VARIETIES OF LATTICE ORDERED GROUPS

by

Roger Wroblewski

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics

© Roger Wroblewski 1979

SIMON FRASER UNIVERSITY

April 1979

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.
APPROVAL

Name: Roger Wroblewski
Degree: Master of Science
Title of Thesis: Varieties of Lattice Ordered Groups

Examining Committee:
Chairman: Dr. S.K. Thomason

N.R. Reilly
Senior Supervisor

A.R. Freeman

J. Sember

A. Das
External Examiner

Date Approved: April 26, 1979
PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis or dissertation (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this thesis for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis

VARIETIES OF LATTICE-ORDERED GROUPS

Author:

(signature)

ROGER WROBLEWSKI

(name)

MAY 1/79

(date)
ABSTRACT

In approaching the study of 1-groups, there are two well known methods. First, the study of the internal structure (the subgroups, the elements, etc.) of 1-groups. By this method the answers are given to questions such as: "When does the triangle of inequality hold in a 1-group?"

The second method is concerned with an external view of 1-groups, i.e. the study of collections of 1-groups which have some common attributes.

In Chapter I, the study of 1-groups begins with a consideration of the internal structure of 1-groups, and then (via Birkhoff's Theorem) progresses to a specification of the collections (varieties) of 1-groups and the common attributes (laws).

In Chapter II the mathematical tools (Holland's Representation Theorem, wreath products of ordered permutation groups) utilized in the study of varieties of 1-groups are presented and examined.

Several examples of varieties of 1-groups are presented in Chapter III. The containment relationships between the varieties are also examined.

Finally, in Chapter IV a mathematical structure is defined on the collection of all varieties of 1-groups. The structure consists of a binary operation called varietal multiplication, and an order relationship induced by the containment relationship presented in Chapter III. Then the connection between these two relationships is examined in the conclusion of Chapter IV.
I would like to thank Dr. N.R. Reilly for his supervision. Also, thanks to my dearest C.C..
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title page</td>
<td>i</td>
</tr>
<tr>
<td>Approval</td>
<td>ii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iv</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>v</td>
</tr>
<tr>
<td>I. L-groups and Universal Algebra</td>
<td>1</td>
</tr>
<tr>
<td>1. Lattice Ordered Groups</td>
<td>1</td>
</tr>
<tr>
<td>i) Riesz Decomposition Theorem</td>
<td>4</td>
</tr>
<tr>
<td>ii) Prime and Regular Subgroups</td>
<td>8</td>
</tr>
<tr>
<td>iii) L-homomorphisms</td>
<td>12</td>
</tr>
<tr>
<td>2. Birkhoff's Theorem</td>
<td>14</td>
</tr>
<tr>
<td>i) Varieties</td>
<td>17</td>
</tr>
<tr>
<td>ii) Laws of a Variety</td>
<td>22</td>
</tr>
<tr>
<td>iii) Birkhoff's Theorem</td>
<td>24</td>
</tr>
<tr>
<td>II. Ordered Permutation Groups</td>
<td>26</td>
</tr>
<tr>
<td>1. Lattice Ordered Permutation Groups</td>
<td>26</td>
</tr>
<tr>
<td>i) 0-k transitive</td>
<td>27</td>
</tr>
<tr>
<td>ii) 0-primitive</td>
<td>27</td>
</tr>
<tr>
<td>2. Holland's Representation Theorem</td>
<td>28</td>
</tr>
<tr>
<td>3. Wreath Products of Ordered Permutation Groups</td>
<td>32</td>
</tr>
<tr>
<td>i) Properties of the Wreath Product</td>
<td>37</td>
</tr>
<tr>
<td>ii) An Example</td>
<td>38</td>
</tr>
<tr>
<td>iii) An Embedding Theorem</td>
<td>39</td>
</tr>
</tbody>
</table>
4. Wreath Products of the Real Numbers

i) 0-primitive Components

ii) Wreath Product of 0-primitive Components

iii) Holder's Theorem

iv) An Application of Holder's Theorem and Wreath Products

v) Orbits

vi) Paired Orbits

vii) 0-2 transitive subgroups

III. Examples of Varieties of Lattice Ordered Groups

1. Examples

i) Abelian l-groups

ii) Representable l-groups

iii) Normal Valued l-groups

iv) Scrimger Varieties

2. Containment Relationships

i) The smallest proper variety of l-groups

ii) Varieties of l-groups which cover \( \mathbb{A} \)

iii) The largest proper variety of l-groups

IV. The Lattice \( \mathbb{L} \) of Varieties of Lattice Ordered Groups

1. Torsion Classes

i) Torsion Radical

ii) The Torsion Class

iii) Every l-group Variety is a Torsion Class
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Varietal Products</td>
<td>81</td>
</tr>
<tr>
<td>i) Multiplication is Associative</td>
<td>83</td>
</tr>
<tr>
<td>ii) The Idempotent</td>
<td>86</td>
</tr>
<tr>
<td>3. The Generation of Varieties</td>
<td>89</td>
</tr>
<tr>
<td>i) Definition of Mimics</td>
<td>90</td>
</tr>
<tr>
<td>ii) The Main Theorem</td>
<td>91</td>
</tr>
<tr>
<td>iii) $(Z,Z)$ Mimics</td>
<td>94</td>
</tr>
<tr>
<td>4. The Factorization of 1-group Varieties</td>
<td>97</td>
</tr>
</tbody>
</table>

Appendix                                       | 99   |

Bibliography                                   | 101  |
CHAPTER 1

L-groups and Universal Algebra

This chapter is devoted to the discussion of some of the basic properties of lattice ordered groups (1-groups), which are important requirements for the presentation of examples of varieties of 1-groups in Chapter 3. (For a fuller treatment of the properties of 1-groups see Conrad [2], and Fuchs [3].) Then, in the second section of this chapter the reader is introduced to the background material for Birkhoff's Theorem and to the proof of this theorem.

Section 1. Lattice Ordered Groups

A group \((G,+)\) endowed with a partial order \(\leq\) is a partially ordered group if for all \(a, b, x, y \in G\)

\[a \leq b \implies x + a + y \leq x + b + y.\]

If the partial order \(\leq\) is a lattice order, then \(G\) is called a lattice ordered group (1-group). If the partial order is a total order then \(G\) is called a totally ordered group (0-group). If, for \(a, b \in G\), an 1-group, one denotes by \(a \lor b\) (\(a \land b\)) the least upper bound or join (greatest lower bound or meet) of \(a\) and \(b\) then, for \(a, b, x, y \in G\),

\[x + (a \lor b) + y = (x + a + y) \lor (x + b + y)\]

and

\[x + (a \land b) + y = (x + a + y) \land (x + b + y).\]

(see Appendix)

With this introduction, we now exhibit some of the rich structure of 1-groups.
Proposition 1.1.1. A partially ordered group $G$ is an l-group, if for all $g \in G$, $g \lor 0 \in G$.

Proof: If $G$ is an l-group, then it is evident from the definition of $G$, that $g \lor 0 \in G$ for all $g \in G$.

Conversely, if $g \lor 0 \in G$ for all $g \in G$, then it is straightforward to show that

$$[(a-b) \lor 0] + b = a \lor b$$

and,

$$-[a + [(a-b) \lor 0]] = a \land b.$$  

This second equality follows from the identity

$$-(a \lor b) = (-a) \land (-b).$$  

(See Appendix)

Lemma 1.1.2. For an l-group $G$, and a positive integer $n$, if $ng \geq 0$ then $g \geq 0$.

Proof: By an induction argument it may be shown that

$$n(a \land 0) = na \land (n-1)a \land \ldots \land a \land 0.$$  

(See Appendix)

Now if $na \geq 0$ then $na \land 0 = 0$, and hence,

$$n(a \land 0) = na \land (n-1)a \land \ldots \land a \land 0$$

$$= (n-1)a \land \ldots \land a \land 0$$

$$= (n-1)(a \land 0).$$

Therefore

$$n(a \land 0) - (n-1)(a \land 0) = 0$$

or,

$$(a \land 0) = 0.$$
Proposition 1.1.3. For an 1-group $G$, and a positive integer $n$,

$$n(a \land 0) = (na) \land 0$$

and,

$$n(a \lor 0) = (na) \lor 0.$$ 

Proof: One should first note that for $k$ and $m$ non-negative integers, where $0 \leq k \leq m$; and $a \in G$

$$m[(m-k)a \lor -ka] = \ldots \lor [k(m-k)a + (m-k)(-ka)] \lor \ldots \geq 0.$$

By lemma 1.1.2 we then have

$$(m-k)a \lor (-ka) \geq 0 \quad (0 \leq k \leq m),$$

or

$$(ma) \lor 0 \geq ka \quad (0 \leq k \leq m).$$

Therefore

$$n(a \lor 0) = na \lor (n-1)a \lor \ldots \lor a \lor 0 = na \lor 0$$

and,

$$n(a \land 0) = -n(-a \lor 0)$$

$$= -((-na) \lor 0)$$

$$= na \land 0.$$ 

Corollary 1.1.4. For an 1-group $G$, and $a,b \in G$; if $a + b = b + a$

then

$$n(a \lor b) = na \lor nb$$

and

$$n(a \land b) = na \land nb.$$
Proof: Notice first that

\[ n(a \lor b) = n[(a-b) \lor 0] + b, \]

and since \( a \) and \( b \) commute, this implies that

\[ [(a-b) \lor 0] + b = b + [(a-b) \lor 0]. \]

Hence, \( n(a \lor b) = na \lor nb \), and \( n(a \land b) = na \land nb \).

Proposition 1.1.5. If \( G \) is an \( l \)-group, the cardinality of \( G \) is at least countable, \( (G \neq 0) \).

Proposition 1.1.6. For an \( l \)-group \( G \) and \( a,b,c \in G \), if \( a \lor c = b \lor c \) and \( a \land c = b \land c \) then \( a = b \).

Proof: Since \( a \lor c = a \land (a \land c) + c \), we have

\[ a = (a \lor c) = c + (a \land c) = (b \lor c) - c + (b \land c) = b. \]

Consequently, \( G \) is a distributive lattice since it does not contain a sublattice of the form

\[ \begin{align*}
&\text{or} \\
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet
\end{array}
\end{align*} \]

Proposition 1.1.7. Riesz Decomposition. For an \( l \)-group \( G \), if

\[ 0 < x \leq d_1 + \ldots + d_n, \quad \text{where} \quad 0 \leq d_i \quad (i=1,\ldots,n), \]

then there exist \( c_1,\ldots,c_n \) such that \( 0 \leq x = c_1 + c_2 + \ldots + c_n \), where \( 0 \leq c_i \leq d_i \)

for \( i = 1,2,\ldots,n \).

Proof: (By Induction). If \( 0 < x \leq d_1 + d_2 \), let \( c_1 = x \land d_1 \geq 0 \) and \( c_2 = -c_1 + x \geq 0 \). Then,
\[ 0 \leq x = c_1 + c_2 \]

and,

\[
c_2 = -c_1 + x = -(x \land d_1) + x = 0 \lor (-d_1 + x) \leq d_2.
\]

Now assume that the statement of the proposition is true for all positive integers \( k \) such that \( 0 \leq k \leq n \). Then for

\[ 0 \leq x \leq d_1 + d_2 + \ldots + d_n + d_{n+1}, \]

there exist \( c_1, c_2, \ldots, c_{n-1}, g \) such that,

\[
0 \leq x = c_1 + c_2 + \ldots + c_{n-1} + g 
\leq d_1 + d_2 + \ldots + d_{n-1} + (d_n + d_{n+1})
\]

where \( 0 \leq c_i \leq d_i \ (1 \leq i \leq n-1) \) and \( 0 \leq g \leq d_n + d_{n+1} \). By another application of the induction hypothesis, there exist \( c_n \) and \( c_{n+1} \) such that \( 0 \leq g = c_n + c_{n+1} \leq d_n + d_{n+1} \) where \( 0 \leq c_n \leq d_n \) and \( 0 \leq c_{n+1} \leq d_{n+1} \).

Now we consider the subgroups of an \( l \)-group \( G \). These play an important role in varieties of \( l \)-groups, and representations of \( l \)-groups.

A subgroup \( S \) of an \( l \)-group \( G \) is an \textbf{\( l \)-subgroup} of \( G \) if \( S \) is a sublattice of \( G \). An \( l \)-subgroup \( S \) of an \( l \)-group \( G \) is called a \textbf{convex \( l \)-subgroup} if \( 0 \leq g \leq c \in S \) and \( g \in G \) imply \( g \in S \).

A normal convex \( l \)-subgroup \( S \) of an \( l \)-group \( G \) is called an \textbf{\( l \)-ideal} of \( G \). An \( l \)-ideal plays the same role in the theory of \( l \)-groups as a normal subgroup plays in the theory of groups.
If one denotes by $G(a)$, $(a \in G, a \geq 0)$, the smallest convex $l$-subgroup of $G$ containing $a$, then $G(a) \land G(b) = G(a \land b)$. The inclusion one way is trivial. The other inclusion, $G(a) \land G(b) \subseteq G(a \land b)$, may be seen by noticing that for non-negative elements $a, b, c \in G$

$a \land (b + c) \leq (a \land b) + (a \land c)$. A more general result which will be used in later parts of the thesis is: for $M$ a convex $l$-subgroup of $G$, and two non-negative elements $a$ and $b$ in $G$, $G(M,a) \land G(M,b) = G(M,a \land b)$ where $G(M,x)$ denotes the smallest convex $l$-subgroup of $G$ containing $M$ and the non-negative element $x$.

For $C$ a convex $l$-subgroup of $G$, the right cosets of $C$ may be ordered as follows: $C + x \leq C + y$ if and only if there exists an element $b$ in $C$ such that $b + x \leq y$. It can be shown that $\leq$ is a partial order on the right cosets of $C$ in $G$, which we shall call the induced order.

The absolute value of an element $a \in G$ denoted by $|a|$ is equal to $a \lor -a$.

Theorem 1.1.8. For an $l$-subgroup $C$ of $G$, the following are equivalent.

1) $C$ is a convex $l$-subgroup of $G$.

2) The set of right cosets of $C$ form a distributive lattice, under the induced order with

$(C + x) \lor (C + y) = C + (x \lor y)$

and,

$(C + x) \land (C + y) = C + (x \land y)$.

3) If $g \in G$, $s \in C$ and $|g| \leq |s|$, then $g \in C$. 

6.
Proof: \((1 \rightarrow 2)\). Since \(x \lor y \geq x, y\); let \(b = 0\) in the definition of the partial order on the cosets of \(C\). Then,

\[ C + (x \lor y) \geq C + x, C + y. \]

Now if \(C + d \geq (C+x), (C+y)\), then there exist \(a, b \in C\) such that \(a + d \geq x\) and \(b + d \geq y\). Let \(s\) be an upper bound in \(C\) for both \(a\) and \(b\). Then \(s + d \geq x, y\), and hence \(s + d \geq x \lor y\). Consequently, \(C + d \geq C + (x \lor y)\). Therefore, the join \((C+x) \lor (C+y)\) exists and is equal to \(C + (x \lor y)\). Similarly, \((C+x) \land (C+y) = C + (x \land y)\).

To see that the right cosets of \(C\) form a distributive lattice, we use the above results, and the fact that \(G\) is a distributive lattice.

\[
[(C+x) \land (C+y)] \lor (C+z) = [C + (x \land y)] \lor (C+z)
\]

\[
= C + [(x \land y) \lor z]
\]

\[
= [C + (x \lor z)] \land [C + (y \lor z)]
\]

\[
= [(C+x) \lor (C+z)] \land [(C+y) \lor (C+z)]
\]

The dual is similarly proved.

\((2 \rightarrow 3)\). For \(g \in G\) and \(s \in C\) if \(|g| \leq |s|\) then,

\[-|s| \leq g \leq |s| \quad (|g| \geq -g, g). \]

Hence,

\[ C = C + (-|s|) \leq C + g \leq C + |s| = C, \]

and then \(g \in C\).
If \( 0 \leq g \leq s \) where \( g \in G \) and \( s \in C \), then \( |g| \leq |s| \) and by the hypothesis, this implies \( g \in C \). Likewise for \( s \in C \), \( 0 \leq |s \lor 0| \leq |s| \) so that \( s \lor 0 \in C \), and consequently \( C \) is an \( 1 \)-subgroup of \( G \).

