikhtman (1966) analyzed the conditions for acoustic oscillation in thin platelets with parallel faces. Good review articles were written by McFee (1966) and Gurevich (1969). Acousto-electric oscillation was observed in ZnO (Maines et al. 1968). Working with photoconducting CdS, Maines and Paige (1969) found good agreement between a small signal theory (based on the theory of White (1962)) and experimental measurements of the electric fields below which and above which acousto-electric oscillations were not possible (threshold and extinction). The frequency of oscillation at threshold and the incubation time required for oscillation to develop at various electric fields were also measured and again good agreement with theory was found. In another development very thin (~25 μm.) oscillators with very low threshold voltages (~2.5 V) were produced from CdS (Marshall 1970) and ZnO (Janus 1970) crystals. Extensive light diffraction studies of oscillating crystals (Vrba and Haering 1973 a, b, c, 1974) have revealed that the acoustic waves do not in general propagate along the normal to the cavity as previously assumed, but rather travel at an angle to the cavity axis. However, for certain crystal orientations quasi-one dimensional behavior occurs.
1.2 Topics Considered

With the objective of further understanding acousto-electric oscillations in piezoelectric semiconductor platelets, this dissertation presents electrical impedance studies and light diffraction studies of photoconducting CdS platelets. The main interest was in the case of acoustic waves travelling approximately along the cavity axis for which a quasi-one dimensional treatment is possible.

Light diffraction from acoustic waves inside the platelets (or "active" cavities) has some drawbacks: the limited size of the platelets implies poor resolution of diffraction, the probing light (which is diffracted by acoustic waves) perturbs the conductivity of the crystal and light diffraction is weak or non-existent for some crystal orientations and acoustic polarizations considered. Therefore light diffraction, in this study, was generally performed in buffers or "passive" cavities coupled to the platelets.

Other than the use of transducers to monitor the acoustic signals (White and Wang 1966, Marshall 1969) no previous attempt to couple acoustic fields of high strain amplitude out of an oscillating phonon maser (acousto-electric oscillator) has been reported (to the knowledge of the author). The coupling of active and passive cavities is of some interest as this opens the possibility of using
phonon masers as a tool in studying the acoustic properties of the passive material.

In Chapter 2 a study of a single cavity is presented. The small signal electrical impedance of a one dimensional conducting piezoelectric resonator has been considered by a number of authors. Arlt (1965) derived quite simple expressions for the impedance near an acoustic resonance and has given an equivalent circuit. However, his treatment included only acoustic waves, ignored electrical diffusion effects and is valid only in the absence of an applied d.c. electrical field. A treatment by Greebe (1965) considers both diffusion effects and carrier density waves but also assumes no applied d.c. electrical field. Adler and Farnell (1966) present a very general treatment but the resulting expressions are best suited to numerical analysis. Using physical arguments Marshall (1969) proposed a relatively simple equivalent circuit which was very successful in some applications but not in others (in particular, shear wave oscillations.)

An attempt is made in Chapter 2 to give a coherent discussion of the impedance of a piezoelectric semiconductor under general conditions. Various limiting cases are developed and Marshall's equivalent circuit (Marshall 1969) is found under certain conditions. Experimentally measured electrical impedances are used to determine acoustic dispersion and acoustic loss in a single cavity under a variety of
voltages, frequencies and conductivities.

Since very strong acoustic coupling between cavities (of multiple cavity systems) was employed, the interactions among cavities cannot be considered as small perturbations. The cavity system must be treated as a whole; hence the term composite cavity will be used. In Chapter 3 simple theories are developed for the acoustical resonances of composite cavities. The electrical impedance of an active cavity bonded to a passive cavity is calculated in the limit of zero conductivity. Experimental measurements of the electrical impedance and resonant frequencies of double cavities agree very well with theory.

In Chapter 4 the results of light diffraction studies of oscillating composite cavities are presented. Acoustic strain amplitude measurements were made, acoustic diffraction was observed and various types of modes of oscillation were noted. Evidence was found for an acoustic spiking phenomenon.

A summary of results and suggestions for future experiments are presented in Chapter 5.
CHAPTER 2

IMPEDANCE ANALYSIS OF A PIEZOELECTRIC SEMICONDUCTOR

2.1 Introduction

In the study of a piezoelectric semiconductor three types of waves (or modes of propagation) will be considered: electromagnetic waves, acoustic waves and carrier density waves (related to Debye-Hückel screening). In a non-piezoelectric insulator the carrier density waves are non-existent, while the electromagnetic waves and acoustic waves are essentially uncoupled and travel independently at the velocity of light in the dielectric and at the acoustical velocity respectively. In a piezoelectric insulator, however, the electric field and the acoustic strain are coupled through the piezoelectric tensor. This introduces such effects as the piezoelectric stiffening of the material (an increase in the acoustic velocity). In a semiconductor the coupling between the electron system and the electromagnetic wave introduces velocity dispersion and loss for the electromagnetic wave.

For the case of a piezoelectric semiconductor all three waves are coupled. A general feature of the coupling is a perturbation of the free waves (complex dispersion as mentioned above) and the introduction of forced waves (e.g., an electron density wave forced to travel at the acoustic velocity or an acoustic wave forced
to travel at the velocity of light. Piezoelectric semiconductors are of particular interest as under certain conditions (which will be developed) the coupling between the waves can lead to acoustic gain.

The problem of coupled waves in a piezoelectric semiconductor was first treated by Kyame (1949, 1954). Hudson and White (1962) have shown that the contribution of the transverse electromagnetic waves to the velocity dispersion of the acoustic waves is smaller by a factor of order \((s_0/c)^2 \approx 10^{-10}\) (the square of the ratio of the acoustic velocity to the velocity of light) than the contribution of the longitudinal electric fields (electrostatic in nature). In fact, if one makes the quasi-static approximation (the velocity of light is assumed infinite) the transverse electromagnetic waves are decoupled from both the acoustic waves and the carrier density waves.

A treatment which considers only the coupling between acoustic waves, carrier density waves and longitudinal electric fields is developed in the following sections.
2.2 Coupled Wave Theory

Under the quasi-static approximation the relevant equations of state for a piezoelectric semiconductor are:

(i) Hooke's Law given by

\[ T_{ij} = c_{ijkl} S_{kl} - \beta_{kji} E_k \]  

(2.1a)

where \( T_{ij} \) is the stress tensor, \( S_{kl} \) is the strain tensor, \( E_k \) is the electric field, \( c_{ijkl} \) is the elastic tensor and \( \beta_{kji} \) is the piezoelectric tensor. The strain tensor is defined by

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(2.1b)

where \( u_i \) is the mass displacement vector.

(ii) Newton's Law given by
where $\rho$ is the density of the medium.

(iii) The electric displacement, $D_i$, given by

$$D_i = 4\pi \beta_{ijk} S_{jk} + \varepsilon_{ij} E_j$$

where $\varepsilon_{ij}$ is the dielectric permittivity tensor.

(iv) Poisson's Equation given by

$$\frac{\partial^2 D_i}{\partial x_i} = -4\pi \rho$$
where $e$ is the magnitude of the electronic charge and $n$ is the excess electron concentration over the equilibrium electron concentration, $n_0$. An $n$-type impurity semiconductor or a photoconductor with a short lifetime for holes (e.g., CdS) is assumed.

(v) The electrical current density, $J_i$, given by

$$J_i = \sigma_{ij} E_j + e D_{ij} \frac{\partial n}{\partial x_j}$$  \hspace{1cm} (2.5)

where $\sigma_{ij}$ is the conductivity tensor and $D_{ij}$ is the diffusion tensor for electrons. It has been assumed that all the excess electron concentration is mobile (i.e., no electron trapping). Trapping effects are considered in Section 2.4.

(vi) The equation of continuity for electrons is given by
This set of equations (Eqs. 2.1 - 2.6) contains the implicit assumptions that all wavelengths are much longer than the electron mean free path and that the frequency of operation is much smaller than the reciprocal of the electron scattering time. For the frequencies of interest (< 1 GHz.) these assumptions are valid.

Assume a plane wave propagating along the $x_1$ axis. The $x_1$-axis need not correspond to a crystal symmetry axis, if the tensor quantities ($c_{ijkl}$, $\epsilon_{ij}$, etc.) are modified appropriately so that they refer to the $x_1$-axes.

Equations 2.1 to 2.6 can be reduced to

\[
\frac{\partial J_i}{\partial x_i} = \epsilon \frac{\partial n}{\partial t} \tag{2.6}
\]

\[
T_{i1} = c_{i1} \frac{\partial u_i}{\partial x_1} - \beta_{i11} E_1 \tag{2.7a}
\]

\[
\frac{\partial T_{i1}}{\partial x_1} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{2.7b}
\]

\[
D_1 = 4\pi \beta_{11j} \frac{\partial u_j}{\partial x_1} + \epsilon_{11} E_1 \tag{2.7c}
\]
The solution of the set of coupled equations (Eqs. 2.7) is algebraically complex as it contains all three possible acoustic polarizations (one quasi-longitudinal and two quasi-shear). The problem can be simplified considerably if one assumes the possible polarizations of the perturbed acoustic waves are known. To a good approximation they are equal to the polarizations of the free acoustic wave ($\beta_{ijk} = 0$). For the x-axis along a direction of high crystal symmetry there is an exact correspondence between the polarizations of the perturbed and free acoustic waves. Defining $\hat{r}_i$ as a unit vector along one of the possible polarizations Eq. 2.7 can be reduced to the one-dimensional form

$$T = c \frac{\partial u}{\partial x} - \beta E \quad (2.8a)$$
where \( x = x_i u = u_i \hat{r}_i \), \( T = T_i \hat{r}_i \), \( C = c_{ij} \hat{r}_i \hat{r}_j \), \( \beta = \beta_{ij} \hat{r}_i \hat{r}_j \), \( \sigma = \sigma_{ij} \), \( \mathcal{D} = \mathcal{D}_{ij} \), \( E_u = E_{ij} \), \( D = D_{ij} \), and \( J = J_i \). If \( \beta \neq 0 \) the quasi-acoustic wave is termed active; if \( \beta = 0 \) the wave is termed inactive.

In an n-type semiconductor without traps the conductivity, \( \sigma \), in Eq. 2.8d can be written as

\[
\sigma = \mu e (n_0 + n) = \sigma_0 + \mu e n \quad (2.9)
\]
where \( \mu \) is the electron mobility.

Using Eq. 2.9, the Eq. Set 2.8 represents six equations in the six variables: \( T, u, E, D, n, \) and \( J \). Assume each variable has a time independent value plus a plane wave component as illustrated for the electric field by

\[
E(x,t) = E_0 + E_a e^{i(\omega t-kx)} + \text{complex conjugate (2.10)}
\]

Note that by definition the deviation from equilibrium electron density, \( n \), has no time independent component.

The time invariant quantities \( T, E_0, D_0 \) and \( J_0 \) will also be assumed to be spatially invariant. The quantities \( T_a, u_a, E_a, D_a, n_a \) and \( J_a \) are the plane wave amplitudes where the subscript \( a \) will take on values from 1 to 5 to represent the different possible propagation modes as discussed later.

If the time independent values of the variables are separated from the Eq. Set 2.8 and the non-linear cross term of order \( n_a E_a \) in Eq. 2.8e (using Eq. 2.9) is dropped under the assumption \( E_a \ll E_0 \) and \( n_a \ll n_0 \) (small signal theory), one is left with a set of linear homogeneous equations for the plane wave amplitudes. In matrix
form the equations are given by

\[
\begin{bmatrix}
1 & i \kappa k & \beta & 0 & 0 & 0 \\
-i \kappa & \rho \omega^2 & 0 & 0 & 0 & 0 \\
0 & i4\pi\beta k & -\kappa & 1 & 0 & 0 \\
0 & 0 & 0 & -ik & 4\pi & 0 \\
0 & 0 & \sigma_0 & 0 & -\frac{\mu E_0}{\kappa D} & -1 \\
0 & 0 & 0 & 0 & \omega & \kappa \\
\end{bmatrix}
\begin{bmatrix}
T_a \\
T_a \\
E_a \\
D_a \\
W_a \\
J_a \\
\end{bmatrix}
= 0
\]

(2.11)

The condition for a solution is that the determinant of the coefficient matrix be equal to zero. The resulting quintic secular equation is given by

\[
\kappa \left( k^2 - \omega^2 / s_0^2 \right) (1 + k^2 R^2 + i\omega + ik\tau v) + \chi k^2 (k^2 R^2 + i\omega + ik\tau v) = 0
\]

(2.12a)

where

\[
s_0 = \frac{\sqrt{c/\rho}}{\kappa} \quad (2.12b) \quad R = \frac{\sqrt{D}}{4\pi\beta} \quad (2.12e)
\]

\[
v = \frac{\mu E_0}{\kappa} \quad (2.12c) \quad \chi = \frac{4\pi\beta^2}{\epsilon c} \quad (2.12f)
\]

\[
\tau = \frac{F}{4\pi\rho_0} \quad (2.12d)
\]

correspond respectively to the unstiffened (uncoupled) velocity of sound, the electron drift velocity, the di-
electric relaxation time, the Debye-Hückel screening radius, and the piezoelectric coupling constant.

The diffusion constant may be related to the mobility through the Einstein relation given by (Kittel 1966)

\[ D = \frac{\mu k_B T}{e} \]  \hspace{1cm} (2.13)

where \( k_B \) is the Boltzmann constant and \( T \) is the absolute temperature. Therefore the Debye-Hückel radius (Eq. 2.12e) becomes (using Eqs. 2.9, 2.12d, 2.12e and 2.13)

\[ R = \left( \frac{\mu k_B T}{e} \right)^{\frac{1}{2}} \]

\[ = \left( \frac{\varepsilon k_B T}{4\pi \varepsilon_0 n_0} \right)^{\frac{1}{2}} \]  \hspace{1cm} (2.14)

Before commenting on the solutions of the secular equation (Eq. 2.12) some useful relations will be derived.

Using Eq. 2.11 the plane wave amplitudes may be expressed as functions of the electric field amplitude.

\[ T_a = -\varepsilon \left[ 1 + \frac{(l+R^2 k^2 + i\omega t + i\nu k)}{\chi (R^2 k^2 + i\omega t + i\nu k)} \right] E_a \]  \hspace{1cm} (2.15a)
Returning now to the secular equation (Eq. 2.11) the trivial solution, \( k = 0 \), corresponds to a longitudinal electric wave (with infinite phase velocity) for which Eqs. 2.11 (Eqs. 2.15 involve division by \( k \)) reduce to

\[ T = - \beta E \]  
\[ u_1 = 0 \]  
\[ D_1 = \epsilon E_1 \]  
\[ n_1 = 0 \]  
\[ J_1 = \sigma E_1 \]
where the subscript 1 represents the solution k = 0 (uniform mode). As a one dimension treatment is now considered, subscripts no longer refer to vector components unless specifically noted. The remaining quartic equation, (equivalent to Eq. 12 of Adler and Farnell 1966) given by

\[
\left[k^2 - \frac{\omega^2}{\omega_0^2}\right]\left[1 + k^2 R^2 + i\tau(\omega + kv)\right] + \chi k^2 \left[k^2 R^2 + i\tau(\omega + kv)\right] = 0
\]

(2.17)

has four solutions: two quasi-acoustic waves travelling forward and backward and two quasi-carrier density waves. In the limit as \( \chi \to 0 \) the acoustic waves and the carrier density waves decouple. The solutions of Eq. 2.17 for \( \chi = 0 \) will be termed the zero order solutions and are given by

\[
\left(k_0^2\right)^2 - \left(\frac{\omega}{\omega_0}\right)^2 = 0
\]

(2.18a)

and

\[1 + \left(k_0^2 R^2 + i\tau(\omega + k_0 v)\right) = 0\]

(2.18b)

Eq. 2.18a is the dispersion relation for loss-free unstiffened acoustic waves. The zero order propagation constants for the forward (propagation along \( x_1 \) direction) and the backward (along \(-x_1\) direction) acoustic waves are given respectively by
\[ k_2^0 = q_0 \]  

(2.19a)

\[ k_3^0 = -q_0 \]  

(2.19b)

where \( q_0 = \omega/s_0 \) and the subscripts 2 and 3 refer to the forward and backward quasi-acoustic waves (respectively).

In the limit of zero drift field \((v = 0)\), Eq. 2.18b reduces to the dispersion relation for Debye-Huckel screening given by

\[ k = \pm i \frac{\sqrt{1+i\omega T}}{R} \]  

(2.20)

The general case \((v \neq 0)\) is discussed by Blotekjaer and Quate (1964) and a review is included here.

It is convenient from the point of view of terminology to specify the electron drift direction to be along the \( x_1 \) axis (forward direction). This implies that the electric drift field, \( E_0 \), is a negative quantity. Therefore define the drift velocity magnitude by
with this notation the dispersion relation for carrier density waves (Eq. 2.18b) can be solved for the forward and backward propagation constants given respectively by

\[ k^o_4 = -\frac{i}{2} \frac{1}{\tau^2} \left[ \frac{\tau V_0 - \sqrt{\left(\tau V_0\right)^2 + 4\tau^2 (1 + i\omega \tau)}}{2} \right] \]  

(2.22a)

\[ k^o_5 = -\frac{i}{2} \frac{1}{\tau^2} \left[ \frac{\tau V_0 + \sqrt{\left(\tau V_0\right)^2 + 4\tau^2 (1 + i\omega \tau)}}{2} \right] \]  

(2.22b)

where subscripts 4 and 5 refer to forward and backward quasi-carrier waves.

The complex propagation constants can be expressed in terms of real quantities as

\[ k^o_4 = q^o_4 - i\alpha^o_4 \]  

(2.23a)

\[ k^o_5 = q^o_5 - i\alpha^o_5 \]  

(2.23b)

where the wave vectors are given by

\[ q^o_4 = -q^o_5 = \frac{1}{\sqrt{2}} \frac{\tau V_0}{2\tau^2} \left( \sqrt{1 + \left(\frac{2\tau}{\tau V_0}\right)^2 + \left[ \left(\frac{2\tau}{\tau V_0}\right)^2 \omega \tau \right]^2} - \left[ 1 + \left(\frac{2\tau}{\tau V_0}\right)^2 \right]^{\frac{1}{2}} \right) \]  

(2.24)
and the attenuation constants are given by

\[
\omega_0 = \frac{1}{\sqrt{2}} \frac{V}{2R^2} \left[ \left( \sqrt{1 + \left( \frac{2R}{TV_0} \right)^2} \right)^2 + \left( \frac{2R}{TV_0} \right)^2 \omega T \right]^{\frac{1}{2}} \left[ 1 + \left( \frac{2R}{TV_0} \right)^2 \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} \right]
\]

(2.25a)

\[
\alpha_0 = \frac{1}{2} \frac{TV_0}{2R^2} \left[ \left( \sqrt{1 + \left( \frac{2R}{TV_0} \right)^2} \right)^2 + \left( \frac{2R}{TV_0} \right)^2 \omega T \right]^{\frac{1}{2}} \left[ 1 + \left( \frac{2R}{TV_0} \right)^2 \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} \right]
\]

(2.25b)

Use was made of the identity given by

\[
\sqrt{a + 1b} = \frac{1}{\sqrt{2}} \left[ (a + \sqrt{a^2 + b^2}) + i (a - \sqrt{a^2 + b^2}) \right]^{\frac{1}{2}} \right]
\]

(2.26)

A frequency versus wave vector plot is given in Fig. 2.1. The attenuation per wavelength for forward and backward carrier density waves is plotted, in Figs. 2.2a and 2.2b respectively, as a function of frequency. It is to be noted that the backward wave is very highly attenuated. In Figs. 2.3a and 2.3b both the phase velocity, \( (\omega / q) \), and the attenuation per wavelength of the forward carrier density waves are plotted respectively as functions of the electron drift velocity.

For typical values of \( D = 7 \text{ cm}^2/\text{sec.} \), \( \epsilon = 9 \) and \( V_0 = 1.8 \times 10^5 \text{ cm./sec.} \), the low frequency limit specified by

\[
\left( \frac{2R}{TV_0} \right)^2 \omega T = \frac{8\pi \nu D}{V_0^2} \ll 1
\]

(2.27)

is satisfied for the operating frequency, \( \nu \ll 2 \times 10^6 \text{ Hz} \).
The forward and backward waves propagate with equal and opposite phase velocities. (After Blotekjaer and Quate 1964.)

Fig. 2.2 Attenuation per wavelength for carrier density waves plotted versus frequency: a) the forward wave; b) the backward wave. (After Blotekjaer and Quate 1964.)
Fig. 2.3  

a) Phase velocity versus electron drift velocity for the forward carrier density wave.

b) Attenuation per wavelength versus electron drift velocity for the forward carrier density wave. (After Blotekjaer and Quate 1964.)
In this limit, Eqs. 2.24 and 2.25 reduce to

\[ q^o = -q^o = \frac{\omega}{V_0} \left[ 1 + \left( \frac{2R}{TV_0} \right)^2 \right]^{-\frac{3}{2}} \]  
\[ (2.28a) \]

\[ \phi^o = \frac{V_0}{2R^2} \left( \sqrt{1 + \left( \frac{2R}{TV_0} \right)^2} + 2 \left( \frac{R}{TV_0} \right)^4 (\omega_1)^2 \left[ 1 + \left( \frac{2R}{TV_0} \right)^2 \right]^{-\frac{3}{2}} \right) - 1 \]  
\[ (2.28b) \]

\[ \alpha^o = -\frac{V_0}{2R^2} \left( \sqrt{1 + \left( \frac{2R}{TV_0} \right)^2} + 2 \left( \frac{R}{TV_0} \right)^4 (\omega_1)^2 \left[ 1 + \left( \frac{2R}{TV_0} \right)^2 \right]^{-\frac{3}{2}} \right) + 1 \]  
\[ (2.28c) \]

The low conductivity limit given by

\[ \frac{2R}{TV_0} = \frac{4}{V_0} \sqrt{\frac{\pi D \sigma}{\varepsilon}} < 1 \]  
\[ (2.29) \]

is satisfied for \( \sigma < 9 \times 10^{-4} \Omega^{-1} \text{cm}^{-1} \) \( (\rho = 7 \text{ cm}^2/\text{sec.}, \) \( \varepsilon = 9 \) and \( V_0 = 1.8 \times 10^5 \text{ cm/sec.} \). In the low conductivity limit Eqs. 2.28 reduce to

\[ q^o = -q^o = \frac{\omega}{V_0} \left[ 1 - 2 \left( \frac{R}{TV_0} \right)^2 \right] \]  
\[ (2.30a) \]

\[ \alpha^o = \frac{1}{TV_0} \left[ 1 + R^2 \left( \frac{\omega}{V_0} \right)^2 \right] \]  
\[ (2.30b) \]

\[ \alpha^o = -\frac{1}{TV_0/R^2} = -\frac{V_0}{\rho} \]  
\[ (2.30c) \]

The coupling between the acoustic waves and the carrier density waves will now be considered. The quartic secular equation (Eq. 2.17) can be written in the form
where \( F(k) = 0 \) represents the zero order dispersion relation for either the acoustic waves (Eq. 2.18a) or the carrier density waves (Eq. 2.18b). Realizing that the piezoelectric coupling constant, \( \chi \), is small (\( \chi = 0.0378 \) for strain \( S_1 \), in CdS) one can solve for a propagation constant to first order in \( \chi \) by substituting the zero order value of the propagation constant (Eqs. 2.19 or 2.22) into the right-hand side of Eq. 2.31 and solving the now tractable equation given by

\[
F(k) = -\chi G(k^0) \quad (2.32)
\]

where \( p = 2, 3, 4, 5 \)

The corresponding first order dispersion relations for quasi-acoustic waves are given by

\[
k^2_{2,3} - q^2 = -\frac{q^2_0}{\chi} \frac{(q^2 R^2 + i \omega \gamma_{2,3})}{(1 + R^2 q^2_0 + i \omega \gamma_{2,3})} \quad (2.33)
\]

where \( \gamma_2 = 1 - v_0 / s_0 \) and \( \gamma_3 = 1 + v_0 / s' \) (2.34a,b)

Each propagation constant can again be expressed as

\[
k_2 = q_2 - i \alpha_2 \quad (2.35a)
\]
\[ k_3 = q_3 - i\alpha_3 \] (2.35b)

where to first order in \( \chi \)

\[ q_2 = q_0 \left[ 1 - \frac{\chi}{2} \frac{q^2 R^2 (1 + R^2 q^2) + (\omega \tau \gamma)^2}{(1 + R^2 q^2)^2 + (\omega \tau \gamma)^2} \right] \] (2.36a)

\[ q_3 = -q_0 \left[ 1 - \frac{\chi}{2} \frac{q^2 R^2 (1 + R^2 q^2) + (\omega \tau \gamma)^2}{(1 + R^2 q^2)^2 + (\omega \tau \gamma)^2} \right] \] (2.36b)

\[ \alpha_2 = \frac{\frac{\chi}{2} \omega \tau q_0 \gamma_2}{(1 + R^2 q_0^2)^2 + (\omega \tau \gamma)^2} \] (2.36c)

\[ \alpha_3 = \frac{\frac{\chi}{2} \omega \tau q_0 \gamma_3}{(1 + R^2 q_0^2)^2 + (\omega \tau \gamma)^2} \] (2.36d)

Eq. 2.36 are equivalent to those derived originally by White (1962).

Both \( k_2 \) and \( k_3 \) are functions of the drift velocity \( \nu \) through the functions \( \gamma_2 = 1 - \nu / s_0 \) and \( \gamma_3 = 1 + \nu / s_0 \).

Using Eqs. 2.35 and 2.36 one can show that...
It is interesting to note that the forward attenuation constant, $\alpha_2$, changes sign when the electron drift velocity exceeds the ultrasonic velocity ($v_0 > s_0$) implying ultrasonic gain (negative loss) for the forward wave.

Both $|\alpha_2|$ and $|\alpha_3|$, as functions of frequency, reach a maximum for the condition given by

$$R^2 q^2 = \tau \rho (\omega^2/s_0)^2 = 1$$

(2.38)

The corresponding frequency given by

$$v_m = s_0 \frac{1}{2\pi} \frac{1}{\sqrt{1 + \sqrt{\frac{\rho}{\tau}}}} = s_0 \frac{\sqrt{\rho}}{\pi \sqrt{\tau}}$$

(2.39)

is termed the "frequency of maximum gain." Under the condition of maximum gain, the forward wave attenuation per wavelength and the phase velocity, which is defined by

$$s_2 = \frac{u}{q_2}$$

(2.40)

are plotted as functions of drift velocity in Figs. 2.4a and 2.4b respectively.

The first order relations between the plane wave and amplitudes are found by substituting the zero order acoustic propagation constants (Eq. 2.19) into Eqs. 2.15 and are given by the following equations:

$$k_3(v) = -k_2(-v)$$

(2.37)
Fig. 2.4  a) Attenuation per wavelength versus drift velocity.

b) Phase velocity versus drift velocity.

Both for the forward acoustic wave under the condition of maximum gain.
where in the term \(+\) the minus sign refers to forward waves (subscript \(2\)) while the plus sign refers to backward waves (subscript \(3\)).

For the case of perturbed carrier density waves Eq. 2.32 takes the form

\[
1 + R^2 k^2, 4, 5 + i\omega t - ivk = \chi \frac{(k^0, 4, 5)^2}{(k^0, 4, 5)^2 - q_0^2} \quad (2.42)
\]

The solution of Eq. 2.42 is straightforward but will not be presented as it only adds a correction of order \(\chi\) to
the solutions previously considered (Eqs. 2.20 to 2.30).
The first order plane wave amplitudes are given by (Eqs. 2.15 and 2.17)

\[
T_{4,5} = -\beta \left[ 1 - \frac{k_{4,5}^2}{k_{4,5}^2 - \left( \frac{\omega}{S_0} \right)^2} \right] E_{4,5} \quad (2.43a)
\]

\[
u_{4,5} = \frac{\varepsilon}{4\pi} \frac{k_{4,5}}{k_{4,5}^2 - \left( \frac{\omega}{S_0} \right)^2} E_{4,5} \quad (2.43b)
\]

\[
D_{4,5} = \varepsilon E_{4,5} \quad (2.43c)
\]

\[
\varepsilon n_{4,5} = \frac{\varepsilon}{4\pi} k_{4,5} E_{4,5} \quad (2.43d)
\]

\[
J_{4,5} = -i \frac{\varepsilon}{4\pi} \omega E_{4,5} \quad (2.43e)
\]
2.3 Boundary Conditions

The five possible modes of energy propagation in a piezoelectric semiconductor, as considered in Section 2.2 and consisting of two quasi-acoustic waves, two quasi-carrier density waves and the uniform mode (associated with the uniform electric field), are all independent (to order $\chi$) in an infinite medium. However, the introduction of a physical boundary couples all five modes.

