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SOME DECOMPOSITION PROBLEMS FOR COMPLETE GRAPHS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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of

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SOME DECOMPOSITION PROBLEMS FOR COMPLETE GRAPHS

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ABSTRACT

Given a graph \( G \) and graphs \( H_1, H_2, \ldots, H_s \), if there exists a partition of the edge set \( E(G) \) such that the resulting subgraphs of \( G \) are isomorphic to \( H_1, H_2, \ldots, \) or \( H_s \) we say that the graph \( G \) can be decomposed into the graphs \( H_1, H_2, \ldots, H_s \). In particular if \( H_1 \cong H_2 \cong H_s \cong H \) (say), the decomposition is called an isomorphic factorization of \( G \). In this case we also say that \( H \) divides \( G \).

Similar definitions hold for directed graphs.

In Chapter 1, decompositions of some complete multipartite graphs and some special graphs into cycles of different lengths are obtained. Most of the graphs considered here often appear as factors in the decomposition of complete graphs and complete symmetric digraphs considered in the subsequent chapters. Thus the results of this chapter are not only used in obtaining results in other chapters, but can be used as important tools in many other decomposition problems.

In Chapter 2 it is shown that for \( n \equiv 5 \pmod{6} \) the complete graph \( K_n \) can be decomposed into \( C_3 \)'s and one \( K_5 \) where \( C_r \) denotes a cycle of length \( r \). This together with the fact that a Steiner triple system is known to exist for \( n \equiv 1 \) or \( 3 \pmod{6} \) establishes the existence of a pairwise balanced design \( (n; 5, 3, 1) \) for any odd \( n \).

In Chapter 3 it is shown that the necessary conditions for the cycle \( C_2 \cdot p^\alpha \), \( p \) prime and \( \alpha \) any positive integer, to divide \( K_n \) are also sufficient.
In Chapter 4 it is shown that necessary and sufficient conditions for each orientation of $C_5$ to divide a complete symmetric digraph $DK_n$ are $n \equiv 0$ or $1 \pmod{5}$ and $n \geq 5$.

In Chapter 5 sufficient conditions are found for a self-converse orientation of $C_k$, $k$ odd, to divide a complete symmetric digraph $DK_n$. This result is used to show that the necessary conditions for any self-converse orientation of $C_7$ to divide $DK_n$ are also sufficient.
To Kamal, my wife, who while driving
the car once discovered that $8 + 6 = 14$. 
ACKNOWLEDGEMENTS

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INTRODUCTION

It all began in 1736 when Leonhard Euler initiated the theory of graphs in his famous paper [12], though people had been using combinatorial ideas to solve entertaining puzzles earlier. Thus Oystein Ore has rightly remarked in his book *Graphs and their uses* [27] that the theory of graphs is one of the few fields of mathematics which has a definite birth date. Euler himself started his paper with the discussion of a puzzle, known as Königsberg problem (see [27, page 23]). In the initial stage, since it dealt largely with recreational problems, graph theory could not draw much attention from the scholars in mathematics until 1847 when Kirchhoff [24] counted the number of spanning trees of a labeled graph and applied it to electrical networks. After almost four decades, there came another startling result in graph theory when Cayley [11] gave a count for the labeled trees with \( n \) vertices. He applied this result to the problem of counting chemical isomers. In spite of all these early promising applications, graph theory could not sever its relationship with entertaining puzzles. In 1890, Heawood [18] in an attempt to solve the famous Four color problem* proved the five color theorem, which is regarded as another landmark in the history of graph theory.

*The Four color problem after its long history was finally solved by Appel, Haken and Koch [1] in the summer of 1976.
For many years to come, because of its study of topological properties such as connectedness, planarity and so on, graph theory was regarded as a branch of topology. Thus, when J.H.C. Whitehead once described graph theory as 'the slums of topology', he was expressing the general consensus which prevailed among the mathematicians at that time. Dénes König's book *Theorie der endlichen und unendlichen Graphen* published in 1936 was the first book ever written on graph theory. Though it was an attempt to establish graph theory as a unitary discipline, it failed to satisfy a section of mathematicians who subscribed to the above stated view. We are indeed indebted to Turán who discovered a pioneering theorem in 1941. It not only opened a new avenue in the study of graphs known as extremal graph theory but settled the issue that graph theory has its own entity and can not be looked upon merely as a branch of topology. Turán [36] found the maximum possible number of edges for all graphs on \( n \) vertices which do not contain a triangle. He also showed that there is a unique graph which realizes this number.

Graph theory today is like an old banyan tree with its many branches extending friendly help to the other branches of mathematics, sciences and humanities. To cite a few examples: Menger's separation theorem of 1927 has led to the development of network theory formulated by Ford and Fulkerson in their work *Flows in network* (1962). Graph theory is used in the study of certain engineering and structural systems which involve interrelated components. It has found applications in many areas like assignment...
problems, timetabling problems, storage problems and so on. The study of directed graphs has successfully been applied to the problem of making a road system one-way, ranking participants in a tournament or that of designing an efficient computer drum. Graph Theory has also found applications in economics, psychology and biology.

The problem considered in this dissertation come under the branch called combinatorial designs which has been growing at a fast rate. These designs are used as powerful tools in statistical experiments. Starting with Latin squares, Hadamard matrices, Steiner triple systems and more recently coding theory, it has generated much enthusiasm. Some of these designs such as balanced incomplete block designs, pairwise block designs, balanced cycle or circuit designs or more generally the G-designs are defined in the subsequent chapters. The third chapter deals with the balanced cycle designs which are useful in serology, a science of virus research. To conduct the Ouchterlony gel diffusion test, samples from a number of antigens (virus preparations) are to be arranged around an antiserum on a plate so that every antigen has two others as its neighbours. There are \( n \) kinds of antigens to be arranged on \( b \) plates each containing \( k \) antigens. (see Rees [28] and Hwang [22]).
DEFINITIONS AND NOTATION

We shall use here the terminology commonly used in any standard text on graph theory, for example, by Bondy and Murty [8], by Behzad and Chartrand [2] or by Harary [15]. However, we give here a few definitions which are not in common usage.

By a graph here we mean an ordinary graph, that is, a finite graph with no multiple edges or loops.

We shall use the same notation as used in the above books except for a complete n-partite graph $K_{x,x,...,x}$, which is a graph whose vertex set may be partitioned into sets $X_1,X_2,...,X_n$ where $|X_i| = x$ for each $i = 1,2,...,n$ and two vertices $u$ and $v$ are adjacent if and only if they belong to distinct sets $X_i$ and $X_j$ of the partition. We shall denote this graph by $K_n^x$.

We shall use the notation $K_{|A|,|B|}$ for the complete bipartite graph with maximal independent sets $A$ and $B$.

DEF.: The union $G_1 \cup G_2$ of two graphs $G_1$ and $G_2$ is a graph with

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \cup G_2) = E(G_1) \cup E(G_2).$$

DEF.: The join $G_1 + G_2$ of two graphs $G_1$ and $G_2$ with disjoint vertex sets is their union $G_1 \cup G_2$ together with all edges joining the vertices of $G_1$ and $G_2$. 
(Note: Bondy and Murty [8] use the notation $G_1 + G_2$ for the disjoint union of the graphs $G_1$ and $G_2$. It is easy to see that

$$K_{m,n} = K_m \cup K_n$$

and

$$K_m + K_n = K_{m+n} = K_m \cup K_n \cup K_{m,n},$$

where $\overline{K}_m$ represents the complement of the complete graph $K_m$, that is, a set of $m$ isolated vertices.

DEF.: For a connected graph $G$, $nG$ is a graph with $n$ components, each isomorphic to $G$.

Note that $nG$ itself is not connected if $n > 1$.

Example:

![Diagram](image)

DEF.: The composition $G = G_1[G_2]$ of two graphs $G_1$ and $G_2$ is a graph with $V(G) = V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$
and \( v = (v_1, v_2) \) of \( G \) are adjacent if and only if \( u_1 \) is adjacent to \( v_1 \) in \( G_1 \) or \( u_1 = v_1 \) and \( u_2 \) is adjacent to \( v_2 \) in \( G_2 \).

Example:

\[ G_1: \]

\[ G_2: \]

\[ G_1[G_2]: \]

DEF.: For any graph \( G \), \( DG \) is a directed graph with \( V(G) = V(DG) \) and for each edge \( uv \) of \( G \) there are arcs \( uv \) and \( vu \) in \( DG \).

Example:

\[ G: \]

\[ DG: \]

The digraph \( DK_n \) in particular is called a complete symmetric digraph.
DEF.: The converse $D'$ of a digraph $D$ is a digraph with $V(D') = V(D)$ and the arc $uv$ is in $D'$ if and only if the arc $vu$ is in $D$.

Example:

\[ \text{D:} \quad \begin{array}{c}
\text{D':} \\
\end{array} \]

DEF.: A digraph is said to be self-converse if and only if it is isomorphic to its converse digraph.

Example: The digraph $D$ of the above example is not self-converse, while the following digraphs are self-converse.

\[ \text{D:} \quad \begin{array}{c}
\text{D':} \\
\end{array} \]

DEF.: A decomposition of a graph $G$ into graphs $H_1, H_2, \ldots, H_s$ is a partition of $E(G)$ so that the resulting subgraphs of $G$ are isomorphic to $H_1, H_2, \ldots$, or $H_s$ and we say that $G$ can be decomposed (or $G$ is decomposable) into the graphs $H_1, H_2, \ldots, H_s$. 
In particular, if \( H_1 \cong H_2 \cong \ldots \cong H_s \cong H \), the decomposition is called an *isomorphic factorization* of \( G \) and we write \( H|G \) and read it as \( H \) divides \( G \).

Isomorphic factorizations of digraphs are defined analogously.

In the proof of Lemma 2.3 we use the following concept.

**DEF.** A graph \( G \) with \( V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \) is called a *circulant* if for \( 0 \leq i, j \leq n-1 \), \( i \neq j \), \( v_i v_j \) is an edge if and only if \( v_{i+k} v_{j+k} \), \( 1 \leq k \leq n-1 \), is an edge of \( G \). The set \( S = \{i : v_{0 \cdot i} \in E(G)\} \) is called the *symbol* of \( G \).

A circulant graph is described completely by its symbol. The vertices in the neighbourhood of one vertex determine the neighbourhoods of all the other vertices.
In this chapter we consider the decomposition of some complete multipartite graphs and some other graphs into cycles of different lengths. The graphs considered here generally appear as factors in the decomposition of complete graphs and complete symmetric graphs. These results thus can be used as important tools in other decomposition problems. The main results of the succeeding chapters will illustrate this fact.

1.1 PROPOSITION: $C_3 \mid 3K_x$ for any positive integer $x$.

Proof: Let $\{u_1, u_2, \ldots, u_x\}$, $\{v_1, v_2, \ldots, v_x\}$ and $\{w_1, w_2, \ldots, w_x\}$ be the three maximal independent subsets of $V(3K_x)$. Consider the triangles

$$u_i, v_j, w_k, u_i \quad \ldots \quad (1.1.1)$$

where $1 \leq i, j, k \leq x$ and $i + j + k \equiv 0 \pmod{x}$. Since for any pair of positive integers $i, j$ such that $1 \leq i, j \leq x$, the relation $i + j + k \equiv 0 \pmod{x}$ determines a unique positive integer $k, 1 \leq k \leq x$, the set of triangles in (1.1.1) gives a desired decomposition of $3K_x$.

Prop. 1.1 also follows from a result of Bermond [3].
1.2 PROPOSITION: \( C_3 \mid 5K_{3x} \) for any positive integer \( x \).

Proof: First let \( x = 1 \). Also let \( \{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{d_1, d_2, d_3\} \) and \( \{e_1, e_2, e_3\} \) be the five maximal independent subsets of \( V(5K_3) \). Following is a decomposition of \( 5K_3 \) into disjoint triangles. With subscripts reduced modulo 3,

\[
\begin{align*}
& a_{1}, b_{1}, d_{1+1}, a_{1} \\
\end{align*}
\]

for \( i = 1, 2, 3 \), give six triangles. The rest are obtained by rotating \( a, b, c, d, e \) in the cyclic order \( a \) to \( b \), \( b \) to \( c \), \( c \) to \( d \), \( d \) to \( e \) and \( e \) to \( a \).

Now for any \( 3x \) in general, let \( A_1, A_2, A_3, A_4 \) and \( A_5 \) be the maximal independent subsets of \( V(5K_{3x}) \), each of cardinality \( 3x \). For \( i = 1, 2, 3, 4, 5 \), let

\[
A_i = A_{1i} \cup A_{12} \cup A_{13},
\]

where \( |A_{1i}| = |A_{12}| = |A_{13}| = x \). With \( 5K_{3x} \) we associate a \( 5K_3 \) so that

\[
V(5K_3) = \{A_{1i} : 1 \leq i \leq 5, 1 \leq j \leq 3\},
\]
and an edge incident with $A_{\ell m}$ and $A_{rs}$ corresponds to the set of all edges of the complete bipartite graph $K|A_{\ell m}|,|A_{rs}|$ which is a subgraph of $5K_{3x}$. By the first part $5K_3$ is decomposable into triangles, which amounts to the fact that the graph $5K_{3x}$ is decomposable into $3K_x$'s and the result then follows from Prop. 1.1.0.

1.3 PROPOSITION: For $n \equiv 0$ or $2 \pmod{6}$, $n \geq 6$, $C_3|(K_n - I)$ where $I$ is a 1-factor of $K_n$.

Proof: Adjoin a vertex $u$ to $K_n$. The graph $K_n + u$ is a complete graph $K_{n+1}$ with $n + 1 \equiv 1$ or $3 \pmod{6}$ and hence can be decomposed into triangles by the existence of Steiner triple system [14, Thm. 15.4.3]. The graph $K_n = K_{n+1} - u$ can be decomposed into one 1-factor and the rest triangles. □

1.4 PROPOSITION: $C_3 \mid nK_{2x}$ where $n \equiv 0$ or $4 \pmod{6}$ and $x$ is any positive integer.

Proof: With $nK_{2x}$ we associate a graph $G$ as follows. Let

$$V_i = \{u_i^1, u_i^2, \ldots, u_i^{2x}\}, 1 \leq i \leq n$$

be $n$ maximal independent subsets of $V(G)$. For $i = 1, 2, \ldots, n$,
let

\[ S_i = \{u_1^i, u_2^i, \ldots, u_x^i\}, \]

\[ S_{n+i} = \{u_{x+1}^i, \ldots, u_{2x}^i\}. \]

Then we define

\[ V(G) = \{S_1, S_2, \ldots, S_{2n}\}. \]

and an edge between \( S_i \) and \( S_j \) corresponds to the set of all edges of the complete bipartite subgraph \( K_{|S_i|, |S_j|} \) of \( nK_{2x} \).

We observe that any two vertices \( S_i, S_j \) are adjacent except when \( j = n + i \) \((1 \leq i \leq n)\). Moreover, the edges \( S_i S_{n+i} \) \((1 \leq i \leq n)\) form a 1-factor of \( G \). Thus the graph \( G \) is \( K_{2n} - I \) where \( I \) denotes a 1-factor of \( K_{2n} \). Since \( 2n \equiv 0 \) or \( 2 \) (mod 6), by Prop. 1.3, \( K_{2n} - I \) can be decomposed into \( C_3 \)'s, which amounts to the fact that \( K_{n2x} \) can be decomposed into \( 3x \)'s and the result then follows by Prop. 1.1.0

1.5 PROPOSITION: \( C_5 | C_5[\overline{K_x}] \) for any positive integer \( x \).

Proof: Let \( V(C_5[\overline{K_x}]) = \bigcup_{i=1}^{5} A_i \) where \( |A_i| = x \) for \( i = 1, 2, 3, 4, 5 \) and the vertices \( u \) and \( v \) are adjacent if and only
if \( u \in A_i \) and \( v \in A_{i+1} \) for \( i = 1, 2, 3, 4, 5 \) and subscripts are to be taken modulo 5. For \( i = 1, 2, 3, 4, 5 \) let

\[
A_i = \{a_{i1}, a_{i2}, \ldots, a_{ix}\}
\]

Consider the 5-cycles

\[
a_{i1}, a_{2j}, a_{3i}, a_{4j}, a_{5k}, a_{li}
\]

where \( i = 1, 2, \ldots, x; \ j = 1, 2, \ldots, x \) and \( i + j + k \equiv 0 \pmod{x} \).