**Corollary 1.1.9.** If \( M \) is an \( 1 \)-ideal of \( G \), then the right cosets of \( M \) form an \( 1 \)-group.

A convex \( 1 \)-subgroup \( M \) of an \( 1 \)-group \( G \) is called **regular**, if \( M \) is maximal with respect to not containing an element \( g \neq 0 \) of \( G \). In this case, \( M \) is also called a **value** of \( g \). By Zorn's lemma each non-zero element of \( G \) has a value.

A convex \( 1 \)-subgroup \( P \) of \( G \) is called a **prime subgroup** if for \( A, B \) convex \( 1 \)-subgroups of \( G \) \( M \supseteq A \cap B \) implies that either \( M \supseteq A \) or \( M \supseteq B \).

**Theorem 1.1.10.** For an \( 1 \)-group \( G \) and a convex \( 1 \)-subgroup \( M \) of \( G \), the following are equivalent.

1) \( M \) is **regular**.

2) \( M \notin M^* = \cap \{ C | M \not\subseteq C, \ C \text{ convex} \ 1 \text{-subgroup of} \ G \} \).

3) \( M \) is meet irreducible in the collection of convex \( 1 \)-subgroups of \( G \).

If \( M \) is normal, then each of the above is equivalent to

4) \( G/M \) is an \( O \)-group, with a convex \( 1 \)-subgroup that covers the identity \( M \) in \( G/M \).

**Proof:** (1 \( \Rightarrow \) 2). Suppose that \( M \) is a value of \( g \in G \). Then for
\( C \supseteq M, \ g \in C \). Hence, \( g \in M^* \setminus M \) and \( M^* \) is called the cover of \( M \).

\((2 \rightarrow 3)\). Since \( \wedge \) and \( \wedge \) agree for convex 1-subgroups, 
\( M \subseteq M^* = \bigwedge \{ C \mid C \supseteq M, \ C \text{ convex 1-subgroup of } G \} \). Therefore, \( M \) is meet irreducible in the collection of convex 1-subgroups of \( G \).

\((3 \rightarrow 1)\). For \( M \) a convex 1-subgroup which is not regular, 
\( M = \bigcap \{ K \mid K \supseteq M, \ K \text{ is regular} \} \), since for each nonzero element \( g \) in \( G \) with \( g \) not in \( M \), there exists a value \( K \) of \( g \) with \( K \supseteq M \). Consequently, if \( M \) is meet irreducible, then \( M \) is regular.

For the remaining part of the proof of the theorem assume \( M \) is normal.

\((2 \rightarrow 4)\). Assume by way of contradiction that \( G/M \) is not an 0-group. Then there exist elements \( M + x \) and \( M + y \) strictly greater than \( M \) in \( G/M \) such that \( (M+x) \wedge (M+y) = M \). Consequently, in the group \( G/M \)

\[
\frac{M^*}{M} \subseteq \frac{G/M}{M} (M+x) \wedge \frac{G/M}{M} (M+y) \\
= \frac{G/M}{M} [(M+x) \wedge (M+y)] \\
= \frac{G/M}{M} (M) \\
= M.
\]

But this contradicts \( M \subsetneq M^* \).
Let (4) hold. It is then straightforward to verify that there is a one-to-one correspondence which preserves containment between the convex 1-subgroups of $G/M$ that contain $M$, and the convex 1-subgroups of $G$ that contain $M$. Thus, since the convex 1-subgroups of $G/M$ which contain $M$ form a totally ordered set under inclusion, the corresponding convex 1-subgroups of $G$ which contain $M$ also form a totally ordered set under inclusion. Therefore, by the assumption that there exists a convex 1-subgroup of $G/M$ which covers $M$, we conclude that there is a convex 1-subgroup of $G$ that covers $M$.

Theorem 1.1.11. For an 1-group $G$ and a convex 1-subgroup $M$ of $G$ the following are equivalent.

1) $M$ is prime.

2) If $A$ and $B$ are convex 1-subgroups of $G$ and $M \subseteq A$, $M \not\subseteq B$ then $M \subseteq A \cap B$.

3) If $a, b \in G \setminus M$ and $a, b \geq 0$ then $a \land b \in G \setminus M$.

4) The lattice of right cosets of $M$ is totally ordered.

If $M$ is normal, each of the above is equivalent to

5) $G/M$ is an 0-group.

Proof: $(1 \Rightarrow 2)$. This is simply a restatement of the definition.

$(2 \Rightarrow 3)$. If $a, b \in G \setminus M$ and $a, b > 0$ then,

$$G(M, a \land b) = G(M, a) \cap G(M, b) \not\subseteq M.$$  

Consequently,
a \land b \notin M.

(3 + 4). Consider M + a, M + b with a, b \in G and let

\[ \bar{a} = a - (a \land b); \]
\[ \bar{b} = b - (a \land b). \]

Then, \( \bar{a} \land \bar{b} = 0 \) and by hypothesis (3) either \( \bar{b} \notin M \) or \( \bar{a} \notin M. \)

If \( \bar{b} \in M \), then

\[ M + b = M + [\bar{b} + (a \land b)] = M + (a \land b) \leq M + a. \]

On the other hand, if \( \bar{a} \in M \) then,

\[ M + a = M + [\bar{a} + (a \land b)] = M + (a \land b) \leq M + b. \]

(4 + 1). Suppose by way of contradiction that hypothesis (4) holds, and (1) does not. Then there exist elements \( a \in A \setminus M, b \in B \setminus M \) where \( a, b > 0 \) such that

a \land b \in A \cap B \subseteq M.

Hence,

M + a > M and M + b > M.

However,

\[(M+a) \land (M+b) = M + (a \land b) = M,\]

which contradicts the assumption that the right cosets of M are totally ordered.

(5 \Rightarrow 4). This is clear when M is assumed normal since only then is G/M a group.
Corollary 1.1.12. If $M$ is regular, then $M$ is prime.

Proof: For $a$ and $b$ two strictly positive elements of $G\setminus M$

$$G(M, a \wedge b) = G(M, a) \cap G(M, b) \neq M.$$ 

Hence, $a \wedge b \notin M$ and therefore, $M$ is prime.

This section closes with several properties of 1-homomorphisms which we will not prove. The proofs of these properties are similar to their counterparts in the theory of groups.

A map $f : G \to H$ between two 1-groups $G$ and $H$ is called an 1-homomorphism (1-isomorphism), if $f$ is a group homomorphism (isomorphism) and $f(h \wedge g) = f(h) \wedge f(g)$, or equivalently, $f(h \vee 0) = f(h) \vee 0$. The kernel of an 1-homomorphism, $h : A \to B$ denoted by $\ker h$ is $\{a \in A \mid h(a) = 0\}$.

Theorem 1.1.13. Given $A, B, C, D$ 1-groups and $f, g, h$ 1-homomorphisms such that $f : A \to B$, $g : B \to C$, $h : A \to D$, and the $\ker h$ is mapped into $\ker g$ by $f$, then there exists a unique 1-homomorphism $k$ such that $k \circ h(a) = g \circ f(a)$ for all $a \in A$. Moreover, $k$ is an 1-isomorphism if and only if $\ker h = h^{-1}(\ker g) = \{a \in A \mid f(a) \in \ker g\}$.

Proof: This is a standard theorem of group theory, and we need only to show that $k$ is an 1-homomorphism.

For $d \in D$

$$k(d \vee 0) = k(h(a) \vee 0) = (k \circ h)(a \vee 0) = (g \circ f)(a \vee 0)$$

$$= (g \circ f(a)) \vee 0 = k(d) \vee 0.$$
Theorem 1.1.14. For an $l$-subgroup $C$ of $G$ the following are equivalent.

1) $C$ is an $l$-ideal.

2) $C$ is the kernel of an $l$-homomorphism.

Theorem 1.1.15. If $A$ and $B$ are $l$-ideals of $G$ and $A \subseteq B$, then $B/A$ is an $l$-ideal of $G/A$ and $G/B$ is $l$-isomorphic to $(G/A)/(B/A)$.

Theorem 1.1.16. Suppose $G$ is an $l$-subgroup of an $l$-group $H$, $C$ is a convex $l$-subgroup of $H$, and $C$ is normal in the $l$-group generated by $G \cup C$. Then $C \cap G$ is an $l$-ideal of $G$, $C + G$ is an $l$-subgroup of $H$, and $G/(G \cap C)$ is $l$-isomorphic to $(C + G)/C$.

If $\{G(i) | i \in I\}$ is a collection of $l$-groups, then $\Pi G(i)$ ($\Sigma G(i)$) will denote the cardinal product (cardinal sum) of the $G(i)$'s. This is the direct product (direct sum) of the $G(i)$'s as groups with the order defined componentwise, i.e.,

for $g, h \in \Pi G(i)$, $g \preceq h$ if and only if $g(i) \preceq h(i)$ for all $i \in I$.

If $G$ is an $l$-group and $H$ is an $0$-group, then $G \times H$ will denote the lexicographical product of $G$ and $H$. This is the direct product of $G$ and $H$ as group, with the order defined by:

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$, $(g_1, h_1) \preceq (g_2, h_2)$ if $h_1 \neq h_2$, or $h_1 = h_2$ and $g_1 \leq g_2$.

An $l$-group $G$ is a subdirect product of $l$-groups $\{G(i) | i \in I\}$, if $G$ is $l$-isomorphic to an $l$-subgroup of $\Pi G(i)$ such that $G$'s image in $\Pi G(i)$ projects onto $G(i)$, for each $i \in I$. Equivalently, there
exist l-ideals $N(i)$ ($i \in I$) in $G$ such that $\cap N(i) = \{0\}$ (where 0 is the group identity) and $G/N(i)$ is l-isomorphic to $G(i)$.

An l-group $G$ is subdirectly irreducible, if whenever $G$ is a subdirect product of l-groups $\{G(i)| i \in I\}$, there exists $j \in I$ such that $G$ and $G(j)$ are l-isomorphic. This corresponds to $G$ having a smallest nontrivial l-ideal.

Section 2. Birkhoff's Theorem

In this section we develop the necessary ideas and notation to prove Birkhoff's Theorem. The approach to Birkhoff's Theorem is by way of Universal Algebra. There are two reasons for this approach. First, Universal Algebra in its abstractness allows the reader to apply its results to a wider class of algebras (rather than l-groups alone). Second, the old adage "simplicity in abstractness" could not be more aptly applied in algebra as in this case. For more details concerning Universal Algebra see Cohn [1].

An operator domain is a set $\Omega$ with a mapping $a : \Omega \rightarrow N$ where the elements of $\Omega$ are called operators; and if $\omega \in \Omega$ then $a(\omega)$ is called the arity of $\omega$. If $a(\omega) = n$ then we say $\omega$ is n-ary, and write

$$\Omega(n) = \{\omega \in \Omega \mid a(\omega) = n\}.$$  

Consequently, 

$$\Omega = \bigcup_{n=0}^{\infty} \Omega(n).$$
Let \( A \) be a set and \( \Omega \) an operator domain. Then an abstract algebra denoted by \([A, \Omega]\) is a family of maps

\[ \Omega(n) \to A^n \quad (n \in \mathbb{N}) \].

Thus, with each \( \omega \in \Omega(n) \) there is associated an \( n \)-ary operation on \( A \). The usual notation for this is, for \( (a_1, \ldots, a_n) \in A \) and \( \omega \in \Omega(n) \), \( \omega(a_1, \ldots, a_n) \in A \), i.e. we identify \( \omega \) with its associated \( n \)-ary operation.

A subalgebra \([B, \Omega]\) of \([A, \Omega]\) is an abstract algebra such that \( B \subseteq A \) and for each \( \omega \in \Omega(n) \) if

\[ \omega : A^n \to A \]

then the restriction of \( \omega \) to \( B^n \) agrees with \( \omega \), and maps \( B^n \) to \( B \).

We say that \([A, \Omega]\) and \([B, \Omega']\) are similar abstract algebras when \( \Omega = \Omega' \) and if this is the case, we call a map \( \psi : A \to B \) a homomorphism from \([A, \Omega]\) into \([B, \Omega]\) if and only if for all \( \omega \in \Omega \) and \( a_i \in A \) if \( \omega \in \Omega(n) \) then

\[ \omega(\psi(a_1), \ldots, \psi(a_n)) = \psi(\omega(a_1, \ldots, a_n)) \].

A homomorphism \( \psi \) is called an isomorphism, if \( \psi \) is also one to one and onto.

If \([A, \Omega]\) and \([B, \Omega]\) are similar abstract algebras, then one may form a new algebra (called the direct product of \([A, \Omega]\) and \([B, \Omega]\) denoted by \([A \times B, \Omega]\)) by defining for all \( \omega \in \Omega \), and \([a_i, b_i] \in A \times B \), if \( \omega \in \Omega(n) \), then
\[ \omega([a_1,b_1],\ldots,[a_n,b_n]) = [\omega(a_1,\ldots,a_n),\omega(b_1,\ldots,b_n)] \, . \]

The direct product may be generalized to an arbitrary collection \( \{[A(i),\Omega] \mid i \in I\} \) of similar algebras in the natural way, and the resulting object will be denoted by \( \prod [A(i),\Omega] \).

For an abstract algebra \([A,\Omega]\), an equivalence relation \( \mathcal{C} \) on \( A \) is said to be a congruence relation on \([A,\Omega]\) if, for all \( \omega \in \Omega \), \( a_i, b_i \in A \)

\[ [\omega(a_1,\ldots,a_n),\omega(b_1,\ldots,b_n)] \in \mathcal{C} \]

whenever \( \omega \in \Omega(n) \), and \([a_i,b_i] \in \mathcal{C} \). We can make the set \( A/\mathcal{C} \) of \( \mathcal{C} \)-equivalence classes into an abstract algebra \([A,\Omega]/\mathcal{C}\) similar to \([A,\Omega]\) by defining, for all \( \omega \in \Omega \) and \( a_i \in A \),

\[ \omega(a_1c,\ldots,a_n) = \omega(a_1,\ldots,a_n) \, c \]

if \( \omega \in \Omega(n) \). It can be shown that the operation is well defined.

For \([A,\Omega]\) and \([B,\Omega]\) similar algebras and \( \psi \) a homomorphism from \([A,\Omega]\) to \([B,\Omega]\) the kernel of \( \psi \) (ker \( \psi \)) is defined as

\[ \{ (s,t) \in A^2 \mid \psi(s) = \psi(t) \} \, . \]

Consequently, the kernel of a homomorphism \( \psi \) is a congruence relation on \([A,\Omega]\) and if \( \psi : [A,\Omega] \to [B,\Omega] \), then \([A,\Omega]/\ker \psi \) is isomorphic to a subalgebra of \([B,\Omega]\). This subalgebra is denoted by \( \psi[A,\Omega] \).

The congruence relations \( \{ \mathcal{C}(i) \mid i \in I\} \) in an algebra \([A,\Omega]\) form a complete lattice where

\[ \omega([a_1,b_1],\ldots,[a_n,b_n]) = [\omega(a_1,\ldots,a_n),\omega(b_1,\ldots,b_n)] \, . \]
\( \wedge c(i) = \cap c(i) \)

and

\( \vee c(i) = \cap \{c' \mid c' \supseteq \cup c(i), \ c' \text{ is a congruence relation on } [A, \Omega] \} \).

We may extend the results from the previous paragraph to a family of congruence relations on \([A, \Omega]\), an abstract algebra. For a family \( \{c(i) \mid i \in I\} \) of congruence relations on \([A, \Omega]\), \([A, \Omega] / \cap c(i)\)

is isomorphic to a subalgebra of \( \Pi([A, \Omega]/c(i)) \).

With this background we may now turn our attention to the properties of varieties that are useful in Chapter IV.

A class \( C \) of similar algebras is called abstract, if \([A, \Omega] \in C\)

whenever there exists a \([B, \Omega] \in C\) such that \([A, \Omega] \) is isomorphic to \([B, \Omega]\). A nonempty abstract class \( C \) of algebras is called a variety, if the following hold:

1) If \([A, \Omega] \in C\) and \([B, \Omega] \) is a subalgebra of \([A, \Omega]\) then \([B, \Omega] \in C\).

2) If \([A, \Omega] \in C\) and \( c \) is a congruence relation on \([A, \Omega]\)

then \([A, \Omega]/c \in C\).

3) If \([A(i), \Omega] \in C\) \((i \in I)\) then \( \Pi[A(i), \Omega] \in C\).

If \( C \) is a nonempty class of similar algebras, we say \( S \) is a \( C \)-free set of generators of \([A, \Omega]\) if,

1) \( S \subseteq A \) and the smallest subalgebra of \([A, \Omega]\) containing \( S \) is \([A, \Omega]\) itself.

