FUNDAMENTAL SNR AND SAR LIMITATIONS IN
VERY LOW FIELD MRI

by

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Abstract

New techniques for magnetic resonance imaging (MRI) at low field strength (and hence low frequency) are currently being developed. Two important factors related to the weak electrical conductivity of the human body remain uncharacterized in this regime: (i) the intrinsic signal-to-noise ratio (ISNR) and (ii) the specific absorption rate (SAR) for electromagnetic energy associated with radiofrequency (RF) tipping pulses. This thesis presents experiments – performed over the frequency range 0.01 to 1.25 MHz – in which saline phantoms and human subjects were exposed to low-level oscillating magnetic fields produced by RF coils appropriate for low field MRI. The sample perturbs the quality factor and field map of the coil, and measurements of these properties were used to quantify ISNR and SAR. Results obtained from phantoms show excellent agreement with analytical predictions; results obtained from human subjects are directly applicable to design of low field MRI devices and pulse sequences.

Keywords: signal-to-noise ratio (SNR); specific absorption rate (SAR); low field nuclear magnetic resonance (NMR); magnetic resonance imaging (MRI); very low field imaging; hyperpolarized noble gases.
for Hieg, who keeps me going
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Chapter 1

Introduction

Nuclear magnetic resonance (NMR) experiments are typically performed using strong static magnetic fields (\( \sim 1 \text{T} \) and above), so as to achieve high signal-to-noise ratios (SNR). Recently, new NMR techniques employing much weaker static magnetic fields have been developed. For example, \textit{in vivo} NMR and magnetic resonance imaging (MRI) of human lung airspaces with hyperpolarized noble gases have been performed at 'very low' field strengths of 100 mT [1] and \( \sim 10 \text{ mT} \) [2–5]. In fact, MRI at field strengths as low as \( \sim 100 \mu \text{T} \) has been demonstrated using thermal pre-polarization techniques coupled with SQUID-detection [6–9].

Because very low field MRI is an experimental technique that is in its infancy, design constraints for both hardware (especially transmit and receive coils) and radiofrequency (RF) pulse sequences (which are necessarily applied for manipulation of nuclear spins) have not yet been systematically determined. One issue influencing these constraints is image quality, which is strongly dependent on the SNR. An understanding of SNR is necessary to design very low field MRI receive coils for improved image quality. A second issue that imposes constraints is the safety of the subject undergoing an MRI exam. The safety of the subject is in part determined by the specific absorption rate (SAR) of electromagnetic energy from the RF fields. Health safety limitations on the maximum safe SAR in turn place limits on the design of RF pulse sequences. Characterization of the SNR and SAR in MRI experiments involving human subjects is therefore essential to the development of MRI. Extensive investigations of SNR and SAR have been carried out under conventional high field conditions [10–16]; however these properties have yet to be characterized under
very low field conditions.

In general, many factors affect the SNR and SAR, such as geometry of the transmit and receive coils, the shape of the sample, and the electrical conductivity of the sample. First, consider factors influencing the SNR. Two intrinsic sources of noise in an MRI experiment are the Johnson noise associated with the resistance of the receive coils (used to detect the NMR signal), and the Johnson noise associated with the random motion of charges within the subject. It is well understood that the conductivity of an MRI sample contributes to the Johnson noise in a receiver circuit [11] and that this "body noise" increases with the magnetic field strength, often becoming the dominant noise source in conventional, high field MRI experiments involving human subjects. In spite of this, the SNR is still improved at high fields because the signal increases with frequency faster than does the noise [10]. SNR has been characterized for high field MRI [12, 16]; although the body noise dominates, receiver design is still an important issue and thus cryogenically cooled receivers have been investigated [17, 18]. In very low field MRI, it is suspected that the body noise is in fact smaller than the intrinsic Johnson noise of the receiver circuit itself, indicating that receiver design will be much more important. While some investigations of SNR have been conducted in the context of very low field MRI [18–20] a direct comparison of the body and coil noise contributions has not been reported.

Second, consider factors influencing the SAR. The applied RF magnetic fields deposit energy into the sample in the form of ohmic losses associated with induced eddy currents. SAR safety guidelines [21, 22] for human subjects have been set to ensure patient safety during an MRI exam. These SAR limits place constraints on the strength and duration of RF pulses, which in turn limit the time needed to acquire an image. The image acquisition time is a factor in how hospitals schedule the use of MRI and how technicians choose which sequence balances imaging needs with RF exposure considerations. Many investigations have been conducted to probe SAR in the context of conventional high field MRI [13–16]. The dependence of the SAR on the frequency and strength of the RF fields has been predicted; however we are not aware of any very low field MRI studies characterizing SAR in human subjects. Since the frequency of the RF field is proportional to the applied static field strength, the dependence of SAR on frequency is an especially important consideration. This type of study is required because high frequency data cannot necessarily be extrapolated down to lower frequencies; more importantly, any extrapolations must be
verified with definitive experimental data in order to satisfy national health safety criteria.

The purpose of this thesis is to fill the gap in information about SNR and SAR as it pertains to very low field MRI. I first investigate a model problem consisting of an electrically conducting sphere placed in a uniform oscillating magnetic field. This is followed by a parallel experiment involving human subjects, allowing realistic conclusions regarding SNR and SAR for \textit{in vivo} MRI under very low field conditions to be drawn.

I begin with a classic problem in electrodynamics that involves solving the eddy current distribution in a conducting sphere placed in a uniform oscillating magnetic field. This model problem was first explored in the context of MRI by Hoult and Lauterbur \cite{11} and there exists a large body of MRI literature on this subject \cite{23-25}. The primary value of the problem is that it can be solved analytically – allowing the current density, magnetic field, and power dissipated by ohmic losses to be written in closed form – and represents a well-understood reference against which experiments may be quantitatively compared. I present a method of solution to the problem and give a complete set of expressions for the current density, magnetic field, and power dissipation in the low field limit. The ohmic losses and magnetic field were then measured in two independent experiments that involved exposing spherical phantoms to a weak magnetic field oscillating at frequencies in the range 0.1 to 1.25 MHz. The ohmic losses were measured by detecting the perturbing effect of the phantom on a resonant circuit, in a technique analogous to cavity perturbation \cite{26}. The oscillating magnetic field in the vicinity of the sphere was measured directly using a small probe coil. The results from both experiments are seen to be in excellent agreement with predictions of the model. Previous investigations of this problem have all been at high frequencies; to our knowledge this experiment is the first full characterization of this electrodynamic problem in the low frequency limit.

With an understanding of the model problem in hand, similar experiments involving human subjects were then performed, and yield results that allow conclusions regarding SNR and SAR of low field MRI experiments to be drawn. These experiments involved exposing human subjects to low level oscillating magnetic fields produced by nearby coils, similar to a design used previously as receiver coils for very low field MRI \cite{3, 4}. The ohmic losses in the subject are measured in the same manner as that used for the phantoms, and can be directly related to the noise that the subject would produce in an MRI experiment. In the same way, the noise that would be produced by the receive coils was also effectively mea-
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Comparison of these contributions indicates which of the two is the dominant source of noise under the conditions of the experiment, and the degree to which improvements can be made by reducing the receiver coil resistance.

The experiment probing SAR was similar to that for SNR, except that a transmit coil was used in place of the receive coils. Since the field generated by the transmit coil is very uniform, the power deposition is also relatively uniform within the subject; measurements of ohmic losses thus reflect the average SAR of energy deposited over the volume of the subject exposed to the field.

To our knowledge the experiments presented in this thesis are the first that directly investigate SAR and SNR in human subjects at the frequencies relevant to very low field MRI.

1.1 Outline of thesis

This thesis is structured as follows. In Chapter 2, I present the basics of NMR and MRI, and introduce MRI of hyperpolarized noble gases in both the low and high field limits; I then describe the dependence of SNR and SAR on frequency and the role played by the conductivity of the sample, leading to a detailed motivation for this thesis. Chapter 3 contains a description of the mathematics underlying the problem of a conducting sphere exposed to a uniform oscillating magnetic field. In Chapter 4, I describe the experimental apparatus including the transmit and receive coils; in Chapter 5 I describe the methods used to collect and analyze the data. Data obtained from experiments on the spherical phantoms are presented and discussed in Chapter 6, while the same is done for the experiments involving human subjects in Chapter 7. I present the conclusions from this work and make suggestions for further avenues of research in Chapter 8.
Chapter 2

Background

In this chapter I present background information necessary to put the work described later in this thesis into context. I begin with a brief classical description of nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI) in order to present key concepts. I then introduce aspects of emerging MRI modalities involving hyperpolarized gases and very low field strengths. This material is followed by discussions of signal-to-noise ratio (SNR) and specific absorption rates (SAR) for MRI involving electrically conducting media, which are essential concepts in this thesis. Finally a detailed motivation for the work presented in this thesis is given.

2.1 Nuclear magnetic resonance

Nuclear magnetic resonance was discovered independently by Bloch and Purcell in 1946 [27, 28]. In a static magnetic field $B_0$, a spin-1/2 nucleus tends to align with its magnetic moment parallel or antiparallel to the magnetic field. Since the parallel state has a lower energy than the antiparallel state, more nuclei occupy this favoured state at thermal equilibrium. A macroscopic sample has a net nuclear magnetization per unit volume $M$ parallel to the applied field. Denoting the number of parallel and antiparallel nuclei per unit volume as $N^+$ and $N^-$ respectively, we define the fractional polarization as

$$P = \frac{N^+ - N^-}{N^+ + N^-}. \quad (2.1)$$
CHAPTER 2. BACKGROUND

The magnitude of the magnetization per unit volume is defined as

\[ M = \mu_n (N^+ - N^-) = \mu_n PN \]  

(2.2)

where \( \mu_n \) is the magnetic moment of a single nucleus and \( N = N^+ + N^- \) is the total number of nuclei per unit volume. The population of nuclei in the parallel and antiparallel states is described by a Boltzmann distribution. The resulting magnetization is [29]

\[ M = \mu_n N \tanh \left( \frac{\mu_n B_0}{k_B T} \right), \]  

(2.3)

where \( k_B \) is the Boltzmann constant and \( T \) is the temperature. For hydrogen nuclei at body temperature and experimentally accessible magnetic fields, the argument of Eq. 2.3 is very small; typically the ratio \( \mu_n B_0 / k_B T \sim 1 - 10 \) ppm. As a result, since \( \tanh(x) \approx x \) for small \( x \), the magnetization is proportional to the magnetic field under experimental conditions:

\[ M \approx \mu_n N \left( \frac{\mu_n B_0}{k_B T} \right) = \frac{N \mu_n^2 B_0}{k_B T}. \]  

(2.4)

In a classical sense, the motion of \( M \) is described by

\[ \frac{dM}{dt} = \gamma M \times B_0, \]  

(2.5)

where \( \gamma \) is the gyromagnetic ratio of the nucleus. If \( M \) is aligned at an angle to \( B_0 \), Eq. 2.5 describes a precession of \( M \) about \( B_0 \) with an angular frequency

\[ \omega_0 = \gamma B_0 \]  

(2.6)

where \( B_0 \) is the magnitude of \( B_0 \) and \( \omega \) is the Larmor frequency. In the reference frame rotating about \( B_0 \) at this frequency, \( M \) is stationary and there is effectively no magnetic field.

Consider now the application of a magnetic field \( B_1 \) (magnitude \( B_1 \)), rotating about \( B_0 \) at the Larmor frequency. In the rotating frame, \( B_1 \) is stationary and \( M \) now precesses about \( B_1 \) at an angular frequency \( \omega_1 = \gamma B_1 \). For a \( B_1 \) applied perpendicular to \( B_0 \), \( M \)

\[ \text{\footnotesize{\textsuperscript{1}}Since the field strength and frequency are directly related in this way, field and frequency are often used interchangeably in the NMR literature.} \]
is tipped away from $B_0$ in the plane perpendicular to $B_1$. After $B_1$ has been applied for a period of time $t$, the angle $\alpha$ between $B_0$ and $M$ is given by

$$\alpha = \omega_1 t$$

(2.7)

In practice, a linearly oscillating field may be used in place of the rotating field described above since it can always be broken down into two counter-rotating field components. Commonly the Larmor frequency is in the radiofrequency (RF) range and therefore the $B_1$ field is referred to as the RF field or RF radiation; the RF field is often applied in short pulses. Note also that RF fields that are not applied at the Larmor frequency tend to average to zero in the rotating frame and do not affect the magnetization except under specialized circumstances.

After the application of an RF pulse, relaxation processes associated with energy transfer mechanisms cause the magnetization to return to equilibrium. This behaviour is phenomenologically described by the Bloch Equation:

$$\frac{dM}{dt} = \gamma M \times B_0 - \frac{M_x e_x + M_y e_y}{T_2} - \frac{(M_z - M_0) e_z}{T_1}$$

(2.8)

where $B_0$ is applied in the $z$-direction$^2$, $e_x$, $e_y$, and $e_z$ are the Cartesian unit vectors, and $M_x$, $M_y$ and $M_z$ are the Cartesian components of the magnetization vector $M$. The constant $T_1$ is the spin-lattice relaxation time, which represents the time required for $M_z$ to return to its equilibrium value $M_0$. The constant $T_2$, is the spin-spin relaxation time which represents the time required for the transverse components $M_x$ and $M_y$ to vanish. The Bloch Equation has the following general solution:

$$M_x(t) = [M_x(0) \cos(\omega_0 t) + M_y(0) \sin(\omega_0 t)]e^{-t/T_2}$$

(2.9)

$$M_y(t) = [-M_x(0) \sin(\omega_0 t) + M_y(0) \cos(\omega_0 t)]e^{-t/T_2}$$

(2.10)

$$M_z(t) = M_0 + [M_z(0) - M_0]e^{-t/T_1}.$$ 

(2.11)

To describe how the resonance is detected experimentally, we first define the transverse

$^2$This choice of coordinate system is the normal convention in NMR. However it will only be used in this chapter; following chapters use a slightly different coordinate system in which the $z$-axis is aligned with $B_1$. 
magnetization as
\[ M_T(t) = \sqrt{M_x(t)^2 + M_y(t)^2} \]
\[ = \sqrt{M_x(0)^2 + M_y(0)^2} e^{-t/T_2} \]
\[ = M_T e^{-t/T_2} . \]

The resonance can be detected with a nearby coil oriented with its axis perpendicular to \( B_0 \). At the position of the receive coil, the field component parallel to the coil axis varies sinusoidally as a function of time and is proportional to the transverse magnetization \( M_T(t) \)
\[ B_T(t) \propto M_T(t) \sin(\omega_0 t + \theta) . \]

The magnetic flux \( \Phi \) passing through the coil is proportional to \( B_T \) and the voltage \( v \) induced across the coil – also called an electromotive force (EMF) – is given by Faraday’s Law
\[ v(t) \propto \frac{d\Phi}{dt} \propto \omega_0 \cos(\omega_0 t + \theta) M_T(t) + \sin(\omega_0 t + \theta) \frac{dM_T(t)}{dt} . \]

Since the relaxation time \( T_2 \) is usually much longer than the period of precessional rotation, we can neglect the second term and write
\[ v \propto M_T \omega_0 \cos(\omega_0 t + \theta) e^{-t/T_2} . \]

This equation describes the observed voltage, called a free induction decay (FID). It is also worth noting that since the field of a magnetic dipole drops off as the third power of distance, the coil is most sensitive to nearby spins.

The time constants \( T_1 \) and \( T_2 \) give information about relaxation mechanisms in a material, making NMR a useful technique in materials science. Additionally, the local environment around each nucleus influences its Larmor frequency, making NMR spectroscopy an invaluable technique for chemists to determine molecular structures. Since the amplitude of the FID is proportional to the density of the material, NMR is also well suited to biological samples where the prevalence of water provides a high density of hydrogen nuclei (protons) for NMR. For more detailed information regarding NMR, the reader is directed to references [30, 31].
CHAPTER 2. BACKGROUND

2.2 Magnetic resonance imaging

In the 1970's, Lauterbur and Mansfield independently recognized that NMR could be used to non-invasively image materials and biological samples [32, 33]. This imaging modality is called magnetic resonance imaging (MRI). I give a brief simplified description of a common way to acquire images of "slices" of the sample (tomographic imaging); more detailed descriptions may be found elsewhere [34, 35].

In tomographic imaging, a 'slice' of the sample to be imaged is selected. A gradient in the static field \( B_0 \) is applied across the sample in a direction perpendicular to the desired slice. In a thin slice of sample where the Larmor frequency matches the frequency of the applied RF field \( B_1 \), the nuclear magnetization is tipped away from the axis of the static field. Application of gradients in \( B_0 \) causes the Larmor frequency and precession angle (or phase) of the nuclei within the slice to depend on their position in a precisely defined way.

The voltage induced across the receive coil is a superposition of signals having a form similar to Eq. 2.17 (the exact form will depend on the way the image was acquired) with a range of frequencies \( \omega_0 \) and phase angles \( \theta \). The resulting data set can be analyzed with Fourier transform methods to convert frequency and phase information to a map indicative of physical properties, such as nuclear density, relaxation rates, and diffusivity.

A diagram of the main components of a MRI magnet system are shown in Fig. 2.1. The static field \( B_0 \) is usually parallel to the axis of the scanner, in which case it is generated by a solenoid (shown in the figure as the outermost layer of the scanner). The RF coils are used to manipulate the precession of the nuclear spins as well as to detect their response to applied fields. In some experiments, two separate coil systems are used to generate the RF fields and to detect the NMR signal, while in other cases both functions are served by a single coil system. Since these coils function as antennas that transmit and receive RF radiation, they are often referred to as transmit (\( T_x \)) and receive (\( R_x \)) respectively. The \( T_x \) and \( R_x \) coils are designed to generate and detect fields in the transverse plane. When separate \( T_x \) and \( R_x \) coils are used, they are usually arranged to be sensitive to perpendicular components of the transverse magnetization so as to avoid cross-talk, i.e. detection of the tipping pulses in the \( R_x \) coils.

Figure 2.2 is a diagram illustrating the functions of the transmit and receive coils. The transmit coil, shown in the top diagram, provides the RF pulses that are required to ma-
Figure 2.1: Schematic diagram showing a cross-sectional view of the main components of an MRI magnet. The static field $B_0$ is generated by the outer magnet (usually a solenoid). Field gradients are generated by the next layer of coils, while the RF is transmitted and received by the innermost layer of coils. The sample to be imaged is placed inside the scanner.
CHAPTER 2. BACKGROUND

RF Pulse

Figure 2.2: Block diagram indicating the coupling between the RF coils and the sample. RF fields are transmitted by the $T_x$ coil (top) and received by the $R_x$ coil (bottom). The NMR signal is recorded by a computer. The effective resistance of the $R_x$ coil includes contributions from the sample as well as the coil itself.

nipulate the spins. An unavoidable consequence of the application of RF fields is that they also generate eddy currents in the sample if it is electrically conducting. The receive coil, shown in the bottom diagram, detects the FID. The receive coils are primarily sensitive to two signals: coherent precession of the nuclei (shown by the arrow) and random motions of charges within the sample (shown by the random path of an electron). I will elaborate on these statements in following sections.

2.3 MRI of hyperpolarized noble gases

In MRI, the Larmor frequency is selected to excite the nuclei of interest, typically hydrogen nuclei (*i.e.* protons) in the water molecules of the sample. Conventional, high field MRI is
now a successful, well-established technique for detailed imaging of most of the anatomy. However it is difficult to image cavities such as the lungs where the density of hydrogen atoms is very low. Lung imaging is further complicated by differences in magnetic susceptibility between air and tissues which can give rise to strong, inherent field gradients that accelerate relaxation and impair resolution of images. While specialized techniques do exist to image lung tissue [36–38], emerging new modalities take a different approach by imaging pre-polarized noble gases which are inhaled into the lungs. By using laser optical pumping (OP) techniques [39–41], it is possible to polarize the nuclei of the spin-1/2 noble gases $^3$He and $^{129}$Xe to very high, non-equilibrium levels. This high degree of polarization is called hyperpolarization (HP) and such gases are referred to as hyperpolarized gases (HPG). The fractional polarization obtained with OP techniques is approximately $10^5$ times larger than the polarization obtained in conventional proton imaging, and compensates for the lower density of the gas (approximately 3000 times less dense than protons in the body) which would normally make it unviable for MRI. In addition to basic structural information, images of HP $^3$He can also give information about the function and ventilation of the lungs. For example, $^{129}$Xe actually dissolves into blood and tissue, making it a candidate to study perfusion and oxygenation. Recommended reviews of HPG MRI include [42–44]. The first HPG MRI in a biological setting was obtained in 1994 by introducing polarized $^{129}$Xe into the excised lungs and heart of a mouse [45]. The first in vivo HPG images were of rat and guinea pig lungs in 1995 [46] and 1996 [47]. Today a number of research groups routinely acquire in vivo HPG images at high fields for research purposes [43].

Because the magnetization of hyperpolarized gases depends on OP processes rather than the static field strength, HPG imaging in low magnetic fields should be possible. Because it is possible, and scientifically relevant, low field HPG MRI is currently being pursued by a number of groups. The hyperpolarization of gases eliminates the need for the large, expensive superconducting magnets used in conventional high field imaging. This allows many new configurations and designs for low field scanners including open geometries, which are less costly to make and operate [5].

Low field imaging also presents other advantages. Low field scanners are lighter and therefore more mobile, allowing the orientation of the magnet to be adjusted for imaging a patient in supine or standing positions [5]. Dedicated low field scanners would reduce demands on conventional MRI scanners in hospitals. New possibilities also arise in detection
methods; devices such as SQUIDs and other novel receive coils are being explored [6–9]. Additionally, gradients caused by susceptibility differences between materials are reduced in lower fields. An important advantage, and one that is directly relevant to this thesis, is that the reduced Larmor frequency associated with low frequency operation is expected to result in a corresponding reduction of the specific absorption rate (SAR) of electromagnetic energy deposited into the patient [11], making the technique more safe for subjects and raising possibilities for rapid new RF pulse sequences. This issue is addressed in Sec. 2.5.