Once \( i \) and \( j \) are fixed, the integer \( k \) is uniquely determined by the relation \( i + j + k \equiv 0 \pmod{x} \). This implies that the \( C_5 \)'s in 1.5.1 are all disjoint, and that they cover all the edges of \( C_5[\overline{K_x}] \). Thus 1.5.1 gives a decomposition of \( C_5[\overline{K_x}] \) into \( C_5 \)'s.

1.6 PROPOSITION: \( C_5 \mid 5K_x \) for any positive integer \( x \).

Proof: Let \( V(5K_x) = \bigcup_{i=1}^{5} A_i \) where for \( i = 1, 2, 3, 4, 5, \ A_i \) is a maximal independent subset of vertices and \( |A_i| = x \). With \( 5K_x \) we associate a complete graph \( K_5 \) so that

\[
V(K_5) = \{A_1, A_2, A_3, A_4, A_5\}
\]
and an edge $A_iA_j$, $1 \leq i, j \leq 5$, corresponds to the set of all edges of the complete bipartite subgraph $K_{|A_i|,|A_j|}$ of $K_x$.

We know that $K_5$ can be decomposed into two cycles of length five, which amounts to the fact that the graph $K_x$ can be decomposed into two isomorphic factors $C_5[K_x]$ which can be further decomposed into 5-cycles by Prop. 1.5.0

1.7 PROPOSITION: $C_5|_{3K_5^x}$ for any positive integer $x$.

Proof: First let $x = 1$. Also let \{u_1, u_2, u_3, u_4, u_5\}, \{v_1, v_2, v_3, v_4, v_5\} and \{w_1, w_2, w_3, w_4, w_5\} be the three maximal independent subsets of $V(K_5^x)$. The following gives one decomposition of $3K_5$ into $C_5$'s. The cycles

$$u_1, v_1, w_1, v_2, u_2, w_2, v_3, w_3, u_3, v_3, u_1, w_1, v_4, w_4, u_4, v_4, w_5, u_5, v_5, w_5$$

are five $C_5$'s of the decomposition. The rest then are obtained by rotating $u_i, v_i, w_i$ in the cyclic order $u_i$ to $v_i$, $v_i$ to $w_i$ and $w_i$ to $u_i$.

Now for any $5x$ in general, let $A_1, A_2$ and $A_3$ be the maximal independent subsets of $V(3K_5^x)$, each of cardinality $5x$. 
For $i = 1, 2, 3$, let

$$A_i = A_{i1} \cup A_{i2} \cup A_{i3} \cup A_{i4} \cup A_{i5},$$

where $|A_{i1}| = |A_{i2}| = |A_{i3}| = |A_{i4}| = |A_{i5}| = x$. With $3K_5x$ we associate a $3K_5$ so that

$$V(3K_5) = \{A_{ij} : 1 \leq i \leq 3, 1 \leq j \leq 5\},$$

and an edge incident with $A_{\ell m}$ and $A_{rs}$, $1 \leq \ell, r \leq 3$, $1 \leq m, s \leq 5$, corresponds to the set of all edges of the complete bipartite subgraph $K|A_{\ell m}|, A_{rs}|$ of $3K_5x$. By the first part, $3K_5$ can be decomposed into $C_3$'s, which amounts to the fact that the graph $3K_5x$ can be decomposed into $C_5[K_5]$ and the result then follows from Prop. 1.5.0.

1.8 PROPOSITION: For any odd $n \geq 3$, $K_n$ can be decomposed into 3-cycles and 5-cycles.

Proof: If $n \equiv 1$ or $3$ (mod 6), we know that there exists a Steiner triple system on $n$ objects, which is equivalent to the fact that the complete graph $K_n$ can be decomposed into disjoint triangles. In case $n \equiv 5$ (mod 6), Spencer [35] has shown that $K_n$ can be decomposed into triangles and one $C_4$. Let $a, b, c, d, a'$ be the 4-cycle of one such decomposition. Let $u$
and \( v \) (need not be distinct) be two vertices of \( K_n \) such that \( u, b, d, u \) and \( v, a, c, v \) are two triangles of the decomposition.

![Diagram](image.png)

**FIGURE 1(A)**

**FIGURE 1(B)**

If the two vertices \( u \) and \( v \) are the same (see Figure 1(A)), we get a decomposition of \( K_n \) into triangles and one \( K_5 \) whose vertex set is \( \{a, b, c, d, u\} \). But \( K_5 \) can be decomposed into two \( C_5 \)'s, giving us the desired decomposition. In case the two vertices \( u \) and \( v \) are distinct (see Figure 1(B)) we get a decomposition of \( K_n \) into triangles and the two 5-cycles

\[ u, b, a, c, d, u \quad \text{and} \quad v, a, d, b, c, v. \]

1.9 Proposition: \( C_5 \mid K_5 \) for any odd positive integer \( n \).

**Proof:** With \( K_n \) we associate a complete graph \( K_n \) as follows:

Let \( V_1, V_2, \ldots, V_n \) be the maximal independent subsets of \( V(K_5) \).
Then $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and an edge $V_iV_j$, $1 \leq i, j \leq n$, corresponds to the set of all edges of the complete bipartite subgraph $K_{|V_i|,|V_j|}$ of $K_n$. By Prop. 1.8, $K_n$ can be decomposed into 3-cycles and 5-cycles, which amounts to the fact that the graph $K_n$ can be decomposed into graphs isomorphic to either $3K_5$ or $C_5[K_5]$ and the result then follows from Prop. 1.1 and Prop. 1.5.\p

1.10 PROPOSITION (Bermond and Faber [5]): For any odd positive integer $n \geq 5$, the complete symmetric digraph $DK_n$ can be decomposed into directed cycles of length $n-1$.

Proof: Let $V(DK_n) = \{0, 1, 2, \ldots, n-1\}$. Also let $\phi$ be a permutation on the vertices of $DK_n$ with cycle representation

$$\phi = (0 \ 1 \ 2 \ \cdots \ n-1).$$

Case 1. First let $n \equiv 1 \pmod{4}$, that is, $n = 4m + 1$ for some positive integer $m$. Consider the directed cycle $C$ of length $4m$ (see Figure 1(C)) as given below.

$C: 0, 1, 4m, 2, 4m-1, 3, \ldots, m, 3m+1, m+2, 3m, m+3, \ldots, 2m, 2m+2, 2m+1, 0.$
We say that a vertex $j$ is at distance $d$ from a vertex $i$, $0 \leq i, j \leq 4m$, if and only if $j - i \equiv d \pmod{4m+1}$. We observe that the vertices in the neighbourhood of a vertex $i$, $0 \leq i \leq 4m$, are at distances $1, 2, 3, \ldots, 4m$ from $i$. Moreover, if we consider the distances between the successive vertices of the directed cycle $C$, each of the distances $1, 2, \ldots, 4m$ occur exactly once. Hence, the directed cycles $\phi^k C$, $k = 0, 1, 2, \ldots, 4m$ are all disjoint and cover all the arcs of $DK_{4m+1}$. This gives a desired decomposition of $DK_{4m+1}$.
Case 2: If \( n \equiv 3 \pmod{4} \), let \( n = 4m+3 \) for some positive integer \( m \). Consider the directed cycle \( C' \) of length \( 4m+2 \) (see Figure 1(D))

\[
C' : 0, 1, 4m+2, 2, 4m+1, \ldots, 3m+3, m+1, 3m+1, m+2, 3m, \ldots, 2m, 2m+2, 2m+1, 0.
\]

As in the case of the directed cycle \( C \), here also the distances 1, 2, \ldots, 4m+2 occur exactly once among the distances between the successive vertices of the directed cycle \( C' \). Hence the directed cycles \( \phi^k C' \), \( k = 0, 1, \ldots, 4m+2 \) are all disjoint and cover all the arcs of \( DK_{4m+3} \). Moreover, by an argument similar to the one we used in the case of \( C \), the rotation under the permutation \( \phi \)
preserves the orientation. Thus the directed cycles \( \phi^{k} C' \), 
\( k = 0, 1, 2, ..., 4m+2 \), give a decomposition of \( D_{K_{4m+3}} \).

1.11 PROPOSITION: \( C_{n} \parallel K_{n} \) for any odd positive integer \( n \geq 3 \).

Proof: Let the five maximal independent subsets of \( V(K_{n}) \) be

\[ A = \{a_0, a_1, ..., a_{n-1}\}, \quad B = \{b_0, b_1, ..., b_{n-1}\}, \]
\[ C = \{c_0, c_1, ..., c_{n-1}\}, \quad D = \{d_0, d_1, ..., d_{n-1}\}, \]
\[ E = \{e_0, e_1, ..., e_{n-1}\}. \]

On the \( 5n \) symbols in the union of the sets \( A, B, C, D \) and \( E \), we define a permutation \( \sigma \) whose cycle representation is the product of \( n \) 5-cycles as given below.

\[ \sigma = (a_0 \ b_0 \ c_0 \ d_0 \ e_0)(a_1 \ b_1 \ c_1 \ d_1 \ e_1) ... (a_{n-1} \ b_{n-1} \ c_{n-1} \ d_{n-1} \ e_{n-1}) \]

Also, let \( \tau \) be a permutation on the \( n \) numbers \( 0, 1, 2, ..., n-1 \) whose cycle representation is

\[ \tau = (0 \ 1 \ 2 \ ... \ n-1). \]
We shall show that the $10n$ $n$-cycles of the decomposition are given by

$$\sigma^{i}z \quad \text{and} \quad \sigma^{i}z,' \quad \ldots(1.11.1)$$

for $0 \leq i \leq 4$ and $0 \leq j \leq n-1$, where $Z$ and $Z'$ are two $n$-cycles constructed with the help of the directed cycles $C$ and $C'$ of length $n-1$ of Prop. 1.10, in the following manner. We recall that in the case $n = 4m+1$, $m$ a positive integer, $C$ is the cycle

$$0, 1, 4m, 2, 4m-1, 3, \ldots, m, 3m+1, m+2, 3m, m+3, \ldots, 2m, 2m+2,$$

$$2m+1, 0$$

and in the case $n = 4m+3$, $m$ a positive integer, $C'$ is the cycle

$$0, 1, 4m+2, 2, 4m+3, \ldots, 3m+3, m+1, 3m+1, m+2, 3m, \ldots, 2m,$$

$$2m+2, 2m+1, 0.$$

If we regard the sets $A, B, C, D$ and $E$ as the vertices of a complete graph $K_5$, this complete graph can be decomposed into two disjoint 5-cycles

$$H_1 : A, B, C, D, E, A$$
and \( H_2 : A, C, E, B, D, A. \)

Let \( \alpha \) and \( \beta \) be mappings of the set \( \{0,1,\ldots,n-2\} \) into the set \( \{0,1,\ldots,n-1\} \) defined as follows. For any \( 0 \leq i \leq n-2 \)

\[ \alpha(i) = \text{label of the } i^{\text{th}} \text{ vertex of } C \text{ and} \]

\[ \beta(i) = \text{label of the } i^{\text{th}} \text{ vertex of } C'. \]

Let us first consider \( n \equiv 1 \pmod{10} \). In each case given below the two \( n \)-cycles \( Z \) and \( Z' \) are constructed as follows. In the case of the cycle \( Z \) we first construct a path of length \( n \) by starting from the vertex \( a \) in \( A \) and moving through the sets \( A, B, C, D \) and \( E \) according to the 5-cycle \( H_1 \) while changing the subscripts according to the cycle \( C \) or \( C' \). The cycle \( Z \) is then obtained by joining the end-vertices of this path. The cycle \( Z' \) is also constructed in a similar manner with the difference that this time we use the 5-cycle \( H_2 \) instead of \( H_1 \).

**Case 1:** If \( n \equiv 3 \pmod{20} \), the cycle \( Z \) is

\[ a_0 = a_\beta(0), b_\beta(1), c_\beta(2), d_\beta(3), e_\beta(4), a_\beta(5), \ldots, a_\beta(n-2), b_\beta(n-1), \]

\[ c_\beta(n) = c_0, a_0 \]

and the cycle \( Z' \) is

\[ a_0 = a_\beta(0), c_\beta(1), e_\beta(2), b_\beta(3), d_\beta(4), a_\beta(5), \ldots, a_\beta(n-2), c_\beta(n-1), \]

\[ e_\beta(n) = e_0, a_0 \]
Case 2: If \( n \equiv 5 \pmod{20} \), the cycle \( Z \) is

\[
a_0 = a(0), b(1), c(2), d(3), e(4), a(5), \ldots, e_0 = e_0, a_0
\]

and the cycle \( Z' \) is

\[
a_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, d_0 = d_0, a_0.
\]

Case 3: If \( n \equiv 7 \pmod{20} \), the cycle \( Z \) is

\[
a_0 = a(0), b(1), c(2), d(3), e(4), a(5), \ldots, a_0, a_0
\]

and the cycle \( Z' \) is

\[
a_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, a_0, a_0.
\]

Case 4: If \( n \equiv 9 \pmod{20} \), the cycle \( Z \) is

\[
a_0 = a(0), b(1), c(2), d(3), e(4), a(5), \ldots, a_0, a_0
\]

and the cycle \( Z' \) is

\[
a_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, a_0, a_0.
\]

Case 5: If \( n \equiv 13 \pmod{20} \), the cycle \( Z \) is

\[
a_0 = a(0), b(1), c(2), d(3), e(4), a(5), \ldots, a_0, a_0
\]

and the cycle \( Z' \) is

\[
a_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, a_0, a_0.
\]
and the cycle $Z'$ is
\[ a_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, a(n-2), c(n-1), e(n) = e_0, a_0. \]

Case 6: If $n \equiv 15 \pmod{20}$, the cycle $Z$ is
\[
\begin{align*}
  a_0 &= b(0), b(1), c(2), d(3), e(4), a(5), \ldots, e(n), e_0, a_0, \\
  &\quad \\n  a_0 &= b(0), c(1), e(2), b(3), d(4), a(5), \ldots, d(n) = d_0, a_0.
\end{align*}
\]

Case 7: If $n \equiv 17 \pmod{20}$, the cycle $Z$ is
\[
\begin{align*}
  a_0 &= a(0), b(1), c(2), d(3), e(4), \ldots, a(n), a_0, \\
  &\quad \\n  a_0 &= a(0), c(1), e(2), b(3), d(4), a(5), \ldots, a(n-1), b(n) = b_0, a_0.
\end{align*}
\]

Case 8: If $n \equiv 19 \pmod{20}$, the cycle $Z$ is
\[
\begin{align*}
  a_0 &= b(0), b(1), c(2), d(3), e(4), a(5), \ldots, a(n-3), b(n-2), \\
  &\quad \\n  a_0 &= b(0), c(1), e(2), b(3), d(4), a(5), \ldots, b(n-1), b(n) = b_0, a_0.
\end{align*}
\]

In the case $n \equiv 1 \pmod{10}$, if we use the same construction, the path of length $n$ ends with the last vertex being $a_0$, and thus gives a cycle of length $n-1$. To avoid this we make a different jump.
at the next to last vertex of the path. Thus in the case of the cycle 
Z, we first construct a path of length \( n-1 \) moving along the 5-cycle 
\( H_1 \) as follows

\[
\alpha_0 = a(0), b(1), c(2), d(3), e(4), \ldots, e_{n-2} \]

if \( n \equiv 1 \pmod{20} \)

or

\[
\alpha_0 = b(0), b(1), c(2), d(3), e(4), a(5), \ldots, e_{n-2} \]

if \( n \equiv 11 \pmod{20} \)

and then complete the cycle by joining \( e_{n-2} \) or \( e_{n-2} \) to \( 0 \)

and \( 0 \) to \( a_0 \) according as \( n \equiv 1 \) or \( 11 \pmod{20} \), respectively.