2) For every algebra \([B, \Omega] \in C\) and every mapping \( \psi : S \to B\),
there exists a homomorphism \( \phi \) from \([A, \Omega]\) to \([B, \Omega]\) which when restricted to \( S \) agrees with \( \psi \).

If \([A, \Omega]\) is also an element of \( C \), \([A, \Omega]\) is called a free \( C \)-algebra.

We will now construct a word algebra \( W[X, \Omega] \) over a set \( X \) with cardinality greater than zero. A convention which is tacitly assumed throughout the construction is:

if \( \Omega(n) = \phi \) then \( A(i) \times \Omega(n) = \phi \).

With a word algebra \( W[X, \Omega] \), it will be proved that each variety \( V \) has a free \( V \)-algebra as a member. And this result is of great importance in the proof of Birkhoff's Theorem.

For a set \( X \) with cardinality greater than zero let \( A(0) = X \) and define \( A(i) \) recursively as follows: for \( i \geq 1 \) let

\[
A(i+1) = A(i) \cup ( \bigcup_{n=0}^{\infty} A(i)^n \times \Omega(n) ).
\]

Note that the elements of \( \bigcup_{n=0}^{\infty} A(i)^n \times \Omega(n) \) are of the form

\[
((a_1, a_2, \ldots, a_n), \omega) \text{ where } \omega \in \Omega(n) \text{ and } a_k \in A(i) \quad (1 \leq k \leq n), \text{ and that } A(0) \subseteq A(1) \subseteq \ldots \text{. Let}
\]

\[
A = U A(i).
\]

By defining for all \( b_1, b_2, \ldots, b_n \in A \) and \( \omega \in \Omega(n) \)

\[
\omega(b_1, b_2, \ldots, b_n) = ((b_1, \ldots, b_n), \omega) ,
\]

\([A, \Omega]\) becomes an abstract algebra, since if \( b_1, b_2, \ldots, b_n \in A \) and \( \omega \in \Omega(n) \), then \( b_1, b_2, \ldots, b_n \in A(k) \) for some fixed \( k \) and
This algebra is denoted by $W[X, \Omega]$.

Lemma 1.2.1. If $X$ is a set of cardinality greater than zero and $C$ is a class of abstract algebras with $W[X, \Omega] \in C$, then $X$ is a set of $C$-free generators of $W[X, \Omega]$.

Proof: For $[B, \Omega] \in C$ and $\theta$ a map from $X$ to $B$, it will be shown that $\theta$ may be extended to a homomorphism $\phi$ from $W[X, \Omega]$ to $[B, \Omega]$ which when restricted to $X$ agrees with $\theta$.

Let $\phi(0) = \theta$ and define $\phi(i+1) : A(i+1) \to B$ recursively as follows: for $i \geq 1$

$$\phi(i+1)(a) = \begin{cases} 
\phi(i)(a) & \text{if } a \in A(i) \\
\omega[\phi(i)(a_1), \ldots, \phi(i)(a_n)] & \text{if } \mathcal{A} = (((a_1, \ldots, a_n), \omega) \in \bigcup_{n=0}^{\infty} A(i)^n \times \Omega(n). 
\end{cases}$$

Consequently, $\phi(i+1)$, when restricted to $A(i)$, is $\phi(i)$. And if $\phi$ denotes the union of the $\phi(i)$ ($i=0,1,2,\ldots$), then for each $i$ such that $a \in A(i)$

$$\phi(a) = \phi(i)(a).$$

The map $\phi$ is an extension of $\theta$, and $\phi$ is a homomorphism from $W[X, \Omega]$ to $[B, \Omega]$ since, if $a_1, a_2, \ldots, a_n \in A$, then $a_1, a_2, \ldots, a_n \in A(k)$ for some fixed $k$, and for $\omega \in \Omega(n)$,

$$\phi(\omega(a_1, \ldots, a_n)) = \phi((a_1, \ldots, a_n), \omega) = \omega(\phi(k)(a_1), \ldots, \phi(k)(a_n)) = \omega(\phi(a_1), \ldots, \phi(a_n)).$$
Finally, by the construction of $W[X,\Omega]$ it is clear that the smallest subalgebra of $W[X,\Omega]$ containing $X$ is $W[X,\Omega]$.

**Theorem 1.2.2.** Let $C$ be a variety of algebras, which contains an algebra of cardinality greater than zero. Then for any cardinal $m$ greater than zero, there exists a free $C$-algebra with $C$-free generating set of cardinality $m$.

**Proof:** Consider $W[X,\Omega]$ where the cardinality of $X$ is $m$. Let $c = \cap c(i)$ where $\{c(i) \mid i \in I\}$ is the collection of all congruence relations on $W[X,\Omega]$ such that $W[X,\Omega]/c(i) \in C$. The algebra $W[X,\Omega]/c$ is isomorphic to a subalgebra of $\prod W[X,\Omega]/c(i)$. Since $C$ is a variety, $W[X,\Omega]/c \in C$. The set $X_c = \{x \in X \mid x \in X\}$ generates $W[X,\Omega]/c$. To see this let $\chi$ denote the algebra generated by $X_c$. We first show that $\chi \supset W[X,\Omega]/c$.

The algebra $\chi$, contains $A(0)c$. If $\chi$ contains $A(i)c$, then for $a_1^c,\ldots,a_n^c \in A(i)c$ and $\omega \in \Omega(n)$

$$((a_1^c,a_2^c,\ldots,a_n^c),\omega)c = \omega(a_1^c,a_2^c,\ldots,a_n^c)c$$

$$= \omega(a_1^c,a_2^c,\ldots,a_n^c)$$

$$\in \chi.$$ 

Hence $A(i+1)c \subseteq \chi$, and so $A^c \subseteq \chi$. Therefore, $W[X,\Omega]/c \subseteq \chi$. Conversely, it is clear that $\chi \subseteq W[X,\Omega]/c$.

To show that the set $X_c$ has cardinality $m$, it suffices to show that if $x,y \in X$ and $x \neq y$ then $xc \neq yc$. Let $[B,\Omega]$ be an element of $C$ such that there exist $a,b \in B$ with $a \neq b$. Define a map $\theta$. 


from $X$ to $B$ such that

$$\theta(x) = a \quad \text{and} \quad \theta(y) = b$$

Extend $\theta$ to a homomorphism $\psi$ from $W[X,\Omega]$ to $[B,\Omega]$. Then, if $c(i)$ is the kernel of this homomorphism, $W[X,\Omega]/c(i) \in C$. Consequently $c \subseteq c(i)$ and therefore $xc \neq yc$.

To see that $Xc$ is a $C$-free set of generators, let $[B,\Omega] \in C$ and $\theta'$ a map from $Xc$ to $B$. Define a map $\phi'$ from $X$ to $B$ by,

$$\phi'(x) = \theta'(xc).$$

Since $X$ is a $C$-free set of generators for $W[X,\Omega]$, $\phi'$ may be extended to a homomorphism $\phi$ from $W[X,\Omega]$ to $[B,\Omega]$. Hence, $W[X,\Omega]/\ker \phi$ is isomorphic to a subalgebra of $[B,\Omega]$, and consequently, $W[X,\Omega]/\ker \phi \in C$. Therefore, $c \subseteq \ker \phi$. Finally, define a map $\theta$ from $W[X,\Omega]/c$ into $[B,\Omega]$ by,

$$\theta \circ \eta = \phi$$

where $\eta$ is the natural homomorphism from $W[X,\Omega]$ to $W[X,\Omega]/c$. The map $\theta$ is a homomorphism which extends $\theta'$.

**Corollary 1.2.3.** For a variety $\Psi$ and $[B,\Omega] \in \Psi$ with $B \neq \phi$, there exists a homomorphism from a suitably chosen free $C$-algebra onto $[B,\Omega]$.

**Proof:** For $[B,\Omega] \in C$, chose a free $C$-algebra, with $C$-free set of generators with cardinality equal to the cardinality of $B$. 

Then the bijection between the $C$-free set of generators and $B$ may be extended to a homomorphism onto $[B,\Omega]$.

By a law or identity in $W[X,\Omega]$ we mean an ordered pair $(\omega_1,\omega_2) \in W[X,\Omega] \times W[X,\Omega]$. Sometimes this ordered pair is written as $\omega_1 = \omega_2$. We say this law holds in an algebra $[A,\Omega]$, if under every homomorphism $\phi$ from $W[X,\Omega]$ to $[A,\Omega]$, $\phi(\omega_1) = \phi(\omega_2)$. Or, in other words, the ordered pair $(\omega_1,\omega_2)$ is in the kernel of every homomorphism from $W[X,\Omega]$ to $[A,\Omega]$. When a law holds in every element in a class $C$ of abstract algebras, we say that the law holds in $C$, or that $C$ satisfies the law.

**Theorem 1.2.4.** If there exists a set of laws $\Sigma$ such that $C$ is the class of abstract algebras which satisfy the laws of $\Sigma$, then $C$ is a variety.

**Proof:** Let $[B,\Omega]$ be a subalgebra of $[A,\Omega] \in C$. If $\phi$ is a homomorphism from $W[X,\Omega]$ to $[B,\Omega]$, then $\phi$ is a homomorphism from $W[X,\Omega]$ to $[A,\Omega]$, since $[B,\Omega]$ is a subalgebra of $[A,\Omega]$. Consequently, for any law $\omega_1 = \omega_2$ in $\Sigma$, $\phi(\omega_1) = \phi(\omega_2)$. Hence, $[B,\Omega]$ satisfies the laws of $\Sigma$, and therefore, $C$ is closed under the formation of subalgebras.

Let $\{[A(i),\Omega] \mid i \in I\}$ be a collection of algebras from $C$. If $\phi$ is a homomorphism between $W[X,\Omega]$ and $\Pi[A(i)\Omega]$, and $\omega_1 = \omega_2$ is a law in $\Sigma$ such that $\phi(\omega_1) \neq \phi(\omega_2)$ in $\Pi[A(i)\Omega]$, then for at least one $i \in I$ the law will not hold in $[A(i),\Omega]$ under the composition of $\phi$ and the projection map from the product onto $[A(i),\Omega]$. Con-
sequently, the hypothesis \([A(i), \Omega] \in C\) for all \(i \in I\) is contradicted, and therefore one concludes that \(\Pi [A(i), \Omega] \in C\).

Finally, for \([A, \Omega] \in C\) and \(c\) a congruence relation on \([A, \Omega]\), let \(\phi\) be a homomorphism from \(W[X, \Omega]\) to \([A, \Omega]/c\). Let \(\psi\) be a homomorphism from \(W[X, \Omega]\) to \([A, \Omega]\) defined by \(\eta \circ \psi(x) = \phi(x)\), where \(\eta\) is the natural map between \([A, \Omega]\) and \([A, \Omega]/c\). Then for any law \(\omega_1 = \omega_2\) in \(\Sigma\), \(\phi(\omega_1) = \eta \circ \psi(\omega_1) = \eta(\psi(\omega_1)) = \eta(\psi(\omega_2)) = \eta \circ \psi(\omega_2) = \phi(\omega_2)\). Therefore, \([A, \Omega]/c \in C\). And thus \(C\) is a variety.

Theorem 1.2.5. If \(C\) is a variety and \(\Sigma\) is the set of all laws which hold in \(C\), then \(C'\), the collection of all abstract algebras satisfying the laws in \(\Sigma\), is \(C\).

Proof. Clearly \(C' \supseteq C\). Conversely, if \([A, \Omega] \in C'\), then there exists a word algebra \(W[X, \Omega]\) such that \(W[X, \Omega]\) is large enough to write all the laws of \(\Sigma\), and \([A, \Omega]\) is a homomorphic image of \(W[X, \Omega]\) by the homomorphism \(\psi\). Since \(C\) is a variety, it has a free \(C\)-algebra \(F\) on \(X\), and the identity map \(i : X \rightarrow X\) may be extended to a surjection \(I\),

\[ I : W[X, \Omega] \rightarrow F. \]

If \(\omega_1, \omega_2 \in W[X, \Omega]\) and \(I(\omega_1) = I(\omega_2)\) then \(\omega_1 = \omega_2\) in \(F\) and consequently since every algebra in \(C\) is a homomorphic image of a free \(C\)-algebra, \(\omega_1 = \omega_2\) is in \(\Sigma\). Therefore \(\omega_1 = \omega_2\) is satisfied by all elements of \(C'\) and in particular \([A, \Omega]\).
This implies that the law $\omega_1 = \omega_2$ is in the kernel of $\psi$ and, therefore, there exists an epimorphism $\psi^*$:

$$\psi^*: W[X,\Omega]/\ker I \to [A,\Omega].$$

But $W[X,\Omega]/\ker I$ is isomorphic to $F$. Thus $[A,\Omega]$ is a homomorphic image of an element of $C$ and so $[A,\Omega] \in C$. Therefore, $C' \subseteq C$.

Summing up Theorems 1.2.4 and 1.2.5, we have Birkhoff's Theorem.

Theorem 1.2.6. $C$ is a variety if and only if there exists a set of laws $\Sigma$ such that $C$ is the class of algebras satisfying the laws of $\Sigma$.

We end this chapter with several notes on varieties, and 1-group varieties.

Every variety is generated by its subdirectly irreducible members. Consequently, it is sufficient, in many of the theorems concerning varieties, to show that these theorems hold for the subdirectly irreducible members of a variety.

For every variety $V$, the free word algebra $w[X,\Omega]$ supplies an alphabet for the laws of $V$. The laws of $V$ may then be written using a set $X$ which is countably infinite. Hence, since the collection of all 1-groups forms a variety we have:

Theorem 1.2.7. For the free group $F$ on a countably infinite set $X$, and $F$ the free 1-group over the trivially ordered free group $F$, each law of a variety of 1-groups has the form

$$\omega(x) = \bigwedge_{I,J} \omega_{ij}(x) = \bigwedge_{I,K} \Pi_{i,j,k} e$$
where the index sets are finite, \( e \) is the identity of the free 1-group and \( x_{ijk} \in X \cup \{e\} \cup \{x^{-1} \mid x \in X\} \). Moreover, \( \omega_{ij}(x) \) is in reduced form as an element of \( F \).

The form \( \omega(x) \) will be called a **word**, and we denote by \( \omega_{ij}^k(x) \) the product \( \prod_{m=1}^{k} x_{ijm} \). If \( \psi : F \rightarrow H \) is an 1-homomorphism from the free 1-group \( F \) to an 1-group \( H \) defined by \( \psi(x_{ijk}) = h_{ijk} \),

\[
\{x_{ijk} \in X \cup \{e\} \cup \{x^{-1} \mid x \in X\},
\]

then

1. \( x_{ijk} = x_{i',j'k'} \) implies \( h_{ijk} = h_{i',j'k'} \)
2. \( x_{i',j'k'} = (x_{ijk})^{-1} \) implies \( h_{i',j'k'} = (h_{ijk})^{-1} \)
3. \( x_{ijk} = e \) implies \( h_{ijk} = e \).

In this case, we write

\[
\psi(\omega(x)) = \omega(h) = \bigwedge \prod_{ijk} h_{ijk},
\]

and say that \( \psi \) is a **substitution** which will be written simply as \( x \mapsto h \).

An 1-group variety \( V \) is **generated** by a collection

\[
\{G(s) \mid s \in S, \ G(s) \in V\}
\]

of 1-groups, if \( V \) is the smallest variety containing each element of \( \{G(s) \mid s \in S, \ G(s) \in V\} \). Equivalently, \( V \) is generated by \( \{G(s) \mid s \in S, \ G(s) \in V\} \), if for every word \( \omega(x) \) for which there is a substitution \( x \mapsto h \) such that \( \omega(h) \neq e \) in \( H \in V \) there exists a substitution \( x \mapsto g \) such that \( \omega(g) \neq e \) in some \( G(s) \).

In this instance \( V \) is denoted by \( 1\text{-var} \{G(s) \mid s \in S\} \).
CHAPTER 2

Ordered Permutation Groups

This chapter introduces ordered permutation groups. In particular the discussion focuses on two fundamental tools utilized in examining 1-groups and varieties of 1-groups. These are: Holland's Representation Theorem and ordered wreath products of ordered permutation groups. For an extensive treatement of ordered permutation groups see Glass [4].

Section 1. Lattice Ordered Permutation Groups

An ordered permutation group \((G, \Omega)\) is a permutation group \((G, \cdot)\) (where \(1\) denotes the identity) acting on a totally ordered set \(\Omega\) where

1) for all \(\alpha, \beta \in \Omega\), \(\alpha < \beta\) if and only if \(\alpha g < \beta g\) for all \(g \in G\)

The group \(G\) is then a partially ordered group with respect to the partial order given by:

1) for \(g, h \in G\), \(g \leq h\) if and only if \(\alpha g \leq \alpha h\) for all \(\alpha \in \Omega\).