Until recently, adaptations of pre-existing conventional high-field MRI technology have been used exclusively in HPG studies. Low field HPG imaging has been demonstrated in specialized custom built apparatus. The first published in vivo low field images of HPG in human lungs below 100 mT were acquired by Venkatesh et al. [2] at a static field strength of 15 mT, Bidinosti et al. [4] at 3 mT, and Mair et al. [5] at 3.8 mT. The images obtained by Bidinosti et al. currently represent the state-of-the-art in low field HPG MRI. Currently, five research groups are actively pursuing low field MRI. These groups are based at SFU, Carlton University and the Robarts Institute, the Ecole Normale Supérieure (LKB/Orsay), Harvard University (Harvard-Smithsonian and Brigham Women's Hospital), and the University of California at Berkeley. Since low field HPG MRI is a technique in its infancy, many of the issues that are well characterized at high fields have not been studied in detail at low fields. The quality of low field images lags behind that of high field images of HP gases; however it is believed that this situation is largely a result of the use of technology that has not been optimized for this type of imaging, rather than a fundamental limitation on the SNR. To clarify why this is the case, I now discuss some of the factors influencing the intrinsic SNR of an MRI image.

2.4 SNR Considerations in MRI

The image quality in MRI experiments is clearly linked to SNR. While the average SNR of an image ultimately depends on many factors, here we only consider the intrinsic SNR limited by fundamental factors. These factors were recognized when MRI was originally developed [10, 11].

The intrinsic SNR is the ratio of the detected signal amplitude to the intrinsic noise level. The signal is the amplitude of the FID induced in the receiver coil and the intrinsic
noise is the thermal noise associated with random motion of charges in the receiver coil and sample. As described in Section 2.1, the FID amplitude is proportional to both the magnetization and the Larmor frequency:

\[ |v| \propto Mf \]  \hspace{1cm} (2.18)

where we are now using the frequency \( f = \omega_0/(2\pi) \) and it is implied from now on that frequencies refer to Larmor frequencies.

The frequency dependence of the magnetization is very different in the cases of conventional and HPG imaging. In conventional imaging, the nuclear magnetization is established by the static magnetic field \( B_0 \). As described in Section 2.1, the equilibrium magnetization is proportional to the nuclear density \( N \) and static field strength \( B_0 \), which in turn determines the Larmor frequency (Eq. 2.6); thus we obtain the dependence of the conventional, equilibrium magnetization on the frequency:

\[ M_{\text{conv}} \approx \frac{2\pi N \mu_n^2 \gamma}{k_B T} f . \]  \hspace{1cm} (2.19)

Since it is not possible to change the temperature of the subject, increasing the Larmor frequency is the only way to increase the magnetization in conventional imaging. In contrast, HP gases are magnetized outside the body to very high non-equilibrium levels. The effect of the static field \( B_0 \) on the magnetization is negligible in comparison, and the magnetization of an HPG is given by

\[ M_{\text{HPG}} = \mu_n PN \]  \hspace{1cm} (2.20)

where \( P \) depends on the details of the OP processes used to generate it, rather than \( B_0 \).

Returning to the thermal noise, its behaviour can be understood if we now consider the RF detection circuit. If the sample is conducting, random motions of charges in the sample induce corresponding voltages in the receiver circuit. This type of noise in a circuit is well understood through the fluctuation-dissipation theorem and is called Johnson noise. The Johnson noise of a resistor \( R \) has a root mean-square voltage amplitude of

\[ (\overline{V_J^2})^{1/2} = \sqrt{4k_BT R \Delta f} \]  \hspace{1cm} (2.21)

where \( \Delta f \) is the receiver bandwidth. Because the conductivity of the sample results in Johnson noise in the receiver, the effect of the sample may be modeled as an additional
Figure 2.3: Cross-sectional view of the receiver coil wire. In the limit where the skin depth is much smaller than the wire radius, the conducting area is approximated by a narrow layer (of width $\delta$) at the surface of the wire.

resistance in the receiver circuit. This additional resistance is called the effective sample resistance and denoted $R_s$.

Combining the fact that the total effective resistance in the receive coil is the sum of the resistance of the receive coil itself ($R_{coil}$) and the effective sample resistance $R_s$ with Eqs. 2.18 and 2.21, we obtain

$$\text{SNR} \propto \frac{Mf}{\sqrt{R_{coil} + R_s}}.$$  (2.22)

The resistances $R_{coil}$ and $R_s$ have frequency dependencies that can be understood by considering the good conductor and weak conductor limits. The resistances depend on the relative sizes of the sample or the wires in the coil compared to the skin depth:

$$\delta = \sqrt{\frac{2}{\mu \omega \sigma}}.$$  (2.23)

for a material with bulk conductivity $\sigma$ and permeability $\mu$. When the radius $b$ of the wire making up the coil is thick compared to $\delta$, (the "thick limit") the conducting cross-sectional area may be approximated as a circle of thickness $\delta$ at the outer edge of the wire as shown Fig. 2.3. The resistance per unit length of the wire is then given by

$$\frac{R}{L} = \frac{1}{\sigma 2\pi b \delta} = \frac{1}{2b} \sqrt{\frac{\mu f}{\pi \sigma}}.$$  (2.24)
CHAPTER 2. BACKGROUND

Figure 2.4: Frequency dependence of the resistance of the body or sample (dashed curve) and receive coil (solid curve) on a log-log plot. The sample resistance has an \( f^2 \) behaviour and the coil resistance has an \( f^{1/2} \) behaviour. The high field regime encompasses conventional MRI techniques. The crossover (low field) and very low field regimes are currently being investigated.

The behaviour of the resistance in the opposite limit when the sample is thin compared to the skin depth ("thin limit") has been considered by Landau and Lifshitz [48] (in the case of a wire), who show that the resistance presented to inductively driven currents is proportional to \( f^2 \). In Chapter 3 this result will be shown in detail for a conducting sphere. The idealized frequency dependencies of \( R_{coil} \) and \( R_s \) are thus

\[
R_{coil} \propto f^{1/2} \quad \text{(2.25)}
\]

\[
R_s \propto f^2 \quad \text{(2.26)}
\]

as shown in Fig. 2.4. In practical high field situations (\( \sim 1.5 \) T), the body noise usually dominates over the coil noise. The term 'low field' generally refers to the crossover region, where the two resistances are comparable, (roughly in the range 0.1 to 1 T) and the term 'very low field' usually refers to the studies conducted in the range 1 to 100mT. 'Ultra
low field’ designates the region being explored by SQUID-detected techniques at μT fields. Now we may summarize the frequency dependence of the SNR in both conventional and HPG imaging. We define two limiting cases, the body-dominated regime and the coil-dominated regime, indicating which source of Johnson noise is most prevalent. Combining Eqs. 2.22 and 2.25, the SNR under coil-dominated conditions is given by

\[
\text{SNR}^{\text{coil}} \propto \frac{M f}{\sqrt{R_{\text{coil}}}} \propto M f^{3/4}
\]

while under body-dominated conditions, Eqs. 2.26 and 2.22 give

\[
\text{SNR}^{\text{body}} \propto \frac{M f}{\sqrt{R_s}} \propto M .
\]

Making use of Eqs. 2.19 and 2.20 we may write the frequency dependence of the intrinsic SNR for both conventional and HPG imaging, as summarized in Tab. 2.1. The relative sizes of \( R_s \) and \( R_{\text{coil}} \) determine the operating regime (and hence frequency dependence), while their absolute sizes influence the overall magnitude of the SNR. Clearly the SNR in conventional imaging benefits from an increased Larmor frequency; however in HPG imaging, the SNR increases only until the crossover to the body-dominated regime has been made, above which improvements are negligible.

The crossover between the coil-dominated and the body-dominated regimes may be a favourable operating point for imaging. Many calculations, simulations and measurements have been carried out to characterize SNR [10–12, 16, 49] at high fields but only a few investigations have probed SNR at low frequencies [18–20].

One way in which SNR might be improved can be understood by considering Fig. 2.4. The body resistance cannot be changed, but at low fields the resistance of the coil may

<table>
<thead>
<tr>
<th></th>
<th>Intrinsic SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Conventional</td>
</tr>
<tr>
<td>Body-dominated</td>
<td>( \propto f )</td>
</tr>
<tr>
<td>Coil-dominated</td>
<td>( \propto f^{3/4} )</td>
</tr>
</tbody>
</table>

Table 2.1: Dependence of the intrinsic SNR on frequency in the coil-dominated and body-dominated noise regimes, for both conventional and HPG MRI.
Figure 2.5: Frequency dependence of the resistance of various RF receivers for MRI, as it would appear on a log-log plot. As in Fig. 2.4, the resistance of the body (dashed curve) and receive coil (solid curve) are shown for comparison. Additionally the potential behaviour of cooled resistive coils, superconducting coils, and SQUID sensors are shown. The frequency at which the crossover between the body resistance dominated regime and coil resistance dominated regime occurs is reduced when alternative designs are used.

become the dominant noise source. Gains may be made to the SNR by redesigning the receive coil to have a lower resistance. This might be done by cryogenically cooling the windings and/or using superconducting or Litz wire. Alternatively, novel new sensors based on SQUIDs are being investigated.

In fact, combinations of these ideas have been used in numerous studies; for example cold normal conductors have been used at high frequency [17], superconducting coils are available commercially and have been used at low fields [50], and the use of superconducting coils coupled to SQUIDs have been demonstrated [51]. Litz wire has been used in both transmit coils [52, 53] and receive coils [54]. A detailed discussion of cryogenic RF coils is given in [18].

The idealized frequency dependence of the effective resistance of cooled resistive coils, superconducting coils, and SQUID detectors have been sketched in Fig. 2.5 along with that


\begin{table}
\begin{center}
\begin{tabular}{|l|c|}
\hline
Tissue & Low frequency $\sigma$ (S/m) \\
\hline
Muscle & 0.07 (perpendicular to fibres) \\
 & 0.86 (parallel to fibres) \\
Fat & 0.04 \\
Bone & 0.04 \\
Blood & 0.60 \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 2.2: Electrical conductivities of various tissues [13].

of resistive room-temperature coils, and the sample (or body) resistance. Comparison of these curves indicates the potential gains to SNR that might be made through appropriate coil design. The crossover between the body resistance-dominated and coil resistance-dominated regimes is expected to occur at lower fields if designs other than conventional room temperature resistive coil designs are used.

\section{2.5 SAR Considerations in MRI}

We now turn to the question of the SAR in conducting samples. The RF and other time-varying applied fields used in MRI drive eddy currents in the sample, resulting in the dissipation of energy. The weak electrical conductivity of the body is the dominant mechanism for the absorption of electromagnetic energy in human subjects. Measured values of various tissue conductivities have been reported [13, 55], and most tissues are seen to have a conductivity less than 1 S/m (Tab. 2.2).

The SAR is defined as the time-averaged rate of energy absorption per kilogram of sample – \textit{i.e.} power per kilogram,

$$\text{SAR} = \frac{<P>}{m}$$ \hspace{1cm} (2.29)

where $P$ now represents the total average power and $m$ is the mass of the sample. It will be shown in Chapter 3 that the power absorbed from $B_1$ is

$$<P> \propto B_1^2 R_s.$$ \hspace{1cm} (2.30)
CHAPTER 2. BACKGROUND

<table>
<thead>
<tr>
<th>Site</th>
<th>Dose</th>
<th>Time (min) equal to or greater than:</th>
<th>SAR (W/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>whole body</td>
<td>averaged over</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>head</td>
<td>averaged over</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>head or torso</td>
<td>per gram of tissue</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>extremities</td>
<td>per gram of tissue</td>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 2.3: SAR guidelines for magnetic resonance diagnostic devices, according to the US Food and Drug Administration [21].

In the thin limit, $R_s$ is proportional to $f^2$ by Eq. 2.26, which we combine with Eqs. 2.29 and 2.30 to give

$$\text{SAR} \propto B_1^2 R_s \propto B_1^2 f^2. \quad (2.31)$$

For a given operating frequency, SAR safety guidelines effectively place constraints on the strength and duration of the RF pulses. This in turn limits the speed at which an image can be acquired. In conventional imaging, $180^\circ$ tipping pulses (also called $\pi$-pulses) may be as short as a few milliseconds. In very low field applications, it should be possible to increase the strength of $B_1$ to rotate nuclei through the same tip angle on a much shorter timescale while keeping the same SAR. A summary of SAR recommendations for magnetic resonance imaging as given by the United States Food and Drug Administration [21] is presented in Tab. 2.3. These values indicate levels that put the patient at significant risk. Recommendations made by Health Canada [22] are less explicit, but consistent with those of the USFDA. SAR in MRI has been extensively investigated at high frequencies [13–16].

2.6 Experimental motivation

The motivation behind this thesis is twofold. First, since the SNR of an MRI image depends on the sizes of $R_s$ and $R_{coil}$, choices of coil design and operating frequencies rely heavily on an understanding of their behaviour. This thesis presents a measurement of $R_s$ for human subjects, characterizing the dependence on frequency and sample size. These results will help in the design of receive coils as well as optimization of the operating frequency for very low field MRI.
CHAPTER 2. BACKGROUND

Second, the SAR of human subjects during an MRI exam directly affects the RF pulse sequence design. An understanding of $R_s$ is thus fundamental to characterization of the SAR in conducting samples. The frequency dependence of $R_s$ as it pertains to SAR is also examined as part of the investigations presented in later chapters of this thesis. In both cases (i.e. SNR and SAR), this work is believed to be the first of its kind in the context of very low field MRI.

Before experiments on human subjects can be done, we must first understand a model problem and demonstrate that we can successfully measure $R_s$ of a model system, or phantom. This work presents a complete investigation of the model problem of a spherical conducting sample placed in a uniform oscillating magnetic field, focusing on the frequency range 0.1 - 1.25 MHz. First, I present a mathematical analysis of the analytic solution to this problem, which gives predictions for the current density, magnetic field, and power dissipation, (which in turn is related to $R_s$). Next, measurements of the sample resistance $R_s$ and magnetic field perturbation of spherical phantoms are presented and compared to the prediction of the model. These measurements were performed by exposing samples to a weak uniform oscillating magnetic field produced by a very low field MRI transmit coil. The coil constitutes the inductive element in a tuned resonant circuit, and the sample perturbs the quality factor and magnetic field map of the circuit due to the induced eddy currents. The induced losses in the sample are modeled as an additional resistance $R_s$ in the resonant circuit, determined from measuring the change in the quality factor when the sample enters the vicinity of the circuit. The oscillating magnetic field in the vicinity of a spherical phantom was also measured in several locations using a small probe coil. Both the results of $R_s$ and magnetic field measurements are compared to the model.

With a complete understanding of the model problem in hand, measurements of $R_s$ for human subjects become meaningful. Measurements of the effective sample resistance $R_s$ of human subjects are presented in two variations. The experimental setup is exactly the same as that used for measuring $R_s$ in spherical samples; that is, by detecting the perturbing effect of a sample on a resonant circuit.

To investigate SNR, we used receive coils as the inductive element. These coils were designed to mimic those used in a very low field MRI experiment [3, 4]. These coils are useful because they couple closely to the body and thus have a high sensitivity. This experiment gives practical values of the resistances $R_s$ and $R_{coil}$ over the frequency range
0.01 - 1.25 MHz. The absolute size and the relative frequency dependence of each term is indicative of the SNR encountered in very low field MRI experiments.

To investigate SAR, the same experiment was repeated using a uniform $B_1$ coil (i.e. a transmit coil) rather than the receive coils, to measure $R_s$ over the frequency range 0.1 - 1.25 MHz. Since the field is uniform, the value of $R_s$ that we measure is much more indicative of the overall average SAR that is encountered in low field MRI of human subjects.
Chapter 3

Spherical model

In this chapter, I present a mathematical analysis of the problem of a conducting sphere placed in a uniform oscillating magnetic field. Determination of the eddy current distribution in the sphere and the resulting field profile give predictions which may be compared to experiments.

The sphere is often used as an example in textbook problems in electromagnetism because it is a geometrically simple object, leading to tractable solutions. Investigation of an analytically convenient problem provides a foundation upon which to study more complicated systems. The problem of finding the eddy current distribution and magnetic field when a uniform conducting sphere is placed in a uniform oscillating magnetic field is relevant to a wide range of applications. A correspondingly wide range of approaches have been taken to find solutions to the problem. It is addressed in general electromagnetism textbooks [56, 57] but also appears in the literature of specific disciplines where the problem is relevant. Analytical expressions for eddy-current distributions in conducting media are of interest to engineers and geophysicists; inducing eddy currents in a material allows non-destructive testing for defects [58], and eddy-current probes can give information about the materials inside geological bore holes [59]. Among other applications, understanding the role played by eddy currents in determining the performance and efficiency of motors constitutes a sizeable field in engineering [60]. Further discussions of eddy-current losses in spheres from these viewpoints can be found in References [61–65].

Hoult and Lauterbur were the first to investigate eddy-current losses in conducting samples [11] from the viewpoint of MRI. Their 1979 paper in which intrinsic SNR limitations
were recognized and estimated represents a landmark in the development of modern MRI. They considered both the dielectric losses associated with the distributed capacitance of the RF coils, and the inductive (or magnetic) losses associated with induced eddy currents. While it is possible to reduce dielectric losses through coil design and specialized shields, the inductive losses are fundamental and constitute the focus of this thesis.

Hoult and Lauterbur calculated the rate at which energy is absorbed by a homogeneous, conducting sphere (conductivity $\sigma$, permittivity $\epsilon$, permeability $\mu$) in a uniform RF magnetic field (amplitude $B_1$, angular frequency $\omega$) in the limit when the wavelength of the RF field inside the material is much larger than the sample. Another assumption in the analysis is that the skin depth of the RF field inside the sphere is so large compared to the size of the sample that its effect can be neglected completely. This second assumption makes their analysis extremely simple in comparison to most of the references cited above.

More advanced analyses of the sphere problem in the context of MRI include effects of the skin depth in the limit when it is larger than the sample but cannot necessarily be neglected (the thin limit). Carlson [66] solved the problem for the non-magnetic sphere ($\mu = \mu_0$, the permeability of free space) in the thin limit to obtain more complete expressions of the current density and magnetic field. It is worth noting that despite the widely recognized importance of Hoult and Lauterbur’s results (their paper has been cited some 400 times to date) and the numerous related articles in the MRI literature [23-25], many of the results published after Hoult and Lauterbur’s work contain errors. Particularly lacking are consistent, explicit expressions for the absorbed power and magnetic field. In this chapter I provide a summary of the correct versions of some results of this solution, under the conditions relevant to MRI. Complete results including some additional ones we have not found in existing literature are being prepared for publication [67].

### 3.1 Method of solution

Figure 3.1 illustrates the problem at hand: a uniform conducting sphere of radius $a$ is placed in an applied, uniform oscillating magnetic field $B_1 = B_1 e^{-\mu_0 \omega t} e_z$. We use spherical coordinates, where $r$ is the radial position, $\phi$ is the azimuthal angle, $\theta$ is the polar angle, and $e_r, e_\phi, e_\theta$ are the unit vectors in each of these directions, respectively. Electromagnetism in homogeneous linear media is handled with Maxwell’s Equations for the magnetic
CHAPTER 3. SPHERICAL MODEL

Figure 3.1: Schematic representation of a conducting sphere in a uniform oscillating magnetic field. The field is uniformly polarized in the $z$-direction.

Field $B$ and $E$ in such materials, which are

$$\mathbf{\nabla} \cdot \mathbf{E} = \frac{\rho_f}{\epsilon}$$  \hspace{1cm} (3.1)
$$\mathbf{\nabla} \cdot \mathbf{B} = 0$$  \hspace{1cm} (3.2)
$$\mathbf{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$  \hspace{1cm} (3.3)
$$\mathbf{\nabla} \times \mathbf{B} = \mu_0 \mathbf{\sigma} \mathbf{J} + \mu_0 \epsilon \frac{\partial \mathbf{E}}{\partial t}$$  \hspace{1cm} (3.4)

where $\rho_f$ is the free charge density and $\mathbf{J}$ is the free current density. The complete problem is specified by including Ohm's Law

$$\mathbf{J} = \mathbf{\sigma} \mathbf{E}.$$  \hspace{1cm} (3.5)

The free charge density $\rho_f$ is essentially zero.$^1$

$$\mathbf{\nabla} \cdot \mathbf{E} = 0.$$  \hspace{1cm} (3.6)

---

$^1$Even if there were a net charge on the sphere, it would flow to the surface on a timescale $\tau = \epsilon/\sigma$. In a physiological context, $\sigma \sim 1 \text{ S/m}$ and $\epsilon_{\text{max}} \sim 90 \text{ e}_0$ which corresponds to $\tau^{-1} \sim 80 \text{ GHz}$. Our experiments are performed at frequencies up to $\sim 1 \text{ MHz}$ and thus it is perfectly reasonable to ignore free charge in the interior of the sphere. Furthermore, the total number of ions introduced into our samples is so large that the sphere would have to be charged to a very high potential before free charges would influence the ohmic losses.
CHAPTER 3. SPHERICAL MODEL

There are many methods for solving Maxwell’s equations in the context of this problem. Jackson [68] and Carlson [66] use an expansion in vector spherical harmonics to determine \( B \); Harpen [25, 69] uses the vector potential. A more direct approach is provided by London [70], in which the current density \( J \) is solved and used to calculate the fields. I will follow London’s discussion; first I will follow the discussion presented by Griffiths [71] to derive the appropriate wave equation.

We begin with substitution of Eq. 3.5 into Eq. 3.3. Taking the curl of the resulting equation gives

\[
\nabla \times (\nabla \times J) = -\sigma \frac{\partial}{\partial t} (\nabla \times B).
\]  

(3.7)

Making use of Eq. 3.4, Eq. 3.6 and the vector identity

\[
\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A,
\]  

(3.8)

we obtain

\[
\nabla^2 J = \mu \sigma \frac{\partial J}{\partial t} + \mu \frac{\partial^2 J}{\partial t^2}.
\]  

(3.9)

We look for time harmonic solutions to the current density, reflecting the harmonic nature of the applied field:

\[
J(r, t) = J(r) e^{-i\omega t}
\]  

(3.10)

and thus Eq. 3.9 yields the wave equation

\[
(\nabla^2 + k^2)J(r) = 0
\]  

(3.11)

where the wavenumber \( k \) is defined as

\[
k^2 = \mu \varepsilon \omega^2 + i \sigma \mu \omega.
\]  

(3.12)

The symmetry of the problem determines the general form of allowed solutions for the current density. The applied field breaks the symmetry in the polar direction, but the currents cannot depend on the azimuthal angle \( \phi \). The currents must have closed paths as implied by Eq. 3.1 and the finite nature of the sample. These facts combined with Eq. 3.3 imply that the currents can only flow in the azimuthal direction:

\[
J(r) = J_\phi e^{i\phi}.
\]  

(3.13)
It makes sense that the currents all go in the same direction but they could vary according to the polar angle $\theta$, so we make the ansatz

$$J_\phi = Af(r) \sin(\theta) .$$

(3.14)

The form of the function $f(r)$ is determined from substitution of $J_\phi$ into the wave equation, which yields

$$\left( \frac{d^2}{dt^2} + \frac{2}{r} \frac{d}{dt} - \left( \frac{2}{r^2} + k^2 \right) \right) f(r) = 0 .$$

(3.15)

This equation is solved by the first order spherical Bessel functions $j_1(kr)$ and $n_1(kr)$; however $n_1(kr)$ diverges at the origin, so the current density has the form

$$J_\phi = A j_1(kr) \sin \theta .$$

(3.16)

Now that the current density is known, the magnetic field inside the sphere can be calculated. Assuming time harmonic solutions $B(r, t) = B(r) e^{-i\omega t}$, substitution of this form of the field into Eqs. 3.3 and making use of Eq. 3.5, we get

$$B(r) = \frac{1}{iωσ} \nabla \times J(r) .$$

(3.17)

Using the form of the current density implied by Eq. 3.16, we have the following magnetic field inside the sphere:

$$B_r = -A \frac{j_1'(kr)}{iωσr} \cos \theta$$

$$B_\theta = A \frac{j_1(kr) + kr j_1'(kr)}{iωσr} \sin \theta$$

(3.18)

(3.19)

where the prime represents the derivative with respect to $kr$ and $A$ is a coefficient determined by the boundary conditions.