The cycle \( Z' \) is constructed in the similar manner by first constructing

a path of length \( n-1 \) moving along the 5-cycle \( H_2 \) as

\[
\alpha_0 = a(0), c(1), e(2), b(3), d(4), a(5), \ldots, d_{n-2} \]

if \( n \equiv 1 \pmod{20} \)

or

\[
\alpha_0 = b(0), b(1), e(2), b(3), d(4), c(5), \ldots, d_{n-2} \]

if \( n \equiv 11 \pmod{20} \)

and then completing the cycle by joining \( d_{n-2} \) or \( d_{n-2} \) to \( 0 \)

and \( 0 \) to \( a_0 \) according as \( n \equiv 1 \) or \( 11 \pmod{20} \), respectively.

Noting that the permutation \( \tau \) acts on the vertex subscripts
and the permutation \( \sigma \) on the vertices it follows then that

\[
\sigma^{-1} \tau^j z \quad \text{and} \quad \sigma^{-1} \tau^j z'
\]

for \( 0 \leq i \leq 4 \) and \( 0 \leq j \leq n-1 \), are all edge-disjoint, have length \( n \)

and thus cover all the edges of \( K_n \). This completes the proof.\( \Box \)
In Prop. 1.8 of the previous chapter we have shown that for any odd positive integer \( n \geq 3 \), a complete graph \( K_n \) can be decomposed into cycles of length 3 and 5. For \( n \equiv 1 \) or 3 (mod 6) the result follows immediately from the existence of Steiner triple systems. For the case \( n \equiv 5 \) (mod 6), in the proof, using Spencer's [35] result, we show that \( K_n \) can be decomposed into either one \( K_5 \) and the rest triangles or two disjoint \( C_5 \)'s and the rest triangles. Since \( K_5 \) itself is a union of two disjoint \( C_5 \)'s, it is natural to ask: For \( n \equiv 5 \) (mod 6), is it possible to decompose \( K_n \) into exactly one \( K_5 \) and the rest triangles? In this chapter we answer this question in the affirmative. Moreover, such a decomposition is a pairwise balanced design, which gives another direction to look at this problem.

Pairwise balanced designs were introduced by R.C. Bose and S.S. Shrikhande [9] and applied by them [9,10] to construct sets of pairwise orthogonal Latin squares and thus prove the falsity of an old conjecture of Euler [13] on Latin squares.

**DEF.**: A pairwise balanced design \((v; k_1, k_2, \ldots, k_m; \lambda)\) of index \( \lambda \) is an arrangement of \( v \) elements into subsets (called blocks), such that each block contains either \( k_1 \) or \( k_2 \) \ldots or \( k_m \) distinct elements \((k_i \leq v)\), and such that every pair of distinct elements occurs in precisely \( \lambda \) blocks of the design.
Unless the size of the blocks is important, we shall denote a pairwise balanced design of index \( \lambda \), by \( \lambda \)-PBD or simply by PBD. In graph theory the existence of a pairwise balanced design \((v;k_1,k_2,\ldots,k_m;1)\) of index unity is equivalent to partitioning the edge set of \( K_v \) so that the resulting subgraphs are isomorphic to a complete graph \( K_{k_1}, K_{k_2}, \ldots, K_{k_m} \). In geometry 1-PBD's have also been known as linear spaces (in which case the blocks are called lines). A \( \lambda \)-PBD on \( v \) elements in which all blocks have the same size \( k \) is traditionally known as a BIBD(v,k,\lambda) \((balanced \ incomplete \ block \ design; \ see \ [14])\). Thus, pairwise balanced designs of index \( \lambda \) are generalizations of BIBD(v,k,\lambda)-designs.

R.M. Wilson has considered the problem of existence of balanced incomplete block designs and pairwise balanced designs. He has shown \([37,38,40]\) that the necessary conditions for the existence of BIBD's or that of PBD's on \( v \) objects are sufficient for sufficiently large values of \( v \). He calls such conditions 'necessary and "asymptotically sufficient" conditions'. The problem of constructing balanced incomplete block designs or pairwise balanced designs has generated a great deal of interest. Rosa, Hell, Huang, Bermond, Mendelsohn and many others have considered the problem of constructing these designs for different values of the parameters \( v, k \) and \( \lambda \).

Spencer's result \([35]\) that for \( n \equiv 5 \pmod{6} \), the edge set of \( K_n \) can be partitioned into triangles and one \( C_4 \), the cycle of length 4, corresponds to the existence of a pairwise balanced design.
if each edge of \( C_4 \) is taken as a block of size two. However, if trivial blocks of size two are not allowed, the decomposition is not a pairwise balanced design and simple arithmetic shows that the best one can obtain is one block of size 5 and the rest triangles. Thus the main result of this chapter is equivalent to showing the existence of a PBD\( (n;3,5;1) \) for any odd positive integer \( n \). We have learned that the existence of this design also follows as a special case from a result of C. Huang, E. Mendelsohn and A. Rosa which shall appear in a paper under preparation. However, the result proved here shows something more than the existence of a PBD\( (n;3,5;1) \). It shows that there exists a PBD\( (n;3,5;1) \) in which the blocks of size 5 are at most one in number. This result as such also follows from a result due to R. Wilson [39]. His proof is based on group theoretic concepts. Here we give an independent proof which is consistent with the unified proof techniques used in obtaining other results of this dissertation.

2.1 LEMMA: If \( K_{m/m} \), \( m \) odd can be decomposed into one \( K_5 \) and \( C_3 \)'s, then so can \( tK_{m+1} + K_m \) where \( t \) is any non-negative integer.

Proof. We write

\[
tK_{m+1} + K_m = tK_{m+1} \cup K_m \cup tK_{m,m+1}
\]

where the vertex set of each of the complete bipartite graphs \( K_{m,m+1} \) is chosen appropriately. Since \( K_m \) can be decomposed into one \( K_5 \) and \( C_3 \)'s, it is enough to show that the graph
can be decomposed into $C_3$'s. Moreover, since this graph is a union of $t$ copies of the graph

$$K_{m+1} \cup K_{m,m+1}$$

it is sufficient to show that $K_{m+1} \cup K_{m,m+1}$ can be decomposed into $C_3$'s.

Since $m+1$ is even, we know that $K_{m+1}$ can be decomposed into $m$ 1-factors [15, Thm. 9.1]. Take one such 1-factor of $K_{m+1}$ and one vertex, say $u$ of $\overline{K}_m$. The edges of this 1-factor of $K_{m+1}$ together with the edges of $K_{m,m+1}$ which are incident with the vertices of $K_{m+1}$ and the vertex $u$ of $\overline{K}_m$ give triangles. This is shown in the Figure 2(A).
Since the number of 1-factors of \( K_{m+1} \) equals \( m \), we can associate each 1-factor with a distinct vertex of \( K_m \). From the above it then follows that \( C_3 | K_m \cup K_{m+1,m} \) and hence the result. □

2.2 THEOREM: If \( n \equiv 0 \) or \( 4 \) (mod 6), \( K_{6n+5} \) can be decomposed into one \( K_5 \) and \( C_3 \)'s.

Proof: We write

\[
K_{6n+5} = (nK_6 + K_5) \cup \underset{n}{K_6}.
\]

where the vertex set of the complete multipartite graph \( K_6 \) is chosen appropriately. By Lemma 2.1, \( nK_6 + K_5 \) can be decomposed into one \( K_5 \) and \( C_3 \)'s. Also, since \( n \equiv 0 \) or \( 4 \) (mod 6), \( nK_6 \) can be decomposed into \( C_3 \)'s by Prop. 1.4. □

To prove the result for \( K_{6n+5} \), \( n \equiv 2 \) (mod 6) we need the following lemmas.

2.3 LEMMA: \( K_{17} \) can be decomposed into one \( K_5 \) and \( C_3 \)'s.

Proof: We write \( K_{17} = K_5 + K_{12} \). We know \( K_{12} \) can be decomposed into eleven 1-factors. Moreover, \( K_{12} \) can be regarded as a circulant graph with symbol \( \{ \pm1, \pm2, \pm3, \pm4, \pm5, \pm6 \} \). The circulant subgraph with symbol \( \{ \pm1, \pm2, 6 \} \) is the union of five 1-factors of \( K_{12} \), which together with the edges joining the vertices of \( K_{12} \) and the vertices
of $K_5$ give $C_3$'s. The union of the remaining six 1-factors of $K_{12}$ is a circulant subgraph with symbol $\{±3,±4,±5\}$. We now show that it can be decomposed into $C_3$'s. Let $V(K_{12}) = \{v_1, v_2, \ldots, v_{12}\}$, let $C = \{v_1, v_4, v_8\}$, and let $\phi$ be the permutation whose cycle decomposition is $(v_1 \ v_2 \ \ldots \ v_{12})$. Then $C$ and $\phi^k C$ $(1 \leq k \leq 11)$ are the disjoint 3-cycles of the decomposition. 

2.4 LEMMA: Let $K_{16}$ be the complete graph with

$$V(K_{16}) = \{u_i : 1 \leq i \leq 16\},$$

and $I = \{u_i u_{i+8} : 1 \leq i \leq 8\}$ a 1-factor of $K_{16}$. Then $K_{16} - I$ can be decomposed into four $K_5$'s and $C_3$'s.

Proof: A desired decomposition of $K_{16} - I$ can be constructed as follows: Let $A = \{u_1, u_4, u_7, u_{10}, u_{13}\}$ and $\phi$ be the permutation whose cycle decomposition is $(u_1 \ u_2 \ \ldots \ u_{16})$. Then $A$, $\phi^4 A$, $\phi^8 A$ and $\phi^{12} A$ are the vertex sets of the four $K_5$'s. The twenty-four triangles in the remainder of the graph are given by

$$C_1 = \{u_1, u_2, u_6\} \quad C_4 = \{u_2, u_4, u_8\},$$
$$C_2 = \{u_1, u_3, u_{12}\} \quad C_5 = \{u_3, u_8, u_{10}\},$$
$$C_3 = \{u_1, u_{15}, u_{16}\} \quad C_6 = \{u_9, u_{10}, u_{14}\}.$$
\[ \phi^4 c_i, \phi^8 c_i, \phi^{12} c_i \quad \text{for} \quad i = 2, 3, 4, 5, \]
\[ \phi^3 c_i, \phi^4 c_i, \phi^5 c_i \quad \text{for} \quad i = 1, 6. \]

2.5 **Lemma:** \( C_3 | K_{8 \cdot 6^x} \) for any positive integer \( x \).

**Proof:** Using the method used in the proof of Prop. 1.4., we associate with \( K_{8 \cdot 6^x} \) a \( K_{16} - I \) where the vertices of \( K_{16} \) are independent subsets of \( V(K_{8 \cdot 6^x}) \) each of cardinality \( 3x \) and \( I \) is a 1-factor of \( K_{16} \). Moreover, by construction this 1-factor is the same as that of \( K_{16} \) in Lemma 2.4. Hence, by Lemma 2.4 \( K_{16} - I \) can be decomposed into \( C_3 \)'s and four \( K_5 \)'s which amounts to the fact that \( K_{8 \cdot 6^x} \) can be decomposed into four \( K_{3x} \)'s and the rest \( 3K_{3x} \)'s. The result then follows from Prop. 1.1 and Prop. 1.2.0.

2.6 **Lemma:** \( K_{6 \cdot 3n-1} \) can be decomposed into one \( K_5 \) and \( C_3 \)'s.

**Proof:** We shall use induction on \( n \). The result is true for \( n = 0 \) and 1 using Lemma 2.3. Suppose that \( n \geq 2 \) and that the result is true for all \( k < n \). Consider \( K_{6 \cdot 3n-1} \) and write

\[ K_{6 \cdot 3n-1} = K_{8 \cdot 6 \cdot 3n-2 + 6 \cdot 3n-2} - 1 \]

\[ = (8K_{6 \cdot 3n-2} + K_{6 \cdot 3n-2} - 1) \cup K_{6 \cdot 3n-2} , \]
where the vertex set of the complete multipartite graph is chosen appropriately. By Lemma 2.5, $8K_{6\cdot3n-2}$ can be decomposed into $C_3$'s.

By the induction hypothesis $K_{6\cdot3n-2-1}$ is decomposable into one $K_5$ and $C_3$'s and since $6\cdot3^{n-2} - 1$ is odd, by Lemma 2.1

$8K_{6\cdot3n-2} + K_{6\cdot3n-2-1}$ is decomposable into one $K_5$ and $C_3$'s.

2.7 Theorem: If $n \equiv 2 \pmod{6}$, $K_{6n+5}$ can be decomposed into one $K_5$ and $C_3$'s.

Proof: Let $n = 6m_1 + 2$ for some non-negative integer $m_1$. We write

$$K_{6n+5} = K_{36m_1+17} = (2m_1K_{18} + K_{17}) \cup 2m_1K_{18}$$

By Lemma 2.3 and Lemma 2.1, $2m_1K_{18} + K_{17}$ can be decomposed into one $K_5$ and $C_3$'s. Also, if $2m_1 \equiv 0$ or $4 \pmod{6}$, by Prop. 1.4 $2m_1K_{18}$ can be decomposed into all $C_3$'s, and hence the result. In case $2m_1 \equiv 2 \pmod{6}$, we write $2m_1 = 6m_2 + 2$ for some non-negative integer $m_2$ and

$$K_{6n+5} = K_{54\cdot2m_2+53} = (2m_2K_{54} + K_{53}) \cup 2m_2K_{54}$$

Since $53 = 6 \cdot 3^2 - 1$, by Lemma 2.6, $K_{53}$ can be decomposed into one $K_5$ and $C_3$'s and, hence, by Lemma 2.1, $2m_2K_{54} + K_{53}$ also. Moreover,
$2m_2 K_{54}$ can be decomposed into $C_3$'s by Prop. 1.4, provided $2m_2 \equiv 0$ or 4 (mod 6). In case $2m_2 \equiv 2$ (mod 6) we write $2m_2 = 6m_3 + 2$ for some non-negative integer $m_3$ and

$$K_{6n+5} = K_{162} \cdot 2m_3 + 161 = (2m_3 K_{162} + K_{161}) \cup 2m_3 K_{162}.$$ 

Repeating the above argument, at the $p^{th}$ stage we get

$$K_{6n+5} = K_{6 \cdot 3^p \cdot 2m_3 + 6 \cdot 3^p - 1} = (2m_3 K_{6 \cdot 3^p} + K_{6 \cdot 3^p - 1}) \cup 2m_3 K_{6 \cdot 3^p},$$

and as long as $2m_3 \equiv 0$ or 4 (mod 6) we get the desired decomposition of $K_{6n+5}$ into one $K_5$ and $C_3$'s. In case $2m_3 \equiv 2$ (mod 6) we write $2m_3 = 6m_{p+1} + 2$ and

$$K_{6n+5} = K_{6 \cdot 3^p + 1 \cdot 2m_{p+1} + 6 \cdot 3^p + 1 - 1}$$

$$= (2m_{p+1} K_{6 \cdot 3^p + 1} + K_{6 \cdot 3^p + 1 - 1}) \cup 2m_{p+1} K_{6 \cdot 3^p + 1}.$$

Continuing like this, at the final stage, say the $t^{th}$, we have

$$K_{6n+5} = (2m_t K_{6 \cdot 3^t} + K_{6 \cdot 3^t - 1}) \cup 2m_t K_{6 \cdot 3^t},$$

where $2m_t$ is either 0, 2 or 4. In case $2m_t = 4$, Lemmas 2.6, 2.1 and Prop. 1.4 give the result.
If $2m_t = 0$ or 2 then $K_{6n+5}$ is $K_{6 \cdot 3t - 1}$ or $K_{6 \cdot 3t + 1 - 1}$, respectively, and the result then follows from Lemma 2.6.

2.8 THEOREM: $K_{6n+5}$, n odd, can be decomposed into one $K_5$ and $C_3$'s.

Proof: We shall use induction on n. For $n = 1$ we write

$$K_{11} = K_{6+5} = K_6 + K_5,$$

and the result follows from Lemma 2.1. Suppose that the result is true for all odd $k < n$. Consider $K_{6n+5}$ and write

$$K_{6n+5} = (nK_6 + K_5) \cup nK_6.$$

By Lemma 2.1, $nK_6 + K_5$ can be decomposed into one $K_5$ and $C_3$'s.