Consider a platelet of large (compared to the relevant wavelengths) lateral extent and acoustically free boundaries at the planes $x = 0$ and $x = d$. Assume that there are electrical contacts and appropriate electrical leads at each boundary. The condition of current continuity will be considered first. Eliminating the electron density between Eqs. 2.4 and 2.6, the resulting expression given by

$$\nabla \cdot \left( \vec{J} + \frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t} \right) = 0 \quad (2.44)$$

is very general applying both inside and outside the platelet. Using the divergence theorem Eq. 2.44 can be converted into a surface integral given by

$$\iint_S \left( \vec{J} + \frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{n} \, d\sigma = 0 \quad (2.45)$$
where \( \hat{n} \) is a unit vector normal to the surface element \( d\sigma \).

Choose a surface, \( S \), which lies between the boundary planes \( (x = 0 \text{ and } x = d) \) and encloses one of the boundaries as illustrated in Fig. 2.5. If fringing fields are neglected, the only contributions to the integral (Eq. 2.45) come from the plane waves inside the crystal and from the current in the electrical lead. As there is no displacement current in the electrical lead Eq. 2.45 takes the form

\[
I = A \sum_{a=1}^{5} \left( J_a + \frac{1}{4\pi} \frac{\partial \mathbf{D}_a}{\partial t} \right) \tag{2.46}
\]

where \( I \) is the external current in the electrical lead and \( A \) is the area of the platelet.

From Eqs. 2.16 it can be shown that

\[
J_1 + \frac{1}{4\pi} \frac{\partial \mathbf{D}_1}{\partial t} = (\sigma_0 + i \frac{E_\omega}{4\pi} \mathbf{E}_1 \tag{2.47}
\]

\[
= \sigma_0 (1 + i\omega t) \mathbf{E}_1
\]

which is the current density-field relationship for a lossy capacitor.

Referring to Eqs. 2.15 it is easily shown that the electron current and the displacement current cancel for the acoustic and carrier density waves as given by
Fig. 2.5 Application of the divergence theorem to a piezoelectric platelet with electrical contacts to the major faces.
\[ J_a + \frac{1}{4\pi} \frac{\partial D_a}{\partial t} = 0 \text{ for } a = 2, 3, 4, 5 \] (2.48)

In fact this is a necessary condition for any one dimensional mode which has spatial variation (Eq. 2.44). Therefore the current inside the crystal is carried solely by the uniform mode and Eq. 2.46 reduces to

\[ I = A \sigma (1 + i\omega t) E \] (2.49)

The conditions of stress free boundaries at \( x = 0 \) and \( x = d \) are given respectively by

\[ \sum_{a=1}^{5} T_a = 0 \] \hspace{1cm} (2.50a)

and

\[ \sum_{a=1}^{5} T_a e^{-ikad} = 0 \] \hspace{1cm} (2.50b)

There exists some controversy over what are appropriate boundary conditions for a metallic contact to a semiconductor (Falnes 1968). Three possible electrical boundary conditions are to require that the net electrical current density vanish (Adler and Farnell 1966), the net electrical
space charge vanish (Greebe 1965, Adler and Farnell 1966) or the net electric field vanish (McCumber and Chynoweth 1966, Butcher and Janus 1972) at both boundaries \((x = 0\) and \(x = d\)) as given respectively by

\[
\begin{align*}
\sum_{a=1}^{5} J_a &= 0 \quad (2.51a) \\
\sum_{a=1}^{5} n_a e^{-ik_a d} &= 0 \quad (2.52a)
\end{align*}
\]

\[
\begin{align*}
\sum_{a=1}^{5} J_a e^{-ik_a d} &= 0 \quad (2.51b) \\
\sum_{a=1}^{5} n_a e^{-ik_a d} &= 0 \quad (2.52b)
\end{align*}
\]

or

\[
\begin{align*}
\sum_{a=1}^{5} E_a &= 0 \quad (2.53a) \\
\sum_{a=1}^{5} E_a e^{-ik_a d} &= 0 \quad (2.53b)
\end{align*}
\]

The vanishing of the current density (Eq. 2.51) is an appropriate condition for a blocking (capacitive) contact while the vanishing of either the electric field (Eq. 2.52) or the space charge (Eq. 2.53) can be appropriate for an ohmic contact between a metal and a semiconductor.

Using Eqs. 2.16, 2.41 and 2.43, the stress free (Eq. 2.50) and the electrical (Eq. 2.51, 2.52 or 2.53) boundary conditions can be expressed as relations between the electric field amplitudes as given by
\[ E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 + a_5 E_5 = 0 \]  
(2.54a)

\[ \begin{align*} 
E_1 &+ a_2 E_2 e^{-ik_2 d} + a_3 E_3 e^{-ik_3 d} + a_4 E_4 e^{-ik_4 d} + a_5 E_5 e^{-ik_5 d} = 0 \\
\end{align*} \]  
(2.54b)

\[ b_1 E_1 + b_2 E_2 + b_3 E_3 + b_4 E_4 + b_5 E_5 = 0 \]  
(2.54c)

\[ \begin{align*} 
b_1 E_1 &+ b_2 E_2 e^{-ik_2 d} + b_3 E_3 e^{-ik_3 d} + b_4 E_4 e^{-ik_4 d} + b_5 E_5 e^{-ik_5 d} = 0 \\
\end{align*} \]  
(2.54d)

where

\[ a_{7, 3} = \frac{1}{\chi} \frac{(1 + R^2 q_0^2 + i \omega \gamma_{2, 3})}{(R^2 q_0^2 + i \omega \gamma_{2, 3})} \]  
(2.55a)

\[ a_{4, 5} = \frac{q_0^2}{q_0^2 - k_{4, 5}^2} \]  
(2.55b)

\[ \begin{bmatrix} 
b_1 \\
b_2 \\
b_4, 5 \\
\end{bmatrix} = \begin{bmatrix} 
\frac{1}{i \omega T} \\
\frac{1}{R^2 q_0^2 + i \omega \gamma_{2, 3}} \\
-1 \\
\end{bmatrix} \begin{bmatrix} 
0 \\
\pm q \frac{1}{R^2 q_0^2 + i \omega \gamma_{2, 3}} \\
k_{4, 5} \\
\end{bmatrix} \begin{bmatrix} 
1 \\
or 1 \\
\end{bmatrix} \]  
(2.55c, d, e)
where the three values for $b_1, b_2, b_3, b_4, b_5$ correspond to the vanishing of current density, of charge density and of electric field respectively.

Considerable simplification of Eqs. 2.54 can be achieved by judicious pruning. In the low frequency ($\frac{6\pi vD}{v^2} << 1$) and the low conductivity ($\frac{2R}{(\pi v_0)} << 1$) limits and for an electron drift velocity of the order of the velocity of sound the attenuation of the backward carrier density wave is of the order of (using $D = 7$ cm$^2$/sec., $v_0 = 1.8 \times 10^5$ cm./sec.) (Eq. 2.30) $|a_5| = 2.6 \times 10^4$ Np./cm. (even for zero drift velocity (Eq. 2.20) the attenuation is still of the order of $a_5 \approx 10^3$ Np./cm. under typical conditions). This extremely strong attenuation implies that the backward carrier density wave can be completely neglected at the $x = 0$ surface ($E_5 << e^{-ik_5d} E_5$). Therefore the terms involving $E_5$ may be dropped in Eqs. 2.54a and c. A second result of the strong attenuation is that the coefficient $a_5 << a_2, a_3$ and $a_4$. This condition allows the dropping of the term $a_5 e^{-ik_5d} E_5$ in Eq. 2.54b (provided $\omega << \frac{v_0^2}{D} = 2\pi \times 7 \times 10^8$ Hz).

The coefficients $a_2$ and $a_3$ (Eq. 2.55a) have magnitudes greater than $\frac{1}{\chi_2} - 26$. However, the coefficient $a_4$ (Eq. 2.55b) can also have a large magnitude whenever $\varphi_0^2 - k_4^2$ is small. In the low frequency and low conductivity limit $a$ is given by (Eqs. 2.30 and 2.55b).
The magnitude of $a$ (as given by Eq. 2.56a) reaches a maximum as a function of frequency for the condition given by

$$R^2 \left( \frac{\omega}{v_0} \right)^2 = 1$$  \hspace{1cm} (2.57)

Under the condition $v_0 = s_0$, Eq. 2.57 represents the condition of maximum acoustic gain (Eq. 2.38). In Fig. 2.6 the magnitude of $a$, as calculated from the general equations (Eq. 2.22a and 2.55b), is plotted as a function of drift field. The magnitude of $a_s$ may be obtained from the relation

$$a_s(v_0) = a_u(-v_0)$$  \hspace{1cm} (2.58)
Fig. 2.6 Magnitude of the coefficient, $a_i$, versus drift velocity under the conditions $\sigma = 3 \times 10^{-6} \Omega^{-1} \text{cm.}^{-1}$, $D = 7 \text{ cm.}^2/\text{sec.}$, $s_0 = 1.8 \times 10^5 \text{ cm./sec.}$
and is seen to be very small (see Fig. 2.6).

The dropping of all terms involving $E_5$ is termed the three wave approximation (Adler and Farnell 1966) under which Eqs. 2.54a, b and c reduce to the inhomogenous set expressed in matrix form by

$$
\begin{bmatrix}
a_2 e^{-ik_2d} & a_3 e^{-ik_3d} & a_4 e^{-ik_4d} \\

b_2 & b_3 & b_4
\end{bmatrix}
\begin{bmatrix}
E_2 \\
E_3
\end{bmatrix}
= -
\begin{bmatrix}
E_1 \\
b_1 E_1
\end{bmatrix}
$$

(2.59)

The plane wave amplitudes $E_2$, $E_3$ and $E_4$ can be solved for in terms of $E_1$ as given by

$$
E_2 = -\frac{E_1}{H} \left[ a_1 b_4 (e^{-ik_1d} - 1) - a_4 b_3 (e^{-ik_4d} - 1) + b_1 a_3 a_4 (e^{-ik_4d} - e^{-ik_3d}) \right]
$$

(2.60a)

$$
E_3 = \frac{E_1}{H} \left[ a_2 b_4 (e^{-ik_2d} - 1) - a_4 b_2 (e^{-ik_4d} - 1) + b_1 a_2 a_4 (e^{-ik_3d} - e^{-ik_3d}) \right]
$$

(2.60b)

$$
E_4 = -\frac{E_1}{H} \left[ a_2 b_3 (e^{-ik_2d} - 1) - a_3 b_2 (e^{-ik_3d} - 1) + a_2 a_3 b_1 (e^{-ik_3d} - e^{-ik_2d}) \right]
$$

(2.60c)
In the three wave approximation the amplitude of the voltage across the crystal is given by

\[ V = \int_0^d (E_1 e^{-ik_2 x} + E_3 e^{-ik_3 x} + E_4 e^{-ik_4 x}) dx \]

\[ = E_1 d + i \sum_{m=2}^{4} \frac{E_m}{k_m} (e^{-ik_m d} - 1) \]  

(2.61)

Using the expression for current derived in Eq. 2.49 the electrical impedance of the crystal is given by

\[ Z = \frac{V}{I} = \frac{d + i \sum_{m=2}^{4} \left( \frac{E_m}{E_1} \right) (e^{-ik_m d} - 1) / k_m}{A \sigma_0 (1 + i\omega t)} \]

(2.62)

where the values of \( \frac{E_m}{E_1} \) are found in Eq. 2.60.

The general expression for the impedance (Eq. 2.62) can be separated into a passive impedance \( Z_{\text{pas}} \) in series...
with an active impedance $Z_{\text{act}}$ as illustrated in Fig. 2.7.

![Diagram of an active impedance circuit](image)

**Fig. 2.7** The separation of the impedance, $Z$, into passive and active parts.

$Z_{\text{pas}}$ and $Z_{\text{act}}$ are given by (Eq. 2.62)

$$Z_{\text{pas}} = \frac{R_0}{1+i\omega C_0 R_0} \quad (2.63a)$$

$$Z_{\text{act}} = \frac{i R_0}{1+i\omega C_0 R_0} \sum_{m=2}^{4} \frac{\hat{E}_m}{E_1 k_m} (e^{-ik_m d} - 1) \quad (2.63b)$$
where the static resistance of the platelet $R_0$, the static capacitance of the platelet $C_0$ are given respectively by

$$R_0 = \frac{d}{\sigma_0 A}$$  \hspace{1cm} (2.63c)

$$C_0 = \frac{A\varepsilon}{4\pi d}$$  \hspace{1cm} (2.63d)

The dielectric relaxation time (Eq. 2.12d) is given by

$$\tau = \frac{\varepsilon}{4\pi \sigma_0} = R_0 C_0$$  \hspace{1cm} (2.64)

Eq. 2.63b is best treated by numerical techniques. However, if the forward carrier density wave can be neglected an algebraic solution is feasible. In the following treatment, the assumption that conditions are such that carrier density wave effects may be neglected is made. A detailed discussion of this approximation is given in Appendix D, where the validity of the approximation is established in terms of a number of inequalities involving the frequency and certain parameters characterizing the piezoelectric material.

When carrier density waves can be ignored the active electric impedance $Z_{act}$ (Eq. 2.63b) is given by
\[
\begin{align*}
\Xi_{\text{act}} &= \frac{-iR_0}{1+i\omega C R_0} \left[ \frac{1}{q_0 d \left( \frac{1}{a_2} + \frac{1}{a_3} \right)} \left( \frac{1}{e^{-ik_2d} - 1} \right) \left( \frac{1}{e^{-ik_3d} - 1} \right) \right] \\
&= \frac{1}{a_2 + a_3} \left( \frac{1}{e^{-ik_2d} - 1} \right) \left( \frac{1}{e^{-ik_3d} - 1} \right)
\end{align*}
\]
(2.65a)

where the factor \((1/a_2 + 1/a_3)\) is given by

\[
\begin{align*}
\frac{1}{a_2} + \frac{1}{a_3} &= \chi \left[ \frac{R^2 q_0^2 + i\omega \gamma_2}{1 + R^2 q_0^2 + i\omega \gamma_2} + \frac{R^2 q_0^2 + i\omega \gamma_3}{1 + R^2 q_0^2 + i\omega \gamma_3} \right]
\end{align*}
\]
(2.65b)

For an insulating platelet \((\tau \to \infty)\) the impedance (Eq. 2.65) reduces to a form of the standard expression for an insulating piezoelectric with free surfaces as given by (Beyer and Letcher 1969)

\[
\Xi = \frac{1}{i\omega C_0} \left( 1 - \frac{2\chi}{\pi N} \frac{1 - \cos kd}{\sin kd} \right)
\]
(2.66a)

where \(k = q_0 \left( 1 - \chi/2 \right) - i\alpha_L\)
(2.66b)
with $\alpha_L$ representing lattice losses, and where $N$ is the harmonic number defined by

$$N = \frac{2d\nu}{s_0} = \frac{q_0 d}{\pi} \quad (2.66c)$$

The condition of acoustic resonance will now be developed. For a conducting piezoelectric, the resonant term in Eq. 2.65a defined by

$$R. \ T. = \frac{(e^{-ik_2 d} - 1)(e^{-ik_3 d} - 1)}{(e^{-ik_3 d} - e^{-ik_2 d})} \quad (2.67)$$

may be written in terms of wave vectors and attenuation constants (Eq. 2.35) as given by

$$R. \ T. = \left\{ \left[ (e^{-2\Gamma_1} - e^{-2\Gamma_2}) + e^{\Gamma_3} \cos \beta_3 (1 - e^{-2\Gamma_2}) - e^{-\Gamma_2} \cos \beta_2 \right] \right\}$$

$$\times \left[ e^{2\Gamma_3} + e^{-2\Gamma_2} - 2e^{(\Gamma_3 - \Gamma_2)} \cos (\beta_3 + \beta_2) \right]^{-1} \quad (2.68)$$
where the net phase change and the net acoustic loss for waves travelling a distance \(d\) in the forward and in the backward direction are given respectively by

\[
\beta_2 = q_2 d \quad (2.69a)
\]

\[
\beta_3 = -q_3 d \quad (2.69b)
\]

\[
\Gamma_2 = \alpha_2 d \quad (2.69c)
\]

\[
\Gamma_3 = -\alpha_3 d \quad (2.69d)
\]

The denominator of the resonant term, \(R.T\), given by

\[
e^{2\Gamma_3 + e^{-2\Gamma_2} - 2 e^{\Gamma_3 - \Gamma_2} \cos(\beta_3 + \beta_2)} \quad (2.70)
\]

is positive definite and reaches a minimum for the condition

\[
\cos(\beta_3 + \beta_2) = 1 \quad (2.71a)
\]

or

\[
\beta_2 + \beta_3 = 2\pi N \quad (2.71b)
\]
The condition expressed in Eq. 2.71b corresponds to the round trip phase change of an acoustic wave being an integral multiple of $2\pi$. This condition is equivalent to the following expression for the $N$th acoustic resonant frequency:

$$
\nu_N = \frac{N}{d} \left[ \frac{|s_{\bar{s}}s_2|}{|s_3| + |s_2|} \right] \quad (2.72a)
$$

where

$$
|s_{\bar{s}}, 3| = s_0 \left[ 1 + \chi \left( \frac{R^2q_0^2(1 + R^2q_0^2) + (\omega \gamma_{2,3})^2}{(1 + R^2q_0^2)^2 + (\omega \gamma_{2,3})^2} \right) \right] \quad (2.72b)
$$

Note that the frequency of acoustic resonance is a function of drift voltage (through $\gamma_2$, $\gamma_3$) and of conductivity through (through $R_{3T}$). Fig. 2.8 contains a plot of $|s_{\bar{s}}s_2| / (|s_3| + |s_2|)$ as a function of drift velocity under the condition of maximum gain (Eq. 2.38).

If the acoustic loss (or gain) per transit is much less than unity (i.e., the resonance is "sharp") relatively simple expressions for the impedance of a platelet can be found in the neighbourhood of an acoustical resonance.

Under these conditions ($|\Gamma_2|, |\Gamma_3| \ll 1$) the resonant term, $R_{3T}$, (Eq. 2.68) of the active electrical impedance (Eq. 2.65a) can be expanded about the $N$th reso-
Fig. 2.8 "Effective" acoustic velocity versus drift velocity under the condition of maximum gain.

Fig. 2.9 Dispersion parameter, $\delta$, versus drift velocity under the condition of maximum gain.
nant frequency in powers of $\Gamma_2$, $\Gamma_3$ and $\Delta \nu$ where

$$\Delta \nu = \nu - \nu_N \quad (2.73)$$

In general there exists a velocity dispersion between the forward and the backward acoustic waves. A convenient way of expressing the forward and the backward velocities is given by

$$s_2 = \bar{s}(1-\delta) \quad (2.74a)$$

$$s_3 = \bar{s}(1+\delta) \quad (2.74b)$$

where the mean velocity magnitude, $\bar{s}$, and the dimensionless dispersion parameter, $\delta$, are given by (Eq. 2.72)

$$\bar{s} = \frac{|s_2| + |s_3|}{2} \quad (2.75a)$$

$$\delta = \frac{|s_3| - |s_2|}{|s_3| + |s_2|}$$

$$\frac{\chi}{4} = \frac{R^2 q_0^2 (1+R^2 q_0^2) + (\omega \gamma_3)^2}{(1+R^2 q_0^2)^2 + (\omega \gamma_3)^2} - \frac{R^2 q_0^2 (1+R^2 q_0^2) + (\omega \gamma_2)^2}{(1+R^2 q_0^2)^2 + (\omega \gamma_2)^2} \quad (2.75b)$$
A plot of the dispersion parameter, \( \delta \), versus drift velocity, \( v_0 \), is found in Fig. 2.9. Using Eqs. 2.72, 2.73 and 2.74 the following expressions are found,

\[
\nu_N = \frac{N}{2d} s (1-\delta) (1+\delta) = \frac{N s}{2d} = N v \tag{2.76a}
\]

\[
\beta_2 = q \frac{d}{2} \frac{2\pi (\nu_0 + \Delta \nu) d}{s (1-\delta)} = \pi N + \frac{\pi \Delta \nu}{v} + \pi \delta N \tag{2.76b}
\]

\[
\beta_3 = -q \frac{d}{3} \frac{2\pi (\nu_0 + \Delta \nu) d}{s (1+\delta)} = \pi N + \frac{\pi \Delta \nu}{v} - \pi \delta N \tag{2.76c}
\]

Keeping only the leading terms, the resonant term, R.T., (Eq. 2.68) can be expanded as a function of \( \Gamma_2 \), \( \Gamma_3 \) and \( \Delta \nu \) as follows (assuming \( \Delta \nu / \sqrt{v} \ll 1 \))

\[
R.T. = 2 \left[ 1 - (-1)^N \cos(N \pi \delta) \right] \left( \frac{-(\Gamma_2 + \Gamma_3) + i(2\pi \Delta \nu)}{(\Gamma_2 + \Gamma_3)^2 + (2\pi \Delta \nu)^2} \right) \tag{2.77}
\]

If the dispersion parameter is zero, \( \delta = 0 \), (e.g., for \( v_0 = 0 \) or \( +\infty \)) the term in square brackets in Eq. 2.77 reduces to \([1 - (-1)^N] \) which is equal to 2 for odd harmonics \( (N = 1, 3, \ldots) \) but equal to 0 for even harmonics \( (N = 0, 2, \ldots) \). Even harmonics are not electrically active if no for-
ward-backward-velocity dispersion is present (the inclusion of carrier density waves can provide some electrical coupling though).

For the case of $\delta \neq 0$ odd harmonics can be electrically active. For $N\delta = 1/2$ even and odd harmonics produce nearly equal responses. The roles of even and odd harmonics can reverse in the sense that odd harmonics become electrically inactive for $N\delta = 1$. For frequencies $v << 7 \times 10^8$ Hz. and at the condition of maximum gain (Eq. 2.38) the dispersion parameter reaches a maximum value given by $\chi/8 = 0.005$ (under the condition $v_0 = s_0$). Therefore the condition for equal response of odd and even harmonics ($N\delta = 1/2$) corresponds to a harmonic number $N \geq 100$. The effect of velocity dispersion on the electrical activity of odd and even harmonics was first pointed out by Maines and Paige (1969) in connection with the voltage developed across a crystal. Putting Eq. 2.77 into 2.65a, the active impedance can be expressed, in the neighbourhood of the frequency of the $N^{th}$ harmonic (for $|\Gamma_2|$, $|\Gamma_3| < 1$), by

$$Z_{act} = \frac{iR_0}{1 + i\omega C R_0} \frac{2\chi}{\pi N} \left[ \frac{R^2q_0^2 + i\omega \tau_2}{1 + R^2q_0^2 + i\omega \tau_2} + \frac{R^2q_0^2 + i\omega \tau_3}{1 + R^2q_0^2 + i\omega \tau_3} \right] \times [1 - (-1)^N \cos(\pi N\delta)]$$

$$\text{(2.78)}$$
In the limit $\omega t \gg 1$ (and sufficiently far from $v_0 = s_0$, i.e., $\omega v_2 \gg 1$) Eq. 2.78 reduces to

\[
\Xi = \frac{V}{\omega C_0} \frac{8X}{\pi N} \left[ \frac{(\gamma + \Gamma_2) - i \left( \frac{2\pi \Delta v}{\nu} \right)}{(\gamma + \Gamma_2)^2 + \left( \frac{2\pi \Delta v}{\nu} \right)^2} \right]
\]

(2.79)

Under the conditions for which Eq. 2.79 is valid the total impedance can be represented by a simple equivalent circuit as illustrated in Fig. 2.10.

![Equivalent circuit diagram](image)

- $C_0 = \frac{c A}{4\pi d}$
- $R_0 = \frac{d}{A_\theta}$
- $C_s = \frac{\pi^2 N^2}{8X} C_0$
- $L_s = \frac{8X}{\pi N^2 \omega_0 C_0}$
- $R_s = \frac{8X}{\pi N(\gamma + \Gamma_2) \omega_0 C_0}$

**Fig. 2.10** Series equivalent circuit for the impedance $\Xi$ for $\omega t, \omega v_2 \gg 1$ where $C_0$ is expressed in c.g.s. units.
Returning briefly to the case of the general impedance (including the forward carrier density wave) of Eq. 2.62, for experimental reasons (Section 2.5) the impedance of a piezoelectric cavity is more conveniently expressed in terms of a passive admittance, $Y_{\text{pas}}$, in parallel with an active admittance, $Y_{\text{act}}$, as illustrated in Fig. 2.11 as opposed to two impedances in series (Fig. 2.7).

\[ \frac{1}{Y_{\text{act}}} = \frac{1}{Y_{\text{pas}}} \left( 1 + \frac{Y_{\text{pas}}}{Y_{\text{act}}} \right) \]  

(2.80)
If the carrier density waves can be ignored and if the single transit loss (or gain) is small (\(| \Gamma_2 |, | \Gamma_3 | \ll 1 \)), the active admittance can be expressed, in the neighbourhood of the \(N^{th}\) harmonic frequency, by (Eqs. 2.65a, 2.77 and 2.80)

\[
y_{\text{act}} = \frac{-R_0}{1 + i \omega R_0 C_0} \left\{ 1 + \left( \frac{\pi N}{2} \right) \frac{2 \pi \Delta \nu}{\nu} \frac{-i \left( \Gamma_2 + \Gamma_3 \right)}{\left[ 1 - (-1)^N \cos(\pi N) \right] \left( \frac{1}{a_2} + \frac{1}{a_3} \right)} \right\}
\]

(2.81)

To consider the importance of the term 1 in the braces of Eq. 2.81, the coefficient \((1/a_2 + 1/a_3)\) (Eq. 2.65b) (see Fig. 2.12) can be expressed in terms of the resonant frequency (Eqs. 2.72 and 2.76a) and the round trip acoustic loss (Eqs. 2.69 and 2.36) as given by

\[
\left( \frac{1}{a_2} + \frac{1}{a_3} \right) = 4 \frac{(\nu_N - \nu N S_0/d)}{(\nu_N S_0/d)} + 2i \frac{(\Gamma_2 + \Gamma_3)}{\pi N}
\]

(2.82)
Fig. 2.12 Real and imaginary parts of the coefficient \( \frac{1}{a_2} + \frac{1}{a_3} \) versus drift velocity, plotted in a) and b) respectively under the condition of maximum gain.
where the term \((v_N - 1/2N \omega_0 /d)\) is the difference between the first order (i.e., piezoelectric effects included to first order in \(\chi\)) resonant frequency and the zero order \((\chi = 0)\) resonant frequency. Applying Eq. 2.82 to Eq. 2.81, the term \(1\) in the braces of Eq. 2.81 can be dropped with negligible correction to the resonant frequency and to the round trip loss if the following condition is satisfied

\[ N \gg \frac{8}{\pi^2} \quad (2.83) \]

Under this condition the active admittance is given by

\[
Y_{\text{act}} = -\frac{(1 + i \omega R C_0)}{R_0} \left( \frac{1}{\pi N} \right) \left[ 1 - (-1)^N \cos(\pi N) \right] \left( \frac{1}{a_2} + \frac{1}{a_3} \right) - \left[ \frac{i(\Gamma_2 + \Gamma_3) + (2\pi \frac{\Delta \nu}{\nu})}{(\Gamma_2 + \Gamma_3)^2 + (2\pi \frac{\Delta \nu}{\nu})^2} \right] \quad (2.84)
\]

For the condition \(\omega T, \omega T Y_2 \gg 1\) Eq. 2.84 reduces to

\[
\frac{1}{Y_{\text{act}}} = \frac{\pi N}{8 \chi \omega C_0} \left[ (\Gamma_2 + \Gamma_3) + i 2\pi \frac{\Delta \nu}{\nu} \right] \quad (2.85)
\]
If Eq. 2.83 does not hold but the condition \( \omega \tau_{2} \gg 1 \) is valid, the total impedance of the piezoelectric semiconducting platelet can be represented by the equivalent circuit illustrated in Fig. 2.13.

![Equivalent Circuit Diagram](image)

\[
\begin{align*}
C_0 &= \frac{A}{4\pi d} \\
R_0 &= \frac{d}{A\alpha_0} \\
C_p &= \frac{8X}{\pi^2 N^2} C_0 \\
L_p &= \left(\frac{d}{g}\right)^2 \frac{1}{8X C_0} \\
R_p &= \left(\frac{d}{g}\right) \frac{(\Gamma_2 + \Gamma_3)}{8XC_0}
\end{align*}
\]

**Fig. 2.13** Parallel equivalent circuit for the impedance \( Z \) for \( \omega \tau, \omega \tau_{2} \gg 1 \) where \( C_0 \) is expressed in c.g.s. units.
If the condition specified by Eq. 2.83 holds, the impedances \(-R_0\) and \(-1/(i\omega C_0)\) can be dropped. The resulting equivalent circuit is equivalent to one used by Marshall (1969).
2.4 Electron Trapping

In Sections 2 and 3 the absence of electron trapping effects was tacitly assumed. In an actual semiconductor there always exists some trapping of electrons (e.g., in a compensated semiconductor the ionized donor impurities are potential traps). Trapped electrons are bound to an impurity site and contribute to the electron space charge but are not free to contribute to the conduction current. This can produce a considerable reduction in the possible acoustic gain and thus is of importance to the operation of a phonon maser. A simple accounting of traps will be considered with the express purpose of deriving the appropriate expressions for acoustic gain and dispersion under conditions of trapping.