Outside the sphere, we can use the quasistatic/long wavelength approximation — that the wavelength of the electromagnetic field is much larger than the size of the sphere — to drop the terms on the righthand side of Eq. 3.4:

$$\nabla \times B = 0 .$$

(3.20)

Considering the solutions of a spinning charged spherical shell and a uniformly magnetized sphere, it seems reasonable that the field outside may have the form of a dipole field, which
is irrotational. Thus we make the ansatz that the field outside is the combination of the applied field $B_1$ and an induced dipole field of dipole moment $m$:

$$B(r \geq a) = B_1 + \frac{\mu_0 m}{4\pi r^3}(2 \cos \theta e_r + \sin \theta e_\theta)$$

(3.21)

which may also be written as

$$B_r = \left(B_1 + \frac{2\mu_0 m}{4\pi r^3}\right) \cos \theta$$

$$B_\theta = \left(-B_1 + \frac{\mu_0 m}{4\pi r^3}\right) \sin \theta.$$  

(3.22)

To find $A$ and $m$, we use the boundary conditions on $B$

$$B(a)_{in} \cdot n = B(a)_{out} \cdot n$$

$$B(a)_{in} \times n = \frac{\mu_{in}}{\mu_{out}} B(a)_{out} \times n$$

(3.23)

where $n$ is the unit vector normal to the surface of the sphere. Outside the sphere we have free space ($\mu_{out} = \mu_0$); combining the field (Eq. 3.22) with the boundary conditions (Eq. 3.23) we now have

$$\frac{2A}{i\omega \sigma a} j_1(ka) \cos \theta = \left(B_1 + \frac{2\mu_0 m}{4\pi a^3}\right) \cos \theta$$

(3.24)

$$\frac{-A}{i\omega \sigma a} \left(j_1(ka) + ka j_1'(ka)\right) \sin \theta = \left(-B_1 + \frac{\mu_0 m}{4\pi a^3}\right) \sin \theta$$

(3.25)

which may be simplified by making use of two properties of spherical Bessel functions [72]

$$j_1'(x) = \frac{1}{3} (j_0(x) - 2j_2(x))$$

$$j_1(x) = \frac{x}{3} (j_0(x) + j_2(x)),$$

(3.26)

(3.27)

resulting in

$$A = \frac{9B_1}{2k} \frac{i\mu \sigma \omega}{(\mu + 2\mu_0)j_0(ka) + (\mu - \mu_0)j_2(ka)}$$

$$m = \frac{2\pi a^3 B_1}{\mu_0} \frac{2(\mu - \mu_0)j_0(ka) + (2\mu + \mu_0)j_2(ka)}{(\mu + 2\mu_0)j_0(ka) + (\mu - \mu_0)j_2(ka)}.$$
In the case of a non-magnetic sphere \((\mu = \mu_0)\), Eqs. 3.28 and 3.29 reduce to

\[
A = \frac{3i\omega\sigma}{2kj_0(ka)}
\]
\[
m = \frac{2\pi a^3 B_1 j_2(ka)}{\mu_0 j_0(ka)}.
\]

Now that we have the coefficients \(A\) and \(m\), we may write expressions for the radial and polar components of the magnetic field inside the sphere \(^2\)

\[
B_r = \frac{B_1 \cos \theta}{j_0(ka)} \left( j_0(kr) + j_2(kr) \right)
\]
\[
B_\theta = -\frac{B_1 \sin \theta}{j_0(ka)} \left( j_0(kr) - \frac{1}{2} j_2(kr) \right)
\]

and outside the sphere

\[
B_r = B_1 \cos \theta \left[ 1 + \frac{j_2(ka)}{j_0(ka)} \frac{a^3}{r^3} \right]
\]
\[
B_\theta = -B_1 \sin \theta \left[ 1 - \frac{1}{2} \frac{j_2(ka)}{j_0(ka)} \frac{a^3}{r^3} \right].
\]

In Cartesian coordinates, the field inside the sphere is

\[
B_x = \frac{3B_1 j_2(ka) x z}{2 j_0(ka) r^2}
\]
\[
B_y = \frac{3B_1 j_2(ka) y z}{2 j_0(ka) r^2}
\]
\[
B_z = \frac{3B_1}{2j_0(ka)} \left[ \frac{2j_1(kr)}{kr} - \frac{x^2 + y^2}{r^2} \frac{j_2(kr)}{r} \right],
\]

while outside the sphere

\[
B_x = \frac{3B_1 j_2(ka) a^3 x z}{2 j_0(ka) r^5}
\]
\[
B_y = \frac{3B_1 j_2(ka) a^3 y z}{2 j_0(ka) r^5}
\]
\[
B_z = B_1 \left[ 1 - \frac{1}{2} \frac{j_2(ka)}{j_0(ka)} \left( \frac{a^3}{r^3} - \frac{3a^3 z^2}{r^5} \right) \right].
\]

\(^2\)It can be seen that the corresponding result published by Carlson [66] must have an error since \(\nabla \cdot \mathbf{B} \neq 0\) (as pointed out by Petropoulos [24]).
The current density \( \mathbf{J} \) dissipates energy in the form of ohmic losses. The time-averaged power is given by:

\[
< P > = \frac{1}{2 \sigma} \int |\mathbf{J}|^2 dV .
\]  

(3.38)

This integral can be evaluated by recognizing that the spherical Bessel function \( j_{\nu-1/2}(x) \) is related to the ordinary Bessel function \( J_\nu(x) \) by \( j_{\nu-1/2}(x) = \sqrt{\pi/(2x)}J_\nu(x) \); this allows identities involving the Bessel functions to be used. Equation 3.38 can be evaluated and expressed in many ways, depending on which of the following identities is used:

\[
\int_0^\alpha r J_\nu(kr) J_\nu(lr) dr = \frac{a}{l^2 - k^2} \left[ k J_\nu'(kr) J_\nu(lr) - l J_\nu'(lr) J_\nu(kr) \right]
\]

(3.39)

\[
\int_0^\alpha r J_{\nu+1}(kr) J_\nu(lr) dr = \frac{a}{k^2 - l^2} \left[ k J_{\nu+1}'(kr) J_\nu(lr) - l J_{\nu+1}'(lr) J_\nu(kr) \right]
\]

(3.40)

\[
\int_0^\alpha r J_{\nu-1}(kr) J_\nu(lr) dr = \frac{a}{l^2 - k^2} \left[ k J_{\nu-1}'(kr) J_\nu(lr) - l J_{\nu-1}'(lr) J_\nu(kr) \right]
\]

(3.41)

Carlson writes the result as

\[
< P > = \frac{3\pi \omega B_1^2}{\mu |k|^2 j_0(ka)^2} \text{Im} \left[ k^* j_1(ka) j_1^*(ka) \right]
\]

(3.42)

where \( \text{Im}(x) \) denotes the imaginary part of \( x \) and \( z^* \) denotes the complex conjugate of \( z \).

Petropoulos [24] expresses the power in a similar way but has assumed the opposite sign for the exponential \( e^{i\omega t} \); rewriting his result to be consistent with our notation we obtain

\[
< P > = \frac{3\pi \omega B_1^2 a^2}{\mu |k|^2 j_0(ka)^2} \text{Im} \left[ k^* j_2(ka) j_1(k^*a) \right].
\]

(3.43)

While Petropoulos states that these results are not the same, in fact it can be shown that both results are equivalent to

\[
< P > = \frac{\pi \omega B_1^2 a^3}{\mu} \text{Im} \left[ \frac{j_2(ka)}{j_0(ka)} \right]
\]

(3.44)

or, in a form that more directly shows the relationship to the current density,

\[
< P > = \frac{3\pi \omega B_1^2 a^2}{\mu_0} \text{Im} \left[ \frac{j_1(ka)}{k j_0(ka)} \right].
\]

(3.45)
In the low-frequency limit \( \epsilon \omega \ll \sigma \), the real part of \( k^2 \) may be neglected, giving

\[
k = \sqrt{i\mu_0\sigma\omega}
\]

(3.46)

\[
k = \frac{1 + i}{\sqrt{2\mu_0\sigma\omega}}
\]

(3.47)

\[
k = \frac{1 + i}{\delta}
\]

(3.48)

where \( \delta \) is the skin depth (Eq. 2.23). Since we are in the thin limit where \( a \leq \delta \), the Bessel functions may be expanded about \( k = 0 \) to express the power in terms of \( a/\delta \). The series expansion of the Bessel functions appearing in the expressions are

\[
\frac{j_2(ka)}{j_0(ka)} = \frac{a^2k^2}{15} + \frac{2a^4k^4}{315} + \frac{a^6k^6}{1575} + \cdots
\]

(3.49)

\[
= 2i\left(\frac{a}{\delta}\right)^2 - \frac{8}{315}\left(\frac{a}{\delta}\right)^4 - \frac{8i}{1575}\left(\frac{a}{\delta}\right)^6 + \cdots
\]

(3.50)

\[
\frac{j_1(ka)}{kj_0(ka)} = \frac{a}{3} + \frac{a^3k^2}{45} + \frac{2a^5k^4}{945} + \frac{a^7k^6}{4725} + \cdots
\]

(3.51)

\[
= \frac{a}{3} + \frac{2ai}{45}\left(\frac{a}{\delta}\right)^2 - \frac{8a}{945}\left(\frac{a}{\delta}\right)^4 - \frac{8i}{4725}\left(\frac{a}{\delta}\right)^6 + \cdots
\]

(3.52)

\[
\frac{1}{j_0(ka)} = 1 + \frac{a^2k^2}{6} + \frac{7a^4k^4}{360} + \frac{31a^6k^6}{15120} + \cdots
\]

(3.53)

\[
= 1 + \frac{5i}{\delta^2}\left(\frac{a}{\delta}\right)^2 - \frac{8}{315}\left(\frac{a}{\delta}\right)^4 - \frac{8i}{1575}\left(\frac{a}{\delta}\right)^6 + \cdots
\]

(3.54)

Substitution of the first two terms in the expansion given by Eq. 3.50 into Eq. 3.37 allows the magnetic field outside the sphere to be written in terms of \( a/\delta \):

\[
B_x = \frac{3B_1}{2}\left(\frac{2i}{\delta^2} - \frac{8a^2}{315\delta^4}\right)\frac{a^5xz}{r^5}
\]

\[
B_y = \frac{3B_1}{2}\left(\frac{2i}{\delta^2} - \frac{8a^2}{315\delta^4}\right)\frac{a^5yz}{r^5}
\]

(3.55)

\[
B_z = B_1\left[1 - \frac{1}{2}\left(\frac{2i}{\delta^2} - \frac{8a^2}{315\delta^4}\right)\left(\frac{a^5}{r^3} - \frac{3a^5z^2}{r^5}\right)\right]
\]

while inclusion of the third term allows Eq. 3.45 to be written as

\[
<P> = \frac{\pi\sigma\omega^2B_1^2a^5}{15}\left[1 - \frac{4}{105}\left(\frac{a}{\delta}\right)^4 + \cdots\right].
\]

(3.56)
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This expression is equivalent to Eq. (19) in Ref. [66]. The first term is the result obtained by Hoult and Lauterbur when the skin depth is ignored.

This power is linked to the ohmic losses, modeled by including an effective sample resistance $R_s$ in the circuit responsible for generating the magnetic field. If the peak current in that circuit is denoted $I$, then the RMS power dissipated by the effective resistance is:

$$< P > = \frac{I^2 R_s}{2}. \quad (3.57)$$

For a coil that produces a homogeneous field over the sample volume, we may substitute $\chi = B_1/I$, the magnet constant of the coil, which allows us to express $R_s$ in terms of readily accessible experimental parameters:

$$R_s = \frac{2\pi \sigma \omega^2 \chi^2 a^5}{15} \left[ 1 - \frac{4}{105} \left( \frac{a}{\delta} \right)^4 + \cdots \right]. \quad (3.58)$$

Equation 3.58 is the primary expression for the sample resistance against which comparisons with experimental data are made in Chapter 6.

3.2 Interpretation

The magnetic field lines associated with the solution for $B$ (Eqs. 3.36, 3.37) in the case of a weakly conducting sphere are shown in Fig. 3.2 as they depend on the phase of the applied field. The results can be interpreted by keeping in mind that the applied field amplitude is $B_1 = B_1 \cos(\omega t)$. When $\omega t = 0^\circ$, the applied field is at its peak amplitude and the induced field is close to its minimum strength. As the phase increases, the applied field strength drops and the induced field strength increases, hence increasing the field line density inside the sphere. At $90^\circ$, the applied field vanishes and the induced field is nearly a maximum. As the phase passes through $135^\circ$, the applied field is dropping less rapidly; thus the induced field weakens. However as the applied field has changed sign, the overall field strength inside the sphere is reduced. When the applied field changes direction at $180^\circ$, this cycle repeats with the induced currents now flowing in the opposite sense. In general, we expect the induced field to be out of phase with the applied field. The phase is an indication of the impedance of the sphere; if it were perfectly inductive (i.e. lossless), the induced field would be out of phase with the applied field by $180^\circ$. If the sphere was purely resistive the
Figure 3.2: Magnetic field lines associated with the conducting sphere in the thin limit. The top panel shows the normalized amplitude of the applied field as it varies with time. The lower panels correspond to phases of the applied field: a: $\omega t = 0^\circ$, b: $45^\circ$, c: $90^\circ$, and d: $135^\circ$ (left to right). The direction of the field lines in panel (d) is opposite that in panels (a) and (b). Note that these are qualitative representations of the field and that the line density does not represent the same magnetic field strength for all diagrams. Drawings copyright Chris Bidinosti (2006). Used with permission.
induced field would be 90° out of phase with the applied field. From the plots of the field for a weakly conducting sphere shown in Fig. 3.2, the phase lag between the induced and applied fields is close to 90°, showing the sphere is mostly resistive. From the expansions used in the above analysis, we see that the deviation from a 90° phase lag depends on the relative size of $a$ and $\delta$. 
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The effect of the skin depth can be seen by considering the accuracy of the expansion Eq. 3.50 as \( a \) increases relative to \( \delta \). Figure 3.3 shows the real and imaginary parts of the magnetic field at the centre of the sphere:

\[
B_z(0) = \frac{B_1}{j_0(\sqrt{2ja/\delta})} = B_1 \left[ 1 + \frac{ia^2}{3\delta^2} - \frac{7a^4}{90\delta^4} + \cdots \right]
\]  

(3.59)

as a function of \( a/\delta \). From the graph, we see that the first two terms in the expansion describe the field very well, even up to \( a \sim \delta \). This is surprising since we assumed \( a/\delta \ll 1 \) in order to perform the expansion; however it is useful because the results presented in this chapter are accurate even up to \( a \sim \delta \).
Chapter 4

Apparatus

In this chapter I describe the apparatus that were used in the experiments to measure ohmic losses and magnetic fields associated with eddy currents in weakly conducting samples. First, I describe the coils and circuitry that were used to generate and detect oscillating magnetic fields. Second, I describe the samples that were used and how they were positioned inside the coil. Last, I describe the instruments and data acquisition.

4.1 Transmit coil

In Chapter 3, I presented the mathematical analysis of a conducting sphere in a uniform oscillating magnetic field. To do the corresponding experiment, such a magnetic field is a primary requirement. In this section I describe the coil that was used to generate this field.

4.1.1 Coil design

As mentioned in Sec. 2.2, a cylindrical solenoid is usually employed in MRI to generate the static magnetic field $B_0$, aligned with the solenoid axis. The coil producing the $B_1$ field, called the transmit or $B_1$ coil, has to be designed to fit within the $B_0$ solenoid, generate a field in any direction perpendicular to $B_0$, and leave enough room inside for a subject.

It has long been recognized that a cylindrical coil arrangement called a saddle or saddle-shaped coil pair generates a uniform, transverse magnetic field near its midpoint. One saddle coil consists of two parallel wires joined by arcs called the return paths. A saddle coil
pair having the is shown in Fig. 4.1. The highest field homogeneity is achieved when the parallel wires are located at values of the azimuthal angle $\phi$ of $\pm 60^\circ$, $\pm 120^\circ$. By superimposing a number of saddle coils subtending various angles, we can increase the field homogeneity. It is a well known that a wire spacing that is sinusoidal in the azimuthal angle $\phi$ generates a perfectly uniform field (when the cylinder is infinite) [73, 74]; hence by spacing the wires in a sine-phi distribution we obtain a close approximation of a uniform field inside the coil.

In practical applications, the performance of this type of design is excellent when driven with a low frequency current, but degrades at higher frequencies as the wavelength approaches the length of wire in the coil and currents in one region of the coil become out of phase with currents in other regions.

The field produced by this coil is linearly polarized, since it oscillates between pointing in the $+z$-direction and the $-z$-direction. I mentioned in Sec. 2.1 that a circularly polarized field can also be used in NMR (although this requires the phase of the currents in each of
Figure 4.2: This diagram displays the locations of the wires in the transmit coil. The first quadrant of a unit cylinder is shown from the end-on view. The circles show the ideal positions for an infinite cylinder [75]. The diamonds show the positions of the wires as optimized by computer simulation of this truncated coil.

the parallel wires be set appropriately). For a given NMR tip angle $\alpha$, the SAR associated with a circularly polarized field is half that produced by a linearly polarized field.

For the experiments presented in this thesis, we used a sine-phi coil designed specifically to be a $B_1$ transmit coil for very low field MRI [76]. The transmit coil consists of five saddle coil elements constructed from a single continuous length of 1 mm diameter solid copper wire wound on a hollow, cylindrical, fibreglass form. The axial length of the transmit coil is 90 cm and its diameter is 54 cm, suitable dimensions for use in an MRI scanner designed to accommodate a human torso. The total length of wire is $\sim 30$ m and its self-resonance frequency is $\sim 1.9$ MHz. The angular positions of the wires that optimize the field homogeneity can be predicted exactly for an infinite cylinder [75]; truncation of the cylinder affects the homogeneity of the resulting field. To compensate for truncation effects, the field profile of this coil was calculated including contributions from the return paths. The optimal wire positions were determined using a computer algorithm to maximize the field homogeneity.
Figure 4.2 shows the optimal angular position of wires for one quadrant of this transmit coil. Also shown are the wire positions from the analysis of the infinite coil. We define the \( z \)-direction to be the direction of the \( B_1 \) field to be consistent with the analysis presented in Chapter 3. The axial direction is defined as the \( y \)-direction. Note that I have now made a change from the normal convention used in Chapter 2, in which the static field \( B_0 \) defined the \( z \)-direction. In all further discussions, the \( z \)-direction refers to that of \( B_1 \).

### 4.1.2 Coil performance

The transmit coil described above has been characterized extensively. When driven with an oscillating current at frequencies below its self-resonant frequency it produces an oscillating field that is homogeneous to within 1.5% over a 12 litre volume: 15 cm in each transverse direction (\( x \) and \( z \)) and 10 cm in the axial direction (\( y \)) from the centre point of the coil (called the isocentre). Figure 4.3 shows a comparison of the calculated and measured field strength inside the transmit coil. The total normalized calculated magnetic field in the \( xz \)-plane is shown in the top left panel of the figure. Clearly the field is very uniform near the isocentre of the transmit coil. The top right panel in Figure 4.3 shows the \( z \)-component of the magnetic field (the dominant component near the isocentre), in a narrower field of view highlighting the most homogeneous region of the field in more detail. With human imaging in mind, the axial homogeneity is of interest considering the aspect ratio of the volume that will generally be occupied by a subject. The lower left panel in the figure shows the normalized calculated \( z \)-component of the magnetic field in the \( yz \)-plane, which can be directly compared to the lower right panel showing experimental data [76] for the \( z \)-component of the field over a similar range. This comparison shows that the \( z \)-component is homogeneous at the 0.5 % level over a 20 cm \( \times \) 20 cm area in the centre of the \( yz \)-plane. These data demonstrate that the performance of the transmit coil is consistent with design specifications.
Figure 4.3: Field maps for the transmit coil. Top left: Calculated field contours showing the normalized total field strength in the cross-sectional, transverse $xz$-plane. The region occupied by various phantoms in experiments (Chapter 6) is shown by the dashed curves. Top right: Calculated contours of the normalized $z$-component of the field, focusing on the homogeneous area near the coil isocentre. Bottom left: Calculated normalized $z$-component of the magnetic field in one quadrant of the $yz$-plane. Bottom right: Measurements of the $z$-component of the magnetic field in the $yz$-plane at 100 kHz [76].
Figure 4.4: Measurement of the magnet constant $\chi$ for the transmit coil. The slope of the plot of magnetic field as it depends on current through the transmit coil is equal to the magnet constant.

Another property of any coil is its magnet constant, defined as the ratio of the magnetic field strength per unit of current through the wires, $\chi = B/I$. For this transmit coil, $\chi$ has been calculated to have the value $12.86 \mu T/A$. It was measured by driving a current through the transmit coil using a standard power supply, and measuring the field strength at the centre of the coil with a Group3 model MPT-231 Hall probe. Figure 4.4 shows the measured field strength versus current in the transmit coil. From the slope of the data, the value of the magnet constant was measured to be $\chi = 13.1(1) \mu T/A$. This value will become important later in order to make use of Eq. 3.58. The inductance of the coil was measured to be $L = 101(1) \mu H$.