Let $V_1, V_2, ..., V_n$ be n maximal independent subsets of $V(K_6)$.

With $nK_6$ we associate a complete graph $K_n$ with

$$V(K_n) = \{V_1, V_2, \ldots, V_n\},$$

and for all $1 \leq i, j \leq n$, the edge $V_iV_j$ corresponds to the set of all edges of the complete bipartite subgraph $K_{|V_i|, |V_j|}$ of $nK_6$.

Now if $n \equiv 1$ or 3 (mod 6), $K_n$ can be decomposed into $C_3$'s and hence, $nK_6$ can be decomposed into 3$K_6$'s and the result then follows from Prop. 1.1. If $n \equiv 5$ (mod 6), let $n = 6m + 5$ for
some m. We claim $K_n$ can be decomposed into one $K_5$ and $C_3$'s. In case $m$ is even, our claim follows from Theorem 2.2 and Theorem 2.7. If $m$ is odd, the claim follows from the induction hypothesis, as $m < n$. Thus the graph $K_n^6$ can be decomposed into one $5K_6$ and $3K_6$'s. The result then follows from Prop. 1.1 and Prop. 1.2.

Finally the results of Theorems 2.2, 2.7 and 2.8 can be put together into a single theorem.

2.9 THEOREM: If $n \equiv 5 \pmod{6}$, $n \geq 5$, $K_n$ can be decomposed into one $K_5$ and $C_3$'s.

2.10 COROLLARY: For any odd integer $n \geq 3$, $K_n$ can be decomposed into all triangles with the exception of at most one $K_5$.

Proof: For $n \equiv 1$ or $3 \pmod{6}$, we know that a Steiner triple system exists, which is equivalent to the fact that $K_n$ can be decomposed into triangles. In case $n \equiv 5 \pmod{6}$ the result follows from Theorem 2.9.

As we have stated in the beginning of the chapter, Corollary 2.10 can be restated in the following way.

2.11 COROLLARY: For any odd integer $n \geq 3$, there exists a pairwise balanced design $(n; 3, 5; 1)$. 
2.12 COROLLARY: For \( n \geq 10 \) and \( n \equiv 4 \pmod{6} \), \( K_n - I \) can be decomposed into one \( K_5 \) and the rest \( C_3 \)'s where \( I \) denotes a 1-factor of \( K_n \).

Proof: Let \( u \notin V(K_n) \) be any point. The graph \( K_n + u \) is a complete graph \( K_{n+1} \) with \( n + 1 \equiv 5 \pmod{6} \) and hence by Theorem 2.9 can be decomposed into one \( K_5 \) and the rest \( C_3 \)'s. We can relabel the vertices of \( K_{n+1} \) so that \( u \) is not one of the vertices of \( K_5 \). This implies that the graph \( K_n = K_{n+1} - u \) can be decomposed into one 1-factor, one \( K_5 \) and the rest \( C_3 \)'s.

The results of the Prop. 1.3, Corollary 2.12 and the fact that \( K_5 \) is a union of two disjoint \( C_3 \)'s, give the following result.

2.13 COROLLARY: For any even \( n \geq 6 \), \( K_n - I \) can be decomposed into 3-cycles and 5-cycles where \( I \) denotes a 1-factor of \( K_n \).

In Prop. 1.11 we proved that for any odd integer \( n \geq 3 \),
\[ C_n \mid K_n \]. Cor. 2.10 enables us to extend this result.

2.14 THEOREM: \( C_n \mid K_n \) for any odd integers \( m, n \geq 3 \).

Proof: In case \( m \geq 3 \), let \( V_1, V_2, \ldots, V_m \) be \( m \) maximal independent subsets of \( V(K_n) \) and \( |V_i| = n, 1 \leq i \leq m \). With \( m \nmid n \) we
associate a complete graph $K_m$ as follows:

$$V(K_m) = \{V_1, V_2, \ldots, V_m\}$$

and an edge incident with the vertices $V_i$ and $V_j$, $1 \leq i, j \leq m$, corresponds to all the edges of the complete bipartite subgraph $K_{|V_i|, |V_j|}$ of the original graph $K_m$. Since $m$ is odd by Cor. 2.10, $K_m$ can be decomposed into at most one $K_5$ and the rest triangles, which amounts to the fact that the graph $K_m$ can be decomposed into at most one $5K_n$ and the rest $3K_n$. Sotteau [33] has shown that $C_n \nmid K_n$ for any odd $n$. That $C_n \nmid 5K_n$ follows from Prop. 1.11.

This completes the proof. $\square$
CHAPTER 3

In the beginning of Chapter 2 we have described BIBD's (balanced incomplete block designs). These designs are special cases of G-designs introduced by Hell and Rosa [19].

DEF.: Let \( K_n^{(\lambda)} \) denote a graph with \( n \) vertices and any two distinct vertices joined by exactly \( \lambda \) edges. An \((n,k,\lambda)\) G-design is a partition of the edge set of the graph \( K_n^{(\lambda)} \), so that the resulting subgraphs are isomorphic to a given graph \( G \) where \( |V(G)| = k \).

In the particular case where \( G \) is the complete graph \( K_k \), an \((n,k,\lambda)\) \( K_k \)-design is nothing but a BIBD \((n,k,\lambda)\). For a short account of G-designs we refer to a survey article [7] by Bermond and Sotteau.

In case the graph \( G \) is \( C_k \), a cycle of length \( k \), an \((n,k,\lambda)\) \( C_k \)-design is also called a BCD \((n,k,\lambda)\) (balanced cycle design, see [20]).

Similar definitions hold for directed graphs. If \( G \) is a directed graph with \( k \) vertices, an \((n,k,\lambda)\) G-design is a partitioning of the arcs of \( D K_n^{(\lambda)} \) so that the resulting directed subgraphs are isomorphic to \( G \). In case \( G \) is a circuit \( C_k^* \), a \( C_k^* \)-design is called a balanced circuit design. A number of people have considered the problem of existence of BCD's for directed and undirected graphs. We mention here a few of them: Hung and
For the existence of a BCD \((n,k,A)\), it is necessary that \(n \geq k\) and \(k\) divides the number of edges of \(\lambda K_n\), that is, \(k \mid \frac{\lambda n(n-1)}{2}\) and the degree \(\lambda(n-1)\) of each vertex of \(K_n\) is even. Thus we have the following result.

3.1 THEOREM: The necessary conditions for the existence of a BCD \((n,k,\lambda)\) are

(i) \(n \geq k\),

(ii) \(\lambda n(n-1) \equiv 0 \pmod{2k}\) and

(iii) \(\lambda(n-1) \equiv 0 \pmod{2}\).

We wish to know whether the conditions (i), (ii) and (iii) of Theorem 3.1 are also sufficient for the existence of a BCD \((n,k,\lambda)\).

In this Chapter we consider the case for \(k\) even. The most complete results in this case are due to Bermond, Huang and Sotteau [4]. They have shown that the necessary conditions of Theorem 3.1 are also sufficient for even values of \(k\) satisfying \(4 \leq k \leq 16\).

We shall extend the result for an infinite number of even positive integers \(k \geq 4\) where we restrict ourselves to the case \(\lambda = 1\) and prove that for \(k = 2^\alpha p\), where \(p\) is a prime and \(\alpha\) is a positive integer, a BCD \((n,k,1)\) exists if and only if the conditions (i), (ii) and (iii) of Theorem 3.1 hold.
We shall use the following result of Sotteau [34].

3.2 **Lemma:** \( K_{m,n} \) can be decomposed into cycles of length \( k \), \( k \) even, if and only if \( m \) and \( n \) are even, \( m \geq \frac{k}{2}, n \geq \frac{k}{2} \) and \( k \) divides \( m \).

In case \( k \equiv 0 \pmod{4} \), Kotzig [23] has shown that
\( K_{2mk+1} \) can be decomposed into \( C_k \)'s for all positive integers \( m \).

A similar result has been obtained by Rosa [29] for \( k \equiv 2 \pmod{4} \).
We give here an independent proof to cover both the cases together.

3.3 **Lemma:** For any even positive integer \( k \) and any positive integer \( m \), \( C_k \mid K_{2mk+1} \).

**Proof:** We shall use induction on \( m \). For \( m = 1 \), we use a construction to show that the result holds. In case \( k \equiv 0 \pmod{4} \), let \( k = 2l \), \( l \) even. Consider a complete graph \( K_{2k+1} \) with \( V(K_{2k+1}) = \{u_0, u_1, \ldots, u_t, u_{t+1}, \ldots, u_{2k}\} \). Let \( \phi \) be the permutation with cycle representation \((u_0, u_1, u_2, \ldots, u_{2k})\). Consider a cycle \( C \) of length \( k \) as given below (see Figure 3(A))

\[
C = u_0, u_1, u_{2k-1}, u_3, \ldots, u_{2k-l+3}, u_{l-1}, u_{k+l+1}, u_{k-\ell+3}, \ldots
\]

\[
\ldots, u_{k+5}, u_{k-1}, u_{k+3}, u_{k+1}, u_0
\]
If we say that the two vertices $u_i$ and $u_j$ are at a distance $d$ if and only if $|i-j| = d$, we observe that the vertices in the neighbourhood in $K_{2k+1}$ of a vertex $u_i$ for all $i$ are at distances $1, 2, 3, ..., k$ from $u_i$. Moreover, if we consider the distances between the successive vertices of the cycle $C$, the distances $1, 2, ..., k$ occur exactly once. Hence the cycles $\phi^j C$, $j = 0, 1, ..., 2k$ are all disjoint and cover all the edges of $K_{2k+1}$.

In case $k \equiv 2 \pmod{4}$, that is, $k = 2\ell$, $\ell$ odd, $\ell \geq 3$, consider the cycle $C'$ given below (see Figure 3(B))

$$C': u_0, u_1, u_4, u_{2k}, u_6, u_{2k-2}, ..., u_{2k-l+5}, u_{k+l+1}, u_{k+l-3}, ..., u_{k+6}, u_{k-2}, u_{k+4}, u_k, u_{k+2}, u_0$$
Here again among the distances between vertices of the cycle $C'$, the distances $1, 2, \ldots, k$ occur exactly once. Hence the cycles $^jC'$, $j = 0, 1, 2, \ldots, 2k$ are all disjoint and cover all the edges of $K_{2k+1}$.

This shows that the result is true for $m = 1$. Next assume that the result is true for $m$, that is, $K_{2mk+1}$ can be decomposed into $C_k$'s. Consider the graph $K_{2(m+1)k+1}$. We write it as

$$K_{2(m+1)k+1} = K_{2mk+2k+1} = (K_{2mk} + K_1) \cup (K_{2k} + K_1) \cup K_{2k, 2mk}$$

$$= K_{2mk+1} \cup K_{2k+1} \cup K_{2k, 2mk}$$
where the vertex set of each graph is chosen appropriately. Since
\[ K_{2^m+1}, \] by the induction hypothesis, and \[ K_{2^k+1}, \] by the first part
for \( m = 1 \), can be decomposed into cycles \( C_k \), it is enough to show
that \( C_k | K_{2^k, 2mk} \). Since \( 2^k \) and \( 2mk \) are both even and each is
greater than \( \frac{k}{2} \) and also \( 2^k \cdot 2mk \equiv 0 \pmod{k} \), the result then follows
from Lemma 3.2.0.

3.4 Lemma: If \( C_k | K_r \), \( k \) even and \( r \) odd, then \( C_k | K_{2mk+1} \) for
all positive integer \( m \).

Proof: We write

\[
K_{2^m+r} = K_{2^m+(r-1)+1} = (K_{2^m} + K_1) \cup (K_{r-1} + K_1) \cup K_{2^m},(r-1)
\]

\[
= K_{2^m+1} \cup K_r \cup K_{2^m,(r-1)}
\]

where the vertex set of each graph is chosen appropriately. Now \( K_{2^m+1} \)
can be decomposed into \( C_k \)'s by Lemma 3.3. Also \( C_k \) divides \( K_r \) by
the hypothesis of the Lemma. Thus it is enough to show that \( C_k \)
divides \( K_{2^m,(r-1)} \), which follows from Lemma 3.2, as \( 2^m \) and \( r-1 \)
are even, both are greater than \( \frac{k}{2} \) and \( k \) divides \( 2mk(r-1) \).

The following result follows immediately from Lemma 3.4.
3.5 **THEOREM**: Given $k$ even let $n = 2mk + r$ where $m$ is a positive integer and $0 < r < 2k$. Also, let $n, k$ and $\lambda = 1$ satisfy conditions (i), (ii) and (iii) of Theorem 3.1. Then there exists a BCD $(n, k, l)$ if there exists

(a) a BCD $(r, k, l)$ if $k < r < 2k$ or

(b) a BCD $(2k + r, k, l)$ if $0 < r < k$.

Now and henceforth let $n = 2mk + r$, $0 < r < 2k$, for some integer $m$. Furthermore, let $k = 2^p \alpha$ where $p$ is a prime and $\alpha$ is a positive integer. The conditions (ii) and (iii) of Theorem 3.1 are equivalent to

$$r(r-1) \equiv 0 \pmod{2k} \quad \text{and} \quad r-1 \equiv 0 \pmod{2} \quad (*)$$

In case $p = 2$ we have the following theorem.

3.6 **THEOREM**: For any integer $n$ and $k = 2^\alpha$, satisfying the conditions (i), (ii) and (iii) of Theorem 3.1 with $\lambda = 1$ there exists a BCD $(n, 2^\alpha, l)$ where $\alpha$ is a positive integer.

**Proof**: The conditions (*) in this case become

$$r(r-1) \equiv 0 \pmod{2^{\alpha+1}} \quad \text{and} \quad r-1 \equiv 0 \pmod{2}.$$ 

That is, $r \equiv 1 \pmod{2^{\alpha+1}}$. Since $0 < r < 2^{\alpha+1}$ holds, it follows that $r = 1$. Thus, in view of Theorem 3.5, it is enough to show that
there exists a BCD \((2^{a+1}+1, 2^a, 1)\). The result then follows from Lemma 3.3.

In case \(k = 2^p\) where \(p\) is an odd prime and \(a\) is a positive integer, the conditions (*) become

\[
  r(r-1) \equiv 0 \pmod{4p^a} \quad \text{and} \quad r-1 \equiv 0 \pmod{2}.
\]

That is either \(r \equiv 1 \pmod{p^a}\) and \(r \equiv 1 \pmod{4}\)

or \(r \equiv 0 \pmod{p^a}\) and \(r \equiv 1 \pmod{4}\).

But \(r \equiv 1 \pmod{p^a}\) and \(r \equiv 1 \pmod{4}\) imply that \(r \equiv 1 \pmod{4p^a}\).

Since \(0 < r < 4p^a\) we have \(r = 1\). In view of Theorem 3.5, it suffices to show the existence of a BCD \((4^*p^a+1, 2^p, 1)\) which follows from Lemma 3.3.

If \(r \equiv 0 \pmod{p^a}\) and \(r \equiv 1 \pmod{4}\) we claim:

\[
  r = 3p^a \quad \text{if} \quad p^a \equiv 3 \pmod{4}
\]

and \(r = p^a \quad \text{if} \quad p^a \equiv 1 \pmod{4}\).

To show this let \(r = q^*p^a\) for some integer \(q\), so that \(r \equiv 1 \pmod{4}\) implies that \(r-1 = q^*p^a-1 \equiv 0 \pmod{4}\). Hence,

\[
  q \equiv 3 \pmod{4} \quad \text{if} \quad p^a \equiv 3 \pmod{4}
\]

or \(q \equiv 1 \pmod{4} \quad \text{if} \quad p^a \equiv 1 \pmod{4}\).

