ELASTIC PROBLEMS LEADING TO THE BIHARMONIC EQUATION
IN REGIONS OF SECTOR TYPE

by

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ABSTRACT

A number of problems in elasticity may be reduced to solving the biharmonic equation $\nabla^4 \phi = 0$ in two dimensions under appropriate boundary conditions on the function $\phi$. The purpose of this thesis will be to examine certain methods of solution of this equation in regions bounded by lines radiating from the origin and by arcs of circles centered at the origin. The basic region of this type is the sector $r < a, -\omega < \theta < \omega$, where $(r, \theta)$ are polar coordinates. We shall be especially concerned with the particular case when $\omega = \pi$, so that the sector becomes a circular region with a crack lying between the boundary and the centre of the circle.

After reviewing in the first chapter the basic equations of plane elasticity and of the theory of plate bending, and showing how in both cases problems may be reduced to the biharmonic equation, we proceed in Chapter II to examine the solution of a number of particular problems, most of which involve regions of sector type. In this chapter we consider a problem previously examined by Williams [3] concerning a cracked cylinder with imposed tractions on the boundary $r = a$. He constructs a basic set of eigenfunctions satisfying the biharmonic equation and the homogeneous boundary conditions on the crack faces $\theta = \pm \pi$. Unfortunately, these eigenfunctions are not orthogonal which makes it very difficult to determine the unknown coefficients in the expression of the stress function.

In an interesting paper, Gaydon and Shepherd [5] consider the problem of a semi-infinite rectangular strip. Each of the eigenfunctions is
expanded in a series of orthonormal beam functions, thus enabling them to compute numerically the coefficients of the stress function corresponding to any arbitrary distribution of traction on the end of the strip. In an extension of their work, Gopalacharyulu [6] follows the same method in solving the sector problem. He also expands the eigenfunctions in a series of orthonormal beam functions.

In Chapter III, we consider the problem of an infinite cylinder of unit radius cracked along the plane \( \theta = \pi \). Instead of using the more complicated beam eigenfunctions as were used by Gopalacharyulu [6], the set of basic eigenfunctions discussed in Chapter II is expanded in terms of simple Fourier sine and cosine series. An infinite system of simultaneous equations is obtained from which we can compute numerically the coefficients corresponding to arbitrary tractions on the boundary \( r = 1 \). We have computed these coefficients numerically for a particular loading as well as the corresponding stress distribution around the crack tip. The stress intensity factors for different loadings are also computed. Following the same method, the set of simultaneous equations for determining the coefficients of the stress function is obtained in the case of an infinite cylinder having a crack with a rounded tip, and also in the case of a semi-circular cylinder.
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CHAPTER I

ELASTOSTATIC PROBLEMS LEADING TO THE BIHARMONIC EQUATION

In this chapter we shall consider two different types of problems that lead to the biharmonic equation: plane elastostatic problems and pure bending of thin plates.

1.1 Plane elastostatic problems

Plane elastostatic problems include generalized plane stress and plane strain problems. The first of these problems arises when considering a thin plate loaded by forces applied at the boundary, parallel to the plane of the plate. Therefore, if we take the plane of the plate as the xy-plane, the average stress components through the plate thickness, $\bar{\sigma}_{zz}$, $\bar{\sigma}_{xz}$ and $\bar{\sigma}_{yz}$ will all be zero, whereas $\bar{\sigma}_{xx}$, $\bar{\sigma}_{yy}$ and $\bar{\sigma}_{xy}$ are only functions of x and y. On the other hand plane strain problems arise when the dimension of the body in the z-direction is very large. If a long cylindrical or prismatic body is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross-sections are in the same condition. We assume that the end sections are confined between fixed smooth rigid planes. Since there is no axial displacement at the ends, and, by symmetry, at the mid-section, it may be assumed that the same holds at every cross-section.

In this section we shall take the coordinate axes to be $x_1$, $x_2$ and $x_3$. We use the Greek indices $\alpha$ and $\beta$ for the range 1, 2. A repeated
index will represent the sum of all allowable values of that index.

1.1.1 Plane deformation

A body is said to be in a state of plane deformation, or plane strain, parallel to the $x_1x_2$-plane, if the component $u_3$ of the displacement vector $\bar{u}$ vanishes and the components $u_1$ and $u_2$ are functions of the coordinates $x_1$ and $x_2$ but not $x_3$. Thus, a state of plane deformation is characterized by the formulae,

$$\begin{align*}
&u_\alpha = u_\alpha(x_1, x_2), \\
&u_3 = 0.
\end{align*}$$

The stress-displacement relations in this case will be

$$\begin{align*}
\sigma_{\alpha\beta} &= \lambda \bar{\nu} \delta_{\alpha\beta} + G (u_{\alpha,\beta} + u_{\beta,\alpha}), \\
\sigma_{33} &= \lambda \bar{\nu}, \\
\sigma_{13} &= \sigma_{23} = 0
\end{align*}$$

where the dilatation $\bar{\nu} = u_{\alpha,\alpha}$ and $G$ is the modulus of rigidity.

The equilibrium equations are

$$\sigma_{\alpha\beta} = -F_\alpha(x_1, x_2)$$

where $F_\alpha$ are the components of the body force.

If the solutions of these equilibrium equations are to correspond to the state of stress that can exist in an elastic body, the $\sigma_{\alpha\beta}$ must satisfy the Beltrami-Mitchell compatibility equation

$$\nabla^2 \theta_1 = -\frac{2(\lambda + G)}{\lambda + 2G} F_{\alpha,\alpha}.$$

where \( \Theta_1 \equiv \sigma_{11} + \sigma_{22} \).

If the components \( T_\alpha(x_1, x_2) \) of external stresses are specified along the boundary in the form

\[
\sigma_{\alpha\beta} n_\beta = T_\alpha(x_1, x_2),
\]

where the \( n_\beta \) are the components of the exterior unit normal vector to the boundary, the formulation of the problem is complete.

1.1.2 Generalized Plane stress

A body is in the state of plane stress parallel to the \( x_1 x_2 \)-plane when the stress components \( \sigma_{13}, \sigma_{23}, \sigma_{33} \) vanish.

Sokolnikoff [1] has shown that the Beltrami-Mitchell compatibility equation turns out to be

\[
\nabla^2 \bar{\sigma}_1 = -\frac{2(\lambda + G)}{\lambda + 2G} \bar{F}_\alpha, \alpha,
\]

where \( \bar{\sigma}_1 = \sigma_{11} + \sigma_{22}, \ \bar{\lambda} \equiv \frac{2\lambda G}{\lambda + 2G}, \ \bar{\sigma}_{\alpha\beta}(x_1, x_2) \) and \( \bar{F}_\alpha(x_1, x_2) \) are the mean values of \( \sigma_{\alpha\beta} \) and \( F_\alpha \) in a cylinder of thickness \( 2h \), and bases in the planes \( x_3 = \pm h \), i.e.

\[
\bar{\sigma}_{\alpha\beta}(x_1, x_2) \equiv \frac{1}{2h} \int_{-h}^{h} \sigma_{\alpha\beta}(x_1, x_2, x_3) \, dx_3,
\]

\[
\bar{F}_\alpha(x_1, x_2) \equiv \frac{1}{2h} \int_{-h}^{h} F_\alpha(x_1, x_2, x_3) \, dx_3.
\]

Equations (1) and (2) suffice to determine the mean stresses \( \bar{\sigma}_{\alpha\beta} \) when the boundary conditions on the edge are given in the form
1.1.3 Airy's stress function

We shall consider boundary value problems in plane elasticity in which body forces are absent. Accordingly, we consider the equilibrium equations in the form

\[ \sigma_{\alpha \beta} n_\beta = T_\alpha . \]

where \( \sigma_{\alpha \beta} \) satisfy the compatibility equation

\[ \nabla^2 (\sigma_{11} + \sigma_{22}) = 0 , \]

and are given on the boundary by

\[ \sigma_{\alpha \beta} n_\beta = T_\alpha (s) , \]

where the \( T_\alpha (s) \) are known functions of the arc parameter \( S \) on the boundary \( C \).

The equilibrium equations imply the existence of a function \( \phi(x_1, x_2) \) such that

\[ \sigma_{22} = \phi_{,11} , \quad \sigma_{12} = -\phi_{,12} , \quad \sigma_{11} = \phi_{,22} . \]

The compatibility equation implies that \( \phi \) must satisfy the biharmonic equation

\[ \nabla^2 \nabla^2 \phi = 0 , \]

or

\[ \nabla^4 \phi = 0 . \]
in the region $R$.

Every solution of this equation of class $C^4$ is called a biharmonic function, but since we are interested in those states of stress for which the $\sigma_{\alpha\beta}$ are single-valued, we need consider only biharmonic functions with single-valued second partial derivatives.

Expressing the biharmonic equation and the stresses in terms of $x$ and $y$, we obtain

\[ \nabla^4 \phi = \nabla^2 \nabla^2 \phi = 0, \tag{3} \]

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \]

and

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2}, \\
\sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2}, \\
\sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}.
\end{align*} \tag{4}
\]

In plane polar coordinates, these equations become

\[ \nabla^{4\phi} = \nabla^2 \nabla^2 \phi = 0, \]

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \]

and

\[
\begin{align*}
\sigma_{rr} &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}, \\
\sigma_{\theta \theta} &= \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}.
\end{align*} \]
\[
\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2},
\]
\[
\sigma_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial^2 \phi}{\partial \theta \partial \theta}.
\]

Here \( \sigma_{rr} \), \( \sigma_{\theta\theta} \) and \( \sigma_{r\theta} \) are the physical components of stress with respect to the polar coordinates.

1.2 Pure Bending of plates

The classical small-deflection theory of plates, developed by Lagrange, is based on the following assumptions:

i) points which lie on a normal to the mid-plane of the undeflected plate lie on a normal to the mid-plane of the deflected plate;

ii) the stresses normal to the mid-plane of the plate, arising from the applied loading, are negligible in comparison with the stresses in the plane of the plate. Thus, every transverse single loading considered in the thin-plate theory is merely a discontinuity in the magnitude of the shearing forces. If the effect of the surface load becomes of special interest, thick-plate theory has to be used;

iii) the slope of the deflected plate in any direction is small, so that its square may be neglected in comparison with unity;

iv) the mid-plane of the plate remains neutral during bending, i.e. any mid-plane stresses arising from the deflection of the plate section into a non-developable surface may be ignored.

1.2.1 Curvature of slightly bent plates

In discussing small deflections of a plate, we take the middle plane
of the plate, before bending occurs, as the xy-plane. During bending, the particles on this plane undergo small displacements \( w \) perpendicular to the xy-plane and form the middle surface of the deformed plate.

In determining the curvature of the middle surface of the plate we shall be using assumption iii, namely the slope of the tangent to the surface in any direction can be taken equal to the angle that the tangent makes with the xy-plane, and the square of the slope is neglected compared to unity. Thus the curvature of the surface in a plane parallel to the xz-plane (Fig. 1) is equal to

\[
\frac{1}{r_x} = \frac{-\frac{\partial^2 w}{\partial x^2}}{1 + \left(\frac{\partial w}{\partial x}\right)^2}^{3/2} \approx \frac{-\frac{\partial^2 w}{\partial x^2}}{\frac{\partial w}{\partial x}}.
\]  

(6)

Similarly, the curvature of the surface in a plane parallel to the yz-plane is approximately equal to

\[
\frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}.
\]  

(7)

Now, for any direction \( \mathbf{n} \) (Fig. 2)

\[
\frac{1}{r_n} = -\frac{\partial^2 w}{\partial n^2}.
\]  

(8)
But, for any direction $\overrightarrow{an}$ making an angle $\alpha$ with the x-axis, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$,

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha .$$

(9)

Therefore, the curvature in the $\overrightarrow{an}$ direction will be

$$\frac{1}{r_n} = - \frac{\partial}{\partial n} (\frac{\partial w}{\partial n}) = - (\frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha) (\frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha)$$

$$= - (\frac{\partial^2 w}{\partial x^2} \cos^2 \alpha + \frac{\partial^2 w}{\partial x \partial y} \sin \alpha \cos \alpha + \frac{\partial^2 w}{\partial y^2} \sin^2 \alpha)$$

$$= \frac{1}{r_x} \cos^2 \alpha + \frac{1}{r_y} \sin^2 \alpha - \frac{1}{r_{xy}} \sin 2\alpha ,$$

(10)

where the quantity $\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \partial y}$ is called the twist of the surface with respect to the x and y axes.

In the case of the direction $\overrightarrow{at}$, the angle with the x-axis will be $\alpha + \frac{\pi}{2}$, and the curvature in the t-direction will be given by

$$\frac{1}{r_t} = \frac{1}{r_x} \sin^2 \alpha + \frac{1}{r_y} \cos^2 \alpha + \frac{1}{r_{xy}} \sin 2\alpha .$$

(11)

The twist of the surface with respect to the n and t directions is given by

$$\frac{1}{r_{nt}} = \frac{\partial^2 w}{\partial n \partial t} = \frac{\partial}{\partial t} (\frac{\partial w}{\partial n}) .$$
We note that
\[
\frac{\partial}{\partial t} = -\frac{\partial}{\partial x} \sin \alpha + \frac{\partial}{\partial y} \cos \alpha ,
\]
and therefore
\[
\frac{1}{r_{nt}} = (-\frac{\partial}{\partial x} \sin \alpha + \frac{\partial}{\partial y} \cos \alpha)(\frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha)
\]
\[
= \frac{1}{2} \sin 2\alpha (-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) + \frac{\partial^2 w}{\partial x \partial y} \cos 2\alpha
\]
\[
= \frac{1}{2} \sin 2\alpha \left(\frac{1}{r_x} - \frac{1}{r_y}\right) + \frac{1}{r_{xy}} \cos 2\alpha .
\]  (12)

Therefore, if the quantities \( \frac{1}{r_x} , \frac{1}{r_y} \) and \( \frac{1}{r_{xy}} \) are known, we can get the corresponding quantities related to any system of axes inclined at an angle \( \alpha \) to the original system, by using equations (10) - (12).

In order to obtain the principal curvatures of the surface and the corresponding principal directions, we try to find the values of the angle \( \alpha \) for which \( \frac{1}{r_n} \) is an extremum. Thus differentiating equation (10) with respect to \( \alpha \) and equating the result to zero, we find that
\[
\sin 2\alpha \left(\frac{1}{r_y} - \frac{1}{r_x}\right) - \frac{2}{r_{xy}} \cos 2\alpha = 0 ,
\]
or
\[
\tan 2\alpha = -\frac{2}{r_{xy}} \left(\frac{1}{r_x} - \frac{1}{r_y}\right) .
\]  (13)

We denote the roots of equation (13) by \( \alpha_1 \) and \( \alpha_1 + \frac{\pi}{2} \). Substituting these values of \( \alpha \) in equation (10) we obtain the two principal curvatures.

