LIQUIDITY-ADJUSTED VALUE-AT-RISK
FOR PORTFOLIOS OF ASSETS

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ABSTRACT

Over the past decade, Value-at-Risk (VaR) has become the most prevalent technique for measuring maximum portfolio losses over a specific time horizon at a given probability. However, traditional VaR models do not account for liquidity so that they underestimate the magnitude of overall risks and misrepresent the reserved capital for financial institutions. This paper aims to incorporate the liquidity component into the traditional VaR model to demonstrate the importance of liquidity when managing market risk. After reviewing a number of studies, we decided to apply the Liquidity-Adjusted Value-at-Risk model developed by Bangia, Diebold, Schuermann and Stroughair (1998). We first use this model to estimate the Liquidity-Adjusted VaR (LAVaR) for a single asset. We later extend this model to get a general form of LAVaR for portfolios of N assets by adding the covariance between relative spreads of any two assets.

Keywords: Liquidity; Value-at-Risk; Portfolios
DEDICATION

To our dear parents for encouraging and empowering us!
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1 INTRODUCTION

The traditional Value-at-Risk (VaR) model has recently become the most popular tool to manage market risk because it estimates the downside risk due to financial market fluctuations as a single easy-to-understand number. However, the traditional VaR model assumes that trades of any sizes can always be executed at the mid-price. In reality, investors are not able to execute their trades at the mid-price when they liquidate their positions quickly or the market moves against them. In other words, when bid-ask spreads are substantially different, investors can only trade at the bid price, which can be in some cases very far from the mid-price. Since liquidity affects the asset price, several studies attempted to integrate the liquidity component into the traditional VaR model. However, most of these studies have focused on Liquidity-Adjusted VaR (LAVaR) for only a single asset. Therefore, we try to achieve two objectives in this project. First, we study and apply the traditional VaR model based on the Liquidity-Adjusted Value-at-Risk (LAVaR) model developed by Bangia, Diebold, Schuermann and Stroughair (1998) in the one-asset case. Second, we extend this model to incorporate liquidity into the traditional VaR model for portfolios of two and even N assets.

Before we proceed with this project, it is important to understand the concepts and taxonomy of VaR and liquidity risk. Value-at-Risk (VaR) "summarizes the expected maximum loss (or worst loss) over a target horizon within a given confidence interval" (Jorion, 2001, p.22). For example, for a selected time horizon $t$ and a confidence level $c$, 

VaR is the loss in terms of market value, during the horizon \( t \), which is equal to or exceeded within the probability of \( 1 - c \). Simply stated, we are \( c \) percent certain that we will not lose more than \( X \) dollars over the next \( t \) days, where the variable \( X \) is the VaR of the portfolio.

If \( r_t \) is the daily asset return at time \( t \), then \( r_t = \ln(P_t) - \ln(P_{t-1}) = \ln\left(\frac{P_t}{P_{t-1}}\right) \).

When we take a one-day horizon over which the change in asset value is considered and assume that daily asset returns are normally distributed, then the 99% worst value is \( P_{99\%} = P_t e^{(E(r_t) - 2.33\sigma_t)} \), where \( E(r_t) \) and \( \sigma_t^2 \) are the first and second moments of the distribution of the asset returns, respectively, and the multiple of 2.33 for the standard deviation arises from the assumption of normality since \( N(2.33) = 0.01 \).

If we assume that the expected value of daily returns \( E(r_t) \) is zero due to an absence of any loss of generality, then the 99% worst value is \( P_{99\%} = P_t e^{\{0 - 2.33\sigma_t\}} \) so that the potential loss at time \( t \) will be \( P_t - P_{99\%} = P_t (1 - e^{-2.33\sigma_t}) \). Then the standard parametric VaR becomes \( VaR = P_t (1 - e^{-2.33\sigma_t}) \).

From our discussion above, we know that VaR is a function of two variables: the asset value and the volatility of the asset. Additionally, the calculation of VaR is often based on a number of assumptions: the portfolio returns need to be i) independently distributed, i.e. zero auto-correlation, ii) normally distributed, and iii) with constant parameters.
Liquidity risk arises when a transaction cannot be immediately executed at current market price due to the mismatch between the size of the position and market depth. According to Jorion (2001), liquidity risk usually takes two forms: asset liquidity risk and funding liquidity risk. Asset liquidity risk “arises when a transaction cannot be conducted at prevailing market prices due to the size of the position relative to normal trading lots” (Jorion, 2001, p.17). The funding liquidity risk is “also known as cash-flow risk, (which) refers to the inability to meet payments obligations, (this in turn) may force early liquidation, thus transforming project losses into realized losses” (Jorion, 2001, p.18).

Hence, the concept of liquidity risk can be summarized as “the potential that an institution will be unable to meet its obligations as they come due because of an inability to liquidate assets or obtain adequate funding (funding liquidity risk) or that it cannot easily unwind or offset specific exposures without significantly lowering market prices because of inadequate market depth or market disruptions (market liquidity risk)” (Federal Reserve Bank of Chicago, 2006). In this project, we only focus on market liquidity risk.

Furthermore, liquidity risk can be conceptually divided into an endogenous liquidity component and an exogenous liquidity component. The latter is “the result of market characteristics that are common to all market players and unaffected by the actions of any particular trader” (Bangia, Diebold, Schuermann and Stroughair, 1998, p.4). The key factors that determine the exogenous liquidity risk are subject to trading volumes, bid-ask spreads and levels of quote depth. On the other hand, the endogenous liquidity component “is specific to the characteristics of one’s position and therefore varies across market traders” (Bangia, Diebold, Schuermann and Stroughair, 1998, p.4).
Moreover, the manifestation of the liquidity risk is often reflected in the form of important trading costs, a weak number of trades and a wide bid-ask spread or large price movement. These elements mean that the party who wishes to settle a position will have to pay a large sum to achieve it. If an asset cannot be bought or sold quickly enough to prevent or minimize a loss, the party must bear the trading costs, wait for a relatively long period due to the absence of the counterparty, or sell quickly at an unfavourable price. Thus, Bangia, Diebold, Schuermann and Stroughair (1998) quantify liquidity risk in terms of cost of liquidity. Moreover, they focus on the exogenous liquidity risk rather than the endogenous liquidity risk because i) the fluctuations in exogenous liquidity risk are often large and important; and ii) the data needed to quantify the exogenous liquidity risk are widely available.