But \(0 < r < 4^*p^a\) implies that \(q = 3\) if \(p^a \equiv 3 \pmod{4}\) or \(q \equiv 1\) if \(p^a \equiv 1 \pmod{4}\).
In view of Theorem 3.5 and the above discussion, to prove that the conditions (i), (ii) and (iii) of Theorem 3.1 are necessary and sufficient for the existence of a BCD \((n, 2^a p, 1)\) where \(p\) is an odd prime and \(a\) a positive integer, it is enough to show the existence of a BCD \((3^a p^a, 2^a p^a, 1)\) if \(p^a \equiv 3 \text{ (mod 4)}\) and that of a BCD \((5^a p^a, 2^a p^a, 1)\) if \(p^a \equiv 1 \text{ (mod 4)}\).

Thus we have proved the following lemma.

3.7 **Lemma:** Let \(n\) be a positive integer where \(n, 2^a p^a\) and \(\lambda = 1\) satisfy the conditions (i), (ii) and (iii) of Theorem 3.1. Then there exists a BCD \((n, 2^a p^a, 1)\) for \(p\) an odd prime and \(a\) a positive integer if there exists

(a) a BCD \((3^a p^a, 2^a p^a, 1)\) if \(p^a \equiv 3 \text{ (mod 4)}\) or

(b) a BCD \((5^a p^a, 2^a p^a, 1)\) if \(p^a \equiv 1 \text{ (mod 4)}\).

3.8 **Lemma:** There exists a BCD \((3^a p^a, 2^a p^a, 1)\) where \(p\) is a prime, \(a\) is a positive integer and \(p^a \equiv 3 \text{ (mod 4)}\).

**Proof:** Partition the vertex set \(V(K_{3^a p^a})\) into \(p^a\) subsets each of size three. Label them with

\[
T_0: \{u_{00}, u_{01}, u_{02}\}, \ T_1: \{u_{10}, u_{11}, u_{12}\}, \ldots, \ T_{p^a-1}: \{u_{p^a-1,0}, u_{p^a-1,1}, u_{p^a-1,2}\}.
\]

With \(K_{3^a p^a}\) we associate a complete graph \(K_{p^a}\) whose vertices are the vertex sets \(T_0, T_1, \ldots, T_{p^a-1}\) and an edge \(T_i T_j, 0 \leq i, j \leq p^a-1\).
corresponds to all the edges of the bipartite subgraph $K_{T_1, T_3}$ of $K_{3^p\alpha}$. Since $p^\alpha$ is odd, we know that $K_{p^\alpha}$ can be decomposed into $\frac{(p^\alpha-1)}{2}$ disjoint hamiltonian cycles. We obtain one such decomposition using Walecki's construction (see [25], p 162-3) as follows.

Let

\[ \zeta : T_0, T_1, T_2, T_{p\alpha-1}, T_3, T_{p\alpha-2}, \ldots, T_{\frac{(p^\alpha-3)}{2}}, T_0 \]

be one hamiltonian cycle of the decomposition. (see Figure 3(C))

Also let $\phi$ be a permutation on $p^\alpha$ symbols $T_0, T_1, \ldots, T_{p\alpha-1}$ whose cycle representation is

\[ \phi = (T_0)(T_1 T_2 T_3 \cdots T_{p\alpha-1}). \]
Then the \( \left( \frac{p^a - 1}{2} \right) \) disjoint hamiltonian cycles of the decomposition are given by \( \phi^k \zeta, \ k = 0, 1, \ldots, \left( \frac{p^a - 3}{2} \right) \).

Using the cycle \( \zeta \) we define two cycles \( C \) and \( C' \) of length \( 2 \cdot p^a \) in the graph \( K_{3, p^a} \) as follows.

\[
C: \ u_0, u_1, u_1', u_0, u_2, u_2', u_3, u_3', \ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots
\]

\[
\ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots
\]

and

\[
C': \ u_0, u_1, u_2, u_2', u_3, u_3', \ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots
\]

\[
\ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots, u_0, u_1, u_2, u_2', u_3, u_3', \ldots
\]

These cycles are also shown in Figures 3(D) and 3(E) respectively.

\[\text{FIGURE 3(D)}\]
Let $\sigma$ be a permutation on $3p^\alpha$ symbols $u_{00}, u_{01}, u_{02}, \ldots$, $u_{p\alpha-1,0}, u_{p\alpha-1,1}, u_{p\alpha-1,2}$ whose cycle representation is

$$\sigma = (u_{00} u_{01} u_{02})(u_{10} u_{11} u_{12}) \ldots (u_{p\alpha-1,0} u_{p\alpha-1,1} u_{p\alpha-1,2}).$$

The cycles $C, \sigma C$ and $\sigma^2 C$ are three cycles of length $2p^\alpha$ that use all internal edges of the $p^\alpha$ triangles on the vertex sets $T_0, T_1, \ldots, T_{p\alpha-1}$ and all edges $u_{ij} u_{kj}, 0 \leq j \leq 2$ between the successive vertex sets $T_i$ and $T_{k}$ of the cycle $C$. 
Also \( C', \sigma C' \) and \( \sigma^2 C' \) are three more cycles of length \( 2p^\alpha \) and they use the remaining edges \( u_{ir} u_{ks} \), \( 0 \leq r, s \leq 2, r \neq s \), between the successive vertex sets \( T_i \) and \( T_k \) of the cycle \( \zeta \).

Thus we have obtained six cycles \( C, \sigma C, \sigma^2 C, C', \sigma C', \sigma^2 C' \) of length \( 2p^\alpha \) in \( K_{3p^\alpha} \) and we have used:

- edges of \( K_{3p^\alpha} \) which are the internal edges of the vertex sets \( T_0, T_1, \ldots, T_{p^\alpha-1} \) and
- edges of \( K_{3p^\alpha} \) which corresponds to the edges of the cycle \( \zeta \).

In case \( p = 3 \) and \( \alpha = 1 \) this gives a desired decomposition of \( K_9 \) into \( C_6 \)'s. Otherwise, we are left with the edges of \( K_{3p^\alpha} \) which correspond to the edges of the remaining cycles \( \phi^k \zeta, \quad k = 1, 2, \ldots, \frac{(p^\alpha-3)}{2} \) of \( K_{p^\alpha} \). Since \( p^\alpha \equiv 3 \text{(mod 4)} \), the number \( \frac{(p^\alpha-3)}{2} \) is even. We pair these cycles as

\[
\phi^{(2m-1)} \zeta, \quad \phi^{2m} \zeta \quad 1 \leq m \leq p^\alpha-3
\]

To complete the proof of the lemma, because of the rotation under the permutation \( \phi \), it is enough to show that the edges of \( K_{3p^\alpha} \) in the union of one such pair can be decomposed into cycles of length \( 2p^\alpha \). Further, again because of the action of \( \phi \) there is no loss in assuming this pair is \( \zeta, \phi \zeta \). We relabel the vertices of the complete graph \( K_{p^\alpha} \) so that the cycle \( \zeta \) is written as
Under this new labeling the cycle $\phi \zeta$ then becomes

$$\zeta' : T_0, T_1, T_2, \ldots, T_{p^\alpha - 1}.$$  

Let $Z$ be a cycle of length $2^* p^\alpha$ obtained by joining the vertices with labels 0 and 1 alternately between the successive triangles of the cycle $\zeta'$, that is

$$Z : u_00, u_{11}, u_{22}, \ldots, u_{p^\alpha - 1,0}, u_{00}, u_{10}, \ldots, u_{p^\alpha - 1,1}, u_{00}.$$  

The three cycles $Z, \sigma Z$ and $\sigma^2 Z$ use all the edges between the successive vertex sets of the cycle $\zeta'$ except the edges of the form $u_{i,j}, u_{i+1,j}$ for $0 \leq i \leq p^\alpha - 1$ and $0 \leq j \leq 2$ which form a union of three disjoint cycles of length $p^\alpha$. In other words, the set of edges between the successive triangles of the cycle $\zeta'$ can be so partitioned that after pulling out three disjoint cycles of length $p^\alpha$, the remaining edges can be partitioned into three disjoint cycles of length $2^* p^\alpha$. Moreover, this is also true for the other hamiltonian cycles in the decomposition of $K_{p^\alpha}$. Thus, in general, for any hamiltonian cycle of $K_{p^\alpha}$, if from the edges between its successive
triangles we pull out three disjoint cycles of length $p^\alpha$ then the remaining edges can be decomposed into three disjoint cycles of length $2^*p^\alpha$.

If we can find three edge-disjoint cycles of length $2^*p^\alpha$ which use only the edges of $K_{3^2, p^\alpha}$ along the cycle $\zeta'$ or $\zeta''$ and show that their union can be decomposed into three disjoint cycles of length $p^\alpha$ along $\zeta'$ and three disjoint cycles of length $p^\alpha$ along $\zeta''$, then we are done. We regroup the vertices of $K_{3^2, p^\alpha}$ into three groups

$$A_0 = \{u_{00}, u_{10}, u_{20}, \ldots, u_{p^\alpha - 1, 0}\}$$

$$A_1 = \{u_{01}, u_{11}, u_{21}, \ldots, u_{p^\alpha - 1, 1}\}$$

and $$A_2 = \{u_{02}, u_{12}, u_{22}, \ldots, u_{p^\alpha - 1, 2}\}$$

We construct a cycle $\hat{C}$ of length $2^*p^\alpha$ illustrated in Figure 3(F) and described below.

A solid line joining two vertices, say $u$ and $v$, represents an edge incident with those vertices. A dotted line
between two vertices, say \( u \) and \( v \), represents a path obtained by joining the successive vertices, starting at \( u \) and ending at \( v \) along the cycle \( \zeta' \). The jagged line between the two vertices \( u_{60} \) and \( u_{p6-0} \) represents a path obtained by joining the vertices in succession along the cycle \( \zeta'' \). A forward jump from one block to another, for example, from \( A_0 \) to \( A_1 \), \( A_1 \) to \( A_2 \) or \( A_2 \) to \( A_0 \) corresponds to a change in labeling in the second coordinate of the vertices in the cyclic order \( 0 \) to \( 1 \), \( 1 \) to \( 2 \) or \( 2 \) to \( 0 \).

We start at the vertex \( u_{00} \) of the vertex set \( T_0 \) in the block \( A_0 \) and join it to the vertex \( u_{21} \) in block \( A_1 \). Note that this is an edge between the vertex sets \( T_0 \) and \( T_2 \) of cycle \( \zeta'' \). Next we join \( u_{21} \) to \( u_{41} \) (this is an edge between the vertex sets \( T_2 \) and \( T_4 \) of \( \zeta'' \)) and then remain in block \( A_1 \) joining edges along the path

\[ u_{41} u_{51} u_{61} \ldots u_{p2-1} \]

We then join the last vertex \( u_{p2-1} \) of this path to the vertex \( u_{02} \) of the block \( A_2 \). We remain in block \( A_2 \) joining edges along the path

\[ u_{02} u_{12} u_{22} u_{32} u_{42} \]

We join the vertex \( u_{42} \) to the vertex \( u_{10} \) of the block \( A_0 \). We remain in the same block joining edges along the path
and these are the vertices belonging to the successive vertex sets of the cycle $\zeta$. We join the vertex $u_{p^\alpha-6,0}$ to $u_{p^\alpha-1,2}$ and then $u_{p^\alpha-1,2}$ to $u_{p^\alpha-4,0}$. We complete the cycle by joining $u_{p^\alpha-4,0}$ to $u_{p^\alpha-2,0}$, $u_{p^\alpha-2,0}$ to $u_{p^\alpha-1,0}$ and finally $u_{p^\alpha-1,0}$ to $u_{00}$.

In the construction of $\hat{C}$, since we have picked $p^{\alpha-1}$ vertices from the block $A_0$, $p^{\alpha-4}$ vertices from the block $A_1$ and five vertices from the block $A_2$, $\hat{C}$ is a cycle of length $2p^{\alpha}$. We rotate the blocks $A_0$, $A_1$ and $A_2$ two times in cyclic order, that is, $A_0$ to $A_1$, $A_1$ to $A_2$ and $A_2$ to $A_0$ and get two more cycles of length $2p^{\alpha}$. In other words these cycles are nothing but $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$. In view of what we have said earlier, it is enough to show that the union of these three cycles from two sets of three disjoint cycles of length $p^{\alpha}$.

Let us first consider the edges along the cycle $\zeta$. In the construction of $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$ we have used

- the path $u_{00}, u_{10}, u_{20}, u_{30}, u_{40}$ in $\sigma\hat{C}$
- the path $u_{40}, u_{50}, u_{60}, \ldots, u_{p^\alpha-2,0}$ in $\sigma^2\hat{C}$
- and the path $u_{p^\alpha-2,0}, u_{p^\alpha-1,0}, u_{00}$ in $\hat{C}$
Therefore the cycles \( \hat{C} \), \( \sigma \hat{C} \) and \( \sigma^2 \hat{C} \) contain all the edges \( u_{i,j} \), \( u_{i+1,j} \), \( 0 \leq i \leq p^\alpha - 1 \), \( 0 \leq j \leq 2 \), which form three disjoint cycles of length \( p^\alpha \).

As regard to the edges along the cycle \( \zeta^* \), in the construction of \( \hat{C} \), \( \sigma \hat{C} \) and \( \sigma^2 \hat{C} \) we have used:

- the path \( u_{0,0} u_{2,1} u_{4,1} \) in \( \hat{C} \)
- the edge \( u_{4,1} u_{1,2} \) in \( \sigma^2 \hat{C} \)
- the path \( u_{1,2} u_{2,6} \ldots u_{p^\alpha - 6, 2} \) in \( \sigma^2 \hat{C} \)
- the edge \( u_{p^\alpha - 6, 2} u_{p^\alpha - 1, 1} \) in \( \sigma^2 \hat{C} \)
- the edge \( u_{p^\alpha - 1, 1} u_{p^\alpha - 4, 2} \) in \( \sigma^2 \hat{C} \)
- the edge \( u_{p^\alpha - 4, 2} u_{p^\alpha - 2, 2} \) in \( \sigma^2 \hat{C} \)
- and the edge \( u_{p^\alpha - 2, 2} u_{0,0} \) in \( \sigma \hat{C} \)

Therefore, the cycles \( \hat{C} \), \( \sigma \hat{C} \) and \( \sigma^2 \hat{C} \) contain three disjoint cycles of length \( p^\alpha \) whose edges lie along the cycle \( \zeta^* \). This completes the proof. \( \Box \)

3.9 LEMMA: There exists a BCD \( (5^\alpha, 2^\alpha, 1) \) where \( p^\alpha \equiv 1 \pmod{4} \).
Proof: The proof is similar to that of Lemma 3.8. We divide the vertices of $K_{5 \cdot p^\alpha}$ into $p^\alpha$ groups each of size five and label them with

$$S_0 : \{ u_{00}, u_{01}, u_{02}, u_{03}, u_{04} \}, \quad S_1 : \{ u_{10}, u_{11}, u_{12}, u_{13}, u_{14} \}, \ldots$$

$$\ldots, S_{p^\alpha-1} : \{ u_{p^\alpha-1,0}, u_{p^\alpha-1,1}, u_{p^\alpha-1,2}, u_{p^\alpha-1,3}, u_{p^\alpha-1,4} \}.$$ 

As in the proof of Lemma 3.8, with $K_{5 \cdot p^\alpha}$ we associate a complete graph $K_p^\alpha$, so that $V(K_p^\alpha) = \{ S_0, S_1, \ldots, S_{p^\alpha-1} \}$ and an edge $S_i S_j$, $0 \leq i, j \leq p^\alpha-1$ corresponds to all the edges between the vertices of the sets $S_i$ and $S_j$ in the original graph. We decompose $K_p^\alpha$ into $(p^\alpha-1)/2$ disjoint hamiltonian cycles $\Psi^k \eta$, $k = 0, 1, \ldots, (p^\alpha-3)/2$, where

$$\eta : S_0, S_1, S_2, S_{p^\alpha-1}, S_3, S_{p^\alpha-2}, \ldots, \frac{(p^\alpha-1)}{2} S_{(p^\alpha+3)/2}, S_{(p^\alpha+1)/2}, S_0,$$

and $\Psi$ is a permutation on $p^\alpha$ symbols $S_0, S_1, \ldots, S_{p^\alpha-1}$ with cycle representation

$$\Psi = (S_0)(S_1 S_2 S_3 \ldots S_{p^\alpha-1}).$$

Each $K_5$ with the vertex set $S_i$, $0 \leq i \leq p^\alpha-1$ can be decomposed into two disjoint 5-cycles which are obtained by joining the vertices
at distance 1 and vertices at distance 2 respectively. The five cycles $\tau^k C$, $0 \leq k \leq 4$ of length $2p^a$ where we define $C$ as

$$C: u_0, u_1, u_{10}, u_{20}, u_{21}, u_{0p^a-1,0}, \ldots, u_{0(p^a+5)}, u_{1u_{0}(p^a+5)}, 2'$$

$$u_{0}(p^a-1), 2' u_{0}(p^a-1), 3' u_{0}(p^a+3), 3' u_{0}(p^a+3), 4' u_{0}(p^a+1), 4' u_{0}(p^a+1), 0' u_{00}$$

and $\tau$ is the permutation

$$\tau = (u_0, u_1, u_2, u_3, u_4, \ldots) (u_{p^a-1,0}, u_{p^a-1,1}, u_{p^a-1,2}, u_{p^a-1,3}, u_{p^a-1,4})$$

use all the internal edges of $K_5's$ which are at distance one and all edges of the form $u_{ij} u_{kj}, 0 \leq j \leq 4$ between the successive pentagons $S_i$ and $S_j$ of the cycle $\eta$. Further we construct ten more cycles $\tau^k C', \tau^k C'', 0 \leq k \leq 4$, of length $2p^a$ where

$$C': u_0, u_{11}, u_{20}, u_{p^a-1,1}, \ldots, u_{3(p^a+1)}, 0' u_{01}, u_{10}, \ldots, u_{3(p^a+1)}, 1' u_{00}$$

and

$$C'': u_0, u_{12}, u_{20}, u_{p^a-1,2}, \ldots, u_{3(p^a+1)}, 0' u_{02}, u_{10}, \ldots, u_{3(p^a+1)}, 2' u_{00}$$

These cycles use the remaining edges between the successive pentagons of the cycle $\eta$. Next using the internal edges at distance 2 of
each of the $K_5$'s with vertex set $S_i$, $0 \leq i \leq p - 1$ and the edges between the successive vertex sets of the cycle $\psi\eta$ we get fifteen more cycles of length $2^p p^\alpha$ in the same way.