If this partial order on \(G\) is a lattice order so that \(G\) is an 1-group, then \((G, \Omega)\) is called a lattice ordered permutation group (1-permutation group). In this case, for all \(\alpha \in \Omega\) and \(g, h \in G\),

\[\alpha(g \land h) = \alpha g \land \alpha h\]

and

\[\alpha(g \lor h) = \alpha g \lor \alpha h\].
For any totally ordered set \( \Omega \), we shall denote by \( P(\Omega) \) the group of all order preserving permutations of \( \Omega \). Then \( P(\Omega) \) is an 1-permutation group.

For a positive integer \( k \), an ordered permutation group \((G, \Omega)\) is called \( 0-k \) transitive, if whenever

\[
\alpha_1 < \alpha_2 < \ldots < \alpha_k \quad \text{(in} \; \Omega) \quad \text{and} \quad \beta_1 < \beta_2 < \ldots < \beta_k \quad \text{(in} \; \Omega)
\]

there exists \( g \in G \) such that \( \alpha_i g = \beta_i \) \((1 \leq i \leq k)\). If \((G, \Omega)\) is \( 0-k \) transitive, then we say \( G \) acts \( 0-k \) transitively on \( \Omega \).

Also, if \( k \) is one, then we will refer to \((G, \Omega)\) as being transitive rather than \( 0-1 \) transitive. If \((G, \Omega)\) is an 1-permutation group which is \( 0-2 \) transitive, then for any positive integer \( n \) greater than two, \((G, \Omega)\) is \( 0-n \) transitive. Moreover, if one also assumes that the cardinality of \( \Omega \) is greater than two, then \((G, \Omega)\) is transitive.

However, the 1-permutation group \((Z, Z)\), which is the right regular representation of the integers, is transitive, but not \( 0-n \) transitive for \( n \) a positive integer greater than one. Thus not all transitive ordered permutation groups are \( 0-2 \) transitive.

For an ordered permutation group \((G, \Omega)\) and an equivalence relation \( R \) on \( \Omega \), \( R \) is called a convex congruence of \((G, \Omega)\), if each equivalence class is a convex subset of \( \Omega \) and for \( \alpha \) and \( \beta \) in \( \Omega \), \( \alpha R \beta \) implies \( \alpha g R \beta g \) for all \( g \in G \). If \((G, \Omega)\) is a transitive ordered permutation group with no proper convex congruence, it is said to be \( 0 \)-primitive.
Section 2. Holland's Representation Theorem

Holland's Representation Theorem in the theory of 1-groups is an analogue to Cayley's theorem in group theory and it is a very useful tool for studying 1-groups. For more details see Holland [9].

Lemma 2.2.1. For an 1-permutation group \((G, \Omega)\) and \(\alpha \in \Omega\), the set \(G_\alpha = \{g \in G \mid \alpha g = \alpha\}\) is a prime subgroup of \(G\).

Proof: Let \(g, h \in G \setminus G_\alpha\) where \(g\) and \(h\) are nonnegative. Then, \(\alpha g > \alpha\) and \(\alpha h > \alpha\). Hence, \(\alpha (g \wedge h) = \alpha g \wedge \alpha h > \alpha\). Thus, \(g \wedge h \not\in G_\alpha\). By Theorem 1.1.11, and the observation that \(G_\alpha\) is a convex 1-subgroup of \(G\), we conclude that \(G_\alpha\) is prime. The prime subgroup \(G_\alpha\) of \(G\) is called the stabilizer of \(\alpha\).

Lemma 2.2.2. Let \(C\) be a prime subgroup of an 1-group \(G\). Then the map

\[
\psi : G \rightarrow P(\text{G/C})
\]

defined by

\[
(Cx)(\psi g) = C + x + g \quad \text{for all} \quad x \in G,
\]

is an 1-homomorphism of \(G\) into \(P(\text{G/C})\) where \(P(\text{G/C})\) is the 1-group of all order preserving permutations of the set \(\text{G/C}\) of right cosets of \(C\) under the natural order. Moreover, \(g\psi\) acts transitively on \(\text{G/C}\).

Proof: From Lemma 1.1.11 we know that the set of right cosets of \(C\) in \(G\) is totally ordered. Clearly \(g\psi\) is a permutation of \(\text{G/C}\), for all \(g \in G\).
For elements $x$ and $y$ of $G$, suppose $C + x \leq C + y$. Then, there exists an element $s$ in $C$ such that, $s + x \leq y$. Hence, for any element $g$ in $G$, $s + x + g \leq y + g$. Consequently,

$$(C+x)(g\psi) = C + x + g \leq C + y + g = (C+y)(g\psi),$$

and therefore, $g\psi \in P(G/C)$.

The map $\psi$ is clearly a group homomorphism. In order to show that it is an $1$-homomorphism, let $x \in G$. Then

$$(C+x)(g\psi \vee 1) = \max(C+x+g, C+x)$$
$$= (C+x+g) \lor (C+x)$$
$$= C + ((x+g) \lor x)$$
$$= C + (x+(g \lor 0))$$
$$= (C+x)[(g \lor 0)\psi].$$

Hence $\psi$ is an $1$-homomorphism.

In order to see that $G\psi$ acts transitively on $G/C$, let $C + x$ and $C + y$ be two elements of $G/C$ and let $h = -x + y$. Then

$$(C+x)(h\psi) = C + y.$$  

**Lemma 2.2.3.** If $G$ is an $1$-group, then $G$ can be $1$-embedded in a cardinal product of transitive $1$-permutation groups $\{(K_g,\Omega_g) \mid g \in G, \ g \neq 0\}$.

**Proof:** For each $g \in G$, $g \neq 0$, there exists a regular subgroup $H_g$ of $G$ having $g$ as a value. The regular subgroup $H_g$
is a prime subgroup by Corollary 1.2.12. Let \( \Omega_g = G/H_g \).

By Lemma 2.2.2 each map

\[ \psi_g : G \rightarrow P(G/H_g) \]

is an \( 1 \)-homomorphism, where \( G \psi_g \) acts transitively on \( G/H_g \). If we denote by \( K_g \) the \( 1 \)-permutation group \( G \psi_g \), then \( (K_g, \Omega_g) \) is a transitive \( 1 \)-permutation group.

Consider the cardinal product of the \( 1 \)-permutation groups \( \{ (K_g, \Omega_g) \} \). The map

\[ \psi : G \rightarrow \prod_{0 \neq g \in G} K_g \]

defined by

\[ (h \psi)_g = (h)_g \quad (\text{for all } h \in G \text{ and } 0 \neq g \in G) \]

is a \( 1 \)-homomorphism. Moreover, since

\[ \ker \psi = \bigcap_{0 \neq g \in G} \ker \psi_g \subseteq \bigcap_{0 \neq g \in G} H_g = \{0\}, \]

(the last equality, because for each \( g \neq 0 \) in \( G \), \( H_g \) is a value of \( g \)), the map \( \psi \) is injective, and therefore, an \( 1 \)-embedding.

**Holland's Theorem 2.2.4.** If \( G \) is an \( 1 \)-group, \( G \) is \( 1 \)-isomorphic to an \( 1 \)-permutation group.

**Proof:** By Lemma 2.2.3 there exists an \( 1 \)-embedding

\[ \psi : G \rightarrow \prod_{0 \neq g \in G} P(\Omega_g) \].

well order \( G - \{0\} \) by \( \prec \), and let

\[ \Omega = \bigcup_{0 \neq g \in G} \Omega_g \]

be the lexicographic union of the \( \Omega_g \) \((0 \neq g \in G)\). The order on \( \Omega \) is; for \( \alpha, \beta \in \Omega \), \( \alpha < \beta \) if and only if

i) \( \alpha \in \Omega_g \), \( \beta \in \Omega_h \) and \( g \prec h \)

or ii) \( \alpha, \beta \in \Omega_g \) and \( \alpha < \beta \) in \( \Omega_g \).

With this order, \( \Omega \) is a totally ordered set. We may \( 1 \)-embed the cardinal product \( \prod_{0 \neq g \in G} P(\Omega_g) \) in \( P(\Omega) \) via the map

\[ \phi : \prod_{0 \neq g \in G} P(\Omega_g) \to P(\Omega) \]

defined by

\[ \alpha(\phi) = \alpha g \] when \( \alpha \in \Omega_g \).

A representing subgroup \( C \) of an \( 1 \)-group \( G \) is a prime \( 1 \)-subgroup of \( G \) which contains no \( 1 \)-ideals of \( G \) other than \( \{0\} \).

**Corollary 2.2.5.** Let \( G \) be an \( 1 \)-group. For some totally ordered set \( \Omega \), \( (G, \Omega) \) is a transitive \( 1 \)-permutation group, if and only if \( G \) contains a representing subgroup.

**Proof:** If \( (G, \Omega) \) is transitive and \( \alpha \in \Omega \), then by Lemma 2.2.1, \( G_\alpha \) is a prime \( 1 \)-subgroup; and transitivity yields

\[ \bigcap_{g \in G} g^{-1} G_\alpha g = \bigcap_{g \in G} G_\alpha = \bigcap_{\beta \in \Omega} G_\beta = \{1\} \]
However, \( \bigcap_{g \in G} g^{-1} G g \) is the largest \( l \)-ideal of \( G \) contained in \( G_\alpha \). Therefore \( G_\alpha \) is a representing subgroup.

Conversely, if \( C \) is a representing subgroup of \( G \), then the set \( G/C \) of right cosets of \( C \) are totally ordered. The map

\[ \psi : G \to \mathcal{P}(G/C) \]

defined by

\[ (\mathcal{C} + x)(\psi) = \mathcal{C} + x + g \quad \text{(for all} \ x \in G) \]

is an injective \( l \)-homomorphism such that, \( G\psi \) acts transitively on \( G/C \).

Section 3. Wreath Products of Ordered Permutation Groups

The wreath product of ordered permutation groups provides a method for studying the algebraic properties of the lattice of varieties of \( l \)-groups, in much the same way as wreath products of groups aids in determining properties of the algebraic structure of the lattice of varieties of groups. Because the construction of the generalized wreath product of ordered permutation groups is complicated, it will be presented by a series of lemmas. For more details concerning the properties of wreath products of ordered permutation groups see Holland and McCleary [6].
Throughout the construction we will assume that
\[
\{(G_\gamma, \Omega_\gamma) \mid \gamma \in \Gamma, \Gamma \text{ is a totally ordered set}\}
\]
is a collection of ordered permutation groups. Let \(\Lambda = \bigwedge_{\gamma \in \Gamma} \Omega_\gamma\).
Choose a reference point in \(\Lambda\), say \(0\), and for each \(\lambda \in \Lambda\) let
\[
\text{supp } \lambda = \{\gamma \in \Gamma \mid \lambda_\gamma \neq 0_\gamma\}
\]
and
\[
R = \{r \in \Lambda \mid \text{supp } r \text{ is inversely well ordered}\}.
\]

Lemma 2.3.1. \(R\) can be totally ordered.

Proof: Let \(\Gamma(r,s) = \{\gamma \in \Gamma \mid r_\gamma \neq s_\gamma\}\). For \(r \neq s\),
\[
\phi \neq \Gamma(r,s) \subseteq \text{supp } r \cup \text{supp } s,
\]
and since \(\text{supp } r \cup \text{supp } s\) is inversely well ordered, \(\Gamma(r,s)\) is also
inversely well ordered. Let \(\alpha\) be the maximal element in \(\Gamma(r,s)\).
Define an order on \(R\) by,
\[
r < s \text{ if and only if } r_\alpha < s_\alpha.
\]
Since \(\Omega_\alpha\) is totally ordered, the order that has been defined on \(R\)
is likewise a total order.

Lemma 2.3.2. For \(\gamma \in \Gamma\) and \(r,s \in R\), define \(r \equiv^\gamma s\), if and
only if \(r_\alpha = s_\alpha\) for all \(\alpha > \gamma\). Similarly, define \(r \equiv^\gamma s\) if and only
if \(r_\alpha = s_\alpha\) for all \(\alpha > \gamma\). Then \(\equiv^\gamma\) and \(\equiv^\gamma\) are convex equivalence
relations on \(R\).
Proof: It is clear that $\equiv_Y$ and $\equiv_Y$ are equivalence relations. To show that the equivalence classes of $\equiv_Y$ are convex, let $s,r \in R$ be in the same equivalence class, then $r \equiv_Y s$. For $t \in R$ such that $r < t < s$, let $\beta$ and $\delta$ denote the maximal elements of $\Gamma(r,t)$ and $\Gamma(t,s)$ respectively. Then for $\varepsilon = \max(\gamma, \beta, \delta)$, assume $\varepsilon > \gamma$. We then have $r_\varepsilon = s_\varepsilon$, and either

$$r_\varepsilon < t_\varepsilon \quad \text{and} \quad t_\varepsilon \leq s_\varepsilon,$$

or

$$r_\varepsilon \leq t_\varepsilon \quad \text{and} \quad t_\varepsilon < s_\varepsilon.$$

Hence, $\varepsilon > \gamma$ contradicts the fact that $r_\varepsilon = s_\varepsilon$. Thus, $\varepsilon = \gamma$ and $\gamma \geq \beta, \delta$. Consequently, for all $\alpha > \gamma$, $r_\alpha = t_\alpha = s_\alpha$, and therefore $r \equiv_Y t \equiv_Y s$.

Similarly, $\equiv_Y$ can be shown to be a convex equivalence relation.

Lemma 2.3.3. If

$$W' = \{ g \in P(R) \mid \text{for all } r \in \Gamma; r \equiv_Y s, \text{ if and only if } \text{rg} \equiv_Y \text{sg}, \text{ and } r \equiv_Y s \text{ if and only if } \text{rg} \equiv_Y \text{sg} \},$$

then $(W', R)$ is a permutation group. Moreover, for each $\gamma \in \Gamma$

$\equiv_\gamma$ and $\equiv_Y$ are convex congruences on $R$.

Proof: This is a summary of the previous work.

We may characterize each $g$ in $W'$ as a matrix, since each $g$ will induce a permutation $g_\gamma r(\gamma \in \Gamma, r \in R)$ on $\Omega_\gamma$ defined in the
following way. If $\alpha \in \Omega_Y$ there exists $s \in R$ such that $s \equiv_Y r$ and $s_Y = \alpha$. Now define

$$\alpha g_{Y,r} = (sg)_Y.$$ 

The map $g_{Y,r}$ is well defined, since $t \equiv_Y s$ and $t_Y = \alpha$ imply $t \equiv_Y s$. Hence, $t g \equiv_Y s g$ for every $g \in \mathcal{W}$. In other words,

$$(tg)_Y = (sg)_Y.$$ 

To show that $g_{Y,r}$ is one to one, let $\alpha, \beta \in \Omega_Y$ and

$$\alpha g_{Y,r} = \beta g_{Y,r}.$$ 

Let $s, t \in R$ such that

$$s \equiv_Y r \equiv_Y t \quad \text{and} \quad s_Y = \alpha, \quad t_Y = \beta.$$ 

Then

$$(tg)_Y = \beta g_{Y,r} = \alpha g_{Y,r} = (sg)_Y.$$ 

Consequently, $s g \equiv_Y t g$ and so $s \equiv_Y t$. Hence,

$$\alpha = s_Y = t_Y = \beta.$$ 

The map $g_{Y,r}$ ($\gamma \in \Gamma, r \in R$) is onto, since for $\alpha \in \Omega_Y$ choose $s \in R$ such that

$$s \equiv_Y rg \quad \text{and} \quad s_Y = \alpha.$$ 

There exists $t \in R$ such that $tg = s$, and hence $tg \equiv_Y rg$ so that $t \equiv_Y r$. Also
So we see that each \( g \in W' \) is characterized by the matrix
\[
\{ g_{\gamma,r} \mid \gamma \in \Gamma, r \in R \}
\] of its components. The reader may notice that from the definition of \( g_{\gamma,r} \) we have \( r \equiv^\gamma s \) implies that \( g_{\gamma,r} = g_{\gamma,s} \).

**Lemma 2.3.4.** Let \( g \in W' \). Then \( g_{\gamma,r} \) is an order preserving permutation on \( \Omega_\gamma \) for each \( \gamma \in \Gamma \) and \( r \in R \).