This project is organized as follows. Section 2 presents the literature review. Section 3 describes our methodology and the framework of LAVaR in the two-asset case and the N-asset case. Section 4 provides the illustration about our findings. Section 5 is the empirical analysis. Finally we draw our conclusion in Section 6.
2 LITERATURE REVIEW

VaR has become the most prevalent tool to measure market risk in recent years. However, the traditional VaR model neglects liquidity. If liquidity risk is important, this will cause the traditional VaR to underestimate the overall market risk. Because of this drawback, the literature on incorporating liquidity into the traditional VaR model is fairly extensive.

Liquidity is a common concern among investors with portfolios that are comprised of any type of asset. Jarrow and Subramanian (1997) propose a LAVaR model by incorporating a discount of liquidity, the volatility of this discount and the variations of volatility during the horizon of liquidation. However, this model requires a number of estimates, such as the average of discount factor, its standard deviation and the period of execution. Hence, the reliability of this model will depend on the accuracy of these estimates.

Bangia, Diebold, Schuermann and Stroughair (1998) distinguish between endogenous liquidity risk and exogenous liquidity risk. They define endogenous liquidity risk as the liquidity fluctuations driven by individual actions, such as the investor’s position, and the exogenous liquidity risk as the liquidity fluctuations driven by factors out of individual investor’s control. They argue that many markets have an additional liquidity component that exists because traders do not realize the mid-price when liquidating their position. Instead, traders liquidate their positions at the mid-price
minus the bid-ask spread. Bangia, Diebold, Schuermann and Stroughair (1998) state that liquidity risk associated with the relative spread, particularly for illiquid or emerging market securities, is a critical part of overall market risk. Therefore, the VaR model should include a liquidity component. In order to adjust the conventional VaR model, Bangia, Diebold, Schuermann and Stroughair (1998) integrate liquidity into the VaR environment by adding the exogenous liquidity component according to the VaR concept itself. The cost of liquidity, which is based on the average relative spread plus the scaled spread volatility, will be added into the VaR model. In their empirical analysis, Bangia, Diebold, Schuermann and Stroughair (1998) verify that conventional VaR misses almost 16% of the aggregate market risk for the illiquid Thai Baht.

François-Heude and Van Wynendaele (2001) argue that there are some drawbacks in the LAVaR model developed by Bangia, Diebold, Schuermann and Stroughair (1998). They indicate that the spread is not normally distributed. In addition, the LAVaR model only takes the exogenous liquidity risk into consideration and neglects the endogenous component. To minimize these drawbacks, François-Heude and Van Wynendaele (2001) present a new framework for integrating liquidity into the traditional VaR model. This new framework avoids the assumption of perfect correlation between the variations of prices and spreads in the LAVaR model proposed by Bangia, Diebold, Schuermann and Stroughair (1998). The unique feature of this new framework is that François-Heude and Van Wynendaele (2001) quantify the impact of the endogenous liquidity risk by considering any quantity in position along side with a single unit of the asset. They achieve the liquidity adjustment by using the average weighted spreads, which makes the quantity equivalent to the normal market size.
According to Chordia, Roll and Subrahmanyam (2000, p.24), “liquidity is more than just an attribute of a single asset.” They document common determinants of liquidity and correlated movements in liquidity, and their empirical work demonstrates that “individual liquidity measures co-move with each other” (Chordia, Roll and Subrahmanyam, 2000, p.24). Their study helps to explain some market incidents, such as the crash of bond market in the summer of 1998 when Long Term Capital Management was unable to liquidate its position. Following the study of Chordia, Roll and Subrahmanyam (2000), Brockman, Chung and Péignon (2006) investigated the commonality in liquidity using intraday spread and depth data from 47 stock exchanges. Brockman, Chung and Péignon (2006, p.4-5) conclude that “exchange-level commonality is a pervasive phenomenon across the globe” and that “commonality in liquidity spills across national borders”. In other words, the co-movements in the liquidity of all other firms can affect the liquidity of an individual firm if they are traded on the same exchange, and we can observe this kind of result across most exchanges in the world. Given the commonality in liquidity, it is important to find out how co-movements in liquidity impact the level of market risk of a given portfolio.
3 METHODOLOGY

3.1 Liquidity-Adjusted VaR for a Single Asset

Based on the study of Bangia, Diebold, Schuermann and Stroughair (1998), \( \text{LAVaR} \) consists of two terms: the first term is the standard parametric \( \text{VaR} \), the other is the exogenous cost of liquidity (\( \text{COL} \)). The \( \text{LAVaR} \) model is described as follows:

\[
\text{VaR} = P_t (1 - e^{-2.33\sigma_t}) \tag{1}
\]

\[
\text{COL} = \frac{1}{2} \left[ P_t (\bar{S} + a\bar{\sigma}) \right] \tag{2}
\]

\[
\text{LAVaR} = P_t (1 - e^{-2.33\sigma_t}) + \text{COL} \tag{3}
\]

where \( P_t \) is the mid-price of the asset at time \( t \),

\( \sigma_t \) is the volatility (standard deviation) of the asset at time \( t \),

\( \bar{S} \) is the average relative spread (relative spread is defined as \( \frac{\text{Ask} - \text{Bid}}{\text{Mid}} \)),

\( \bar{\sigma} \) is the volatility of relative spread,

\( a \) is the scaling factor that corrects the spread distribution.

Bangia, Diebold, Schuermann and Stroughair (1998, p.6-7) state that the value of “\( a \) ranges from 2.0 to 4.5 depending on the instrument and market in question.” “The
greater the departure (of spread distribution) from normality, the larger \( a \).” We assume that the spread distribution is normally distributed and set \( a \) to 2.

After calculating the \( LAVaR \) from Equation (3), we graph the relationship among

\[
\frac{\text{LAVaR} - \text{VaR}}{\text{VaR}}, \quad S \quad \text{and} \quad \bar{\sigma}
\]

in order to find out the percentage of change due to the impact of \( COL \) on \( VaR \).