In case $p = 5$ and $\alpha = 1$ this gives a desired decomposition of $K_{25}$ into cycles of length ten. Otherwise we are left with the edges of $K_5 p^\alpha$ which correspond to the edges of the remaining cycles $\psi^k \eta$, $k = 2, 3, \ldots, (p^\alpha - 3)/2$ of $K_{p^\alpha}$. Since $p^\alpha \equiv 1 \pmod{4}$, these cycles are even in number. We pair these cycles as

$$\psi^{2m} \eta \cap \psi^{2m+1} \eta \text{ for } 1 \leq m \leq \frac{p^\alpha - 5}{4}.$$

To prove the lemma it suffices to show that the edges of $K_5 p^\alpha$ in the union of any such pair can be decomposed into cycles of length $2^p p^\alpha$. Because of the rotation under the permutation $\psi$, it is enough to show this for the pair $\eta, \psi\eta$. We relabel the vertices of the complete graph $K_{p^\alpha}$ so that the cycle $\eta$ is written as

$$\eta': S_0, S_1, S_2, \ldots, S_{p^\alpha - 1}.$$

Under this new labeling the cycle $\psi\eta'$ then becomes

$$\eta'' : S_0, S_2, S_4, S_6, \ldots, S_{p^\alpha - 1}, S_{p^\alpha - 4}, S_{p^\alpha - 2}, S_0.$$
Using the edges between the successive vertex sets of the cycle \( \eta' \), we obtain ten cycles of length \( 2^p \alpha \) and we are left with the edges of the form \( u_{ij} u_{i+1,j} \) for \( 0 \leq i \leq \alpha - 1 \) and \( 0 \leq j \leq 4 \) which form a union of five disjoint cycles of length \( \alpha \). This, in general, is true for the edges of \( K_{5,\alpha} \) along any hamiltonian cycle of \( K_{\alpha} \). Moreover, this fact can be restated as follows.

The edges of \( K_{5,\alpha} \), between the successive vertex sets along any hamiltonian cycle can be decomposed into five disjoint cycles of length \( \alpha \) and five cycles of length \( 2^p \alpha \).

Our next attempt, therefore, is to pull out five disjoint cycles of length \( \alpha \) along the cycle \( \eta' \) and five disjoint cycles of length \( \alpha \) along the cycle \( \eta'' \) such that their union gives five cycles of length \( 2^p \alpha \). In view of what we have said earlier the remaining edges of \( K_{5,\alpha} \) along the cycles \( \eta' \) and \( \eta'' \) then can be decomposed into twenty cycles of length \( 2^p \alpha \). We regroup the vertices of \( K_{5,\alpha} \) into five groups:

\[ \lambda_i = \{ u_{0i}, u_{1i}, u_{2i}, \ldots, u_{\alpha-1, i} \} \]

for \( i = 0, 1, 2, 3, 4 \).

Using the notations of Figure 3(F), we give here in Figure 3(G) a method of constructing a cycle \( C \) of length \( 2^p \alpha \).
We rotate the blocks $A_0$, $A_1$, ..., $A_4$ four times in cyclic order, that is, $A_0$ to $A_1$, $A_1$ to $A_2$, ..., $A_4$ to $A_0$ and obtain four more cycles of length $2^p \cdot \alpha$. These cycles are nothing but $\tau^k \overline{c}$, $0 \leq k \leq 4$. 
Now to check that \( \tau_k^C \), \( 0 \leq k \leq 4 \), contain five disjoint cycles of length \( p^4 \) whose edges lie along the cycle \( \eta' \). We observe that in the construction of \( \tau_k^C \), \( 0 \leq k \leq 4 \), we have used:

- the path \( u_0, u_{10}, u_{20}, u_{30}, u_{40} \) in \( \tau C \),
- the path \( u_{40}, u_{50}, \ldots, u_{p^4-2}, 0 \) in \( \tau^2 C \),
- and the path \( u_{p^4-2}, 0, u_{p^4-1}, 0, u_0 \) in \( \tau^3 C \).

Their union gives a cycle of length \( p^4 \) whose edges lie along the cycle \( \eta' \). Under the rotations we obtain five disjoint such cycles of length \( p^4 \).

Similarly, we have used the following edges and the paths along the cycle \( \eta'' \):

- the edge \( u_{00}, u_{21} \) in \( \tilde{C} \),
- the edge \( u_{21}, u_{43} \) in \( \tilde{C} \),
- the edge \( u_{43}, u_{14} \) in \( \tau^4 C \),
- the path \( u_{14}, u_{64}, \ldots, u_{p^4-6}, 4 \) in \( \tau^4 C \),
- the edge \( u_{p^4-6}, 4, u_{p^4-1}, 3 \) in \( \tau^4 C \),
- the edge \( u_{p^4-1}, 3, u_{p^4-4}, 4 \) in \( \tau^4 C \),
- the edge \( u_{p^4-4}, 4, u_{p^4-2}, 4 \) in \( \tau^4 C \),
- and the edge \( u_{p^4-2}, 4, u_{00} \) in \( \tau \tilde{C} \).
Their union gives a cycle of length $p^\alpha$. Under the rotations we obtain five disjoint cycles of length $p^\alpha$ whose edges lie along the cycle $\eta''$. This completes the proof. □

3.10 THEOREM: There exists a BCD $(n, 2^\alpha p^\alpha, 1)$ where $p$ is an odd prime and $\alpha$ a positive integer if and only if $n$ is odd, $n > 2^\alpha p^\alpha$ and $n(n-1) \equiv 0 \pmod{4^\alpha p^\alpha}$.

Proof: In view of Lemmas 3.7, the proof follows from the Lemmas 3.8 and 3.9. □

Theorems 3.6 and 3.10 can be put together to give the main result.

3.11 THEOREM: There exists a BCD $(n, 2^\alpha p^\alpha, 1)$ where $p$ is any prime and $\alpha$ a positive integer if and only if $n > 2^\alpha p^\alpha$, $n$ odd and $n(n-1) \equiv 0 \pmod{4^\alpha p^\alpha}$.
As we have mentioned in the beginning of the previous chapter, the problem of decomposing complete symmetric digraphs $D_{K_n}$ into directed cycles has been of equal interest to that of decomposing complete graphs $K_n$ into cycles of different lengths. But much less work has been done in the direction of decomposing complete symmetric digraphs $D_{K_n}$ into orientations of a cycle, other than the directed cycle. Hung and Mendelsohn [21] have found a necessary and sufficient condition for the partitioning of the set of arcs of a complete symmetric digraph into each of the oriented triangles. In a recent paper [16], Harary, Heinrich and Wallis have considered the same problem for each of the four oriented quadrilaterals. In doing so they have made strong use of the fact that each orientation of a triangle and that of a quadrilateral is self-converse. Also, Harary, Palmer and Smith [17] had earlier shown that the only graphs for which every orientation is self-converse are the two smallest complete graphs $K_1$ and $K_2$ and the three smallest cycles $C_3$, $C_4$ and $C_5$. In their paper [16], Harary, Heinrich and Wallis made a concluding remark that it remains to investigate the case of oriented pentagons. In this chapter we consider this problem and show that the necessary conditions for each of the oriented pentagon to divide complete symmetric digraphs $D_{K_n}$ are also sufficient.
A cycle of length five can be oriented in only four different ways and these are shown in Figure 4(A).

![Figure 4(A)](image)

The orientations will be described as:

- $A_1$ contains a path of maximum length two,
- $A_2$ contains a path of maximum length three,
- $A_3$ contains a path of maximum length four,
- and $A_4$ is the directed cycle of length five.

4.1 Theorem: If $A_i | D_{K_n}$, $i = 1, 2, 3, 4$, then $n \equiv 0, 1, 5 \text{ or } 6 \pmod{10}$.

Proof: For $i = 1, 2, 3, 4$ $A_i | D_{K_n}$ implies that $5 | n(n-1)$. That is, $n \equiv 0$ or $1 \pmod{5}$ or equivalently $n \equiv 0, 1, 5 \text{ or } 6 \pmod{10}$. $\square$
Since each oriented pentagon is self-converse, we have the following result.

4.2 Lemma: If $C_5 | K_n$, then $A_i | D_k$ for $i = 1, 2, 3, 4$.

Rosa and Huang [31] and Bermond and Sotteau [6] have shown independently that $K_n$ can be decomposed into $C_5$'s if and only if $n \equiv 1$ or $5 \pmod{10}$. This result together with Lemma 4.2 gives the following result.

4.3 Theorem: If $n \equiv 1$ or $5 \pmod{10}$, then $A_i | D_k$ for $i = 1, 2, 3, 4$.

To prove the result for $n \equiv 0$ or $6 \pmod{10}$, first we show it for $n = 6$ and $n = 10$.

4.4 Lemma: $A_i | D_k$ for $i = 1, 2, 3, 4$.

Proof: Let $V(D_k) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Following is the decomposition of $D_k$ into each of $A_i, i = 1, 2, 3, 4$ by construction.
A_1 | DK_6

FIGURE 4(B)
$A_2 |_{\Delta K_6}$

FIGURE 4(C)
\[ A_3 \mid DK_6 \]

FIGURE 4(D)
\textbf{Figure 4(b)}

\begin{align*}
A_4 &\mid \text{IX}_6
\end{align*}
4.5 Lemma: \( A_{i1} \cap K_{10} \) for \( i = 1, 2, 3, 4 \).

Proof: We write

\[ K_{10} = K_6 + K_4 = K_6 \cup K_4 \cup K_{4,6} \]

Let \( V(K_6) = \{u_0, u_1, u_2, u_3, u_4, u_6\} \) and \( V(K_4) = \{v_1, v_2, v_3, v_4\} \)

where \( V(K_{4,6}) = V(K_4) \cup V(K_6) \), \( V(K_4) \) and \( V(K_6) \) are the maximal independent subsets of \( K_{4,6} \).

The graph \( K_4 \cup K_{4,6} \) can be decomposed into four \( C_5 \)'s,

\[ u_1, v_1, v_4, u_3, v_2, u_1, \quad u_2, v_2, v_1, u_4, v_3, u_2, \]

\[ u_3, v_3, v_4, u_2, v_1, u_3, \quad u_4, v_4, u_1, v_3, v_2, u_4, \]

and a graph \( H \) shown in the Figure 4(F).

\[ \text{Figure 4(F)} \]
Since, for \( i = 1,2,3,4 \), \( A_i \) is self-converse, \( A_i \mid D_{C_5} \). Moreover, \( A_i \mid D_{K_6} \), \( i = 1,2,3,4 \), by Lemma 4.4. Thus to prove that \( A_i \mid D_{X_{10}} \), \( i = 1,2,3,4 \), it is enough to show that \( A_i \mid D_H \) for \( i = 1,2,3,4 \).

Here we give a decomposition of \( D_H \) into each of \( A_i \), \( i = 1,2,3,4 \).

\[
\begin{align*}
A_i & \mid D_H \\
\end{align*}
\]

FIGURE 4(G)
$A_{2|DH}$

FIGURE 4(H)

Figure continued.
4.6 THEOREM: If \( n \equiv 6 \pmod{10} \), \( A_i | D K_n \) for \( i = 1, 2, 3, 4 \).

**Proof:** Let \( n = 10k + 6 \) for some positive integer \( k \). We write
\[
K_n = K_{10k+6} = K_{5(2k+1)+1} = [(2k+1)K_5 + K_1] \cup (2k+1)K_5
\]
where the vertex set of the complete multipartite graph \( (2k+1)K_5 \) is chosen appropriately. By Prop. 1.9 of Chapter 1, \( (2k+1)K_5 \) can be decomposed into \( C_5 \)'s and hence the complete multipartite digraph \( D_{(2k+1)K_5} \) can be decomposed into each of \( A_i \), \( i = 1, 2, 3, 4 \). Moreover, by Lemma 4.4 \( A_i | D K_6 \), \( i = 1, 2, 3, 4 \) and hence the result. \( \square \)

4.7 THEOREM: If \( n \equiv 0 \pmod{10} \) and \( n \not= 20 \), \( A_i | D K_n \) for \( i = 1, 2, 3, 4 \).

**Proof:** The result for \( n = 10 \) has been proved in Lemma 4.5. Hence, let \( n = 10k, k > 2 \). We write
\[
K_n = K_{10k} = kK_{10} \cup k^2K_{10}
\]
where the vertex set of the complete multipartite graph \( kK_{10} \) is chosen appropriately. In view of Lemma 4.5, it is enough to show that \( A_i | D_{kK_{10}} \) for \( i = 1, 2, 3, 4 \). With \( kK_{10} \) we associate a graph \( G \) as follows.
Let $V_i = \{u_{1}^{i}, u_{2}^{i}, \ldots, u_{10}^{i}\}$, $1 \leq i \leq k$, be the maximal independent subsets of vertices in $k\overline{K}_{10}$. Also, for $i = 1, 2, \ldots, k$ let

$$S_i = \{u_{1}^{i}, u_{2}^{i}, u_{3}^{i}, u_{4}^{i}, u_{5}^{i}\},$$

$$S_{i+k} = \{u_{6}^{i}, u_{7}^{i}, u_{8}^{i}, u_{9}^{i}, u_{10}^{i}\}.$$  

We define $V(G) = \{S_1, S_2, \ldots, S_{2k}\}$ and an edge incident with $S_i$ and $S_j$ corresponds to the set of all edges of the complete bipartite subgraph $|S_i|, |S_j|$ of $k\overline{K}_{10}$. We observe that any two vertices $S_i, S_j$ are adjacent except when $j = k + i$. Moreover, the edges $S_i S_{i+k}$, $1 \leq i \leq k$, form a 1-factor of $G$. Thus $G \cong K_{2k} - I$ where $I$ is a 1-factor of $K_{2k}$. By Cor. 2.13 of Chapter 2, $G$ can be decomposed into 3-cycles and 5-cycles, which amounts to the fact that the graph $k\overline{K}_{10}$ can be decomposed into the factors $3K_5$ and $C_5[\overline{K}_5]$. Both these factors can be decomposed into $C_5$'s by Prop. 1.7 and Prop. 1.5 of Chapter 1, respectively. Thus $C_5[k\overline{K}_{10}]$ and the result then follows from the fact that each orientation $A_i$, $i = 1, 2, 3, 4$ of $C_5$ is self-converse. $\diamond$
4.8 THEOREM: \( A_i \mid D(2K_{10}) \) for \( i = 1, 2, 3, 4 \).