We also note that if \( \alpha \) satisfies (13) then from (12)
\[
\frac{1}{r_{nt}} = 0 ,
\]
i.e. the twist of the surface is zero on the principal planes.

1.2.2 Relations between bending moments and deflection

Consider a rectangular plate under uniformly distributed moments along the edges of the plate (Fig. 3).

The xy-plane is taken as the middle plane of the plate before bending. \( M_x \) will denote the bending moment per unit length acting on the edges parallel to the y-axis and \( M_y \) the moment per unit length acting on the edges parallel to the x-axis. These moments are considered positive when they produce compression in the upper surface of the plate and tension in the lower. The thickness \( h \) of the plate is assumed to be small in comparison with other dimensions.

Now, to derive the expressions for the bending moments in terms of the deflection of the plate, we consider an element cut out of the plate by two pairs of planes parallel to the \( xz \) and \( yz \)-planes (Fig. 4).
Using assumption i, p. (6), the lateral sides of the element will remain plane during bending and will rotate about the neutral axes nn so as to remain normal to the deflected middle surface of the plate, and therefore the middle surface will not undergo any extension during bending. The longitudinal strain of an element at a distance \( z \) from the neutral surface in the x-direction (Fig. 5) is

\[ e_{xx} = \frac{z}{r_x} . \]

Similarly \( e_{yy} = \frac{z}{r_y} . \)

Using Hooke's Law, the normal stresses are

\[
\sigma_{xx} = \frac{E}{1-\nu^2}(e_{xx} + \nu e_{yy}) = \frac{Ez}{1-\nu^2} \left( \frac{1}{r_x} + \frac{1}{r_y} \right) = \frac{-Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right),
\]

\[
\sigma_{yy} = \frac{E}{1-\nu^2}(e_{yy} + \nu e_{xx}) = \frac{Ez}{1-\nu^2} \left( \frac{1}{r_y} + \frac{1}{r_x} \right) = \frac{-Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right).
\]

The couples produced by these stresses on the lateral sides should obviously be equal to the external couples \( M_x \, dy \) and \( M_y \, dx \), thus
Substituting equations (14) into (15) for the values of \( \sigma_{xx} \) and \( \sigma_{yy} \), we obtain

\[
M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right),
\]

where

\[
D = \frac{Eh^3}{12(1-\nu^2)}.
\]

This is called the flexural rigidity of the plate.

Now we shall express the moments acting on a section inclined to the \( x \) and \( y \) axes in terms of \( M_x \) and \( M_y \). If we cut the lamina \( abcd \) (Fig. 4) by a plane parallel to the \( z \)-axis and intersecting the lamina along \( ac \) (Fig. 5), we can determine the normal and shear stresses acting on this inclined face in terms of \( \sigma_{xx} \) and \( \sigma_{yy} \). These will be given by the well known equations

\[
\sigma_{nn} = \sigma_{xx} \cos^2 \alpha + \sigma_{yy} \sin^2 \alpha, \quad \text{and} \quad \sigma_{nt} = \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) \sin 2\alpha,
\]

where \( \alpha \) is the angle between the normal \( n \) to the inclined face and the \( x \)-axis.
Considering all laminas, such as acd (Fig. 6), over the thickness of the plate, the normal stresses $\sigma_{nn}$ give the bending moment acting on the inclined plane, the magnitude of which per unit length along ac is

$$M_n = \int_{-h/2}^{h/2} \sigma_{nn} \, dz = \int_{-h/2}^{h/2} (\sigma_{xx} \cos^2 \alpha + \sigma_{yy} \sin^2 \alpha) \, dz$$

$$= M_x \cos^2 \alpha + M_y \sin^2 \alpha$$

$$= D \left[ \left( -\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \cos^2 \alpha + \left( -\frac{\partial^2 w}{\partial y^2} - \nu \frac{\partial^2 w}{\partial x^2} \sin^2 \alpha \right) \right]$$

$$= D \left[ \left( \frac{1}{r_x} \cos^2 \alpha + \frac{1}{r_y} \sin^2 \alpha \right) + \nu \left( \frac{1}{r_y} \cos^2 \alpha + \frac{1}{r_x} \sin^2 \alpha \right) \right]$$

$$= D \left[ \frac{1}{r} + \nu \frac{1}{r} \right] = -D \left( \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right). \quad (19)$$

Similarly

$$M_t = -D \left[ \frac{\partial^2 w}{\partial t^2} + \nu \frac{\partial^2 w}{\partial n^2} \right]. \quad (20)$$

The shearing stresses $\sigma_{nt}$ will give a twisting moment acting on the inclined face, the magnitude of which per unit length of ac is:

$$M_{nt} = \int_{-h/2}^{h/2} \sigma_{nt} \, dz = \frac{1}{2} \sin 2\alpha (M_x - M_y). \quad (21)$$

Here we note that the signs of $M_n$ and $M_{nt}$ are chosen in such a manner that their positive values are represented by vectors in the positive directions of $t$ and $n$ respectively.

To obtain the expression for $M_{nt}$ in terms of the deflection $w$, consider the distortion of a thin lamina efgh with the sides ef and
eh parallel to the n and t directions respectively, and at a distance z from the middle plane (Fig. 7).

During bending of the plate, the points e, f, g and h undergo small displacements. The components of the displacement of the point e in the n and t directions are denoted by u and v respectively. Then the displacement of the adjacent point h in the n direction is \( u + \frac{\partial u}{\partial t} \) dt, and the displacement of the point f in the t direction is \( v + \frac{\partial v}{\partial n} \) dn. Owing to these displacements, the shearing strain will be

\[
e_{nt} = \frac{1}{2} \left( \frac{\partial u}{\partial t} + \frac{\partial v}{\partial n} \right),
\]

and the corresponding shearing stress is

\[
\sigma_{nt} = G \left( \frac{\partial u}{\partial t} + \frac{\partial v}{\partial n} \right),
\]

where G is the modulus of elasticity in shear.

In order to express u and v in terms of the deflection w of the plate, consider a section of the middle surface made by the normal plane through the n-axis. The angle of rotation in the counter-clockwise direction of an element pq, which initially was perpendicular to the xy plane, about an axis perpendicular to the nz plane is equal to \( -\frac{\partial w}{\partial n} \) (Fig. 8). Owing to this rotation a point of the element at a distance z from the neutral surface has a displacement in the n-direction equal to \( u = -z \frac{\partial w}{\partial n} \).
Similarly considering the section through the t-axis, the same point will have a displacement in the t-direction equal to

\[ v = -z \frac{\partial w}{\partial t}. \]

Therefore, the shear stress will be

\[ \sigma_{nt} = -2Gz \frac{\partial^2 w}{\partial n \partial t}, \]

and the corresponding twisting moment from its definition in eqn. (21)

\[ M_{nt} = \frac{Gh}{6} \frac{\partial^2 w}{\partial n \partial t} = D(1-\nu) \frac{\partial^2 w}{\partial n \partial t}. \] (22)

From equation (21), we notice that if \( \alpha = 0 \) or \( \frac{\pi}{2} \), i.e. when the \( n \) and \( t \) directions coincide with the \( x \) and \( y \) axes, \( M_{nt} = 0 \) and there are only bending moments \( M_x \) and \( M_y \) acting on the sections perpendicularly to those axes as was assumed in Fig. 3 and in deriving the equations of this section. From equation (22) we see that the twist of the surface is proportional to \( M_{nt} \), and when \( M_{nt} = 0 \), the twist is zero. Hence the curvatures \( \frac{1}{r_x} \) and \( \frac{1}{r_y} \) are principal curvatures.
1.2.3 The differential equation of the deflection surface of laterally loaded plates

In the last section we expressed the bending moment $M_n$, $M_t$ and $M_{nt'}$, acting on a section parallel to the z-axis and whose normal makes an angle $\alpha$ with the x-axis, in terms of the deflection $w$. The $x$ and $y$-axes were considered to be principal axes. Consider now a plate under the action of loads normal to its surface. We shall assume that, at the boundary, the edges of the plate are free to move in the plane of the plate. This way the reactive forces at the edges will be normal to the plate. Together with the usual assumption that the deflections are small compared to the thickness of the plate, the strain in the middle plane may be neglected during bending.

Consider, as was done in Fig. 4, an element cut out of the plate by two pairs of planes parallel to the $xz$ and $yz$ planes (Fig. 9). We note that $M_x$ and $M_y$ are positive if they produce compression in upper layers and tension in lower layers, whereas $M_{xy}$ and $M_{yx}$ are positive if they produce rotation in the direction of the outward normal. We denote the shearing forces per unit length acting on the planes perpendicular to the $x$ and $y$-axes by $Q_x$ and $Q_y$, respectively. Therefore $Q_x$ and $Q_y$ will be given by

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} \, dz, \quad Q_y = \int_{-h/2}^{h/2} \sigma_{yz} \, dz,$$
The load will be considered distributed over the upper surface of the plate, and has the intensity \( q \, dx \, dy \). Fig. 10 represents the middle plane with the positive directions of the forces and the moments.

For equilibrium of forces in the \( z \)-direction, we have

\[
\frac{\partial Q_x}{\partial x} \, dx \, dy + \frac{\partial Q_y}{\partial y} \, dy \, dx + q \, dx \, dy = 0,
\]

from which it follows that

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0. \quad (23)
\]

Taking moments about the \( x \)-axis

\[
(Q_y + \frac{\partial Q_y}{\partial y} \, dy) \, dx \, dy + \left( -\frac{\partial Q_x}{\partial x} \, dx \right) \, dy \, \frac{dy}{2} + q \, dx \, dy \, \frac{dy}{2} - \frac{\partial M}{\partial y} \, dy \, dx + \frac{\partial M_{xy}}{\partial x} \, dx \, dy = 0.
\]
The moments due to the load \( q \) and change in \( Q_x \) and \( Q_y \) may be neglected because they are of a higher order, so that

\[
Q_y - \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = 0.
\]  

(24)

Similarly taking moments about the \( y \)-axis we obtain

\[
\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0.
\]  

(25)

Equations (23) - (25) completely define the equilibrium of the element. Substituting the values of \( Q_x \) and \( Q_y \) from (24) and (25) into (23) we obtain

\[
\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q.
\]  

(26)

We also have

\[
M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z \, dz \quad \text{and} \quad M_{yx} = -\int_{-h/2}^{h/2} \sigma_{yx} z \, dz,
\]

but since \( \sigma_{xy} = \sigma_{yx} \), it follows that

\[
M_{xy} = -M_{yx}.
\]

Therefore equation (26) reduces to

\[
\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q.
\]  

(27)

If the thickness of the plate is small compared to the other dimensions, the effect of the stress \( \sigma_{zz} \) produced by the load \( q \), and the shearing forces \( Q_x \) and \( Q_y \), on the bending of the plate may be neglected, and
we can make use of the results obtained in the last section for the case of pure bending.

Using this assumption, we can now express equation (27) in terms of the deflection \( w \). From equations (19), (20) and (22) we have

\[
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \quad M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right),
\]

and

\[
M_{xy} = -M_{yx} = D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}.
\]

Substituting these expressions in equation (27) we get

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{Q}{D},
\]

which may be re-written as

\[
\nabla^4 w = \frac{Q}{D}. \tag{29}
\]

Here \( \nabla^4 \equiv \nabla^2 \nabla^2 \) where \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian operator in the rectangular coordinates. In polar coordinates,

\[
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]

From equations (24) and (25), the shearing forces \( Q_x \) and \( Q_y \) will be given by

\[
Q_x = \frac{\partial M_y}{\partial y} + \frac{\partial M_x}{\partial x} = -D \frac{\partial}{\partial x} (\nabla^2 w),
\]

\[
Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = -D \frac{\partial}{\partial y} (\nabla^2 w). \tag{30}
\]
The problem of bending of plates by a lateral load \( q \) therefore reduces to the integration of equation (29). If the solution satisfying the given boundary conditions is found, all the relevant quantities may be computed. They are listed here for convenience.

\[
\begin{align*}
\nabla^4 w &= \frac{q}{D} \\
\sigma_{xx} &= \frac{-Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\
\sigma_{yy} &= \frac{-Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
\sigma_{xy} &= -2Gz \frac{\partial w}{\partial x \partial y} \\
M_{xy} &= D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \\
M_x &= -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\
M_y &= -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\
Q_x &= -D \frac{\partial}{\partial x} (\nabla^2 w) \\
Q_y &= -D \frac{\partial}{\partial y} (\nabla^2 w).
\end{align*}
\]

In polar coordinates

\[
\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial r} \cos \theta - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta},
\]

\[
\frac{\partial}{\partial y} \equiv \frac{\partial}{\partial r} \sin \theta + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}.
\]
We can easily obtain the corresponding values for the second derivatives \( \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial y^2} \) and \( \frac{\partial^2 U}{\partial x \partial y} \). Hence equations (31) lead to the following equivalent results in terms of polar quantities.

\[
\nabla^2 w = \frac{q}{D}
\]

\[
\sigma_{rr} = -\frac{Ez}{1-\nu} \left( \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right)
\]

\[
\sigma_{\theta\theta} = -\frac{Ez}{1-\nu} \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right)
\]

\[
\sigma_{r\theta} = -2Gz \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)
\]

\[
M_r = -D \left( \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right)
\]

\[
M_\theta = -D \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right)
\]

\[
M_{r\theta} = (1-\nu) \ D \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)
\]

\[
Q_r = -D \frac{\partial}{\partial r} (\nabla^2 w)
\]

\[
Q_\theta = - \frac{D}{r} \frac{\partial}{\partial \theta} (\nabla^2 w)
\]

1.2.4 Boundary conditions

In this section we shall discuss several types of boundary conditions for straight boundaries. In the case of rectangular plates, we assume that the x and y-axes are taken parallel to the sides of the plate. From the results for the rectangular plate, we shall obtain the
corresponding ones for boundaries of the type $\theta = \text{constant}$ in polar coordinates, and express them in terms of the deflection $w$ of the plate using the appropriate relations from (32).