### 3.2 Liquidity-Adjusted VaR for Portfolios of Two Assets

Using the framework of \( LAVaR \) for a single asset, we then extend the model developed by Bangia, Diebold, Schuermann and Stroughair (1998) for portfolios of two assets by taking into account of the commonality of liquidity, especially the covariance between relative spreads of two assets. Our framework of \( LAVaR \) for portfolios of two assets is as follows:

\[
\begin{align*}
\text{VaR}_1 &= P_u (1 - e^{-2.33\bar{\sigma}_1}) \quad \text{and} \quad \text{VaR}_2 = P_u (1 - e^{-2.33\bar{\sigma}_2}) \\
\text{VaR}_p &= \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho \text{VaR}_1 \text{VaR}_2} \\
\text{COL}_p &= \frac{1}{2} [w_1 P_u (\bar{S}_1 + a_1 \bar{\sigma}_1) + w_2 P_u (\bar{S}_2 + a_2 \bar{\sigma}_2) + 2(w_1 P_1 + w_2 P_2)b \bar{\sigma}_{1,2}] \\
\text{LAVaR}_p &= \text{VaR}_p + \text{COL}_p
\end{align*}
\]

where \( \text{VaR}_p \) is the portfolio \( \text{VaR} \),
LAVaR, is the liquidity-adjusted portfolio VaR,

\( \rho \) is the correlation coefficient between Asset_1 and Asset_2.

\( w_1 \) is the weight of Asset_1 in the portfolio,

\( w_2 \) is the weight of Asset_2 in the portfolio,

\( P_{1t} \) is the mid-price of the Asset_1 at time \( t \),

\( P_{2t} \) is the mid-price of the Asset_2 at time \( t \),

\( \sigma_{1t} \) is the volatility (standard deviation) of the Asset_1 at time \( t \),

\( \sigma_{2t} \) is the volatility (standard deviation) of the Asset_2 at time \( t \),

\( \overline{S}_1 \) is the average relative spread for Asset_1,

\( \overline{S}_2 \) is the average relative spread for Asset_2,

\( \overline{\sigma}_1 \) is the volatility of relative spread for Asset_1,

\( \overline{\sigma}_2 \) is the volatility of relative spread for Asset_2,

\( a_1 \) is the scaling factor that corrects the spread distribution for Asset_1,

\( a_2 \) is the scaling factor that corrects the spread distribution for Asset_2,

\( \overline{\sigma}_{1,2} \) is the covariance between the relative spreads of Asset_1 and Asset_2,
\( b \) is the scaling factor of covariance between the relative spreads of Asset_1 and Asset_2.

For the simplicity of illustration, we assume that the two assets are equally weighted. Then \( \omega_1 \) and \( \omega_2 \) are equal to \( \frac{1}{2} \). We set \( \alpha_1 = \alpha_2 = 2 \) because we assume that the spreads of assets are normally distributed. We also set \( b = 1 \) in our illustration so that we can determine the range of \( b \) after we test the impact of \( \bar{\sigma}_{1,2} \) on the \( \text{LAVaR}_p \).

After calculating the value of \( \text{LAVaR}_p \) from Equation (7), we graph the relationship among \( \dfrac{\text{LAVaR}_p - \text{VaR}_p}{\text{VaR}_p} \), \( \bar{\sigma} \) and \( \bar{\sigma}_{1,2} \) or among \( \dfrac{\text{LAVaR}_p - \text{VaR}_p}{\text{VaR}_p} \), \( \bar{S} \) and \( \bar{\sigma}_{1,2} \) in order to find out the percentage of change due to the impact of \( \text{COL}_p \) on \( \text{VaR}_p \).

### 3.3 Liquidity-Adjusted VaR for Portfolios of N Assets

According to the framework of \( \text{LAVaR} \) for portfolios of two assets, we can generalize our framework of \( \text{LAVaR} \) for portfolios of \( N \) assets as follows:

\[
\text{VaR}_i = P_e (1 - e^{-2.33\sigma_i}), \quad i = 1, 2, ..., N
\]  

(8)

\[
\text{VaR}_p = \sqrt{\sum_{i=1}^{N} \text{VaR}_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{i,j} \text{VaR}_i \text{VaR}_j}, \quad i = 1, 2, ..., N, \quad j = 1, 2, ..., N
\]  

(9)
i.e. \( \text{VaR}_p = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{i,j} \text{VaR}_i \text{VaR}_j} \), \( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, N \)

\[
\text{COL}_p = \frac{1}{2} \left[ \sum_{i=1}^{N} w_i P_i \left( \bar{S}_i + a_i \bar{\sigma}_i \right) + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} (w_i P_i + w_j P_j) b_{i,j} \bar{\sigma}_{i,j} \right],
\]

(10)

\( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, N \)

\[
\text{LAVaR}_p = \text{VaR}_p + \text{COL}_p
\]

(11)

where \( \text{VaR}_p \) is the portfolio \( \text{VaR} \),

\( \text{LAVaR}_p \) is the liquidity-adjusted portfolio \( \text{VaR} \),

\( \rho_{i,j} \) is the correlation coefficient between Asset\(_i\) and Asset\(_j\),

\( w_i \) is the weight of Asset\(_i\) in the portfolio such that \( w_i = \frac{P_i}{N} \sum_{i=1}^{N} P_i \),

\( w_j \) is the weight of Asset\(_j\) in the portfolio such that \( w_j = \frac{P_j}{N} \sum_{i=1}^{N} P_j \),

\( P_i \) is the mid-price of the Asset\(_i\) at time \( t \),

\( P_j \) is the mid-price of the Asset\(_j\) at time \( t \),

\( \sigma_i \) is the volatility (standard deviation) of the Asset\(_i\) at time \( t \),

\( \bar{S}_i \) is the average relative spread for Asset\(_i\),
$\tilde{\sigma}_i$ is the volatility of relative spread for Asset$_i$,

$a_i$ is the scaling factor that corrects the spread distribution for Asset$_i$,

$\tilde{\sigma}_{i,j}$ is the covariance between the relative spreads of Asset$_i$ and Asset$_j$,

$b_{i,j}$ is the scaling factor of covariance between the relative spreads of Asset$_i$ and Asset$_j$. 
We use the simulation method to illustrate that liquidity is one of the critical components in managing the aggregate market risk. First, we simulate 50 values for $\bar{S}$, $\bar{\sigma}$ and $\bar{\sigma}_{1,2}$. The 50 values of $\bar{S}$ are from zero to 50%. Each of them increases by 1% from the former value. The 50 values of $\bar{\sigma}$ range from zero to 40%, and each value increases by 0.8% incrementally. The 50 values of $\bar{\sigma}_{1,2}$ range from -1 to +1, and each value increases by 0.04. We subsequently use these simulated values to graph a three-dimensional diagram to illustrate the impact of $\bar{S}$ and $\bar{\sigma}$ on the percentage of change between $\text{LAVaR}$ and the traditional $\text{VaR}$ for the one-asset case. Hereafter, we simply call the percentage of change between $\text{LAVaR}$ and the traditional $\text{VaR}$ as the Percentage Change. Later, we use a similar technique to see the impact of the $\bar{S}$ (or $\bar{\sigma}$) and $\bar{\sigma}_{1,2}$ on the Percentage Change in the two-asset case.