Traps at different energy levels below the conduction band will be assumed to be independent. The net rate at which electrons enter a trap level is equal to the difference between the rate at which electrons are captured by traps and the rate at which electrons "boil" out of the traps as given by (Haering 1964)

\[
\frac{dn_i}{dt} = n_c (N_i - n_i) S_i v_{th} - n_i N_c e^{-E_i/(k_B T)} S_i v_{th} \quad (2.86)
\]
where \( n_c \) is the electron concentration in the conduction band, \( n_i \) is the electron concentration in the \( i \)th trap level, \( S_i \) is the capture cross-section of the \( i \)th trap level, \( E_i \) is the ionization energy of the \( i \)th trap level, \( v_{th} \) is the thermal velocity of conduction electrons, \( N_i \) is the concentration of traps and \( N_c \) is the effective density of states in the conduction band given by (Henrich and Weinreich 1969)

\[
N_c = 2 \left( \frac{m^* e B}{2 \pi \hbar^2} \right)^{3/2} \tag{2.87}
\]

In Eq. 2.87 \( m^* \) is the effective mass of the electrons and \( \hbar \) is Planck's constant \( /2\pi \).

In steady state, \( \frac{dn_i}{dt} = 0 \) and Eq. 2.86 reduces to

\[
\frac{n_i}{N_i} = \frac{n_{c0}}{n_{c0} + N_c^*} \tag{2.88a}
\]

where

\[
N_c^* = N_c e^{-E_i / (k_B T)} \tag{2.88b}
\]

and the subscript 0 indicates the steady state solution.

To investigate the response of the traps, at an angular frequency \( \omega \), assume a solution of the form
for which Eq. 2.86 has the solution (keeping only linear terms)

\[ n_{i1} = \frac{(N_i - n_{i0})}{i\omega + n_{c0} + N'_c} \]  

(2.90)

The net space charge electron concentration is the sum of the conduction band and the trapped electron concentrations as given by

\[ n + n_0 = n_c + \sum n_i \]  

(2.91)

where \( n \) is the deviation from the equilibrium value \( n_0 \) and the sum is over all trap levels. The fraction of the space charge electrons with time variation \( e^{i\omega t} \) that are in the conduction band is given by

\[ n_i = n_{i0} + n_{i1} e^{i\omega t} \]  

(2.89a)

\[ n_c = n_{c0} + n_{c1} e^{i\omega t} \]  

(2.89b)
For \( \omega \rightarrow 0 \), Eq. 2.92 can be reduced to

\[
f_0 = \left[ 1 + \sum_j \frac{n_{j_0} (N_j + n_{j_0})}{N_j n_{c_0}} \right]^{-1}
\]  

(2.93)

which is equivalent to the fraction of mobile carriers used by Hudson and White (1962).

Specializing to a single trap level (labelled by subscript \( t \)) and using the equilibrium condition (Eq. 2.88a), Eq. 2.92 may be written as

\[
f = \frac{f_0 + i\omega t}{1 + i\omega t}
\]

(2.94a)
where

\[ t = \frac{f_b}{S_t V_{th} (n_{c_0} + N'_C)} \]  \hspace{1cm} (2.94b)

and an equivalent form of Eq. 2.93 (for one trap level) is given by

\[ f_b = \left[ 1 + \frac{N_t N'_C}{(N'_C + n_{c_0})^2} \right]^{-1} \]  \hspace{1cm} (2.94c)

Eq. 2.94 is equivalent to an expression used by Greebe (1966) although a different formalism is used. If the condition given by

\[ \frac{n_{c_0}}{n_{t_0}} \ll 1 \]  \hspace{1cm} (2.95)

is satisfied, the expression for the trapping time (Eq. 2.94b) reduces to
Under these conditions, Eqs. 2.94a, 2.93 and 2.96 are equivalent to expressions used by Uchida et al. (1964). The condition expressed by Eq. 2.95 is reasonable for an n-type semiconductor at low temperatures with donor levels acting as traps, but is not appropriate for a compensated photoconductor. A plot of the trapping function $f$ and the trapping time $\tau$ as functions of the electron concentration in the conduction band $n_c$ is found in Fig. 2.14.

When trapping effects are included, the conduction electron deviation from equilibrium concentration, $n_{c1}$, becomes a complex (i.e., there exists a phase shift) fraction, $f$, of the space charge deviation from equilibrium concentration, $n$. Therefore Eqs. 2.8e and 2.9 in the one dimensional coupled wave theory (Section 2.2) must be modified by replacing $n$ by $fn$ as given by:

$$t = \frac{1}{S_{tvth} \left( \frac{n_{c0} + N'}{N_c N_i} \right)} = \frac{1}{S_{tvth} \left( N_c - n_{t0} \right)} \quad (2.96)$$
The quintic secular equation for the propagation constants of the two quasi-acoustic waves and the two quasi-carrier density waves becomes

\[ J = \frac{1}{2}(\sigma_0 + u\text{efn})E + e Df \frac{\partial n}{\partial x} \quad (2.97) \]

The inclusion of electron trapping does not affect the zero order solution for acoustic waves. The general solution for the zero order carrier density propagation constants becomes (the trapping equivalent of Eqs. 2.22)

\[
[k^2 - \left( \frac{w}{s} \right)^2 \left( 1 + i k^2 R^2 + i \tau (\omega + f \nu) \right) + \sum k^2 [f k^2 R^2 + i \tau (\omega + f \nu)] = 0
\]

\[ \mathcal{Z} \quad (2.98) \]

The effect of trapping can be taken into account in all equations relating to the perturbed acoustic waves by
making the following substitutions (Eq. 2.98)

\[
\gamma'_{2,3} = 1 + f_r \frac{v_0}{s_0} + f_1 \frac{\omega D}{s_0^2} \quad (2.100a)
\]

\[
R^2 q_0^2 \rightarrow R'_{2,3} q_0^2 = R^2 q_0^2 \left( f_r + f_1 \omega \right) \frac{v_0}{s_0} \quad (2.100b)
\]

where the upper sign in (±) or (±) refers to the forward waves (2) and the lower sign refers to backward waves (3). The quantities \( f_r \) and \( f_1 \) are the real and imaginary parts of \( f \) respectively. In particular expressions for the acoustic attenuation (Eq. 2.36) and the acoustic velocity (Eq. 2.72b) corrected for trapping effects are given by

\[
\alpha_{2,3} = \frac{\pm \frac{1}{2} \chi \omega q_0 (1 + f_r v_0 / s_0 + f_1 \omega D / s_0^2)}{(1 + R^2 q_0^2 f_r + f_1 \omega t v_0 / s_0)^2 + (\omega t)^2 (1 + f_r v_0 / s_0 + f_1 \omega D / s_0^2)^2} \quad (2.101a)
\]
In Fig. 2.15 the forward acoustic attenuation $\alpha^2$ is plotted as a function of drift velocity $v_0$ for typical conditions. The forward wave acoustic velocity is plotted as a function of $v_0$ in Fig. 2.16. Trapping can effect a considerable reduction in acoustic loss (and gain) as displayed in Fig. 2.15. The extrema in the loss versus drift velocity curves (the maximum loss and maximum gain) are given by

$$\alpha^2_{\text{max}} = \frac{\pm \chi q_0 f_0}{4 (1 + R^2 q_0^2 |f|^2 / f_0^2 + \omega f_0 / f_0^2) [f + i f_0]}$$

(2.102a)
Fig. 2.15 Acoustic attenuation of the forward wave versus drift velocity under conditions of "maximum" gain ($R^2q_0^2 = 1$), $\omega \tau = 10$, $\chi = 0.0378$. For solid line (---) $f_r = 0.9$, $f_i = 0.05$. For dashed line (----) $f_r = 1$, $f_i = 0$.

Fig. 2.16 Phase velocity of the forward acoustic wave versus drift velocity under conditions of "maximum" gain ($R^2q_0^2 = 1$), $\omega \tau = 10$, $\chi = 0.0378$. For solid line (---) $f_r = 0.9$, $f_i = 0.05$. For dashed line (----) $f_r = 1$, $f_i = 0$. 
at drift velocities of

\[ v_0 \bigg|_{\text{max}} = \frac{s_0}{f_{r\omega}} \left\{ (\omega I + f_{r0}R^2q_0) + \left[ R^2 q_0^2 |f| + \frac{(f_{r0}^2 + f_{r0}^2 \omega I)}{|f|} \right] \right\} \]

(2.102b)

where the upper sign in \((\pm)\) and \((\pm)\) refers to attenuation and the lower sign refers to gain. Eqs. 2.101a and 2.102 are equivalent to expressions used by Southgate and Spector (1965) (a different time development is used).
2.5 Experimental Results

2.5.1 Measuring Techniques

The results of measuring the real and the imaginary components of the admittance of CdS platelets (phonon masers) under a variety of experimental conditions are presented. All measurements were performed by using a wide band admittance bridge which is described in detail in Appendix A. The design of the bridge allowed one to cancel out both the passive admittance (see Fig. 2.11) and any stray capacitances due to the sample holder, etc. Therefore the active admittance (Eqs. 2.81, 2.84 and 2.85) was measured directly.

The admittance measurements were performed on shear wave "b-crystals" which had the crystallographic b-axis (Burbank 1971) oriented to lie along the cavity axis (i.e., the b-axis normal to the major faces of the platelets). A one-dimensional linear approach has been quite successful in describing incubation times (Maines and Paige 1969) and threshold frequencies (Burbank 1971) for this orientation. Platelets of varying dimensions were studied and detailed results are presented for the typical crystal SC 24.01.02-04 E. The crystal specifications (e.g., thickness, surface finish, etc.) are found in Appendix B.
2.52 Measurements in Dark

To establish the background loss (e.g., lattice loss, end reflection loss) for typical phonon masers the admittance of a number of cavities was measured in dark (\( \nu_{\text{dark}} < 10^{-10} \Omega^{-1} \text{cm}^{-1} \)) in the absence of a drift field (\( \gamma_2, \gamma_3 = 1 \)). The maximum frequency (110 MHz.) which was used in the measurements satisfied the criterion of Eq. D.2 for completely neglecting carrier density waves while the high dark resistance implies \( \omega \tau \gg 1 \). Therefore Eq. 2.85 for the active admittance is applicable (in the one dimensional limit) with real and imaginary parts given respectively by

\[
\begin{align*}
\{ G_{\text{act}} \} &= \frac{8 \chi S}{\rho_0} \frac{\Gamma_c}{d} \left( \frac{\Gamma_c}{\Gamma_c^2 + (2\pi \Delta v / \nu)^2} \right) \\
\{ B_{\text{act}} \} &= -\frac{8 \chi S}{\rho_0} \frac{(2\pi \Delta v / \nu)}{d} \left( \frac{\Gamma_c}{\Gamma_c^2 + (2\pi \Delta v / \nu)^2} \right)
\end{align*}
\]  

(2.103a)

(2.103b)

where \( \Gamma_c \) replaces the expression \((\Gamma_2 + \Gamma_3)\) and takes on the meaning of the total round trip cavity loss in dark including lattice loss, end reflection loss and coupling to transverse modes. The acoustoelectric loss (Eq. 2.36c, d) is negligible for \( \gamma_2, \gamma_3 = 1 \) and \( \omega \tau \gg 1 \).
Typical experimental traces of the real and of the imaginary parts of the admittance for a cavity in dark are displayed in Fig. 2.17. At frequencies above the primary resonance, subsidiary features due to the finite lateral extent of the platelet are evident (see Fig. 2.17). These subsidiary resonances can be understood in terms of resonant modes of the three dimensional cavity which have an off-axis character (i.e., the acoustic waves do not travel along the major cavity axis) (Bolef and Miller 1971) and will be termed "transverse modes." Similar effects are found in quartz resonators where the subsidiary resonances are termed "the inharmonic series" (Shockley, Curran and Koneval 1966).

The primary (quasi-one dimensional) resonance (Fig. 2.17) will now be considered. Analysis of Eq. 2.103 shows that the real part of the admittance reaches a maximum given by

\[
\{G_{\text{act}}\}_{\text{max}} = \frac{8\chi_s C}{d \Gamma_c} \frac{1}{\Delta \nu}
\]  

(2.104)

when \( \Delta \nu = 0 \).
Fig. 2.17 Experimental traces of the real, a), and the imaginary, b), admittance versus frequency of Cavity SC24 01.02.04 E under condition of "light" clamping.
The real part reaches its half maximum value for

\[ 2\pi \frac{\Delta \nu}{\nu} = \Gamma_c \quad (2.105) \]

The imaginary part of the admittance reaches a maximum and a minimum value given by

\[ \{ B_{\text{act}} \}_{\text{max/min}} = \pm \frac{8\chi s C_0}{d} \left( \frac{\Gamma_c}{2} \right) \quad (2.106a) \]

under the condition

\[ 2\pi \frac{\Delta \nu}{\nu} = \mp \Gamma_c \quad (2.106b) \]

Therefore the net round trip acoustic loss can be determined by experimentally measuring the full width half maximum of real active admittance or the frequency separation of the maximum and the minimum of the imaginary active admittance.

A plot of the round trip acoustic loss as calculated from the experimentally measured extrema of the imaginary part of the admittance (Eq. 2.106b) for cavity SC24.01.02.04 E is given in Fig. 2.18 for two conditions of crystal mounting. The mounting of the crystal was found to be of considerable importance to the operation of a phonon maser. In Fig. 2.18
Fig. 2.18 Round trip acoustic loss in dark for odd harmonics of Cavity SC24.01.02.04 E under conditions of free mounting (I) and light clamping (+).
the lower curve corresponds to "free" mounting where the crystal was held loosely between two electrodes (the top electrode was not in physical contact). The upper curve of Fig. 2.18 corresponded to a condition of "light" clamping where the electrodes were accurately parallel (adjusted for no optical interference bands) and a pressure of \(\sim 1 \text{ Kg./cm.}^2\) was applied to the crystal.

The high acoustic loss at low frequencies exhibited by both curves of Fig. 2.18 is a result of the cavity having finite lateral extent. For crystal SC24.01.02.04 E the lateral dimensions are only a factor \(\sim 2\) greater than the thickness (see Appendix B). Once the acoustic wavelength is much smaller than the dimensions of the cavity (i.e., \(N >> 1\)) the loss drops significantly (Fig. 2.18).

The net loss for both free mounting and light clamping is not a smooth function of frequency (Fig. 2.18) but jumps from one harmonic to the next (only odd harmonics are plotted as even harmonics are electrically inactive). The "scatter" in the points shown in Fig. 2.18 is not of instrumental origin, but represents a real fluctuation of the loss curve. The variations in loss from resonance to resonance for the free mounting case are due to interference between on-axis and off-axis waves (Bolef and Miller 1971). This effect is also responsible for some of the variations observed in the lightly clamped crystal (e.g., harmonic
numbers 35 and 43 in Fig. 2.18). However, in the latter case there are some variations (e.g., harmonic numbers 71, 83 and 93) which have no counterpart in the freely mounted case. These jumps are likely due to acoustic resonance in the conducting glass electrode. (The effects of coupling to a second "cavity" are discussed in Chapter 3).

If care were taken in cleaning all surfaces and in aligning the electrodes of the crystal holder (no optical fringe visible) and a condition of "firm" clamping (a pressure of \(~ 10 \text{ Kg./cm.}^2\) was used, the acoustic loss could be significantly increased. For harmonic number 45 (Fig. 2.18) the round trip acoustic loss went from \(\Gamma_c = 0.0155 \text{ Np.} \) (equivalent to a reflection coefficient \( r = 0.992 \)) for free mounting to \(\Gamma_c = 0.135 \text{ Np.} \) (equivalent to a reflection coefficient \( r = 0.935 \)) for firm clamping. As a secondary effect the resonant frequency of the 45th harmonic shifted from 33.702 MHz. to 33.746 MHz. in going from free to firm mounting, which is 1 - 2 orders of magnitude larger than expected from physical compression of the sample. In Section 3.4 a mechanism is proposed by which coupling to a second cavity (e.g., the glass electrode) can produce tuning of this magnitude.

The minimum loss for a freely mounted cavity (e.g., \(\Gamma_{c\text{, min}} = 0.0110 \text{ Np.} \) for crystal SC24.01.02.04 E in Fig. 2.18) was found not to vary significantly for an increase of contact area of \(\approx 2 \) (for crystal SC24.01.02.01,
see Appendix B) but was found to change by a factor 5.4 for a change in cavity thickness by a factor .38 (for crystal SC24.07.11.03, see Appendix B). The fact that the loss decreased as thickness decreased, but not proportionately, indicates the round trip loss consists partly of a loss per unit thickness (effective lattice loss $\alpha_L \approx 3 \text{ Np./cm}^2$) and partly of a thickness independent loss (effective reflection coefficient $r \approx .9986$).

In a separate mounting of crystal SC24.01.02.04 E (the loss for each harmonic is similar to the upper curve of Fig. 2.18) the maximum amplitude of the real admittance was measured for various harmonics and is plotted in Fig. 2.19a. The maximum amplitude is inversely proportional to $\Gamma_C$ (Eq. 2.104) while the line width parameter, $\Delta \nu$, is proportional to $\Gamma_C$ (Eq. 2.105). The line width-amplitude product should be therefore a constant with respect to frequency given by

$$\{G_{\text{act}}\}_{\text{max}} \Delta \nu = \frac{\nu}{\pi} \frac{\chi_s C_0}{d}$$ (2.107)

The experimentally determined relative line width-amplitude product is displayed in Fig. 2.19b. Above harmonic number 70 there exists a progressive decrease in amplitude (the line width is relatively constant; see Fig. 2.18) which can be projected to zero amplitude for harmonic number $N = 185$ ($\nu = 140 \text{ MHz}$). This behavior can be explained by assuming that
Fig. 2.19  a) Peak amplitude of the real admittance for individual harmonics of cavity SC24.01.02.04 E versus harmonic number.

b) Peak amplitude multiplied by the line width of the real admittance for individual harmonics of cavity SC24.01.02.04 E versus harmonic number. An amplitude reduction factor (Eq. 2.108) is fitted to the data.
the diffused indium contacts tend to short out the electric field in the neighbourhood of the contacts. Consider a very simple model where the acoustic boundary conditions are unaffected but the ohmic diffusion produces an electrical short for an "effective" diffusion depth, $d_D$, and then drops off rapidly (an abrupt diffusion). To calculate the voltage across the crystal (Eq. 2.61) the piezoelectric field must be integrated only from $x = d_D$ to $x = d - d_D$ as illustrated in Fig. 2.20.

![Piezoelectric Field](image)

**Fig. 2.20** The effect of a diffusion depth, $d_D$, illustrated for the third harmonic where $\Lambda$ is the acoustic wavelength.
The effect of the reduced integration length is to introduce an "amplitude reduction factor" (which reduces both the active impedance (Eq. 2.62) and the active admittance (Eq. 2.80)) given by

$$A.R. = \cos \left( \frac{2\pi d_D}{\Lambda} \right)$$

(2.108)

Eq. 2.108 makes a good fit to the data in Fig. 2.19b (see Fig. 2.19b) when an effective diffusion depth of $d_D = 3\mu m$ is used.
2.53 Measurement Under Illumination Without a Drift Field

The round trip acoustoelectric loss in the absence of a drift field ($\gamma_2, \gamma_3 = 1$) but including trapping effects is given by (Eq. 2.101a).

\[
\Gamma_2 + \Gamma_3 = \frac{\chi \omega \tau q_0 d (1 + f_1 \omega D/s_0^2)}{(1 + R^2 q_0^2 f_r)^2 + (\omega \tau)^2 (1 + f_1 \omega D/s_0^2)^2} \quad (2.109)
\]

Again the low operating frequency ($v << 110 \text{ MHz}$) satisfies the condition for ignoring carrier density waves as given by (Eq. D.1)

\[
\frac{R^2 q_0^2}{\omega \tau} = \frac{\omega D}{s_0^2} \ll 1 \quad (2.110a)
\]

or equivalently (Eq. D.2) $v << 700 \text{ MHz}$. (2.110b)

This condition (Eq. 2.110a) also allows the dropping of the terms $q_0^2 R^2 f_r$ and $f_1 \omega D/s_0^2$ and Eq. 2.109 reduces to

\[
\Gamma_2 + \Gamma_3 = (\alpha_2 - \alpha_3) d = \frac{\chi \omega \tau q_0 d}{1 + (\omega \tau)^2} \quad (2.111a)
\]
\[ \frac{x_d}{\tau} \quad \text{for } \omega T >> 1 \quad (2.111b) \]

For all measurements the condition, $\omega T >> 1$, was satisfied; therefore Eq. 2.103 is still a valid expression for the admittance provided that the cavity loss, $\Gamma_c$, is replaced by the sum of the cavity loss, $\Gamma_c$, plus the acoustoelectric loss, $\Gamma_2^2 + \Gamma_3$. In Fig. 2.21 the experimentally determined acoustoelectric loss per unit length for a condition of "light" clamping (the cavity loss has been subtracted off) is plotted versus conductivity for a frequency of 45.70 MHz. ($N = 61$): The theoretical round trip acoustoelectric loss per unit length as calculated from Eq. 2.111 with no fitting involved (i.e., all quantities are experimentally determined), which is plotted as a solid line in Fig. 2.21, lies slightly below the experimentally measured points. The experimental determination of the average bulk electrical conductivity is considered poor. Accurate determination of the "effective" area of cavity SC24.01.02.04 E was not possible as the edges were not uniform, having been cut by a wire saw (Appendix B). Also the non-uniform electric field profile, which exists in the neighbourhood of the electrical contacts (Burbank 1971), has not been accounted for (a two probe method is used). Excellent agreement between experiment and theory is obtained if the bulk conductivity is assumed to be 1.27 times the estimated conductivity, which will be termed the
Fig. 2.21 Acoustoelectric loss (per cm.) versus conductivity. △ - experimental. --- - theoretical for conditions $\chi = 0.0378$, $s_0 = 1.76 \times 10^5$ cm.

$\rho / \text{sec.}, \varepsilon = 9, \nu = 45.6 \text{ MHz}$. 
"corrected" conductivity.

A comment on the measuring technique used to obtain the data in Fig. 2.21 is in order. The oscillator used (H.P. 8601A) had a minimum sweep width of .1 MHz.; therefore accurate measurements of line widths less than 10 KHz. was found to be difficult. To avoid this problem the loss for narrow lines was determined by measuring the amplitude of the response which is inversely proportional to loss (Eqs. 2.104 and 2.106). The proportionality constant was evaluated in conductivity range (ν I x 10⁻⁶ Ω⁻¹cm⁻¹) where both line width and amplitude could be accurately measured. At the opposite extreme of very broad responses (high loss) amplitude measurements were again used to determine loss as noise fluctuations made determination of line widths difficult.

In Fig. 2.22 the acoustoelectric loss per centimeter is plotted over the frequency range, 10 - 110 MHz.
Fig. 2.22 Acoustoelectric loss (per cm.) versus frequency:
- experimental.
- theoretical for conditions $\chi = 0.0378$, $s_0 = 1.76 \times 10^5$ cm./sec., $E_0 = 0$
$\varepsilon = 9$ and "corrected" conductivity $\sigma_0 = 2.0$
$\times 10^{-6}$ $\Omega^{-1}$ cm.$^{-1}$.
2.54 Measurement Under Illumination With a Drift Field

When a drift field is employed the general expressions, with trapping effects included, for the acoustoelectric loss and for the acoustic velocity are given by Eqs. 2.101a and 2.101b. The experimental conditions were such that the condition, \( \omega t \gg 1 \), was satisfied; therefore the general expression for the active admittance in the two wave approximation (Eq. 2.84) reduces to

\[
Y_{\text{act}} = \frac{SC}{d} \left[ 1 - (-1)^N \cos (N\pi) \right]
\]

\[
\times \left( \frac{-1}{a_2} + \frac{1}{a_3} \right) \left[ \frac{\Gamma - i(2\pi \frac{\Delta v}{V})}{\Gamma^2 + (2\pi \frac{\Delta v}{V})^2} \right]
\]

\[
(2.112a)
\]

where \( \Gamma = \Gamma_2 + \Gamma_3 + \Gamma_c \) \( (2.112b) \)

The factors \( [1 - (-1)^N \cos (N\pi)] \) (see Fig. 2.9) and \( (1/a_2 + 1/a_3) \) (see Fig. 2.12) are functions of drift velocity.

The coefficient \( (1/a_2 + 1/a_3) \) can be expressed in terms of a real and an imaginary component as given by

\[
a^\dagger = a^\dagger_R + ia^\dagger_I = \left( \frac{1}{a_2} + \frac{1}{a_3} \right)
\]

\[
(2.113)
\]
The fact that $a^+$ has a significant imaginary component for
a drift velocity near the sound velocity (see Fig. 2.12)
implies that the real and imaginary parts of the active
admittance (Eq. 2.112a) are no longer respectively symmetric
and antisymmetric functions of $\Delta v$ (Eq. 2.103 or Fig. 2.17). Symmetric and antisymmetric functions can be achieved by an
appropriate mixing of the real and imaginary parts of the
active admittance. However, use can be made of the fact
that $a^+_R > a^+_I$ (Fig. 2.12 or Eq. 2.65b). Analysis of
Eq. 2.112a shows that the maxima and minima of the imaginary
part of the active admittance have a frequency separation
given to first order in $a^+_R/a^+_I$ by

$$\Delta v_{\text{min}} - \Delta v_{\text{max}} = \frac{\Gamma}{\pi v} + O\left(\left(\frac{a^+_R}{a^+_I}\right)^2\right)$$

(2.114)

This latter approach was applied to all measurements.

For a crystal thickness of $d = .1194$ cm, the cri-
terion for ignoring carrier density waves for the case of
a drift velocity of the order of the acoustic velocity
($\bar{v}_o \approx s$) becomes (Eq. D-13)

$$\sigma \gg 2.4 \times 10^{-6} \Omega^{-1} \text{ cm}^{-1}$$

(2.115)
which was not always met experimentally. However, for all cases a reasonable fit between theoretical acoustoelectric loss (Eq. 2.101a) and experiment was found by using an "effective" trapping factor, $f$, as a fitting parameter. Agreement between the theoretical shift in the frequency of oscillation as calculated from the acoustic velocity (Eqs. 2.72a and 2.101b) and experiment was poor, but from the discussion in Appendix D a good fit is not expected due to carrier density waves. Typical results are illustrated in Fig. 2.23 where the data ends abruptly at a field of 613 V/cm. due to the initiation of oscillation at a frequency of 20.96 MHz.

If the conductivity is low enough no oscillation occurs as the acoustic gain cannot overcome the cavity losses. Typical plots of the experimentally determined and theoretical generated loss versus drift velocity for a conductivity slightly below the conductivity threshold is displayed in Fig. 2.24 (using a different experimental technique Southgate and Spector (1965) report similar results). Note that this conductivity, $\sigma = 4.4 \times 10^{-7} \text{ cm}^{-1}$, definitely does not satisfy the condition of Eq. 2.115.

Referring to Figs. 2.23a and 2.24, the acoustoelectric loss reaches a maximum in the neighbourhood of 450 V/cm. As displayed in Fig. 2.25, this peak loss was established for a variety of conductivities and frequencies. Two sets of
Fig. 2.23  a) Acoustoelectric loss versus electric drift field.

b) Frequency tuning of the acoustic resonance versus electric drift field.

Experimental measurements (Δ) were made under conditions of "light" clamping for a corrected conductivity, $\sigma_0 = 4.4 \times 10^{-6} \Omega^{-1} \text{cm.}^{-1}$ and $\nu = 33.7 \text{ Mhz}$. Theoretical curves, (--.--.--) and (-----), are for no trapping ($f_0 = 1$) and for trapping factors, $f_r = 0.98$ and $f_i = 0.067$ ($f = 0.7$, $\tau_t = 2 \times 10^{-6} \text{ sec.}$) respectively, where the factors, $T = 300^0 \text{ K.}$, $\mu = 340 \text{ cm}^2 \text{ sec}^{-1} \text{ V}^{-1}$, $\sigma_0 = 1.76 \times 10^5 \text{ cm./sec.}$, $\epsilon = 9$ were used.