**Proof:** Suppose \( g \in W' \). Let \( \alpha, \beta \in \Omega_\gamma \) with \( \alpha < \beta \). Choose \( r, s, t \in R \) such that \( r \equiv^\gamma s \equiv^\gamma t \) and \( s_\gamma = \alpha, t_\gamma = \beta \). Then \( s < t \), and consequently \( sg < tg \). But since \( s \equiv^\gamma t \), we also have \( sg \equiv^\gamma tg \).

Therefore,
\[
\alpha g_{\gamma,r} = (sg)_\gamma < (tg)_\gamma = \beta g_{\gamma,r},
\]
as required.

Recalling that \( \{ (G_\gamma, \Omega_\gamma) \mid \gamma \in \Gamma \} \) is a collection of ordered permutation groups, we have the following corollary.

**Corollary 2.3.5.** For \( W = \{ g \in W' \mid g_{\gamma,r} \in G_\gamma \) for all \( \gamma \in \Gamma, r \in R \}, (W,R) \) is an ordered permutation group. If each \( (G_\gamma, \Omega_\gamma) \) is an 1-permutation group, then \( (W,R) \) is an 1-permutation group.
Proof: For convenience let $\Xi^\gamma = C^\gamma$ and let $1$ denote the group identity of $W$. Then for $g \in W$ $g \cdot 1 = h$ where

$$h_{\gamma, r} = \begin{cases} g_{\gamma, r} & \text{if } (rg)c^\gamma > rc^\gamma \\ g_{\gamma, r} \cdot 1^\gamma & \text{if } (rg)c^\gamma = rc \\ 1^\gamma & \text{if } (rg)c^\gamma < rc^\gamma. \end{cases}$$

The ordered permutation group $(W, R)$ is called the \textbf{Wreath product} of $\{(G^\gamma, \Omega^\gamma) \mid \gamma \in \Gamma\}$.

We now list several properties of the Wreath product which will be useful in the later chapters.

1) $g_{\gamma, r} = g_{\gamma, s}$ if $r \equiv^\gamma s$.

2) $(r)g_{\gamma, r} = (rg)_\gamma$.

3) $(gh)_{\gamma, r} = g_{\gamma, r}h_{\gamma, rg}$.

4) $(g^{-1})_{\gamma, r} = (g_{\gamma, rg})^{-1}$.

5) If $(W, R)$ and $(V, S)$ are two wreath products of the collection $\{(G^\gamma, \Omega^\gamma) \mid \gamma \in \Gamma$ a totally ordered set} of transitive 1-permutation groups, then $(W, R)$ and $(V, S)$ are also transitive. Moreover, if $0$ and $0^*$ are the reference points used in the constructions of $(W, R)$ and $(V, S)$ respectively, then $(W, R)$ and $(V, S)$ are 1-isomorphic.

6) The \textbf{restricted wreath product} or \textbf{small wreath product} of the collection $\{(G^\gamma, \Omega^\gamma) \mid \gamma \in \Gamma$ a totally ordered set} of ordered permutation groups is $\{g \in (W, R) \mid g_{\gamma, r} = 1^\gamma \text{ except on finitely many } \equiv^\gamma \text{ classes}\}$. The wreath product of $\{(G^\gamma, \Omega^\gamma) \mid \gamma \in \Gamma$ a totally ordered set} will be denoted by $W\{(G^\gamma, \Omega^\gamma)\}$. The small wreath product will be denoted by $w\{(G^\gamma, \Omega^\gamma)\}$. 
For a transitive 1-permutation group \((G, \Omega)\), and a convex congruence \(\mathcal{C}\) of \((G, \Omega)\), each \(g \in G\) will map a \(C\)-class \(A\) to itself or to another \(C\)-class \(B\). Since the \(C\)-classes are convex, there is a natural order on them, which forms a total order on \(\Omega/C\). The lazy subgroup of \(G\) with respect to the convex congruence \(\mathcal{C}\) is \(L(C) = \{g \in G \mid A = Ag\}\) for all \(C\)-classes \(A\). The pair \((G/L(C), \Omega/C)\) is a transitive 1-permutation group. For a \(C\)-class \(A\), let \(G(A) = \{g \in G \mid Ag = A\}\) and let \(L(A)\) denote the set \(\{g \in G(A) \mid ag = a\ \text{for all } a \in A\}\). Then \((G(A)/L(A), A)\) is an 1-permutation group.

Let us apply the previous results of wreath products to a wreath product of \((G(A)/L(A), A)\) and \((G/L(C), \Omega/C)\). A permutation \(g\) in \((G(A)/L(A), A)\) \(\text{Wr}(G/L(C), \Omega/C)\) will be written as an ordered pair \((\hat{g}, \bar{g})\).

For an element \(\alpha = (a, b) \in (A \times \Omega/C)\),

\[\alpha g = (a, b)(\hat{g}, \bar{g}) = (a\hat{g}_b, b\bar{g})\]

where \(\hat{g}\) is a map from \(\Omega/C\) to \(G(A)/L(A)\) and \(\hat{g}_b\) is the image of \(b\) in \(G(A)/L(A)\). Also, \(\bar{g}\) is a permutation in \(G/L(C)\).

The product \(gh\) of two elements \(g = (\hat{g}, \bar{g})\) and \(h = (\hat{h}, \bar{h})\) of the wreath product is determined by,

\[\alpha(gh) = (\alpha g)h = (a\hat{g}_b, b\bar{g})h = (a\hat{g}_b\hat{h}_b, b\bar{g}\bar{h}).\]

Hence, for all \(\alpha \in A \times \Omega/C\)

\[\alpha g^{-1} = (a, b)(\hat{g}, \bar{g})^{-1} = (a(\hat{g}_{b^{-1}})^{-1}, b\bar{g}_{b^{-1}}^{-1})\].
Finally, for an element \( g \) of the wreath product and the identity \( 1 \) of the wreath product, \( g \cdot 1 = h \), where for all \( a \in A \times \Omega/C \).

\[
(a)h = (a,b)(\bar{h},\bar{h}) = \begin{cases} 
(a,b) & \text{if } bg < b \\
(a\bar{g}_b,b\bar{g}) & \text{if } bg > b \\
(a(\bar{g}_b \vee 1_b),b) & \text{if } bg = b
\end{cases}
\]

An 0-isomorphism from a totally ordered set \( X \) to a totally ordered set \( Y \) is a map \( f \) from \( X \) to \( Y \) such that \( f \) is one to one and such that \( a < b \) in \( X \) if and only if \( f(a) < f(b) \) in \( Y \).

One says that an 1-permutation group \((G,\Omega)\) may be 1-embedded in an 1-permutation group \((H,\Gamma)\), if there exists an 0-isomorphism \( \phi \) from \( \Omega \) to \( \Gamma \), and a monomorphism \( \psi \) from \( G \) into \( H \), such that for \( \alpha \in \Omega \) and \( g \in G \),

\[
(\alpha g)\phi = (\alpha\phi)(g\psi)
\]

The following procedure is a method by which a transitive 1-permutation group \((G,\Omega)\) may be 1-embedded in a wreath product. More precisely, it will be shown that \((G,\Omega)\) may be 1-embedded in a wreath product of the ordered permutation groups \((G/L(C),\Omega/C)\) and \((G(A)/L(A),A)\).

**Theorem 2.3.6.** If \((G,\Omega)\) is a transitive 1-permutation group, \(C\) a convex congruence of \((G,\Omega)\) with \(A\) a C-class, then \((G,\Omega)\) can be 1-embedded in \((G(A)/L(A),A)\text{Wr}(G/L(C),\Omega/C) = (W,R)\).

**Proof:** For each C-class \(B\), choose a permutation \( k_B \in G \) with \( k_B = e \), if \( B = A \), so that \( Ak_B = B \). This is possible since
\((G, \Omega)\) is transitive.

The map \(\phi: \Omega \rightarrow A \times \Omega/C\), defined by

\[ \alpha \phi = (\alpha k^{-1}_C, \alpha C) \]

is an \(O\)-isomorphism.

The map \(\psi: G \rightarrow W\), defined by

\[ g\psi = (\bar{g}, \bar{g}) \]

is a monomorphism, where \(\bar{g}\) is the image of \(g\) in \(G/L(C)\) under the canonical map, and \(\hat{g}_B\) acting on \(A\) is given by

\[ \hat{g}_B = k_B g(k_B^{-1}) \] (restricted to \(A\)) \(\in G(A)/L(A)\).

To see that the homomorphism \(\psi\) is one to one let \(g \in G\) be mapped to \((\hat{g}, \bar{g}) \in W\), where \(\bar{g}\) is the image of \(g\) in \(G/L(C)\) under the canonical map, and \(\hat{g}_B = k_B g(k_B^{-1})\) restricted to \(A\), and so \(\hat{g}_B \in G(A)/L(A)\). Then, if for all \((a, B) \in A \times \Omega/L(C)\), \((a, B) (\hat{g}, \bar{g}) = (a, B)\) then

\[ (a\hat{g}_B, B\bar{g}) = (a, B) \] for all \((a, B) \in A \times \Omega/L(C)\).

Since \(\bar{g}\) fixes every \(C\)-class, \(g \in L(C)\). Consequently,

\[ \hat{g}_B = k_B g(k_B^{-1}) = k_B g(k_B^{-1}). \] Since \(a\hat{g}_B = a\) for all \(a \in A\) and \(B \in \Omega/L(C)\), we have \(ak_B g(k_B^{-1}) = a\) or \(ag = a\) for all \(a \in A\) and \(B \in \Omega/L(C)\). Thus \(g\) must be the identity of \(G\), and so \(\psi\) is one to one.
Section 4. Wreath Products of the Real Numbers

For a transitive 1-permutation group \((G, \Omega)\) for which every value \(M\) in \(G\) is normal in its cover \(M^*\), it will be shown that \((G, \Omega)\) is 1-embedded in a wreath product of subgroups of the real numbers. Also, if \(G\) does not have the above property, then it will be shown that \(G\) contains an 1-subgroup which is 0-2 transitive on some totally ordered set. The results in this section may be found in Holland and McCleary [6].

For a transitive 1-permutation group \((G, \Omega)\), let \(C\) and \(K\) be a pair of convex congruences on \(\Omega\) such that \(C \subseteq K\), and let \(A\) be a \(K\)-class. The set of \(C\)-classes contained in the \(K\)-class \(A\) is denoted by \(C|A\), \(G(A) = \{g \in G \mid Ag = A\}\), and \(L(A) = \{g \in G \mid ag = a\ \text{for all} \ a \in A\}\). In this context, the 1-permutation group \((G(A)/L(A), A/(C|A)\) is called the \((C,K)\) component of \((G,\Omega)\). It is 0-primitive if there does not exist a convex congruence \(R\) on \(\Omega\) such that \(C \subseteq R \subseteq K\). In this case the \((C,K)\) component of \((G,\Omega)\) will be called an 0-primitive component.

**Lemma 2.4.1.** If \((G,\Omega)\) is a transitive 1-permutation group, the set of all convex congruences of \(G\) is a totally ordered set under inclusion.

Consequently, if we denote by \((C_i,C^i)\) the pairs of convex congruences on \(\Omega\) for which the \((C_i,C^i)\) component of \((G,\Omega)\) is 0-primitive, then \(\{(C_i,C^i)\}\) is totally ordered. We will write \(i = (C_i,C^i)\) and \(\{(C_i,C^i)\} = I\). Thus \(I\) is a totally ordered set, and we shall write \((G_i,\Omega_i)\) as the \((C_i,C^i)\) component of \((G,\Omega)\).
Lemma 2.4.2. If $H$ is a group, there exists a set $\{T(K) \mid K$ is a subgroup of $H\}$ such that $T(K)$ is a set consisting of exactly one element from each of the right cosets of $K$ in $H$, $T(K) \cap K$ is the identity of $H$, and if $G$ and $K$ are subgroups of $H$ with $G \subseteq K$, then $T(K) \subseteq T(G)$.

Proof: Well order the elements of $H$ so that the identity is the smallest element in the well ordering. Let $K$ be a subgroup of $H$ and $h \in H$. Choose for the unique element of $T(K)$ in $Kh$ the smallest element in the well ordering of $Kh$. Then clearly $T(K) \cap K$ is the identity of $H$. Also if $G$ and $K$ are subgroups of $H$ with $G \subseteq H$, then $T(H) \subseteq T(G)$. The function $T$ is called a transversal function.

Theorem 2.4.3. Let $(G,\Omega)$ be a transitive 1-permutation group with O-primitive components $\{G_i,\Omega_i\}_{i \in I}$. There exists an 1-embedding of $(G,\Omega)$ into $\text{Wr}\{(G_i,\Omega_i) \mid i \in I\} = (W,R)$.

Proof: Fix $\alpha_0 \in \Omega$, and for each $i = (C_i^1, C_i^1) \in I$ let $\Omega_i = (\alpha_0 C_i^1)/C_i^1$. The O-primitive components of $(G,\Omega)$ are $(G_i,\Omega_i)$. Let $T$ be a transversal function for the subgroups of $G$. For each $\alpha \in \Omega$, let $g(\alpha, i)$ be the unique element of $T(G(\alpha C_i^1))$ such that $\alpha_0 g(\alpha, i) C_i^1 \alpha$. (Uniqueness is a consequence of the fact that the right cosets are disjoint.) Also $g(\alpha_0, i)$ is the identity in $G$.

Pick $0 \in \bar{\Omega_i}$ by $0 = \alpha_0 \phi$ where

$$\phi : \Omega \to \bar{\Omega_i}$$

by $(\alpha \phi)_i = \alpha [g(\alpha, i)]^{-1} C_i^1 \in \Omega_i$. 

42.
The map $\phi$ also maps $\Omega$ in a one-to-one, order preserving manner into the subset $R$ of $\Omega_i$ consisting of those points whose supports with respect to $0 = \alpha_0 \phi$ are inversely well ordered.

Let $r \in R$, $i \in I$ and $g \in G$. Define

$$\psi : G \to W$$

by

$$(r(g\psi))_i = \begin{cases} ((\alpha g)\phi)_i & \text{if } \alpha \phi C_i r \text{ for some } \alpha \in \Omega \\ (r)_i & \text{otherwise} \end{cases}$$

Then $\psi$ is a monomorphism from $G$ into $W$. Therefore $(G, \Omega)$ is $1$-embeddable in $(W, R)$.

Hölder's Theorem 2.4.4. For an $O$-group $G$ the following are equivalent.

1) For $a, b \in G$, $0 < a < b$ implies that $b < na$ for some positive integer $n$.

2) $G$ is a subgroup of the real numbers.

3) $G$ has no proper convex subgroups.

Theorem 2.4.5. If a transitive 1-permutation group $(G, \Omega)$ has the property that each value $M$ in $G$ is normal in its cover $M^*$, then $(G, \Omega)$ is 1-embeddable in the wreath product of subgroups of the real numbers (permuting themselves in the right regular representation).

Proof: It will be shown that each $O$-primitive component of $(G, \Omega)$ is 1-isomorphic to a subgroup of the real numbers, permuting
it self in the right regular representation.

Fix \( 0 \in \Omega \) and suppose \( C \) is a convex congruence of \((G,\Omega)\).

Let \( H = \{ g \in G \mid (OC)g = OC \} \) then \( H \) is a convex 1-subgroup of \( G \).

Conversely, each convex 1-subgroup \( H \subseteq G \) (for some \( \alpha \in \Omega \)) of \( G \) defines a convex congruence \( C \) on \((G,\Omega)\) by \( sCt \) if and only if \( s \in \{ th \mid h \in H \} \).

Thus each \((C,K)\)-component of \( G \) which is 0-primitive determines a pair \((A,B)\) of convex 1-subgroups of \( G \) such that \( A \subseteq B \), and there does not exist an 1-subgroup of \( G \) strictly between \( A \) and \( B \). Let \( s \in OK \setminus OC \). Then since \( G \) is transitive, there exists \( g \in G \) such that \( Og = s \). Consequently, \( g \in B \setminus A \). Therefore \( A \) is a value of \( g \), and \( B \) is a cover of \( A \). Moreover, \( A \) is normal in \( B \) and hence \( B/A \) is an 0-group with no non-trivial convex 1-subgroups. Therefore, \( B/A \) is 1-embeddable in a copy of the real numbers.

We now proceed to show that for an 1-group \( G \), if \( G \) contains a value \( M \) which is not normal in its cover \( M^* \), then \( G \) contains an 1-subgroup which is 0-2 transitive on some totally ordered set.