(a) **Built-in edge**

The deflection $w$ along the built-in edge is zero. Furthermore, the tangent plane to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate. Assuming this built-in edge is at $x = a$, the boundary conditions are

$$w\bigg|_{x=a} = 0 , \quad \frac{\partial w}{\partial x}\bigg|_{x=a} = 0 .$$

In the case of a sector, with the built-in edge along $\theta = \alpha$, say, we get

$$w\bigg|_{\theta=\alpha} = 0 \quad \text{and} \quad \frac{\partial w}{\partial \theta}\bigg|_{\theta=\alpha} = 0 . \quad (33)$$

(b) **Simply supported edge**

If the edge $x = a$ of the plate is simply supported, the deflection $w$ along this edge is zero. Also the edge can rotate freely about the edge line, i.e. $M_x = 0$. Therefore

$$w\bigg|_{x=a} = 0 \quad \text{and} \quad \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0 .$$

But we notice that $\frac{\partial^2 w}{\partial y^2}\bigg|_{x=a} = 0$. This implies that $\frac{\partial^2 w}{\partial x^2} = 0$ and hence the boundary conditions can be written as

$$w\bigg|_{x=a} = 0 \quad \text{and} \quad V^2 w\bigg|_{x=a} = 0 .$$
Again in the case of the sector whose edge $\theta = \alpha$ is simply supported we shall obtain

$$w|_{\theta=\alpha} = 0 \quad \text{and} \quad M_{\theta}|_{\theta=\alpha} = \left[ \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right]_{\theta=\alpha} = 0.$$ 

But along $\theta = \alpha$, $\frac{\partial w}{\partial r} = \frac{\partial^2 w}{\partial r^2} = 0$. Therefore $\frac{\partial^2 w}{\partial \theta^2} = 0$, and the boundary conditions are

$$w|_{\theta=\alpha} = 0, \quad \gamma^2 w|_{\theta=\alpha} = 0. \quad (34)$$

(c) **Free edge**

In case the edge $x = a$ of the plate is free, we have no bending and twisting moments and also no vertical shearing forces, so that at first sight it appears that the appropriate boundary conditions are

$$M_x|_{x=a} = 0, \quad M_{xy}|_{x=a} = 0, \quad Q_x|_{x=a} = 0.$$ 

However the second and third conditions should be combined in one condition as follows. Consider the twisting couple $M_{xy} \frac{dy}{dy}$ produced by the horizontal forces and acting on an element of length $dy$ of the edge $x = a$.

![Figure 11](image_url)
We replace this couple by two vertical forces of magnitude $M_{xy}$ and $dy$ apart (Fig. 11). Such a replacement does not change the magnitude of twisting moments and produces only local changes in the stress distribution at the edge of the plate, leaving the stress condition of the rest of the plate unchanged. Considering two adjacent elements of the edge, the distribution of twisting moments $M_{xy}$ is statically equivalent to a distribution of shearing forces of intensity

$$Q_x' = - \left[ \frac{\partial M_{xy}}{\partial y} \right]_{x=a},$$

and therefore the joint requirement regarding $M_{xy}$ and $Q_x$ along $x = a$ becomes

$$V_x = \left[ Q_x - \frac{\partial M_{xy}}{\partial y} \right]_{x=a} = 0.$$

In terms of the deflection $w$, the necessary boundary conditions will be

$$\left[ \frac{\partial^3 w}{\partial x^3} + (2-v) \frac{\partial^3 w}{\partial x \partial y^2} \right]_{x=a} = 0,$$

and for $M_x \big|_{x=a} = 0$

$$\left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]_{x=a} = 0.$$

For a sector with a free edge $\theta = \alpha$, the corresponding boundary conditions are

$$M_{\theta} \big|_{\theta=\alpha} = 0 \text{ and } V_{\theta} = \left[ Q_{\theta} - \frac{\partial M_{x\theta}}{\partial x} \right]_{\theta=\alpha} = 0.$$
or in terms of the deflection \( w \),

\[
\left[ \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right]_{\theta=\alpha} = 0 ,
\]

and

\[
-\left[ \frac{1}{r} \frac{\partial}{\partial \theta}(\nabla^2 w) + (1-\nu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right) \right]_{\theta=\alpha} = 0 .
\]
CHAPTER II

THE SOLUTION OF CERTAIN ELASTOSTATIC PROBLEMS

2.1 Introduction

In this chapter we shall investigate the solution of certain elasto-
static problems leading to the biharmonic equation. The general method
of approach will be separation of variables, leading to the construction
of sets of basic eigenfunctions. The solutions of general boundary value
problems for the types of regions considered are found as linear combin-
ations of these eigenfunctions.

In §2.2 the problem of the cracked cylinder, previously investigated
by Williams [3], is considered. Here the body consists of a cylinder \( r < a \)
deformed in plane strain (or plane stress) by means of tractions imposed on
the boundary \( r = a \). The cylinder contains a crack running from \( r = 0 \) to
\( r = a \) on the plane \( \theta = \pi \), and the two surfaces, \( \theta = \pm \pi \), of the crack
are traction free. In §2.3 the corresponding problem is considered for
a circular plate, cracked along the radial line \( \theta = \pi \), and deformed under
bending loads. In both of these sections the problem considered is solved
to the extent of obtaining general expansions for the stress fields in terms
of the relevant eigenfunctions. The coefficients in these expansions will
be related to the tractions on \( r = a \) in the next chapter.

In section §2.4 the problem of plane deformation of a semi-infinite
strip, \(-1 < y < 1, \ 0 < x < \infty\), under tractions imposed on the end \( x = 0 \),
is discussed. Again an eigenfunction expansion method is used, and in this
section the coefficients in the expansion are related to the imposed
tractions using a method given by Gaydon and Shepherd [5]. Finally in §2.5 the general sector problem \((0 \leq r < a, -w < 0 < w)\) is discussed for the case when the edges \(\theta = \pm w\) are traction-free and given tractions are imposed on \(r = a\). The coefficients in the expansions are again obtained for this case using the method of Gopalacharyulu [6], which is a development of that of Gaydon and Shepherd [5].

2.2 The cracked cylinder

Consider the plane strain deformation of the cylindrical region \(0 \leq r < a, -\pi < \theta < \pi\) under the condition that the parts of the boundary \(\theta = \pm \pi, 0 \leq r < a\) are traction-free. Letting \(\phi(r, \theta)\) denote the Airy stress function, then the corresponding stress components are given by (5). The conditions that the crack faces, \(\theta = \pm \pi\), be traction-free are that \(\sigma_{\theta\theta} = \sigma_{r\theta} = 0\) there, or in other words

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial \theta} \right) = 0 \quad \text{on} \quad \theta = \pm \pi.
\]

Integrating these equations with respect to \(r\) gives therefore

\[
\phi = A(\theta)r + B(\theta), \quad \frac{\partial \phi}{\partial \theta} = C(\theta)r \quad \text{on} \quad \theta = \pm \pi,
\]

where \(A, B, C\) are functions of \(\theta\) only. From the continuity of \(\phi\) and appropriate derivatives at the origin, it follows that these constants are identical on \(\theta = +\pi\) and \(\theta = -\pi\). Now the Airy stress function is undefined up to linear terms in \(x\) and \(y\): if we add the function \((Ax - B + Cy)\) to \(\phi\), the stresses are unchanged, and the new stress function satisfies the boundary conditions
2.2.1 Separable solutions of the biharmonic equation

Let us begin by seeking separable solutions of the biharmonic equation \( \nabla^4 \phi = 0 \) satisfying boundary conditions (36):

\[
\phi = 0, \quad \frac{\partial \phi}{\partial \theta} = 0 \quad \text{on} \quad \theta = \pm \pi.
\]  \hspace{1cm} (36)

where \( R \) is a function of \( r \) only and \( F \) is a function of \( \theta \) only.

Then equation (3) will be:

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( (R'' + \frac{R'}{r})F + \frac{R}{r^2} F'' \right) = 0,
\]

or

\[
f(r)F + \frac{1}{r^2}(2R'' - \frac{2R'}{r} + \frac{4R}{r^2})F'' + \frac{R}{r^4} F^{IV} = 0,
\]

where \( f(r) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)(R'' + \frac{R'}{r}). \)

Multiplying throughout by \( \frac{r^4}{RF} \) and differentiating with respect to \( r \) and \( \theta \) we obtain

\[
\frac{d}{dr} \left( \frac{1}{R} (2r^2 R'' - 2r R' + 4R) \right) \frac{d}{d\theta} (\frac{F''}{F}) = 0.
\]

Thus we have the two possible cases:

(a) \( \frac{F''}{F} \) = const., \( \lambda^2 \) say,

\[
i.e. \quad F'' - \lambda^2 F = 0 \hspace{1cm} (37)
\]

(b) \( \frac{2}{R}(r^2 R'' - r R' + 2R) = \) const., \( 2\mu \) say,

\[
i.e. \quad r^2 R'' - r R' + (2-\mu)R = 0. \hspace{1cm} (38)
\]
The boundary conditions (36) imply that

\[ F(\pi) = F(-\pi) = F'(\pi) = F'(-\pi) = 0. \quad (39) \]

The solution of (37) is

\[ F(\theta) = A e^{\lambda \theta} + B e^{-\lambda \theta}, \quad \lambda \neq 0. \]

It is easy to check that the above boundary conditions will lead only to the trivial solution. Similarly, if \( \lambda = 0 \), the differential equation will be

\[ F''(\theta) = 0, \]

and the solution is \( F(\theta) = A + B \theta \). Again the boundary conditions will give only the trivial solution. Finally the case when \( \lambda^2 \) is negative may also be shown to have only the trivial solution.

Therefore, we have to consider Euler's equation (38). Its associated indicial equation is:

\[ m(m-1) - m + 2 - \mu = 0 \]

or \( m^2 - 2m + 2 - \mu = 0 \). If we let

\[ \lambda = \sqrt{\mu - 1} \quad (40) \]

the two roots will be given by

\[ m_1 = 1 + \lambda, \quad m_2 = 1 - \lambda, \quad (41) \]

and therefore the solution of equation (38) is

\[ R(x) = A x^{1+\lambda} + B x^{1-\lambda}, \quad \lambda \neq 0, 1. \quad (42) \]
The cases \( \lambda = 0, 1 \) will be considered later.

In order to have a finite strain energy in the neighbourhood of the crack-tip \( r = 0 \), we require that the stresses are \( O \left( \frac{1}{r} \right) \) as \( r \to 0 \). This condition will imply that \( B = 0 \) when \( \lambda > 0 \). Therefore the stress function \( \phi \) can be written in the form

\[
\phi = r^{1+\lambda} F(\theta) .
\] (43)

Substituting this expression for \( \phi \) in the biharmonic equation, we obtain

\[
\frac{d^4 F}{d\theta^4} + 2(\lambda^2 + 1) \frac{d^2 F}{d\theta^2} + (\lambda^2 - 1)^2 F = 0 .
\] (44)

The general solution of equation (44) is

\[
F(\theta) = A \cos(\lambda + 1) \theta + B \cos(\lambda - 1) \theta + C \sin(\lambda + 1) \theta + D \sin(\lambda - 1) \theta .
\]

Here, we shall discuss only the symmetric stress distribution, i.e. \( F \) will be an even function of \( \theta \). (For antisymmetric stress distribution, the method of solution will be the same.) Then \( F \) may be written now as

\[
F(\theta) = A \cos(\lambda + 1) \theta + B \cos(\lambda - 1) \theta .
\] (45)

Since we are considering the symmetric solution, only the boundary conditions

\[
F(\pi) = F'(\pi) = 0
\] (46)

will be relevant. Applying the boundary conditions (46) on equation (45), we obtain

\[
(A + B) \cos \lambda \pi = 0 ,
\]
Here we shall have two cases:

Case i) \( A + B \neq 0 \).

This will imply that

\[
\cos \lambda \pi = 0 \quad \text{and} \quad A(\lambda+1) + B(\lambda-1) = 0 ,
\]

or \( \lambda_n^{(1)} = \frac{2n-1}{2}, n = 1,2,3,... \) since \( \lambda_n > 0 \). Substituting these eigenvalues in (45), we get

\[
P_n^{(1)}(\theta) = a_n \left[ \cos \left( n + \frac{1}{2} \right) \theta - \frac{n + \frac{1}{2}}{n - \frac{3}{2}} \cos \left( n - \frac{3}{2} \right) \theta \right]
\]

\[
= a_n \left[ \frac{\cos \left( n + \frac{1}{2} \right) \theta - \cos \left( n - \frac{3}{2} \right) \theta}{n + \frac{1}{2}} \right].
\]

Case ii) \( A + B = 0 \)

This implies that

\[
\sin \lambda \pi = 0
\]

or \( \lambda_n^{(2)} = n \quad n = 2, 3, 4, ... \) and the corresponding solution is

\[
P_n^{(2)}(\theta) = b_n \left[ \cos(n-1)\theta - \cos(n+1)\theta \right].
\]

Now we shall return to the cases \( \lambda = 0, 1 \). For \( \lambda = 0 \), the solution of (38) will be

\[
R(r) = r[a + b ln r].
\]

The corresponding even function \( F(\theta) \), from (44), is \( F(\theta) = A \cos \theta + B \theta \sin \theta \), and the conditions \( F(\pi) = F'(\pi) = 0 \) imply that \( A = B = 0 \).
Thus there is no eigenfunction for $\lambda = 0$. For $\lambda = 1$, $\mu$ will be equal to 2, and equation (38) will be

$$r^2 R'' - r R' = 0.$$ 

Its solution is $R(r) = ar^2 + b$. We must take $b = 0$ to avoid too singular behaviour at $r = 0$. The remaining $r^2$ term leads to an eigenfunction of the same type as case ii) above. Therefore

$$\phi_n^{(2)} = r^{1+\lambda_n^{(2)}} F_n^{(2)}(\theta), \quad \lambda_n^{(2)} = 1, 2, 3, \ldots.$$ Here we note that, for $n = 1$, $F_1^{(2)}(\theta) = b_1[1 - \cos 2\theta]$.