The crystal breaks into oscillation with a frequency of 20.96 MHz for a drift field of 613 V/cm.
Fig. 2.24 Experimentally measured acoustoelectric loss (△) versus drift field for conditions below the conductivity threshold for oscillation. Theoretical curves, (---) and (------), correspond respectively to conditions with no trapping and with trapping factors, \( f_r = 0.85 \) and \( f_t = 0.10 \), where \( T = 300 \, ^{0}\text{K}, \mu = 360 \, \text{cm}^2/\text{sec.} \), \( s_0 = 1.76 \times 10^5 \, \text{cm./sec.} \), \( \epsilon = 9 \), \( \sigma_0 = 4.8 \times 10^{-7} \, \Omega^{-1}\text{cm.}^{-1} \), \( \nu = 45.7 \, \text{MHz} \).
Fig. 2.25 Maximum acoustoelectric loss versus frequency for "corrected" conductivities, $\sigma_0 = 13.2$ x $10^{-6}$ $\Omega^{-1}$ cm.$^{-1}$ (□), $\sigma_0 = 4.4$ x $10^{-6}$ $\Omega^{-1}$ cm.$^{-1}$ (△) and $\sigma_0 = 8.8$ x $10^{-7}$ $\Omega^{-1}$ cm.$^{-1}$ (○). Theoretical curves are given for no trapping (----) and for trapping included (-----), where the following conditions were used: $T = 300^\circ$ K., $\mu = 340$ cm.$^2$ sec.$^{-1}$ V.$^{-1}$, $s_0 = 1.76$ x $10^5$ cm./sec.
curves for the theoretical peak loss are included in Fig. 2.25: one set for which trapping effects are absent (Eqs. 2.36c, d) and one for which trapping effects are included (Eq. 2.101a). where the frequency dependence of the trapping factor, $f$, was determined from Eq. 2.94a (using a different experimental technique Uchida et al. report similar results). The trapping factor, $f_0$, and the trapping time, $\tau_t$, were adjusted for a satisfactory fit with experiment.
2.55 Comments on Oscillation

To illustrate the effects of a finite cavity and of the mounting conditions (see Section 2.52) on the oscillation frequency of a phonon maser, the cavity loss as a function of harmonic number for a condition of "light" clamping (Fig. 2.18) is repeated in Fig. 2.26. All frequencies of oscillation below 60 MHz, which were observed under threshold conditions (conductivity was varied) are marked. In every case (excluding the even harmonics whose cavity loss could not be measured) the oscillation frequency corresponds to a harmonic with a particularly low cavity loss.

Oscillation at both even and odd harmonics was observed (Fig. 2.26). At high harmonic numbers and near the condition, \( v_0 = s_0 \), for which the dispersion parameter, \( \delta \), (Fig. 2.9) is large, even harmonics did become electrically active in accordance with Eq. 2.81 as illustrated in Fig. 2.27.

In Fig. 2.28 the threshold frequency of oscillation is plotted as a function of conductivity. At high conductivities the agreement between the measured frequencies and the frequency of maximum gain (solid line, Fig. 2.28) of Eq. 2.39 is good. However, at low conductivities the agreement is quite poor. The deviations from the frequency of maximum gain (trap free theory) can come from three sources: the increase in cavity loss at low frequencies (Fig. 2.18), trapping effects which affect the acoustoelectric loss and gain
Fig. 2.26 Cavity loss for light clamping (Fig. 2.18) with frequencies of threshold oscillation marked; even and odd harmonics are noted.
Fig. 2.27  

a) Trace of real part of active admittance for conditions of zero electric field and in dark for which even modes are inactive.

b) Trace of real part of active admittance (gain increased by 2 from a)) for conditions of $\sigma_0 = 2 \times 10^{-5} \ \Omega^{-1} \ \text{cm}^{-1}$, $E_0 = 627 \ \text{V/cm}$. (slightly below threshold). The even mode, harmonic number, $N = 90$, is now electrically active.
Fig. 2.28 Experimentally determined threshold frequency (△) versus conductivity for a condition of "light" clamping. Theoretical plot (---) corresponding to the condition of maximum gain, $R^2 q_o^2 = 1$, where $D = 7 \text{ cm.}^2/\text{sec.}$, $\varepsilon = 9$ and $s_o = 1.76 \times 10^5$ cm./sec.
(Fig. 2.25) and carrier density effects which can play an important role at low conductivities (Eq. 2.115).
CHAPTER 3

ACOUSTIC RESONANCE IN COMPOSITE CAVITIES

3.1 Introduction

Composite resonators consisting of two or more layers of material with parallel major faces (one dimensional acoustic cavity) bonded together have become of interest in a number of fields. Use of evaporated layers of CdS on substrates of uniform thickness (e.g., polished quartz or silicon) to manufacture high frequency and integrated-circuit-compatible crystal filters have been investigated (Slifer and Roberts 1967, Page 1968). In continuous wave studies of the acoustic properties of solids (Bolef and Miller 1971) the combination of one or more transducers and the sample under study inevitably forms a composite resonator. In this connection, the difference between the acoustic resonance frequency of a composite resonator (transducer and sample) and the resonance frequency of an isolated sample has been of recent interest (Ringermacher, Moerner and Miller 1974). In this dissertation the use of composite cavities allows coupling of acoustic power out of oscillating phonon masers.

In single acoustic cavities the acoustic resonance frequencies are integral multiples of a fundamental (lowest)
frequency. In general this is not true of composite cavities where the acoustic resonance frequencies are found by solving a transcendental equation. In this chapter expressions for both the resonance frequencies and the electrical impedance (for insulating crystals) of a double cavity are presented. An alternate approach using ray summation is developed and applied to a double cavity mounted to a substrate.
3.2 Normal Modes of a Double Cavity

Consider two passive Fabry-Perot type acoustic cavities (labelled A and B) intimately bonded together as illustrated in Fig. 3.1.

![Diagram of two cavities](image)

**Fig. 3.1** Notation for double cavity.

For a one dimensional treatment with the x-axis along the cavity axes, each cavity can support a forward and a backward on-axis acoustic plane wave labelled with + and - superscripts respectively. The acoustic waves can be either all longitudinal or all shear as long as the velocity and elastic constants are defined appropriately. The net acoustic displacement in Cavity A and Cavity B will be given respectively by
where the propagation constants $k_A$ and $k_B$ will be assumed to be real (loss-free case). The propagation constants can be expressed as $k_A = \omega/s_A$ and $k_B = \omega/s_B$ where $s_A$ and $s_B$ are the appropriate acoustic velocities in Cavities A and B respectively.

The two free boundaries at $x = 0$ and $x = d_A + d_B$ and the common boundary (intimate bonding) at $x = d_A$ all satisfy the condition of zero net stress for which (Eq. 2.8a)

$$u_A^+ - u_A^- = 0 \quad (3.2a)$$

and

$$k_A c_A \left( u_A^+ e^{-ik_A d_A} - u_A^- e^{ik_A d_A} \right) \quad \text{and} \quad -k_B c_B \left( u_B^+ e^{-ik_B d_A} - u_B^- e^{ik_B d_A} \right) = 0 \quad (3.2b)$$
where $c_A$ and $c_B$ are the elastic constants of the two cavities.

At the boundary, $x = d_A$, the condition of continuity of displacement (the two materials are bonded together) must also be satisfied as given by

$$u_B^+ e^{-ik_B(d_A + d_B)} - u_B^- e^{ik_B(d_A + d_B)} = 0 \quad (3.2c)$$

$$\begin{pmatrix} u_A^+ e^{-ik_A d_A} + u_A^- e^{ik_A d_A} \\ u_B^+ e^{-ik_B d_A} + u_B^- e^{ik_B d_A} \end{pmatrix} - \begin{pmatrix} u_B^+ e^{-ik_B d_A} + u_B^- e^{ik_B d_A} \\ u_B^+ e^{-ik_B d_A} + u_B^- e^{ik_B d_A} \end{pmatrix} = 0 \quad (3.3)$$

Eqs. 3.2 and 3.3 represent four independent homogeneous equations in the plane wave amplitudes, $u_A^+$, $u_A^-$, $u_B^+$ and $u_B^-$. The condition for solution (the determinant of the coefficient matrix equal to zero) is given by

$$Z_A \sin (k_A d_A) \cos (k_B d_B) + Z_B \cos (k_A d_A) \sin (k_B d_B) = 0 \quad (3.4)$$

where the acoustic impedances, $Z_A$ and $Z_B$, are defined by
The terms, $k_A d_A$ and $k_B d_B$, can be expressed in terms of angular frequency by

$$
k_A d_A = \omega d_A / s_A = \omega t_A \quad (3.6a)
$$
$$
k_B d_B = \omega d_B / s_B = \omega t_B \quad (3.6b)
$$

where $t_A$ and $t_B$ are the acoustic transit times. Using Eq. 3.6 and trigonometric identities, Eq. 3.4 can be cast into the following equivalent forms:

$$
Z_A \sin (\omega t_A) \cos (\omega t_B) + Z_B \cos (\omega t_A) \sin (\omega t_B) = 0 \quad (3.7a)
$$
$$
\tan (\omega t_A) = - \frac{Z_B}{Z_A} \tan (\omega t_B) \quad (3.7b)
$$
\[
\sin \omega (t_A + t_B) = - \left( \frac{Z_A - Z_B}{Z_A + Z_B} \right) \sin \omega (t_A - t_B)
\]  
(3.7c)

Eqs. 3.7b and c are equivalent to expressions given by Ringermacher, Moerner and Miller (1974).

The acoustic amplitude reflection coefficient for an acoustic wave propagating in a medium of acoustic impedance \(Z_A\) and impinging normally on a boundary common with a medium of acoustic impedance \(Z_B\) is easily shown to be (Beyer and Letcher 1969)

\[
r_A = \frac{Z_A - Z_B}{Z_A + Z_B}
\]  
(3.8a)

Note that \(r_B = -r_A\).  
(3.8b)

The case of two cavities very weakly coupled is best considered by using Eq. 3.7a. If \(Z_A > Z_B\) \((r_A \approx 1)\), Eq. 3.7a becomes the condition

\[
\sin (\omega t_A) \cos (\omega t_B) \approx 0
\]  
(3.9)

which has solutions given by
where \( v = \omega/2\pi \) and \( n \) and \( m \) are integers. In the very weak coupling limit (with \( z_A \gg z_B \)) the resonances of the composite cavity correspond closely to the combination of the resonances of Cavity A with two free surfaces and of Cavity B with one free surface and one fixed surface as illustrated in Fig. 3.2.

![Composite cavity modes](image)

Fig. 3.2 Normal modes of a composite cavity in zero coupling limit.
Weak coupling between the cavities only produces significant perturbations on the solutions of Eq. 3.10 when there is a near coincidence (degeneracy) in frequency between a mode of Cavity A and a mode of Cavity B (in the sense of Eqs. 3.10a and 3.10b). Coupling of the cavities tends to split degeneracies. This can best be seen if Eq. 3.7a is written in the form

$$\sin (\omega t_A) = -\frac{Z_B}{Z_A} \frac{\sin (\omega t_B) \cos (\omega t_A)}{\cos (\omega t_B)}$$ \hspace{1cm} (3.11)

The term on the left hand side of Eq. 3.11 can no longer be approximated by zero if the term $\cos \omega t_B = 0$, which occurs near the modes of Cavity B, in the sense of Eq. 3.10b.

For conditions of arbitrary coupling Eq. 3.7c given by

$$\sin \omega(t_A + t_B) = -r_A' \sin \omega(t_A - t_B)$$ \hspace{1cm} (3.12)

is the most useful form of the composite cavity normal mode condition. As the absolute value of the reflection coefficient is necessarily less than or equal to 1 ($-1 \leq r_A \leq 1$) one and only one solution of Eq. 3.12 is found for $\omega(t_A + t_B)$ within the limits given by

$$(N - 1/2)\pi \leq \omega(t_A + t_B) < (N + 1/2)\pi$$ \hspace{1cm} (3.13)
where \( N \) is the integer labelling the solution. For the two special cases, \( z_A = z_B \) (i.e., \( r_A = 0 \)) and \( t_A = t_B \), the frequency of the \( N^{th} \) normal mode of the composite cavity is given by

\[
\tilde{\nu}_N = \frac{N}{2(t_A + t_B)} \quad \text{(3.14a)}
\]

and

\[
\frac{\bar{\nu}_N}{N} = \frac{1}{2(t_A + t_B)} = \frac{\bar{\nu}}{N} \quad \text{(3.14b)}
\]

In the general case the average (over all \( N \)) of \( \frac{\nu_N}{N} \) is equal to \( \bar{\nu} \). It is useful to express the \( N^{th} \) frequency solution of the general case by

\[
\nu_N = N \bar{\nu}_1 + \delta \nu_N \quad \text{(3.15)}
\]

where \( \delta \nu_N \) is the deviation of the actual solution from the "unity" coupling (\( z_A/z_B = 1, r_A = 0 \)) solution. Using Eq. 3.15, Eq. 3.12 can be written as

\[
\sin \left( \frac{\pi \frac{\delta \nu_N}{\nu_1}}{\bar{\nu}_1} \right) = -r_A (-1)^N \sin \omega(t_A - t_B) \quad \text{(3.16a)}
\]

\[
= -r_A \sin \left( 2\omega t_A - \pi \frac{\delta \nu_N}{\nu_1} \right) \quad \text{(3.16b)}
\]
\[ r_A \sin \left( 2\omega t_B - \pi \frac{\delta \nu_N}{\nu_1} \right) \] 

(3.16c)

The deviation from the unity coupling solution, \( \delta \nu_N \), has a maximum given by

\[ \sin \left( \pi \frac{\delta \nu_N}{\nu_1} \right) = 1 + r_A \]

(3.17a)

for

\[ \sin \omega (t_A - t_B) = 1 \]

(3.17b)

If the maximum value of \( \delta \nu_N \) can be measured experimentally, the reflection coefficient, \( r_A \), can easily be determined. \( \delta \nu_N \) is equal to zero under the conditions given by

\[ \omega (t_A - t_B) = \pi m \]

(3.18a)

\[ 2\omega t_A = \pi m \]

(3.18b)

or

\[ 2\omega t_B = \pi m \]

(3.18c)

where \( m \) is an integer.
Because of the discrete nature of the frequency solutions, the condition for \( \delta \nu_N \) being either an extrema (Eq. 3.17) or equal to zero (Eq. 3.18) and the condition for a normal mode are seldom met simultaneously. Often though the extremal values of \( \delta \nu_N \) can be accurately estimated by interpolation as illustrated in Fig. 3.3 which is a plot of experimentally measured deviations from unity coupling. An expanded plot of the data is compared with theoretically calculated values in Fig. 3.4.

The normal mode frequencies (Figs. 3.3 and 3.4) were measured using an electrical resonance technique (described in the next section) on a double cavity, DC5, consisting of a relatively thick (1.194 mm.) platelet of CdS bonded intimately to a thin (176. \mu m.) platelet of sapphire. (Detailed specifications of the dimensions, orientations, surface finishes, etc. are found in Appendix B.) The method of bonding the platelets consisted of a modification of a metallic cold welding process by Sittig and Cook (1968). The process is described in Appendix E.

Referring to Fig. 3.3, the average fundamental frequency, \( \overline{\delta \nu_1} = .7169 \) MHz., was found by averaging the value of \( \nu_N/N \) from \( N = 85 \) to \( N = 133 \). The average value of the maximum deviation is found to be

\[
|\delta \nu|_{\text{max}} = .116 \text{ MHz.} \quad (3.19)
\]
Fig. 3.3  Experimentally measured frequency deviations from the "unity" coupling solution versus normal mode number.
Fig. 3.4 Experimentally measured, ▲, and calculated, △, frequency deviations from "unity" coupling versus normal mode number. Parameters used in the calculation: $\frac{z_B}{z_A} = 2.81$, $t_A = 0.6681 \, \mu\text{sec}$, $t_B = 0.0294 \, \mu\text{sec}$. 
Using Eq. 3.17a the magnitude of the acoustic amplitude reflection coefficient is calculated to be

\[ |r_A| = 0.487 \]  \hspace{1cm} (3.20)

From Appendix B accepted values of the acoustic impedance of the materials of Cavity A and Cavity B are given respectively by \( Z_A = \frac{c_A}{s_A} = 8.63 \times 10^5 \text{ g.cm}^{-2}\text{sec}^{-1} \) and \( Z_B = \frac{c_B}{s_B} = 24.28 \times 10^5 \text{ g.cm}^{-2}\text{sec}^{-1} \). Using these values and Eq. 3.8a the theoretical amplitude reflection constant is calculated to be

\[ |r_A| = \left| \frac{Z_A - Z_B}{Z_A + Z_B} \right| = 0.476 \]  \hspace{1cm} (3.21)

The excellent agreement between the values of \( |r_A| \) from Eqs. 3.20 and 3.21 indicates that the acoustic bond is nearly perfect in the frequency range measured.
3.3 Electrical Impedance of a Double Cavity

A convenient technique for the study of the acoustic resonant frequencies (normal modes) of a double cavity is to have one (or both) cavity of the composite resonator consist of a piezoelectric material. Then electrical impedance (or admittance) measurements, similar to those used for single cavities in Chapter 2, can be employed. With this objective in mind, assume Cavity A (of Fig. 3.1) is a piezoelectric insulator. Acoustic loss in both Cavity A and Cavity B will also be considered.

The relevant equations of state for a one dimensional treatment of a piezoelectric insulator are found in Chapter 2 (Eq. 2.8 with \( n \) and \( \sigma \) set equal to zero). Eq. 3.4 for the net acoustic displacement in Cavities A and B will still hold. Because of the assumption of a small acoustic loss the propagation constants, \( k_A \) and \( k_B \), have small imaginary components and can be expressed as

\[
k_A = \sigma_A - i\alpha_A = \omega/s_A - i\alpha_A \quad (3.22a)
\]

\[
k_B = \sigma_B - i\alpha_B = \omega/s_B - i\alpha_B \quad (3.22b)
\]
The relevant acoustic velocity in Cavity A is the piezoelectrically stiffened velocity (found from Eq. 2.36a in the limit \( \tau \to \infty \)) given by

\[
\begin{align*}
\delta_A &= \sqrt{\frac{c_A}{\rho_A}} (1 + \chi/2) \\
&= \left( \frac{c_A}{\rho_A} \right)^{1/2} (1 + \chi/2) \\
\end{align*}
\]

where \( c_A, \rho_A \) and \( \chi \) are the elastic constant, density and piezoelectric coupling constant appropriate for Cavity A.

As was shown in Chapter 2, the piezoelectric effects couple a uniform electric field (uniform mode) to the two piezoelectrically stiffened acoustic waves in Cavity A (the carrier density waves are absent due to the assumption of an insulator).

The new boundary conditions corresponding to Eqs. 3.2 and 3.3 are given by

\[
\begin{align*}
Z_A \left( u_A^+ - u_A^- \right) - i \frac{\beta}{\omega} E_u &= 0 \\
&= 0 \\
Z_A \left( u_A^+ e^{-ik_A d_A} - u_A^- e^{ik_A d_A} \right) \\
&\quad - Z_B \left( u_B^+ e^{-ik_B d_A} - u_B^- e^{ik_B d_A} \right) - i \frac{\beta}{\omega} E_u &= 0 \\
\end{align*}
\]
The extra term, $-\beta E_u$, is the contribution to the net stress of the uniform electric field, $E_u$, (Eq. 2.16) which is coupled to the acoustic waves in Cavity A. Due to the driving term, $\beta E_u$, Eq. 3.24 is not a homogeneous set.

The electric field due to the displacement, $u_A$, is given by (Eq. 2.15b with $\tau + \infty$)

$$E_A = i \frac{4\pi\beta}{\epsilon} k_A \left( u_A e^{-ik_A x} - u_A e^{ik_A x} \right)$$  \hspace{1cm} (3.25)

The amplitude of the voltage across the piezoelectric crystal is given by

$$V = \int_0^{d_A} (E_u + E_A) \, dx$$  \hspace{1cm} (3.26)
As was shown in Chapter 2 (Eq. 2.46) the external electrical current (electrical contacts are assumed at the planes, \( x = 0 \) and \( x = d_A \), Fig. 3.1) is given by

\[
I = i \frac{A \varepsilon}{4\pi} \omega E_u
\]

(3.27)

where \( A \) and \( \varepsilon \) are the surface area and dielectric constant of Cavity A. The electrical impedance for the double cavity is given by

\[
\tilde{Z}_D = \frac{V}{I}
\]

\[
= \frac{1}{i\omega C_A} + i \left( \frac{4\pi}{\varepsilon} \right)^2 \frac{\beta}{\omega A E_u} \left[ u_A^+ (e^{-i k_A d_A} - 1) + u_A^- (e^{i k_A d_A} - 1) \right]
\]

(3.28a)

where the capacitance of the crystal, \( C_A \), is given by

\[
C_A = \frac{\varepsilon A}{4\pi d}
\]

(3.28b)

Solving Eq. 3.24 for \( u_A^+ \) and \( u_A^- \) in terms of \( E_u \) the electrical impedance can now be expressed as (Eq. 3.28a)
If \( z_B = 0 \) (i.e., Cavity A has two free boundaries) Eq. 3.29 reduces to an expression equivalent to Eq. 2.66a for the impedance of a piezoelectric insulating single cavity.

Again (as in Chapter 2 for a single cavity) the net impedance of a double cavity (Eq. 3.28) can be expressed in terms of a passive impedance in series with an active impedance given respectively by

\[
\Xi_{\text{pas}} = \frac{1}{i\omega C_A} \quad (3.30a)
\]

\[
\Xi_{\text{act}} = \frac{-1}{i\omega C_A} \left( \frac{\chi}{\omega t_A} \right)
\]

\[
\left[ 2z_A (1 - \cos k_A d_A) \cos k_B d_B + z_B \sin k_A d_A \sin k_B d_B \right] \\
\left[ z_A \sin k_A d_A \cos k_B d_B + z_B \sin k_B d_B \cos k_A d_A \right]
\]

\[
(3.30b)
\]
It is to be noted that the denominator of the active impedance is resonant under the conditions of a normal mode (acoustic resonance) (Eqs. 3.4 and 3.7). If the single transit acoustic losses given respectively for Cavities A and B by

\[ \Gamma_A = \alpha_A d_A \]  
\[ \Gamma_B = \alpha_B d_B \]  

are much less than unity, the active impedance (Eq. 3.30b) can be expanded (to first order in the denominator and to zero order in the numerator) in terms of \( \Gamma_A \), \( \Gamma_B \) and the frequency deviation from the \( N^{th} \) normal mode, \( \Delta \nu \) (assumed small) as given by

\[ \sum_{act} = \frac{1}{\omega C_A} \left( \frac{\chi}{\omega t_A} \right) \frac{(1 - \cos \omega t_A)^2}{[\Gamma_A + \Gamma_B P_N] + i2\pi \Delta \nu \left[ t_A + t_B P_N \right]} \]  

(3.32a)

where

\[ P_N = Z_A/Z_B + (Z_B/Z_A - Z_A/Z_B) \cos^2 \omega t_A \]  

(3.32b)
with \( v_N = \omega_N / 2\pi \) being the frequency of the \( N^{th} \) normal mode (solution of Eq. 3.7). The factor, \( F_N \), has a value which lies between \( z_A / z_B \) and \( z_B / z_A \); also \( F_N + 1 \) as \( z_B + z_A \) (unity coupling). The impedance of the double cavity (Eq. 3.32a) is equivalent to the single cavity impedance (Eq. 2.79) rationalized when the following substitutions are made:

\[
2[\Gamma_A + \Gamma_B F_N] + [\Gamma_2 + \Gamma_3], \quad 2[t_A + t_B F_N] + 1/\nu \quad \text{and} \quad \omega t_A + \pi N
\]

except for the term \((1 - \cos \omega t_A)^2/4\) in Eq. 3.32a which is due to the fact that the impedance is sampled across only part of the composite cavity (i.e., only across Cavity A).

It is again convenient for experimental reasons to transform the series combination of the passive impedance, \( \Xi_{\text{pas}} \), and the active impedance, \( \Xi_{\text{act}} \), into a passive admittance, \( 1/\Xi_{\text{pas}} \), in parallel with an active admittance, \( Y_{\text{act}} \). (See Fig. 3.5.)
Fig. 3.5 Equivalent parallel and series representations of the impedance of a double cavity.

The condition for equivalence of the two representations is given by

\[ Y_{\text{act}} + i\omega C_A = \frac{1}{\varepsilon_{\text{act}} + 1/i\omega C_A} \]  
(3.33a)

or

\[ Y_{\text{act}} = i\omega C_A \left[ \frac{i\omega C_A \varepsilon_{\text{act}}}{1 + i\omega C_A \varepsilon_{\text{act}}} \right] \]  
(3.33b)
Referring to Eq. 3.32a, the maximum magnitude of \( i\omega C_A E_{\text{act}} \) is given by

\[
|i\omega C_A E_{\text{act}}|_{\text{max}} = \frac{\chi}{\omega t_A} \frac{4}{\Gamma_A + \Gamma_B (Z_B/Z_A)}
\]  

(3.34)

For typical values of \( \chi = 0.0378 \), \( t_A = 7 \times 10^{-7} \) sec., \( \Gamma_A \), \( \Gamma_B \approx 5 \times 10^{-2} \), and \( Z_B/Z_A \approx 1 \), \( |i\omega C_A E_{\text{act}}|_{\text{max}} \ll 1 \) under the condition

\[
u >> 3 \times 10^6 \text{ Hz.}
\]

(3.35)

When this condition is satisfied Eq. 3.33b can be reduced to (Eq. 3.32)

\[
Y_{\text{act}} = \chi \frac{C_A}{t_A} \frac{(1 - \cos \omega N t_A)^2}{(\Gamma_A + \Gamma_B F_N) + i2\pi \Delta \nu (t_A + t_B F_N)}
\]  

(3.36)

The normal mode frequencies (Figs. 3.3 and 3.4) were experimentally established by measuring the real part of \( Y_{\text{act}} \) using the admittance bridge described in Appendix A.

Under acoustic resonance (normal mode condition) the real
part reaches a maximum given by

\[
\{ G_{\text{act}} \}_{\text{max}} = \chi \frac{C_A (1 - \cos \omega N t_A)^2}{t_A (\Gamma_A + \Gamma_B F_N)}
\]  

(3.37)

Measurements were performed on the double cavity, DC4, which consisted of a piezoelectrically active CdS platelet (1.194 mm. thick), Cavity A, bonded to a passive CdS buffer (6.95 mm. thick), Cavity B. Details of the composite cavity are found in Appendix B. In the cavity, DC4, the close matching of the acoustic impedances of the two cavities (there is a small difference due to different crystal orientations) implies (Eqs. 3.32b and 3.37)

\[
F_N = 1
\]  

(3.38a)

and

\[
\{ G_{\text{act}} \}_{\text{max}} \propto (1 - \cos \omega N t_A)^2
\]  

(3.38b)

Typical plots of the experimentally measured real part, \( G_{\text{act}} \), of the active admittance versus frequency are displayed in Fig. 3.6 (similar displays have been published elsewhere (Bolef and Miller 1971)). An envelope function joining the peaks of the individual responses agrees well with Eq. 3.38b.
Fig. 3.6 Experimental traces of the real part of the active admittance versus frequency for double cavity DC4 for various scan widths. The frequency spacing of the fine structure (≈ 0.044 MHz.) corresponds to the reciprocal of the round trip transit time of the combined cavities, while the frequency spacing of the groups (≈ 1.5 MHz.) corresponds to the reciprocal of the one way transit time of active cavity. The vertical scale is ≈ 1.6 × 10^{-6} \, \Omega^{-1} \, \text{cm}^{-1} \, \text{per cm.}
The case where Cavity B is much thinner than Cavity A \((d_B << d_A)\) will now be considered. Due to the differences in thickness it is reasonable to make the assumption \(\Gamma_B \cdot F_N << \Gamma_A\) for which Eq. 3.37 reduces to

\[
\{G_{\text{act}}\}_{\text{max}} = \chi_c \frac{C_A}{t_A \Gamma_A} (1 - \cos \omega N t_A)^2 \quad (3.39)
\]

As the acoustic resonance frequencies can be calculated (Eq. 3.7) or experimentally measured (Fig. 3.3) the maximum amplitude, \(\{G_{\text{act}}\}_{\text{max}}\), of the real part of the active admittance at resonance can be evaluated from Eq. 3.39 at discrete frequencies. However, envelope functions can be found for two limiting cases.

The limit of no coupling, \(Z_A/Z_B = 0\), will be considered first. If Cavity B were non-existent, \(\{G_{\text{act}}\}_{\text{max}}\) would be equal to \(4 \chi c A / (t_A \Gamma_A)\) for odd modes (acoustic resonances) of Cavity A \((n\text{-odd})\) (Eq. 2.104 where \(\Gamma_C + 2 \Gamma_A\)) and to zero for even modes (Eq. 2.77). However, if Cavity B is included in the composite resonator (but still with no coupling, \(Z_A/Z_B = 0\)), the sequential numbering of the normal modes of the composite system must include the modes of Cavity B (see Fig. 3.2). Therefore whenever a mode of Cavity B occurs (the spacing of the modes of Cavity B is
much wider than the spacing of the modes of Cavity A as $d_B << d_A$, the addition of an extra mode will reverse the roles of odd and even modes (in terms of $|G_{act}|_{max}$) as illustrated in Fig. 3.7.

---

**Fig. 3.7** Maximum amplitude of a composite cavity in the no coupling limit. The open dots refer to "odd" normal modes while the solid dots refer to "even" normal modes of the composite cavity.