4.2 Receive coils

To characterize the SNR in very low field MRI of human subjects, MRI receive coils make a more natural choice than the transmit coil described in the previous section. As stated in Sec. 2.1, it is possible either to use a single coil to both transmit and receive RF signals, or to use two separate systems for transmission and reception. A transmit coil for whole-body MRI scanners is large enough to generate a uniform field over the volume of the subject. Consider the consequences for SNR of this design. NMR signals originate from only a
small region of the sample (i.e. the excited slice) while the noise originates from the entire sample. A transmit coil is thus sensitive to noise from the entire sample and has a poor SNR. If we consider the filling factor $\eta$, defined as the ratio of the volume of the region being imaged to the sensitive volume of the coil, we see that a transmit coil has a low filling factor.

Commonly, SNR can be improved by using a higher filling factor coil. This can be done with a separate receive coil (or coils) that is smaller (than the transmit coil), sits very close to the sample, and is most sensitive to spins near the surface of the sample. Consequently, this design is also most sensitive to noise from nearby spins rather than the entire sample. Measurements obtained from receive coils with larger $\eta$ therefore have a higher SNR than those obtained with lower $\eta$ transmit coils.

For the experiments described in this thesis the main receive coils used by Bidinosti et al. [3, 4] were duplicated: two 32 cm × 40 cm rectangular wooden formers were wound with ten turns each of stranded, insulated 18 AWG copper wire to create our prototype. Hence the total length of wire is $\sim 30$ m. Stranded wire was used to partially avoid the skin effect described in Sec. 2.4. To more effectively reduce the skin effect, Litz wire could be employed. The two coils were connected in series and arranged in front and in back of the subject's thorax, separated by 28 cm. A drawing of these coils is shown in Fig. 4.5. The inductance of the coils when connected in series was measured to be $197.3(1) \mu$H.

### 4.3 Loss measurement circuitry

Ohmic losses in samples can be measured by detecting their loading effect on a resonant circuit (see Chapter 5). To create a resonant circuit, the coil relevant to the appropriate experiment – either the transmit coil or the receive coil pair – was used as the inductor, and tuned with an adjustable air gap capacitor.

We chose to excite the resonance indirectly by driving an alternating current through a small coil which is weakly inductively coupled to the resonant circuit. Similarly, the response of the circuit was detected with a pair of weakly coupled receive coils. Using this configuration of separate coupling loops for drive and detection allows the resonance to be measured in transmission. That is, the drive loop transmits signal to the detection loop almost exclusively via their coupling to the transmit coil. This can be contrasted to
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Figure 4.5: Illustration of the receive coils used when measuring losses in human subjects. The small drive (DRV) and detection (DET) coils that are shown here are discussed in Sec. 4.3.

reflection measurements in which one coupling loop is used to both drive and detect the resonance. The benefit of a transmission measurement is that far from resonance, the signal in the detection loop is zero, whereas the signal from a reflection measurement is not.

Three nine-turn coils with a 4.75 cm diameter were used as coupling loops; one of which served as the drive loop (DRV) while the remaining two were connected in series to form the detection loop (DET). The two coils in the detection loop were counter-wound with respect to each other to reduce undesirable direct coupling between DRV and DET. For experiments involving the transmit coil, the drive and detection loops were coupled to the transmit coil by positioning them such that the $B_1$ field lines pass perpendicularly through their cross sectional area as shown in Fig. 4.6. When using the receive coils, the drive and detection loops were coupled to one of the receive coils as shown in Fig. 4.5. Again, this arrangement allows the field lines produced by the receive coil to pass perpendicularly through the coupling loops. Additionally, DRV and DET were placed in perpendicular
Figure 4.6: Schematic diagram of transmit coil (centre), tuning capacitance ($C$), drive loop (DRV, right) and detection loops (DET, left). The coupling of the $B_1$ field lines to the detection loops is shown in the detail. Drawings copyright Mike Hayden (2006). Used with permission.
Figure 4.7: Circuit used to measure the effective resistance of a sample. A small coil $L_1$ is used to drive current in the resonant cavity, the LRC circuit on the right. The induced EMF in the receive coils $L_3$ is measured with a preamplifier and a lock-in amplifier.

planes to further reduce direct coupling, as can be seen in Fig. 4.5.

To conduct measurements of the quality factor of the tuned coil, the apparatus were arranged as described above. The drive loop was driven with a function generator (FG), in series with a dropping resistor $R_d$ to maintain a constant current in the drive coil. The voltage across the detection coils was detected with a preamplifier followed by a phase sensitive detector (also called a lock-in amplifier). The apparatus may be modeled with the circuit shown in Fig. 4.7. The inductors $L_1$, $L_2$ and $L_3$ represent the drive, resonant, and detection coils respectively; $R_1$, $R_2$ and $R_3$ represent their resistances. The $M$'s represent the mutual inductances: $M_{12}$ is that between the drive loop and $L_2$, $M_{23}$ between $L_2$ and the detection loop, and $M_{13}$ between drive and detection loops directly. The capacitances $C_1$ and $C_3$ represent the capacitances of the cables and instruments used in the experiment, while $C_2$ is the tuning capacitance. Each of these quantities (except for $R_2$), including the frequency dependence of $R_1$ and $R_3$, were measured directly using standard techniques. The resistance $R_2$ is the quantity that is extracted from measurements of the quality factor of the resonance.


### 4.4 Field mapping coil and circuitry

Measurements of the magnetic field inside the transmit coil were obtained using a small probe coil. The probe coil has a self-resonance at $\sim 1$ MHz, has 28 turns and is 2.1 cm in diameter. This is small enough compared to the size of the sphere to be considered a point probe for the purposes of the experiment. The size of the sphere is the relevant length scale because it appears in the expressions for the magnetic field (Chapter 3). To estimate the error introduced by this assumption, the full integral of the magnetic flux over the area of the probe coil was calculated under typical conditions. The calculation shows that the finite size of the probe coil reduces the measured magnetic field by no more than 0.5%.

To repeatably position the probe coil, a wooden jig was secured to the top of the transmit coil. The jig can be rotated to be at any azimuthal angle. The uncertainty in the angle is estimated to be $\sim 0.5^\circ$.

The probe coil was mounted on the end of a long hollow plexiglass rod which passed through a slot and was clamped in place such that it was in the $xz$-plane of the transmit coil for all the measurements. This arrangement is illustrated in Fig. (4.8), which shows a drawing of the apparatus and a photograph representing the experiment.

To place the probe coil into the $xz$-plane, the EMF induced across the probe coil was measured while scanning its vertical position $\sim \pm 2$ cm from the equator of the sphere. The variation in the signal was $\sim 5\%$, indicating that the sensitivity to the vertical position is quite high. The probe coil was placed in the plane of the equator of the sphere by adjusting the vertical position of the probe coil until the signal amplitude was a maximum. It is estimated that the vertical position of the probe coil is within $\sim 0.5$ mm of the equator of the sphere. The radial position of the probe coil within the $xz$-plane, relative to the centre of the sphere, was determined by measuring the distance from the sphere equator to the inner wall of the transmit coil, along the axis of measurement. This distance was measured to a precision of 0.5 mm. Additionally the outer diameter of the sphere in the $xz$-plane was measured to determine the distance between the probe coil and the isocentre of the transmit coil. Slight deviations in the shape of the phantom from a true sphere however prevent us from determining the accuracy of the position of the probe coil at the 0.5 mm level. Rather, given that the uncertainty in the mean sphere radius is $\sim 0.3$ mm, the relative positioning of the coil is more likely accurate to $\sim 1$ mm. Furthermore, the diameter of the transmit coil
Figure 4.8: Left: Drawing of the apparatus used for measuring the oscillating magnetic field inside the transmit coil. The outer cylinder represents the transmit coil former (windings are not shown). The probe coil is mounted on a rod passing through a slot in an alignment jig on the top of the transmit coil form. The jig may be oriented in any direction about the cylinder axis and the probe coil may be rotated in any direction about its axis. Right: Photograph indicating arrangement of the probe coil (on the left) relative to the phantom (on the right). The glass flask is representative of the phantom, although a different phantom was actually used in the experiment (see Sec. 4.5).
former varies as a function of angle by \( \sim 5 \) mm, but this variation was taken into account in the determination of the coil position.

In this experiment the transmit coil was not driven with the DRV coil or tuned with an external capacitor, but rather was driven via a direct connection to the secondary windings of a transformer; the FG being connected across the primary windings. In this way a larger oscillating field is generated inside the transmit coil. The reason for including the transformer – rather than having a direct connection between the FG and transmit coil – is to reduce electric field effects arising from the fact that each of the wires in the transmit coil is at a slightly different potential. Therefore the transmit coil acts as a collection of weak, oscillating electric multipoles. Consider two distinct effects associated with the electric fields generated by these multipoles: (i) the capacitive coupling between the probe coil and the transmit coil serves as a mechanism for the oscillating electric fields to impose potential differences across the probe coil; (ii) the electric fields drive both free and bound charges in the sample. Each one of these effects influences the measurement.

The reason for including the transformer becomes apparent if we consider the potentials involved in the experiment. Since the function generator provides an oscillating potential that is referenced to ground, if it is connected directly across the transmit coil, one end of the coil windings is grounded. Thus the mean potential of the transmit coil is forced to swing from negative to positive with each period of oscillation. This large asymmetric potential swing causes capacitively coupled EMFs to appear across the probe coil, unless an isolation transformer is used. Additionally, a centre-tap was employed to ground the midpoint of the windings to ensure that the potential is evenly balanced between the two halves of the transmit coil. An improved transmit coil could be made by interleaving the windings of the saddle coils to further reduce the electric multipole fields inside the transmit coil.

Evidence for reduction of the electric fields inside the transmit coil was observed by performing two types of measurements, before and after the changes described above were made. First, using a short electric dipole antenna, the electric field within the transmit coil was directly measured. Second, the change in the EMF across the probe coil was monitored as the empty fibreglass container (i.e. a sample of nominally lossless dielectric) was placed inside the transmit coil. When the FG was connected directly across the terminals of the transmit coil, the magnitude of the EMF across the probe coil was observed to change by
The transmit coil was grounded at the centre of the windings. The signal from the function generator was used to drive a transformer, connected directly to the transmit coil. These attempts to balance the drive circuit reduce capacitive coupling effects in the probe coil.

Figure 4.9: Instruments and circuitry used to measure the field in the vicinity of the sphere. The transmit coil was grounded at the centre of the windings. The signal from the function generator was used to drive a transformer, connected directly to the transmit coil. These attempts to balance the drive circuit reduce capacitive coupling effects in the probe coil.

a maximum of $\sim 1\%$ when the empty fibreglass container was placed inside the transmit coil. When the FG was floated and the midpoint of the transmit coil was grounded, the change in the EMF was reduced to $\sim 0.1\%$, which is at the same level as the noise in the measurement. We therefore infer that the balancing of the electric potential between the two halves of the coil reduced capacitive coupling effects to the probe coil by a factor of ten. Furthermore, magnetic field perturbations due to the dielectric container are not measurable after this procedure was done. While these observations do not tell us if electric fields are responsible for driving significant currents in the conducting sphere, it is possible to make arguments based on dimensional analysis that such effects should be much smaller than magnetically induced losses [11].

The circuits involved in the field mapping measurement are illustrated in Fig. 4.9. In this experiment we may still refer to transmitting and receiving signals; the function generator drives the transmit ($B_1$) coil (via the transformer) and the circuit response is detected with the probe coil. Therefore 'transmit' in this experiment refers to the transmit ($B_1$) coil and 'receive' refers to the probe coil.
Table 4.1: Characteristics of the phantoms used in these experiments. The ‘loss’ refers to the measurement of ohmic losses in spherical samples (Sec. 5.1) and the ‘field map’ refers to the measurement of the magnetic field near the sphere (Sec. 5.2).

4.5 Phantom preparation and placement

All of the phantoms used in this experiment were spherical containers filled with NaCl solutions. One of these containers was made by coating a rubber ball with fibreglass and then drilling holes for filling and draining. This particular phantom was used both in the measurement of losses and the measurement of magnetic field perturbations. When the loss experiment was expanded to include the dependence on the sphere radius, additional roundbottom Pyrex glass flasks of varying radii were used. A list of the properties of the phantoms and their functions is provided in Tab. 4.1. The uncertainties in the sphere radii represent measured deviations from non-uniformity.

The use of NaCl solutions is appealing because they are cheaply and readily prepared, and the electrical conductivity is readily deduced from tables of published data [77]. The concentration of the solution used in this work was approximately 2.3 M, because a measurable effect was seen when the fibreglass container was filled with a solution of this concentration.

Independent determinations of the conductivity of solutions were performed in two ways. The first calculation uses the mass of salt and the mass or volume of water used to prepare the solutions. The molarity $M$ of the solution is determined from these values:

$$M = \frac{m_s \mu}{V}$$

(4.1)

where $m_s$ is the mass of salt used (in grams) and $V$ is the volume of the solution. Two solutions were prepared. One was prepared by weighing the mass of salt and the mass of
Figure 4.10: Conductivities of NaCl solutions, as a function of concentration at 20°C [77]. The symbols show data points, and the solid line shows a straight line fit (Eq. 4.2).

water to meet the target concentration; seven litres of this solution was prepared and used in sphere #2. The other solution was prepared by weighing the required mass of salt and using a volumetric flask to measure the volume of total solution to meet the target concentration. Eleven litres of this solution was prepared so there would be enough to use in the largest sphere. From tabulated values of concentration and conductivity [77] – shown in Fig. 4.10 – we determine a relationship between the molarity $M$ and the conductivity $\sigma$ locally near the target concentration:

$$\sigma = a + bM$$  \hspace{1cm} (4.2)

where $a = 4.55(11)$ S/m and $b = 4.44(48)$ g S L$^{-1}$ m$^{-1}$. In this way the nominal conductivity of each solution was calculated; the solution used in sphere #2 has a nominal conductivity of $\sigma = 14.89(21)$ S/m, and the solution used in all other spheres has a nominal conductivity of $\sigma = 14.90(16)$ S/m. The second determination of the electrical conductivity is obtained from assays of the solutions, in which a specified volume of solution was weighed and evaporated. From the mass of remaining salt, the concentration of the solution is determined. As before, the nominal conductivity of the solution is obtained from comparisons with previous work [77] (Eq. 4.2 and Fig 4.10). The assay calculations of the nominal solution conductivities were $\sigma = 14.63(36)$ S/m and $\sigma = 14.74(37)$ S/m for the
solutions used in sphere #2, and all others, respectively.

The two values of conductivity obtained for each solution can be averaged to yield a 'preparation' conductivity. For the solution used in sphere #2, we obtain a value of \( \sigma_{\text{prep}} = 14.76(21) \text{ S/m} \). For the solution used in all other spheres, we obtain a value of \( \sigma_{\text{prep}} = 14.82(20) \text{ S/m} \). Since these two values overlap, they were averaged to give the preparation conductivity representing the conductivity of all the spherical samples used in this thesis. This value is \( \sigma_{\text{prep}} = 14.79(15) \text{ S/m} \).

Note that this value of conductivity allows us to calculate the skin depth \( \delta \) of the samples. At a frequency of 10 kHz, the skin depth is \( \delta = 1.3 \text{ m} \), while at 1 MHz the skin depth is \( \delta = 13 \text{ cm} \). For the largest sphere (#1) at 1 MHz, the skin depth is the same size as the radius of the sphere. In other cases, the skin depth is much larger than the radius. This situation allows us to probe the thin limit up to the point where \( a \sim \delta \).

Several additional details regarding the preparation and assay of solutions are listed below.

- Adsorption and absorption of water vapour onto the glassware and into the salt respectively were determined to have a total effect of \( \sim 0.1 \% \) on the measured salt mass. The salt was not baked prior to being weighed. However during assays the mass of the salt was taken while the salt and glassware were still hot.

- The temperature of the water used to prepare the solution for sphere #2 was not recorded. The temperature of the water used to prepare the solution for other spheres was \( 20.0(1) ^\circ \text{C} \), consistent with the data used from the CRC handbook [77]. Therefore temperature variations were not accounted for in the values of the concentrations \( \sigma_{\text{prep}} \) for either solution.

- Variations in the solution temperature during experiments due to changes in the ambient temperature were included by considering a 2.5\(^\circ\) temperature uncertainty to contribute to the uncertainty in the concentration \( \sigma_{\text{prep}} \) for both solutions.

Repeatable, centred placement of the fibreglass sphere inside the transmit coil was achieved using an annular disc of foam fitted around the sphere. The diameter of the disc matched the inner diameter of the coil form, and a section of the disc was cut away to make room for the probe coil. The sphere was moved into position using a rope and pulley system. In other cases the sphere was placed on a platform inside the transmit coil and centred to within \( \sim 1 \text{ cm} \) by measuring the distance to the transmit coil walls. Placement
at the exact isocentre during the loss measurement is not necessary given the homogeneity of the transmit coil. Recalling Fig. 4.3, we can see that a 1 cm error in the \(xz\)-plane of the placement of a sphere having a \(\sim 12\) cm radius causes the magnetic field to vary by less than 0.5\% over the volume of the sphere. For the field measurement however, the measurement is very sensitive to the relative placement of the sphere and probe coil, and that is why the centring disc was used. The vertical position was set by aligning the edge of the foam with a fiducial mark on the inside of the transmit coil wall, to within \(\sim 2\) mm.

4.6 Data acquisition

In both the loss measurement and the field mapping measurement, a transmit coil needs to be driven with a current, and the signal across a receive coil needs to be measured. In both cases, the driving current was provided by an Agilent 33120A function generator and the signal in the coil was measured using a preamplifier (either Princeton Applied Research model 113 or Stanford Research Systems model 560) followed by an EG & G PAR model 5202 lock-in amplifier which allows for phase-sensitive detection. The lock-in outputs were fed into two HP model 3457A digital multimeters (DMMs). The transmit and receive circuits are shown in Fig. 4.11.

The resonance measurements were automated using a GPIB interface to connect the instruments to a computer (PC). LABVIEW software was written to control the function generator drive frequency and capture the DMM measurements of the lock-in outputs (the lock-in did not have GPIB capabilities). The frequency sweep must be broad enough to capture the essential features of the resonance curve, including the resonance peak and the outer ‘wings’. Scans were typically 7-15 times the width of the resonance peak, depending on the resonance frequency. It is important to collect the entire frequency scan relatively quickly for two reasons. First, the resistance of the coil includes effects due to electromagnetic coupling with its environment, which is subject to change. The resistance of the coil has also been observed to vary with the ambient temperature in a manner consistent with the temperature dependence of the resistivity of copper. Specifically, a change in the resistance of \(\sim 1\%\) is seen to correlate with a temperature change of \(\sim 1\%\), from which we conclude that \(dR/R \sim dT/T\). Because of these two effects, the time between scans must not be so great that a measurable change the resistance occurs due to these factors. Second, when
Figure 4.11: Schematic illustration of computer control used in the experiments. Top: The current in the transmit coil was driven with a function generator. Middle: The signal in the receive coils or probe coil was amplified first with a pre-amplifier, then with a lock-in amplifier. The in-phase ($X$) and out-of-phase ($Y$) components of the signal were measured with digital multimeters. Bottom: Block diagram of the computer control. During measurements of the resonance curve, the function generator was computer-controlled and the reading on each DMM was recorded by computer using a GPIB interface.
conducting experiments with human subjects, the time to comfortably stand in the coil and the time to wait while the background is scanned are logistical considerations.

The speed of the scan is limited by the speed of the instruments to acquire measurements, and that of the GPIB interface to capture those values from the instruments. We found that best balance between speed and number of data points resulted in typical resonance scan times of ~ three minutes and comprising ~ 350 evenly spaced data points. The time constant and integration time on the lock-in and DMMs respectively were set short enough (100 ms or less) for the scans to be this fast with a reasonable resolution. An additional wait time between setting the frequency and capturing the DMM values was included to allow averaging performed by the lock-in and DMMs to take place. The GPIB connections are shown schematically in Fig. 4.11. Measurements of the magnetic field did not require automation. A longer time constant was used on the lock-in, and a 250 ms filter was applied. A longer integration time on the DMMs were used since fewer measurements were needed and measurements could be slower with more averaging for more precision. These instrumentation parameters are summarized in Tab. 4.2.
<table>
<thead>
<tr>
<th>When measuring resonance:</th>
<th>sphere #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lock-in time constant:</td>
<td>100 ms</td>
</tr>
<tr>
<td>Wait time between measurements:</td>
<td>500 ms</td>
</tr>
<tr>
<td>DMM averaging time (multiples of 1/60):</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When measuring resonance:</th>
<th>all other spheres</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lock-in time constant:</td>
<td>100 ms</td>
</tr>
<tr>
<td>Wait time between measurements:</td>
<td>500 ms</td>
</tr>
<tr>
<td>DMM averaging time (multiples of 1/60):</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When measuring fields:</th>
<th>sphere #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lock-in time constant:</td>
<td>100 ms</td>
</tr>
<tr>
<td>Lock-in filter:</td>
<td>250 ms</td>
</tr>
<tr>
<td>DMM averaging time (multiples of 1/60):</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When measuring resonance:</th>
<th>humans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lock-in time constant:</td>
<td>10 ms</td>
</tr>
<tr>
<td>Wait time between measurements:</td>
<td>100 ms</td>
</tr>
<tr>
<td>DMM averaging time (multiples of 1/60):</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.2: Instrumentation settings during each of the experiments. The speed at which the experiment can be performed depends on these values.
Chapter 5

Methods

In this chapter I describe two experimental techniques used to study weakly conducting samples: (i) the measurement of losses associated with induced currents in samples and (ii) the measurement of magnetic fields associated with the induced currents in a sample. The procedures to collect the data and the methods employed to analyze the data are presented.

5.1 Measurement of ohmic losses

The sample resistance of spherical phantoms and human subjects was measured at low frequencies by characterizing the perturbation to the quality factor of a tuned LRC circuit. The change in the quality factor due to the presence of the sphere is proportional to the effective sample resistance.