It follows that the general even solution of the biharmonic equation (3) satisfying the boundary conditions (46) is

$$\phi(r, \theta) = \sum_{n=1}^{\infty} \sum_{j=1,2} r^{1+\lambda_n^{(j)}} F_n^{(j)}(\theta) = \sum_{n=1}^{\infty} \phi_n(r, \theta)$$

where

$$\phi_n(r, \theta) = a_n r^{n+\frac{1}{2}} \left[ \cos \left( \frac{n+1}{2} \theta \right) - \cos \left( \frac{n-3}{2} \theta \right) \right] + b_n r^{n+1} \left[ \cos \left( n \theta - \cos (n+1) \theta \right) \right]$$

$n = 1, 2, 3, \ldots$

2.2.2 Expansion of stresses in terms of the basic eigenfunctions

By using equations (5), the stresses will be

$$\sigma_{rr}(r, \theta) = \sum_{n=1}^{\infty} \left\{ a_n r^{n-3} \left[ \frac{1-(n+1)}{2} \cos \left( \frac{n+1}{2} \theta \right) + \frac{n+1}{2} \cos \left( \frac{n-3}{2} \theta \right) \right] 
+ b_n r^{n-1} \left[ (n+1)-(n-1) \cdot \cos (n-1) \theta + (n+1)^2 \cdot \cos (n+1) \theta \right] \right\}$$

$$= \sum_{n=1}^{\infty} \left\{ a_n r^{n-3} \left[ \frac{1}{2-n} \left[ \cos \left( \frac{n+1}{2} \theta \right) - \frac{n-7}{n-2} \cos (n-3) \theta \right] 
+ b_n r^{n-1} \left[ -3 \cos (n-1) \theta + (n+1) \cos (n+1) \theta \right] \right\}.$$
Let

\[ A_n = a_n \left( n - \frac{1}{2} \right), \]
\[ B_n = n b_n. \]  

Then the normal stress at any point \((r, \theta)\) in the considered domain will be given by

\[
\sigma_{rr}(r, \theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{n-3} \left[ -\cos \left( n + \frac{1}{2} \right) \theta + \frac{n - \frac{1}{2}}{n - \frac{3}{2}} \cos \left( n - \frac{3}{2} \right) \theta \right] \right. \\
\left. + B_n r^{n-1} \left[ (n+1) \cos(n+1)\theta - (n-3) \cos(n-1)\theta \right] \right\}. 
\]

(49)

Similarly the shear and tangential stresses are given by

\[
\sigma_{r\theta}(r, \theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{n-3} \left[ \sin \left( n + \frac{1}{2} \right) \theta - \sin \left( n - \frac{3}{2} \right) \theta \right] \right. \\
\left. + B_n r^{n-1} \left[ (n-1) \sin(n-1)\theta - (n+1) \sin(n+1)\theta \right] \right\}. 
\]

(50)

\[
\sigma_{\theta\theta}(r, \theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{n-3} \left[ \cos \left( n + \frac{1}{2} \right) \theta - \frac{n + \frac{1}{2}}{n - \frac{3}{2}} \cos \left( n - \frac{3}{2} \right) \theta \right] \right. \\
\left. + B_n r^{n-1} \left[ (n+1) \cos(n+1)\theta - (n-1) \cos(n-1)\theta \right] \right\}. 
\]

(51)

The constants \(A_n\) and \(B_n\), \(n = 1, 2, \ldots\), are determined from the boundary conditions on \(r = a\), where \(a\) is a fixed radius in the case of a finite domain, or at infinity in the case of an infinite domain.

It may be interesting at this point to consider the term

\[ B_1 r^2 \left[ 1 - \cos 2\theta \right] \]

appearing in the stress function (47). This term corresponds to the case where \(\lambda = 1\). In rectangular coordinates this term may be written as \(B_1 r^2 \left[ 1 - \cos 2\theta \right] = 2B_1 r^2 \sin^2 \theta = 2B_1 y^2\).
The corresponding terms for the stresses, from equation (4), are given by

\[ \sigma_{xx} = 4B_1, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0. \]

If \( \sigma_{xx} = 0 \) along some straight boundary \( x = -x_0 \) (Fig. 12), the constant \( B_1 \) will be equal to zero.

![Figure 12](image)

2.2.3 **Radial stress variations near the crack tip**

Equations (49) - (51) will all be of the form \( r^{-\frac{1}{2}} + O(r^{\frac{1}{2}}) \) with respect to the radial variation. The local stress variations in the vicinity of the base of the crack, \( r \to 0 \), are dominated by the contribution of the first term. It is also noted that along the line of propagation of the crack, \( \theta = 0 \), the shear stress is zero. Hence \( \sigma_{rr}(r, 0) \) and \( \sigma_{\theta\theta}(r, 0) \) are principal stresses; we denote them by \( \sigma_1 \), \( \sigma_2 \) respectively, so that

\[ \sigma_1(r, 0), \sigma_2(r, 0) \sim 4A_1 r^{-\frac{1}{2}} \text{ as } r \to 0. \]

In other words, at the base of the crack there exists a strong tendency toward a state of two-dimensional hydrostatic tension which consequently
may permit the elastic analysis to apply close to the crack-tip, notwithstanding the square root stress singularity. It has been suggested that this feature would tend to reduce the amount or area of plastic flow at the crack-tip which might ordinarily be expected to exist under such high stress magnitudes and lead therefore toward more of a brittle type failure.

2.2.4 Angular variations of the principal stresses and the distortional strain energy density

From (49) - (51), the singular terms in the stresses as \( r \to 0 \) are

\[
\sigma_{rr} \sim A_1 r^{-\frac{1}{2}} \left[ -\cos \frac{3\theta}{2} + 5\cos \frac{\theta}{2} \right],
\]
\[
\sigma_{r\theta} \sim A_1 r^{-\frac{1}{2}} \left[ \sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right],
\]
\[
\sigma_{\theta\theta} \sim A_1 r^{-\frac{1}{2}} \left[ \cos \frac{3\theta}{2} + 3\cos \frac{\theta}{2} \right].
\]

The principal stresses are given by the expression

\[
\sigma_{1,2} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} \pm \sqrt{\left( \frac{\sigma_{rr} - \sigma_{\theta\theta}}{2} \right)^2 + \sigma_{r\theta}^2}
\]
\[
\sim 4A_1 r^{-\frac{1}{2}} \cos \frac{\theta}{2} \left[ 1 \pm \sin \frac{\theta}{2} \right].
\]

The maximum principal stress occurs when \( \frac{\partial \sigma_1}{\partial \theta} = 0 \). This condition implies that \( \cos \theta = \sin \frac{\theta}{2} \), i.e. the maximum principal stress occurs at \( \theta = \frac{\pi}{3} \). The value of this maximum stress is

\[
\sigma_{1,\text{max}} = \frac{A_1}{r^2} \cdot 3\sqrt{3} \approx 5.2 \frac{A_1}{r^2}.
\]

Also the expression for the distortion strain energy density, i.e. the total strain energy less that due to change in volume, per unit volume,
is given by

\[ W_d(\theta) = \frac{1+\nu}{6E} \left[ \left( \sigma_1 - \sigma_2 \right)^2 + \sigma_1^2 + \sigma_2^2 \right] \]

\[ = \frac{4A_1^2}{Er} \cos^2 \frac{\theta}{2} \left[ (1+\nu) \sin^2 \frac{\theta}{2} + \frac{1+\nu}{3} \right]. \]

From this expression we obtain

\[ W_d(0) = \frac{4A_1^2}{3Er} (1+\nu), \]

and

\[ \frac{W_d(\theta)}{W_d(0)} = \frac{3}{1+\nu} \cos^2 \frac{\theta}{2} \left[ (1+\nu) \sin^2 \frac{\theta}{2} + \frac{1+\nu}{3} \right]. \]

Williams [3] shows that the two principal stresses \( \sigma_1 \) and \( \sigma_2 \) have the angular variation shown in Fig. 13, and that the distortional strain energy density is as shown in Fig. 14.

It is interesting to note that because of the hydrostatic tendency, the maximum energy of distortion does not occur along the line of crack.
direction, \( \theta = 0 \), but rather at \( \theta^* = \pm \cos \frac{1}{3} \approx \pm 70 \text{ deg.} \), where it is one-third higher.

2.3 The bending stress distribution at the base of a stationary crack

Following the same procedure as for the extensional stress distribution, Williams [4] studied the stresses around a crack point owing to bending loads.

The problem is formulated as follows. We have to satisfy the differential equation

\[
\nabla^4 w(r, \theta) = \frac{q(r, \theta)}{D},
\]

where \( q \) is the applied load on the plate. For a plate subject to edge loading only, \( q = 0 \). We shall take the edges \( \theta = \pm \pi \) as free edges. Then, from (35)

\[
\begin{bmatrix}
\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \\
\frac{1}{r} \frac{\partial (\nu^2 w)}{\partial \theta} + (1-\nu) \left( \frac{1}{r} \frac{\partial \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} \right)}{\partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta^2} \right)
\end{bmatrix} = 0,
\]

Again, as section 2.2, the symmetric characteristic solutions are of the form

\[
w_n(r, \theta) = R(r) F_n(\theta) = r^{1+\lambda_n} \left[ A \cos(\lambda_n + 1) \theta + B \cos(\lambda_n - 1) \theta \right].
\]

In order to satisfy the physical boundary condition of finite slope at the origin, \( \lambda_n > 0 \). Applying the boundary conditions (53), we have

\[
\begin{bmatrix}
\frac{1}{r} \frac{\partial R'}{\partial r} F + \frac{1}{r^2} \frac{\partial R}{\partial r} F'' + \nu R'' F
\end{bmatrix} = 0.
\]
or, denoting \( \lambda_n \) by \( \lambda \),

\[
\left[ r^{\lambda-1} F'' + (1+\lambda)(1+\lambda\nu) F \right]_{\theta=\pi} = 0
\]

or

\[
F''(\pi) + (1+\lambda)(1+\lambda\nu) F(\pi) = 0 . \quad (54)
\]

The second of equations (53) will be

\[
\left\{ \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) RF + (1-\nu) \frac{\partial}{\partial r} \left( \frac{1}{r} R' \right) F' - \frac{1}{r^2} R F' \right\}_{\theta=\pi} = 0
\]

or

\[
F'(\pi) \left\{ (\lambda+1)^2 + (1-\nu) \lambda(\lambda-1) \right\} + F'''(\pi) = 0 . \quad (55)
\]

Now

\[
F(\theta) = A \cos(\lambda+1)\theta + B \cos(\lambda-1)\theta .
\]

Therefore

\[
F(\pi) = -(A+B) \cos \lambda \pi ,
\]

\[
F'(\pi) = [(\lambda+1) A + (\lambda-1) B] \sin \lambda \pi ,
\]

\[
F''(\pi) = [(\lambda+1)^2 A + (\lambda-1)^2 B] \cos \lambda \pi ,
\]

\[
F'''(\pi) = -[(\lambda+1)^3 A + (\lambda-1)^3 B] \sin \lambda \pi .
\]

Substituting these values in (54) we get

\[
[-(1+\lambda)(1+\lambda\nu)(A+B) + (\lambda+1)^2 A + (\lambda-1)^2 B] \cos \lambda \pi = 0 ,
\]

which reduces to

\[
\left\{ (\lambda+1)(1-\nu) A + [\lambda(1-\nu) - (3+\nu)] B \right\} \cos \lambda \pi = 0 . \quad (56)
\]
Equation (55) will give
\[
\{(\lambda+1)A + (\lambda-1)B\}[(\lambda+1)^2 + (1-\nu)\lambda(\lambda-1)] - (\lambda+1)^3A - (\lambda-1)^3B\} \sin \lambda \pi = 0,
\]
which reduces to
\[
\{A(\lambda+1)(1-\nu) + B[4 + (1-\nu)(\lambda-1)]\} \sin \lambda \pi = 0 .
\]  
(57)

Now if \(\cos \lambda \pi = 0,\) or \(\lambda^{(1)}_n = \frac{2n-1}{2},\) \(n = 1, 2, 3, \ldots,\) this will imply that
\[
A(\lambda^{(1)}_{n+1})(1-\nu) + B[4 + (1-\nu)(\lambda^{(1)}_n - 1)] = 0 ,
\]
or
\[
B^{(1)} = \frac{-A^{(1)}}{4(1-\nu)(n-\frac{3}{2})} \frac{(n+\frac{1}{2})(1-\nu)}{2}. \]

Therefore
\[
\omega^{(1)}_n = a_n \sqrt{\frac{1}{2}} \left[\cos(n+\frac{1}{2})\theta - \frac{(n+\frac{1}{2})(1-\nu)}{4(1-\nu)(n-\frac{3}{2})} \cos(n-\frac{3}{2})\theta\right].
\]

On the other hand if \(\sin \lambda \pi = 0,\) or \(\lambda^{(2)}_n = n,\) \(n = 1, 2, 3, \ldots,\) this will imply that
\[
(n+1)(1-\nu) A + [n(1-\nu) - (3+\nu)] B = 0
\]
or
\[
B^{(2)} = \frac{-A^{(2)}}{n(1-\nu)-(3+\nu)} \frac{(n+1)(1-\nu)}{n}. \]

and hence
\[
\omega^{(2)}_n = b_n \left[\cos(n+1)\theta - \frac{(n+1)(1-\nu)}{n(1-\nu)-(3+\nu)} \cos(n-1)\theta\right].
\]

From the above, the general solution of (52) satisfying (53) will be
This equation may also be written in the form

\[
W(r, \theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{n+\frac{1}{2}} \left[ \cos \left( n + \frac{1}{2} \right) \theta - \frac{(n + \frac{1}{2}) (1 - \nu)}{4 + (1 - \nu) \left( n - \frac{3}{2} \right)} \cos \left( n - \frac{3}{2} \right) \theta \right] + B_n r^{n+1} \left[ \cos (n+1) \theta \cos \left( n - \frac{3}{2} \right) \theta \right] \right\}.
\]
\[ \sigma_{\theta\theta} = -2Gz \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right) \]

\[ = -2Gz \sum_{n=1} A_n r^{n-2} \left[ -\sin\left( n+\frac{1}{2} \right) \theta + \frac{(n-\frac{3}{2}) (1-\nu)}{(n+\frac{5}{2})-\nu(n-\frac{3}{2})} \sin(n-\frac{3}{2}) \theta \right] \]

\[ + B_n r^{n-1} \left[ -\sin(n+1) \theta + \frac{(n-1)(1-\nu)}{(n-3)-\nu(n+1)} \sin(n-1) \theta \right] \]

Using the relation \( G = \frac{E}{2(1+\nu)} \), where \( G \) is the modulus of elasticity in shear, the stresses will be

\[ \sigma_{rr} = -2Gz \sum_{n=1} A_n r^{n-2} \left[ \cos\left( n+\frac{1}{2} \right) \theta - \frac{(n+\frac{1}{2})-\nu(n-\frac{7}{2})}{(n+\frac{5}{2})-\nu(n-\frac{3}{2})} \cos(n-\frac{3}{2}) \theta \right] \]

\[ + B_n r^{n-1} \left[ \cos(n+1) \theta - \frac{(n+1)-\nu(n-3)}{(n-3)-\nu(n+1)} \cos(n-1) \theta \right] \], (59)

\[ \sigma_{\theta\theta} = -2Gz \sum_{n=1} A_n r^{n-2} \left[ -\cos\left( n+\frac{1}{2} \right) \theta + \frac{(n-\frac{7}{2})-\nu(n+\frac{1}{2})}{(n+\frac{5}{2})-\nu(n-\frac{3}{2})} \cos(n-\frac{3}{2}) \theta \right] \]

\[ + B_n r^{n-1} \left[ -\cos(n+1) \theta + \cos(n-1) \theta \right] \] , (60)

\[ \sigma_{r\theta} = -2Gz \sum_{n=1} A_n r^{n-2} \left[ -\sin\left( n+\frac{1}{2} \right) \theta + \frac{(n-\frac{3}{2}) (1-\nu)}{(n+\frac{5}{2})-\nu(n-\frac{3}{2})} \sin(n-\frac{3}{2}) \theta \right] \]

\[ + B_n r^{n-1} \left[ -\sin(n+1) \theta + \frac{(n-1)(1-\nu)}{(n-3)-\nu(n+1)} \sin(n-1) \theta \right] \]. (61)

In equations (59) - (61), we notice that the stresses have the same characteristic square root singularity as in the case of extension considered in §2.2.
2.4 Plane stresses in a semi-infinite strip

In the two problems considered so far, Williams expressed the stresses in a series of non-orthogonal eigenfunctions. This creates considerable difficulty when it comes to obtaining the coefficients in the expansions in terms of the prescribed boundary values. In solving a similar problem for a semi-infinite strip, Gaydon and Shepherd [5] expanded each of the eigenfunctions in a series of orthogonal functions. This way it was possible to obtain the coefficients of the stress function corresponding to any arbitrary distribution of traction on the end of the strip directly from two numerical matrices.