In the second limit of unity coupling, $z_A = z_B$, one can show (Eq. 3.7)

$$\cos \omega_N t_A = t \cos \omega_N t_B$$  \hspace{1cm} (3.40)
where the + and - refer to even and odd modes of the composite cavity respectively. Eq. 3.39 can then be written as the envelope function given by

$$\{G_{act}\}_{max} = \frac{C_A}{t_A' A} [1 + (i) \cos \omega_N t_B]^2$$  \hspace{1cm} (3.41)

In Fig. 3.8, for double cavity DC5 ($Z_A/Z_B = .36$) the experimentally measured and the calculated (Eq. 3.39 and Fig. 3.3) maximum real response, $\{G_{act}\}_{max}$, are plotted versus normal mode number of the composite cavity. The two envelope functions (Fig. 3.7 and Eq. 3.41) are included in Fig. 3.8.
Fig. 3.8 Experimental (▲ and □) and theoretical (▲ and △) maximum amplitudes of the real part of the active admittance of Cavity DC5 versus normal mode number where even modes are denoted by ■ and □ while odd modes are denoted by ▲ and △. The envelope functions for the "zero" coupling limit (----) and for the "unity" coupling limit (-----) are included.
3.4 Acoustic Interference Mirror

The acoustic equivalent of a dielectric mirror will be developed with the objective of providing a controlled coupling of acoustic power out of an oscillating phonon maser. Similar techniques have been investigated in connection with piezoelectric resonators (Newell 1965).

An alternate approach of ray summation will be used rather than the wave treatment employed earlier. Consider three media labelled A, B and C with common boundaries labelled 1 and 2 (see Fig. 3.9).

![Diagram](image)

**Fig. 3.9** Notation for acoustic interference mirror.
Let the amplitude reflection and transmission coefficients for boundaries 1 and 2 be denoted respectively by $r_1$, $t_1$ and $r_2$, $t_2$ where the plus sign refers to waves propagating from left to right and the minus sign refers to waves propagating in the reverse direction. For an incident wave travelling from the left in Medium A, multiple reflections occur inside Cavity B which all contribute to the reflected wave travelling to the left from Boundary 1. Assuming that there is no loss in any medium (loss will be allowed in the boundaries), the phase change each wave undergoes in a single transit of Cavity B can be expressed as $e^{-i\beta}$. Medium C is assumed to be an infinite half space; therefore any energy leaked into Medium C is assumed lost.

Summing all contributions (for steady state conditions) the net reflection coefficient for waves in Medium A normally incident on Boundary 1 is given by

$$r'e^{i\phi} = r_1 + t_1^2 r_2 e^{-i2\beta} \sum_{n=0}^{\infty} \left(-r_1 r_2 e^{-i2\beta}\right)^n$$

$$= r_1 + \frac{t_1^2 r_2 e^{-i2\beta}}{1 + r_1 r_2 e^{-i2\beta}}$$

where $\beta = q_B d_B$
The effective energy reflection coefficient and the phase of reflection are given by

\[ R' = (r')^2 \]

\[ R' = \frac{\Delta r_1 r_2 + r_2^2 (r_1^2 + t_1^2)^2 + 2r_1 r_2 (r_1^2 + t_1^2) \cos 2\beta}{1 + 2r_1 r_2 \cos 2\beta + r_1^2 r_2^2} \quad (3.43a) \]

and

\[ \tan \phi = -\frac{r_2 t_1^2 \sin 2\beta}{r_2 (2r_1^2 + t_1^2) \cos 2\beta + r_1 [1 + r_2^2 (r_1^2 + t_1^2)]} \quad (3.43b) \]

Eqs. 3.52 and 3.53 are the acoustic equivalent of light reflected by a dielectric layer (Born and Wolf 1959). If Boundary 1 is assumed to contribute no loss (i.e., \((r_1^2 + t_1^2) = 1\)) Eqs. 3.43 reduce to

\[ R' = \frac{r_1^2 + r_2^2 + 2r_1 r_2 \cos 2\beta}{1 + 2r_1 r_2 \cos 2\beta + r_1^2 r_2^2} \quad (3.44a) \]
For the case of an isolated double cavity with \( r_2 = 1 \) the energy reflection and phase coefficient are given by

\[
R' = 1 \quad (3.45a)
\]

\[
\tan \phi = - \frac{(1 - r_1^2) \sin 2\beta}{(1 + r_1^2) \cos 2\beta + 2r_1} \quad (3.45b)
\]

Therefore a second cavity (Cavity B) bonded to a primary cavity (Cavity A) acts as a loss-free mirror with a frequency dependent phase of reflection. This analogy is particularly useful when Cavity B is much thinner than Cavity A as in Figs. 3.3 and 3.4. In Fig. 3.10 the phase of reflection, \( \phi \), is plotted versus the single transit phase change, \( \beta_B \), for a reflection coefficient, \( r_2 = -0.476 \).

In the more general case for \( |r_2| \leq 1 \), the net energy reflection coefficient, \( R' \), (Eq. 3.44a) is no longer equal to unity but reaches a maximum for the condition given by
Single Transit Phase Change

Fig. 3.10 Phase of reflection, $\phi$, versus the single transit phase change, $\beta$, of Cavity B for a reflection coefficient, $r_1 = -0.476$, for acoustic waves in Medium A at the A-B interface; $r_2 = 1$. 
\[
\cos 2\beta = \frac{r_{12}^*}{|r_{12}|} \tag{3.46}
\]

for which (Eq. 3.44a)

\[
R'_{\text{max}} = \frac{(|r_1| + |r_2|)^2}{(1 + |r_{12}|)^2} \tag{3.47}
\]

For a CdS phonon maser bonded to a thin platelet of sapphire which is in turn bonded to a fused silica buffer the reflection coefficients calculated from Eq. 3.8a (physical data in Appendix B) are \(r_1 = -0.476\) and \(r_2 = 0.492\) for which

\[
R'_{\text{max}} = 0.615 \tag{3.48a}
\]

and

\[
r'_{\text{max}} = \sqrt{R'_{\text{max}}} = 0.784 \tag{3.48b}
\]

If the silica buffer can be treated as an infinite half space the minimum fractional loss of energy from the CdS-silica system would be \(1 - 0.615 = 0.385\) which may be too high for the system to oscillate. Lower loss can be achieved by using more layers (Newell 1965). Experiments utilizing these approaches are discussed in Chapter 4.
CHAPTER 4

LIGHT DIFFRACTION IN COMPOSITE ACOUSTIC CAVITIES

4.1 Introduction

Light diffraction can be a very powerful tool for the study of acoustic waves. In transparent media, the frequency, wavelength, velocity, polarization, propagation direction and spatial extent of acoustic waves can be determined using light diffraction techniques. A review of light diffraction by acoustic waves is presented in Section 4.2 in order to provide a background for the experimental techniques employed.

Experiments were performed on four separate double cavity systems. Three of the double cavities, DC1, DC2 and DC3, each consisted of a single crystal CdS platelet bonded to a fused silica buffer which had parallel faces. The orientation of each CdS crystal was chosen such that the c-axis formed an angle of 30° (rotation about the a-axis) with the normal to the main faces of the platelet (i.e., cavity axis). For crystals of this orientation (30° crystal), the on-axis piezoelectric coupling constant is a maximum for quasi-shear waves and is low for quasi-longitudinal waves.
(Vrba and Haering 1973b). The fourth double cavity, DC4, consisted of a CdS platelet, with orientation such that the b-axis lies parallel to the cavity axis (b-crystal), bonded to a CdS buffer. For b-crystals the on-axis piezoelectric coupling constant is a local maximum for shear waves and is zero for longitudinal waves (Vrba and Haering 1973b). The CdS buffer is of interest since the diffraction efficiency can be higher than that of silica (Dixon 1967) and since a better acoustic match to the active crystal is possible. Further details on the construction of the composite cavities are found in Appendix B.

Exploratory measurements were made on two triple cavities, TC1 and TC2, each consisting of a 30° crystal bonded to a sapphire platelet which was in turn bonded to a fused silica buffer. For triple cavity TC2 the main faces were not parallel. The non-bonded face was polished such that its normal formed an angle of approximately 4° with the cavity axis.
4.2 Diffraction Theory

Diffraction of light by acoustic waves was predicted by Brillouin (1922) and the first experimental observations were reported in 1932 (Debye and Sears 1932, Lucas and Biquard 1932).

The general theories of light diffraction by acoustic waves are quite complex (Born and Wolf 1959). However two tractable limits exist: the Raman-Nath limit for low acoustic frequencies and narrow acoustic beams and the Bragg limit for high frequencies. These limits have been put on a quantitative basis (Extermann and Wannier 1936, Willard 1949, Klein and Cook 1967). Two dimensionless parameters are defined by

\[ Q = \frac{2\pi \lambda L}{n_0 \Lambda^2} \]  \hspace{1cm} (4.1a)

\[ v = \frac{2\pi L \Delta n}{\lambda} \]  \hspace{1cm} (4.1b)

where \( \lambda \) is the wavelength of light in vacuo, \( \Lambda \) is the acoustic wavelength, \( n_0 \) is the undisturbed index of refraction of the medium, \( \Delta n \) is the amplitude of the modulation of the index of refraction due to the acoustic wave and \( L \) is the thickness of the acoustic beam (Fig. 4.1). The Raman-Nath limit occurs for:
Fig. 4.1 Notation used in describing diffraction of light by an acoustic beam in a medium of refractive index, \( n_0 \).
The Bragg limit for diffraction occurs for:

\[ Q << 2 \]  \hspace{1cm} (4.2a)

and \( Qv \lesssim 2 \) \hspace{1cm} (4.2b)

For the diffraction experiments presented, typical operating values for fused silica at a frequency of 40 MHz.

\[ \lambda = 10^{-2} \text{ cm}, \quad \lambda = 6.3 \times 10^{-5} \text{ cm}, \quad L = 2 \times 10^{-1} \text{ cm}. \]

are: \( L = 1.5 \) which give \( Q \approx 0.5 \). The condition \( Qv \leq 2 \)

is very easily satisfied at all reasonable values of strain.

(The connection between \( \Delta n \) and strain is derived in Section 4.3.) Therefore only Raman-Nath theory (Raman and Nath 1935 a, b, 1936 a, b) will be developed.

Consider a monochromatic beam of light with plane wave fronts incident normally on an ultrasonic beam of thickness \( L \) (Fig. 4.1). If the acoustic wave is of sinusoidal form with frequency \( \Omega \) and if the wave propagates in the \( x \) direction then the index of refraction will be of the form

\[ n = n_0 - \Delta n \sin(\Omega t \frac{2\pi x}{\lambda}) \]  \hspace{1cm} (4.4a)
where the minus sign in Eq. 4.4a is due to the convention of having $\Delta n > 0$ for longitudinal waves. As shown in Section 4.3 this necessitates the defining of $\Delta n$ by

$$\Delta n = n - n' \quad (4.4b)$$

where $n'$ is the acoustically perturbed index of refraction.

Assume the incident light propagates in the $z$-direction. If the acoustic beam is thin (in the sense of Eqs. 4.2a and 4.2b) then the light after crossing the acoustic beam will have a phase modulation due to the variation of the index of refraction. Taking the co-ordinate origin at the exit side of the acoustic beam, and ignoring the fact that the acoustic wave is moving while the light crosses it, the amplitude of the light at the $z=0$ surface can be written as

$$A = A_0 e^{i (\omega t + \frac{2\pi L \Delta n}{\lambda} \sin(\Omega t - \frac{2\pi x}{\lambda})]) + c.c. \quad (4.5)$$

where $\omega$ is the light frequency. Using the Bessel relation given by

$$e^{ia \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(a)e^{im\theta} \quad (4.6)$$

and using the definition of the Raman-Nath parameter, $\nu$, 

Eq. 4.5 becomes

\[ A = A_n \sum_{m=-\infty}^{\infty} J_m(v) e^{i(\omega_0 + m\Omega)t} e^{-i2\pi mx/\Lambda} + c.c. \]  \hspace{1cm} (4.7)

To find the amplitude distribution of the diffracted light as a function of spherical polar co-ordinate angles one must evaluate the Fraunhofer integral given by (Raman and Nath 1935a)

\[ A(\phi, \psi) = A_0 e^{i\omega t} \frac{1}{S} \iint_S e^{ikd + \psi \sin(\phi t - 2\pi x/\Lambda)} d\phi d\psi + c.c. \]

\[ = A_0 \sum_{m=-\infty}^{\infty} J_m(v) e^{i(\omega_0 + m\Omega)t} \frac{1}{S} \iiint_S e^{ikd} e^{-2\pi mx/\Lambda} d\phi d\psi + c.c. \]  \hspace{1cm} (4.8)

where \( S \) is the area of the aperture of the light beam at \( z=0 \), \( k \) is the wave vector of the light in the medium given by

\[ k = \frac{(2\pi n_0)}{\lambda} \]  \hspace{1cm} (4.9a)
and
\[ d = x \sin \theta \cos \phi + y \sin \theta \sin \phi \] (4.9b)

is the path difference between parallel beams leaving from
points \((0,0)\) and \((x,y)\) in the \(z=0\) plane.

If the dimensions of the aperture grow infinitely
large one finds

\[ \frac{1}{S} \int_{S} e^{ikd} e^{-2\pi mx/A} dx dy = \delta \sin \theta \cos \phi, m2\pi/(kA) \] (4.10a)

with the condition \(\phi = 0\) or \(\pi\). (4.10b)

Choose \(\phi\) to be equal to \(\theta\), then the \(m\)th order diffracted
light propagates at an angle, \(\theta_m\), to the incident light
propagation direction, specified by

\[ \sin \theta_m = \frac{m\lambda}{nA} \quad m = 0, \pm 1, \pm 2, \ldots \] (4.11)

The amplitude and time averaged intensity of the light in
the \(m\)th order become respectively

\[ A_m = A_m (v) e^{i(\omega_0 + m\Omega)t} + c.c. \] (4.12a)
and

\[ I_m = 2A^2 \sum \frac{J_m^2(v)}{2} \]  \hspace{1cm} (4.12b)

From the Bessel function sum rule given by (Gradshteyn and Ryzhik 1965)

\[ \sum_{m=1}^{\infty} J_m^2(v) = 1 \]  \hspace{1cm} (4.12c)

\[ \sum_{m=-\infty}^{\infty} I_m = 2A^2 \]  \hspace{1cm} (4.12d)

\[ \sin(\theta_m + \theta_0) - \sin\theta_0 = \frac{m\lambda}{n_0\Lambda} \]  \hspace{1cm} (4.13)

To this point only light incident normally to the acoustic beam has been considered. Quantitative calculations for an arbitrary incident angle, \( \theta_0 \), measured from the normal to the acoustic beam, show that Eq. 4.11 becomes (Raman and Nath 1935b)
where $\theta_m$ is still the angle between the $m^\text{th}$ order and the incident light (or the $0^\text{th}$ order). For small incident angles $\theta_m$ is independent of $\theta_0$.

$$\sin \theta_m = \frac{m\lambda}{n_t \Lambda} \quad (4.14)$$

The amplitude of the $m^\text{th}$ order becomes (Klein and Cook 1967)

$$A_m = A_0 J_m \left[ \sin \left( \frac{\pi L}{\Lambda} \sin \theta_0 \right) \right] e^{i(\omega t + m\Omega)t + c.c.} \quad (4.15)$$

$A_m$ has its maximum value for $\theta_0 = 0$ (normal incidence).

The diffraction pattern is symmetrical for any $\theta_0$ as

$$|A_m| = |A_{-m}| \quad (4.16)$$

However $A_m = 0$ for all $m \neq 0$ for the condition given by

$$\sin \left( \frac{\pi L}{\Lambda} \sin \theta_0 \right) = 0 \quad (4.17)$$

The smallest value of $\theta_0$ for which $A_m = 0$ is given by

$$\overline{\theta}_0 = \frac{\Lambda}{L} \quad (4.18)$$
which geometrically corresponds to the incident light traversing equal regions of increase ($\Delta n < 0$) and decrease ($\Delta n > 0$) of index of refraction. The effects of these regions cancel giving no diffraction.

For $\lambda = 10^{-2}$ cm. and $L = 2 \times 10^{-1}$ cm. Eq. 4.18 gives

$$\theta_0 = 3^\circ$$  \hspace{1cm} (4.19)

Therefore to use Eqs. 4.12a and b for the amplitude and intensity of diffraction one must have the incident angle $\theta_0 \ll 3^\circ$.

If the ultrasonic beam is formed by a standing acoustic wave the amplitude of the $m^{th}$ order is given by (Cook and Hiedemann 1961)

$$A_m = A_0 \sum_{r=-\infty}^{\infty} J_r(v)J_{r-m}(\frac{v}{a})e^{i[\omega_0 + (2r-m)\Omega]t} + c.c.$$  \hspace{1cm} (4.20)

where $a$ is the ratio of the amplitudes of the forward acoustic wave to the return wave. The time averaged
intensity is given by

\[ I_m = 2\Lambda_0^2 \sum_{r=-\infty}^{\infty} J_r^2(v) J_{r-m}(\frac{v}{a}) \]  

(4.21a)

Double application of the sum rule (Eq. 4.12c) gives

\[ \sum_{m=-\infty}^{\infty} I_m = 2\Lambda_0^2 \]

(4.21b)

If the acoustic field consists of two running waves of frequencies \( \Omega_1 \) and \( \Omega_2 \) and propagation directions \( \beta_1 \) and \( \beta_2 \) in a plane normal to the light beam (Fig. 4.2), then from Vrba and Haering (1973a) the directions of the diffracted light are the solutions of the two equations given by

\[ \sin \theta \cos \phi - m_1 \frac{\lambda}{n_0} \cos \beta_1 - n_1 \frac{\lambda}{n_0} \cos \beta_2 = 0 \]  

(4.22a)

\[ \sin \theta \sin \phi - m_1 \frac{\lambda}{n_0} \sin \beta_1 - n_1 \frac{\lambda}{n_0} \sin \beta_2 = 0 \]  

(4.22b)

where \( m, n = 0, \pm 1, \pm 2, \ldots \)
Fig. 4.2 Notation used in describing diffraction from a two-dimensional superposition of two plane sound waves. (After Vrba and Haering 1973a.)

Fig. 4.3 Two-dimensional diffraction pattern resulting from the situation shown in Fig. 4.2. The angles, $\beta_1$ and $\beta_2$, in the diffraction pattern are identical with the angles between the two sound waves and the x-axis. The distances, $a_1$ and $a_2$, are proportional to $\frac{\lambda n}{10}$ and $\frac{\lambda n}{20}$. (After Vrba and Haering 1973a.)
For observation at a distance \( z = R \gg D_1, D_2 \) and for a small diffraction angle \( \theta \ll \pi \), the rectangular co-ordinates of the diffraction maxima of order \((m, n)\) are given by

\[
x_{mn} = R\theta \cos \phi = R\left( m \frac{\lambda}{n_1} \cos \beta_1 + n \frac{\lambda}{n_2} \cos \beta_2 \right) \quad (4.23a)
\]

\[
y_{mn} = R\theta \sin \phi = R\left( m \frac{\lambda}{n_1} \sin \beta_1 + n \frac{\lambda}{n_2} \sin \beta_2 \right) \quad (4.23b)
\]

as shown in Fig. 4.3.

The amplitude and time averaged intensity of the \((m, n)\) order diffraction spot are given respectively by

\[
A_{mn} = A_0 J_m(v_1) J_n(v_2) e^{i(\omega_0 + m\Omega_1 + n\Omega_2)t} + c.c. \quad (4.24a)
\]

\[
I_{mn} = 2 A_0^2 J^2_m(v_1) J^2_n(v_2) \quad (4.24b)
\]
and

\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{mn} = 2 A_0^2 \]  

(4.24c)

In the situation just considered, if the acoustic field consists of two standing waves, the diffraction pattern is the same but the amplitude and time averaged intensity of the \((m,n)\) order diffraction spot are given respectively by

\[ A_{mn} = A_0 \sum_{s} \sum_{p} J_{m+s}^2(v_1) J_s(v_1) J_{m+p}^2(v_2) J_p(v_2) \times \]

\[ e^{i[\omega_t + (2s+m)\Omega_1 + (2p+n)\Omega_2]} + c.c. \]  

(4.25a)

and

\[ I_{mn} = 2I_0 \sum_{s} \sum_{p} J_{m+s}^2(v_1) J_s^2(v_1) J_{m+p}^2(v_2) J_p^2(v_2) \]

(4.25b)

and

\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{mn} = 2 A_0^2 \]  

(4.25c)
4.3 Strain Index of Refraction Relationship

For the general case of an anisotropic medium (but ignoring piezoelectric effects) the acoustic properties are determined by first solving the elastic equations (Eqs. 2.1 and 2.2)

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{jkl} c_{ijkl} \frac{\partial^2 u_j}{\partial x_j \partial x_k} \]  

(4.26)

where \( u_i \) is the \( i \)th component of mass displacement, \( \rho \) is the density and \( c_{ijkl} \) is the elastic tensor. For a travelling wave solution with acoustic frequency, \( \Omega \), propagation vector, \( \hat{k} \), and a displacement given by

\[ u_i = A_i \sin(\Omega t - \hat{k} \cdot \hat{x}_j) \]  

(4.27)

Eq. 4.26 produces three homogeneous equations for the amplitude components \( A_1, A_2 \) and \( A_3 \). The condition for a solution is a secular equation cubic in \( |\hat{k}|^2 \).

For each of the three possible values of \( |\hat{k}|^2 \) one can solve for an amplitude vector \( \hat{A} \), where \( |\hat{A}| \) is undetermined.

The strain tensor is related to the displacements by (Eq. 2.16)
Consider now the problem of light propagation in an anisotropic medium. For a given wave normal, light can propagate in two modes, each plane polarized but with polarization directions normal to each other (Nye 1964). In general the phase velocity of each polarization is different (i.e., they have different indices of refraction). The polarizations and indices of refraction for both modes can be specified by construction of the indicatrix ellipsoid defined by (Nye 1964)

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_j} + \frac{\partial u_1}{\partial x_i} \right) \]  \hspace{1cm} (4.28)

Consider now the problem of light propagation in an anisotropic medium. For a given wave normal, light can propagate in two modes, each plane polarized but with polarization directions normal to each other (Nye 1964). In general the phase velocity of each polarization is different (i.e., they have different indices of refraction). The polarizations and indices of refraction for both modes can be specified by construction of the indicatrix ellipsoid defined by (Nye 1964)

\[ B_{ij} x_i x_j = 1 \]  \hspace{1cm} (4.29)

where the relative dielectric impermeability, \( B_{ij} \), is defined by

\[ B_{ij} = \frac{\partial E_i}{\partial D_j} \]  \hspace{1cm} (4.30)

where \( E_i \) and \( D_j \) are components of the electric field and electric displacement respectively. If the co-ordinate axes, \( x_1 \), \( x_2 \) and \( x_3 \), are taken to lie along the principal axes of the dielectric constant tensor, Eq. 4.29 reduces to
\[
x^2 \cdot \frac{1}{n_1^2} + x^2 \cdot \frac{2}{n_2^2} + x^2 \cdot \frac{1}{n_3^2} = 1
\]  \hspace{1cm} (4.31)

where \( n_1, n_2 \) and \( n_3 \) are the principal refractive indices.

For an arbitrary direction of light wave normal, the propagation characteristics are described by that central section of the indicatrix ellipsoid which is perpendicular to the wave normal. This section is an ellipse (the index ellipse), the semi-axes of which specify the indices of refraction for the two possible wave fronts. Each polarization (i.e., the direction of the displacement vector \( \mathbf{D} \)) lies parallel to the semi-axis which specifies the corresponding index of refraction.

Condensed matrix notation (Nye 1964) will now be used. Tensor indices will be contracted as follows:

\[
\begin{array}{ccc}
11 \rightarrow 1 & 32, 23 \rightarrow 4 \\
22 \rightarrow 2 & 31, 13 \rightarrow 5 \\
33 \rightarrow 3 & 21, 12 \rightarrow 6 \\
\end{array}
\]  \hspace{1cm} (4.32)

Note that a redefinition of strain is necessary.
For a crystal under strain the indicatrix ellipsoid in general changes orientation, size and shape (the photo-elastic effect). The new ellipsoid will be described by

\[ S_1 = \frac{\partial u}{\partial x_1}, \quad S_2 = \frac{\partial u}{\partial x_2}, \quad S_3 = \frac{\partial u}{\partial x_3}, \]

\[ S_4 = \left( \frac{\partial u}{\partial x_4} \right), \quad S_5 = \left( \frac{\partial u}{\partial x_5} \right), \quad S_6 = \left( \frac{\partial u}{\partial x_6} \right). \]

(4.33)

The difference between the components of the impermeability matrix in the presence and in the absence of strain can be related to the strain by (Nye 1964)
where $p_{ij}$ are the components of the photo-elastic matrix.

If the strain is known, Eqs. 4.34 and 4.35 can be used to find the new indicatrix ellipsoid and hence the new polarizations and indices of refraction.

Consider now the case of an optically and acoustically isotropic solid for which the unstrained impermeability matrix becomes

$$B_1 = B_2 = B_3 = \left(\frac{1}{n_0}\right)^2$$

(4.36a)

and

$$B_4 = B_5 = B_6 = 0$$

(4.36b)

In the absence of strain the indicatrix is a sphere (the orientations of the $x_1$, $x_2$ and $x_3$ axes are arbitrary) with any central section being a circle. Therefore any light
polarization can propagate as a normal mode. For isotropic material the photo-elastic matrix becomes (Nye 1964)

\[
\begin{bmatrix}
  p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\
  p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\
  p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & p_{66} & 0 & 0 \\
  0 & 0 & 0 & 0 & p_{66} & 0 \\
  0 & 0 & 0 & 0 & 0 & p_{66}
\end{bmatrix}
\] (4.37a)

with \( p_{66} = \frac{1}{2}(p_{11} - p_{12}) \) (4.37b)

Solving the elastic wave equation, Eq. 4.26, for the isotropic case, gives a longitudinal wave with velocity given by

\[
s_L = \sqrt{\frac{c_{11}}{\rho}}
\] (4.38)

and two shear waves each with a velocity given by

\[
s_S = \sqrt{\frac{c_{66}}{\rho}} = \frac{1}{2} \sqrt{\frac{c_{11} - c_{12}}{\rho}}
\] (4.39)
The polarizations of the shear waves are normal to each other but can be anywhere in a plane perpendicular to the wave normal.

Choose the co-ordinate system such that the acoustic wave normal lies along the $x_1$-axis. For the case of a shear wave let the mass displacement lie along the $x_2$-axis. The cases of longitudinal and shear waves will be considered separately.

Case 1. Longitudinal wave

The only non-zero strain component is $S_1$; therefore Eq. 4.35 reduces to

\[
\begin{bmatrix}
\Delta B_1 \\
\Delta B_2 \\
\Delta B_3 \\
\Delta B_4 \\
\Delta B_5 \\
\Delta B_6
\end{bmatrix} = \begin{bmatrix}
p_{11}S_1 \\
p_{12}S_1 \\
p_{12}S_1 \\
0 \\
0 \\
0
\end{bmatrix}
\]  

(4.40)

Using Eq. 4.40, Eq. 4.34 becomes

\[
x_1^2(p_{11}S_1 + 1/n_0^2) + (x_2^2 + x_3^2)(p_{12}S_1 + 1/n_6^2) = 0
\]  

(4.41)
which represents an ellipsoid of revolution. For incident light falling normally on the acoustic column, the new indices of refraction become

\[ n'_I = n_0 (1 + p_{11} n_0^2 S_1)^{-\frac{1}{2}} \]  \hspace{1cm} (4.42)

with polarization along the acoustic propagation direction and

\[ n'_{II} = n_0 (1 + p_{12} n_0^2 S_1)^{-\frac{1}{2}} \]  \hspace{1cm} (4.43)

with polarization normal to the acoustic propagation direction.