Figure 5.1 shows an LRC circuit, which consists of lumped resistive \((R)\), capacitive \((C)\), and inductive \((L)\) elements that are driven by a sinusoidal voltage of amplitude \(V\) and angular frequency \(\omega\). The drive voltage is represented by the complex phasor \(V = Ve^{-i\omega t}\). The total impedance viewed by the source is

\[
Z_T = R + i\left(\omega L - \frac{1}{\omega C}\right)
\]

\[= R + i\omega L\left(1 - \frac{1}{\omega^2 LC}\right),\]  

which has a magnitude

\[
|Z_T| = \sqrt{R^2 + \omega^2 L^2\left(1 - \frac{1}{\omega^2 LC}\right)^2}
\]
that is minimum at the resonance frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$  \hspace{1cm} (5.4)

The current in the circuit is represented by the complex phasor $I = I e^{-i\omega t + \theta}$ and is given by Ohm's Law

$$I = \frac{V}{Z_T}. \hspace{1cm} (5.5)$$

The time-average power dissipated by the circuit is

$$\langle P \rangle = Re (VI^*) = I_{rms}^2 R \hspace{1cm} (5.6)$$

$$= \frac{1}{2} \frac{V^2}{\sqrt{R^2 + \omega^2 L^2 \left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}}, \hspace{1cm} (5.7)$$

which reaches a maximum when $\omega = \omega_0$:

$$\langle P(\omega_0) \rangle = \frac{V^2}{2R^2}. \hspace{1cm} (5.8)$$

The bandwidth of the resonant circuit is defined as

$$\Delta \omega = \omega^+ - \omega^- \hspace{1cm} (5.9)$$
where \( \omega^+ \) and \( \omega^- \) are the frequencies at which the power has dropped to half of its maximum value (in the positive and negative directions from the peak value, respectively). These points are given by the solution to

\[
\frac{1}{2} < P(\omega_0) > = < P(\omega) >
\]

\[
\frac{V^2}{4R^2} = \frac{1}{2} \frac{V^2}{R^2 + \omega^2 L^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}.
\]  

The quality factor, defined as

\[
Q \equiv \frac{\omega_0}{\Delta \omega},
\]

provides a measure of the sharpness of the resonance. In the case of weak damping \((R \ll \omega L)\), the bandwidth is \(\Delta \omega \approx R/L\), which leads to

\[
Q = \frac{\omega_0 L}{R}.
\]

At resonance, the magnitude of the voltage across the resistor is \(V\), while the magnitude of the voltage across both the inductor and capacitor is \(VQ\).

When a sample of material is allowed to interact with the electromagnetic fields generated by the circuit elements, the response of the circuit is perturbed in a manner that depends on the electromagnetic properties of the sample. In particular, as is the case for the work described in this thesis, the perturbation to the quality factor can be modeled as an effective sample resistance \(R_s\) in the circuit. If the perturbation is small, the quality factor is decreased to

\[
Q' = \frac{\omega_0 L}{R + R_s}.
\]

The values of \(\omega_0\) and \(Q\) are determined from measurements of the unloaded circuit. If the circuit is re-tuned to the same resonance frequency after addition of the sample (by adjusting the tuning capacitor), a second measurement of the loaded quality factor \(Q'\) allows Eq. 5.15 to be applied directly to determine \(R_s\). Figure 5.2 shows an example of two resonant curves with the same resonant frequency but different quality factors.

### 5.1.1 Procedure

To measure \(R_s\), two separate measurements of the quality factor \(Q\) are needed: one when the circuit is unloaded, and one when the circuit is loaded by the sample. To obtain these
measurements, we made a resonant circuit using transmit coil and additional circuitry described in Sec. 4.3. The resonant behaviour of the circuit was characterized in transmission. The EMF across the receive coils was measured while sweeping the frequency of the drive current through the resonance.

The procedure to collect data was as follows. The tuning capacitor was adjusted to set the circuit to the desired resonance frequency, then the voltage across the receive coils was measured over a frequency range centred on the resonance frequency (usually several times the bandwidth). The data were curve fit to a model function — presented in the next section — yielding the resonance frequency and the resistance of the circuit. Upon the addition/removal of the sample into/from the coil, both properties are changed. The tuning capacitor was then adjusted to match the resonance frequency of the previous run as closely as possible, and the resonance was scanned and curve fit again. The scan when the coil was empty is called the ‘background’ scan and the scan when the sample is present inside the coil is called the ‘foreground’ scan. The difference in resistance between foreground
and background scans determines the value of $R_s$ at the given resonance frequency.\(^1\) This procedure was then repeated at a number of discrete resonance frequencies over the range 0.1 - 1.25 MHz to determine the frequency dependence of $R_s$.

### 5.1.2 Method of data analysis

A simple expression for the voltage expected at the input to the preamplifier can be estimated as follows. Recalling the impedance seen by the source, (Eq. 5.3) and Ohm’s Law (Eq. 5.5), the magnitude of the current in the circuit is

$$I = \frac{V}{\sqrt{R^2 + \omega^2 L^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}. \quad (5.16)$$

As described in Sec. 4.3, the resonant circuit was driven inductively, rather than directly as shown in Fig. 5.1. Therefore, the magnitude of the EMF induced across the inductor (and hence the LRC circuit) is

$$V \propto \frac{|dI_D|}{dt} \propto \omega |I_D| \propto v \omega \quad (5.17)$$

where $v$ is the voltage across the drive loop, and $I_D$ is the current in the drive loop. Therefore the magnitude of the current in the resonant circuit is related to $V$ by Ohm’s Law

$$I \propto \frac{v \omega}{\sqrt{R^2 + \omega^2 L^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}. \quad (5.18)$$

Now consider the detection loops. The EMF $v_0$ across the detection loops is given by

$$v_0 \propto \frac{|dI|}{dt} \propto \omega I = \alpha \frac{v \omega^2}{\sqrt{R^2 + \omega^2 L^2 \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}. \quad (5.19)$$
where $\alpha$ is a scale factor describing the proportionality.

By fitting the resonance curves to Eq. 5.19, the values of $\alpha$, $R$ and $f_0 = \omega_0/(2\pi)$ are obtained by letting them be free parameters in the fit. The argument presented above allowed us to quickly obtain a model function for the data that captures all the important features; however systematic residuals (the fit values subtracted from data values) from these fits were observed. For example, Fig. 5.3 shows a background data set 750 kHz, the curve fit to Eq 5.19, and the residuals that result from the fit. Generally, we see there is a slight overall mismatch between the data and fit – the fit tends to undershoot on the low frequency side of resonance and overshoot on the high frequency side. The feature at the centre of the scan is associated with the resonance. I pointed out that the voltage across the inductor and capacitor are multiplied by $Q$ at the resonance. Small deviations of the circuit from the model become amplified near the resonance by this effect. Any inadequacies of the circuit model to describe the real apparatus will tend to play a more important role near resonance and so the fit function will tend to describe the data more poorly. Notice that the width of the feature visible in Fig. 5.3 is $\sim 10$ kHz while the width of the resonance is $\sim$ 10 kHz.

---

1While $R_s$ and $R_{coil}$ each in fact depend on frequency, we are ignoring variations of their values over the width of the resonance scan. Inclusion of the appropriate frequency dependence influences the resulting fit values by approximately 0.05 %. 

---

Figure 5.3: Left: An example of resonance data amplitude (normalized such that '1' represents full scale) and the curve fit to the simple model function, Eq. 5.19. Right: The residuals resulting from this fit are clearly not random.
7 kHz. The size of the residuals is approximately on the 1% level. This is not particularly large but the systematic nature of the residuals indicates that a more realistic model function is required if we are to reliably and accurately detect and monitor small perturbations of the resonance curve. In an attempt to reduce the residuals of the fits to the data, I conducted a more in-depth analysis of the circuits involved in the apparatus. This analysis is now presented.

General case

A detailed model for the electrical circuit was presented in Sec. 4.3. For convenience, the circuit diagram is shown again in Fig. 5.4. Recall that $R_2$, $L_2$ and $C_2$ represent the components of the resonant circuit. Thus $R_2$ is the quantity that will ultimately be measured. We begin by defining the following symbols to simplify the notation.
CHAPTER 5. METHODS

Impedance of drive coupling loop:  \[ Z_1 = R_1 + i\omega L_1 \]

Resonance frequency:  \[ \omega_0 = \frac{1}{\sqrt{L_2 C_2}} \]

Impedance of circuit 2:  \[ Z_2 = R_2 + i\omega L_2 \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \]

Input impedance of preamp:  \[ Z_L = \frac{R_L}{(1 + i\omega R_L C_3)} \]

Impedance of circuit 3:  \[ Z_3 = R_3 + i\omega L_3 + Z_L \]

The overall \( e^{-i\omega t} \) time dependence present in the phasors \( V \) and \( I \) propagates through the analysis of the circuit, and will be suppressed to simplify the notation. It is important to note that these factors are still complex quantities and that the relative phase between \( V \) and \( I \) is significant.

For circuit 1 (the drive circuit), the Kirchoff voltage equation may be written as:

\[ v = I_1' R_d + I_1 Z_1 - i\omega(M_{12} I_2 + M_{13} I_3) \]  \hspace{1cm} (5.20)

where \( I_1' \) is the current through the resistor \( R_d \) and is determined from a second Kirchoff voltage equation for this circuit

\[ I_1 Z_1 - i\omega(M_{12} I_2 + M_{13} I_3) = (I_1' - I_1) Z_{C1} \]  \hspace{1cm} (5.21)

where \( Z_{C1} = -i/(\omega C_1) \) is the impedance of the capacitance \( C_1 \). Combining Eqs. 5.20 and 5.21, we have, for circuit 1

\[ v = \left( I_1 Z_1 - i\omega(M_{12} I_2 + M_{13} I_3) \right) \left( 1 + \frac{R_d}{Z_{C1}} \right) + I_1 R_d \]  \hspace{1cm} (5.22)

while the circuit equations describing circuits 2 and 3 require that

\[ 0 = I_2 Z_2 - i\omega(M_{12} I_1 + M_{23} I_2) \]  \hspace{1cm} (5.23)

\[ 0 = I_3 Z_3 - i\omega(M_{13} I_1 + M_{23} I_2) \]  \hspace{1cm} (5.24)

The magnitude of the voltage at the preamp input is given by

\[ v_0 = |I_3 Z_L| \]  \hspace{1cm} (5.25)

and \( I_3 \) is found by solving Eqs. 5.22, 5.23, and 5.24. One way to write the solution in a compact form is to define the following quantities

\[ A = R_d + Z_{C1} \]  \hspace{1cm} (5.26)

\[ B = Z_1 Z_{C1} + R_d Z_1 + R_d Z_{C1} \]
This allows the solution to be written as

\[ v_0^{\text{full}} = \left| \frac{i\omega v Z_{C1} Z_L (i\omega M_{12} M_{23} + M_{13} Z_2)}{f + g} \right| \]  \hspace{1cm} (5.27)

where \( f \) is the part of the denominator that is independent of the direct coupling \( M_{13} \):

\[ f = A(\omega M_{12})^2 Z_3 + B((\omega M_{23})^2 + Z_2 Z_3) \]  \hspace{1cm} (5.28)

while \( g \) is the part of the denominator that does depend on \( M_{13} \):

\[ g = A(\omega M_{13})^2 Z_2 + 2i\omega^3 M_{12} M_{23} M_{13} \]  \hspace{1cm} (5.29)

Note that in practical terms, the dominant functional form is still that provided by Eq. 5.19. The additional parameters in Eq. 5.27 are only expected to account for minor features near the resonance.

**Simplifications**

Before discussing the quality of the fits that are obtained when the full circuit model function is used, I will show that it reduces to simpler forms under certain conditions. The analysis up to this point is completely general, however many of the components shown in the circuit diagram (Fig. 5.4) are likely to have a small or even negligible effect on the form of the voltage \( v_0 \). If these components do not give a quantifiably better fit, then the function has only become unnecessarily complicated. Simpler functions are less cumbersome to manage and program into the computer to do the curve fits. By making a series of approximations, as described below, the model function for \( v_0 \) may be simplified; a discussion of the residuals resulting from fits to model functions with various levels of sophistication follows.

From inspection of the measured parameter values listed in Tab. 5.1, we see that the direct coupling constant \( M_{13} \) is much smaller than \( M_{12}, M_{23} \). The first approximation is simply to set \( M_{13} \) equal to zero, which allows us to remove the second term from the numerator of Eq. 5.27 and the term \( g \) appearing in the denominator:

\[ v_0^{(1)} = \left| \frac{-v Z_{C1} Z_L \omega^2 M_{12} M_{23}}{A(\omega M_{12})^2 Z_3 + B((\omega M_{23})^2 + Z_2 Z_3)} \right| \]  \hspace{1cm} (5.30)

where the superscript denotes the 1\(^{st}\) level of approximation.
Table 5.1: Measured values of the components shown in the circuit diagram, Fig. 5.4. See Fig. 5.5 for the frequency dependence of $R_1$ and $R_3$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Measured Value</th>
<th>Quantity</th>
<th>Measured Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{12}$</td>
<td>0.43 $\mu$H</td>
<td>$L_1$</td>
<td>9.44 $\mu$H</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>1.04 $\mu$H</td>
<td>$L_2$: $T_x$ coil</td>
<td>101 $\mu$H</td>
</tr>
<tr>
<td>$M_{13}$</td>
<td>3.4 pH</td>
<td>$L_2$: $R_x$ coil</td>
<td>197 $\mu$H</td>
</tr>
<tr>
<td>$R_d$</td>
<td>9.93 k$\Omega$</td>
<td>$L_3$</td>
<td>18.5 $\mu$H</td>
</tr>
<tr>
<td>$R_1$</td>
<td>$\sim 1$ $\Omega$</td>
<td>$C_1, C_3$</td>
<td>400 pF</td>
</tr>
<tr>
<td>$R_3$</td>
<td>$\sim 1$ $\Omega$</td>
<td>$R_L$</td>
<td>100 M$\Omega$</td>
</tr>
</tbody>
</table>

Figure 5.5: Measured frequency dependence of the resistances of the drive and detection loops. Each was fit to a 5-term polynomial to obtain a functional form for the overall resonance fit function.
CHAPTER 5. METHODS

Next consider the relative size of the terms in the denominator of Eq. 5.30 involving the mutual inductances \( M_{12} \) and \( M_{23} \) as compared to the term involving \( Z_2Z_3 \). Consider the first term \( (\omega M_{23})^2 + Z_2Z_3 \). Making use of the component values listed in Tab. 5.1, and knowing that \( Z_2 \) is typically of order \( 1 - 10 \Omega \) in our apparatus, it is clear that \( (\omega M_{23})^2 \ll Z_2Z_3 \) and thus we may drop \( (\omega M_{23})^2 \) relative to \( Z_2Z_3 \), giving

\[
v_0^{(2)} = \left| \frac{-v Z_{C1} Z_L \omega^2 M_{12} M_{23}}{A(\omega M_{12})^2 Z_3 + B Z_2 Z_3} \right|.
\] (5.31)

Next we substitute Eq. 5.26 into the denominator; this allows us to write it as \( |(R_d + Z_{C1} Z_3)(\omega^2 M_{12}^2 + Z_1 Z_2) + R_d Z_{C1} Z_2 Z_3| \). Again making use of the values from Tab. 5.1, \( \omega^2 M_{12}^2 \ll Z_1 Z_2 \) and thus it may be dropped, giving

\[
v_0^{(3)} = \left| \frac{-v Z_{C1} Z_L \omega^2 M_{12} M_{23}}{B Z_2 Z_3} \right|.
\] (5.32)

Now we recognize that \( Z_1 \ll Z_{C1} \) since \( R_1 \ll \omega L_1 \) and \( \omega L_1 \ll 1/(\omega C_1) \) as apparent from the values listed in Tab. 5.1. This yields

\[
v_0^{(4)} = \left| \frac{-v R_L \omega^2 M_{12} M_{23}}{R_d Z_2 R_L} \right|.
\] (5.33)

If we have an ideal preamplifier and negligible capacitance associated with the cables, the impedance of \( C_3 \) can be ignored relative to \( R_L \), and thus \( Z_L = Z_3 = R_L \) and:

\[
v_0^{(5)} = \left| \frac{-v R_L \omega^2 M_{12} M_{23}}{R_d Z_2 R_L} \right|.
\] (5.34)

In this limit \( R_L \) drops out; recalling that \( Z_2 \) is the series combination of \( R_2, L_2, \) and \( C_2 \), we can use Eq. 5.3 to conclude

\[
v_0^{(5)} = \frac{v \omega^2 M_{12} M_{23}}{R_d \sqrt{R_2 + \omega^2 L_2^2 \left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}}.
\] (5.35)

This function is identical to Eq. 5.19, where the previously defined scale factor is \( \alpha = M_{12} M_{23}/R_d \). Therefore, the complete model circuit simplifies to give the voltage that we had predicted from a simple argument. Table 5.2 summarizes the approximations that were necessary to arrive at this conclusion.
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<table>
<thead>
<tr>
<th></th>
<th>Approximation</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>weak direct coupling, $M_{13}$ very small</td>
<td>$M_{13} \ll M_{12}, M_{23}$</td>
</tr>
<tr>
<td>2</td>
<td>effect of $M_{23}$ negligible</td>
<td>$\omega^2 M_{23}^2 \ll Z_2 Z_3$</td>
</tr>
<tr>
<td>3</td>
<td>effect of $M_{12}$ negligible</td>
<td>$\omega^2 M_{12}^2 \ll Z_1 Z_2$</td>
</tr>
<tr>
<td>4</td>
<td>$R_1$ small compared to impedance of $L_1$</td>
<td>$R_1 \ll \omega L_1$</td>
</tr>
<tr>
<td>5</td>
<td>impedance of $L_1$ small compared to that of $C_1$</td>
<td>$\omega L_1 \ll 1/\omega C_1$</td>
</tr>
<tr>
<td>6</td>
<td>preamp input impedance</td>
<td>$R_L, 1/\omega C_3 \gg R_3, \omega L_3$</td>
</tr>
<tr>
<td>7</td>
<td>$R_L$ large compared to impedance of $C_3$</td>
<td>$1/\omega C_3 \ll R_L$</td>
</tr>
</tbody>
</table>

Table 5.2: Approximations that simplify the fitting function, in order of application.

**Comparisons**

At this point the question is to identify the level of detail required to adequately represent our apparatus. The process that was adopted was to fit the data to the models arising from the successive levels of approximation described above, and look for any change to the residuals of the fit. The performance of the fits is discussed here.

The first approximation listed in Tab. 5.2 had no effect on the residuals of the fits to the data at a detectable level. This indicates that the direct coupling between the drive and detection loops is extremely weak, as expected given the design of the experiment. Further approximations have a small but detectable effect. The last approximation is in fact not at all valid for our apparatus, since $R_L \gg 1/(\omega C_3)$ especially at high frequencies. So instead, the last approximation was replaced with $Z_L \approx Z_{C3}$. Therefore, the most complete model function that reduces the residuals can be written by substituting $Z_L = Z_3 = Z_{C3}$ into Eq. 5.30:

$$v_0^{(1a)} = \frac{-v Z_{C1} Z_{C3} \omega^2 M_{12} M_{23}}{A(\omega M_{12})^2 Z_{C3} + B ((\omega M_{23})^2 + Z_2 Z_{C3})}.$$  \hspace{1cm} (5.36)

Since there is no improvement to the quality of fits obtained with the full solution (Eq. 5.27) over those obtained with (Eq. 5.36) the 'full' solution now refers to Eq. 5.36 from this point forward.

At best, the full model function reduces the amplitude of the residuals by approximately a factor of two, relative to that of the simple model function. For example, Fig. 5.6 shows
Figure 5.6: Examples of the residuals obtained from the curve fit to the measured resonance curve. The residuals from the fit to the full model (Eq. 5.36) are smaller than those from the simple model (Eq. 5.19), indicating a better fit.

the residuals obtained from the full and simple fits to the data set that was shown previously in Fig. 5.3. In this case, the residuals are reduced by approximately a factor of three when the full model function is used.

Improvements to the residuals obtained with fits to the full function can be illustrated in a more compact way. The standard figure-of-merit for curve fits is the reduced chi-squared value, \( \chi^2_\nu \). When fitting a data set consisting of \( n \) data points \( (x_i, y_i) \) with uncertainties \( \sigma_i \) to a model function \( y_{fit}(x) \), \( \chi^2_\nu \) is defined as

\[
\chi^2_\nu = \frac{1}{\nu} \sum_{i=1}^{n} \frac{(y_{fit}(x_i) - y_i)^2}{\sigma_i^2}
\]

(5.37)

where \( \nu \) is the number of degrees of freedom. If each data point falls within the uncertainty \( \sigma_i \), \( \chi^2_\nu \approx 1 \).

The value of \( \chi^2_\nu \) was obtained for each function by estimating the uncertainty \( \sigma_i \) from the scatter in the data at the edges of the scan. Figure 5.7 shows the dependence of \( \chi^2_\nu \) on the function used to curve fit the data. Clearly the fits obtained using the full function
CHAPTER 5. METHODS

Figure 5.7: Comparison of the average $\chi^2_v$ values obtained from averaging fits to five sets of background data. Average values obtained from fits to each of the simple and full model functions are shown. The $\chi^2_v$ value is smaller when using the full function, indicating a better quality of fit.

Exhibit a smaller value of $\chi^2_v$. Below $\sim 300$ kHz, the $\chi^2_v$ values from the full fit are $\sim 10$; above $300$ kHz they increase to as much as $10^4$. In contrast, the full fit gives smaller values. Below $\sim 300$ kHz, the $\chi^2_v$ values from the full fit are $\sim 1$; above $300$ kHz they increase to $\sim 10^3$. Above $\sim 300$ kHz, the residuals of the fits exhibit systematic trends regardless of which fit function is used, as clearly seen in Fig. 5.7. This is indicative that some feature of the circuit behaviour is not captured by the models.

Another quantitative comparison between the two fits can be made by considering the value of the resistance that is extracted from the fit. The value obtained from the full function is not significantly different from that given by the simple fit, as can be seen in Fig. 5.8. The difference between the sample resistances obtained from the simple and full fit is typically less than 1%. Much more important is the improvement in the uncertainty in the sample resistance that is obtained when using the full fit function over the simple fit function. Figure 5.8 also shows that the uncertainty in the value of the resistance is typically reduced by 60% when using the full fit. For the smaller volume samples this
Figure 5.8: Top: Fractional difference between the values of $R_s$ obtained from the simple and full models. Bottom: Fractional difference between the uncertainty in values of $R_s$ obtained from the simple and full models.
becomes important in distinguishing the small change in resistance that occurs.

It is suspected that the residuals in these fits are a consequence of the distributed nature of the inductance and capacitance of the $B_1$ coil, which have been modeled here as lumped elements. In this context, it is worth noting that the free space wavelength of electromagnetic radiation at 1 MHz field is 300 m, which is only ten times the total length of wire in the coil ($\sim 30$ m). As a result, at higher frequencies, the current at one end of the wire in the coil is not perfectly in phase with the current at the other end. This effect is not modeled in the circuit analysis and likely contributes to the systematic residuals. Considering the comparison between simple and full models of the circuits, we postulate that further improvements to the circuit model incorporating distributed elements and finite wavelength effects might further reduce the uncertainty in $R_s$, but that they will not likely change the value of $R_s$ that is extracted from the data in any significant way. In spite of the small systematic trends in the residuals, the important point remains that they are typically less than 1% of the signal amplitude, and that adding further complexity to the model function is unlikely to yield significant improvements in accuracy.