Proof: We write

\[
K_{20} = 2K_{10} \cup 2K_{10}
\]

with the vertex set of the complete bipartite graph \( 2K_{10} \) chosen appropriately. We shall show that \( A_i \mid D(K_{10} \cup 2K_{10}) \) for \( i = 1, 2, 3, 4 \). This together with Lemma 4.5 will prove the result.

Let the vertex sets of the two \( K_{10} \)'s be \( \{u_1, u_2, \ldots, u_{10}\} \) and \( \{v_1, v_2, \ldots, v_{10}\} \). The graph \( K_{10} \cup 2K_{10} \) can be decomposed into a graph \( \Lambda \), as shown in the Figure 4(K).

![Figure 4(K)](image-url)
and twenty five 5-cycles given by

\[ C: u_1, v_1, u_2, v_3, v_4, u_1, \quad C': u_1, v_3, u_7, v_2, v_5, u_1, \]

\[ C'': u_1, v_8, u_10, v_7, v_9, u_1, \]

\[ \phi^k C, \quad \phi^k C' \quad (1 \leq k \leq 9) \quad \text{and} \quad \phi^{2k} C'' \quad (1 \leq k \leq 4) \]

where \( \phi = (1 \ 2 \ 3 ... \ 10) \) is a cyclic permutation acting on the ten subscripts.

Here we give a decomposition of \( D_9 \) into each of \( A_2, A_3 \) and \( A_4 \).

Figure continued.
Figure continued.
$A_3 \upharpoonright \Delta A$

FIGURE 4(M)

Figure continued.
Now to show that $A_1 | D(K_{10} \cup 2K_{10})$, let $\Gamma$ be a graph defined by

$$\Gamma = \Lambda - C^* + C' + \phi^4 C^*$$

where $C^*$ is the 5-cycle $v_2, v_6, v_{10}, v_4, v_8, v_2$. Then $K_{10} \cup 2K_{10}$
can be decomposed into the graph $\Gamma$ and twenty four disjoint 5-cycles:

$$C^*, C, C'', \phi^k C, \phi^k C' \quad (1 \leq k \leq 9), \phi^2 C'', \phi^6 C'' \text{ and } \phi^8 C''.$$ 

Since $A_1$ is self-converse, it is enough to show that $A_1 | D\Gamma$. Following is one such decomposition:

Figure continued.
\[ A_1 | D \Gamma \]

FIGURE 40
Finally, the results of Theorems 4.1, 4.3, 4.6, 4.7 and 4.8 can be put together into a single theorem.

4.9 Theorem: \( a_i | D_k n \), \( n \geq 5 \), for \( i = 1, 2, 3, 4 \) if and only if \( n \equiv 0 \) or 1 (mod 5).
CHAPTER 5

In Chapter 4 we have seen that each orientation of \( C_5 \), the cycle of length five divides the complete symmetric digraph \( D_{K_n} \) for all \( n \)'s which satisfy the necessary conditions. In this chapter we consider the case of the cycle \( C_k \) where \( k \) is any odd positive integer and give some sufficient conditions so that any self-converse orientation \( X \) of \( C_k \) divides \( D_{K_n} \) for \( n \equiv 0 \) or 1 (mod \( k \)). As an application of these results, we show that the necessary conditions are also sufficient for any self-converse orientation of \( C_7 \), the cycle of length seven, to divide \( D_{K_n} \).

Theorem 5.2 does not include the case for \( k = 3 \), as we know that the directed cycle \( C_3 \) does not divide \( D_{K_6} \) (see [21]) which is one of the conditions of the hypothesis. We prove the result for odd \( k > 3 \) and for the case \( k = 3 \) we refer to Hung and Mendelsohn [21] who have shown that each orientation \( X \) of \( C_3 \) divides \( D_{K_n} \) if and only if \( n \equiv 0 \) or 1 (mod 3) and \( n \geq 3 \) with the exception of \( n = 6 \) in the case when \( X \) is the directed cycle \( C_3 \).

5.1 LEMMA: Let \( X \) be a self-converse orientation of \( C_k \), the cycle of length \( k \), and \( G \) be any graph. If \( C_k | G \), then \( X | DG \).

Proof: The proof follows from the fact that the orientation \( X \) of the cycle \( C_k \) is self-converse.
5.2 THEOREM: Let \( x \) be a self-converse orientation of \( C_k \), a cycle of length \( k \) where \( k \) is an odd positive integer greater than 3, such that

(i) \( x | D_{k+1} \)
(ii) \( x | D_{2k} \)

and (iii) \( x | D_{k,k,2k} \). Then

\( x | D_k \) if \( n \equiv 0 \) or \( 1 \) (mod \( k \)) and \( n \geq k \).

Proof: To prove that the conditions \( n \equiv 0 \) or \( 1 \) (mod \( k \)) and \( n \geq k \) are sufficient, we consider the three cases \( n \equiv 1 \), \( k \) (mod \( 2k \)), \( n \equiv k+1 \) (mod \( 2k \)), and \( n \equiv 0 \) (mod \( 2k \)) separately.

Case 1: Let \( n \equiv 1 \) or \( k \) (mod \( 2k \)). In this case Rosa [30] has shown that \( C_k | K_n \). Since \( x \) is a self-converse orientation of \( C_k \), the result then follows from Lemma 5.1.

Case 2: Let \( n \equiv k+1 \) (mod \( 2k \)), that is, \( n = 2mk + (k + 1) \) for some non-negative integer \( m \). We write

\[
K_n = K_{(2m+1)k+1} = [(2m+1)K_{k} + K_1] \cup 2m+1K_k
\]

\[
= (2m+1)K_{k+1} \cup 2m+1K_k
\]

where the vertex set of the complete multipartite graph \( 2m+1K_k \) is appropriately chosen. Since \( x | D_{k+1} \) by hypothesis, it is enough to show that \( x | D(2m+1K_k) \). By Theorem 2.14, \( C_k | 2m+1K_k \) and the result then follows from Lemma 5.1.
Case 3: Finally let \( n \equiv 0 \pmod{2k} \), that is, \( n = 2mk \) for some non-negative integer \( m \). For \( m = 1 \), the result is true by hypothesis as \( X|DK_{2k} \). In case \( m = 2 \), we write

\[
K_{4k} = 2K_{k} \cup K_{2k} \cup K_{k,k,2k}
\]

where the vertex set of each graph is appropriately chosen. Since \( k \) is odd, we know \( K_{k} \) can be decomposed into hamiltonian cycles, that is, \( C_{k}|K_{k} \) and hence by Lemma 5.1 \( X|DK_{k} \). Moreover, \( X|DK_{2k} \) and \( X|DK_{k,k,2k} \) by hypothesis and hence the result.

Now let \( m \geq 3 \). We write

\[
K_{n} = K_{2mk} = mK_{2k} \cup mK_{2k}
\]

where the vertex set of the complete multipartite graph is appropriately chosen. Since \( X|DK_{2k} \) by hypothesis, it is enough to show that \( X|D(K_{m2k}) \). We will show that \( C_{k}|mK_{2k} \) and the result then will follow from Lemma 5.1. With the graph \( mK_{2k} \) we associate a graph \( G \) as follows.

Let \( V_{1}, V_{2}, \ldots, V_{m} \) be \( m \) maximal independent subsets of \( V(K_{m}) \). Clearly \( |V_{i}| = 2k \) for \( 1 \leq i \leq m \). Also let

\[
V_{i} = \{ u_{i,1}^{i}, u_{i,2}^{i}, \ldots, u_{i,k}^{i}, u_{i,k+1}^{i}, \ldots, u_{i,2k}^{i} \} \text{ for } 1 \leq i \leq m.
\]

We define

\[
S_{i} = \{ u_{i,1}^{i}, u_{i,2}^{i}, \ldots, u_{i,k}^{i} \}
\]

and \( S_{m+1} = \{ u_{i,k+1}^{i}, u_{i,k+2}^{i}, \ldots, u_{i,2k}^{i} \} \).
for $1 \leq i \leq m$: Then

$$V(G) = \{S_1, S_2, \ldots, S_{2m}\}$$

and an edge $S_i S_j$ corresponds to the set of all edges in the original graph $K_{2m}$ which join vertices in $S_i$ with vertices in $S_j$. We observe, $S_i S_j$ is an edge for all $1 \leq i, j \leq 2m$, $i \neq j$ except when $j = m + i$ for $1 \leq i \leq m$. But $S_i S_{m+i}$, $1 \leq i \leq m$ are not edges of $G$, and form a 1-factor of $\overline{G}$, the complement of $G$. Thus, $G$ is isomorphic to $K_{2m-I}$ where $I$ is a 1-factor of $K_{2m}$. If $2m \equiv 0 \mod 2$ or $2 \mod 6$, then $2m \geq 6$, as $m \geq 3$ and hence by Prop. 1.3, $G$ can be decomposed into $C_3$'s. If $2m \equiv 4 \mod 6$, then $2m \geq 10$, as $m \geq 3$ and hence by Cor. 2.12 $G$ can be decomposed into one $K_5$ and the rest $C_3$'s. Hence for $m \geq 3$, $G$ can be decomposed into $C_3$'s and at most one $K_5$, which amounts to the fact that the graph $K_{2m}$ can be decomposed into $3K$'s and at most one $5K$. Since $k$ is odd, $C_k \upharpoonright 3K_k$ and $C_k \upharpoonright 5K_k$ by Theorem 2.14. This completes the proof.

Let $C_k^*$, $k$ odd, denote the directed cycle of length $k$.

Bermond and Faber [5] have shown the existence of a balanced circuit design $(k+1,k,1)$ for any odd $k \geq 3$. Also Bermond and Sotteau [6] have shown the existence of a balanced circuit design $(2k,k,1)$ for $k \geq 5$. In other words, $C_k^* \upharpoonright DK_{k+1}$ for any odd $k \geq 3$ and $C_k^* \upharpoonright DK_{2k}$ for any odd $k \geq 5$. Moreover, Sotteau [33] has shown that $C_k^* \upharpoonright DK_{k,k,2k}$ for any odd $k \geq 5$. In view of the fact,
that the directed cycle $C^*_k$ is a self-converse orientation of $C_k$, the cycle of length $k$. Theorem 5.2 gives the following result of Sotteau [33].

5.3 THEOREM: $DK_n$ can be decomposed into $k$-circuits (directed cycles of length $k$), $k$ odd, $k \geq 5$ if $n \equiv 0$ or $1 \pmod{k}$ and $n \geq k$.

In case $k = p^a$ where $p$ is an odd prime and $a$ a positive integer, the necessary conditions for a self-converse orientation $X$ of $C_{p^a}$ to divide $DK_n$ are $n \equiv 0$ or $1 \pmod{p^a}$ and $n > p^a$. Thus Theorem 5.2 gives the following in this case.

5.4 THEOREM: Let $X$ be a self-converse orientation of $C_{p^a}$, where $p$ is an odd prime and $a$ a positive integer, such that

(i) $X|DK_{p^a+1}$

(ii) $X|DK_{2 \cdot p^a}$

and (iii) $X|DK_{p^a, p^a, 2 \cdot p^a}$. Then $X|DK_n$ if and only if $n \equiv 0$ or $1 \pmod{p^a}$ and $n > p^a$.

As an application of Theorem 5.4, we show that a self-converse orientation $X$ of $C_7$ divides $DK_n$ if and only if $n \equiv 0$ or $1 \pmod{7}$ and $n \geq 7$. We know that $C_7$ has ten orientations and they are shown in Figure 5(A).
The orientations $A_{21}$ and $A_{22}$ have a path of maximum length 2; $A_{31}$, $A_{32}$, and $A_{33}$ have a path of maximum length 3; $A_{41}$ and $A_{42}$ have a path of maximum length 4; $A_{51}$ and $A_{61}$ have paths of maximum length 5 and 6 respectively and $A_{71}$ is the directed cycle.
Out of these ten orientations of $C_7$, $A_{21}$, $A_{22}$, $A_{31}$, $A_{41}$, $A_{42}$, $A_{51}$, $A_{61}$ and $A_{71}$ are the only self-converse orientations.

5.5 THEOREM: Let $x$ be one of the eight self-converse orientations of $C_7$, then $x|DK_n$ if and only if $n \equiv 0$ or $1 \pmod{7}$ and $n \geq 7$.

Proof: In view of the Theorem 5.4, it is enough to show that $X|DK_8$, $X|DK_{14}$ and $X|DK_{7,7,14}$. Henceforth we shall write the eight self-converse orientations as:

\begin{align*}
A_{21} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{22} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{31} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{41} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{42} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{51} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{61} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a \\
A_{71} & : a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow a
\end{align*}

(i) Let $V(DK_8) = \{u_1, u_2, u_3, \ldots, u_8\}$. We list below a decomposition of $DK_8$ into each $X$. The direction of an edge is as given by the top cycle.
| $A_{21}$: $a \to b \to c \to d \to e \to f \to g \to a$ |
|---|---|---|---|---|---|---|---|
| $u_2$ | $u_3$ | $u_5$ | $u_8$ | $u_4$ | $u_6$ | $u_2$ |
| $u_5$ | $u_8$ | $u_1$ | $u_6$ | $u_4$ | $u_3$ | $u_7$ |
| $u_4$ | $u_5$ | $u_7$ | $u_2$ | $u_1$ | $u_6$ | $u_8$ |
| $u_3$ | $u_2$ | $u_7$ | $u_1$ | $u_5$ | $u_6$ | $u_8$ |
| $u_7$ | $u_6$ | $u_2$ | $u_8$ | $u_3$ | $u_1$ | $u_7$ |
| $u_4$ | $u_3$ | $u_4$ | $u_1$ | $u_7$ | $u_8$ | $u_2$ |
| $u_3$ | $u_1$ | $u_8$ | $u_4$ | $u_5$ | $u_6$ | $u_3$ |
| $u_6$ | $u_7$ | $u_1$ | $u_4$ | $u_5$ | $u_3$ | $u_6$ |

| $A_{22}$: $a \to b \to c \to d \to e \to f \to g \to a$ |
|---|---|---|---|---|---|---|---|
| $u_2$ | $u_3$ | $u_5$ | $u_8$ | $u_7$ | $u_4$ | $u_6$ | $u_2$ |
| $u_7$ | $u_5$ | $u_8$ | $u_1$ | $u_6$ | $u_4$ | $u_3$ | $u_7$ |
| $u_4$ | $u_5$ | $u_7$ | $u_2$ | $u_1$ | $u_6$ | $u_8$ | $u_4$ |
| $u_1$ | $u_7$ | $u_2$ | $u_3$ | $u_8$ | $u_6$ | $u_5$ | $u_1$ |
| $u_6$ | $u_7$ | $u_1$ | $u_4$ | $u_3$ | $u_8$ | $u_2$ | $u_6$ |
| $u_3$ | $u_1$ | $u_4$ | $u_5$ | $u_2$ | $u_8$ | $u_7$ | $u_3$ |
| $u_8$ | $u_1$ | $u_3$ | $u_6$ | $u_5$ | $u_2$ | $u_4$ | $u_8$ |
| $u_5$ | $u_3$ | $u_6$ | $u_7$ | $u_4$ | $u_2$ | $u_1$ | $u_5$ |

| $A_{31}$: $a \to b \to c \to d \to e \to f \to g \to a$ |
|---|---|---|---|---|---|---|---|
| $u_6$ | $u_8$ | $u_1$ | $u_2$ | $u_4$ | $u_7$ | $u_3$ | $u_6$ |
| $u_5$ | $u_3$ | $u_2$ | $u_1$ | $u_7$ | $u_4$ | $u_8$ | $u_5$ |
| $u_8$ | $u_2$ | $u_3$ | $u_4$ | $u_6$ | $u_1$ | $u_5$ | $u_8$ |
| $u_7$ | $u_5$ | $u_4$ | $u_3$ | $u_1$ | $u_6$ | $u_2$ | $u_7$ |
| $u_2$ | $u_4$ | $u_5$ | $u_6$ | $u_3$ | $u_7$ | $u_2$ | $u_4$ |
| $u_1$ | $u_7$ | $u_6$ | $u_5$ | $u_3$ | $u_8$ | $u_4$ | $u_1$ |
| $u_4$ | $u_6$ | $u_7$ | $u_8$ | $u_2$ | $u_5$ | $u_1$ | $u_4$ |
| $u_3$ | $u_1$ | $u_8$ | $u_7$ | $u_5$ | $u_2$ | $u_6$ | $u_3$ |

| $A_{41}$: $a \to b \to c \to d \to e \to f \to g \to a$ |
|---|---|---|---|---|---|---|---|
| $u_4$ | $u_7$ | $u_8$ | $u_5$ | $u_3$ | $u_2$ | $u_6$ | $u_4$ |
| $u_1$ | $u_6$ | $u_4$ | $u_3$ | $u_7$ | $u_5$ | $u_8$ | $u_1$ |
| $u_6$ | $u_1$ | $u_2$ | $u_7$ | $u_5$ | $u_4$ | $u_8$ | $u_6$ |
| $u_3$ | $u_8$ | $u_6$ | $u_5$ | $u_1$ | $u_7$ | $u_2$ | $u_3$ |
| $u_8$ | $u_3$ | $u_4$ | $u_1$ | $u_7$ | $u_6$ | $u_2$ | $u_8$ |
| $u_5$ | $u_2$ | $u_8$ | $u_7$ | $u_3$ | $u_1$ | $u_4$ | $u_5$ |
| $u_2$ | $u_5$ | $u_6$ | $u_3$ | $u_1$ | $u_8$ | $u_4$ | $u_2$ |
| $u_7$ | $u_4$ | $u_2$ | $u_1$ | $u_5$ | $u_3$ | $u_6$ | $u_7$ |
The following directed cycles give a decomposition of $DK_8$ into $A$'s.