In the one-asset case, we set $P_t$ as $100$, $\sigma_t$ as 0.3 and the value of the scaling factor, $a$, as 2. We use the Equations (1) and (3) to calculate the $\text{VaR}$ and $\text{LAVaR}$ for the one-asset case, respectively. Then, the Percentage Change is calculated as $rac{\text{LAVaR} - \text{VaR}}{\text{VaR}}$. When graphing a three-dimensional diagram, we set $\bar{S}$ as the X-axis, $\bar{\sigma}$ as the Y-axis and the Percentage Change as the Z-axis. We can observe from Figure 4.1 that the Percentage Change increases monotonically with the growth of both $\bar{S}$ and $\bar{\sigma}$. Moreover, because all original points are at zero, we can observe that the slope of $\bar{\sigma}$ is
around 2, and the slope of $S$ is around 1. In other words, $\bar{\sigma}$ has as twice as the impact on the Percentage Change compared to $\tilde{S}$. This is because there is a scaling factor $a$, which has value of 2, for $\bar{\sigma}$ in the $COL$ equation. This scaling factor enlarges the impact of $\bar{\sigma}$ on the Percentage Change.

**Figure 4.1: The Percentage Change for a single asset**

In the two-asset case, we set $P_1 = P_2 = $100, $\sigma_1 = 0.3$ and $\sigma_2 = 0.5$. Then we can use Equation (4) to calculate the individual $VaR$ for $Asset_1$ and $Asset_2$. We later set $\rho$ at three different predetermined levels (-1, 0 and 1). Thus, we can calculate the $VaR_{p}$.
according to Equation (5) with different values of $\rho$. We also assume that $\bar{S}_1 = \bar{S}_2 = \bar{S}$, $\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}$, $w_1 = w_2 = \frac{1}{2}$, $a_1 = a_2 = 2$ and $b = 1$. By using the simulated values of $\bar{S}$, $\bar{\sigma}$ and $\bar{\sigma}_{i,2}$, we can calculate $LAVaR_\rho$ from Equation (7) and the Percentage Change will be $\frac{LAVaR_\rho - VaR_\rho}{VaR_\rho}$. When graphing a three-dimensional diagram, we set $\bar{\sigma}$ (or $\bar{S}$) as the X-axis, $\bar{\sigma}_{i,2}$ as the Y-axis and the Percentage Change as the Z-axis. If $\bar{\sigma}$ is set as the X-axis, we change $\bar{S}$ at low, medium and high levels to equal 2%, 25%, and 48%, respectively. On the other hand, if $\bar{S}$ is set as the X-axis, the low, medium and high levels of $\bar{\sigma}$ are 1.6%, 20% and 39.2%, respectively. Hence, with different combinations of $\bar{S}$, $\bar{\sigma}$ and $\rho$, we can generate 18 different three-dimensional graphs for portfolios of two assets. Moreover, we can categorize these 18 graphs into two large groups. The first group (Group$_1$) is with different levels of $\bar{S}$, and the second group (Group$_2$) is with different levels of $\bar{\sigma}$. Then we can further divide each group into three sub-groups with different values for $\rho$, such as sub$_1$ with $\rho = -1$, sub$_2$ with $\rho = 0$ and sub$_3$ with $\rho = 1$.

Group$_1$-sub$_1$: set $\bar{S}$ at low, medium or high levels and $\rho = -1$. We find that the Percentage Change increases monotonically with the growth of both $\bar{\sigma}_{i,2}$ and $\bar{\sigma}$, regardless of what levels we set for $\bar{S}$. From Figures 4.2, 4.3 and 4.4, we can observe that $\bar{\sigma}_{i,2}$ has a larger impact on the Percentage Change in comparison to $\bar{\sigma}$. However, when $\bar{S}$ increases from 2% to 25% and then to 48%, the scale of the impact of $\bar{\sigma}_{i,2}$ and
\( \bar{\sigma} \) on the Percentage Change also increases. In other words, the growth of \( \bar{\sigma} \) will increase the degree of the impact of \( \bar{\sigma}_{1,2} \) and \( \bar{\sigma} \) on the Percentage Change.

Figure 4.2: The Percentage Change for portfolios of two assets with perfectly negative correlation when Spread-Bar is low

\begin{align*}
\text{Spread-Bar} &= 2\%, \quad \text{Rho} = -1, \quad P1 = P2 = 100, \quad a1 = a2 = 2, \quad b = 1
\end{align*}
Figure 4.3: The Percentage Change for portfolios of two assets with perfectly negative correlation when Spread-Bar is medium

Spread-Bar = 25%, Rho = -1, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.4: The Percentage Change for portfolios of two assets with perfectly negative correlation when Spread-Bar is high.

Spread-Bar = 48%, Rho = -1, P1 = P2 = 100, a1 = a2 = 2, b = 1

Group1-sub2 (Figure 4.5, 4.6 and 4.7) and Group1-sub3 (Figure 4.8, 4.9 and 4.10) have similar features as those of Group1-sub1. The only difference is that the scale of the Percentage Change as a whole is smaller as $\rho$ increases. As we know, the Percentage Change is \[ \frac{LAVaR_{p} - VaR_{p}}{VaR_{p}} \]. Then the denominator, $VaR_{p}$, increases proportional to $\rho$. Meanwhile, the numerator decreases as $VaR_{p}$ increases. Therefore, the larger the value of $\rho$, the smaller the Percentage Change.
Figure 4.5: The Percentage Change for portfolios of two assets without correlation when Spread-Bar is low

Spread-Bar = 2%, Rho = 0, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.6: The Percentage Change for portfolios of two assets without correlation when Spread-Bar is medium