**Additional details**

In our experiments, the recorded voltage is amplified by a preamplifier followed by a lock-in amplifier. We may combine the effect of both amplifiers into a single gain factor and express the measured voltage as

$$|v_m| = g(f)|v_0|$$

(5.38)

where the function $g(f)$ was characterized by measuring the output signal when a fixed, known signal was input to the preamp/lock-in combination. The magnitude of all experimental data were corrected by the factor $g(f)$ before any curve fitting was done. The magnitude of all data were also normalized to a peak value of unity before performing further analysis.

Another procedure that was applied to the data was to compensate for the slight mismatch between the resonance frequencies of the foreground and background resonance scans. Figure 5.9 illustrates how this was done. The two points in the lower part of the sketch represent two background data points, at the coordinates $(f_{b1}, R_{b1})$ and $(f_{b2}, b_{f2})$. The upper point represents a nearby foreground data point, $(f_f, R_f)$. If the difference $R_f - R_{b1}$
Figure 5.9: Definitions of quantities used to interpolate the data and correct for small frequency shifts.

is used as the value of the sample resistance $R_s$, the resulting value has a slight systematic error due to the frequency shift. This error was reduced by interpolating the background data to estimate the background resistance at the foreground frequency $f_f$. A linear interpolation scheme was employed; this interpolated value, $R_i$, was found by determining the local slope $b$ of the background data from the adjacent data point; the value of $R_i$ at the foreground frequency $f_f$ is

$$R_i = R_{b1} + b(f_f - f_{b1}) .$$

(5.39)

It is worth noting that the frequency shift between foreground and background data is exaggerated in Fig. 5.9. Typically the shift $f_f - f_{b1}$ is only 0.5% of the frequency difference $f_{b2} - f_{b1}$ between one data point and the next. In most cases, the effect of interpolation was to change the sample resistance by 1 % of its value. In some cases, the effect was $\sim$ 3-4 %, and in rare cases the correction was as much as 8 %.

5.2 Measurement of induced magnetic field

The magnetic field in the vicinity of the sphere can be directly measured using a probe coil, and thus can be compared with the predictions listed in Chapter. 3. This experiment was performed using sphere # 2 (radius 12 cm) placed inside the transmit coil. In this section I
describe the procedure used to collect and analyze the relevant data.

Recalling the curvature of the field lines that are sketched in Fig. 3.2, we chose to measure the magnetic field along two different radial directions in the $xz$-plane. First, we measured the $z$-component of the field along the $z$-axis. Another convenient choice was to measure the $x$-component of the field along the $45^\circ$ line between $z$- and $x$-axes. These coordinates are depicted in Fig. 5.10.

5.2.1 Procedure

The sample was positioned at the centre of the transmit coil. The probe coil was then positioned near the sphere, either along the $z$-axis or along the diagonal direction midway between $z$- and $x$-axes. Perturbations to the magnetic field due to the currents in the probe coil may be neglected because the input impedance of the preamplifier is very high. When measuring the $z$-component of the field, the probe coil was aligned to within $\sim 0.5$ mm of the central $xz$-plane by adjusting the height of the coil and finding the maximum in the EMF across the probe coil, as described in Sec. 4.4. Then to orient the axis of the probe coil to measure the correct field component, it was rotated until the EMF across it was observed to be a maximum. The distance from the sphere to the centre of the probe coil was measured for each coil placement.
The EMF across the probe coil was recorded at twelve frequencies from 0.1 - 1.2 MHz for a given probe coil position. Data above 1 MHz were not used in the analysis however because the probe coil self-resonates at ~ 1 MHz. The spherical phantom was lowered to the bottom of the transmit coil and the measurement was repeated to give the unperturbed background field $B_1$. The probe coil was then moved to a new position, and the procedure was repeated in the opposite order. This procedure was repeated for several probe coil placements in the central $xz$-plane relative to the sphere: five positions along the $z$-axis and four positions along the $45^\circ$ diagonal between the $z$- and $x$-axes.

5.2.2 Method of data analysis

To determine how the data from this experiment are to be analyzed, we first return to the predictions made in Chapter 3. In this problem, the sphere is exposed to a uniform oscillating magnetic field $B_1(t) = B_1e^{-i\omega t}\hat{e}_z$. From Chapter 3, the $z$-component of the field generated by eddy currents in the sphere is

$$B_s^z(r,t) = \frac{1}{2} \frac{j_2(ka)}{j_0(ka)} \left[ \frac{a^3}{r^3} - \frac{3a^3z^2}{r^5} \right] B_1e^{-i\omega t}. \quad (5.40)$$

where $r = (x,y,z)$ is the position vector. Under the conditions of our experiment, where we measure the $z$-component of the field along $z = r$, we have $r = (0,0,r)$ and $B_s^z$ simplifies to

$$B_s^z(0,0,r,t) = \frac{j_2(ka)}{j_0(ka)} \frac{a^3}{r^3} B_1e^{-i\omega t}. \quad (5.41)$$

Since the propagation constant $k$ is complex, $B_s^z(r,t)$ will in general be part in-phase and part out-of-phase with the applied field $B_1(t)$. Defining the applied field as $Re(B_1(t)) = B_1 \cos \omega t$, the field due to the sphere may be expressed in the following way:

$$Re(B_s^z(0,0,r,t)) = B_1(\beta_1^z \sin \omega t + \beta_2^z \cos \omega t) \quad (5.42)$$

where $\beta_1^z$ and $\beta_2^z$ are the coefficients of the signal that are out-of-phase and in-phase with the applied field, respectively.

The total magnetic field is the superposition of the background $B_1$ field and the perturbation due to the presence of the conducting sphere. The $z$-component of the total field is thus

$$B^z = B_1(t) + B_s^z(t). \quad (5.43)$$
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To isolate $B^z$ from the total field, we take the difference between measurements performed when the sample is present (‘foreground’) and when it is absent (‘background’).

The voltage $v$ induced across the probe coil is $v = -\kappa dB/dt$ where $\kappa$ is a constant of proportionality. When the sphere is not present in the $B_1$ coil we measure the background field:

$$v_b = -\kappa \frac{dB_1(t)}{dt} = \kappa \omega B_1 \sin \omega t ;$$

when the sphere is placed inside the coil, we measure the voltage

$$v_f = -\kappa \frac{dB^z(t)}{dt} = \kappa \omega B_1 \sin \omega t (1 + \beta_2^z) - \kappa \omega B_1 \beta_1^z \cos \omega t .$$

In the measurement of the magnetic field, we want to make use of both the magnitude and phase information of the data. The lock-in amplifier outputs two channels, the in-phase signal ($X$) and the quadrature signal ($Y$) defined:

$$X \propto DC(v(t) \cos(\omega t - \phi))$$

$$Y \propto DC(v(t) \sin(\omega t - \phi)) .$$

where the function $DC()$ refers to the DC component of its argument and $\phi$ is the phase angle of the lock-in’s internal reference. Thus the ‘in-phase’ nature of $X$ refers to the fact that it is in phase with the reference signal. If we now include the lock-in gain factors (represented by the proportionality above) in the parameter $\kappa$, we have the following sets of measurements. For the foreground:

$$X_f = \frac{\kappa \omega B_1}{2} \left[ (1 + \beta_2^z) \sin \phi - \beta_1^z \cos \phi \right]$$

$$Y_f = \frac{\kappa \omega B_1}{2} \left[ (1 + \beta_2^z) \cos \phi + \beta_1^z \sin \phi \right] ,$$

and for the background:

$$X_b = \frac{\kappa \omega B_1}{2} \sin \phi$$

$$Y_b = \frac{\kappa \omega B_1}{2} \cos \phi .$$


The difference between the foreground and the background measurements are thus

\[ \Delta X = X_f - X_b = \frac{\kappa \omega B_1}{2} (\beta^2 \sin \phi - \beta^1 \cos \phi) \]

\[ \Delta Y = Y_f - Y_b = \frac{\kappa \omega B_1}{2} (\beta^2 \cos \phi + \beta^1 \sin \phi) . \]

We may now rearrange equations 5.50 to express \( \beta^1 \) and \( \beta^2 \) in terms of \( \Delta X \) and \( \Delta Y \). An extra measurement of \( X_b \) and \( Y_b \) was taken with the probe coil at the centre of the \( B_1 \) coil for each frequency and was used to determine the parameters \( \kappa \) and \( \phi \). Denoting these measurements \( X_c \) and \( Y_c \), we may now express \( \beta^1 \) and \( \beta^2 \) in terms of the measured quantities:

\[ \beta^1 = \frac{X_c \Delta Y - Y_c \Delta X}{X_c^2 + Y_c^2} \]  \hspace{1cm} (5.51)

\[ \beta^2 = \frac{X_c \Delta X + Y_c \Delta Y}{X_c^2 + Y_c^2} . \]  \hspace{1cm} (5.52)

We now return to our spherical model (Chapter 3) and write down the predicted behaviour of \( \beta^1 \) and \( \beta^2 \). Recalling the power series expansion of the spherical Bessel Functions presented in Chapter 3, we may write the \( z \) component of the sphere field as:

\[ Re(B^z_s(t)) = Re \left( \left( \frac{ia^2 \sigma \mu \omega}{15} - \frac{2a^4 \sigma^2 \mu^2 \omega^2}{315} \right) \frac{a^3}{r^3} \right) \]

\[ = \left( \frac{a^2 \sigma \mu \omega}{15} \sin \omega t - \frac{2(\sigma \mu a^2 \omega)^2}{315} \cos \omega t \right) B_1 \frac{a^3}{r^3} . \]  \hspace{1cm} (5.54)

The first and second terms correspond to the parts of the field that are out-of-phase and in-phase with the applied field, respectively. Comparing Eq. 5.54 to Eq. 5.42, we obtain the following expressions for the coefficients \( \beta^1 \) and \( \beta^2 \):

\[ \beta^1 = \frac{\sigma \mu \omega a^5}{15 \frac{r^3}{}} \]

\[ \beta^2 = -\frac{2\sigma^2 \mu^2 \omega^2 a^7}{315 \frac{r^3}{}} . \]  \hspace{1cm} (5.55)

A similar analysis follows for the \( x \)-component of the field. The \( x \)-component of the sphere field is

\[ B^x_s(r, t) = \frac{3B_1 j_2(ka) a^3 x z}{2 \ j_0(ka) \ r^5} . \]  \hspace{1cm} (5.57)
The experiment was done under the conditions \( x = z = r/\sqrt{2} \), so \( r = (r/\sqrt{2}, 0, r/\sqrt{2}) \) and thus
\[
B_z^z\left(\frac{r}{\sqrt{2}}, 0, \frac{r}{\sqrt{2}}\right) = \frac{3B_1 j_2(ka)}{4 j_0(ka)} a^3 r^{-3}.
\] (5.58)

In other words, the \( x \)-component of the field measured along the \( 45^\circ \) diagonal is identical to the \( z \)-component of the field measured along the \( z \)-axis, apart from a factor of \( \frac{3}{4} \). It follows that the coefficients \( \beta \) for these measurements are
\[
\beta_1^x = \frac{\sigma \mu \omega a^5}{20 r^3}, \quad \beta_2^x = -\frac{\sigma^2 \mu^2 \omega^2 a^7}{210 r^3}.
\] (5.59) (5.60)

and that the out-of-phase and in-phase components can be calculated from the data in exactly the same way as for the \( z \)-component:
\[
\beta_1^x = \frac{X_c \Delta Y - Y_c \Delta X}{X_c^2 + Y_c^2}, \quad \beta_2^x = \frac{X_c \Delta X + Y_c \Delta Y}{X_c^2 + Y_c^2}.
\] (5.61) (5.62)

where \( X \) and \( Y \) now refer to the measurements acquired when the coil is arranged so as to measure the \( x \)-component of the oscillating magnetic field.

It follows that each of the measurements may be compared to the theoretical predictions by calculating the out-of-phase and in-phase components of the induced voltage, defined by Eqs. 5.51-5.52 and 5.61-5.62 and comparing the result to the expressions 5.55-5.56, 5.59-5.60.

One further detail which deserves mention is the fact that the difference in foreground and background measurements nominally gives the contribution from eddy currents flowing in the sphere. The background measurement in the experiment was actually performed while the sphere was located at the bottom of the coil, \( y_0 \sim 52 \text{ cm} \) below the central plane, rather than being completely removed from the coil. Since this distance is known, we may calculate the size of the perturbation to the field caused by the sphere during the background measurement. This in turn allows us to correct the data appropriately.

We can approximate the background measurement as the field at a position 52 cm above the sphere. Therefore we consider the background (\( b \)) and foreground (\( f \)) positions of the probe coil relative to the sphere, for each field component that was measured: \( r_b^x = \)
(0, y₀, z), \( r_f^j = (0, 0, z); \) \( r_b^x = (z, y₀, z), r_f^x = (z, 0, z). \) The values of \( \beta \) that we actually measure are given by the difference between values measured with the sphere in these two positions:

\[
\beta_i^z = \frac{1}{B_1} B_i^z(r_f^j) - \frac{1}{B_1} B_i^z(r_b^x) \tag{5.63}
\]

\[
\beta_1^x = \frac{1}{B_1} B_1^x(r_f^j) - \frac{1}{B_1} B_1^x(r_b^x). \tag{5.64}
\]

Substituting in the position vectors into Eqs. 5.40 and 5.58, we have

\[
B_s^z(r_b^x) = -\frac{1}{2} \frac{j_2(ka)}{j_0(ka)} \left[ \frac{a^3}{(y_0^2 + z^2)^{3/2}} - \frac{3a^3z^2}{(y_0^2 + z^2)^{5/2}} \right] \tag{5.65}
\]

\[
B_s^z(r_f^j) = -\frac{1}{2} \frac{j_2(ka)}{j_0(ka)} \left[ -2 \frac{a^3}{z^3} \right] \tag{5.66}
\]

\[
B_s^x(r_b^x) = \frac{3B_1}{j_0(ka)} \frac{j_2(ka)}{2} \frac{a^3z^2}{(2z^2 + y_0^2)^{5/2}} \tag{5.67}
\]

\[
B_s^x(r_f^j) = \frac{3B_1}{j_0(ka)} \frac{j_2(ka)}{2} \frac{a^3}{4\sqrt{2}z^3}. \tag{5.68}
\]

The relative deviation of the measurement from the value calculated earlier is

\[
\frac{\Delta \beta_i^x}{\beta_i^x} = \frac{B_s^z(r_b^x)}{B_s^z(r_f^j)} \tag{5.70}
\]

\[
= -\left[ \frac{a^3}{(y_0^2 + z^2)^{3/2}} - \frac{3a^3z^2}{(y_0^2 + z^2)^{5/2}} \right] \left[ -2 \frac{a^3}{z^3} \right]^{-1} \tag{5.71}
\]

\[
= \frac{z^3}{-2(y_0^2 + z^2)^{3/2}} \left[ 1 - \frac{3z^2}{(y_0^2 + z^2)} \right]. \tag{5.72}
\]

Similarly, for the \( x \)-component of the field,

\[
\frac{\Delta \beta_1^x}{\beta_1^x} = \frac{B_s^z(r_b^x)}{B_s^z(r_f^j)} \tag{5.73}
\]

\[
= \frac{a^3z^2}{(2z^2 + y_0^2)^{5/2}} \left( \frac{a^3}{4\sqrt{2}z^3} \right)^{-1}. \tag{5.74}
\]

\[
= \frac{4\sqrt{2}z^5}{(2z^2 + y_0^2)^{5/2}}. \tag{5.75}
\]
For the geometry of the experiment, the size of these corrections is \( \sim -0.9 \% \) for the \( z \)-component and \( \sim -0.8 \% \) for the \( x \)-component.

Having presented the methods to collect and analyze the data for both the measurement of ohmic losses and the induced magnetic field in spherical phantoms, I discuss the results of these experiments in the following chapter.
Chapter 6

Results: Spherical phantoms

In this chapter I present results from two experiments performed with the goal of measuring the ohmic losses and magnetic field associated with eddy currents induced in spherical phantoms. Experimental results are compared to predictions derived in Chapter 3. These experiments function as controls to characterize the apparatus in detail; having done so, experiments involving human subjects become possible.

6.1 Measurement of ohmic losses

In this section I present the measurements of the ohmic losses in spherical conducting samples. As described in Chapter 5, these losses can be measured by detecting the loading effect when a sample is introduced into a resonant circuit. Analysis of the perturbation of the resonance curve gives the effective sample resistance $R_s$, which is in turn related to the power by Eq. 3.57. The sample resistance of five spheres having the same electrical conductivity was measured at fifteen frequencies over the range 0.1 - 1.25 MHz. By varying the radius of the spheres and the frequency of the applied field, we explored the functional dependence of the sample resistance on two of the variables involved in the spherical model presented in Chapter 3.

First of all, it is interesting to look at the resistance of the coil alone, $R_{coil}$. An example of one such data set is shown in Fig. 6.1. At low frequencies, the resistance exhibits an $f^{1/2}$ dependence, as expected when the skin depth limits the resistance (Sec. 2.4). The power-law frequency dependence of the resistance increases with increasing frequency;
at some point, the resistance is dominated by the radiation resistance which has an $f^2$ dependence [78]. Another interesting feature of the coil resistance is the amount by which it changes in response to environmental fluctuations. In Sec. 4.6, I mentioned that the resistance of the coil partially depends on its electromagnetic and thermal environment, which is why the time between foreground and background scans at a given frequency was minimized. Figure 6.2 shows five different measurements of $R_{\text{coil}}$, from which we can see that the extent of environmental effects can be as much as $\sim 40\%$. Each data set was collected at distinctly different times; i.e. there is no reason to believe that these measurements should have the same value each time. In fact, the apparatus was moved to another room between measurements of sphere #2 and all other phantoms, consistent with the large discrepancy between the sphere #2 background and the other backgrounds. The variations seen in Fig. 6.2 demonstrate the necessity of: (i) measuring $R_{\text{coil}}$ again every time it is needed, and (ii) minimizing the time between the foreground and background measurements. Using these strategies, we were able to reduce the effect of these changes.
Figure 6.2: Frequency dependence of the resistance of the transmit coil, for five distinct data sets corresponding to the background data set for the listed sphere number. The variation between each data set is attributed to electromagnetic coupling of the coil to its environment and thermal variations in the resistivity of the wires. The background corresponding to sphere #2 is different because it was collected in a different room. Error bars are not shown because they are not visible on this scale (smaller than the symbols).

to obtain accurate measurements of the sample resistance $R_s$.

The sample resistance is extracted from the measurements of the resonance curves as described in Sec. 5.1.2, using both the simple and full fit functions, followed by interpolation and subtraction. The sample resistance obtained from using the full fit function is shown in Fig. 6.3 (top) as a function of the square of the resonance frequency. At a glance, we see that the data form straight lines on this plot, indicating that frequency dependence is well described by a quadratic function, as expected from the spherical model (Chapter 3) if the conductivity is independent of frequency. By plotting the slope of each line in the top graph against the fifth power of sphere radius (Fig. 6.3) we see that the results are well described by an $a^5$ dependence, also predicted by the spherical model.

The approach to conducting a thorough analysis of the data is as follows. First, it is important to note that this system is completely constrained. That is, the conductivity of the solution, the radii of the spheres, and all other experimental parameters have already been
Figure 6.3: First glance at the sample resistance $R_s$ for five spherical conducting samples. Top: Demonstration of the $f^2$ behaviour. Bottom: Demonstration of the $a^5$ behaviour. Error bars have not been shown as they are typically about the same size as the symbols.
measured independently. Therefore it should be possible to make quantitative comparisons of the results for $R_s$ with predictions of the model for losses in a sphere presented in Chapter 3. We chose to make this comparison by fitting the loss data to the dependence predicted by the spherical model to extract a value of sample conductivity. In this way conductivity provides a means for comparing of the loss measurement method with other techniques of measuring conductivity. Ultimately, it provides: (i) an indication of how well the apparatus is understood, and (ii) a strict test of the spherical model.

The data for $R_s$ can be compared to the predicted dependence given by Eq. 3.58, which is given as an expansion in powers of $a/\delta$:

$$ R_s = \frac{2\pi\sigma^2 \chi^2 a^5}{15} \left[ 1 - \frac{4}{105} \left( \frac{a}{\delta} \right)^4 + \cdots \right]. $$

(6.1)

To begin, we consider performing a weighted fit of the data to the first term in the expansion; that is, we define a function for the one-term form of the sample resistance:

$$ R_s^{(1)} = \frac{2\pi\sigma^2 \chi^2 a^5}{15}, $$

(6.2)

and since $\chi$ is known for our coil and $a$ is known for each sphere, we can let $\sigma$ be a free parameter in the fit. The resulting value of $\sigma$ extracted from the data is denoted $\sigma_{\text{loss}}$.

Figure 6.4 shows the data with the individual fit to each data set included. The data in this figure are now shown on a log-log plot to make several features apparent that were not visible in Fig. 6.3. First of all, notice that we are able to measure resistances as small as $\sim 0.001 \, \Omega$. Resistances of this magnitude are measurable but also have a significant relative uncertainty, and represent the sensitivity limit of the apparatus. A related feature is that the relative uncertainty in $R_s$ increases as the sample size decreases. This reflects the $a^5$ dependence of $R_s$ – as the radius of the sphere decreases, the resistance decreases very rapidly. However the absolute uncertainty in $R_s$ is limited by the ability of the nonlinear curve fitting algorithm to extract this parameter from the resonance curve, which is approximately the same for each data set. Therefore, low resistances have the largest relative uncertainty.

Consider now Fig. 6.5 which shows the sample resistance as a percentage of the coil resistance; i.e., the size of the perturbation relative to the coil resistance. The lower limit of $\sim 0.001 \, \Omega$ corresponds to $\sim 0.5$% of the coil resistance. Therefore our apparatus is very sensitive to the perturbations caused by the sample. What is also visible from Figure 6.4 is
Figure 6.4: Measured effective sample resistance of spherical vessels filled with sodium chloride solution. Symbols indicate data points, according to the sphere number as listed in Tab. 4.1. Solid lines indicate a least squares fit to Eq. 6.2.

that the scatter in the data increases at small resistances. This indicates that measurements are less repeatable when $R_s$ is small because it becomes more difficult to measure small perturbations to the resonance curve. Since the fit is weighted by the uncertainty in the data, these data should not significantly bias the fit; thus all of the data points were included in all fits.