\begin{align*}
{A}_{42}: & \quad a + b + c + d + e + f + g + a \\
\text{(Cycle)}: & \quad u_6 \ u_3 \ u_7 \ u_4 \ u_2 \ u_1 \ u_8 \ u_6 \\
{A}_{51}: & \quad a + b + c + d + e + f + g + a \\
\text{(Cycle)}: & \quad u_4 \ u_7 \ u_8 \ u_5 \ u_3 \ u_2 \ u_6 \ u_4 \\
{A}_{61}: & \quad a + b + c + d + e + f + g + a \\
\text{(Cycle)}: & \quad u_4 \ u_7 \ u_8 \ u_5 \ u_3 \ u_2 \ u_6 \ u_4 \\
{A}_{71}: & \quad a + b + c + d + e + f + g + a \\
\text{(Cycle)}: & \quad u_8 \ u_3 \ u_4 \ u_1 \ u_7 \ u_6 \ u_2 \ u_8 \\
\text{The following directed cycles give a decomposition of } DK_8 \text{ into } A_{71}'s. \\
\text{(Cycle)}: & \quad u_1 + u_5 + u_2 + u_6 + u_3 + u_7 + u_4 + u_1
\end{align*}
and

\[ u_8 = u_i = u_{i+6} = u_{i+1} = u_{i+4} = u_{i+2} = u_{i+3} + u_8 \]

for \( i = 1, 2, \ldots, 7 \) and the subscripts are taken modulo 7.

(ii) Let \( V(DK_{14}) = \{u_0, u_1, \ldots, u_{13}\} \). The following gives a decomposition of \( DK_{14} \) into \( A_2 \)

\[ u_0 = u_i + u_{i+1} + u_{i+2} + u_{i+3} + u_{i+4} + u_{i+5} + u_{i+6} + u_{i+7} + u_{i+8} + u_{i+9} + u_{i+10} + u_{i+11} + u_{i+12} + u_{i+13} \]

and

\[ u_i + u_{i+1} + u_{i+2} + u_{i+3} + u_{i+4} + u_{i+5} + u_{i+6} + u_{i+7} + u_{i+8} + u_{i+9} + u_{i+10} \]

for \( i = 1, 2, \ldots, 13 \) where the subscripts are taken modulo 13.

To get the decomposition of \( DK_{14} \) into the other self-converse orientations of \( C_7 \) we write

\[ K_{14} = K_8 + K_6 \]

where \( V(K_8) = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_\infty\} \)

and \( V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \)

using all the edges of the form \( u_i v_j, 1 \leq i, j \leq 6 \) and the six edges of the 6-cycle
we construct the following six 7-cycles

\[ C_i : v_i, u_i, v_{i+1}, u_{i+2}, v_{i+4}, u_{i+1}, v_{i+5}, v_i \]

for \( 1 \leq i \leq 6 \) where the subscripts are taken modulo 6. Then \( K_{14} \) can be decomposed into six 7-cycles \( C_i, 1 \leq i \leq 6 \), and a graph \( H \) shown in Figure 5(B).
In view of Lemma 5.1, it is enough to show that each self-converse orientation \( X \) divides \( \text{DH} \). Here we list a decomposition of \( \text{DH} \)

into, \( \lambda_{22}, \lambda_{31}, \lambda_{41}, \lambda_{42}, \lambda_{51}, \lambda_{61} \) and \( \lambda_{71} \).

In each decomposition below the subscripts are to be taken modulo 6 except the subscript 0 and \( \infty \) which remain the same in each orientation. The cycles

\[
v_i + u_\infty + v_{i+5} + v_{i+1} + v_{i+4} + u_0 + v_{i+2} + v_i
\]

for \( 1 \leq i \leq 6 \) give a decomposition of \( \text{DH} \) into \( \lambda_{22} \). The cycles

\[
v_i + u_\infty + v_{i+3} + v_{i+1} + v_{i+4} + u_0 + v_{i+2} + v_i
\]

for \( 1 \leq i \leq 6 \) give a decomposition of \( \text{DH} \) into \( \lambda_{31} \). The cycles

\[
v_i + u_\infty + v_{i+2} + u_0 + v_{i+3} + v_{i+1} + v_{i+4} + v_i
\]

for \( 1 \leq i \leq 6 \) give a decomposition of \( \text{DH} \) into \( \lambda_{41} \). The cycles

\[
v_i + u_0 + v_{i+2} + v_{i+5} + v_{i+1} + u_\infty + v_{i+4} + v_i
\]

for \( 1 \leq i \leq 6 \) give a decomposition of \( \text{DH} \) into \( \lambda_{42} \). The cycles

\[
v_i + u_0 + v_{i+4} + v_{i+2} + u_\infty + v_{i+5} + v_{i+3} + v_i
\]

for \( 1 \leq i \leq 6 \) give a decomposition of \( \text{DH} \) into \( \lambda_{51} \). The cycles

\[
v_i + u_\infty + v_{i+3} + v_{i+1} + v_{i+4} + u_0 + v_{i+2} + v_i
\]
for $1 \leq i \leq 6$ give a decomposition of $DH$ into $A_{61}$ and 

finally, the cycles 

$$u_0 + v_i + v_{i+4} + v_{i+1} + v_{i+3} + u_\infty + v_{i+5} + u_0$$

for $1 \leq i \leq 6$ give a decomposition of $DH$ into $A_{71}$.

(iii) Let $V(DK_{7,7,14}) = \{u_1, u_2, \ldots, u_7\} \cup \{v_1, v_2, \ldots, v_7\} \cup \{w_1, w_2, \ldots, w_7, z_1, z_2, \ldots, z_7\}$

where the three sets in the union are the maximal independent subsets of $V(DK_{7,7,14})$. The undirected graph $K_{7,7,14}$ with $V(K_{7,7,14}) = V(DK_{7,7,14})$ can be decomposed into twenty-eight 7-cycles given below and graph $G$, which is the subgraph induced by the remaining edges of $K_{7,7,14}$. The cycles are

$$u_i, w_i, u_{i+1}, w_{i+2}, u_{i+4}, v_{i+5}, z_i, u_i$$

$$v_i, z_i, v_{i+1}, z_{i+2}, v_{i+4}, u_{i+5}, w_i, v_i$$

$$u_i, v_{i+2}, u_{i+2}, v_{i+5}, w_i, v_{i+6}, v_{i+4}, z_{i+1}, u_i$$

$$u_i, z_{i+2}, u_{i+4}, z_{i+3}, v_i, w_{i+3}, v_i, v_{i+4}, u_i$$

for $1 \leq i \leq 7$, where the subscripts are taken modulo 7. In view of Lemma 5.1, to prove that a self-converse orientation $X$ divides...
it is enough to show that \( X \) divides \( DG \). Here we list a decomposition of \( DG \) into the orientations \( A_{21}, A_{22}, A_{31} \) and \( A_{41} \).

In each decomposition below the subscripts are to be taken modulo 7. The cycles

\[
\begin{align*}
&\quad w_i + u_{i+3} + z_i + u_{i+4} + w_{i+1} + u_{i+5} + v_{i+3} + w_i \\
\end{align*}
\]

and

\[
\begin{align*}
&\quad w_i + v_{i+2} + w_{i+6} + v_{i+1} + u_{i+3} + z_i + u_{i+4} + w_i \\
\end{align*}
\]

for \( 1 \leq i \leq 7 \) give a decomposition of \( DG \) into \( A_{21} \). The cycles

\[
\begin{align*}
&\quad z_i + u_{i+3} + w_i + v_{i+3} + u_{i+5} + w_{i+1} + u_{i+4} + z_i \\
\end{align*}
\]

and

\[
\begin{align*}
&\quad w_i + u_{i+4} + z_i + u_{i+3} + v_{i+1} + w_{i+6} + v_{i+2} + w_i \\
\end{align*}
\]

for \( 1 \leq i \leq 7 \) give a decomposition of \( DG \) into \( A_{22} \). The cycles

\[
\begin{align*}
&\quad v_{i+3} + u_{i+5} + w_{i+1} + u_{i+4} + z_i + u_{i+3} + w_i + v_{i+3} \\
\end{align*}
\]

and

\[
\begin{align*}
&\quad u_{i+3} + v_{i+1} + w_{i+6} + v_{i+2} + w_i + u_{i+4} + z_i + u_{i+3} \\
\end{align*}
\]
for $1 \leq i \leq 7$ give a decomposition of $DG$ into $A_{31}$. The cycles

$$v_{i+3} \rightarrow u_{i+5} \rightarrow w_{i+1} \rightarrow u_{i+4} \rightarrow z_i \rightarrow u_{i+3} \rightarrow w_i \rightarrow v_{i+3}$$

and

$$z_i \rightarrow u_{i+3} \rightarrow v_{i+1} \rightarrow w_{i+6} \rightarrow v_{i+2} \rightarrow w_i \rightarrow u_{i+4} \rightarrow z_i$$

for $1 \leq i \leq 7$ give a decomposition of $DG$ into $A_{41}$.

Now we give a decomposition of $DK_{7,7,14}$ into each of the remaining orientations $A_{42}$, $A_{51}$, $A_{61}$ and $A_{71}$. This time we decompose the undirected graph $K_{7,7,14}$ into fourteen 7-cycles given below and a graph $G'$, which is the subgraph induced by the remaining edges of $K_{7,7,14}$. The cycles are

$$v_i, w_{i+6}, v_{i+1}, v_{i+5}, v_{i+2}, w_{i+4}, u_{i+4}, v_i$$

and

$$w_i, u_{i+6}, v_{i+1}, u_{i+5}, w_{i+2}, u_{i+4}, v_{i+6}, w_i$$

for $1 \leq i \leq 7$ where the subscripts are taken modulo 7. To prove that a self-converse orientation $X$ divides $DK_{7,7,14}$ because of Lemma 5.1 it is enough to show that $X$ divides $DG'$. Here we list a decomposition of $DG'$ into each of $A_{42}$, $A_{51}$, $A_{61}$ and $A_{71}$. In each decomposition the subscripts are to be taken modulo 7. The cycles
for \(1 \leq i \leq 7\) give a decomposition of \(D_G'\) into \(A_{42}\). The cycles

\[
v_i \rightarrow z_1 \rightarrow v_{i+1} \rightarrow z_2 \rightarrow v_{i+2} \rightarrow z_3 \rightarrow u_{i+3} \rightarrow v_i
\]

\[
u_i \rightarrow v_i \rightarrow u_{i+2} \rightarrow z_1 \rightarrow u_{i+6} \rightarrow v_{i+5} \rightarrow z_7 \rightarrow u_i
\]

\[
v_{i+1} \rightarrow z_6 \rightarrow v_i \rightarrow w_i \rightarrow u_{i+1} \rightarrow z_2 \rightarrow u_i \rightarrow v_i \rightarrow v_{i+1}
\]

\[
v_i \rightarrow z_4 \rightarrow v_{i+1} \rightarrow z_5 \rightarrow v_{i+2} \rightarrow u_{i+5} \rightarrow z_3 \rightarrow v_i
\]

\[
u_{i+6} \rightarrow z_4 \rightarrow u_{i+2} \rightarrow v_i \rightarrow u_i \rightarrow z_7 \rightarrow v_{i+5} \rightarrow u_{i+6}
\]

\[
u_{i+2} \rightarrow z_5 \rightarrow u_{i+1} \rightarrow w_i \rightarrow v_i \rightarrow u_{i+6} \rightarrow z_6 \rightarrow u_{i+2}
\]

for \(1 \leq i \leq 7\) give a decomposition of \(D_G'\) into \(A_{51}\). The cycles

\[
v_i \rightarrow z_1 \rightarrow v_{i+1} \rightarrow z_2 \rightarrow v_{i+2} \rightarrow z_3 \rightarrow u_{i+3} \rightarrow v_i
\]

\[
u_i \rightarrow v_i \rightarrow u_{i+2} \rightarrow z_1 \rightarrow u_{i+6} \rightarrow v_{i+5} \rightarrow z_7 \rightarrow u_i
\]

\[
v_i \rightarrow z_6 \rightarrow v_{i+1} \rightarrow u_i \rightarrow z_2 \rightarrow u_{i+1} \rightarrow w_i \rightarrow v_i
\]

\[
z_3 \rightarrow v_i \rightarrow z_4 \rightarrow v_{i+1} \rightarrow z_5 \rightarrow v_{i+2} \rightarrow u_{i+5} \rightarrow z_3
\]

\[
v_{i+5} \rightarrow u_{i+6} \rightarrow z_4 \rightarrow u_{i+2} \rightarrow v_i \rightarrow u_i \rightarrow z_7 \rightarrow v_{i+5}
\]

\[
u_{i+1} \rightarrow z_5 \rightarrow u_{i+2} \rightarrow z_6 \rightarrow u_{i+6} \rightarrow v_i \rightarrow w_i \rightarrow u_{i+1}
\]
for $1 \leq i \leq 7$ give a decomposition of $DG'$ into $A_{61}$. The cycles

$$v_i + z_1 + v_{i+1} + z_2 + v_{i+2} + z_3 + u_{i+3} + v_i$$
$$u_i + v_i + u_{i+2} + z_1 + u_{i+6} + v_{i+5} + z_7 + u_i$$
$$v_i + z_6 + v_{i+1} + u_i + z_2 + u_{i+1} + w_i + v_i$$
$$u_{i+5} + z_3 + v_i + z_4 + v_{i+1} + z_5 + v_{i+2} + u_{i+5}$$
$$z_7 + v_{i+5} + u_{i+6} + z_4 + u_{i+2} + v_i + u_i + z_7$$
$$w_i + u_{i+1} + z_5 + u_{i+2} + z_6 + u_{i+6} + v_i + w_i$$

for $1 \leq i \leq 7$ give a decomposition of $DG'$ into $A_{71}$. This completes the proof of the theorem.
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