2.4.1 Solution of the biharmonic equation

The problem is formulated as follows. We have to determine the stress function \( \phi \) which is the solution of the biharmonic equation

\[
\nabla^4 \phi = 0 \quad (-1 < y < 1, \ x > 0),
\]

and corresponds to zero tractions on \( y = \pm 1 \). The tractions on \( x = 0 \) are prescribed, so that \( \phi \) satisfies the boundary conditions

\[
\left\{ \begin{align*}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial y} &= 0 \quad \text{on} \quad y = \pm 1 \\
\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} &= p(y) , \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = s(y) \quad \text{on} \quad x = 0.
\end{align*} \right. \]

We further assume that \( \phi \to 0 \) as \( x \to \infty \). Here we shall consider \( p(y) \) and \( s(y) \) to be even and odd functions of \( y \) respectively.
First of all, we look for separable solutions,

$$\phi(x, y) = X(x) Y(y),$$

(64)

where $X(x)$, $Y(y)$ are respectively functions of $x$ only and $y$ only. $Y(y)$ must be an even function, satisfying the boundary conditions $Y(1) = Y'(1) = 0$. Then equation (62) will reduce to

$$X^{iv} Y + 2X^{iv} Y'' + X Y^{iv} = 0.$$

(65)

Dividing (65) by $X Y$ and differentiating with respect to $x$ and $y$ we obtain

$$\frac{\partial}{\partial x} \left( \frac{X''}{X} \right) \frac{\partial}{\partial y} \left( \frac{Y''}{Y} \right) = 0.$$

This implies that either

i) $$\frac{X''}{X} = \text{const.} \lambda^2$$ say

or

ii) $$\frac{Y''}{Y} = \text{const.} \mu^2$$ say.
If $Y'' - \mu^2 Y = 0$, the even solution is

$$Y = A \cosh \mu y.$$ 

By applying the boundary conditions $Y(1) = Y'(1) = 0$, we obtain only the trivial solution. In the same way, if $\mu^2 \leq 0$ we obtain no non-trivial solutions.

Therefore consider the differential equation

$$X'' - \lambda^2 X = 0.$$ 

Its general solution is

$$X = A e^{\lambda x} + B e^{-\lambda x}, \quad \text{Re} \lambda > 0.$$ 

Considering the requirement that $X \to 0$ as $x \to \infty$, we must have $A = 0$. This requirement will also exclude the possibility that $\text{Re} \lambda = 0$. Hence

$$X(x) = B e^{-\lambda x}, \quad \text{Re} \lambda > 0.$$ \hspace{1cm} (66)

Substituting (66) in (65) we obtain the fourth order ordinary differential equation

$$\frac{d^4 Y}{dy^4} + 2\lambda^2 \frac{d^2 Y}{dy^2} + \lambda^4 Y = 0.$$ 

Its general solution is

$$Y(y) = (A+By) \cos \lambda y + (C+Dy) \sin \lambda y.$$ 

Considering the even solution, we obtain

$$Y(y) = A \cos \lambda y + Dy \sin \lambda y.$$ \hspace{1cm} (67)

Now applying the boundary conditions $Y(1) = Y'(1) = 0$ on (67) we get
\sin 2\lambda + 2\lambda = 0 \quad (68)

and \( \frac{A}{D} = -\tan \lambda \).

If we take \( D = -2\lambda \), the constant \( A \) will be

\[ A = \cos 2\lambda - 1 \ . \]

The stress function may now be written as

\[ \phi = \sum_{\lambda} \left\{ (A_\lambda + iB_\lambda) \frac{e^{-\lambda x}}{\lambda^2} Y_\lambda(y) + (A_\lambda - iB_\lambda) \frac{e^{-\lambda x}}{\lambda^2} \overline{Y}_\lambda(y) \right\} , \quad (69) \]

where

\[ Y_\lambda(y) = (\cos 2\lambda - 1) \cos \lambda y - 2ky \sin \lambda y \, , \quad (70) \]

and

\[ \lambda = a_\lambda + ib_\lambda \quad (a_\lambda > 0 \, , \, b_\lambda > 0) \quad (71) \]

are the roots of equation (68). The constants \( A_\lambda, B_\lambda \) are real and are determined from the end stress distributions \( p(y) \) and \( s(y) \), i.e.

\[ p(y) = \frac{\partial^2 \phi}{\partial y^2} \bigg|_{x=0} = \sum_{\lambda} \left\{ \frac{A_\lambda + iB_\lambda}{\lambda^2} Y''_\lambda(y) + \frac{A_\lambda - iB_\lambda}{\lambda^2} \overline{Y}'_\lambda(y) \right\} , \quad (72) \]

and

\[ s(y) = -\frac{\partial^2 \phi}{\partial x \partial y} \bigg|_{x=0} = \sum_{\lambda} \left\{ \frac{A_\lambda + iB_\lambda}{\lambda} Y'_\lambda(y) + \frac{A_\lambda - iB_\lambda}{\lambda} \overline{Y}'_\lambda(y) \right\} . \quad (73) \]

2.4.2 Expansion of the stress function in terms of orthogonal beam functions

The functions \( Y_\lambda(y) \) are not orthogonal, so we expand each of them in terms of the beam functions \( F_k(y) \) which are complete and possess orthogonal
properties in the range $-1 < y < 1$. $F_k(y)$ satisfies the equation
\begin{equation}
F_k^{iv}(y) - k^4 F_k(y) = 0 ,
\end{equation}
and the boundary conditions
\begin{equation}
F_k = F'_k = 0 \quad \text{on} \quad y = \pm 1 .
\end{equation}

The general even solution of equation (74) is
\begin{equation}
F_k^{(1)}(y) = A \cos ky + b \cosh ky .
\end{equation}

Applying the boundary conditions (75), we obtain
\begin{equation}
\tanh k + \tanh k = 0 \quad \text{and} \quad \frac{A}{B} = - \frac{\cosh k}{\cos k} .
\end{equation}

Therefore
\begin{equation}
F_k^{(i)}(y) = \cos k \cosh k \left( \frac{\cos ky}{\cos k} - \frac{\cosh ky}{\cosh k} \right) .
\end{equation}

The normalized solution will be
\begin{equation}
F_k(y) = \frac{1}{\sqrt{2}} \left( \frac{\cos ky}{\cos k} - \frac{\cosh ky}{\cosh k} \right) ,
\end{equation}
where the norm is
\begin{equation}
\left\{ \int_{-1}^{1} [F_k^{(i)}(y)]^2 dy \right\}^{1/2} .
\end{equation}

It is easy to check the following orthogonality relations which will be useful in the expansions
\begin{equation}
\int_{-1}^{1} F_m F_n dy = \delta_{mn} ,
\end{equation}
\begin{equation}
\int_{-1}^{1} F''_m F''_n dy = k_n^4 \delta_{mn} ,
\end{equation}
\begin{equation}
\int_{-1}^{1} F'_m F''_n dy = -k_n^4 \delta_{mn} .
\end{equation}
where $\delta_{mn}$ is the Kronecker delta.

The functions $F_k(y)$ are real, which simplifies the evaluation of the coefficients; furthermore, since they satisfy a fourth-order differential equation, with the same four boundary conditions, as do the original functions $Y_{\lambda}(y)$, they give rise to expansions of the latter which are considerably more convergent than would be obtained by Fourier series.

The expansion of $Y_{\lambda}(y)$ will be

$$Y_{\lambda}(y) = (\cos 2\lambda - 1) \cos \lambda y - 2\lambda \sin \lambda y = \sum_{i=1}^{\infty} a_{i\lambda} F_i(y).$$

Multiplying both sides by $F_j$ and integrating from $-1$ to $+1$, then

$$a_{j\lambda} = 4\sqrt{2} \lambda^2 \left[ \lambda \sin \lambda + k_j \cos \lambda \tanh k \right] \left[ \frac{1}{(k_j^2 + \lambda^2)} - \frac{1}{(k_j^2 - \lambda^2)^2} \right]. \quad (79)$$

We now have

$$\phi = \sum_{i=1}^{\infty} \sum_{\lambda} \left\{ \frac{A_{\lambda} + iB_{\lambda}}{\lambda^2} e^{-\lambda x} a_{i\lambda} + \frac{A_{\lambda} - iB_{\lambda}}{\lambda^2} e^{-\lambda x} a_{i\lambda} \right\} F_i(y),$$

and the stresses will be

$$\sigma_{xx} = \sum_{i=1}^{\infty} \sum_{\lambda} \left\{ \frac{A_{\lambda} + iB_{\lambda}}{\lambda^2} e^{-\lambda x} a_{i\lambda} + \frac{A_{\lambda} - iB_{\lambda}}{\lambda^2} e^{-\lambda x} a_{i\lambda} \right\} F_i''(y), \quad (80)$$

$$\sigma_{xy} = \sum_{i=1}^{\infty} \sum_{\lambda} \left\{ \frac{A_{\lambda} + iB_{\lambda}}{\lambda} e^{-\lambda x} a_{i\lambda} + \frac{A_{\lambda} - iB_{\lambda}}{\lambda} e^{-\lambda x} a_{i\lambda} \right\} F_i'(y). \quad (81)$$

2.4.3 Evaluation of the constants $A_{\lambda}$ and $B_{\lambda}$

Now in order to evaluate the constants $A_{\lambda}$ and $B_{\lambda}$, we expand the given functions $p(y)$ and $s(y)$ in terms of $F''$ and $F'$ respectively. Thus
\[ p(y) = \sum_{i=1}^{\infty} \alpha_i F_i'(y), \]

where
\[ \alpha_i = \frac{1}{k_i^4} \int_{-1}^{1} p(y) F_i''(y) \, dy. \]  

Similarly
\[ s(y) = \sum_{i=1}^{\infty} \beta_i F_i'(y), \]

where
\[ -\beta_i = \frac{1}{k_i^4} \int_{-1}^{1} s(y) F_i''(y) \, dy. \]

From equations (80) and (81), with \( x = 0 \), we obtain
\[ \sum_{i=1}^{\infty} \alpha_i F_i'' = \sum_{i=1}^{\infty} \left\{ \frac{A_{\lambda+iB_\lambda}}{\lambda^2} a_{i\lambda} + \frac{A_{\lambda-iB_\lambda}}{\lambda^2} \frac{a_{i\lambda}}{a_{i\lambda}} \right\} F_i'' , \]

and
\[ \sum_{i=1}^{\infty} \beta_i F_i' = \sum_{i=1}^{\infty} \left\{ \frac{A_{\lambda+iB_\lambda}}{\lambda} a_{i\lambda} + \frac{A_{\lambda-iB_\lambda}}{\lambda} \frac{a_{i\lambda}}{a_{i\lambda}} \right\} F_i'. \]

These are satisfied if
\[ \sum_{\lambda} \left\{ \frac{A_{\lambda+iB_\lambda}}{\lambda^2} a_{i\lambda} + \frac{A_{\lambda-iB_\lambda}}{\lambda^2} \frac{a_{i\lambda}}{a_{i\lambda}} \right\} = \alpha_i \quad i = 1, 2, \ldots , \]  

and
\[ \sum_{\lambda} \left\{ \frac{A_{\lambda+iB_\lambda}}{\lambda} a_{i\lambda} + \frac{A_{\lambda-iB_\lambda}}{\lambda} \frac{a_{i\lambda}}{a_{i\lambda}} \right\} = \beta_i \quad i = 1, 2, 3, \ldots . \]

Now
\[ \frac{a_{i\lambda}}{\lambda^2} = 4\sqrt{2} \left[ \lambda \sin \lambda + k_i \cos \lambda \tanh k_i \right] \left[ \frac{1}{(k_i^2 + \lambda^2)^2} - \frac{1}{(k_i^2 - \lambda^2)^2} \right] \]

\[ = a'_{i\lambda} + ib'_{i\lambda} \text{ say .} \]

Also let

\[ \frac{1}{(k_i^2 + \lambda^2)^2} = \gamma_{i\lambda} + i\delta_{i\lambda} \]

and

\[ \frac{1}{(k_i^2 - \lambda^2)^2} = \epsilon_{i\lambda} + i\zeta_{i\lambda} . \]