The value of conductivity extracted from this experiment ($\sigma_{\text{loss}}$) can also be compared to independent determinations based on tabulated values [77]. These determinations yield a 'preparation' conductivity $\sigma_{\text{prep}} = 14.79(15)$, discussed in Sec. 4.5. The ratio of $\sigma_{\text{loss}}$ to $\sigma_{\text{prep}}$ is listed in Tab. 6.1 for each sphere. As a further comparison between the full and simple model functions, this ratio was also calculated using $R_s$ values from fits to both functions.

Several remarks concerning the data presented in Tab. 6.1 can be made. All the ratios come very close to each other, indicating that the results for each sphere are nearly identical. The ratios also come very close to unity, which deserves some discussion; however I will withhold this discussion until more detailed analysis has been presented. Another feature
Figure 6.5: The sample resistance of spherical phantoms, as a percentage of the resistance of the transmit coil. The lower plot shows an expanded view of the lower left corner of the upper plot, showing the region representing the sensitivity limit.
### Table 6.1

<table>
<thead>
<tr>
<th>sphere number</th>
<th>$\sigma_{\text{loss}} : \sigma_{\text{prep}}$, simple fit</th>
<th>$\sigma_{\text{loss}} : \sigma_{\text{prep}}$, full fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.006(17)</td>
<td>1.003(12)</td>
</tr>
<tr>
<td>2</td>
<td>0.988(17)</td>
<td>0.997(12)</td>
</tr>
<tr>
<td>3</td>
<td>0.962(41)</td>
<td>0.969(17)</td>
</tr>
<tr>
<td>4</td>
<td>1.04(18)</td>
<td>0.987(58)</td>
</tr>
<tr>
<td>5</td>
<td>1.4(6)</td>
<td>1.3(2)</td>
</tr>
</tbody>
</table>

Table 6.1: Ratio of the conductivity, as measured by the ohmic loss experiment, to the conductivity as determined from preparation of the solutions and comparison to tabulated values.

Clear from Tab. 6.1 is that the uncertainty in the ratio gets larger as the sample volume decreases, consistent with the larger relative uncertainty in $R_s$ seen at these smaller volumes (Fig. 6.4, Fig. 5.8), as discussed above. Since the uncertainty in $R_s$ is reduced when using the full circuit model, the full circuit model is used to fit all of the resonance curves from this point forward. This shows that it was beneficial to put the effort into solving the full circuit model.

Given the good agreement between the separate fits to each data set, all of the data for sample resistance may be combined together into a global result summarizing the experiment as a whole. The radial dependence can be eliminated from the data if we consider rearranging Eq. 6.2:

$$\frac{15}{2\pi \chi^2 a^5} = \sigma \omega^2,$$

The quantity on the left-hand side of Eq. 6.3, which we will denote as $\xi$, can be calculated from the data and the experimentally accessible parameters; then the slope of a plot of $\xi$ as a function of the square of the frequency should be equal to the conductivity. Additionally, note that $\xi$ includes the experimental uncertainties in $R_s$, $\chi$, and $a$. Now we may perform a linear least squares fit to $\ln \xi$ vs $\ln f$ for the entire data set, with $\sigma$ as the only free parameter. The value of $\sigma$ obtained from this analysis is $\sigma_{\text{loss}} = 14.72(7)$ S/m. The ratio of $\sigma_{\text{loss}}$ to $\sigma_{\text{prep}}$ from the global analysis is $\sigma_{\text{loss}} : \sigma_{\text{prep}} = 0.995(11)$.

A last point about the measurements of losses in spherical samples is that of higher order terms in the model. The predicted form of the sample resistance, Eq. 3.58, also has
CHAPTER 6. RESULTS: SPHERICAL PHANTOMS

A 1 term fit

-4

V

2 term fit

I I I

II II

0.6 0.7 0.8 0.9 1.0 1.1 1.2x10⁶

Frequency (Hz)

Figure 6.6: Residuals of the curve fits to one and two terms of the derived form of the sample resistance \( R_s \). Residuals are shown as a percentage of the measured value of \( R_s \).

higher order terms in \( a/\delta \). Since \( \delta \) involves conductivity and frequency, some deviation in the frequency dependence of the data from the one-term function \( R_s^{(1)} \) is expected. A suggestion of some curvature is visible in the data set for the largest sphere (Fig. 6.3). Consider a function now consisting of the first two terms of the series expansion for \( R_s \) (Eq. 3.58 or Eq. 6.1)

\[
R_s^{(2)} = \frac{2\pi \sigma \chi^2 a^5 \omega^2}{15} \left(1 - \frac{a^4}{105}(\mu \sigma \omega)^2\right).
\]  (6.4)

Note that this equation shows the expected curvature is downwards, in agreement with the data. Additionally, \( \sigma \) still constitutes the only parameter in the equation; both the slope and curvature are set by \( \sigma \) so it is not possible to adjust them both arbitrarily. Using Eq. 6.4, the size of the second term is predicted to be 7% that of the first term at 1.25 MHz for the largest phantom, sphere #1. The data set for this phantom was fit to Eq. 6.4, and the residuals obtained from this fit are compared to those obtained from the fit to the one-term function (Eq. 6.2) in Fig. 6.6.
CHAPTER 6. RESULTS: SPHERICAL PHANTOMS

Note that the residuals from the one-term fit show only a $\sim 3\%$ deviation at 1.25 MHz rather than the 7\% value predicted from 6.4. Generally the residuals from the two-term fit appear marginally more consistent with zero than the one-term fit. The resolution of resistance in the experiment is not high enough to state definitively that we are sensitive to the second term; however the results of this experiment are suggestive that with larger samples and higher conductivity, the second term may become large enough to make quantitative comparisons with the model function. What is relevant for the purposes of this thesis is that there is no clear gain to using the two-term function; fits to one term are sufficient and are used from this point forward.

6.1.1 Discussion

The measurements of ohmic losses in spherical phantoms were fit to the predictions of the spherical model and the resulting conductivity was extracted. The measurement of conductivity from this perturbation technique yields a result that is fully consistent with the nominal value of conductivity determined from the preparation of the solution (obtained with the aid of tabulated values of conductivity vs concentration, as measured with other methods). The implication here is that the eddy current method is actually a very accurate way to measure absolute electrical conductivity of samples. The method has an inherent simplicity in that the boundary conditions are well defined. The standard method for measuring the conductivity of solutions involves a conductivity bridge: the solution is placed in a cell incorporating specialized electrodes. A Wheatstone bridge circuit is then used to obtain the cell impedance by adjusting a resistor in the bridge to balance the currents through the cell and the resistor [79]. Usually a standardized solution with a well known conductivity is used to calibrate the apparatus by obtaining the cell constant, because the current profile is not exactly known. Therefore conventional measurements typically only provide relative comparisons between conductivities of solutions. Our electrodeless determination of conductivity requires no calibration with a reference solution and gives an absolute measurement of the conductivity. The current profile inside the sample is completely specified if the sample is contained in a spherical vessel and immersed in a uniform oscillating magnetic field. Therefore this method provides a very clean, straightforward way of measuring the conductivity of weakly conducting samples.
CHAPTER 6. RESULTS: SPHERICAL PHANTOMS

The fact that $\sigma_{\text{loss}} : \sigma_{\text{prep}} \sim 1$ as measured by the experiment presented in this chapter deserves more commentary. The results of the spherical model were used to extract $\sigma_{\text{loss}}$ from the data. The fact that $\sigma_{\text{loss}}$ agrees so well with that predicted using previously reported measurements of conductivity shows that the spherical model does indeed describe the losses in samples very well. A final point is that this level of agreement indicates that our apparatus is understood to a high degree of accuracy.

6.2 Measurement of induced magnetic field

Further corroboration of the spherical model was investigated by measuring perturbations of an oscillating magnetic field as a function of position in the vicinity of a conducting spherical phantom. The phantom used in this experiment has a 12 cm radius (sphere #2 according to Tab. 4.1). Measurements of the magnetic field near the sphere were obtained using the apparatus described in Chapter 4 and analyzed using expressions given in Chapter 5 to extract a measurement of the conductivity $\sigma$. The $z$-component of the magnetic field was measured at five positions along the $z$-axis; the $x$-component of the magnetic field was measured at four positions along the 45° diagonal between $z$- and $x$-axes. The unperturbed field at the centre of the coil was measured when the sphere was not present. For each placement of the probe coil, the induced field was measured from 0.1 - 1 MHz in 0.1 MHz increments.

Part of each field component is out-of-phase with the applied field $B_1$ and part is in-phase. The portions of each field component ($z$ and $x$) that are out-of-phase and in-phase were separated by calculating the quantities $\beta_1$ and $\beta_2$ as defined by Eqs. 5.51 - 5.52 and Eqs. 5.61 - 5.62. Since the field perturbation is predominantly out-of-phase, we expect the term $\beta_1$ to dominate the response. In fact it was observed that at most, $|\beta_2| \sim 0.08|\beta_1|$, and hence $\beta_1$ data have a lower relative uncertainty than $\beta_2$ data. As a result, only fits to the expression for $\beta_1$ were used to extract conductivity. In a process similar to that followed for the sample resistance (Eqs. 6.2 - 6.3), we define new functions that involve normalizing $\beta$ to eliminate the radial dependence

$$\beta'_1 = \frac{\beta_1 r^3}{a^5}$$  \hspace{1cm} (6.5)

$$\beta'_2 = \frac{\beta_2 r^3}{a^7}.$$  \hspace{1cm} (6.6)
When the data are normalized in this manner, the data sets for the $z$ component collapse onto a single curve, and the data sets for the $x$ component collapse onto another single curve. The new quantity $\beta'$ has an uncertainty which includes the uncertainties in both $r$ and $a$ in addition to the uncertainty from the field measurement itself. Figure 6.7 shows measured values of $\beta_1$ and $\beta'_1$ for each field component.

As in the experiment to measure ohmic losses, we take the same viewpoint here with respect to the data analysis. The system is still completely constrained; there are no free variables. The conductivity $\sigma$ can be extracted from measurements of the field perturbation near the sphere, providing a link between this method and other methods of measuring conductivity. This comparison likewise serves as a measure of the validity of the spherical model.

The process of analyzing this data is identical to that described for the $R_s$ data in the previous section. The measurements of $\beta'_1$ were fit to normalized versions of Eqs. 5.55 and 5.59

$$\beta_1^{(z)} = \frac{\sigma \mu \omega}{15}$$
$$\beta_1^{(x)} = \frac{\sigma \mu \omega}{20}$$

(6.7)

(6.8)

to obtain a value for the conductivity $\sigma$. First, the $\beta'_1$ data set at each coil position was fit; the resulting values of $\sigma$ are presented in Tab. 6.2. Then all of the $\beta'_1$ data sets for each component were combined into one larger data set for each component, (since all data sets for a given component collapse onto the same curve) and were fit together to give an average result. The average results for the conductivity obtained from each field component are shown in Tab. 6.2.

As with the measurements of ohmic losses, the extracted value of conductivity ($\sigma_{\text{field}}$) can be compared with the independent measurement of the solution conductivity, $\sigma_{\text{prep}}$ (see Secs. 4.5 or 6.1). The $z$-component results yield a value of $\sigma_{\text{field}}$ having excellent agreement with $\sigma_{\text{prep}}$. The $x$-component results yield a value of $\sigma_{\text{field}}$ are the correct order of magnitude but do not agree with $\sigma_{\text{prep}}$. This is because the amplification of the lock-in was set to a higher value when measuring the $x$-component than that used when measuring the $z$-component of the field since the $x$-component is much weaker. The relative difference between amplification levels from its nominal value was measured at the time of data collection. A correction factor resulting from these data was used to take the difference into
Figure 6.7: Measurements of the coefficient of the out-of-phase component ($\beta_1$) of the magnetic field generated by eddy currents in the sphere. Top: The $z$-component result is shown on the left and the $x$-component result is shown on the right. Bottom: Normalized data $\beta'_1 = r^3 \beta_1 / a^5$. Error bars are not shown as they are typically less than the size of the symbols.
CHAPTER 6. RESULTS: SPHERICAL PHANTOMS

Table 6.2: Values of solution conductivities extracted from measurements of each component of the magnetic field. The $z$-component values have been corrected for the background deviation from the nominal $B_1$ field.

<table>
<thead>
<tr>
<th>$r$ (m)</th>
<th>$\sigma_{field} : \sigma_{prep}$ (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.148(1)</td>
<td>0.992(12)</td>
</tr>
<tr>
<td>.157(1)</td>
<td>1.002(11)</td>
</tr>
<tr>
<td>.165(1)</td>
<td>0.996(11)</td>
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<tr>
<td>.171(1)</td>
<td>1.003(14)</td>
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<tr>
<td>fit to all data sets:</td>
<td>0.999(11)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r$ (m)</th>
<th>$\sigma_{field} : \sigma_{prep}$ (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.146(1)</td>
<td>0.774(10)</td>
</tr>
<tr>
<td>.157(1)</td>
<td>0.771(10)</td>
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<tr>
<td>.167(1)</td>
<td>0.777(9)</td>
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<td>.168(1)</td>
<td>0.811(12)</td>
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<tr>
<td>fit to all data sets:</td>
<td>0.774(8)</td>
</tr>
<tr>
<td>corrected for amplification factor:</td>
<td>$\sigma = 1.008(11)$ S/m</td>
</tr>
</tbody>
</table>

$z$-component results

account. When this is done the ratio becomes $\sigma_{field} : \sigma_{prep} = 1.008(11)$, consistent with the ratio derived from the $z$-component field data. While correction for the amplification was applied in this case, this complication should be avoided in future measurements by maintaining the same amplification for all measurements where possible.

Measurements of the normalized parameters $\beta_z^z$ and $\beta_z^x$ – representing the portion of the magnetic field that is in-phase with the applied field – are shown in Figure 6.8. The values of conductivity obtained from fits to $\beta_z^1$ were substituted into Eqs. 5.56 and 5.60 and are shown on the same plots as the data for comparison. The agreement of the $x$-component data with the prediction is very good. Agreement of the $z$-component data with the prediction is good for three of the data sets, albeit with a slight systematic shift.

It is not known why the data set with the largest value of $r$ shows such a large systematic shift. Measurements of $\beta_z^z$ however are more difficult than $\beta_z^x$ because they require the detection of a small difference between two large signals, and are thus less likely to be as accurate as measurements of $\beta_z^x$. For example, at 500 kHz, the voltage on one lock-in channel was $\sim$200 mV when measuring the $z$-component, and the shift due to the presence of the sphere was $\sim$ 10 mV (a 5 \% effect); however when measuring the $x$-component, one lock-in channel was $\sim$100 mV and the shift was $\sim$20 mV (a 20 \% effect). Therefore the
Figure 6.8: Normalized data of the in-phase coefficient ($\beta_2$) of the magnetic field generated by eddy currents in the sphere. The $z$-component result is shown on the top graph and the $x$-component result is shown on the bottom graph. Theoretical curves are generated from the value of $\sigma$ calculated from the $\beta_1$ coefficients. The outlying data set on the top plot is that taken at the largest radial position along the $z$-axis.
percent change in the signal due to the presence of the sphere is four times smaller for the $z$-component than the $x$-component.

Additionally, at larger values of $r$ the field map inside the transmit coil becomes less homogeneous (see Sec. 4.1) making these data more susceptible to systematic errors in coil placement between measurements. For example, at the probe coil position of 14.76 cm from the isocentre along the $z$-axis, the field gradient is $5 \times 10^{-4} B_1 \text{cm}^{-1}$. However at the probe coil position of 17.11 cm, the field gradient increases to $1.5 \times 10^{-3} B_1 \text{cm}^{-1}$. This effect is not large enough to explain the largest systematic shift. Considering the level of agreement between the prediction and all other data sets, this data set is treated as an outlier.

In summary, separating the in-phase component from the out-of-phase component allows calculation of more reliable values of conductivity from the dominant out-of-phase component. The values of conductivity calculated from the out-of-phase component then yield excellent predictions for the in-phase component with one exception; thus the measurements of $\beta'_1$ are consistent with the measurements of $\beta'_2$.

6.2.1 Discussion

In this section I presented measurements of the magnetic field generated by eddy currents induced in a conducting sphere. The electrical conductivity of the sphere was extracted from the data to serve as an indication of the validity of the measurement technique, as well as the validity of the spherical model. The conductivity obtained from these experiments shows complete agreement with independent determinations utilizing tabulated values of conductivity. While both the loss and field mapping experiments yield excellent results, the field mapping has several drawbacks since it involves accurate placement of the probe coil, is generally more complicated to do, and extra work must be done to minimize the electric field effects. However the value of the experiment is clear because it demonstrates (i) agreement with the spherical model and (ii) corroboration with the results obtained in Sec. 6.1. Furthermore, it is again demonstrated that the apparatus that was used is fully understood. This allows us to move from spherical system having well-defined boundary conditions to examining induced losses in human subjects.
6.3 Summary

In this chapter I presented measurements of the ohmic losses and field perturbation generated by eddy currents induced in conducting spherical phantoms. The results of each experiment can be characterized by a value for the electrical conductivity of the phantom. These extracted values agree with tabulated values for conductivity obtained previously using other experimental methods. The results of both experiments are consistent with the spherical model presented in Chapter 3. More importantly, the experiments demonstrate that the apparatus used is extremely well characterized. Because there are no discrepancies between the measured properties (losses and magnetic field) and their predicted forms given by the spherical model, we may conclude that there are no ambiguities in the apparatus and procedures to collect and analyze these data. Having complete confidence in our apparatus and methods from these phantom experiments, attention may now be turned to the problem of measuring ohmic losses in human subjects using this same technique, allowing us to make statements about the SNR and SAR in very low field MRI experiments. These experiments are presented in the following chapter.
Chapter 7

Results: Human subjects

In the previous chapter, I presented the results of experiments that were performed to characterize ohmic losses in conducting spherical phantoms over the frequency range 0.01 - 1 MHz. These experiments were repeated to characterize the ohmic losses in human subjects over the same frequency range. In this chapter I describe these experiments and present the results.

Two variations of the experiment were carried out. In the first variation, the receive coils described in Sec. 4.2 were used to measure the effective sample resistance of human subjects. A similar set of coils has been used to collect low field HPG lung images [3, 4], and gives information regarding SNR in low field MRI. In the second variation, which was designed to extract information relevant to SAR, we used the transmit coil described in Sec. 4.1 and used in the phantom experiments (Chapter 6).

Each subject was positioned such that the sternum was located in the central xz-plane of the coil. When using the transmit coil, this ensures that the thorax of the subject occupies the uniform volume at the centre, as would be the case for MRI of the lungs. Subjects were positioned as accurately and reproducibly as possible within the coils. The uncertainty in placement is estimated to be ~ 1-2 cm. The relative positioning of subjects and coils is shown in Fig. 7.1. I will present and discuss the results of the two experiments separately.

Four male and three female subjects participated in the experiments described in this

---

1 The RMS magnetic field strength inside the coil is ~ 7 nT and the maximum frequency was 1.25 MHz. As a comparison, the magnetic field 30 cm from a CRT monitor is ~ 5 nT at 500 kHz. These experiments were deemed 'minimal risk' and approved by the SFU Office of Research Ethics.
Figure 7.1: Positioning of the subject inside the receive coils (left) and inside the transmit coil (right). When using the transmit coil, the shoulders of the subject were oriented parallel to the direction of $B_1$. The sternum of the subject was located in the central $xz$-plane of the coil such that the thorax of the subject occupied the central volume of the coil in each case.

*Drawings copyright Mike Hayden (2006). Used with permission.*

<table>
<thead>
<tr>
<th>Subject number</th>
<th>Sex</th>
<th>Height (cm)</th>
<th>Mass (kg)</th>
</tr>
</thead>
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<td>male</td>
<td>180</td>
<td>91</td>
</tr>
<tr>
<td>6</td>
<td>male</td>
<td>174</td>
<td>77</td>
</tr>
<tr>
<td>7</td>
<td>female</td>
<td>164</td>
<td>61</td>
</tr>
</tbody>
</table>

Table 7.1: Summary of physical characteristics of subjects.
chapter; they ranged in height from 160 - 185 cm, and in mass from 55kg - 91kg. A list summarizing the physical characteristics of the subjects is provided in Tab. 7.1. Subjects will be referred to according to the numbers given in the table.

7.1 Measurement of SNR

The measurement of $R_s$ for human subjects obtained using the receive coils gives the best indication of the SNR in low field MRI experiments because the filling factor of such a coil is much higher than that of the transmit coil, and is therefore more sensitive to small resistances. This increase in sensitivity allows us to probe lower frequencies than is possible using the transmit coil.

To acquire the data, the subject was positioned between the two receive coils. The coils were placed 28 cm apart, one in front of and one behind the subject’s thorax, as would be the case for lung imaging. This arrangement is illustrated in Fig. 7.1. The procedure to collect data was identical to that described for the loss measurement in the previous chapter; the resonant response of the tuned coil was recorded with and without the subject present inside the coil, matching the resonance frequency in each case as well as possible. The resonance curves were fit to the full circuit model function described in Sec. 5.1.2 (Eq. 5.30) to extract $R_s$ in each case.

Data were collected for male subjects #1 and #2. Before commenting on the measured values of $R_s$, one interesting feature of these experiments is that motion associated with breathing is apparent in the residuals of fits to the scan, as demonstrated by the example shown in Fig. 7.2. Since the scans were swept from low frequency to high frequency, the $x$-axis can be interpreted as a time axis in this figure. In this particular example the time to collect the scan was approximately $\sim 110$ seconds. There are approximately $\sim 26$ cycles evident in the residuals, indicating an average time of $\sim 4.2$ seconds per breath. Not only are the timescales consistent with a normal respiratory cycle, this effect was observed directly during the acquisition of this data. When the subject held his breath, the oscillations stopped, confirming that they are correlated with breathing motions.
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Figure 7.2: Residuals from fitting the data collected with the receive coils to the full fit function. Residuals are expressed as a fraction of the peak value of the data, which has been normalized to unity.

Figure 7.3: Normalized amplitudes of the residuals associated with breathing motions. The peak-to-peak amplitude corresponding to each breath is seen to be related to the slope of the resonance curve.
Figure 7.4: The uncertainty in the circuit resistance extracted by the resonance curve fit, as a percentage of the resistance. The foreground and background are both shown. The uncertainty in the foreground data is slightly larger than that of the background data, due to the motions associated with the subject's breathing.