The changes in the index of refraction used in the diffraction theory of Section 4.2 are given by

\[ \Delta n_I = n_0 - n_0 (1 + p_{11} n_0^2 S_1)^{-\frac{1}{2}} \]  \hspace{1cm} (4.44a)

\[ \Delta n_{II} = n_0 - n_0 (1 + p_{12} n_0^2 S_1)^{-\frac{1}{2}} \]  \hspace{1cm} (4.44b)

If \( S \ll \frac{1}{p_{11} n_0^2} \), Eqs. 4.44a, b reduce to

\[ \Delta n_I \approx \frac{p_{11} S n_0^3}{1} \]  \hspace{1cm} (4.45a)
Case 2. Shear waves

The only non-zero strain component is $S_6$; therefore Eq. 4.35 reduces to

$$
\Delta n_{II} = \frac{4}{3} p_{12} S_1 n^3
$$

(4.45b)

$$
\begin{bmatrix}
\Delta B_1 \\
\Delta B_2 \\
\Delta B_3 \\
\Delta B_4 \\
\Delta B_5 \\
\Delta B_6
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
P_{66} S_6
\end{bmatrix}
$$

(4.46)

Eq. 4.34 becomes

$$
\frac{1}{n^2} \left( x_1^2 + x_2^2 + x_3^2 \right)/n^2 + 2x_1x_2 P_{66} S_6 = 1
$$

(4.47)

Changing to new co-ordinates defined by $(\pi/4$ rotation)

$$
x'_1 = \sqrt{2} (x_1 + x_2)
$$

(4.48a)
\[ x_2' = \sqrt{2}(x_1 - x_2) \]  
\[ x_3' = x_3 \]  

Eq. 4.47 becomes,

\[
(x_1')^2 \frac{1}{n_0^2} + p_{\varepsilon\varepsilon} S_6 + (x_2')^2 \frac{1}{n_0^2} - p_{\varepsilon\varepsilon} S_6 + (x_3')^2 \frac{1}{n_0^2} = 1
\]  

(4.49)

For light incident along the \(x_5\) direction the central section of the indicatrix ellipsoid is an ellipse with semi-axes \(n_1'\) and \(n_{11}'\) at angles of ±45° to the acoustic propagation direction and given by

\[ n_1' = n_0 (1 + p_{\varepsilon\varepsilon} S_6 n_0^2)^{-\frac{1}{2}} \]  
\[ n_{11}' = n_0 (1 - p_{\varepsilon\varepsilon} S_6 n_0^2)^{-\frac{1}{2}} \]  

(4.50a)

(4.50b)

For \(S_6 << \frac{1}{p_{\varepsilon\varepsilon} n_0^2}\) the corresponding changes in index of refraction are given by

\[ \Delta n_1' = \frac{1}{2} p_{\varepsilon\varepsilon} S_6 n_0^3 \]  
\[ \Delta n_{11}' = -\frac{1}{2} p_{\varepsilon\varepsilon} S_6 n_0^3 \]  

(4.51a)

(4.51b)
Incident light of arbitrary polarization is resolved into two components with polarizations along the semi-axes of the central section of the indicatrix ellipsoid, which is normal to the phase propagation direction (Mueller 1937). If the incident light propagating along the \( x_1 \)-axis is linearly polarized with the electric displacement, \( D \), at an angle \( \alpha \) to the \( x_1 \)-axis (in the \( x_1, x_2 \) plane) and the semi-axes of the index ellipse are at angles \( \beta \) and \( \beta + \pi/2 \) (Fig. 4.4), the light is resolved into two components given by

\[
D_I = D \cos (\alpha - \beta) \quad (4.52a)
\]

\[
D_{II} = D \sin (\alpha - \beta) \quad (4.52b)
\]

Light diffracted into the \( m \)th order will have displacement vector components along the semi-axes given by

\[
D_{I,m} = A_m (v_I) D_I \quad (4.53a)
\]

\[
D_{II,m} = A_m (v_{II}) D_{II} \quad (4.53b)
\]

where \( A_m \) is now the diffracted amplitude for unit incident amplitude and \( v_I \) and \( v_{II} \) are the appropriate Raman-Nath parameters.
Fig. 4.4 Decomposition of incident light with electric displacement, $\vec{D}$, into two components, $D_I$ and $D_{II}$, having polarizations along the semi-axes of the index ellipse.
If one uses an exit polarizer at an angle $\gamma$ to the $x_1$-axis the amplitude of the exit light will be proportional to (Fig. 4.5)

$$D_{\gamma,m} = D_{I,m} \cos(\gamma - \beta) + D_{II,m} \sin(\gamma - \beta). \quad (4.54)$$

Using Eqs. 4.52 and 4.53 this becomes

$$D_{\gamma,m} = D \left[ A_m(v_I) \cos(\alpha - \beta) \cos(\gamma - \beta) + A_m(v_{II}) \sin(\alpha - \beta) \sin(\gamma - \beta) \right]. \quad (4.55)$$

For longitudinal acoustic waves $\beta = 0$ and for shear waves $\beta = \pi/4$.

For shear waves Eqs. 4.1b, 4.51a, b give

$$v_I = -v_{II} \quad (4.56)$$

Using the Bessel relation given by

$$J_m(-v) = (-1)^m J_m(v) \quad (4.57)$$

one can show that for both running waves (Eq. 4.12a) and standing waves (Eq. 4.20)
Fig. 4.5 Construction of the electric displacement, \( \mathbf{d}_{\gamma,m} \), of the \( m \)-th order diffracted light which passes through an exit polarizer aligned at angle \( \gamma \) to the x-axis.
Eqs. 4.55 and 4.56 give

\[ A_m(-\nu) = (-1)^m A_m(\nu) \quad (4.58) \]

\[ D_{\gamma,m}(\text{shear}) = D A_m(\nu) \cos(\alpha - \gamma) \text{ for } m \text{ even} \quad (4.59a) \]

\[ D_{\gamma,m}(\text{shear}) = D A_m(\nu) \sin(\alpha + \gamma) \text{ for } m \text{ odd} \quad (4.59b) \]

The polarization effects embodied in Eqs. 4.59 can be very useful. For the case of weak diffraction from shear waves, crossed polarizers (\( \alpha = 0, \gamma = \pi/2 \) or \( \alpha = \pi/2, \gamma = 0 \)) eliminate the strong 0 order spot allowing better visibility of the first (and all odd) order(s). Longitudinal and shear acoustic waves can be distinguished by their different polarization effects.
4.4 Experimental Technique

The composite cavities used for the light diffraction experiments were mounted in a water-cooled holder functionally similar to that described in Appendix A. The electrode which was in contact with the active crystal was transparent in order to pass the illumination used to control the conductivity of the crystal. The a.c. current through the crystal was monitored as a voltage across a 50Ω resistor, by either a spectrum analyzer or a sampling oscilloscope as shown in Fig. 4.6.

Simultaneous light diffraction was carried out in an active 30°-crystal and a fused silica buffer by using a double beam experimental set-up as illustrated in Fig. 4.7. The intensity of a diffraction spot was measured by allowing it to pass through an aperture in a screen in front of a photomultiplier tube.

For diffraction in a 30°-crystal the light propagated along the a-axis and the polarization of the incident light was adjusted to be along the b-axis (60° to the acoustic propagation direction). The exit polarizer was adjusted parallel to the incident polarization. From Vrba and Haering (1973a), the appropriate central section of the indicatrix ellipsoid is the index ellipse with semi-axes $n_1$ and $n_2$ approximately along the c and b-axes respectively.

For an active quasi-shear wave propagating at 30° to the
**Fig. 4.6 Electrical connections to the active crystal.**

**Fig. 4.7 Experimental arrangement for simultaneous diffraction from sound in the buffer and sound in the CdS crystal.**
c-axis the polarization is at $-64.04^\circ$ (Appendix B) to the c-axis and the semi-axes have values given by (Appendix B)

\[
n'_I = n_I(1 + 0.0268 s_{30^\circ})^{-\frac{1}{2}} \quad (4.60a)
\]

\[
n'_II = n_{II}(1 - 0.0222 s_{30^\circ})^{-\frac{1}{2}} \quad (4.60b)
\]

where $n_I$ and $n_{II}$ are the extraordinary and ordinary unstrained, indices of refraction respectively. For the chosen polarization (along the b-axis) the Raman-Nath parameter (Eq. 4.1b) is given by

\[
v_{30^\circ} = 0.022 x \frac{n_{II}s_{30^\circ}}{\lambda} \quad (4.61)
\]

where the thickness of the acoustic column is taken to be equal to the dimension of the active crystal along the a-axis (Appendix B).

When simultaneous diffraction on shear waves was performed, the polarization of the light beam incident on the buffer was adjusted to lie parallel to the c-axis of the active crystal ($30^\circ$ to the acoustic column and $90^\circ$ to the exit polarizer). From Eq. 4.1b and 4.51a the Raman-Nath parameter is given by

\[
v_s = \rho_{66} \frac{n_{II}^3 s}{\lambda n_0} \quad (4.62)
\]
For this case of incident and exit polarizers at an angle to the semi-axes of index ellipse, Eq. 4.59 gives corrections to the amplitudes of diffracted light calculated in Section 4.2 (Eqs. 4.12a and 4.20). For odd and even order diffraction amplitudes the multiplicative correction factors are respectively \( \sin (30^\circ - 60^\circ) = -0.5 \) and \( \cos (30^\circ + 60^\circ) = 0 \) (i.e., the odd order diffraction intensity is multiplied by \( \frac{1}{4} \) and no even order diffraction is seen).

When detailed diffraction intensity measurements were made on shear waves in either silica or CdS buffers the light beam in the active crystal was turned off and the polarization of the light incident on the buffer was adjusted to be parallel to the acoustic column. The exit polarizer was adjusted to be perpendicular to the incident polarization. For this case there is no correction (Eq. 4.59) to odd order intensities but again no even order diffraction is seen.

The Raman-Nath parameter for fused silica is again given by Eq. 4.62. For the CdS buffer with light propagating along the c-axis (for which Eq. 4.59 holds) and acoustic propagation along the b-axis with mass vibration along the a-axis the Raman-Nath parameter is given by (Vrba and Haering 1973a)

\[
v_6 = \frac{1}{2} (p_{11} - p_{12}) \frac{\pi L}{\lambda} n_I^3 S_6
\]  

(4.63)
For detailed diffraction intensity measurements, a narrow light beam (\( \sim 2 \text{ mm in diameter} \)) was used in the buffer to ensure that all incident light passed through the acoustic beam. The light beam passing through the 30º crystal was adjusted to have a width of the order of the crystal thicknesses (Appendix B). The light beams had divergences inside the crystal or buffer of less than 0.2º. This leads to a minor underestimation of strain (Eq. 4.15). As a result of using narrow light beams the diffraction did not come to a sharp focus at the photomultiplier. Therefore standing acoustic waves could not be distinguished from two running waves travelling at a small angle to the cavity axis. This has no effect on estimates of strain from intensity measurements since the time averaged intensities for standing waves (Eq. 4.21a) and for running waves (Eq. 4.24b) give identical results (due to the lack of interference between diffracted light from two acoustic waves propagating in opposite directions).

As the largest value of the Raman-Nath parameter measured was \( V = 0.2 \) the Bessel function of order \( m \) can be approximated by (Mathews and Walker 1965)

\[
J_m (v) \approx \frac{1}{m!} \left( \frac{v}{2} \right)^m \left[ 1 - \frac{1}{m+1} \left( \frac{v}{2} \right)^2 \right] \quad \text{for } m > 0 \quad (4.64a)
\]

\[
J_m (v) = (-1)^m J_{-m} (v) \quad \text{for } m < 0 \quad (4.64b)
\]
In this approximation the diffraction intensities given by Eq. 4.21 or Eq. 4.24b for the zero order and first orders are given respectively by

\[ I_0 = 2A_0^2 \left[ 1 - \frac{v^2}{2} \left(1 + \frac{1}{a^2}\right) \right] \]  \hspace{1cm} (4.65a)

\[ I_\pm = 2A_0^2 \left(\frac{v}{2}\right)^2 \left(1 + \frac{1}{a^2}\right) \]  \hspace{1cm} (4.65b)

\[ I_0 + I_{+1} + I_{-1} = 2A_0^2 \]  \hspace{1cm} (4.65c)

For all calculations of strain the amplitudes of the forward and the return acoustic waves are taken to be equal in magnitude. With \( a = 1 \), the Raman-Nath parameter can be evaluated from measured intensities of the zero order and first order diffraction spots by

\[ |v| = \frac{1}{\beta} \sqrt{2I_{\pm1}/I_0} \]  \hspace{1cm} (4.66)

where \( \beta \) is the correction due to the particular choice of polarization as discussed in connection with Eq. 4.62. As crossed polarizers were used for diffraction in the buffers the zero order intensity in the buffer was measured with the exit polarizer rotated by \( 90^\circ \). Using Eqs. 4.61, 4.62, 4.63 and 4.66 acoustic strain amplitudes are readily calculated.

For accurate measurement of wavelength in the buffers a fine focus experimental set-up was used as illustrated in Fig. 4.8a. A screen was used to shield the oscil-
Fig. 4.8 a Experimental arrangement for maximum resolution (fine focus).

Fig. 4.8 b Experimental arrangement for visualization of the acoustic field.
lating crystal from laser light to avoid influencing its conductivity. Again the polarization of the incident light was along the acoustic propagation direction while the exit polarizer was set at 90°. Eq. 4.11

\[ \sin \theta_m = m \frac{\lambda}{n_0} \]  

(4.67)

gives the relationship between diffraction angle and acoustic wavelength inside the crystal or buffer. Assuming the windows of the crystal/buffer are normal to the incident light one can find the diffraction angle outside the crystal/buffer through application of Snell's law given by

\[ \sin \theta'_m = n_0 \sin \theta_m \]  

(4.68)

Eq. 4.67 becomes

\[ \sin \theta'_m = m \frac{\lambda}{A} \]  

(4.69)

At a distance \( R \) from the crystal/buffer the \( m^{th} \) order spot is laterally displaced a distance \( d_m \) from the 0 order spot

\[ d_m = R \tan \theta'_m \simeq R \sin \theta'_m \]

\[ \simeq m \frac{RA}{\lambda} \]  

(4.70)
which gives:

\[
\Lambda = \frac{m R \lambda}{d_m}
\]

(4.71)

In actual experiments to avoid complications such as camera magnifications, etc. a ruled grating with line spacing \(5.00 \times 10^{-3}\) cm. was placed in close proximity to the crystal/buffer and calibration diffraction pictures were taken. If \(d_C\) is the calibration spot separation then the ultrasonic wavelength is given by

\[
\Lambda = 5.00 \times 10^{-3} m \left( \frac{d_C}{d_m} \right)
\]

(4.72)

To visualize the distribution in the buffer of the acoustic field a technique of focal plane spatial filtering (similar to the Schlieren techniques, Born and Wolf 1959) was used. The experimental set-up is very similar to that of the fine focus case (Fig. 4.8a) and is illustrated in Fig. 4.8b. The diffraction pattern is in focus at the focal plane of lens, L3 (Fig. 4.8b). A screen with a small aperture was used to block all light other than that of the diffraction spot of interest which was allowed to continue to the image plane. The distance \(d_2\) from the focal plane to the image plane is found from ray optics.
where the focal length \( f = 15 \text{ cm} \) and \( d_1 + f \) is the distance from the lens to the crystal/buffer. Sometimes some of the light which came around the buffer was allowed through for reference purposes. The zero-order spot gives the magnified image of the crystal and buffer.

To establish the frequency of the acoustic wave one can perform optical heterodyning (Lastovka and Benedek 1966, Dixon and Gordon 1966) where one beats the diffracted light with a portion of the incident light in a square law detector such as a photomultiplier or photodiode. The detected signal will be proportional to

\[
H = [B + A_m(v)]^2
\]

(4.74)

where \( B \) is a reference signal proportional to the incident light and \( A_m(v) \) is the amplitude of the \( m \)th order diffracted light. To avoid lateral incoherence of the two superimposed signals the light must be mixed in a co-linear manner. The experimental set-up used to achieve parallel beams is shown in Fig. 4.9. Use was made of the fact that wave fronts of gaussian beams (e.g. laser beams) are plane at beam waists.
Fig. 4.9 Experimental arrangement used for optical heterodyning. Mirror $M_2$, beam splitter $BS_2$ and the photodiode are adjusted to achieve colinear mixing of the diffracted light (of the desired order) and the reference light.
In Appendix C, Eq. 4.74 is evaluated for standing and running waves. In both cases (Eqs. C.6 and C.9), for first order spots \( m = \pm 1 \) and small Raman-Nath parameter \( (\nu, \nu/a \ll 1) \) the dominant frequency component of the detected signal is \( \Omega \), the acoustic frequency. Measurement of the acoustic frequency and wavelength, \( \Lambda \), (Eq. 4.72) allows calculation of the wave phase velocity given by

\[
s = \frac{\Lambda \Omega}{2\pi} \quad \text{(4.75)}
\]

The laser used (Spectra-Physics 120 He-Ne) operates in a number of longitudinal modes with mode spacing \( \Delta \omega = 376 \text{ MHz} \). This fact is used in Appendix C to derive a simple method of evaluating the ratio of the amplitude of the forward acoustic wave to the amplitude of the return wave (Eq. C.14)

\[
a = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1} \quad \text{(4.76)}
\]

where \( a \) is the ratio of frequency component \( \Delta \omega \) to that of \( \Delta \omega + \Omega \) or \( \Delta \omega - \Omega \).
4.5 Behavior of Double Cavity

4.5.1 Velocity Measurements

The parameters of the composite cavities were chosen to optimize on-axis (wave fronts parallel to major faces) shear wave acoustic oscillations and to reduce the coupling to longitudinal waves. In all four double cavities (DC1 - DC4) no strong longitudinal waves were observed and the predominant form of oscillation consisted of shear waves propagating nearly on-axis. The averages of velocity measurements made in the fused silica and CdS buffers and also in the active 30°-crystals are presented in Table 4.1 along with published values from Appendix B.

Table 4.1 Shear Wave Velocities

<table>
<thead>
<tr>
<th>Material</th>
<th>Propagation Direction</th>
<th>Displacement Direction</th>
<th>Measured Velocity (cm./sec.)</th>
<th>Published Velocity (cm./sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fused Silica</td>
<td>-</td>
<td>-</td>
<td>3.77(±.06) x 10^5</td>
<td>3.76 x 10^5</td>
</tr>
<tr>
<td>CdS</td>
<td>b-axis</td>
<td>a-axis</td>
<td>1.81(±.05) x 10^5</td>
<td>1.84 x 10^5</td>
</tr>
<tr>
<td>CdS</td>
<td>30° to c-axis in b-c plane</td>
<td>64.04° to c-axis in b-c plane</td>
<td>2.09(±.07) x 10^5</td>
<td>2.04 - 2.11 x 10^5***</td>
</tr>
</tbody>
</table>

*** The two values represent the unstiffened and stiffened values. The experimental value is expected to fall between these values.
The agreement between the measured and expected velocities establishes the waves as shear or quasi-shear waves. This is confirmed by observation of the rotation of the polarization of first order diffracted light by $90^\circ$ as predicted by Eq. 4.59b (for incident polarization along the acoustic beam).

4.52 Threshold Behavior

For an electric field, $E_0$, in the active crystal slightly above the threshold field, $E_t$, needed for acoustic oscillation the frequency spectrum of the current through the crystal consists of a fundamental frequency with a few harmonics (Fig. 4.10b). The spectrum is dominated by either the fundamental or the second harmonic (odd or even type modes). The light diffraction pattern in the buffer is very simple with spots due to higher harmonics rapidly falling off in intensity (Fig. 4.10a). Optical heterodyning and acoustic velocity measurements establish that the primary diffraction spots are associated with the fundamental frequency of the current spectrum no matter whether the fundamental or the second harmonic is dominant.

The threshold fundamental frequency of oscillation was found to be a function of conductivity as shown in
Fig. 4.10 a) Typical light diffraction pattern for operating conditions near the threshold for oscillation, where $m$ refers to the diffraction order.

b) Corresponding frequency spectrum of the acousto-electric current (monitored across 50Ω). Operating conditions: $E_0 = 812$ V/cm., $J_0 = 9.4$ mA/cm$^2$.

Threshold $E_0 = 738$ V/cm.
Fig. 4.11. The linear theory frequency of maximum gain $\sqrt{Y}$ (Eq. 2.39) is plotted for comparison. At high conductivities the threshold frequency follows the linear theory. The lack of agreement at low conductivity is felt to be due to electron traps and cavity effects as discussed in Chapter 2.

4.53 Nonlinear Regime

Increasing the electric field above threshold ($E_0 \gtrsim 1.1E_t$) caused the threshold mode to evolve into a new mode with a fundamental lower in frequency and with a much higher harmonic content in both the frequency spectrum of the current and the light diffraction pattern. Further increases in electric field usually caused further downshifting of the fundamental frequency and additional increase in the harmonic content. Typical spectra are shown in Figs. 4.12, 4.13 and 4.14. Very often in the downshifting process the new mode had a fundamental frequency near the subharmonic of the fundamental of the previous mode. Similar behavior has been reported for a single cavity CdS oscillator (White and Wang 1966, Ø Tuuma 1970).

Continued increase of the electric field produces a further downshifting of the fundamental frequency and eventually results in a multi-mode operation with frequency
Fig. 4.11 Threshold frequency versus conductivity.

Symbols o and □ correspond to independent measurements of Cavity DC3 and symbols △ and ▽ correspond to independent measurements of Cavity DC4. The solid line corresponds to the trap free frequency of maximum gain with parameters, \( D = 8 \text{ cm}^2 \text{ sec}^{-1} \), \( \epsilon = 9 \), \( s_0 = 1.76 \times 10^5 \text{ cm./sec} \).
Fig. 4.12 a) Light diffraction pattern for intermediate behaviour, #1. b) Corresponding frequency spectrum of acoustoelectric current. c) Corresponding time display of acoustoelectric current. Operating conditions: $E_0 = 1032 \text{ V/cm}$, $J_0 = 11.3 \text{ mA/cm}^2$. 
Fig. 4.13  a) Light diffraction pattern for intermediate behaviour, #2.  b) Corresponding frequency spectrum of acoustoelectric current. Operating conditions:

\[ E_0 = 837 \text{ V/cm}, \quad J_0 = 12.5 \text{ mA/cm}^2. \]

Fig. 4.14  Light diffraction pattern for intermediate behaviour, #3. Fundamental frequency of oscillation, \( \nu_1 = 6 \text{ MHz}. \)
components near all the acoustical resonance frequencies of the composite cavity within a certain frequency range. Similar effects have been reported for single cavities (Maines and Paige 1970, Janus and Meyer 1970) with an accompanying spiking of the electric current (with a repetition time equal to the round trip acoustic transit time) whenever the multi-modes are phase-locked.

For double cavities the multi-mode frequency spectrum had strong components near multiples of the reciprocal of the round trip transit time of the active cavity. Each component had a fine structure with frequency separation of the order of the reciprocal of the composite cavities' round trip transit time. This behavior is discussed in Section 3.3 and illustrated in Figs. 4.15. In Fig. 4.15a the separation between the spots which one can just resolve corresponds to approximately .75 MHz which is to within experimental error the reciprocal of the active cavity round trip transit time.

Further increases in electric field finally caused all oscillation to cease.

In the intermediate regime between threshold and fully developed multi-mode behavior it was found that modes with their fundamental frequency in the region from 15 to 20 MHz were particularly stable under changes in conductivity and electric field in the active cavity. The intensities of
Fig. 4.15 a) Light diffraction pattern for multimode behaviour.
b) Corresponding frequency spectrum of acoustoelectric current. c) Typical fine structure of the frequency spectrum of b) (under slightly different operating conditions). General operating conditions: $E_0 = 1380 \text{ V/cm.}, J_0 = 4.3 \text{ mA/cm}.$
light diffracted into the different order spots was measured. The ratio of the intensity of the \( m+1 \) order spot to the \( m \) order spot for \( m \geq 1 \) is greater than 0.4 in all cases. If the intensities of higher order spots were due solely to high order Raman-Nath diffraction from the fundamental of the acoustic wave (Eq. 4.21) the ratio \( I_{m+1} / I_m \) would be less than 10\(^{-3} \) in all cases. Therefore one must conclude that the light intensity in the higher order spots is almost entirely due to acoustic harmonics. In calculating the strain each spot was treated as a first Raman-Nath order spot of the appropriate acoustic harmonic.

In Fig. 4.16 the amplitude of the acoustic shear strain as calculated from diffraction intensity measurements is plotted for different acoustic harmonics as a function of electric field for two conductivities. The current density is also shown as a function of field. Strong current saturation is evident.

With reference to Fig. 4.16, it is to be noted that diffraction spots of order greater than 4 have values of \( Q \) (Eq. 4.1a) greater than 2 and hence do not fall into the Raman-Nath limit of diffraction (Eq. 4.2). Klein and Cook (1967) have shown with numerical calculations that for this intermediate regime and for small values of the Raman-Nath parameter, \( v \), the diffraction is similar to Raman-Nath diffraction but the intensities are weaker than predicted.
Fig. 4.16  (a) Current density in crystal versus electric drift field. (b) Acoustic strain (measured in the buffer) versus electric drift field. \( N \) refers to the harmonic number. Symbols \( \triangle \) and \( \Box \) refer to two different conduction electron densities while symbol \( \circ \) refers to a reversal of the drift field direction for conditions equivalent to \( \triangle \). Throughout the measurements the same "mode" of oscillation was maintained (i.e., the fundamental frequency remained the same).
Electric Drift Field (KV cm⁻¹)

- **a)**
  - Plot showing \( J_0 \) (mA cm⁻²) as a function of the electric drift field.
  - Data points and line segments indicate varying parameters or conditions.

- **b)**
  - Graph displaying acoustic strain \( \times 10^6 \) as a function of the electric drift field.
  - Multiple curves and data points for different conditions or samples labeled with \( N = 1 \) to \( N = 11 \).
by Raman-Nath theory. This implies that for high diffraction orders the calculated strains (Fig. 4.16) are underestimates.

Except where noted all diffraction measurements were made with the electrons in the active crystal drifting in the direction away from the buffer (i.e., the voltage was of positive sign at the free surface). Reversing the polarity does not have a very dramatic effect (see Fig. 4.16). The fact that the strain is slightly lower for the reversed polarity may be due to the diffused indium contacts (Appendix B) of the active crystal which show a polarity effect.

Heterodyning measurements made on the first four diffraction spots (Fig. 4.12a) show that the acoustic frequency corresponding to the $m^{th}$ diffraction spot is equal (to better than 1 part in $10^4$) to the frequency of the $m^{th}$ harmonic in the current spectrum (Fig. 4.12b). With the same accuracy the $m^{th}$ harmonic was established to be equal to $m\nu_0$ where $\nu_0$ is the fundamental frequency. This indicates the absence of linear dispersion. The fact that the corresponding current versus time plot (Figs. 4.12c and 4.17b) is periodic with a repetition time equal to the reciprocal of the fundamental frequency, indicates that the harmonics are phase-locked.
Fig. 4.17  

a) Frequency spectrum of acoustoelectric current for intermediate behaviour, #4.

b) Corresponding time display of acoustoelectric current.

c) Fourier amplitude coefficients from a numerical Fourier transform of the time display b).

d) Fourier phase angles relative to the fundamental frequency component from a numerical Fourier transform of the time display b).

Operating conditions: \( E_0 = 1330 \text{ V/cm.}, \ J_0 = 12.6 \text{ mA/cm}^2 \)
The current versus time plot displayed in Fig. 4.17b (taken with a sampling oscilloscope) was digitized and a numerical Fourier transform was performed. The absolute value and the phase (relative to the fundamental) of each calculated Fourier coefficient are plotted in Figs. 4.17c and d. The agreement with the amplitudes of each harmonic as measured on a spectrum analyzer (Fig. 4.17a) is probably better than shown since considerable time elapsed between the measurements shown in Fig. 4.17a and Fig. 4.17b (the spectrum analyzer and the sampling oscilloscope both use the same main frame hence simultaneous measurements were not possible). If one makes the assumption that the phases of the harmonics of the current signal are equal (apart from a constant phase difference) to the phases of the harmonics of the acoustic waveform, then one can construct a model of the acoustic waveform. Using the strain values from Fig. 4.16 (for 1330 V/cm.) and the phases from Fig. 4.17d for the first eleven harmonics, a calculated acoustic waveform is plotted in Fig. 4.18.

Simultaneous light diffraction in the buffer and in the active cavity was performed on double cavity DC3. A typical diffraction pattern is reproduced in Fig. 4.19. Intensity measurements were made to determine the ratio of the strain in the buffer to the strain in the active crystal. The accuracy of the measurements is poor as simultaneous alignment of the two light beams was difficult. The ratio of the
Fig. 4.18 Strain amplitude versus time wave form as calculated from the first eleven harmonics, with amplitudes taken from Fig. 4.16 and relative phases taken from Fig. 4.17d.
Cavity DC3

Fig. 4.19 Light diffraction pattern for simultaneous diffraction in the buffer and the active crystal. Operating conditions: $E_0 = 1330$ V/cm, $J_0 = 17.6$ mA/cm$^2$.

Cavity DC4

amplitude (2 mV/div.)

Frequency (10 MHz/div.)

acoustic fundamental

Fig. 4.20  

a) Light diffraction pattern for pure behaviour.

b) Corresponding frequency spectrum of the acousto-electric current. Operating conditions: $E_0 = 1430$ V/cm, $J_0 = 30.9$ mA/cm$^2$. 
calculated strains (from the first order spots) for the buffer and for the active crystal was found in three measurements to be 1.4, .8 and 1.2. The calculated strains in higher order spots from the active crystal fall off in much the same manner as illustrated in Fig. 4.16.

To this point the evolution of modes of oscillation has been described for conditions of constant conductivity but increasing electric field. In double cavity DC4 a new type of mode could be excited in a narrow range of electric field values by increasing the conductivity at constant field from a low value. This mode is of a very pure nature having a high strain ($\approx 2 \times 10^{-5}$) in the first order spot but very little higher harmonic content as shown in Fig. 4.20.