Then \( a'_{i\lambda} , b'_{i\lambda} \) can be written as

\[ a'_{i\lambda} = 4\sqrt{2} \left\{ (p_\lambda + q_\lambda) \left[ k_i \tanh k_i \right] (\gamma_{i\lambda} - \epsilon_{i\lambda}) - (q_\lambda + s_\lambda) k_i \tanh k_i \left( \delta_{i\lambda} - \zeta_{i\lambda} \right) \right\} , \]

\[ b'_{i\lambda} = 4\sqrt{2} \left\{ (q_\lambda + s_\lambda) k_i \tanh k_i \left( \gamma_{i\lambda} - \epsilon_{i\lambda} \right) + (p_\lambda + r_\lambda) k_i \tanh k_i \left( \delta_{i\lambda} - \zeta_{i\lambda} \right) \right\} , \]

where

\[ p_\lambda = a_\lambda \sin a_\lambda \cosh b_\lambda - b_\lambda \cos a_\lambda \sinh b_\lambda \]

\[ q_\lambda = -b_\lambda \sin a_\lambda \cosh b_\lambda - a_\lambda \cos a_\lambda \sinh b_\lambda \]

\[ r_\lambda = \cos a_\lambda \cosh b_\lambda \]

\[ s_\lambda = \sin a_\lambda \sinh b_\lambda \]

Equations (84) and (85) now reduce to

\[ \sum_{\lambda} \left\{ A_\lambda (2a'_{i\lambda}) - B_\lambda (2b'_{i\lambda}) \right\} = \alpha_i \]

and
\[ \sum_{\lambda} \{ A_{\lambda} (2a_{i\lambda} a'_{i\lambda} + 2b_{i\lambda} b'_{i\lambda}) + B_{\lambda} (2b_{i\lambda} a'_{i\lambda} - 2a_{i\lambda} b'_{i\lambda}) \} = -\beta_i. \]

If we put \( 2a_{i\lambda}' = C_{i\lambda}, \ -2b_{i\lambda}' = D_{i\lambda}, \)

\[ 2a_{i\lambda} a'_{i\lambda} + 2b_{i\lambda} b'_{i\lambda} = E_{i\lambda}, \quad 2b_{i\lambda} a'_{i\lambda} - 2a_{i\lambda} b'_{i\lambda} = F_{i\lambda}, \]

then

\[ \sum_{\lambda} (A_{\lambda} C_{i\lambda} + B_{\lambda} D_{i\lambda}) = \alpha_i \quad i = 1, 2, 3, ... \tag{86} \]

\[ \sum_{\lambda} (A_{\lambda} E_{i\lambda} + B_{\lambda} F_{i\lambda}) = -\beta_i \quad i = 1, 2, 3, ... \tag{87} \]

The matrices \((C_{i\lambda}), (D_{i\lambda}), (E_{i\lambda}), (F_{i\lambda})\) are known, and so (86) and (87) provide an infinite system of linear equations for the unknown coefficients \((A_{\lambda}), (B_{\lambda})\). Gaydon and Shepherd [5] computed numerically an approximate matrix \(M\) such that

\[
\begin{bmatrix}
A_{\lambda} \\
B_{\lambda}
\end{bmatrix} = M \begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix}.
\]

Therefore for any boundary conditions \(p(y)\) and \(s(y)\), the constants \(A_{\lambda}\) and \(B_{\lambda}\) can be evaluated. Knowing these constants, the stresses at any point \((x, y)\) can now be easily evaluated.

2.5 The sector problem

Gopalacharyulu [6] determined the stress field for the plane deformation of a sector with stress-free radial edges and given self-equilibrating loads on the circular boundary. The method followed was similar to that given by Gaydon and Shepherd [5] in solving the rectangular strip problem.
The stress function satisfying the biharmonic equation and the traction-free conditions on the radial edges is initially determined as a series of non-orthogonal eigenfunctions. Each of these eigenfunctions is again expanded in a series of orthogonal functions satisfying a fourth-order differential equation and the same boundary conditions.

2.5.1 Solution of the biharmonic equation

The sector occupies the region \(-w < \theta < w\), \(0 < r < 1\). The stress function \(\phi\) has to satisfy the biharmonic equation

\[ \nabla^4 \phi = 0. \]

From equation (43), the separable solutions are of the form

\[ \phi_k = r^{1+\lambda_k} F_k(\theta), \quad \text{Re}\lambda_k > 0, \]

and

\[ F_k(\theta) = A_k \cos(\lambda_k+1)\theta + B_k \cos(\lambda_k-1)\theta. \]

(Here we are considering again only the symmetric solution.) The function \(F(\theta)\) has to satisfy the boundary conditions

\[ F(w) = F'(w) = 0, \quad (88) \]

which represent the traction-free conditions along the radial edges. The shear and normal stresses are specified along the circular boundary,

\[ \sigma_{rr}\bigg|_{r=1} = \sigma(\theta) \quad \text{and} \quad \sigma_{r\theta}\bigg|_{r=1} = \tau(\theta). \quad (89) \]

The boundary conditions (88) imply that
where the eigenvalues \( \lambda_k \) are determined from the transcendental equation

\[
(\lambda_k + 1) \sin(\lambda_k + 1)w \cos(\lambda_k - 1)w - (\lambda_k - 1) \sin(\lambda_k - 1)w \cos(\lambda_k + 1)w = 0.
\]

This equation may be written as

\[
(\lambda_k + 1) \sin 2w + 2 \sin(\lambda_k - 1)w \cos(\lambda_k + 1)w = 0.
\]

Let \( \rho_k = \lambda_k + 1 \), so that the general stress function \( \phi \) will therefore be the linear combination of the eigenfunctions \( \phi_k \)

\[
\phi = \sum_k \rho_k A_k r F_k(\theta) + \bar{\rho}_k A_k r \bar{F}_k(\theta)
\]

(90)

where \( F_k(\theta) \) is given by

\[
F_k(\theta) = \cos(\rho_k - 2)w \cos \rho_k \theta - \cos \rho_k w \cos(\rho_k - 2)\theta.
\]

(91)

The eigenvalues \( \rho_k \) are themselves determined from

\[
\rho_k \sin 2w + 2 \sin(\rho_k - 2)w \cos \rho_k w = 0.
\]

(92)

2.5.2 Expansion of the arc tractions in terms of the beam functions

The constants \( A_k \) are determined from the applied loads on the circular boundary \( \sigma(\theta) \) and \( \tau(\theta) \), equation (89). Due to the lack of orthogonality of the functions \( F_k(\theta) \), each \( F_k(\theta) \) is expanded in terms of the orthogonal beam functions \( \psi_m(\theta) \) which are the solutions of the fourth order differential equation
\[ \frac{d^4 \psi}{d \theta^4} - \left( \frac{\mu}{w} \right)^4 \psi = 0 \]  \hspace{1cm} (93)

and satisfy the boundary conditions

\[ \psi(w) = \psi'(w) = 0 . \]  \hspace{1cm} (94)

Solutions of the differential equation (93), satisfying the boundary conditions (94) are

\[ \psi_m(\theta) = (2w)^{-1/2} \left\{ \frac{\cos(\mu_m \theta/w)}{\cos \mu_m} - \frac{\cosh(\mu_m \theta/w)}{\cosh \mu_m} \right\} \]  \hspace{1cm} (95)

where

\[ \int_{-w}^{w} [\psi_m(\theta)]^2 d\theta = 1 \]

and the eigenvalues \( \mu_m \) satisfy the equation

\[ \tan \mu_m + \tanh \mu_m = 0 . \]  \hspace{1cm} (96)

There exist orthogonality relations similar to those given in (78), namely

\[ \int_{-w}^{w} \psi_m(\theta) \psi_n(\theta) d\theta = \delta_{mn} \]

\[ \int_{-w}^{w} \psi_m''(\theta) \psi_n''(\theta) d\theta = \left( \frac{\mu_m}{w} \right)^4 \delta_{mn} \]  \hspace{1cm} (97)

\[ \int_{-w}^{w} \psi_m'(\theta) \psi_n'(\theta) d\theta = -\left( \frac{\mu_m}{w} \right)^4 \delta_{mn} \]

where \( \delta_{mn} \) is the Kronecker delta.

Now expanding the functions \( F_k(\theta) \) in a series of the orthogonal
functions $\psi_m(\theta)$, we obtain

$$F_k(\theta) = \sum_m a_{km} \psi_m(\psi)$$

$a_{km}$ are the Fourier coefficients given by

$$a_{km} = \int_{-w}^{w} F_k(\theta) \psi_m(\theta) \, d\theta$$

$$a_{km} = (2w)^{-1} \int_{-w}^{w} \left[ \cos(\rho_k - 2)\theta \cos\rho_k \theta \cos(\rho_k - 2)\theta \right] \left[ \frac{\cos(\mu_m \theta/w)}{\cos\mu_m} - \frac{\cosh \frac{\mu_m \theta}{w}}{\cosh \frac{\mu_m}{w}} \right] \, d\theta$$

$$= (2w)^{-1} \left[ \frac{2}{\rho_k^2 - \mu_m^2/2} \right] \left[ \rho_k \sin \rho_k \theta \cos(\rho_k - 2)\theta \cos \rho_k \theta \cos(\rho_k - 2)\theta \right] \left[ \frac{\sin(\rho_k - 2)\theta \cos \rho_k \theta \cos(\rho_k - 2)\theta \tan \mu_m}{\sin(\rho_k - 2)\theta \cos \rho_k \theta \cos(\rho_k - 2)\theta \tan \mu_m} \right]$$

$$+ (2w)^{-1} \left[ \frac{2}{(\rho_k - 2)^2 + \mu_m/w^2} \right] \left[ (\rho_k - 2) \sin(\rho_k - 2)\theta \cos \rho_k \theta \cos(\rho_k - 2)\theta \right] \left[ (\rho_k - 2) \sin(\rho_k - 2)\theta \cos \rho_k \theta \cos(\rho_k - 2)\theta \tan \mu_m \right]$$

$$+ \frac{\mu_m}{w} \cos \rho_k \theta \cos(\rho_k - 2)\theta \tan \mu_m \right\} . \quad (98)$$

In deriving this expression we have made use of equation (96).

The stress function given in (90) can now be written as

$$\phi = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left[ a_k a_{km} \rho_k + \overline{a_k} \overline{a_{km}} \rho_k^* \right] \psi_m(\theta) .$$

Using the expressions for the stresses in (5), we get on the arc $r = 1$

$$\sigma(\theta) = \sum_{m=1}^{\infty} \left[ a_k a_{km} + \overline{a_k} \overline{a_{km}} \right] \psi''_m(\theta) + \left[ a_k a_{km} \rho_k + \overline{a_k} \overline{a_{km}} \rho_k^* \right] \psi_m(\theta) \right\} \quad (99)$$

and
Let

\[ c_m = \sum_k (\lambda_k a_{km} (1-\rho_k) + \bar{\lambda}_k \bar{a}_{km} (1-\bar{\rho}_k)) \]  

and

\[ d_m = \sum_k (\lambda_k a_{km} \rho_k + \bar{\lambda}_k \bar{a}_{km} \bar{\rho}_k) . \]

Then equations (99) and (100) may be written as

\[ \sigma(\theta) = \sum_{m=1}^{\infty} \left[ c_m \psi_m''(\theta) + d_m \psi_m(\theta) \right] \]

\[ \tau(\theta) = \sum_{m=1}^{\infty} \left[ (c_m - d_m) \psi_m'(\theta) \right] . \]

Multiplying both sides of (104) by \( \psi_m''(\theta) \) and integrating between \(-w\) to \(+w\)

\[ \int_{-w}^{w} \tau(\theta) \psi_m''(\theta) \, d\theta = (c_n - d_n) \left[ -\left( \frac{\mu_n}{w} \right)^4 \right] . \]

From which

\[ c_n - d_n = -\left( \frac{\mu_n}{w} \right)^4 \int_{-w}^{w} \tau(\theta) \psi_m''(\theta) \, d\theta = K_n^{(1)} \text{ say .} \]

To overcome the difficulty of non-orthogonality of \( \psi_m \) and \( \psi_m'' \) in (103) Gopalacharyulu [6] multiplied both sides of this equation by \( (\psi_m - \psi_m'') \) and integrated between \(-w\) to \(+w\) to get one more relation between the constants \( c_n \) and \( d_n \)
Thus, we have

\[-C_n \left( \frac{\mu}{n} \right)^4 + D_n = \int_{-w}^{w} \sigma(\theta) (\psi_n - \psi''_n) \, d\theta + \int_{-w}^{w} \tau(\theta) \psi'_n \, d\theta = K_n^{(2)}.\]

(106)

From (103) and (104), the coefficients $C_n$ and $D_n$ are

\[C_n = \frac{w^4}{w^4 - \mu_n^4} \left[ K_n^{(1)} + K_n^{(2)} \right],\]

(107)

\[D_n = \frac{1}{w^4 - \mu_n^4} \left[ \mu_n^4 K_n^{(1)} + w^4 K_n^{(2)} \right].\]

(108)

2.5.3 Determination of $A_k$

In order to determine $A_k$ let

\[A_k = e_k + i f_k\]

\[a_{km} = g_{km} + i h_{km}\]

(109)

\[\rho_k = \alpha_k + i \beta_k.\]
Hence, expressions (101) and (102) will be

\[ \sum_{k} (e_k g_{km} - f_k h_{km}) = \frac{1}{2} c_m, \]

\[ \sum_{k} \{e_k (\alpha_k g_{km} + \beta_k h_{km}) - f_k (\alpha_k h_{km} + \beta_k g_{km})\} = \frac{1}{2} d_m. \]

(110)

The right hand sides of these two equations are known quantities, from (107) and (108), and the coefficients \( a_{km} \) (and hence \( g_{km} \) and \( h_{km} \)) are given explicitly in (98). Hence equations (110) provide an infinite system of linear algebraic equations for \( e_k \) and \( f_k \).

Gopalacharyulu [6] obtains approximate numerical solutions of this system of equations by retaining only a finite number of the coefficients \( A_k \), and shows that quite good results are obtained by using only very few non-zero \( A_k \)'s.
CHAPTER III

SOLUTION OF THE CRACKED CYLINDER AND SEMI-CIRCLE PROBLEMS

In this chapter we shall complete the solution of the cracked cylinder problem described in §2.2. The method used will be simpler than that of Gaydon and Shepherd [5] and Gopalacharyulu [6] in that Fourier cosine and sine series are used rather than beam eigenfunctions. It does have the disadvantage of leading to quite slow convergence of the series expansions employed, but, as we shall see, reasonably accurate approximate results are obtained by retaining only a few terms in the expansions.

In later sections of the chapter we examine the problem of a crack with a rounded tip and the semi-circle problem using essentially the same method. Numerical solutions have not been obtained however for these problems.