An interesting question is whether the peaks in the residuals correspond to complete inhalation or exhalation. During inhalation, the lungs expand, increasing the filling factor and hence decreasing the quality factor. However upon examination of the amplitude of the oscillations it appears that a shift in the resonant frequency is the important consideration. Figure 7.3 shows the amplitude of the peak-to-peak residuals (normalized to the largest value) as a function of breath number. A 180° phase shift is seen at \(~0.995\) MHz, near the centre of the resonance, hence amplitudes above the resonance frequency are negative. Since this graph is clearly proportional to the slope of the resonance curve, it is more likely that the frequency shift gives rise to these residuals rather than the shift in quality factor. This hypothesis could easily be tested in a future experiment.

Another question that comes to mind when looking at Fig. 7.2 is whether or not the oscillations in the foreground data affect the values of the parameters that are extracted from the curve fit. Figure 7.4 shows the uncertainty in the resistance extracted from the curve fits of the foreground and background data, as a percentage of the resistance. While the uncer-
Figure 7.5: Measurements of the effective resistance of two human subjects as measured with two rectangular receive coils placed in front and back of the thorax. The resistance of the coil itself is also shown.

Figure 7.5 shows the resistances of the receive coils themselves and of each subject. While an accurate prediction for the frequency dependence of these resistances would involve combining a detailed model of the human body — including any tissues having a frequency-dependent conductivity — and the field profile of the coil, it is clear that the re-
results of human subjects obeys an $f^2$ relationship very well, just as was the case for the sphere. Considering the discussion in Sec. 2.4 that a conductor in the thin limit has an effective resistance proportional to $f^2$, this experiment demonstrates that the body appears to be a weak conductor with a frequency independent conductivity over the two decades of frequency examined here. The average resistance of the two subjects was fit to a quadratic relation

$$R_s = kf^2$$

(7.1)

to demonstrate the $f^2$ dependence of the resistance. This can be seen in Fig. 7.5 which shows this fit to the average subject resistance.

The results of this experiment can be compared to those described in Sec. 6.1 – involving phantom measurements with the transmit coils – in several ways. First of all, we see that the receive coils are extremely sensitive in comparison to the transmit coil. The minimum sample resistance that we were able to detect with the transmit coils was ~10 mΩ; this value indicates the point at which the uncertainty in the resistance became large and the scatter also increased in a manner similar to that described in Chapter 6. In contrast, the receive coil was sensitive to sample resistances of 1 mΩ. The receive coils were sensitive down to correspondingly lower frequencies as a result; the transmit coil was sensitive to samples down to 100 kHz, while the receive coils were sensitive down to 30 kHz.

The measurements obtained for the resistance of the receive coils and the effective sample resistance of the human body allow an estimate to be made of their contributions to the noise in the receiver circuit. Recalling the Johnson noise (Eq. 2.21), it is now possible to estimate the noise due to the receiver coil and body in very low field MRI employing these coils. The intrinsic SNR could then be estimated from Eq.2.22 if the magnetization is known.

The primary result of this experiment is that the resistance of the coil dominates the subject resistance by at least a factor of ten over the entire frequency range studied here. If we recall that Bidinosti et al. [4] acquired human lung images at 3 mT, corresponding to a $^3$He Larmor frequency of 100 kHz, from Fig. 7.5 it can be deduced that $R_{coil} : R_{body} \approx 250$, neglecting the resistance of the compensation coils that were used. The Simon Fraser group magnet can operate up to 10 mT (300 kHz), in which case $R_{coil} : R_{body} \approx 50$. Therefore the SNR is clearly coil dominated with this receiver; while the SNR is improved by raising the magnetic field strength (and frequency), careful design of the receiver would allow
improvements to the SNR at a given frequency. Keeping in mind that \( \text{SAR} \propto (R_{\text{coil}} + R_s)^{-1/2} \), we see that the SNR of low field \textit{in vivo} HPG images could potentially be improved by approximately a factor of seven over those that have been acquired to date by redesigning the receive coils. Implementation of various coil designs involving superconducting or Litz wire, cryogenic cooling, and/or SQUID sensors (Sec. 2.4) should be investigated as techniques for reducing the resistance of the receive coils, possibly even to levels below that of the body.

### 7.2 Measurement of SAR

Measurements of the sample resistances for seven subjects listed in Tab. 7.1 were obtained over the frequency range 0.1 - 1.25 MHz using the transmit coil. Data were collected using the same procedure described in the previous section. Once again, we see oscillations in the resonance curve due to the breathing motion of the subjects. An example of this is shown in Fig. 7.6. The time to collect the scan shown was approximately \( \sim 80 \) seconds, while there are approximately \( \sim 18 \) cycles in the residuals, indicating an average time of \( \sim 4.5 \) seconds per breath, consistent with the result seen in the receive coil experiment. Clearly the size of the effect is reduced because the filling factor is much smaller. Due to the smaller size of the residuals, and the small effect seen on the uncertainty in \( R_s \) with the receive coils, the effect of oscillations was not investigated as part of this experiment.

The sample resistances of all the subjects are shown as a function of frequency in Fig. 7.7. The measured effective resistances of the subjects falls within a range of values; the variations in the resistance appears to correlate with the physical characteristics of the subjects (see Tab. 7.1). The resistance of the subjects is again accurately quadratic in frequency; the data were fit to Eq. 7.1 to make quantitative comparisons of the range of resistance values with the physical characteristics of the subjects. The manner in which the resistance of the subjects correlate with sex, mass, and height were investigated. The resistance of each male was higher than the resistance of all of the females; however the mass of each male was also higher than that of all of the females. So since sex and mass are not independent, we will simply regard sex as a convenient way of separating higher mass subjects from lower mass subjects in this subject pool. When all the data from male subjects are averaged and compared to the average of all the data from female subjects, we
clearly see the average male resistance is greater (see Fig. 7.8). The ratio of the coefficient $k$ is $k_{\text{male}} : k_{\text{female}} = 1.48(5)$; therefore losses in male subjects were nearly 50% greater than losses in female subjects, on average. The correlation of the coefficient $k$ with mass and height is illustrated in Fig. 7.8. Correlation with both properties is evident, with mass providing the strongest correlation. A straight line fit of $k$ with the mass was used to calculate the slope, which yields a value of $1.54(6) \times 10^{14} \Omega \text{Hz}^{-2} \text{kg}^{-1}$. This fit is shown in Fig. 7.9. The intercept of this straight line fit is nonzero, indicating that a power law fit might be more appropriate, but there is not enough data to perform a fit that yields a meaningful relation. While the mechanism for variations in resistance is in fact the volume and number of available eddy current paths, the data are seen to correlate strongly with mass, which is readily obtained.

The correlation with height is not as strong as that with mass. It is not immediately obvious if taller subjects should show a larger or smaller resistance than shorter subjects. Because the sternum of the subject was positioned in the $xz$-plane of the coil, taller subjects actually have a smaller fraction of their body inside the coil. However the central homo-
Figure 7.7: Top: Measurements of the effective resistance of seven human subjects in the presence of a uniform $B_1$ field. Bottom: Expanded graph showing the higher frequency data, where error bars have been removed for the sake of clarity. The separation between subjects is clear. Physical characteristics are listed in Tab. 7.1.
Figure 7.8: Top: Plot of the average measured values of effective resistance for males and females. Bottom: Correlation of the coefficient $k$ with the mass and height of the subjects.
Figure 7.9: Top: Mass correlation including straight line fit. Bottom: The graph has been extended to include the origin.
geneous volume of the coil is always well filled regardless of the height of the subject. For the time being, the height correlation is not regarded as an important result and more meaningful correlations are expected to involve mass, volume and body composition.

Deductions may be made about the SAR in the following way. Recalling the time-averaged power, Eqn. 3.57, and the magnet constant \( \chi = B_1/I \), we can write the time-average power absorbed by the subject as

\[
< P > = \left( \frac{B_1}{\chi} \right)^2 \frac{R_s}{2}
\]

since \( B_1 \) is the peak value of the magnetic field. This becomes

\[
< P > = \left( \frac{B_1}{\chi} \right)^2 \frac{k f^2}{2}
\]

when combined with Eq. 7.1. Finally since the SAR is the power per unit mass of tissue exposed to the fields,

\[
\text{SAR} = \frac{< P >}{m} = \frac{k}{2m\chi^2} f^2 B_1^2.
\]

where \( m \) is the total mass exposed to the fields, which are assumed to be uniform.

To derive a conservative estimate of SAR, we use the value of \( k \) extracted from the data set for the largest male, having the highest resistance (subject # 5). Additionally, we can approximate the filling factor with a simple calculation to arrive at an estimate for the relevant mass. To make a lower estimate of the mass, the cross-sectional area of subject # 4 - having nearly the lowest mass - was measured as a function of height. The normalized volume average of the square of the \( B_1 \) field was estimated to have a value of 76(5)%. Therefore we use 76 % of the largest subject's mass is involved in the losses giving rise to the largest subject's. The magnet constant \( \chi \) is known, so all of these values may be substituted into Eq. 7.4. In this case the coefficient of \( f^2 B_1^2 \) is found to be

\[
\frac{k}{2m\chi^2} = 1.9 \times 10^{-5}\text{Wkg}^{-1}\text{Hz}^{-2}\text{T}^{-2}.
\]

To put this value into terms relevant to MRI, we recall the RF pulses discussed in Sec. 2.2. The angle \( \alpha \) between the magnetization vector \( M \) and static magnetic field \( B_0 \) is given by Eq. 2.7, which may be rewritten as

\[
\alpha = \gamma B_1 t.
\]
Therefore $B_1$ can be expressed in terms of the duration of a $\pi$-pulse, which tips $M$ by $180^\circ$:

$$B_1 = \frac{\pi}{\gamma t_\pi}.$$  \hfill (7.7)

Using this relation, the SAR may be calculated as a function of pulse duration and static field strength so that the result may be compared to the safety guidelines [21, 22]. Combining Eq. 7.4 with Eq. 7.7, we have

$$\text{SAR} = \frac{k}{2m\chi^2} f^2 \left(\frac{\pi}{\gamma t_\pi}\right)^2,$$  \hfill (7.8)

and since the frequency is related to the static field $B_0$ by the Larmor equation $2\pi f = \gamma B_0$, we obtain

$$\text{SAR} \propto \frac{R_s}{t_\pi^2},$$  \hfill (7.9)

Using the measured result for $k/2m\chi^2$ (Eq. 7.5), we obtain a relation for the SAR as it depends on the static field strength $B_0$ and the duration of a $\pi$-pulse

$$\text{SAR} = \frac{k}{2m\chi^2} \left(\frac{\gamma B_0}{2\pi}\right)^2 B_1^2 \left(\frac{\pi}{\gamma t_\pi}\right)^2$$  \hfill (7.10)

$$= \frac{k}{8m\chi^2} \left(\frac{B_0}{t_\pi}\right)^2.$$  \hfill (7.11)

Using the measured result for $k/2m\chi^2$ (Eq. 7.5), we obtain a relation for the SAR as it depends on the static field strength $B_0$ and the duration of a $\pi$-pulse

$$\text{SAR} = 1.9 \times 10^{-5} \text{Wkg}^{-1} \text{Hz}^{-2} \text{T}^{-2} \left(\frac{B_0}{t_\pi}\right)^2.$$  \hfill (7.12)

This result is the primary conclusion from experiments of losses measured with the transmit coil. Equation 7.12 is an semi-empirical relation for the SAR during a $\pi$-pulse, which we have measured to be valid for uniform fields in the frequency range 0.01 - 1.25 MHz. To demonstrate what it means, some examples are shown in Fig. 7.10. If we select a field strength $B_0$, Eq. 7.12 gives SAR as a function of the pulse duration $t_\pi$ alone. This function is presented in the case of three static field strengths at which low field images have been acquired [1, 2, 4]. Note that the corresponding Larmor frequencies used in Refs. [2–4] were probed by this experiment, while our result has been extrapolated up to the Larmor frequency used in Ref. [1]. Also shown in Fig. 7.10 is the USFDA whole-body SAR exposure guideline, 4 W/kg. From the figure, we can see that very rapid,
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Figure 7.10: Calculation of the SAR in human subjects as it depends on the duration of a $\pi$ tipping pulse in milliseconds (which sets the $B_1$ field amplitude) and the strength of the static field $B_0$ (which sets the Larmor frequency). Values of SAR extracted from our data are shown for the $B_0$ values used by Refs. [2-4] and an extrapolation of the result up to the value of $B_0$ used by Ref. [1]. The 4 W/kg level is a maximum whole-body exposure guideline recommended by the USFDA [21].

Sub-millisecond $\pi$-pulses may be applied without exceeding this limit if a sufficiently low static field strength $B_0$ is employed. For example, at 15 mT a $\pi$-pulse lasting 0.02 ms corresponds to an SAR of 4 W/kg. However if we extrapolate our relation for low field SAR (Eq. 7.12) to a $B_0$ value of 1.5 T, we reach an SAR of 4 W/kg when the $\pi$-pulse duration is 1.7 ms. This implies that stronger RF pulses may be applied when using low static fields, allowing a corresponding reduction in image acquisition time.

Furthermore, note that the result of this experiment shows that a long sequence of many $\pi$-pulses at a very rapid repetition rate (i.e. a high duty cycle) can be safely performed in very low field MRI. This is an important conclusion because $\pi$-pulses are avoided in high field imaging due to the associated SAR. The function of $\pi$-pulses is to refocus the transverse magnetization; however an equivalent effect is achieved with a bipolar field gradient.
At the high static field strengths employed in conventional imaging, bipolar gradients are used to refocus the transverse magnetization, however these gradients are problematic in very low field imaging where concomitant gradient effects may become important [80]. Therefore we may conjecture that the use of \( \pi \)-pulses may be a safe way to avoid concomitant gradient effects in very low field imaging, and unorthodox pulse sequences employing more RF pulses may be designed for very low field imaging.

Some care must be taken when drawing conclusions from the whole-body SAR limit. Since the energy deposition is of course not uniform in a subject, 'hot spots' may form. Therefore the local maximum SAR is also a consideration, as suggested by [21] (Tab. 2.3). Our measurement of whole-body SAR does not allow conclusions to be drawn regarding local maxima in SAR.

### 7.3 Discussion

The work presented in this chapter constitutes the core of this thesis. The main goal was to obtain an experimental investigation of factors pertaining to SNR and SAR in low field MRI of human subjects. Experiments probing the SNR show that the simple receive coil design used by Bidinosti et al. [3, 4] has significantly a higher resistance than that of the human body. Therefore we conclude that the images they obtained were certainly in the coil-dominated noise regime. If a focused effort is placed in design of a lower resistance receive coil for very low field HPG MRI, the advantages of the predicted frequency-independent SNR in the body-dominated noise regime might yet be reached. Our results imply that an order of magnitude improvement in SNR could potentially be obtained if the coil resistance is reduced. Clearly receive coil design will be a crucial consideration for very low field MRI and must be undertaken if optimum signal-to-noise ratios are to be achieved. Experiments probing SAR show unequivocally that SAR depends on the square of the frequency of the applied RF radiation, over the range explored in this experiment. This dependence was expected from both theoretical predictions based on a spherical model, and from low frequency extrapolations of high frequency data. The key result of the SAR experiment is that MRI experiments in which very low static fields are employed offer the possibility for using stronger oscillating RF fields and hence more complex and more rapid RF pulse sequences than MRI experiments at high static fields.
One potential benefit of the technique to measure ohmic losses employed in this thesis is that it may be a sensitive method for measuring SARs in human subjects due to exposure to other sources of electromagnetic fields. Since general exposure guidelines for all types of electromagnetic exposure exist [81], this method may be useful in conducting SAR experiments in a much broader context.
Chapter 8

Conclusion

In this thesis I have presented a collection of experiments examining the signal-to-noise ratio (SNR) and specific absorption rates (SAR) for electromagnetic energy deposition associated with MRI experiments. The focus was on the very low field regime (\(\sim 1 \) to 100 mT) relevant to emerging MRI techniques such as those employing hyperpolarized noble gases. The first set of experiments that were presented involve spherical phantoms; these experiments serve as controls to validate the measurement techniques. Subsequent experiments involving human subjects yield information regarding fundamental SNR and SAR limits for real MRI experiments. Ultimately, this information allows design constraints to be established for very low field MRI hardware and radiofrequency (RF) pulse sequences. The experimental results presented in this thesis are believed to be the first of their kind in the context of very low field MRI.

The first part of this investigation was to perform the control experiments. Uniform electrically conducting spherical phantoms were exposed to a uniform magnetic field \(B_1\) oscillating at frequencies in the range \(f = 0.1 - 1.25\) MHz. The field was generated by an MRI transmit coil designed for very low field hyperpolarized \(^3\)He lung imaging. Measurements of the ohmic losses associated with induced eddy currents in the phantom were obtained by observing the loading effect caused when the phantom was introduced into the transmit coil. Five phantoms of varying radii were used in order to examine the size dependence of the losses. Measurements of the spatial variation of magnetic field perturbations associated with induced eddy currents were obtained for one of these phantoms using a small probe coil.
Data from both experiments (ohmic losses and magnetic field perturbations) were fit to appropriate functions derived from an electrodynamic model of the conducting sphere to extract a value for electrical conductivity. These values are shown to be in excellent agreement with tabulated reference values. This agreement allows two conclusions to be drawn. First, because there are absolutely no free parameters in our experiments, we conclude that the experimental method is valid and that the apparatus is well understood. It also suggests that a similar level of confidence may be attributed to subsequent measurements involving human subjects. Second, we infer that the electrodynamic model for a weakly conducting sphere makes successful predictions over the frequency range of interest.

The next set of experiments in the investigation were those involving human subjects. As was the case for the spherical phantoms, subjects were exposed to a uniform oscillating magnetic field, spanning the same frequency range. Two experiments were carried out; in the first, the field was generated by an MRI receive coil with a design similar to those used to capture the current state-of-the-art very low field in vivo images of human lung airspaces [4]. In the second, the field was generated by the same MRI transmit coil as that used for experiments involving the spherical phantoms. Again, measurements of the ohmic losses associated with induced eddy currents in the subjects were obtained by observing the loading effect on the appropriate coil.

The first experiment involving human subjects provides information regarding the SNR in very low field MRI experiments that is essential in order to improve image quality. The measurement allows a direct comparison of the noise generated by a human subject to the intrinsic noise in the coil at low frequencies. These measurements show that the subject is not the limiting source of Johnson noise for the particular receive coil used here. Specifically, the noise contribution from the coils dominates that of the subject by at least a factor of ten in the frequency range that was explored. Therefore our measurements demonstrate that approximately a factor of ten improvement to the SNR may be obtained by pursuing designs for lower resistance receive coils or alternate detection methods. The clear dominance of the coil resistance indicates that the implementation of cryogenic, Litz or superconducting wire, and other coil designs will be crucial to the success of low field imaging.

The second experiment yields a direct measurement indicative of the SAR under realistic imaging conditions. This in turn provides design constraints on RF pulse sequences. We deduce a whole-body average SAR of $1.9(3) \times 10^{-5} f^2 B_1^2$ W/kg (MKS units), using con-
servative estimates. By expressing the magnetic field strength $B_1$ in terms of the duration of an RF pulse that tips spins by $180^\circ$, we show that continuous application of sub-millisecond $\pi$-pulses (which would normally be unsafe under high field imaging conditions) does not exceed recommended whole-body SAR levels at sufficiently low static field strengths. This result leads us to the first of two principal conclusions of this thesis: that innovative and complex new pulse sequences involving many strong and rapid RF pulses may be safely applied for use in very low field HPG imaging.

Suggestions for technical and methodological improvements to the experiments presented in this thesis include the following. A more thorough field mapping experiment could be done; measurements could be taken at larger number of positions, including points out of the $xz$-plane, measurements of the $y$-component of the field, and possibly measurements employing a pair of probe coils, to take advantage of the symmetry of the induced magnetic fields. A calibrated system could be developed to more accurately position the probe coil. Furthermore, the use of network analyzers is common in cavity perturbation experiments, for example those involving microwave resonators. A network analyzer could be used in place of the lock-in to potentially simplify the experiment.

Experiments could also be done using other transmit coils. Based on our observation of electric field effects, it would be worth pursuing a new design of transmit coil, that has interleaved windings to balance electric potentials inside the coil. Also, experiments employing the actively shielded design recently developed for use as a very low field MRI transmit coil [73, 75] could be performed. This coil would also be less sensitive to its environment and require less space to conduct the experiments. It is suspected that combination of interleaved windings and active shielding would be a very robust transmit coil design.

Numerous experiments related to the work presented in this thesis are of scientific interest. First, there are many parameters that could be varied in the study of spherical phantoms. The conductivity and radius of the phantoms could be varied such that the second term of the series expansion of the sample resistance becomes more apparent. Materials having a frequency-dependent conductivity would certainly be interesting to study and would provide a comparison with physiologically relevant systems. Second, other shapes of phantoms might be explored. While mathematically more complicated, approximations do exist for shapes such as a hemisphere [82] and a cylinder [65, 83]. Hemispheres have been examined in the high frequency limit [82] while experiments involving cylinders have
been examined in the low frequency limit [84]; corroboration of these results would be a worthwhile exercise.

A large number of further experiments involving human subjects could be performed. A few examples include the following. Measurements could be collected involving many different subjects having a range of physical characteristics (such as mass and body type) to investigate in more detail the relation between these properties and the observed loading effect on the coil. The oscillations associated with the subject's breathing that were evident could also be studied in more detail. Interesting topics include the study of the possible phase shift at resonance, qualitative correlation of the oscillations in the data with exhalation and inhalation, and quantitative determination of the changes to both the quality factor and resonance frequency associated with breathing motions.

The second principal conclusion of this thesis is that a focused effort should be devoted to the design of specialized receive coils for very low field MRI. It is certainly possible to reduce the radiation resistance of receive coils with an appropriate arrangement of multiple coils. However a greater reduction of the resistance is anticipated for designs utilizing superconducting or Litz wire and/or cryogenically cooled coils. In order to compare the performance of such coils to that of resistive, room temperature coils, experiments must be carefully designed to take into consideration the complicated dependence of the SNR on the shape, size, temperature and conductivity of the coil. Another route to develop a high SNR receiver is to investigate novel new sensors such as those incorporating SQUIDs. Since a properly designed high SNR receiver is essential for the acquisition of high quality images and will have a significant impact on very low field MRI in years to come, these studies should begin immediately.

Finally, the methods employed in this thesis to measure ohmic losses in conducting bodies might be applied to other fields. It has been demonstrated that this method provides a sensitive, absolute measurement of solution conductivities, with an inherent simplicity that arises because of extremely well-defined boundary conditions and geometric factors that are lacking in standard approaches. In addition, the apparatus provides information regarding specific absorption rates that in principle could be applicable to a broad range of other fields. In particular, this method may be useful in evaluating the physiological effects associated with very low frequency electromagnetic radiation.
Bibliography