Spread-Bar = 25\%, \rho = 0, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.7: The Percentage Change for portfolios of two assets without correlation when Spread-Bar is high

Spread-Bar = 48%, Rho = 0, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.8: The Percentage Change for portfolios of two assets with perfectly positive correlation when Spread-Bar is low

Spread-Bar = 2%, Rho = 1, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.9: The Percentage Change for portfolios of two assets with perfectly positive correlation when Spread-Bar is medium

Spread-Bar = 25%, Rho = 1, P1 = P2 = 100, a1 = a2 = 2, b = 1
Group2-sub1: fix $\overline{\sigma}$ at the low, medium or high levels and $\rho = -1$. The Percentage Change in Group2-sub1 also increases monotonically in proportion to $\overline{\sigma}_{1,2}$ and $\overline{S}$, no matter what levels we set for $\overline{\sigma}$. However, the increase of $\overline{\sigma}$ from 1.6% to 20% and then to 39.2% will enhance the degree of the impact of $\overline{\sigma}_{1,2}$ and $\overline{S}$ on the Percentage Change. Furthermore, Figures 4.11, 4.12 and 4.13 reveal that $\overline{\sigma}_{1,2}$ has a larger impact on the Percentage Change than does $\overline{S}$.
Figure 4.11: The Percentage Change for portfolios of two assets with perfectly negative correlation when Sigma-S is low

Sigma-S = 1.6%, Rho = -1, P1 = P2 = 100, a1 = a2 = 2, b = 1

spread s
covariance-S

Percentage of Change

0.5 0.5 0.0 0.2 0.3 0.4 0.5

0.5

0

-1

0.1

0.2

0.3

0.4

0.5

0.5

0.0

-1

s

spread s
Figure 4.12: The Percentage Change for portfolios of two assets with perfectly negative correlation when Sigma-S is medium

$$\text{Sigma-S} = 20\%, \text{ Rho} = -1, \text{ P1} = \text{ P2} = 100, a1 = a2 = 2, b = 1$$
Figure 4.13: The Percentage Change for portfolios of two assets with perfectly negative correlation when Sigma-S is high.

\[
\text{Sigma-S} = 39.2\%, \ Rho = -1, \ P1 = P2 = 100, \ a1 = a2 = 2, \ b = 1
\]

Group2-sub2 (Figure 4.14, 4.15 and 4.16) and Group2-sub3 (Figure 4.17, 4.18 and 4.19) inherit similar features as those of Group2-sub1. The only discrepancy is that the scale of the Percentage Change as a whole is smaller when \( \rho \) increases. This result is ascribed to the same reason as mentioned above. As \( \rho \) increases, the denominator in the Percentage Change equation also increases; however, the numerator decreases. Therefore, the scale of the Percentage Change will be smaller as a whole.
Figure 4.14: The Percentage Change for portfolios of two assets without correlation when Sigma-S is low

\[ \text{Sigma-S} = 1.6\%, \ Rho = 0, \ P1 = P2 = 100, \ a1 = a2 = 2, \ b = 1 \]
Figure 4.15: The Percentage Change for portfolios of two assets without correlation when Sigma-S is medium

Sigma-S = 20%, Rho = 0, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.16: The Percentage Change for portfolios of two assets without correlation when Sigma-S is high

Sigma-S = 39.2%, Rho = 0, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.17: The Percentage Change for portfolios of two assets with perfectly positive correlation when Sigma-S is low

Sigma-S = 1.6%, Rho = 1, P1 = P2 = 100, a1 = a2 = 2, b = 1
Figure 4.18: The Percentage Change for portfolios of two assets with perfectly positive correlation when Sigma-S is medium

Sigma-S = 20%, Rho = 1, \( P1 = P2 = 100 \), \( a1 = a2 = 2 \), \( b = 1 \)
When considering portfolios of three assets, we first assume that $S_1 = S_2 = S_3 = S$, $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_3 = \tilde{\sigma}$, $\tilde{\sigma}_{1,2} = \tilde{\sigma}_{1,3} = \tilde{\sigma}_{2,3} = \tilde{\sigma}_{i,j}$, $P_1 = P_2 = P_3 = $100, $a_1 = a_2 = a_3 = 2$, $\rho_{1,2} = 0.3$, $\rho_{1,3} = -0.5$, $\rho_{2,3} = 0.2$ and $b_{1,2} = b_{1,3} = b_{2,3} = 1$. Given these assumptions, we find out that the three-asset case has the same features as those of the two-asset case. For example, the level of the Percentage Change increases with the growth of $\tilde{\sigma}_{i,j}$ and $\tilde{\sigma}$ (or $S$). Increasing the value of $S$ (or $\tilde{\sigma}$) only increases the degree of the impact of each of these parameters on the Percentage Change at a given value of $\tilde{\sigma}$.
Therefore, we can generalize $LAVaR$, for portfolios of $N$ assets based on the framework of $LAVaR$, for the two-asset case.

**Figure 4.20: The Percentage Change for portfolios of three assets without correlation when Spread-Bar is medium**

Spread-Bar = 25%
Rho1,2 = 0.3, Rho1,3 = -0.5, Rho2,3 = 0.2
P1 = P2 = P3 = 100, a1 = a2 = a3 = 2, b1,2 = b1,3 = b2,3 = 1
However, in the two-asset case, different graphs have different scales on the Z-axis; therefore it is difficult to determine which component has the larger impact on the Percentage Change by observing different graphs. In order to correctly determine which component has the larger impact on the Percentage Change from a mathematical point of view, we need to calculate the slopes of $\bar{\sigma}_{1,2}$ and $\bar{\sigma}$ (or $\bar{S}$).
From Table 4.1 Panel A to C, we can see that $\tilde{\sigma}_{1,2}$ has the same slope at a given value of $\rho$ even if we change $\tilde{S}$ from 2% to 25% and then to 48%. However, the slope of $\tilde{\sigma}_{1,2}$ decreases as $\rho$ increases. This indicates that the impact of $\tilde{\sigma}_{1,2}$ on the Percentage Change diminishes as the value of $\rho$ increases. The slope of $\tilde{\sigma}$ has the same characteristics as those of $\tilde{\sigma}_{1,2}$. Additionally, Table 4.1 shows that the slope of $\tilde{\sigma}_{1,2}$ is equal to the slope of $\tilde{\sigma}$ at each value of $\rho$, regardless the variation of $\tilde{S}$ . This coincidence is because we assume that $a = 2$ and $b = 1$. In other words, when the scaling factor of $\tilde{\sigma}_{1,2}$ is a half of the scaling factor of $\tilde{\sigma}$, their slopes tend to be the same at a given value of $\rho$. Therefore, from our study we can conclude that $\tilde{\sigma}_{1,2}$ and $\tilde{\sigma}$ have
the same impact on the Percentage Change because they have the same slopes at different scenarios.