For an 1-permutation group \((G,\Omega)\), \( \alpha \in \Omega \), and an 1-subgroup \( H \) of \( G \) we call \( \alpha H = \{ \alpha h \mid h \in H \} \) an orbit of \( H \) containing \( \alpha \).

It may be easily shown that for orbits \( \alpha H \) and \( \beta H \) either \( \alpha H = \beta H \) or \( \alpha H \cap \beta H = \emptyset \).

For an 1-permutation group \((G,\Omega)\) let the Dedekind completion of \( \Omega \) be denoted by \( \bar{\Omega} \). Then, for each \( g \in G \) and \( \bar{\alpha} \in \bar{\Omega} \setminus \Omega \) we define

\[ \bar{\alpha}g = \sup\{ ag \mid \alpha \in \Omega, \alpha < \bar{\alpha} \} \]
For a totally ordered set $X$ we say that $Y \subseteq X$ is dense in $X$ if whenever $a, b \in X$ and $a < b$ then there is a $y \in Y$ such that $a < y < b$.

Lemma 2.4.6. Let $(G,\Omega)$ be a transitive 1-permutation group. Then for $\alpha \in \Omega$ the orbits of the stabilizer $G_\alpha$ are convex.

Proof: For $\beta \in \Omega$ and $\delta_1, \delta_2 \in \beta G_\alpha$, there exists $g \in G_\alpha$ such that $\delta_1 g = \delta_2$. If $\delta_1 \leq \sigma \leq \delta_2$ then by transitivity there exists $f \in G$ such that $\delta_1 f = \sigma$. Let $h = (f \lor 1) \land (g \lor 1)$. Then, $\delta_1 h = \sigma$, and since $G_\alpha$ is convex, $1 \leq h \leq g \lor 1$ implies that $h \in G_\alpha$.

Lemma 2.4.7. Let $(G,\Omega)$ be an $\Omega$-primitive 1-permutation group. Then, for $\tilde{\alpha} \in \tilde{\Omega}$ either $\tilde{\alpha}G$ is dense in $\tilde{\Omega}$ or $\tilde{\Omega} = \Omega$ and $\Omega$ may be taken to be the integers.

Proof: If an element $\beta \in \Omega$ has an immediate successor, then by transitivity, every element of $\Omega$ has an immediate successor. Consequently there would be an element of $\tilde{\Omega}$ with an immediate predecessor. Again by transitivity, this implies that every element of $\tilde{\Omega}$ would have an immediate predecessor.

Define an equivalence relation $C$ on $\Omega$ by

$\alpha C \beta$ if and only if there are only a finite number of elements in $\Omega$ between $\alpha$ and $\beta$.

Then it is easily shown that $C$ is a convex congruence on $\Omega$. Since, by assumption, $\Omega$ has an element with an immediate successor, the
classes of $C$ are not all singletons. Thus, since $(G,\Omega)$ is $O$-primitive, $C$ has only one class i.e. $\Omega$. Therefore, $\Omega$ may be taken as the integers and so $\Omega = \tilde{\Omega}$.

If no element of $\Omega$ has an immediate successor then by the above argument it follows that $\Omega$ is dense in itself. Since $G$ is transitive on $\Omega$, if $\tilde{\alpha} \in \Omega$, then $\tilde{\alpha}G = \Omega$. Thus, $\tilde{\alpha}G$ is dense in $\tilde{\Omega}$.

On the other hand, if $\tilde{\alpha} \in \tilde{\Omega} \setminus \Omega$, then define an equivalence relation $C$ on $\Omega$ by

$$\alpha C \beta \text{ if and only if there is no } g \in G \text{ such that } \tilde{\alpha}g \text{ lies strictly between } \alpha \text{ and } \beta.$$  

Then, it is straight forward to show that $C$ is a convex congruence on $\Omega$.

Since $\tilde{\alpha} \in \tilde{\Omega} \setminus \Omega$, $\Omega = \{ \beta \in \Omega \mid \beta < \tilde{\alpha} \} \cup \{ \beta \in \Omega \mid \beta > \tilde{\alpha} \}$. But $(G,\Omega)$ is $O$-primitive, which implies that the $C$-classes are all singletons. In other words $\tilde{\alpha}G$ is dense in $\tilde{\Omega}$.

For a transitive 1-permutation group $(G,\Omega)$ and each orbit $\Lambda$ of $G_{\alpha}$, we define the reflection of $\Lambda$ in $\alpha$ as $\Lambda' = \alpha K$ where $K = \{ g \in G \mid \alpha \in \Delta g \}$.

Lemma 2.4.8. If $(G,\Omega)$ is a transitive 1-permutation group and $\Lambda$ is an orbit of $G_{\alpha}$, then $\Lambda'$ is an orbit of $G_{\alpha}$ and $(\Lambda')' = \Lambda$.

Proof: Clearly $\Delta'G_{\alpha} \supset \Delta'$. Conversely, since $K G_{\alpha} \subseteq K$ we have $\Delta'G_{\alpha} = \alpha K G_{\alpha} \subseteq \alpha K = \Delta'$.

Thus, $\Delta'G_{\alpha} = \Delta'$.
We will now show that given $\beta \in \Delta'$ then for each $\sigma \in \Delta'$ there exists $g \in G_\alpha$ such that $\beta g = \sigma$. From this fact and the above argument we may conclude that $\beta G_\alpha = \Delta'$.

Since $\beta, \alpha \in \Delta'$ there exist $h, k \in K$ such that $\beta = \alpha h$ and $\sigma = \alpha k$. Thus $\alpha h^{-1}$ and $\alpha k^{-1}$ are elements of $\Delta$. Consequently, there exists $f \in G_\alpha$ such that $\alpha h^{-1} f = \alpha k^{-1}$. Let $g = h^{-1} f k^{-1}$.

Then $g \in G_\alpha$ and $\beta g = \sigma$.

Finally, let us note that from the definition of $\Delta'$ we have

$\alpha g \in \Delta'$ if and only if $\alpha g^{-1} \in \Delta$.

Thus, $\alpha g \in (\Delta')'$ if and only if $\alpha g \in \Delta$. Therefore, $(\Delta')' = \Delta$.

For the following definitions it is assumed that $(G, \Omega)$ is a transitive 1-permutation group.

Since $\Delta'$ is an orbit of $G_\alpha$ wherever $\Delta$ is an orbit, we will call $\Delta'$ a paired orbit of $\Delta$. Also, since the orbits of $G_\alpha$ partition $\Omega$ into convex classes, the orbits of $G_\alpha$ may be totally ordered in the natural way i.e. $\Delta_1 < \Delta_2$ when there exists $\delta_1 \in \Delta_1$, $\delta_2 \in \Delta_2$ such that $\delta_1 < \delta_2$. We will call an orbit $\Delta$ of $G_\alpha$ positive if $\{\alpha\} < \Delta$ and negative if $\{\alpha\} > \Delta$.

We note that for orbits $\Delta_1$ and $\Delta_2$ of $G_\alpha$, $\Delta_1 < \Delta_2$ if and only if $\Delta_1' > \Delta_2'$. Moreover, the map between the orbits of $G_\alpha$ and the paired orbits, denoted by $\Delta \rightarrow \Delta'$, is a bijection.

An element $\beta \in \Omega$ is called a fixed point of $G_\alpha$ if $\beta G_\alpha = \{\beta\}$. If $\beta$ is not a fixed point then $\beta G_\alpha$ is called a long orbit. If the paired orbit $\{\beta\}'$ of the fixed point $\beta$ of $G_\alpha$ is a fixed
point of $G_\alpha$ then $\beta$ is called a strongly fixed point of $G_\alpha$. If every fixed point of $G_\alpha$ is a strong fixed point, then $G$ is called balanced.

**Lemma 2.4.9.** If $(G,\Omega)$ is an $0$-primitive $1$-permutation group and $\beta$ is a fixed point of $G_\alpha$ then $\beta$ is a strongly fixed point i.e. $G$ is balanced.

**Proof:** Suppose that $\{\beta\}'$ is not a fixed point of $G_\alpha$. Then for each $g \in G_\alpha$, $\{\beta\}'g = \{\beta\}'$, and for each $g \in G \setminus G_\alpha$, $\{\beta\}'g \cap \{\beta\}' = \emptyset$.

Define a relation $C$ on $\Omega$ as follows

$\alpha C \beta$ if and only if $\alpha, \beta \in \{\beta\}'g$ for some $g \in G$.

The relation $C$ is an equivalence relation, since for classes $\{\beta\}'g$ and $\{\beta\}'h$, if $\{\beta\}'g \cap \{\beta\}'h \neq \emptyset$ then $\{\beta\}'gh^{-1} \cap \{\beta\}' \neq \emptyset$.

Thus, from the first paragraph, $\{\beta\}'gh^{-1} = \{\beta\}'$ or $\{\beta\}'g = \{\beta\}'h$.

Since $\{\beta\}'$ is convex, each class of $C$ is convex, and hence $C$ is a convex congruence on $\Omega$ which contains the non-trivial class $\{\beta\}'$. This contradicts the fact that $(G,\Omega)$ is $0$-primitive.

Let $X$ be a totally ordered set. We say that $Y \subseteq X$ is cofinal in $X$ if for all $x \in X$ there exist $y \in Y$ such that $y \geq x$.

**Lemma 2.4.10.** If $(G,\Omega)$ is $0$-primitive and $G_\alpha \neq \{1\}$ for some $\alpha \in \Omega$, then $G_\alpha$ has a positive long orbit $\Lambda_1$, and $\alpha$ is the only point of $\Omega$ between $\Lambda_1$ and $\Lambda_1'$.
Proof: Since $G_\alpha \neq \{1\}$ for some $\alpha \in \Omega$, $G_\alpha$ has a long orbit $\Delta$. Since $(G,\Omega)$ is balanced, we assume $\Delta$ is negative and so not cofinal in $\Omega$. Let $\bar{\alpha} = \sup\{\beta \in \Delta\} \in \bar{\Omega}$. Choose $g \in G$ so that $\alpha^{-1} \in \Delta$. Define $\Delta_1 = \{\beta \in \Omega \mid \bar{\alpha}gk \leq \beta \leq \bar{\alpha}gh, k, h \in G_\alpha\}$. To see that $\Delta_1$ is a positive orbit of $G_\alpha$, we first show that there are no points of $\Omega$ between $\alpha$ and $\Delta_1$. The reader may then convince himself that $\Delta_1$ is the first positive long orbit.

Since $\Omega$ is not the integers, and $\alpha^{-1} < \bar{\alpha}$, there exists $\beta \in \Omega$ such that $\alpha < \beta < \bar{\alpha}$. Also $\bar{\alpha}G$ is dense in $\bar{\Omega}$ so there exists a positive element $h < g$ in $G$, such that $\alpha < \bar{\alpha}h \leq \beta$. Consequently, $\alpha^{-1} < \bar{\alpha}h^{-1} < \bar{\alpha}$, and so $\alpha h^{-1} \in \Delta$. Choose a non negative $f \in G_\alpha$ such that $\alpha^{-1} < \bar{\alpha}h^{-1} < \alpha^{-1} f < \bar{\alpha}$. Then, $\alpha < \alpha g^{-1}fh$ and $(g^{-1}fh \land 1) \in G_\alpha$. Moreover, since $G_\alpha \subset G_\alpha$ we have $\bar{\alpha}g(g^{-1}fh \land 1) = \bar{\alpha}h \land \bar{\alpha}g < \beta$. Thus, $\bar{\alpha}g(g^{-1}fh \land 1) \leq \beta < \bar{\alpha}$ and therefore, $\beta \in \Delta_1$.

Lemma 2.4.11. A transitive 1-permutation group $(G,\Omega)$ is 0-2 transitive if and only if whenever $\alpha < \beta < \sigma$ there exists a non negative $g \in G_\alpha$ such that $\alpha g = \alpha$ and $\beta g = \sigma$.

Proof: If $(G,\Omega)$ is 0-2 transitive then there exists $g \in G$ such that $\alpha g = \alpha$ and $\beta g = \sigma$. Let $h = g \lor 1$, then $h$ is the required order preserving permutation.

Conversely, since $(G,\Omega)$ is transitive, if $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, then there exists $f \in G$ such that $\alpha_1 f = \beta_1$ and without loss of
generality \( \alpha_2 f < \beta_2 \). Hence, \( \beta_1 = \alpha_1 f < \alpha_2 f < \beta_2 \). By the hypothesis there exists a non-negative \( h \in G_{\beta_1} \), such that \( \alpha_1 fh = \alpha_1 f \) and \( \alpha_2 fh = \beta_2 \). Thus \( \alpha_1 fh = \beta_1 \) and \( \alpha_2 fh = \beta_2 \) as required.

Theorem 2.4.12. Let \((G, \Omega)\) be an \( O \)-primitive \( 1 \)-permutation group with \( G_{\alpha} \neq \{1\} \) for some \( \alpha \in \Omega \). Then, \( G \) contains an \( 1 \)-subgroup which acts \( O-2 \) transitively on some totally ordered set.

Proof. Let \( \Delta \) be the first long positive orbit of \( G_{\alpha} \). We will show that \( G_{\alpha} \) acts \( O-2 \) transitively on \( \Delta \) by using lemma 2.4.10. Let \( \beta_1, \beta_2, \beta_3 \in \Delta \) such that \( \alpha < \beta_1 < \beta_2 < \beta_3 \). Since \( \Delta \) is an orbit of \( G_{\alpha} \) there exists a non-negative \( h \in G_{\alpha} \) such that \( \beta_2 h = \beta_3 \). Now \( \beta_1 < \beta_1 h < \beta_2 h = \beta_3 \), so therefore \( \beta_1 h \in \Delta \).

We will now show that there exists a non-positive \( g \in G_{\beta_3} \) such that \( \beta_1 hg = \beta_1 \). It is then easy to demonstrate that \( (h g \lor 1) \in G_{\alpha} \), \( \beta_1 (h g \lor 1) = \beta_1 \), and \( \beta_2 (h g \lor 1) = \beta_3 \).

Let \( f \in G \) such that \( \alpha f = \beta_3 \) since \( \alpha < \beta_1 < \beta_2 < \beta_3 \), we have \( \alpha f^{-1} < \beta_1 f^{-1} < \beta_1 h f^{-1} < \beta_3 f^{-1} = \alpha \). Now, by definition, \( \alpha f \in \Delta \) if and only if \( \alpha f^{-1} \in \Delta' \) (the last negative orbit of \( G_{\alpha} \)). By lemma 2.4.9., \( \beta_1 f^{-1}, \beta_1 h f^{-1} \in \Delta' \). Thus \( G_{\alpha} \) does not fix any point \( \sigma \) such that \( \beta_1 f^{-1} \leq \sigma \leq \beta_1 h f^{-1} \). Now since \( f^{-1} G_{\alpha} f = G_{\beta_3} \), it is clear that \( \beta_1 \) and \( \beta_1 h \) are in the last negative orbit \( \Delta' f \) of \( G_{\beta_3} \). Thus the result follows.
Theorem 2.4.13. If $G$ is an $l$-group which has a value $M$ not normal in its cover $M^*$, then every variety containing $G$ must contain an $l$-group which is $0$-$2$ transitive on some totally ordered set.

Proof: Let $M$ be a value in $G$ which fulfills the conditions of the theorem. Then $\bigcap M + g$ is an $l$-ideal of $M^*$. The $l$-group $H = M^*/(\bigcap gM + g)$ is $l$-isomorphic to a primitive $l$-subgroup of order $g \in M^*$ preserving permutations acting on the totally ordered set $M^*/M$ of right cosets of $M$ in $M^*$. Since $M$ is not normal in $M^*$, the stabilizer $H_M \not\supset (\bigcap gM + g)$. By Theorem 2.4.12, $H$ contains an $l$-subgroup which acts $0$-$2$ transitively on some totally ordered set as required.

We end this chapter by introducing some notation which will be used in Chapters III and IV. Let $Z$ denote the integers, than $(Z,Z)$ will signify the right regular representation of the integers. Likewise, if $R$ denotes the real numbers then $(R,R)$ is the right regular representation of the real numbers. The wreath product of $n$ copies of $(Z,Z)$ will be denoted simply by $\text{Wr}^n Z$ where $n$ is any finite cardinal. For $\Gamma = Z^+ (Z^-)$ we write $\text{Wr}\{(G_{\gamma}, \Omega_{\gamma}) \mid \gamma \in \Gamma\} = \text{Wr}^\infty (G_{\gamma}, \Omega_{\gamma}) \text{ Wr}^{-\infty} (G_{\gamma}, \Omega_{\gamma})$.