3.1 The cracked cylinder problem

We wish to solve the problem of the plane strain deformation of a cylinder \( 0 \leq r < 1 \) under given tractions on \( r = 1 \) and in the case when there is a plane crack running from the axis of the cylinder to the boundary on the plane \( \theta = \pm \pi \). In §2.2 it was shown that, if the two faces of the crack are traction-free, the Airy stress function has an expansion of the form (47) and the stress components \( \sigma_{rr}(r, \theta) \), \( \sigma_{r\theta}(r, \theta) \) and \( \sigma_{\theta\theta}(r, \theta) \) are given by (48) - (51).

The constants \( A_n \) and \( B_n \) have yet to be determined from the given
tractions on the circular boundary \( r = 1 \). The difficulty here is that these stresses are expanded in a non-orthogonal series of functions of \( \theta \), which does not allow us immediately to determine the constants \( A_n \) and \( B_n \). Therefore we shall expand each of these functions in a simple Fourier cosine series in the interval \( [-\pi, \pi] \), in the case of the normal stresses, and sine series in the case of shear stresses. Hence the stresses on the circular boundary \( r = 1 \) will be written as

\[
\sigma_{rr}(1, \theta) = \frac{\sigma_0}{2} + \sum_{k=1}^{\infty} \sigma_k \cos k\theta ,
\]

\[
\sigma_{r\theta}(1, \theta) = \sum_{k=1}^{\infty} \tau_k \sin k\theta .
\]

The coefficients \( \sigma_k \) and \( \tau_k \) are the Fourier coefficients defined by

\[
\sigma_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{rr}(1, \theta) \cos k\theta \, d\theta , \quad k = 0, 1, 2, \ldots ,
\]

\[
\tau_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{r\theta}(1, \theta) \sin k\theta \, d\theta , \quad k = 1, 2, 3, \ldots .
\]

\( \sigma_k \) and \( \tau_k \) are known once the boundary tractions are prescribed.

Using (49), (51) these coefficients are also given as

\[
\frac{\sigma_0}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ (-1)^{n+1} A_n \frac{i}{(n+\frac{1}{2})(n-\frac{3}{2})^2} \right\} + 2B_1
\]

\[
\sigma_k = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ (-1)^{n+k+1} A_n \left[ \frac{n+\frac{1}{2}}{n+\frac{1}{2}^2-k^2} - \frac{n-\frac{7}{2}}{n-\frac{3}{2}^2-k^2} \right] \right\} + k B_{k-1} - (k-2) B_{k+1} ,
\]

\( k = 1, 2, 3, \ldots . \)
\[ \tau_k = \frac{8}{\pi} k \sum_{n=1}^\infty \left\{ (-1)^{n+k+1} A_n \frac{n - \frac{1}{2}}{\left[(n + \frac{1}{2})^2 - k^2\right] \left[(n - \frac{3}{2})^2 - k^2\right]} \right\} + kB_{k+1} - kB_{k-1}, \]

\[ k = 1, 2, \ldots, \]  \hspace{1cm} (117)

where the right hand sides have been obtained by expanding \( \cos(n + \frac{1}{2}) \theta \) and \( \cos(n - \frac{3}{2}) \theta \) as Fourier cosine series and \( \sin(n + \frac{1}{2}) \theta \) and \( \sin(n - \frac{3}{2}) \theta \) as Fourier sine series and then extracting the coefficients of \( \cos k \theta \) in \( \sigma_{rr}(1, \theta) \) and of \( \sin \theta \) in \( \sigma_{r\theta}(r, \theta) \). In these equations, we define \( B_0 = 0 \).

### 3.1.1 Satisfaction of the conditions of overall self-equilibrium

We assume that the cylinder is in self-equilibrium under the prescribed stresses on the circular boundary \( r = 1 \). The three conditions of equil-

\[ \begin{align*}
\text{i)} \quad & \int_{-\pi}^{\pi} \left[ \sigma_{rr}(1, \theta) \cos \theta - \sigma_{r\theta}(1, \theta) \sin \theta \right] d\theta = 0, \\
\text{ii)} \quad & \int_{-\pi}^{\pi} \left[ \sigma_{rr}(1, \theta) \sin \theta + \sigma_{r\theta}(1, \theta) \cos \theta \right] d\theta = 0, \\
\text{iii)} \quad & \int_{-\pi}^{\pi} \sigma_{r\theta}(1, \theta) d\theta = 0.
\end{align*} \]

Since \( \sigma_{rr}(1, \theta) \) and \( \sigma_{r\theta}(1, \theta) \) are even and odd functions of \( \theta \) respectively, conditions (ii) and (iii) are readily satisfied.

Condition (i) is equivalent to \( \sigma_1 = \tau_1 \). From (116)

\[ \sigma_1 = \frac{8}{\pi} \sum_{n=1}^\infty (-1)^n A_n \frac{n - \frac{1}{2}}{\left[(n + \frac{1}{2})^2 - 1\right] \left[(n - \frac{3}{2})^2 - 1\right]} + B_2, \]
and from (117)
\[
\tau_1 = \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{A_n}{(n + \frac{1}{2})^2 - 1} \left( \frac{n - \frac{3}{2}}{2} - 1 \right) + B_2 .
\]

Thus all the self-equilibrating conditions are satisfied.

3.1.2 Separation of the constants $A_n$ and $B_n$

In equations (115) – (117) the constants $A_n$ and $B_n$ are mixed. It is possible though to separate them and this simplifies by a great amount the determination of these constants. In separating these constants, we shall express $B_n$ in terms of $A_n$. Thus, from (115),
\[
B_1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ (-1)^n \frac{A_n}{(n + \frac{1}{2})(n - \frac{3}{2})(2)} \right\} + \frac{\sigma_0}{4} .
\]

Adding (116) and (117)
\[
\sigma_k + \tau_k = 2 \left\{ \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k+1} \frac{(k+1)A_n}{(n + \frac{1}{2} + k) \left( \frac{n - \frac{3}{2}}{2} - k \right)^2} + B_{k+1} \right\} , k=1,2,3,\ldots .
\]

Let $k \rightarrow k - 2$, we obtain
\[
\sigma_{k-2} + \tau_{k-2} = 2 \left\{ \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k+1} \frac{(k-1)A_n}{(n - \frac{3}{2} + k) \left( \frac{n - \frac{7}{2} + k}{2} \right)(n + \frac{1}{2} - k)} + B_{k-1} \right\} , k=3,4,\ldots .
\]

Therefore $B_{k+1}$ and $B_{k-1}$ will be given by
\[
B_{k+1} = \frac{\sigma_{k+1}}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k} \frac{(k+1)A_n}{(n + \frac{1}{2} + k) \left( \frac{n - \frac{3}{2}}{2} - k \right)^2} , k=1,2,3,\ldots ,
\]

(119)
By substituting back in (116) we obtain

\[
B_{k-1} = \frac{\sigma_{k-2} + \tau_{k-2}}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k} \frac{(k-1)A_n}{(n-\frac{3}{2}+k)(n-\frac{7}{2}+k)(n+\frac{1}{2}-k)} , \quad k = 3, 4, \ldots .
\]  

(120)

This equation may be written as

\[
\sigma_k = \frac{16}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k+1} A_n \left[ \frac{k(1-k)}{(n+\frac{1}{2}-k)^2} \right] \frac{1}{(n-\frac{3}{2}+k)(n-\frac{7}{2}+k)} + \frac{k}{2} (\sigma_{k-2} + \tau_{k-2}) - \frac{k-2}{2} (\sigma_k + \tau_k) , \quad k = 3, 4, \ldots .
\]

(121)

From equation (116) with \( k = 2 \)

\[
\sigma_2 = \left\{ \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n A_n \left[ \frac{n-\frac{7}{2}}{(n+\frac{5}{2})(n-\frac{3}{2})(n+\frac{1}{2})(n-\frac{7}{2})} \right] - 2B_1 \right\} .
\]

Using equation (118), we obtain

\[
\sigma_2 = \frac{\sigma_0}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \left[ \frac{-2A_n}{(n+\frac{5}{2})(n-\frac{3}{2})(n+\frac{1}{2})} + \frac{A_n}{(n-\frac{3}{2})(n+\frac{1}{2})} \right],
\]

or

\[
\sigma_2 - \frac{\sigma_0}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n A_n \frac{-n+\frac{11}{2}}{(n-\frac{3}{2})^2(n+\frac{1}{2})(n+\frac{5}{2})} .
\]

(122)

Equations (121) and (122) constitute an infinite system of simultaneous equations from which \( A_n \) may be determined. Once these have been determined, the set of constants \( B_n \) can be obtained from (118) and (119).

In determining the coefficients \( A_n \), Gauss-Seidel iteration method has
been used.

3.1.3 Stress intensity factor

The stress intensity factor is defined by

\[ K_I = \lim_{r \to 0} r^\frac{1}{2} \sigma_{\theta\theta}(r, 0). \]

Using expression (51) for the tangential stress we obtain:

\[ K_I = A_1 \frac{3/2}{1/2} = 4A_1. \] (123)

3.1.4 Examples

i) On the boundary \( r = 1 \), we apply a constant normal stress of unit magnitude, i.e. \( \frac{\sigma_0}{2} = 1.0, \sigma_k = 0, k = 1, 2, \ldots \). We assume that there is no shear on the boundary, in other words \( T_k = 0, k = 1, 2, 3, \ldots \). In equations (121) and (122) we shall retain 100 terms \( (n = 1, 2, \ldots, 100) \) and use the equations with \( k = 2, 3, \ldots, 101 \), so that we have 100 simultaneous equations with a 100x100 matrix of coefficients. An IBM 370 computer model 155 is used to determine \( \{A^n\}_{n=1}^{100} \). \( \{B^n\}_{n=1}^{100} \) are determined from (118) and (119).

Having found \( \{A^n\} \) and \( \{B^n\} \), the stress distribution may be calculated from (49) - (51). We let the radius \( r \) vary with 0.1 intervals and the angle \( \theta \) with five degrees intervals in these expressions for the stresses, thus getting the stress distribution throughout the cylinder. The resulting stress-fields are drawn in Fig. 16, which shows lines of constant \( \sigma_{rr}, \sigma_{r\theta} \) and \( \sigma_{\theta\theta} \). It can be seen that \( \sigma_{r\theta} \) and \( \sigma_{rr} \) approach their respective given boundary values on \( r = 1 \) and \( \theta = \pm \pi \), while they diverge as \( r \to 0 \).
LINES OF CONSTANT NORMAL STRESS $\sigma_{rr}$

FIGURE 16(i)
LINES OF CONSTANT SHEAR STRESS \( \sigma_{r\theta} \)

FIGURE 16(ii)
LINES OF CONSTANT TANGENTIAL STRESS $\sigma_{\theta\theta}$

FIGURE 16(iii)
ii) Secondly we shall calculate the stress intensity factor \( K_I \) provided by each of the Fourier coefficients \( \{\sigma_k, \tau_k\} \) of the applied stress distribution. Therefore we keep the shear as zero, and take each of the \( \sigma_k \)'s equal to unity in turn while keeping the remaining ones as zeros, i.e.

\[
\sigma_i = \delta_{ir}, \quad \tau_i = 0 \quad (r = 0, 2, 3, \ldots, 24; i = 0, 1, 2, \ldots, 101).
\]

In the case of \( r = 1 \), for self-equilibrium we had \( \sigma_1 = \tau_1 \), hence this condition may be written as

\[
\sigma_i = \delta_{i1}, \quad \tau_i = \delta_{i1} \quad (i = 0, 1, 2, \ldots, 101).
\]

For each \( r \), we have computed the corresponding stress intensity factor using (123), and they are listed in Table I. We have only considered the first 25 values of \( r \), since for higher \( \sigma_r \) and \( \tau_r \) it becomes increasingly necessary to include more than 100 non-zero coefficients \( \{A_n, B_n\} \).

In exactly the same way, we take the normal stress to be zero, and take each \( \tau_k \) equal to one in turn, with the provision that \( \sigma_1 = \tau_1 \), i.e.

\[
\tau_i = \delta_{ir}, \quad \sigma_r = 0, \quad r = 0, 2, \ldots, 25
\]

\[
\tau_i = \delta_{i1}, \quad \sigma_1 = \delta_{i1} \quad \text{with} \quad i = 1, 2, \ldots, 101.
\]

The corresponding stress intensity factors are listed in Table II.

3.2 Stress distribution around a crack with a rounded tip

It is more realistic to allow the crack to have a rounded tip, rather
than the infinitely sharp tip considered so far. We model this situation by supposing that the crack extends from a cylinder of small radius $R$ around the infinite cylinder axis, to the external boundary with unit radius. Thus the singularity arising at the cylinder axis in the previous problem will not show up. The region under consideration is $R < r < 1$, $-\pi < \theta < \pi$. The boundary conditions are

$$
\sigma_{r\theta} = \sigma_{r\theta} = 0 \quad \text{on} \quad \theta = \pm \pi,
$$

$$
\sigma_{rr} = \sigma_{r\theta} = 0 \quad \text{on} \quad r = R,
$$

$\sigma_{rr}$ and $\sigma_{r\theta}$ are prescribed on $r = 1$.

The stresses expressed in terms of Airy's stress function are given by (5) with the function $\phi$ satisfying the biharmonic equation $\nabla^4 \phi = 0$.