The ratio, $a_r$, of the amplitude of the forward acoustic wave to the amplitude of the return wave was calculated (Eq. 4.76) from self beating signals in first order diffraction spots. Typical self beating signals and the corresponding frequency spectrum of the crystal current are shown in Fig. 4.21 where the center frequency of 376 MHz corresponds to the longitudinal mode spacing of the laser while the separation between the fundamental and the side-bands ($\approx 38$ MHz. in Fig. 4.21a and $\approx 30$ MHz. in Fig. 4.21c) corresponds to twice the fundamental oscillation frequency (see Figs. 4.21b and 4.21d). The acoustic amplitude ratio calculated from Fig. 4.21a is $a_r = 1.37$. 
Fig. 4.21  

a) Frequency spectrum of self-beating signal. Operating conditions:  $E_0 = 920 \text{ V/cm.}, J_0 = 13.3 \text{ mA/cm}^2$.

b) Frequency spectrum of acoustoelectric current for operating conditions corresponding to a).

c) Frequency spectrum of self-beating signal. Operating conditions:  $E_0 = 1370 \text{ V/cm.}, J_0 = 12.1 \text{ mA/cm}^2$.

d) Frequency spectrum of acoustoelectric current for operating conditions corresponding to c).

The longitudinal modes of the laser have a separation of 376 MHz.
The asymmetry evident in Fig. 4.21c is not predicted by the theory of Appendix C. This asymmetry is felt to be due to high strain levels, not high acoustic harmonic content, as the harmonically very pure mode of Fig. 4.20 showed a similar lack of symmetry for similar strain levels. The acoustic amplitude ratio calculated from the average of the side-band amplitudes of Fig. 4.21c is $a = 1.24$. For the mode of Fig. 4.20 a similar calculation gives $a = 1.05 \pm 0.05$.

The deviation of the calculated acoustic amplitude ratio from a value of unity is felt to be due to the off-axis nature to the acoustic oscillations (i.e., the forward and return waves are not exactly colinear). An actual amplitude difference is unlikely as in Section 2.6 free surfaces were found to have a reflection coefficient of $r \approx 0.995$. Further evidence for off-axis oscillation is presented in the next section.
4.54 Transverse Effects

A variety of acoustic effects are evident in the highly exposed light diffraction photograph displayed in Fig. 4.22 (the corresponding current spectra are displayed in Fig. 4.17). The vertical array of spots is the normal acoustic harmonic series considered previously (Fig. 4.12). However, diffraction from the seventh or higher harmonic clearly can be resolved into two spots implying that the acoustic waves are travelling at a slight angle (≈ 8°) to the cavity axis. Off-axis oscillation in acoustoelectric cavities is a common phenomenon (Vrba and Haering 1973c, 1974) and under special conditions can be quite pronounced (Fig. 4.23).

In Fig. 4.22 the bright circular arcs centered on the zero order spot and passing through the main diffraction spots (due to acoustic harmonics) are due to acoustic diffraction. The small active cavity (≈ 3 mm.) effectively acts as an aperture for acoustic waves radiating into the buffer. The Fraunhofer diffraction amplitude (as a function of angle) for a plane wave incident normally on a slit of width w is given by (Born and Wolf 1959)

\[
A(\phi) = \frac{\sin(\frac{w}{\lambda} \sin \phi)}{\frac{w}{\lambda} \sin \phi}
\]  

(4.77)
Fig. 4.22  Highly exposed light diffraction pattern for intermediate behaviour, #4. Operating conditions are identical to those given in Fig. 4.17.

Fig. 4.23  Light diffraction pattern for simultaneous diffraction in the buffer and active crystal, exhibiting off-axis oscillation.
where \( \phi \) is the angle from the incident propagation direction.

The amplitude of Eq. 4.77 has a principal maximum at \( \phi = 0 \) and secondary maxima at the roots of the equation

\[
\tan(\pi \frac{W}{\lambda} \sin \phi) - \frac{W}{\lambda} \sin \phi = 0
\]  

(4.78)

The first five roots are given by (Born and Wolf 1959)

\[
\frac{W}{\lambda} \sin \phi = 0, 1.430, 2.459, 3.470, 4.479
\]  

(4.79)

The discrete nature of the acoustic diffraction as predicted by Eq. 4.77 is easily seen for the higher acoustic harmonics in Fig. 4.22. For the sixth harmonic \( \sin \phi \), for the first secondary maximum (\( \frac{W}{\lambda} \sin \phi = 1.430 \)), was measured to be

\[
\sin \phi_1 = 8.0 \times 10^{-2}
\]  

(4.80)
From a measured wavelength of \( \lambda = 4.1 \times 10^{-3} \text{cm.} \), the effective column width was found to be (Eqs. 4.79 and 4.80)

\[
w = 7.3 \times 10^{-2} \text{ cm.} \quad (4.81)
\]

which is much smaller than the relevant dimension of the active cavity (3.2 \( \times 10^{-1} \) cm.). This discrepancy can be resolved by assuming that the acoustic column is narrower than the active crystal. (Lensing effects due to the cavity walls are discounted because of the high degree of flatness as listed in Appendix B.)

The intensity distribution of acoustic waves throughout the buffer was measured (using focal plane filtering) for the different acoustic harmonics as illustrated for cavity DC3 in Fig. 4.24a (the operating conditions were not identical to those corresponding to Fig. 4.22 but the mode type and diffraction were very similar). It is evident in Fig. 4.24a that at higher harmonics the crystal is oscillating in two regions with each acoustic column (at the fourth harmonic) having a width of order 8 \( \times 10^{-2} \) cm., which is in good agreement with Eq. 4.81. As illustrated in Fig. 4.24a the width of the acoustic column is greater for lower harmonics as found in Fig. 4.22. This behavior is better illustrated in Fig. 4.25, a highly exposed multi-mode
Fig. 4.24 Visualization of acoustic intensity distributions in the buffer for the first few acoustic harmonics (i.e., diffraction orders) where \( m \) is the harmonic number. The case \( m = 0 \) corresponds to a transmission picture of the double cavity mounted between two electrodes. Dark spots correspond to damage on the windows of the buffer.

**Operating conditions:**

a) \( E_0 = 937 \text{ V/cm}, \ J_0 = 14.6 \text{ mA/cm}^2 \)
b) \( E_0 = 856 \text{ V/cm}, \ J_0 = 18.7 \text{ mA/cm}^2 \)
c) \( E_0 = 837 \text{ V/cm}, \ J_0 = 12.5 \text{ mA/cm}^2 \)
Fig. 4.25 Highly exposed light diffraction pattern for multimode behaviour. Operating conditions: \( E_0 = 1480 \text{ V/cm} \), \( J_0 = 13.3 \text{ mA/cm}^2 \).

Fig. 4.26 Light diffraction patterns from intermediate behaviour using a probe beam in buffer directly underneath active crystal, a), and off to one side, b). Operating conditions: \( E_0 = 1030 \text{ V/cm} \), \( J_0 = 11.7 \text{ mA/cm}^2 \).
diffraction picture (double cavity DC3), where the first secondary maxima form fine lines running along both sides of the principal diffraction. If the acoustic column width were constant for all frequencies the secondary maxima would lie on two lines parallel to the principal maxima. The fact that the secondary maxima draw closer together at low frequencies (i.e., small light diffraction angle) as seen in Fig. 4.25 indicates a widening of the acoustic beam.

More dramatic transverse localization of oscillation in composite cavities is shown in Fig. 4.24b for cavity DC3 oscillating in a mode near threshold. Localization can also be less pronounced as shown in Fig. 4.24c for composite cavity DC4 (corresponding diffraction and current spectra in Fig. 4.14). Transverse localization of modes has been observed in thin single cavities (Marshall 1971).

To further investigate the acoustic diffraction from an aperture, a narrow light beam (≈2 mm. in diameter) was used to probe the acoustic field in the buffer. Fig. 4.26a shows a typical light diffraction pattern taken directly underneath the active crystal (in cavity DC3). For the light beam well off to the side of the active cavity one observes diffraction mainly from the secondary maxima as illustrated in Fig. 4.26b.

By critical adjustment of either the conductivity or the electric field in the active cavity one can excite a
number of transverse diffraction patterns (nearly perpendicular to the main diffraction directions) as illustrated in Fig. 4.27a, b. It is believed that these patterns correspond to transverse acoustic resonances of the buffer (the polished windows can form a cavity). The adjustment of conductivity or electric field varies the oscillation frequency and facilitates tuning into a transverse resonance. This effect can be quite strong as illustrated by the diffraction pattern in Fig. 4.27c, taken with the high resolution experimental set-up (Fig. 4.7).

Returning to Fig. 4.22, light diffraction from longitudinal acoustic waves is evident as circular arcs both inside (fundamental longitudinal) and weakly just outside (second harmonic longitudinal) the diffraction arcs from the fundamental shear waves. From Fig. 4.22 the ratio of the longitudinal acoustic velocity to the shear acoustic velocity in fused silica was estimated to be $1.58 \pm .02$, in excellent agreement with the established value of 1.58 (Appendix B).

The diffraction arcs (Fig. 4.22) due to both longitudinal and shear acoustic waves do not form complete circles due to the use of crossed polarizers (incident polarization along the principal acoustic propagation direction). Light diffraction is absent both for longitudinal acoustic waves propagating parallel to either the incident or exit polarizers.
Fig. 4.27  Light diffraction patterns exhibiting transverse resonance in buffer using a probe beam not directly underneath the crystal, a) and b), and using a high resolution experimental set-up, c).

Fig. 4.28  Highly exposed light diffraction pattern for intermediate behaviour, #2. Operating conditions: $E_0 = 1000$ V/cm, $J_0 = 11.9$ mA/cm$^2$. 
and for shear acoustic waves propagating at ± 45° to either polarizer as predicted by Eq. 4.55.

An explanation of the presence of the longitudinal acoustic waves (Fig. 4.22) lies in the fact that for the 30° active crystal of composite cavity DC3, the active quasi-shear waves propagating along the cavity axis have a mass displacement direction at 94.04° (Appendix B) to the cavity axis. The mass displacement direction for shear waves in the acoustically isotropic buffer is normal to the propagation direction. Therefore at the CdS-silica interface a longitudinal wave is created to satisfy boundary conditions (i.e., mode conversion). In composite cavity DC4, consisting of two b-axis CdS crystals, the on-axis shear waves in both cavities have mass displacement directions normal to the propagation direction and no longitudinal wave is created (potentially the slight off-axis nature of some oscillations can produce longitudinal waves but at the angles observed, ~ 8°, this is a small effect). A highly exposed diffraction photograph from the buffer of DC4 shows no evidence of longitudinal waves (Fig. 4.28).

The transverse effects in the buffers of double cavities are felt to be well understood. In general, they have been found to be of only secondary importance; therefore the one dimensional approximations used previously appear justified. However the transverse localization of the acous-
tic beams is likely due to strong transverse effects in the active cavity (e.g., transverse modes, crystal inhomogeneities) and requires further investigation.
4.6. Behaviour of Triple Cavity

With the objective of coupling out significant acoustic intensity from an oscillating phonon maser, an acoustic interference mirror approach (see Section 3.4) was tried. In triple cavity TC1 a thin (219. μm.) intermediate layer of sapphire was bonded between a 30° CdS crystal and a fused silica buffer (see Appendix B). As all faces normal to the cavity axis were accurately parallel all acoustic energy (minus reflection and other losses) entering the buffer was returned to the active CdS crystal. Under conditions of oscillation the behaviour of the composite cavity was in most respects similar to the behaviour of the double cavities discussed in Section 4.5. Strong light diffraction by acoustic waves in the buffer was observed.

The construction of triple cavity TC2 was similar to that of TC1 (Appendix B) except the bottom surface of the fused silica buffer (SC-FS3) was polished at an angle = 3.6° to the top surface. This angle was introduced to ensure that acoustic energy entering the buffer could not return to the active crystal along the cavity axis and thus was effectively lost. Triple cavity TC2 did experimentally oscillate but at a very high threshold field (> 2000. V/cm.) and no acoustic field was detected in the buffer. Studies of light diffraction from the acoustic field inside the active crystal
showed that the acoustic waves travelled at large angles to the cavity normal (in the cavity normal, a-axis plane). A typical diffraction pattern (only the top half) is illustrated in Fig. 4.29 along with theoretical curves of the inverse acoustic velocities ("slowness" curves) which represent the only possible positions for diffraction spots (Vrba and Haering 1974).

Using approximate values for shear velocities in sapphire of $6.0 \times 10^5$ (Farnell 1961) and in CdS of $2.2 \times 10^5 \text{cm/s}$ (Vrba and Haering 1974) the incident angle for total internal reflection of active shear waves propagating in the CdS crystal and incident to a CdS-sapphire interface is $\approx 21^\circ$. However, the active shear waves in triple cavity TC2 were experimentally found to propagate at a larger angle, $\approx 33^\circ$, to the cavity normal, which corresponds to the minimum incident angle for which mode conversion from an active shear wave to a longitudinal wave is not possible.

The inactive shear waves which produce the diffraction spots $\triangle$ and $\Box$ in Fig. 4.29 are produced by mode conversion (from the active wave) at the CdS-sapphire interface ($\triangle$) and at the edge of the CdS crystal ($\Box$) and are well understood (Vrba and Haering 1974).
Fig. 4.29 Experimentally observed diffraction spots (△, ○, □) for light travelling at 60° to the c-axis in triple cavity, TC2, plotted on curves of the inverse of the acoustic velocities in CdS ("slowness curves" after Vrba and Haering 1974).
CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 Comments: Chapter 2

A detailed one dimensional linear theory which considers the piezoelectric coupling between acoustic waves and charge carriers was derived. Expressions for the loss and the dispersion of perturbed acoustic waves and perturbed carrier density waves under conditions of carrier drift and carrier trapping were presented. For piezoelectric semiconducting platelets (phonon masers), application of boundary conditions to the linear theory resulted in the determination of the general electrical impedance. Conditions were established for which the complications due to carrier density waves could be ignored in electrical impedance considerations. For shear acoustic waves in a typical CdS phonon maser these conditions took the form (Eqs. D.2 and D.8) \( 7 \times 10^8 \text{ Hz.} \gg \nu \gg 5 \times 10^6 \text{ Hz.} \) and (Eq. 2.115) \( \sigma_0 \gg 2.4 \times 10^{-6} \Omega^{-1} \text{ cm.}^{-1} \) if \( \nu_0 = s_0 \). When carrier density waves could be ignored relatively simple expressions for the impedance and conductance in the neighbourhood of an acoustic resonance were developed. Under the conditions \( \omega t \gg 1 \) and \( \nu_0 \neq s_0 \) equivalent electrical circuits of a piezoelectric semiconductor were presented.
A broad band admittance bridge was developed which could measure the real and the imaginary components of the electrical admittance of a phonon maser over the frequency range of 1 MHz. to 100 MHz. Simple methods of determining acoustic loss (valid only when carrier density waves can be ignored) by experimentally measuring the electrical admittance of phonon masers were considered. By applying these techniques the losses due to the three dimensional nature of the platelets, due to the mounting of the platelets in water-cooled holders and due to the introduction of charge carriers (acoustoelectric loss) were investigated. The acoustoelectric loss as a function of drift velocity was experimentally determined at various conductivities and frequencies.

In all cases the experimentally determined acoustoelectric loss could be fitted reasonably well by the theoretical expressions previously developed, where the complex trapping factor, $f$, was used as a fitting parameter. The fit was good even for cases where carrier density waves were expected to contribute significantly and the simple approach used was no longer expected to be strictly valid. Field dependent tuning of the resonances could not be explained satisfactorily, however. The peak acoustoelectric loss as a function of frequency could be accounted for over the range of 10 - 110 MHz.
The conventional technique used to determine acoustoelectric loss consists of measuring the amplitude of an acoustic pulse transmitted through a thick sample (Hudson, McFee and White 1961). By comparison the technique used here of determining acoustoelectric loss through measurements of the electrical admittance was only found to be useful over a narrow range of conductivity. The strong loss which accompanies high conductivities produces a weak and a broad response (amplitude \(1/\text{loss}\), line width \(\alpha\text{loss}\)). Interference by transverse modes which are not so strongly attenuated was found to "wash out" the primary resonance.

Improved measurement capabilities could be achieved by using techniques aimed at eliminating the transverse modes such as "energy trapping" (Shockley, Carran and Koneval 1966) or "waxing" (Bolef and Miller 1971). At low conductivities (with an appropriate sweep oscillator) the measurement is straightforward but the interpretation is difficult due to the potential interference of carrier density waves.

For studying phonon masers electrical admittance measurements have a distinct advantage over the conventional technique in that a realistic phonon maser can be studied rather than a long sample with transducers attached to each end. This aspect will be discussed in Section 5.4.
As a final comment on the contents of Chapter 2, the impedance discussion which was presented is valid only in a one dimensional limit. The crystal orientation (b-axis) was chosen to satisfy this condition. However, for general orientations, crystals tend to oscillate with the acoustic waves propagating at large angles to the cavity axis (Vrba and Haering 1973c, 1974). The electrical impedance expressions need to be extended to cover the general case.
5.2 Comments: Chapter 3

A simple but very successful treatment of quasi-one dimensional normal modes of a double (composite) cavity was presented. By applying physical boundary conditions to acoustic waves in two intimately bonded loss free cavities, algebraic conditions for the normal modes of the composite cavity were derived. Expressions were given which allowed the normal modes (acoustic resonance frequencies) to be accurately predicted. Alternatively experimental measurements can be used to establish the transit time (i.e., the velocity can be determined if the thickness is known or vice versa) of both cavities. Agreement between theory and experiment was excellent.

One of the more useful results found was a method of evaluating reflection coefficients at an interface (i.e., the acoustic bond can be evaluated) by experimentally measuring deviations of the resonant frequencies from a norm. The cold-welded metallic bond which was employed was found to provide almost perfect acoustic coupling in the frequency range investigated (10 MHz. < \nu < 110 MHz.).

The electrical impedance and admittance of a piezoelectric insulating platelet bonded to a second acoustic cavity was derived. Admittance measurements were used to measure resonant frequencies. The theoretical admittance
explained satisfactorily the observed amplitudes of the electrical responses near resonant frequencies.

The effective complex reflection coefficient of an acoustic layer was also presented. The possibility of an acoustic interference mirror was briefly explored.

In the future, the admittance theory could be extended to the case of arbitrary conductivity and arbitrary drift field (as was done for a single cavity in Chapter 2). All the complications mentioned with respect to the single cavity treatment (Section 5.1) would need to be considered in such an extension.
5.3 Comments: Chapter 4

The theoretical groundwork for a light diffraction study of acoustic waves in multiple acoustic cavities was presented in the Raman-Nath diffraction limit. Techniques for determining the amplitudes, frequencies, wavelengths, phase velocities, propagation directions and spatial extent of acoustic waves were discussed.

Oscillating double and triple cavities, where one cavity consists of a phonon maser (piezoelectric semiconductor platelet), were found to behave in terms of frequencies of oscillation and development of modes in a very similar manner to single cavity phonon masers. Many of the conclusions arrived at in Chapter 5 apply to phonon masers in general.

Extensive strain measurements performed on acoustic waves in the buffers (inactive part of composite resonator) were presented. Simultaneous measurements of strain in both cavities of a double cavity system indicate that the strains in the buffer were nearly equal to the strains in the active cavities. The harmonic frequency content of the acoustic strain was found to correspond exactly to the harmonic content of the acoustoelectric current. However, there was very little correlation between the amplitudes of the harmonics of the strain and acoustoelectric current.

As a general feature of an oscillating composite resonator, increases in drift field produced increases in strain levels with discrete mode changes at intervals. The
mode changes consisted of a downshifting of the fundamental frequency of operation accompanied by an increase in the harmonic content (diffraction due to at least 16 harmonics with a fundamental frequency of \( \approx 20 \text{ MHz} \) were observed by eye). For the case of high harmonic content the accompanying acoustoelectric current showed current spiking (in time domain). Assuming that the phase relationships among the acoustic harmonics were equal to the phase relationships of the harmonics of the acoustoelectric current, an acoustic waveform which also exhibited spiking was constructed. The peak strain in the reconstructed waveform was \( \approx 4 \times 10^{-5} \) with a pulse width of \( \approx 1 \times 10^{-4} \) sec. the highest strain measured for an individual harmonic (fundamental of "pure" mode) was \( \approx 2 \times 10^{-5} \).

The acoustic cavities were observed to oscillate in a quasi-one dimensional fashion although some small angle off-axis oscillation was noted. Acoustic diffraction in the buffers due to the small transverse dimensions of the oscillating crystal was observed and successfully explained. Generation of longitudinal waves by mode conversion at a boundary was also observed and explained. However, a quantitative account of the dramatic lateral localization of oscillation which was observed is lacking.

Except for very high harmonic numbers (where the acoustic strains were underestimated) the experiment condi-
tions were such that the Raman-Nath diffraction theory which was developed was applicable. One point that remains unexplained is the unexpected asymmetry of the homodyne (self-beating) signal at high strain levels. Further study is required to check the validity of the phase relationship assumed in constructing the acoustic wave form. Perhaps stroboscopic techniques could be used to visualize the predicted acoustic spiking.
5.4 Nonlinear Effects

In this study non-linear effects have largely been ignored. However they play an important role in the oscillation of phonon masers. In any stable oscillating system there must always exist a non-linear mechanism which limits the continuous growth of acoustic waves predicted by linear theories (i.e., gain saturation). One highly non-linear phenomenon in oscillating phonon masers is the progressive downshifting of the fundamental frequency with increasing drift field which has been observed. The high harmonic content (phase-locked) of both the acoustic signal and the acoustoelectric current is also due to non-linear effects.

Considerable theoretical work has been done on the problem of non-linear interactions in piezoelectric semiconductors. A partial list of the more significant papers is included at the end of the bibliography (page 265). However, very little related experimental work has been done and an understanding of the dominant non-linear effects associated with phonon maser action is presently lacking.

A simultaneous determination of the acoustic strains of oscillating cavity modes (by light diffraction) and of the acoustoelectric loss of inactive cavity modes (by electrical admittance measurements) could be a very powerful technique for investigating non-linear interactions.
Preliminary observations have been made and will be commented upon. For drift fields slightly above those required for oscillation the acoustoelectric loss in cavity modes neighbouring the oscillating mode was observed to increase as the strain of the acoustic oscillations was increased (by raising the drift field). Quantitative measurements could give valuable information as to gain saturation mechanisms. When the drift field was increased to a value just below the field required to force the pattern of acoustic oscillation to downshift to a new pattern with a lower frequency fundamental, dramatic decreases in the acoustoelectric losses of the new modes were observed. In fact, prior to multimode operation the loss in all the cavity modes (over a wide frequency range) was observed to drop. An extensive nonlinear study following the lines suggested could be very productive.
APPENDIX A

ADMITTANCE BRIDGE

A bridge circuit which was capable of measuring the absolute value, the absolute value squared, the real component or the imaginary component of the active admittance of a piezoelectric semiconductor (under conditions of illumination and of an applied drift voltage) was developed. The electrical circuit is displayed in Fig. A.1. The wideband transformer $T_1$ (Relcom Model BT6) was chosen for an accurate balance in amplitude ($0.1$ dB over the range $1 - 100$ MHz) and an accurate $180^\circ$ phase difference ($0.5^\circ$ over the range $1 - 100$ MHz). Special care was taken to achieve a symmetrical layout. The water-cooled crystal mount and the variable capacitor, $C_v$, were identical units (illustrated in Fig. A.2) which were mounted directly onto a printed circuit board. For a high degree of isolation of the input and output of the bridge, grounding was found to be critical. Use of a double-sided printed circuit board with strip line techniques was found to work well.

1Relcom, 3333 Hillview Ave., Palo Alto, California 94304
Fig. A.1 Schematic diagram of admittance bridge where:

- $T_1$ - Relcom BT6 50 Ω to 100 Ω center tapped transformer.
- $T_2$ - Relcom T4 50 Ω to 200 Ω transformer.
- $C_v$ - variable capacitor - See Fig. A.2.
- $R_v$ - 100 Ω subminiature variable resistor.
Filtered Light

One of 3 Alignment Screws

Conducting Glass

Top Electrode

Water

Sample

Bottom Electrode

Pressure Spring

Circuit Board
(copper both sides)

Water Cooled Sleeve

Sliding Piston

Return Spring

Fig. A.2 Cross-sectional view of water-cooled crystal holder.
The bridge could be balanced such that the output signal was 72 dB down from the input signal over the full range of 1 - 100 MHz. (The major lack of balance is due to capacitance associated with the variable resistor, \( R_v \).) For a center frequency anywhere within this range (1 - 100 MHz) the bridge could be balanced for an isolation better than 100 dB over a 1 MHz band width.

In Fig. A.1 the .005 μfd. capacitors were included to provide d.c. isolation and can be ignored in the limit of high frequencies (\( \nu > 1 \) MHz) and low admittance (\( \omega C_v, |Y| < 10^{-2} \Omega^{-1} \)). In this limit the electrical circuit (Fig. A.1) takes the simplified form illustrated in Fig. A.3 where \( V_i \) is the input voltage and the factor \( \frac{1}{\sqrt{2}} \) accounts for the transformer, \( /T_1 \).

In the low admittance limit (\( Y, \omega C_v \ll 10^{-2} \)) the output voltage is given by

\[
V_o = 50\Omega \frac{V_i}{\sqrt{2}} (G_v + Y - i\omega C_v)
\]  

(A.1)

where \( G_v \) is the net conductance due to the two 1 kΩ resistors and the 100Ω variable resistor (\( R_v \)). \( G_v \) can vary between the two approximate limits of \( \pm 10^{-4} \Omega^{-1} \).

If the admittance, \( Y \), of the sample consists of the sum of an active admittance, \( Y_{act} \), and a passive admittance, \( Y_{pas} \), the passive part can be cancelled for a frequency off the
Fig. A.3 High frequency and low admittance equivalent circuit of admittance bridge assuming that the output is connected to a 50 Ohm load. $v_1$ and $v_o$ are the input and output voltages respectively.
resonance condition (where $Y_{\text{act}} << Y_{\text{pas}}$) by adjusting the variable conductance, $G_v$, and the variable susceptance, $\omega C_v$ and the active admittance can then be measured directly.

If a diode R.F. detector is used (after amplification) the interpretation of the signal depends on the input power to the detector (see Fig. A.4). For small signals ($\sim -15$ dBm.) the diode is in the square law region (one measures $|Y_{\text{act}}|^2$) while for large signals ($> +7$ dBm.) the diode is in the linear region (one measures $|Y_{\text{act}}|$).

However, if a controlled off-balance conductance is introduced as given by

$$V_o = 50\Omega \frac{V_i}{\sqrt{2}} (G_v + Y_{\text{act}})$$

(A.2)

the detected signal is given by

$$V_d = A\left(K \times 50\Omega \frac{V_i}{\sqrt{2}}\right)^a |G_v + Y_{\text{act}}|^a$$

(A.3)

where $A$ and $a$ are respectively the proportionality constant and effective exponent characteristic of the detector (Fig. A.4) which are functions of the signal level. The exponent $a$ can vary from 1 to 2. The coefficient, $K$, is the gain of an amplifier between the bridge and the detector.

If $G_v \gg Y_{\text{act}}$ then keeping only leading terms the detected signal is given by (Eq. A.3)
Fig. A.4 Output voltage versus input power for crystal detector (After Hewlett-Packard RF Detector 8471A Operating Note).
where $G_{\text{act}}$ is the real part of $Y_{\text{act}}$. The detection signal (Eq. A.4) consists of a constant part (which can be subtracted off) and a part which is proportional to the real part of the active admittance. The detected real part of the active admittance increases as the offset $G_v$ increases but saturates when the signal level is high enough for $a = 1$ (linear region). The noise due to the constant part of the detected signal increases continuously as $G_v$ increases so a poor signal to noise ratio is found by increasing $G_v$ too much. Knowing the gain of the amplifier and the detection characteristics of the diode (i.e., $a$ and $A$ from Fig. A.3) absolute calibration is possible.

A similar procedure can be followed by using a susceptance offset instead of a conductance offset for which the imaginary part, $B_{\text{act}}$, of the active admittance, $Y_{\text{act}}$, is detected rather than the real part, $G_{\text{act}}$. However, the susceptance offset given by (Eq. A.1), $B_v = -\omega C_v$ is a function of frequency; therefore there would exist a constant slope (negligible over a .1 MHz. band width) to the "constant" term. If this slope is corrected for, and $a = 1$ the line shape ($B_{\text{act}}$ vs $V$) is not affected.
One technical problem which was encountered was that the variable resistor, $R_v$, (Figs. A.1, A.3) always has an imaginary component due to its physical capacitance. However, the variable capacitor, $C_v$, (Figs. A.1, A.3) had very little real component. When adjusting the bridge for a real (conductance) offset the variable capacitor, $C_v$, was always adjusted for a minimum signal after the variable resistor was adjusted (to neutralize the physical capacitance). When the bridge was adjusted for an imaginary (susceptance) offset the conditions of complete cancellation was first adjusted and then the variable capacitor, $C_v$, was adjusted for the required offset.

Typical plots of the real and the imaginary parts of the active admittance versus frequency are given in Fig. A.5 a, b for double cavity DC5. An H.P. 8601A Sweep Oscillator was used with signal level of 0 dBm. input to the bridge. An amplifier of 66 dB. gain was used ahead of an H.P. 8471A R.F. Detector. The detected offset amplitude of 200 mV. (which was used consistently throughout this study) corresponds (Fig. A.4) to $a = 1.3$.