The natural $l$-ideal of $(H, \Gamma) \text{ Wr}(G, \Omega) [(H, \Gamma) \text{ wr}(G, \Omega)]$ is the cardinal product $\prod H^\alpha$ (cardinal sum $\sum H^\alpha$) of copies of $H$, where $H^\alpha = \{(\hat{h}, e) \mid \hat{h}_\beta = e \text{ for all } \beta \neq \alpha\}$. 

51.
CHAPTER III

Examples of Varieties of Lattice Ordered Groups

In this chapter several examples of varieties of 1-groups will be presented, as well as their positions in the lattice of varieties of 1-groups. The notation used for naming varieties of 1-groups is the same notation used in the paper by Glass, Holland and McCleary [5]. The material in this chapter may be found in Holland [7], Martinez [10], Fuchs [3], Smith [14], Weinberg [15], Wolfenstein [14] and Scrimger [13].

Section 1. The Examples

Example 3.1.1. The variety $E$ consists of those 1-groups with one element. It is clear that the equation defining this variety is: For each $G \in E$,

$$x = y \text{ for all } x, y \in G.$$  

Moreover, any two elements of $E$ are 1-isomorphic and consequently, up to isomorphism, this variety contains one element.

Example 3.1.2. On the other hand, the variety $L$ consisting of all 1-groups has as its defining equation the following: For each $G \in L$,

$$x = x \text{ for all } x \in G.$$
Example 3.1.3. A variety of $l$-groups with some interesting properties, which will be discussed later, is the variety $A$ consisting of all abelian $l$-groups. The equation defining this variety is: For each $G \in A$,

$$x + y = y + x \text{ for all } x, y \in G.$$ 

Example 3.1.4. A lattice ordered group $G$ is called representable if and only if $G$ is $l$-embeddable in a cardinal product of totally ordered groups. The collection of all representable $l$-groups is a variety, which is denoted by $R$. The proof of this claim is obtained from Lemma 3.1.5, Lemma 3.1.6 and Theorem 3.1.7.

Lemma 3.1.5. The class $C$ of all $l$-groups $G$ such that $a \wedge b = 0$ implies that $a \wedge (-y+b+y) = 0$ is equationally definable. The equation which defines this class is $(x \lor 0) \wedge (-y + [(-x) \lor 0] + y) = 0$.

Proof. For $a, b, y \in B$ let $x = a - b$. Then, $a \wedge b = 0$ if and only if $a = x \lor 0$ and $b$ is equal to $(-x) \lor 0$.

Consequently, if $a \wedge b = 0$ implies that $a \wedge (-y+b+y) = 0$, then by replacing $a$ and $b$ by $x \lor 0$ and $(-x) \lor 0$ respectively we have

$$(x \lor 0) \wedge (-y + [(-x) \lor 0] + y) = 0.$$ 

Conversely, if $(x \lor 0) \wedge (-y + [(-x) \lor 0] + y) = 0$ for all $x$ and $y \in G$, then upon setting $x = a - b$, we have, $a \wedge b = 0$ implies
that $a = x \lor 0$ and $b = (-x) \lor 0$. Hence,

$$a \land (-y+b+y) = 0 \quad \text{since} \quad (x \lor 0) \land (-y + [(-x) \lor 0] + y] = 0$$

**Lemma 3.1.6.** For $G$ an $1$-group and for any $X$ a subset of $G$, the set

$$X' = \{g \in G \mid |g| \land |a| = 0 \text{ for all } a \in X\}$$

is an $1$-ideal if and only if $a \land b = 0$ implies $a \land (-y+b+y) = 0$ for all $a, b, y \in G$.

**Proof:** If $X'$ is an $1$-ideal for each $X \subseteq G$ then $\{a\}'$ is an $1$-ideal. Also, if $a \land b = 0$, then $a, b \geq 0$, and consequently

$$0 = a \land b = |a| \land |b|.$$  
Hence, $b \in \{a\}'$. Consequently, for any $y \in G$,

$$-y+b+y \in \{a\}' \quad \text{since} \quad \{a\}' \text{ is normal. Therefore, since } a, b \geq 0,$$

$$0 = |a| \land |-y+b+y| = a \land (-y+b+y).$$

Conversely, it is easily shown that $X'$ is a convex $1$-subgroup of $G$. To show that $X'$ is normal in $G$ let $b \in X'$ and $y \in G$. Then $|a| \land |b| = 0$ for all $a \in X$, which implies that

$$|a| \land -y + |b| + y = 0.$$  
But, $|a| \land |-y+b+y| = |a| \land -y+|b|+y$ so therefore $-y+b+y \in X'$, and $X'$ is normal in $G$.

**Theorem 3.1.7.** An $1$-group $G$ is representable, if and only if $a \land b = 0$ implies $a \land (-y+b+y) = 0$ for all $a, b, y \in G$.

**Proof:** If $G$ is representable it is $1$-embeddable in a cardinal product of $0$-groups $\{G(i) \mid i \in I\}$. If $a \land b = 0$ in each $G(i)$,
then $a(i) = 0$ or $b(i) = 0$. Hence, $a(i) \land (-y(i) + b(i) + y(i)) = 0$
in each $G(i)$. Consequently, $a \land b = 0$ in $G$ implies $a \land (-y+b+y) = 0$
in $G$.

Conversely, the class $C$ of all $1$-groups $\{G(i) \mid i \in I\}$ such that
$a \land b = 0$ implies $a \land (-y+b+y) = 0$, for all $a$, $b$ and $y$ members
of $G$, is an equationally definable class by Lemma 3.1.5. Hence, any
$1$-group $G$ in this class may be $1$-embedded in a subdirect product of
subdirectly irreducible $1$-groups from $C$.

Suppose one of these subdirectly irreducible $1$-groups is not an
$0$-group. Call it $H$. Then there exist strictly positive elements
$a, b \in H$ such that $a \land b = 0$. Let
\[
B = \{g \in H \mid |g| \land |b| = 0\}
\]
and
\[
C = \{k \in H \mid |k| \land |g| = 0 \text{ for all } g \in B\}.
\]
Then, by Lemma 3.1.6, $B$ and $C$ are $1$-ideals of $H$. However, $a \in B$
and $b \in C$, and $B \cap C = \{0\}$. This contradicts the assumption that
$H$ is subdirectly irreducible. Therefore, $G$ is representable.

Summing up, the class $C$ consisting of all $1$-groups satisfying
\[
[x \land (-y-x+y)] \lor 0 = 0 \text{ for all } x \text{ and } y,
\]
is the variety $R$ of all
representable $1$-groups.

If $M$ is normal in $M^*$, then $M$ is called a normal value.

We shall now show that the class $N$ of all $1$-groups $G$ such
that each value $M$ in $G$ is a normal value is a variety. Any
element $G$ of $N$ is called a normal valued $1$-group. The defining
equation may be determined by examining the following theorem, which may be found in Wolfenstein [15].

**Theorem 3.1.9.** For $G$ an $l$-group and $\Gamma$ the collection of all values in $G$, the following are equivalent.

1) $G \in N$.
2) For all positive elements $a, b \in G$, $a + b \leq 2b + 2a$.
3) For all pairs of convex $l$-subgroups $A$ and $B$ of $G$, $A + B = B + A$.

**Proof:** (1 \to 2) If $a$ or $b$ is zero, then evidently $2b + 2a \geq a + b$.

Hence, assume $a, b > 0$ and $M$ is a value of $a + b$. Then, since $M$ is normal valued, $M*/M$ is $l$-isomorphic to an $l$-subgroup of the real numbers by Hölder's Theorem. Consequently, $M*/M$ is an abelian $l$-group, and $M + (2b+2a) = M + 2(a+b) > M + (a+b)$. The strict inequality follows from the assumption that $a + b \notin M$.

Suppose now, by way of contradiction, that $2b + 2a \notin a + b$.

Then $(-2b+a+b-2a) \vee 0 > 0$. Hence, for $N$ a value of $(-2b+a+b-2a) \vee 0$ $N + |(-2b+a+b-2b) \vee 0| > N$. Consequently, $N + (a+b) \geq N + (2b+2a)$. Thus, for each value $M$ of $a + b$ which contains $N$, $M + (a+b) \geq M + (2b+2a)$ which is a contraction.

(2 \to 3) For $a \in A$ and $b \in B$, $|a| + |b| \leq 2|b| + 2|a|$. Consequently,

$$-2|b| - 2|a| \leq -|a| - |b| \leq a + b \leq |a| + |b| \leq 2|b| + 2|a|.$$
Hence,

\[ 0 \leq 2|b| + a + b + 2|a| \leq 4|b| + 4|a|. \]

By the Reiz Decomposition Theorem, there exist \( a' \in A \) and \( b' \in B \) such that, \( 0 \leq a' \leq 4|a|, \ 0 \leq b' \leq 4|b| \) and

\[ 2|b| + (a+b) + 2|a| = b' + a'. \]

This implies,

\[ a + b = (-2|b|+b') + (a'-2|a|) \]

\[ \in B + A. \]

Similarly, it may be shown that for \( c \in A \) and \( d \in B \)

\[ d + c \in A + B. \]

Therefore, \( A + B = B + A. \)

\( (3 + 1) \) Suppose, by way on contradiction, there exists a value \( M \) in \( G \) and a positive element \( y \in M* \) such that \(-y+M+y \neq M\). Then,

\( 0 < -y + m + y \in M* \setminus M \) for some positive \( m \in M \). Since \( M* = G(M,-y+m+y) \) and \( G(M,-y+m+y) + M = M + G(M,-y+m+y) \), we claim,

\[ M* = \{ g \in G \mid |g| \leq m* + n*(-y+m+y), \text{ for some } m* \in M, n* \in \mathbb{Z} \}. \]

To see this, let \( g \in G(M,-y+m+y) \). Then

\[ |g| \leq m_1 + (-y+m+y) + m_2 + \ldots + m_n + (-y+m+y) \]

\[ = m* + (-y+m+y)_1 + \ldots + (-y+m+y)_n. \]
for some \( m, m^* \in \mathcal{M} \) and where \((-y^*+y)^i \in G(-y^*+y)\). Hence there exists \( l_i \) a non-negative integer such that

\[
(-y^*+y)^i \leq l_i(-y^*+y).
\]

Consequently,

\[
|g| \leq m^* + l_1(-y^*+y) + \ldots + l_n(-y^*+y).
\]

Let

\[
n_1 = \max\{l_i\}, \text{ then }\]

\[
|g| \leq m^* + l_1(-y^*+y) + \ldots + l_1(-y^*+y)
\]

\[
= m^* + n_1(-y^*+y).
\]

Upon letting \( n_1 = n^* \), the claim has been proved.

Returning to the proof of the theorem, \( 0 < y \in \mathcal{M}^* \setminus \mathcal{M} \), hence,

\[
0 < y \leq m + n(-y^*+y)
\]

for some \( m \in \mathcal{M} \) and an integer \( n \). Thus, since \( y \notin \mathcal{M} \),

\[
\mathcal{M} < \mathcal{M} + y \leq \mathcal{M} + n(-y^*+y) = \mathcal{M} + (-y^*+y) + y.
\]

However, \( \mathcal{M} + y \leq \mathcal{M} + (-y^*+y) \) implies that there exists an \( a \in \mathcal{M} \), such that,

\[
a + y \leq -y + nm + y
\]

or,

\[
y \leq nm - a.
\]
Thus, $M + y \leq M$, contradicting the fact that $M < M + y$. Therefore, $-y + M + y = M$, and $M$ is normal.

From statement (3) in the theorem, the defining equation for $N$ can be determined. For any $G \in N$, and $a, b \in G$,

$$(a \lor 0) + (b \lor 0) = [(a \lor 0) + (b \lor 0)] \lor [2(b \lor 0) + 2(a \lor 0)].$$

The reader may also note that this equation is equivalent to

$$(a \lor 0) + (b \lor 0) = [(a \lor 0) + (b \lor 0)] \lor [n(b \lor 0) + n(a \lor 0)]$$

for any integer $n$ greater than two.

**Example 3.1.10.** For each positive integer $n$, a variety $S(n)$ of $1$-groups will be constructed. In recent literature, this variety is called the Scrimger variety. It was first introduced by Martinez [10]; Smith [14] and Scrimger [13] extended this work.

**Lemma 3.1.11.** Let the set of integers be denoted by $Z$. Then,

$$G(n) = \{(F, k) \mid F(i) = F(j) \text{ if } i \equiv j \text{ (mod } n), k \in Z, F \in P(Z)\} \subseteq Z \wr Z$$

is an $1$-subgroup of $Z \wr Z$.

**Proof:** The binary operation on $Z \wr Z$ is defined as follows:

For $(F, k)$ and $(G, \ell)$ in $Z \wr Z$

$$(F, k) + (G, \ell) = (F + G^k, k + \ell),$$

where $G^k(z) = G(k + z)$ for all $z \in Z$. 
The inverse of \((F,k)\) is \((-F^{-k},-k)\), and the identity element of
\(Z \wr Z\) is \((\bar{0},0)\) where \(\bar{0}(z) = z\) for all \(z \in Z\).

With this information, the reader may convince himself that \(G(n)\)
is a subgroup of \(Z \wr Z\).

The last item to check, is that \(G(n)\) is a sublattice of \(Z \wr Z\).
For \((F,k) \in G(n)\), the least upper bound of \((F,k)\) and \((\bar{0},0)\) in
\(Z \wr Z\) is: \((F,k) \lor (\bar{0},0) = (H,h)\) where for \((a,b) \in Z \times Z\)

\[
(a,b) = \begin{cases} 
(a,b) & \text{if } k < 0 \\
(F \uparrow^b(a), b+k) & \text{if } k > 0 \\
(F \uparrow^b(a) \lor \bar{0}^b(a),b) & \text{if } k = 0 .
\end{cases}
\]

But, \((H,h)\) is again in \(G(n)\). Therefore, \(G(n)\) is an \(l\)-subgroup of
\(Z \wr Z\).

The variety generated by \(G(n)\) is the Scrimger variety \(S(n)\).

**Example 3.1.12.** Let \(n\) be a positive integer. Then \(L(n)\) denotes
the variety of all \(l\)-groups \(G\), such that,

\[nx + ny = ny + nx \text{ for all } x,y \in G.\]

Section 2. Containment Relationships

In this section the containment relationships between the varieties
\(E, A, R, N, S(n), L(n)\) and \(L\) will be exhibited. A variety \(V\) is
contained in a variety \(U\) if each element of \(V\) is also an element of \(U\).
The variety \( E \) is the smallest variety of \( l \)-groups, since it is contained in every other variety of \( l \)-groups.

The variety \( L \) is the largest variety of \( l \)-groups since it contains every other variety of \( l \)-groups.

**Theorem 3.2.1.** If \( G \) is an element of \( A \), then \( G \) is an element of every variety \( V \) of \( l \)-groups other than \( E \).

Weinberg [15] constructed the free abelian \( l \)-group, and characterized it as a subdirect sum of a family consisting of copies of the integers.

The free abelian \( l \)-group, with \( \alpha \) free generators denoted by \( A_\alpha \), is described as follows:

For \( \alpha \), let \( J_\alpha \) denote the free abelian group of rank \( \alpha \), and let \( \Gamma \) denote the family of total orders on \( J_\alpha \), \( \alpha \) free group or free abelian group can always be totally ordered). Then \( A_\alpha \) is a sublattice of the cardinal product \( P_\alpha \) of the family of \( O \)-groups \( \{[J_\alpha,T], T \in \Gamma\} \). The sublattice \( A_\alpha \) is generated by the diagonal of the cardinal product. Thus every abelian \( l \)-group is a subdirect sum of a family of copies of the ordered group of integers.

**Proof of Theorem 3.2.1:** Let \( V \) be any variety of \( l \)-groups other than \( E \), and let \( G \) be an element of \( V \). Then \( G \) contains a copy of the integers \( \mathbb{Z} \). Therefore \( V \) contains, as an element, the \( O \)-group \( (\mathbb{Z},+) \). Moreover, this implies that \( V \) contains a subdirect sum of a family of \( O \)-groups \( (\mathbb{Z},+) \). Consequently, \( V \) contains the free abelian \( l \)-groups; and since each \( G \in A \) is the \( l \)-homomorphic image of some suitable chosen free abelian \( l \)-group, \( A \subseteq V \).
Theorem 3.2.1 also implies that the variety $A$ covers the variety $E$, i.e. there are no varieties strictly contained between $A$ and $E$.