Table 4.2: Slope of $\tilde{\sigma}_{1,2}$ and slope of $\tilde{\sigma}$ when $b$ is equal to 0.75

Fixed $\tilde{S}$ at predetermined values, as 2%, 25% and 48%, respectively

<table>
<thead>
<tr>
<th>Panel A: $\rho = -1$, $a = 2$, $b = 0.75$, $\tilde{\sigma} = 0$</th>
<th>Panel D: $\rho = -1$, $a = 2$, $b = 0.75$, $\tilde{\sigma}_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{S}$</td>
<td>$\tilde{\sigma}<em>{1,2} = 1$, $\tilde{\sigma}</em>{1,2} = -1$, % Change</td>
</tr>
<tr>
<td>2%</td>
<td>4.1046</td>
</tr>
<tr>
<td>25%</td>
<td>4.7257</td>
</tr>
<tr>
<td>48%</td>
<td>5.3467</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: $\rho = 0$, $a = 2$, $b = 0.75$, $\tilde{\sigma} = 0$</th>
<th>Panel E: $\rho = 0$, $a = 2$, $b = 0.75$, $\tilde{\sigma}_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{S}$</td>
<td>$\tilde{\sigma}<em>{1,2} = 1$, $\tilde{\sigma}</em>{1,2} = -1$, % Change</td>
</tr>
<tr>
<td>2%</td>
<td>0.8917</td>
</tr>
<tr>
<td>25%</td>
<td>1.0287</td>
</tr>
<tr>
<td>48%</td>
<td>1.1616</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: $\rho = 1$, $a = 2$, $b = 0.75$, $\tilde{\sigma} = 0$</th>
<th>Panel F: $\rho = 1$, $a = 2$, $b = 0.75$, $\tilde{\sigma}_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{S}$</td>
<td>$\tilde{\sigma}<em>{1,2} = 1$, $\tilde{\sigma}</em>{1,2} = -1$, % Change</td>
</tr>
<tr>
<td>2%</td>
<td>0.6381</td>
</tr>
<tr>
<td>25%</td>
<td>0.7347</td>
</tr>
<tr>
<td>48%</td>
<td>0.8312</td>
</tr>
</tbody>
</table>

% Change: $(\text{LaVaR} - \text{VaR})/\text{VaR}$.

In Table 4.2, when $a = 2$ and $b = 0.75$, the slope of $\tilde{\sigma}_{1,2}$ has the same values at a given value of $\rho$, no matter what the level of $\tilde{S}$. Similarly, the slope of $\tilde{\sigma}$ is unchanged under the same conditions. However, the slope of $\tilde{\sigma}_{1,2}$ is smaller compared to that of $\tilde{\sigma}$ at a given level of $\rho$. On the contrary, Table 4.3 demonstrates that when $a = 2$ and $b = 2$, the slope of $\tilde{\sigma}_{1,2}$ has the same value at a given value of $\rho$, no matter the level of $\tilde{S}$. However, the slope of $\tilde{\sigma}_{1,2}$ is larger than that of $\tilde{\sigma}$ at a given level of $\rho$. 

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Table 4.3: Slope of $\sigma_{1,2}$ and slope of $\sigma$ when b is equal to 2

Fixed $\bar{S}$ at predetermined values, as 2%, 25% and 48%, respectively.

<table>
<thead>
<tr>
<th></th>
<th>Panel A: $\rho = -1, a = 2, b = 2, \bar{S} = 0$</th>
<th>Panel B: $\rho = 0, a = 2, b = 2, \bar{S} = 0$</th>
<th>Panel C: $\rho = 1, a = 2, b = 2, \bar{S} = 0$</th>
<th>Panel D: $\rho = -1, a = 2, b = 2, \sigma_{1,2} = -1$</th>
<th>Panel E: $\rho = 0, a = 2, b = 2, \sigma_{1,2} = -1$</th>
<th>Panel F: $\rho = 1, a = 2, b = 2, \sigma_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{S}$</td>
<td>$\bar{S}<em>{1,2} = 1, \bar{S}</em>{1,2} = -1,$</td>
<td>$\bar{S}<em>{1,2} = 1, \bar{S}</em>{1,2} = -1,$</td>
<td>$\bar{S}<em>{1,2} = 1, \bar{S}</em>{1,2} = -1,$</td>
<td>$\bar{S}_{1,2} = 0, \bar{S} = 0,$</td>
<td>$\bar{S}_{1,2} = 0, \bar{S} = 0,$</td>
<td>$\bar{S}_{1,2} = 0, \bar{S} = 0,$</td>
</tr>
<tr>
<td>% Change</td>
<td>% Change</td>
<td>% Change</td>
<td>% Change</td>
<td>% Change</td>
<td>% Change</td>
<td>% Change</td>
</tr>
<tr>
<td>2%</td>
<td>10.8015</td>
<td>-1.0.7475</td>
<td>10.8015</td>
<td>-8.5872</td>
<td>-1.0.7475</td>
<td>10.8015</td>
</tr>
<tr>
<td>25%</td>
<td>11.4766</td>
<td>-10.1264</td>
<td>10.8015</td>
<td>-7.9661</td>
<td>-10.1264</td>
<td>10.8015</td>
</tr>
</tbody>
</table>

% Change: $(LAVaR - VaR)/VaR.$

In summary, the level of $\bar{S}$ has no effect on the slopes of $\sigma_{1,2}$ and $\sigma.$ The slopes of $\sigma_{1,2}$ and $\sigma$ only change with $\rho.$ When $\rho$ increases, the slopes of $\sigma_{1,2}$ and $\sigma$ decrease. Moreover, the value of $b$ plays an important role in determining the slope of $\sigma_{1,2}.$ In other words, the larger $b$ is, the larger impact $\sigma_{1,2}$ has on the Percentage Change. According to the study done by Brockman, Chung and Péron (2006), commonality in liquidity is a global phenomenon. Therefore, the value of $b$ should be larger than $\frac{a}{2}$ in order to reflect the importance of $\sigma_{1,2}$ for $LAVaR_{\rho}.$
Table 4.4: Slope of $\tilde{\sigma}_{1,2}$ and slope of $\tilde{S}$ when $b$ is equal to 1