In Fig. A.5 c, d the derivatives of the real and the imaginary parts of the admittance as displayed in Fig. A.5 a, b are plotted versus frequency. The derivative signal was obtained by applying a low frequency (200 Hz) modulation of the sweep oscillator and using phase sensitive
Fig. A.5  

a) Real part of the active admittance of double cavity DC5.

b) Imaginary part of the active admittance of double cavity DC5.

c) Experimental derivative of a).

d) Experimental derivative of b).
detection (locked onto the modulation signal) of the detected R.F. signal.

For the experimental samples of interest the active admittance has the form (Eqs. 2.85 and 3.36):\[ Y_{\text{act}} = \frac{\Delta \nu}{\Gamma - i \frac{\Delta \nu}{2\pi}} \]

\[ r^2 + \left( \frac{\Delta \nu}{\nu 2\pi} \right)^2 \] \hspace{1cm} (A.5)

It is of interest to note that both the real part and the absolute value squared of the active admittance have the same frequency dependence (i.e., line shape) (which is not true of \( |Y_{\text{act}}| \)) as given by

\[ G_{\text{act}} = \frac{\Gamma}{r^2 + \left( \frac{\Delta \nu}{\nu 2\pi} \right)^2} \] \hspace{1cm} (A.6a)

\[ |Y_{\text{act}}|^2 = \frac{1}{r^2 + \left( \frac{\Delta \nu}{\nu 2\pi} \right)^2} \] \hspace{1cm} (A.6b)
This is well illustrated in Fig. A.6 where enough real offset (of the appropriate sign) was added to cancel the response for the primary resonance (but not for the transverse responses which require a different offset for cancellation). This cancellation could be useful if one were interested in studying transverse modes (or eliminating one of them). Referring to the earlier discussion of adjustment of the bridge for an imaginary offset, a slightly purer line shape is achieved by first adjusting for maximum cancellation of the primary response rather than cancellation of the background before adjusting the variable capacitor.

As a final comment, the noise in the detected output was ~0.5mV. For a maximum input signal of +10 dBm and an amplifier gain of 66 dB. Eq. A.4 implies that the minimum detectable admittance (for a signal to noise ratio of 1) would be ~1 \times 10^{-7} \Omega^{-1}. This could possibly be improved by amplitude modulation and phase sensitive detection.
Fig. A.6 Effect of various levels of real offset on detected signal. The $G_{act}$ trace (lowest curve) has been moved over for a clearer presentation.
APPENDIX B

PHYSICAL PROPERTIES

A compilation of relevant physical data for the composite cavities (and their constituents) which were studied is presented.

(a) Material parameters for CdS

Density\(^1\) (g./cm.\(^3\)) \( \rho \neq 4.820 \)

Elastic constants\(^2\) (10\(^{11}\) dynes/cm.\(^2\))
\[c_{11} = 8.590, \quad c_{12} = 5.334, \quad c_{13} = 4.612\]
\[c_{33} = 9.382, \quad c_{44} = 1.492\]

Piezoelectric constants\(^2\) (10\(^4\) statcoul./cm\(^2\))
\[\beta_{15} = -6.37, \quad \beta_{31} = -7.78, \quad \beta_{33} = 14.4\]

Dielectric constants\(^2\)
\[\varepsilon_1 = 9.02, \quad \varepsilon_2 = 9.02, \quad \varepsilon_3 = 9.53\]

Refractive indices\(^1\)
\[n_o = 2.506, \quad n_e = 2.491\]

\(^1\)Neuberger (1969)

\(^2\)McCallum (1969)
Photoelastic constants$^3,^4$

\[ p_{11} = 0.11, \quad p_{12} = 0.051, \quad p_{13} = 0.072 \]
\[ p_{31} = 0.050, \quad p_{33} = 0.13, \quad p_{44} = 0.054 \]

(b) Material parameters for fused silica

Density$^3$ (g./cm.$^3$) \( \rho = 2.2 \)

Refractive index$^3$ \( n_0 = 1.46 \)

Photoelastic constants$^3$ \( p_{44} = 0.075 \)

Shear acoustic velocity$^3$ (\( 10^5 \) cm./sec.) \( s_s = 3.76 \)

Longitudinal velocity$^3$ (\( 10^5 \) cm./sec.) \( s_L = 5.95 \)

(c) Material parameters for artificial sapphire

Density$^5$ \( \rho = 4.000 \)

Elastic constants$^5$ (\( 10^{12} \) dynes/cm.$^2$)

\[ c_{11} = 4.968, \quad c_{33} = 4.981, \quad c_{44} = 1.474 \]
\[ c_{13} = 1.109, \quad c_{14} = -0.235 \]

(d) Using methods outlined by Vrba and Haering (1973a), by Farnell (1961) and in Section 4.3, the following quantities were calculated from the physical parameters previously listed:

$^3$ Dixon (1967)

$^4$ Maloney and Carleton (1967)

$^5$ Farnell (1961)
(i) Acoustic shear velocities in CdS

<table>
<thead>
<tr>
<th>Propagation Direction</th>
<th>Polarization Direction</th>
<th>Velocity (10^5 cm./sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>b-axis</td>
<td>c-axis</td>
<td>1.76</td>
</tr>
<tr>
<td>b-axis</td>
<td>a-axis</td>
<td>1.84</td>
</tr>
<tr>
<td>30° to c-axis</td>
<td>-64.04° to c-axis</td>
<td>2.04</td>
</tr>
</tbody>
</table>

(ii) For light propagating in CdS along the a-axis and active acoustic shear waves propagating at 30° to the c-axis (rotation about a-axis) with strain, S_{30°}, the semi-axes of the perturbed index ellipse lie approximately along the c-axis and the b-axis with lengths given respectively by

\[ n_I' = n_e (1 + 0.0268 \times S_{30°})^{-1/2} \]  \hspace{1cm} (B.1a)

\[ n_{II}' = n_o (1 - 0.0222 \times S_{30°})^{-1/2} \]  \hspace{1cm} (B.1b)
(iii) For shear acoustic waves propagating along the \( x_3 \)-axis of sapphire the acoustic velocity is given by \( s = 6.07 \times 10^5 \) cm./sec.

(e) All experimental samples were in the shape of a parallelogram. The techniques for annealing, alignment, polishing and indium diffusion as listed by Burbank (1971) were followed. The CdS platelets were manufactured from Eagle-Picher Grade B single crystals. The accuracy of all x-ray alignments was \( \pm 0.5^0 \). Surfaces polished to an optical finish had surface flatness better than 2 wavelengths of sodium light for dimensions of order 1 cm. \( \times \) 1 cm. (i.e., a phonon maser of area 1 mm. \( \times \) 1 mm. would have a flatness better than \( \lambda/5 \)). The physical parameters of the samples are listed in Table B.1.

(f) All multiple cavities were bonded together following the technique outlined in Appendix E and were assembled from single cavities as follows:

(1) Double cavity DC1: crystal SC27.06.01.01 bonded in approximately the center of buffer SC-FS1.

(ii) Double cavity DC2: crystal SC27.06.04.01.02 bonded in approximately the center of buffer SC-FS1.
(iii) Double cavity DC3: crystal SC27.07.01.01.02 bonded in approximately the center of buffer SC-FS1.

(iv) Double cavity DC4: crystal SC24.01.02.04 C bonded in approximately the center of a b-axis face of buffer SC20.00.00.00 with the a-axis of SC24.01.02.04 C aligned along the c-axis of SC20.00.00.00.

(v) Double cavity DC5: crystal SC24.01.02.04 A bonded in approximately the center of buffer SC-S1.

(vi) Triple cavity T1: crystal SC27.06.02.02 bonded to buffer SC-S2 which was previously bonded to buffer SC-FS2.

(vii) Triple cavity T2: crystal SC27.06.03.02 bonded to buffer SC-S2 which was previously bonded to one of the 3.6° faces of buffer SC-FS3 where the a-axis of SC27.06.03.02 was aligned along the rotation axis of the 3.6° faces of SC-FS3.
<table>
<thead>
<tr>
<th>Sample</th>
<th>Composition</th>
<th>Separation of faces (mm)</th>
<th>Surface finish</th>
<th>Parallelism</th>
<th>Diffused contacts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Active cavities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SC24.01.02.01</td>
<td>Cds, single crystal</td>
<td>( t=1.194, l=2.90, w=2.60 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC24.01.02.04A</td>
<td></td>
<td>( t=1.194, l=2.05, w=1.90 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC24.01.02.04C</td>
<td></td>
<td>( t=1.194, l=2.35, w=2.00 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC24.01.02.04E</td>
<td></td>
<td>( t=1.194, l=2.00, w=1.85 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC24.07.11.03</td>
<td></td>
<td>( t=0.450, l=2.00, w=1.20 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC27.06.01.01</td>
<td></td>
<td>( t=0.70, l=0.70, w=0.70 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC27.06.02.02</td>
<td></td>
<td>( t=0.698, l=0.698, w=0.698 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
<tr>
<td>SC27.06.03.02</td>
<td></td>
<td>( t=0.820, l=1.75, w=1.35 )</td>
<td>optical, wire saw</td>
<td>&lt;10&quot;</td>
<td>yes</td>
</tr>
</tbody>
</table>
(Table B.1 continued)

| SC27.06.04.01.02 | CdS, single crystal | t = 0.625, 30° to c-axis | optical | <10° | yes |
| SC27.07.01.01.02 | " | l = 1.90, 60° to c-axis | optical | <1° | no |
|Buffers | w = 1.15, a-axis | optical | <1° | no |
| SC20.00.00.00 | " | l = 3.354, 30° to c-axis | optical | <10° | yes |
| SG-FS1 | t = 6.80, a-axis | " | <15° | no |
|Fused silica | l = 7.05, c-axis | " | <15° | " |
| | w = 0.95, b-axis | " | <15° | " |
| SG-FS2 | " | t = 7.85 | " | <1° | " |
| | l = 7.75 | " | <1° | " |
| SG-FS3 | " | t = 3.695 | " | <15° | " |
| | l = 7.20 | " | <1° | " |
| | w = 7.10 | " | <1° | " |
| SG-S1 | " | t = 2.05 to 2.50 | " | 3.6° | " |
| Sapphire, single crystal | l = 7.20 | " | 1° | " |
| | w = 7.10 | " | 1° | " |
| SG-S2 | " | t = 0.176, x-axis | wire saw | <10° | " |
| | l = 4.50 | " | " | " |
| | w = 2.50 | " | " | " |
| SG-S2 | " | t = 0.219, x-axis | " | <10° | " |
| | l = 4.80 | " | " | " |
| | w = 4.25 | " | " | " |
APPENDIX C

HETRODYNE DETECTION

The signal in the photodiode due to the mth order diffracted light mixed colinearly with a coherent reference signal is given by (Eq. 4.74)

\[
H = (B + A_m)^2
\]  \hspace{1cm} (C.1)

For diffraction of light from a single mode laser by a standing acoustic wave one finds (Eq. 4.20)

\[
A_m = A_0 \sum_{r=0}^{\infty} J_r(v) J_{r-m}(\frac{v}{a}) e^{i(\omega_0 + 2\pi v/r)} \Delta t + c.c.
\]  \hspace{1cm} (C.2)

For a laser with \( N + 1 \) longitudinal modes of spacing \( \Delta \omega \), the reference amplitude becomes

\[
B = \sum_{n=0}^{N} A^{(n)} e^{i(\omega_0 + n\Delta \omega) t} + c.c.
\]  \hspace{1cm} (C.3)
where \( A_0^{(n)} \) is the complex amplitude of the \( n^{th} \) laser mode and \( b \) is the complex fraction of the incident light that is used as a reference signal or local oscillator.

Eq. C.2 becomes

\[
A_m = b \sum_{r=-\infty}^{\infty} J_r(v) \sum_{m} \left( \frac{v}{a} \right)^{i(2r-m)\Delta \omega t} e^{i(\omega_0 + n\Delta \omega)t} \] 
\[
\times \frac{A_0^{(n)} e^{i(2r-m)\Delta \omega_0 t} e^{i(2\omega_0 + (n+m)\Delta \omega)t}}{e^{2\Delta \omega t} + e^{-2\Delta \omega t}} + c.c. \tag{C.4}
\]

Using Eqs. C.3 and C.4, Eq. C.1 becomes

\[
H = B^2 + 2BA_m + A_m^2
\]

\[
= \left[ b^2 + \sum_{r=-\infty}^{\infty} J_r(v) \sum_{m} \left( \frac{v}{a} \right)^{i(2r-m)\Delta \omega t} \right] + \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(v) J_{r-m} \left( \frac{v}{a} \right) e^{i(2s+2m)\Delta \omega t} \]
\[
\times J_s(v) J_{s-m} \left( \frac{v}{a} \right) e^{i(2r+2s)\Delta \omega t} \left[ \sum_{n=0}^{N} \sum_{x=0}^{N} A_0^{(n)} A_0^{(x)} e^{i(2\omega_0 + (n+x)\Delta \omega)t} \right] 
\]
\[
+ \left[ bb^* + \sum_{r=-\infty}^{\infty} J_r(v) J_{r-m} \left( \frac{v}{a} \right) e^{-i(2r-m)\Delta \omega t} \right] + \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(v) J_{r-m} \left( \frac{v}{a} \right) J_s(v) 
\]
\[
\times J_{s-m} \left( \frac{v}{a} \right) e^{i(2r-2s)\Delta \omega t} \left[ \sum_{n=0}^{N} \sum_{x=0}^{N} A_0^{(n)} A_0^{(x)} e^{i(n-x)\Delta \omega t} \right] 
\]
\[
+ c.c. \tag{C.5}
\]
Case 1.

Retaining only terms with frequencies much less than $\Delta \omega$ and $\omega$, Eq. C.5 reduces to

$$H_0 = (N \Sigma_{n=0}^{N} |A_0^{(n)}|^2) \left[ b^* b \Sigma_{r=-\infty}^{\infty} J_r(v) J_{r-m}(v/a) e^{i(m-2r)\Omega t} \right]$$

$$+ \Sigma_{r=-\infty}^{\infty} \Sigma_{s=-\infty}^{\infty} J_r(v) J_{r-m}(v/a) J_s(v) J_{s-m}(v/a) e^{i(2r-2s)\Omega t}$$

$$+ \text{ c.c.} \quad \text{(C.6)}$$

The second term in square brackets of Eq. C.6 produces frequencies given by

$$\omega_{m,r} = (m-2r)\Omega \quad \text{for} \quad r = 0, \pm 1, \ldots \quad \text{(C.7a)}$$

with amplitudes proportional to

$$A_{m,r} = J_r(v) J_{r-m}(v/a) \quad \text{(C.7b)}$$

The third term of Eq. C.6 is called the homodyne or self-beating signal as it does not depend on the presence of a
reference signal. It produces frequencies given by

$$\omega_{m,r} = 2\pi \Omega \quad r = 0, \pm 1, \ldots \quad (C.8a)$$

with amplitudes proportional to

$$A_{m,r} = \sum_{s=-\infty}^{\infty} J_{r+s}(v) \frac{v}{a} J_{r+s-m}(v/a) J_s(v) \frac{v}{a} \quad (C.8b)$$

For running acoustic waves ($v/a \to 0$) Eq. C.6 reduces to

$$H_0 = \left( \sum_{n=0}^{N} |A_n|^2 \right) \left[ b b^* + J_m(v)e^{i m \Omega t} \right] + c.c. \quad (C.9)$$

Case 2. Considering only self-beating (i.e., $b = 0$) and retaining only terms with frequencies in the neighborhood of $\pm \Delta \omega$ Eq. C.5 reduces to
\[ H_1 = \sum_{r=\infty}^{\infty} \sum_{s=\infty}^{\infty} J_r(v) J_{r-m}(\frac{v}{a}) J_s(v) J_{s-m}(\frac{v}{a}) e^{i2(r-s)\Omega t} \]

\[ \times \sum_{n=0}^{N} \left[ A_0^{(n+1)}(n) * e^{i\Delta wt} + A_0^{(n)}(A_0^{(n+1)}) * e^{-i\Delta wt} \right] + c.c. \]

(C.10a)

Converting to polar form where

\[ \sum_{n=0}^{N} A_0^{(n+1)}(A_0^{(n)}) * e^{i\Delta wt} = \bar{A}_0^2 e^{i\phi} \]

(C.10b)

Eq. C.10a becomes

\[ H_1 = \bar{A}_0^2 \sum_{r=\infty}^{\infty} \sum_{s=\infty}^{\infty} J_r(v) J_{r+s-m}(\frac{v}{a}) J_s(v) J_{s-m}(\frac{v}{a}) \]

\[ \times \left[ e^{i2r\Omega t} + e^{-i2r\Omega t} \right] \left[ e^{i(\Delta wt+\phi)} + e^{-i(\Delta wt+\phi)} \right] \]

(C.11)
For a first order spot (i.e., $m = 1$) and under the assumption $v, v/a \ll 1$ Eq. C.11, to second order in $v$, becomes

$$H_1 = \tilde{A}_0^2 v^2 (1 + 1/a^2) \cos(\Delta \omega t + \phi) - v^2/a [\cos(\Delta \omega t + \phi + 2\Omega t) + \cos(\Delta \omega t + \phi - 2\Omega t)]$$

(C.12)

Let $\alpha$ be the ratio of the detected amplitude of the fundamental ($\omega = \Delta \omega$) to either of the sidebands ($\omega = \Delta \omega \pm 2\Omega$); then from Eq. C.12

$$\alpha = \frac{(1 + 1/a^2)}{a}$$

(C.13)

Solving for $a$

$$a = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2} - 1}{2}$$

(C.14)
APPENDIX D

TWO WAVE APPROXIMATION

Greebe (1965) has shown that for the boundary conditions given by Eq. 2.52a, b (vanishing of charge density) and in the limit of zero electron drift velocity, carrier density waves can be completely ignored in electrical impedance considerations provided the following condition is satisfied

\[ \frac{R^2 q_0^2}{\omega t} = \frac{\omega D}{\pi^2} \ll 1 \]  (D.1)

For typical values of \( D = 7 \text{ cm}^2/\text{sec.} \) and \( s_0 = 1.76 \times 10^5 \text{ cm./sec.} \) Eq. 2.63 implies

\[ v \ll 7 \times 10^5 \text{Hz.} \]  (D.2)

Identical conclusions can be drawn for the other two choices of boundary conditions corresponding to the vanishing of current density (Eq. 2.51a, b) and of electric field (Eq. 2.53a, b).

When the drift velocity is of the order of the acoustic velocity \( (v_0 \sim s_0) \) the situation is not so straightforward. We now consider this case because it is relevant to the phonon maser. The amplitudes \( E_2, E_3 \) and \( E_4 \) contribute significantly to the impedance, \( Z \) (Eq. 2.62), only when their
common denominator, \( H \), is resonant (at other times they are of order \( \chi/N \) smaller than \( E_1 \), where \( N \) is the harmonic number). Referring to Eq. 2.60d, the purely acoustic term \( (e^{-ik_1d} - e^{-ik_2d}) \) is dominant in \( H \) whenever the condition given by

\[
\left| a_{234} \right| >> \left| a_{342} \right|, \left| a_{243} \right| \quad (D.3)
\]

is satisfied. For each of the boundary conditions corresponding to the vanishing of current density, of charge density or of electric field Eq. D.3 takes the forms given respectively by (Eq. 2.55)

\[
\chi \left| a_0 \right| \frac{1}{1 + R^2 q_0^2 + i\omega \gamma_{2,3}} \ll 1 \quad (D.4a)
\]

\[
\chi \left| a_0 \right| \left| q_0 / k_0 \right| \frac{1}{1 + R^2 q_0^2 + i\omega \gamma_{2,3}} \ll 1 \quad (D.4b)
\]

and

\[
\chi \left| a_0 \right| \frac{R^2 q_0^2 + i\omega \gamma_{2,3}}{1 + R^2 q_0^2 + i\omega \gamma_{2,3}} \ll 1 \quad (D.4c)
\]
The coefficient $a_L$ is seen to play an important role. For $v_o \approx s_o$ it is found that (Eq. 2.22) $|q_o/k_o| \leq 1$; therefore the inequalities (Eq. D.4) can only fail if the magnitude of $a_L > 1/\chi = 26$. Referring to Fig. 2.6 or Eq. 2.56a the magnitude of $a_L$ only exceeds unity in the neighbourhood of $v_o \approx s_o$ and for $\omega \tau > 2$. The peak value of $|a_L|$ can be given to good approximation by evaluating $a_L$ under the condition $v_o = s_o$ as given by (Eq. 2.56b)

$$a_L(v_o=s_o) = -\frac{i\omega \tau}{2(1+R^2q_o^2)} \quad (D.5)$$

Therefore the resonance of the denominator, $N$ (Eq. 2.60d), is dominated by purely acoustic processes (i.e., the inequalities of Eq. D.4 are satisfied), for all three boundary conditions, in the region $v_o \not\approx s_o$ if the following condition is satisfied

$$\frac{x}{\chi \omega \tau} \ll 1 \quad (D.6)$$
The term on the left of the inequality reaches a maximum for the condition of maximum acoustic gain (Eq. 2.38) given by

\[ R^2q_0^2 = 1 \]  \hspace{1cm} (D.7)

Using Eq. (D.7) the inequality of Eq. D.6 is satisfied for

\( (\chi = 0.0378, D = 7 \text{ cm}^2/\text{sec.}, s = 1.76 \times 10^5 \text{ cm}^2/\text{sec.}) \) the frequency given by

\[ \nu >> 5 \times 10^6 \text{ Hz.} \]  \hspace{1cm} (D.8)

For frequencies above or below the frequency of maximum gain (Eq. 2.39 or D.7) the condition expressed by Eq. D.6 is even less restrictive than the condition given by Eq. D.8.

Even when the acoustic term \( e^{-ik_1d} - e^{-ik_2d} \) is dominant in the denominator, \( H \) (Eq. 2.60d), the mixed (acoustic and carrier density) terms \( e^{-ik_1d} - e^{-ik_4d} \) and \( e^{-ik_2d} - e^{-ik_4d} \) can potentially degrade the resonance condition. If velocity dispersion between the forward and backward acoustic wave is ignored the condition for acoustic resonance is given by

\[ e^{-iq_2d} = e^{-iq_3d} = \pm 1 \]  \hspace{1cm} (D.9)
where the plus and minus signs refer to even and odd harmonics respectively. Under acoustic resonance the acoustic term \((e^{-ik_1d} - e^{-ik_2d})\) reduces to

\[ e^{-\alpha_3d} - e^{-\alpha_2d} = (\alpha_2 - \alpha_3)d \]  \hspace{1cm} (D.10)

where the approximate equality holds for \(\alpha_2, \alpha_3 \ll 1\).

Expressions for \(\alpha_2\) and \(\alpha_3\) are found in Eq. 2.36. The round trip acoustic loss for unit cavity length, \(\alpha_2 - \alpha_3\), is plotted versus drift velocity, \(v_0\), in Fig. D.1.

If no degradation of resonance is to occur, the round trip acoustic loss must be greater than all remaining mixed terms in the denominator, \(H\) (Eq. 2.60d). Referring to Fig. D.1 the round trip acoustic loss crosses the horizontal axis (i.e., \((\alpha_2 - \alpha_3) = 0\)) for a drift velocity slightly above the acoustic velocity, due to acoustic gain (negative loss) of the forward acoustic wave. In this region the mixed terms are bound to degrade the resonance. However, if the mixed terms are of small magnitude, their imaginary parts will slightly perturb the resonance frequency while their real parts will just act as an additional loss, effectively lowering the horizontal axis slightly and thereby moving the crossing point \(((\alpha_2 - \alpha_3) = 0\)) to a slightly higher drift field. A reasonable criterion for ignoring the mixed terms in the denominator, \(H\), would be to require that the magnitude
Fig. D.1 Acoustic loss per cm. versus drift velocity for the conditions:

\[ R^2 q_0^2 = 1, \quad \omega t = 10, \quad q_0 = 7 \times 10^2 \text{ cm}^{-1}, \quad X = 0.0378 \]
of the mixed terms is small compared to the peak round trip loss (or gain). The maximum round trip loss is given by

\[ (\alpha_2 - \alpha_3) d \bigg|_{\text{max}} > \frac{xq_0 d}{4(1 + R^2 q^2)} \quad (D.11) \]

Using Eq. D.5 to calculate the maximum amplitude of \( a_n \), the condition for ignoring the mixed terms in the resonance of \( H \) is given by

\[ N >> \frac{2}{\pi} \omega \tau \quad (D.12a) \]

or equivalently

\[ \sigma \gg \frac{\varepsilon s}{2 \pi d} \quad (D.12b) \]

where the harmonic number \( N \) is defined by

\[ N = \frac{2d^2}{s_0} = \frac{q_0 d}{\pi} \quad (D.12c) \]

For typical conditions of \( \varepsilon = 9 \) and \( s_0 = 1.76 \times 10^5 \text{ cm/} \text{sec} \),

Eq. D.12b takes the form
The criterion of Eq. D.12 agrees with numerical calculations of the electrical impedance (Eq. 2.62) by Adler and Farnell (1966) under the conditions of $\omega T = 1$ and $v_0 = 2s_0$, where it was found that the forward carrier density wave produced significant effects for the fundamental ($N = 1$) frequency response but could be ignored for frequencies greater than the 7th harmonic.

If the conditions for neglecting the mixed terms in the denominator $H$ are satisfied (Eq. D.12), one can also neglect the mixed terms in the numerator of the expressions for $E_2$ and $E_3$ (Eq. 2.60). Whenever the condition given by

$$\left| \frac{q_a}{k} \right| << 1$$

(Eq. D.14)

is satisfied, which it is for $\omega T << 1$ (Eq. 2.30), the carrier density wave electric field amplitude, $E$ (Eq. 2.60c), can be neglected in the expression for the impedance, $\Xi$ (Eq. 2.62). However, for $\omega T \gtrsim 1$ in the low conductivity limit (Eq. 2.29) the condition expressed in Eq. D.14 is not satisfied and the
amplitude $E_0$ must be considered in the impedance $Z$. Often one is not interested in the actual amplitude of the impedance but only in the resonance condition and the frequency dependance (e.g., the $Q$ of the response) which are unaffected by the addition of $E_0$. 
APPENDIX E

ACOUSTIC BONDS

Essential to coupling acoustic power out of phonon masers is an appropriate acoustic bond. Due to the use of an electron drift in phonon maser operation, one essential feature of a bonding procedure is the facility to provide an electric contact. Conventional bonding techniques were found unsatisfactory. The greases and liquids commonly used have no electrical conductivity and along with the gallium bond (which is a conductor) often require "wringing on." CdS crystals are too easily scratched and damaged by this mounting procedure. The use of epoxy bonds is complicated by the need to provide electric contact.

One bond commonly used that satisfied most of the criteria is the thermo-compression indium bond. The two surfaces to be bonded are coated with indium and then pressed together at \( \sim 154^\circ \text{C.} \) (\( \sim 2^\circ \text{C.} \) below the melting point of In). However, the elevated temperature was found to introduce electron traps in CdS and when there are large mismatches in expansion coefficients the bonds leave large stresses upon cooling.

A successful cold welding procedure similar to that used by Sittig and Cook (1968) (also see Sittig, Warner and
Cook, 1969) was developed for use with CdS crystals. Lapped and polished CdS crystals were bonded to a variety of lapped and polished buffers made from single crystal SiO₂, Al₂O₃ and CdS and fused SiO₂. The basic procedure employed was to clean thoroughly the two surfaces to be bonded, evaporate appropriate metal layers on each surface and then press them together while still under vacuum.

The most crucial step was found to be the cleaning procedure. For very thin bonds (~3000 Å. thick) all dust particles larger than ~1000 Å. had to be removed. The surfaces were first cleaned chemically and as a final stage wiped with lens tissue, moistened with ether, in a laminar flow box. An optical flat (also cleaned) and interference techniques were used to check for dust. If large dust particles were found, the wiping procedure was repeated.

One has some flexibility in the type of metal film used. Sittig, Warner and Cook report the use of primary layer (~2000 Å.) of either gold (proceeded by a flash of chromium for adhesion), silver (proceeded by a flash of titanium) or aluminum followed by a secondary layer (~2000 Å.) of indium on both samples. One problem encountered (specifically for a gold primary layer) with these recipes was the failure of the indium to produce smooth layers. The indium tended to form islands of order of a few microns in size. This effect may be due to resi-
dual oxygen (Caswell 1961) although a vacuum of \( \sim 2 \times 10^{-6} \) Torr was employed.

A good choice of metals for bonding Cds crystals was found to be: 100 \( \AA \) of Cr and 1000 \( \AA \) of Au on the buffer to form a tough electrical contact and 2000 \( \AA \) of In on the Cds to form a reasonable electric contact with the indium diffused surfaces of the crystal. After evaporation of the metal films the surfaces were pressed together with a pressure \( \approx 200 \) Kg./cm.\(^2\). The bond could be visually inspected. Wherever the crystal had bonded to the buffer noticeable alloying could be observed (the gold backing takes on a silver color). Whenever a bond was broken the indium was cleanly lifted off the Cds. Crude measurements found a bond strength of greater than 100 Kg./cm.\(^2\).


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Nonlinear Interactions - Theoretical


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