Figure 17

Considering again the separable solutions

$$
\phi(r, \theta) = R(r) F(\theta),
$$

we look for symmetric solutions that will satisfy the conditions of zero traction on the crack faces $\theta = \pm \pi$, namely

$$
F(\pi) = F'(\pi) = 0.
$$

Having found this set of solutions, the general problem with prescribed tractions on $r = R$ and $r = 1$ is solved by taking a linear combination of these separable solutions. The coefficients in this linear combination will be found in terms of the given tractions.
3.2.1 Solution of the biharmonic equation

It was determined in §2.2 that the solutions to the biharmonic equation satisfying the zero traction conditions on the crack faces may be written as

\[ \phi(r, \theta) = (Ar^{1+\lambda} + Br^{1-\lambda}) F(\theta), \quad \lambda \neq 0, 1. \]

Using equation (47) we may write the stress function as

\[ \phi(r, \theta) = \sum_{n=1}^{\infty} \sum_{j=1,2} \left( C_j r^n + D_j r^{-n} \right) F_n^{(j)}(\theta) = \sum_{n=1}^{\infty} \phi_n(r, \theta) \]

where

\[ \phi_n(r, \theta) = \left( A_n r^{\frac{1}{2}n} + B_n r^{\frac{-3}{2}n} \right) \left[ \frac{\cos(\frac{1}{2}n+\theta)}{n+\frac{1}{2}} - \frac{\cos(\frac{3}{2}n-\theta)}{n-\frac{3}{2}} \right] \]

\[ + \left( C_n r^{n+1} + D_n r^{-n+1} \right) [\cos(n-1)\theta - \cos(n+1)\theta] \]  \hspace{2cm} (124)

In deriving (124), we have taken as in section 2.2

\[ \lambda_n^{(1)} = \frac{2n-1}{2}, \quad n = 1, 2, 3, \ldots, \]

\[ \lambda_n^{(2)} = n, \quad n = 2, 3, 4, \ldots, \]

\[ F_n^{(1)}(\theta) = a_n \left[ \frac{\cos(\frac{1}{2}n+\theta)}{n+\frac{1}{2}} - \frac{\cos(\frac{3}{2}n-\theta)}{n-\frac{3}{2}} \right], \]

\[ F_n^{(2)}(\theta) = b_n [\cos(n-1)\theta - \cos(n+1)\theta] \]  \hspace{2cm} (124)

Here we note that, as in §2.2, for the case of \( \lambda = 0 \), we get only the trivial solution. It should also be noted that the terms which become singular as \( r \to 0 \) cannot be rejected for the present problem.
From the stress function given by (124), we obtain the stresses (5) as

$$
\sigma_{rr}(r, \theta) = \sum_{n=1}^{\infty} \left\{ \left[ \frac{(n+\frac{1}{2})A'_n r^{n-\frac{3}{2}} + (-n+\frac{3}{2})B'_n r^{-n-\frac{1}{2}}}{r} \right] \left[ \frac{\cos (n+\frac{1}{2}) \theta - \cos (n-\frac{3}{2}) \theta}{n+\frac{1}{2}} \right] \right. \\
+ \left[ A'_n r^{n-\frac{3}{2}} + B'_n r^{-n-\frac{1}{2}} \right] \left[ -(n+\frac{1}{2}) \cos (n+\frac{1}{2}) \theta + (n-\frac{3}{2}) \cos (n-\frac{3}{2}) \theta \right] \\
+ \left[ (n+1)C'_n r^{n-1} + (-n+1)D'_n r^{-n-1} \right] \left[ \cos (n-1) \theta - \cos (n+1) \theta \right] \\
+ \left[ C'_n r^{n-1} + D'_n r^{-n-1} \right] \left[ -(n-1)^2 \cos (n-1) \theta + (n+1)^2 \cos (n+1) \theta \right] \left\} ,
$$

or

$$
\sigma_{rr}(r, \theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{n-\frac{3}{2}} \left[ -\cos (n+\frac{1}{2}) \theta + \frac{n-2}{n-\frac{3}{2}} \cos (n-\frac{3}{2}) \theta \right] \\
+ B_n r^{-n-\frac{1}{2}} \left[ -\frac{n+2}{n+\frac{1}{2}} \cos (n+\frac{1}{2}) \theta + \cos (n-\frac{3}{2}) \theta \right] \\
+ C_n r^{n-1} \left[ -(n-3) \cos (n-1) \theta + (n+1) \cos (n+1) \theta \right] \\
+ D_n r^{-n-1} \left[ -(n-1) \cos (n-1) \theta + (n+3) \cos (n+1) \theta \right] \right\},
$$

(125)

where

$$
A_n = (n-\frac{1}{2}) A'_n , \quad B_n = (n-\frac{1}{2}) B'_n , \\
C_n = n C'_n , \quad D_n = n D'_n ,
$$

and we define $C_0 \equiv D_0 = 0$. 
Similarly the shear stress will be given by

\[
\sigma_{r\theta}(r, \theta) = \sum_{n=1}^{\infty} \left( A_n r^{-\frac{3}{2}} - B_n r^{-\frac{1}{2}} \right) \left( \sin(n+\frac{1}{2})\theta - \sin(n-\frac{3}{2})\theta \right) + \left( C_n r^{n-1} - D_n r^{-n-1} \right) \left( (n-1) \sin(n-1)\theta - (n+1) \sin(n+1)\theta \right).
\]

(126)

3.2.2 Satisfaction of the boundary conditions on the boundaries \( r = R, r = 1 \)

Expanding the normal and shear stresses in Fourier cosine and sine series respectively, we obtain

\[
\sigma_{rr}(r, \theta) = \frac{\sigma_0(r)}{2} + \sum_{k=1}^{\infty} \sigma_k(r) \cos k\theta
\]

where \( \sigma_k, k = 0, 1, 2, \ldots \), are the Fourier coefficients defined by

\[
\sigma_k(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{rr}(r, \theta) \cos k\theta \, d\theta.
\]

Similarly \( \sigma_{r\theta}(r, \theta) = \sum_{k=1}^{\infty} \tau_k(r) \sin k\theta \) where

\[
\tau_k(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{r\theta}(r, \theta) \sin k\theta \, d\theta, \quad k = 1, 2, \ldots
\]

Therefore, for \( k = 1, 2, 3, \ldots \)

\[
\tau_k(r) = \sum_{n=1}^{\infty} \left( A_n r^{-\frac{3}{2}} - B_n r^{-\frac{1}{2}} \right) (-1)^{n+k} \frac{2k}{\pi} \left[ \frac{1}{(n+\frac{1}{2})^2 - k^2} - \frac{1}{(n-\frac{3}{2})^2 - k^2} \right] + \left( C_{k+1} r^k - D_{k+1} r^{-k-2} \right) - k \left( C_{k-1} r^{k-2} - D_{k-1} r^{-k} \right),
\]

or
where

\[ \Delta_{nk} = \left[ \left( n+\frac{1}{2} \right)^2 - k^2 \right] \left[ \left( n-\frac{3}{2} \right)^2 - k^2 \right]. \]

\[ \sigma_k(r) = \sum_{n=1} \left\{ (-1)^{n+k+1} \frac{8}{\pi} \frac{n-3}{n} \frac{1}{\Delta_{nk}} \left( -A_n r \frac{n-3}{2} - B_n r \frac{1}{2} \right) \right\} - (k-2) C_{k+1} r^k + k C_{k-1} r^{k-2} - k D_{k+1} r^{-k-2} + (k+2) D_{k-1} r^{-k} \]  

(128)

For \( k = 0 \), we get

\[ \sigma_0(r) = \sum_{n=1} \left\{ (-1)^n \frac{8}{\pi} \frac{1}{(n+1)^2 (n-3)^2} \left( -A_n r \frac{n-3}{2} \frac{1}{n-3/2} + B_n r \frac{n-1}{2} \frac{1}{n+1/2} \right) \right\} + 4C_1 \]

(129)

In order to complete the solution of the originally posed problem, we now set \( \sigma_k(R) = \tau_k(r) = 0 \), since the tractions on the inner boundary \( r = R \) are given to be zero; and set \( \sigma_k(1) \) and \( \tau_k(1) \) equal to the corresponding Fourier coefficients of the given normal and shear tractions on the external boundary. The resulting system of equations can be solved numerically as in section 3.1.

3.2.3 Satisfaction of conditions of overall equilibrium

Here it will be sufficient to satisfy the condition \( \sigma_1 = \tau_1 \) on \( r = 1 \), since \( \sigma_{rr} = \sigma_0 = 0 \) on \( r = R \). Therefore on \( r = 1 \), using equations (127) and (128), we obtain...
Hence our Fourier coefficients do satisfy the conditions of equilibrium.

3.3 Stress distribution in a semi-circular sector

In the case of a semi-circular boundary, the formulation of the problem will be as follows:

We have to determine a stress function $\phi$ which is biharmonic in the region $-\pi/2 < \theta < \pi/2$, $0 < r < 1$, $\sigma_{rr}, \sigma_{r\theta}$ are prescribed on $r = 1$, $\sigma_{\theta\theta}, \sigma_{r\theta}$ are zero on $\theta = \pm \pi/2$.

\[ \sigma_{rr} = \sigma_{r\theta} = 0 \quad \theta = \pi/2 \]
\[ \sigma_{rr} = \sigma_{r\theta} \text{ prescribed} \]
\[ \theta = -\pi/2 \]

Figure 18

Again, the stress function $\phi$ is written as

$\phi(r, \theta) = r^{1+\lambda} F(\theta)$

where $\lambda > 0$ to satisfy boundedness of the strain energy density in the neighbourhood of the origin.
The function $F$ in the symmetric case (equation (45)) is

$$F(\theta) = A \cos(\lambda+1)\theta + B \cos(\lambda-1)\theta .$$

In order to have zero tractions on $\theta = \pm \pi$, we must have

$$F\left(\frac{\pi}{2}\right) = F'\left(\frac{\pi}{2}\right) = 0 .$$

Applying these boundary conditions on (45), we obtain

$$(-A + B) \sin \frac{\pi}{2} = 0 ,$$

$$[ (\lambda+1)A - (\lambda-1)B ] \cos \frac{\pi}{2} = 0 .$$

Here we have two cases:

(i) If $A \neq B$, this implies that

$$\sin \frac{\pi}{2} = 0 \text{ or } \lambda_n^{(1)} = 2n .$$

This leads to

$$B^{(1)} = \frac{n+1}{n} A^{(1)} .$$

(ii) $A = B$ and this implies that

$$\cos \frac{\pi}{2} = 0 \text{ or } \lambda_n^{(2)} = 2n-1 .$$

Therefore the stress function may now be written as

$$\phi(r, \theta) = \sum_{n=1}^{\infty} \left\{ \begin{array}{c}
A_n^{'} r^{2n+1} \left[ \frac{\cos(2n+1)\theta}{2n+1} + \frac{\cos(2n-1)\theta}{2n-1} \right] \\
B_n^{'} r^{2n} \left[ \cos2n\theta + \cos2(n-1)\theta \right] \end{array} \right\} .$$
On the boundary $r = 1$, the normal and shear stresses are found to be

$$
\sigma_{rr}(1, \theta) = -\sum_{n=1}^{\infty} \left\{ A_n \left[ \cos(2n+1)\theta + \frac{2n-3}{2n-1} \cos(2n-1)\theta \right] + B_n \left[ n \cos 2n\theta + (n-2) \cos 2(n-1)\theta \right] \right\}, \quad (130)
$$

where $A_n = 2n A'_n$ and $B_n = 2(2n-1) B'_n$.

$$
\sigma_{r\theta}(1, \theta) = \sum_{n=1}^{\infty} \left\{ A_n \left[ \sin(2n+1)\theta + \sin(2n-1)\theta \right] + B_n \left[ n \sin 2n\theta + (n-1) \sin 2(n-1)\theta \right] \right\} . \quad (131)
$$

In order to determine $A_n$ and $B_n$, we expand $(130)$ and $(131)$ in Fourier cosine and sine series respectively, the series having ranges $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence

$$
\sigma_{rr}(1, \theta) = \frac{\sigma_0}{2} + \sum_{k=1}^{\infty} \sigma_k \cos 2k\theta ,
$$

where $\sigma_k$, $k = 0, 1, 2, \ldots$ are the Fourier coefficients defined by

$$
\sigma_k = \frac{4}{\pi} \int_{0}^{\pi/2} \sigma_{rr}(1, \theta) \cos 2k\theta \, d\theta .
$$

Also,

$$
\sigma_{r\theta}(1, \theta) = \sum_{k=1}^{\infty} \tau_k \sin 2k\theta ,
$$

where $\tau_k$, $k = 1, 2, 3, \ldots$ are given by

$$
\tau_k = \frac{4}{\pi} \int_{0}^{\pi/2} \sigma_{r\theta}(1, \theta) \sin 2k\theta \, d\theta .
$$

The expressions for $\sigma_k$ and $\tau_k$ are given by
where

\[ D_{nk} = [4k^2 - (2n+1)^2][4k^2 - (2n-1)^2] \]

### 3.3.1 Separation of \( A_n \) and \( B_n \)

From (128), we can express \( B_1 \) in terms of \( A_n \) as

\[
B_1 = \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n A_n \frac{1}{(2n+1)(2n-1)} + \frac{\sigma_0}{2}
\]  

(135)

and from (133) and (134), we obtain

\[
B_{k+1} = (\sigma_k + \tau_k) + \frac{16}{\pi} \sum_{n=1}^{\infty} (-1)^{n+k+1} \frac{2k+1}{(2k+2n+1)[4k^2-(2n-1)^2]} A_n
\]

\[ k = 1,2,\ldots \]

(136)

\[
B_k = (\sigma_{k-1} + \tau_{k-1}) + \frac{16}{\pi} \sum_{n=1}^{\infty} (-1)^{k+n} \frac{2k-1}{(2k+2n-1)(2k+2n-3)(2k-2n-1)}
\]

\[ k = 2,3,4,\ldots \]

(137)

Substituting the expressions for \( B_k \) given by (135) - (137) in equations (132) - (133) and rearranging the terms, we obtain the infinite system of equations (138) - (139):
\[
\sigma_1 + \frac{\sigma_0}{2} = -\frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n A_n \frac{\lambda^{-2n+5}}{(2n+3)(1n+1)(2n-1)^2}, \quad (138)
\]

\[
k[\sigma_k + \sigma_{k-1}] + [(k-1) \tau_k + k \tau_{k-1}]
= (-1)^{k+1} k(2k-1) \frac{64}{\pi} \sum_{n=1}^{\infty} (-1)^n A_n \frac{1}{n[4k^2-(2n+1)^2][2k+2n-1](2k+2n-3)}
\]

\[
k = 2, 3, 4, \ldots . \quad (139)
\]

The system of equations (138) - (139) is solved in the same way as in section 3.1. The constants \( B_k \) are determined from (135), (136) and the stresses are readily evaluated.

3.3.2 Satisfaction of self-equilibrium conditions

In order to satisfy self-equilibrium, we must have

\[
\int_{-\pi/2}^{\pi/2} \left\{ [\sigma_{rr}(1, \theta)] \cos \theta - [\sigma_{r\theta}(1, \theta)] \sin \theta \right\} d\theta = 0 .
\]

This implies that

\[
\frac{\sigma_0}{2} + \sum_{k=1}^{k} (-1)^{k+1} \frac{\lambda}{4k^2-1} [\sigma_k - 2k \tau_k] = 0 . \quad (140)
\]

In posing any boundary value problem, \( \{\sigma_k, \tau_k\} \) must be chosen to be consistent with this condition.

It is easy to check using the trigonometric expansion

\[
\cot \alpha = \frac{1}{\pi} \frac{2 \alpha}{\alpha^2 - \sum_{k=1}^{\infty} \frac{2\alpha}{k^2 - \alpha^2}}
\]

that the Fourier coefficients given by (132) - (134) do satisfy condition (140).
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TABLE II

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BIBLIOGRAPHY