Fixed $\sigma$ at predetermined values, as 1.6%, 20% and 39.2%, respectively

<table>
<thead>
<tr>
<th>Panel A: $\rho = -1$, $a = 2$, $b = 1$, $\tilde{S} = 0$</th>
<th>Panel B: $\rho = 0$, $a = 2$, $b = 1$, $\tilde{S} = 0$</th>
<th>Panel C: $\rho = 1$, $a = 2$, $b = 1$, $\tilde{S} = 0$</th>
<th>Panel D: $\rho = -1$, $a = 2$, $b = 1$, $\tilde{\sigma}_{1,2} = -1$</th>
<th>Panel E: $\rho = 0$, $a = 2$, $b = 1$, $\tilde{\sigma}_{1,2} = -1$</th>
<th>Panel F: $\rho = 1$, $a = 2$, $b = 1$, $\tilde{\sigma}_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\sigma}$</td>
<td>$\tilde{\sigma}<em>{1,2} = 1$, $\tilde{\sigma}</em>{1,2} = -1$, $%$ Change</td>
<td>$%$ Change</td>
<td>$%$ Change</td>
<td>$%$ Change</td>
<td>$%$ Change</td>
</tr>
<tr>
<td>1.6%</td>
<td>5.4872</td>
<td>5.4007</td>
<td>1.1921</td>
<td>-1.1546</td>
<td>0.8531</td>
</tr>
<tr>
<td>20%</td>
<td>6.4809</td>
<td>5.4007</td>
<td>1.4080</td>
<td>-0.9387</td>
<td>1.0076</td>
</tr>
<tr>
<td>39.2%</td>
<td>7.5178</td>
<td>5.4007</td>
<td>1.6303</td>
<td>-0.7834</td>
<td>1.1688</td>
</tr>
</tbody>
</table>

% Change: ($LAVaR - VaR)/VaR.

Table 4.4 shows that the slope of $\tilde{\sigma}_{1,2}$ is the same at a given level of $\rho$, even if we change $\sigma$ from 1.6% to 20% and then to 39.2%. It also shows that the slope of $\tilde{\sigma}_{1,2}$ diminishes as $\rho$ increases. The slope of $\tilde{S}$ also demonstrates the above features. However, the slope of $\tilde{\sigma}_{1,2}$ is always larger than the slope of $\tilde{S}$ in each different scenario because we set a scaling factor, $b$, for $\tilde{\sigma}_{1,2}$. The slope of $\tilde{S}$ can be equal to the slope of $\tilde{\sigma}_{1,2}$ only if $b = \frac{1}{2}$. However, because we find out that the value of $b$ should be larger than $\frac{a}{2}$ in order to reflect the importance of $\tilde{\sigma}_{1,2}$ for $LAVaR_\rho$, then $b$ will always be larger than 1, even if $a$ is at its lower bound. Therefore, $\tilde{\sigma}_{1,2}$ always dominates $\tilde{S}$ in
terms of their impact on the Percentage Change. We can verify the above phenomenon in Table 4.5.

Table 4.5: Slope of $\bar{\sigma}_{1,2}$ and slope of $\bar{S}$ when $b$ is equal to 2

Fixed $\bar{\sigma}$ at predetermined values, as 1.6%, 20% and 39.2%, respectively

<table>
<thead>
<tr>
<th>Panel A: $\rho = -1$, $a = 2$, $b = 2$, $\bar{S} = 0$</th>
<th>Panel B: $\rho = 0$, $a = 2$, $b = 2$, $\bar{S} = 0$</th>
<th>Panel C: $\rho = 1$, $a = 2$, $b = 2$, $\bar{S} = 0$</th>
<th>Panel D: $\rho = -1$, $a = 2$, $b = 2$, $\bar{\sigma}_{1,2} = -1$</th>
<th>Panel E: $\rho = 0$, $a = 2$, $b = 2$, $\bar{\sigma}_{1,2} = -1$</th>
<th>Panel F: $\rho = 1$, $a = 2$, $b = 2$, $\bar{\sigma}_{1,2} = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\sigma}$</td>
<td>$\bar{\sigma}<em>{1,2} = 1$, $\bar{\sigma}</em>{1,2} = -1$,</td>
<td>$\bar{\sigma}<em>{1,2} = 1$, $\bar{\sigma}</em>{1,2} = -1$,</td>
<td>$\bar{\sigma}<em>{1,2} = 1$, $\bar{\sigma}</em>{1,2} = -1$,</td>
<td>$\bar{S} = 0.5$, $\bar{S} = 0$,</td>
<td>$\bar{S} = 0.5$, $\bar{S} = 0$,</td>
</tr>
<tr>
<td>% Change</td>
<td>% Change</td>
<td>Slope $\bar{\sigma}_{1,2}$</td>
<td>% Change</td>
<td>% Change</td>
<td>Slope $\bar{\sigma}_{1,2}$</td>
</tr>
<tr>
<td>1.6%</td>
<td>10.8879</td>
<td>-10.7151</td>
<td>10.8015</td>
<td>1.6%</td>
<td>-9.3649</td>
</tr>
<tr>
<td>20%</td>
<td>11.8816</td>
<td>-9.7213</td>
<td>10.8015</td>
<td>20%</td>
<td>-8.3712</td>
</tr>
<tr>
<td>39.2%</td>
<td>12.9186</td>
<td>-8.6844</td>
<td>10.8015</td>
<td>39.2%</td>
<td>-7.3342</td>
</tr>
</tbody>
</table>

% Change: (LAVaR - VaR)/VaR.
5 EMPIRICAL ANALYSIS

We collected daily bid and ask prices of three different Canadian assets between June 01, 2005 and May 31, 2006 from Bloomberg. These three assets are Group Bikini Village Inc. (GBV), Haemacure Corporation (HAE) and Rex Diamond Mining (REX). Because we are collecting bid and ask prices from active trading days, we will ultimately have 252 data points for bid and ask prices of each asset. We then use these bid and ask prices to calculate the $S, \bar{\sigma}$ and $\bar{\sigma}_{i,j}$ for each of these three assets.

<table>
<thead>
<tr>
<th>Company</th>
<th>GBV</th>
<th>HAE</th>
<th>REX</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_t$ ($)</td>
<td>0.1175</td>
<td>0.2300</td>
<td>0.2175</td>
</tr>
<tr>
<td>$\bar{S}$ (%)</td>
<td>7.6314</td>
<td>7.8560</td>
<td>12.5675</td>
</tr>
<tr>
<td>$\bar{\sigma}$ (%)</td>
<td>4.1156</td>
<td>3.2430</td>
<td>6.3987</td>
</tr>
<tr>
<td>VaR ($)</td>
<td>0.0222</td>
<td>0.0207</td>
<td>0.0131</td>
</tr>
<tr>
<td>COL ($)</td>
<td>0.0093</td>
<td>0.0165</td>
<td>0.0276</td>
</tr>
<tr>
<td>LAVaR ($)</td>
<td>0.0315</td>
<td>0.0372</td>
<td>0.0406</td>
</tr>
<tr>
<td>% Change</td>
<td>41.9449</td>
<td>79.7761</td>
<td>211.1538</td>
</tr>
</tbody>
</table>

Data source: Bloomberg

The results from Table 5.1 verify our illustration in the one-asset case. The Percentage Change undergoes a monotonic increase with the growth of both $\bar{S}$ and $\bar{\sigma}$. For example, since REX has the largest $\bar{S}$ and $\bar{\sigma}$ among these three assets, REX has the largest Percentage Change, which is over 211%. For GBV and HAE, HAE has a larger Percentage Change than GBV, even though the differences between their $\bar{S}$ and $\bar{\sigma}$ are
not very big. This is because HAE has a higher $P_i$ than GBV. Therefore, HAE has a larger $COL$ and the percentage of change between $LAVaR$ and $VaR$ than GBV.

Table 5.2: Two-Asset Case

$(a_{GBV} = a_{HAE} = a_{REX} = 2, b_{GBV} = b_{HAE} = b_{REX} = 1.75)$

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>(GBV + REX)</th>
<th>(GBV + HAE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>0.1102</td>
<td>0.1644</td>
</tr>
<tr>
<td>$VaR_p$ ($)</td>
<td>0.0270</td>
<td>0.0327</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{i,j}$ (%)</td>
<td>0.0389</td>
<td>0.0004</td>
</tr>
<tr>
<td>$COL$ ($)</td>
<td>0.0186</td>
<td>0.0129</td>
</tr>
<tr>
<td>$LAVaR_p$ ($)</td>
<td>0.0456</td>
<td>0.0456</td>
</tr>
<tr>
<td>% Change</td>
<td>68.7967</td>
<td>39.456</td>
</tr>
</tbody>
</table>

Data source: Bloomberg

The $\tilde{\sigma}_{i,j}$ of Portfolio$_1$ (GBV and REX) is 0.0389%, which is larger than that of Portfolio$_2$ (GBV and HAE). Therefore, the Percentage Change for Portfolio$_1$ should be larger than the Percentage Change of Portfolio$_2$. This is illustrated in Table 5.2. The Percentage Change of Portfolio$_1$ is over 68%, which is larger than the 39% change of Portfolio$_2$.

Table 5.3: Three-Asset Case

$(a_{GBV} = a_{HAE} = a_{REX} = 2, b_{GBV} = b_{HAE} = b_{REX} = 1.75)$

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>(GBV + REX + HAE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$VaR_p$ ($)</td>
<td>0.0372</td>
</tr>
<tr>
<td>$COL$ ($)</td>
<td>0.0179</td>
</tr>
<tr>
<td>$LAVaR_p$ ($)</td>
<td>0.0551</td>
</tr>
<tr>
<td>% Change</td>
<td>48.1402</td>
</tr>
</tbody>
</table>

Data source: Bloomberg
In the three-asset case, Table 5.3 demonstrates that the Percentage Change is 48.14%. In other words, when the traditional VaR model neglects liquidity, $VaR_p$ misses 48.14% of the aggregate market risk.
6 CONCLUSION

Liquidity risk is an integral component of market risk. However, it is often neglected by the standard parametric VaR model. Without incorporating liquidity into the standard parametric VaR model, the market risk will be underestimated, as a whole. In order to effectively manage market risk, we need to integrate liquidity into the standard parametric VaR model.

In this project, we first applied the Liquidity-Adjusted Value-at-Risk model proposed by Bangia, Diebold, Schuermann and Stroughair (1998) for the one-asset case. We demonstrate that the percentage of change between $\text{LAVaR}$ and $\text{VaR}$ increases monotonically with the growth of both $\bar{S}$ and $\bar{\sigma}$. Based on the study of Brockman, Chung and Pétrignon (2006), commonality in liquidity is a pervasive phenomenon across the globe. It indicates that $\bar{\sigma}_{i,j}$ plays an important role in determining the $\text{COL}$. With the studies of commonality in liquidity, we later extend the model developed by Bangia, Diebold, Schuermann and Stroughair (1998) into the two-asset case by adding a scaled $\bar{\sigma}_{1,2}$ term in the $\text{COL}$ equation. The percentage of change between $\text{LAVaR}_p$ and $\text{VaR}_p$ in the two-asset case still displays a monotonic increase with increasing values of $\bar{\sigma}_{1,2}$ and $\bar{\sigma}$ (or $\bar{S}$). Moreover, from our illustration, we find out that the value of $b$ should be larger than $\frac{a}{2}$ in order to reflect the importance of the co-movement of liquidity in
Using the same approach, we extend our framework of $LAVaR_p$ for portfolios of $N$ assets.

Finally, Bangia, Diebold, Schuermann and Stroughair (1998) pioneer a framework that integrates liquidity into the traditional VaR model. However, other methods to incorporate liquidity into the traditional VaR model also exist, such as the Historical Simulation. Moreover, there are many elements other than $\bar{S}$ and $\bar{\sigma}$ that can affect liquidity. For example, the negative asset return is correlated with liquidity, and market size and depth can also affect liquidity. For further study, scholars can consider negative asset returns, and the market size and depth when they develop a new model to incorporate liquidity into the traditional VaR model.
REFERENCE LIST