The next set of propositions will show that the variety $S(n)$ is contained in the variety $L(n)$; and $S(n)$ covers $A$, if $n$ is a prime number.

To show $S(n)$ is contained in $L(n)$, it suffices to show that $G(n)$ satisfies the defining law, $nx + ny = ny + nx$ of $L(n)$, since $S(n)$ is generated by the 1-group $G(n)$.

**Theorem 3.2.2.** For $(F,k)$ and $(G,h)$ in $G(n)$, $n(F,k) + n(G,h) = n(G,h) + n(F,k)$.

**Proof:**

\[
n(F,k) = (F+F+...+F^{(n-1)k},nk) \\
n(G,h) = (G+G^h+...+G^{(n-1)h},nh)
\]

Hence, $n(F,k) + n(G,h)$

\[
= (F+F+...+F^{(n-1)k}+G^{nk}+G^{nk}+...+G^{(n-1)h},nk+nh),
\]

\[
= (F+F+...+F^{(n-1)k}+G^{nk}+h+...+G^{(n-1)h},nk+nh)
\]

\[
= (G+G^h+...+G^{(n-1)h}+F^{nh}+...+F^{nk+(n-1)k},nk+nh)
\]

\[
= n(G,h) + n(F,k).
\]

**Corollary 3.2.3.** $S(n) \subseteq L(n)$.

In Smith [14], this result has been extended to $S(n) \subseteq L(n)$ for every composite integer $n$. 
Corollary 3.2.4. For \( n \) and \( m \) positive integers the following statements are equivalent.

1) \( L(n) \subseteq L(m) \)
2) \( G(n) \subseteq L(m) \)
3) \( n \) divides \( m \)
4) \( S(n) \subseteq L(m) \).

In showing that \( S(n) \) covers \( A \) if \( n \) is a prime number, several other results will be derived. Namely, neither \( S(n) \) nor \( L(n) \) (\( n > 1 \)) is contained in \( R \). Also, if \( m \) and \( n \) are relatively prime, then \( S(m) \cap S(n) = L(n) \cap L(m) = A \).

Lemma 3.2.5. If \( C \) is a convex 1-subgroup of \( G \) in \( L(n) \), then \( -nx + C + nx = C \) for all \( x \in G \).

Proof: Let \( c \) be a positive element of \( C \) then

\[
0 < c < 2c < \ldots < nc = -nx + nc + nx
\]

\[
\in -nx + C + nx.
\]

Hence, \( c \in -nx + C + nx \), (since a conjugate of a convex 1-subgroup is another convex 1-subgroup). Therefore, \( C \subseteq -nx + C + nx \). Similarly it can be shown that \( -nx + C + nx \subseteq C \).

Lemma 3.2.6. If \( C \) is a convex 1-subgroup of \( G \in L(n) \) and \( x \in G \), then the number of distinct conjugates, of the form \( -ix + C + ix \) (\( i \in \mathbb{Z} \)), is a divisor of \( n \).
Proof: For i and k integers, write

\[ i = pn + r \quad \text{and} \quad k = qn + s \]

where \( 0 \leq r, s < n \) and \( p, q \in \mathbb{Z} \).

If \( r = s \), then by Lemma 3.2.5

\[
-ix + C + ix = -(pn+r)x + C + (pn+r)x
= -rx + C + rx
= -sx + C + sx
= - (qn+s)x + C + (qn+s)x
= -kx + C + kx.
\]

Consequently, in proving the lemma we restrict our attention to the integers \( k \) such that \( 0 \leq k < n \).

If \(-ix + C + ix = C\), then \( i \) divides \( n \). In order to see this, suppose \(-kx + C + kx = C\) where \( k > 0 \) is the smallest positive integer with this property. If \( k \) does not divide \( n \), then the greatest common divisor \( b \) of \( k \) and \( n \) may be written as \( kp + nq \), for \( p \) and \( q \) some integers. Then,

\[
-bx + C + bx = -(kp+nq)x + C + (kp+nq)x
= -kpx + C + kpx
= C.
\]

But this contradicts the assumption that \( k \) was the smallest such integer; and therefore, \( k \) divides \( n \).

Consequently, if there are \( m \) conjugates of the form \(-ix + C + ix = C\), then \( m \) divides \( n \).
Lemma 3.2.7. \( R \cap L(n) = A \).

Proof: For \( G \in R \cap L(n) \), \( G \) is a product of totally ordered groups \( \{G_i\} \), and for \( a, b \in G \), \( na + nb = nb + na \). Hence, in each of the totally ordered groups \( \{G_i\} \), \( nc + nd = nd + nc \) for \( c \) and \( d \) elements in \( G_i \).

Now if \( c \) and \( d \) do not commute in \( G_i \), then without loss of generality, assume that \( c, d \geq 0 \) and \( c + d < d + c \). This implies that \( n(-d+c+d) = -d + nc + d < nc \). But,

\[
nc = -nd + nc + nd \cdots < -d + nc + d < nc,
\]

a contradiction. Hence, \( c \) and \( d \) commute, and consequently each of the totally ordered groups is abelian. Therefore \( G \in A \).

Corollary 3.2.8. \( R \not\subseteq L(n) \) and \( L(n) \not\subseteq R \) for \( n > 1 \).

Corollary 3.2.9. \( R \not\subseteq S(n) \) and \( S(n) \not\subseteq R \) for \( n > 1 \).

Proof: The variety \( S(n) \) also satisfies the law \( nx + ny = ny + nx \).

Lemma 3.2.10. If \( m \) and \( n \) are relatively prime positive integers, then \( L(m) \cap L(n) = A \).

Proof: Let \( G \in L(m) \cap L(n) \) be a subdirectly irreducible 1-group. This is equivalent to \( G \) having a smallest non-trivial 1-ideal or to \( G \) having a representing subgroup. Thus \( G \) acts transitively on some totally order set \( \Omega \).
Suppose \( g \in G \) fixes \( a \in \Omega\), and \( h \in G \) does not fix \( a \in \Omega\). Choose integers \( r \) and \( s \) such that \( rm + sn = 1 \). Then,

\[
ah = a[mng + (mr+ns)h] \\
= a[mrh + mng + nsh] \\
= a[(mr+ns)h + mng] \\
= a[h + mng] \\
= (ah)mng.
\]

The permutation \( mng \) fixes \( ah \), and consequently \( g \) fixes \( ah \). Hence, since \( G \) is transitive, \( g \) must be the identity. Therefore, if an element \( k \) of \( G \) fixes an element of \( \Omega \), then \( k \) is the identity; and \( G \) must be an 0-group. This implies \( G \in L(n) \cap R = A \). But, since \( G \) is an arbitrary element in the generating set of \( L(m) \cap L(n) \), we conclude that \( L(m) \cap L(n) = A \).

**Corollary 3.2.11.** If \( m \) and \( n \) are relatively prime, then \( S(m) \cap S(n) = A \).

**Lemma 3.2.12.** If \( C \) is a representing subgroup of \( G \in L(n) \), \( G \notin A \), then there exists a strictly positive element \( x \) in \( G \) such that \(-x + C + x \neq C\).

**Proof:** Suppose, for all \( x \) where \( 0 < x \in G \), that \(-x + C + x = C\). Then \( C \) is an \( l \)-ideal of \( G \) as well as a representing subgroup. Consequently, \( C = \{0\} \), and therefore, \( G \) is an 0-group. Thus \( G \in R \) and \( G \in L(n) \), which by Lemma 3.2.8 implies \( G \in A \), a contradiction.
Lemma 3.2.13. If $C$ is a representing subgroup of $G \in L(n)$ which has $n$ distinct conjugates of the form $-ix + C + ix = C$ for some $x \in G$, then $G$ contains an $l$-subgroup which is $l$-isomorphic to $G(n)$.

Proof: An element $a$ will be constructed so that $a \in G$, and $a$ and $x$ correspond under an $l$-isomorphism to the generating elements $(0,1)$ and $(B,0)$ of $G(n)$ where

$$B(z) = \begin{cases} 0 & \text{if } z \not\equiv 0 \pmod{n} \\ 1 & \text{if } z \equiv 0 \pmod{n}. \end{cases}$$

For $0 \leq i \leq n-1$ define $C(i)$ and $D(i)$ by

$$C(i) = -ix + C + ix,$$
$$D(i) = \bigcap_{j \neq i} C(j).$$

Let $\Omega$ be the totally ordered set of right cosets of $C$ in $G$.

For $0 \leq i \leq n-1$, $C + ix$ has $C(i)$ as its stabilizer subgroup. The set $D(i)$ consists of all permutations which fix each $C + jx$ for $j \neq i$.

Since by hypothesis $C(0) \neq C(i)$ ($1 \leq i \leq n-1$), there exist $h(i) \in C(0) \setminus C(i)$ ($1 \leq i \leq n-1$) such that $0 < h(i)$. Whence,

$$0 < g(0) = h(0) + \ldots + h(n-1) \in C(0) \setminus \bigcup_{i \neq 0} C(i).$$

Define $d(0)$ by

$$0 < d(0) = x \land \left( \land_{i \neq 0} (-ix + ng(0) + ix) \right).$$

Since $ng(0)$ fixes only $C$, $-ix + ng(0) + ix$ fixes only $C + ix$ ($1 \leq i \leq n-1$). Thus, $d(0)$ moves $C$ only, and $d(0) < x$, since
x moves C.

Define \( d(i) \) by

\[
d(i) = -ix + d(0) + ix \quad (1 \leq i \leq n-1).
\]

Then \( 0 < d(i) \in D(i) \setminus \bigcup_{j \neq i} D(j) \).

Define \( e(i) \) by

\[
e(i) = \bigvee_{j \neq i} (d(i) \wedge d(j)) \quad (0 \leq i \leq n-1),
\]

which by convexity implies that \( e(i) \in \bigcap_{k=0}^{n-1} D(k) = \bigcap_{k=0}^{n-1} C(k) \).

Now let

\[
a(i) = -ix + a(0) + ix = d(i) = e(i) \quad (0 \leq i \leq n-1).
\]

Then, the following hold:

1) \(-nx + a(i) + n(x) = a(i),\)
2) \(0 < a(0) < x,\)
3) \(0 < a(i) < x,\)
4) \(a(i) \in D(i) \setminus \bigcup_{j \neq i} D(j),\)
5) \(a(i) \wedge a(j) = 0 \quad (i \neq j)\)
6) \(a(i) + a(j) = a(j) + a(i).\)

The map \( f : G(n) \to G \), defined by

\[
f(F,k) = F(0)a(0) + \ldots + F(n-1)a(n-1) + kx
\]

is a group of homomorphism, since for \((F,k)\) and \((G,h) \in G(n)\)
\[ f((F,k)+(G,h)) = f(F+G^k,k+h) \]
\[ = [F+G^k(0)]a(0) + \ldots + [F+G^k(n-1)]a(n-1) + (k+h)x \]
\[ = F(0)a(0) + \ldots + F(n-1)a(n-1) - kx + kx + G^k(0)a(0) + \]
\[ k(x) - k(x) + \ldots + k(x) - k(x) + G^k(n-1)a(n-1) + (k+h)x \]
\[ = F(0)a(0) + \ldots + F(n-1)a(n-1) + k(x) + G^k(a(k) + \ldots + \]
\[ G^k(k+n-l)a(n-l+k) + h(x) \]
\[ = f(F,k) + f(G,h) . \]

The homomorphism \( f \) is also an 1-homomorphism. This will be shown by first letting \( k = 0 \), and then \( k \neq 0 \).

For \((F,0) \in G(n)\), let \( T = \{ i \mid F(i) \geq 0 \} \), then since \( a(j) \wedge a(i) = 0 \) for \( i \neq j \),

\[ f(F,0) \wedge f(0,0) = [F(0)a(0) + \ldots + F(n-1)a(n-1)] \wedge 0 \]
\[ = \sum_{i \in T} F(i)a(i) + (\sum_{i \in T} F(i)a(i) \wedge \sum_{i \notin T} F(i)a(i)) \]
\[ = \sum_{i \in T} F(i)a(i) - (\sum_{i \notin T} F(i)a(i) \wedge \sum_{i \in T} F(i)a(i)) \]
\[ = \sum_{i \in T} F(i)a(i) = f((F,0) \wedge (0,0)) . \]

We now show that for \((F,k) \in G(n)\) with \( k \neq 0 \),

\[ f(F,k) \wedge f(0,0) = f((F,k) \wedge (0,0)) . \]

To do so, it will be shown that \( ma(0) < x \) for \( m \in \mathbb{Z} \). This is true for \( m \leq 1 \). Assume for \( 2 \leq i < m \) that \( ia(0) < x \). Since,

\[ ma(0) = 0 \wedge m[(d(0)-d(j))] , \]
\[ j \neq 0 \]

it suffices to show that one of the terms in \( ma(0) \) is less than \( x \), namely the term
\[ d(0) - d(n-1) + [0 \lor m(A \land (d(0) - d(j)))] + d(0) - d(1) \]

Now,
\[ d(0) - d(n-1) + [0 \lor (m-2) \land (d(0) - d(j))] + d(0) - d(1) < x \]

if and only if,
\[ [0 \lor m(A \land (d(0) - d(j)))] < d(n-1) - d(0) + x + d(1) - d(0). \]

But, \( d(n-1) - d(0) + x + d(1) - d(0) = x \). Therefore, \( m a(0) < x \), for all \( m \in \mathbb{Z} \). Moreover, \( m a(j) < x \), \( (m \in \mathbb{Z}) \) \( (j = 0, 1, \ldots, n-1) \).

With this result we have,
\[ F(0)a(0) + \ldots + F(n-1)a(n-1) + mx < 0 \text{ if } m < 0, \]
and
\[ F(0)a(0) + \ldots + F(n-1)a(n-1) + mx > 0 \text{ if } m > 0. \]

Thus, \( f[(F,k) \lor (0,0)] = f(F,k) \lor f(0,0). \)

Finally, \( f \) is one-to-one, since \( f(F,k) = 0 \) if and only if \( F = 0 \) and \( k = 0. \)

**Theorem 3.2.14.** If \( n \) is a prime number then \( S(n) \) covers \( A \).

**Proof:** If \( G \) is a subdirectly irreducible member of \( S(n) \), but not an element of \( A \), then \( G \) contains a representing subgroup \( C \), which is not an 1-ideal of \( G \). Thus, there exists a strictly positive element \( x \) in \( G \) such that \(-x + C + x \neq C\). By Lemma 3.2.6, since \( n \) is prime, \( C \) must have \( n \) distinct conjugates of the form \(-ix + C + ix. \) Thus, \( G \) contains an 1-subgroup 1-isomorphic to \( G(n) \). Consequently, if \( V \) is any variety contained in \( S(n) \) which has a
nonabelian member, then $V$ contains $S(n)$. Thus, $S(n)$ covers $A$.

The variety $N$ consisting of normal valued 1-groups will be shown to be the largest non-trivial variety. This result was established in Holland [7].

**Theorem 3.2.15.** If $H$ is a nontrivial 0-2 transitive 1-group of order preserving permutations on a totally ordered set $S$, if $F$ is the free 1-group on a countable set $X$, and $w \in F$ is not the identity element of $F$ then $H$ does not satisfy the law $w = e$.

**Proof:** By Hollands' Representation Theorem, it may be assumed that $F$ is an 1-permutation group on some totally ordered set $T$.

Since $w \in F$ is not the identity element, there exists an element $t$ in $T$ such that $tw \neq t$. Let $w = \bigwedge_{IJK} x_{ijk}$, where $I$ and $J$ are finite sets, and $K = \{1, 2, \ldots, n\}$. For each $(i, j) \in I \times J$, define

$$t(i, j, 0) = t;$$

and for $1 \leq k \leq n$, define

$$t(i, j, k) = t(i, j, k-1)x_{i,j,k}.$$

For each element $x$ of $X$ occurring in $w$, and each pair $(i, j)$ let

$$P_{ij}(x) = \{k \in K \mid x = x_{i,j,k}\}$$

and

$$N_{ij}(x) = \{k \in K \mid x^{-1} = x_{i,j,k}\}.$$
If \( k \in P_{ij}(x) \), then

\[
t(i,j,k-1)x = t(i,j,k);
\]

and if \( k \in N_{ij}(x) \), then

\[
t(i,j,k)x = t(i,j,k-1).
\]

The set \( T' = \{t(i,j,k) \mid (i,j) \in I \times J, \ 0 \leq k \leq n\} \) is a finite subset of \( T \). Choose and label any subset \( \{s(i,j,k) \mid (i,j) \in I \times J, \ 0 \leq k \leq n\} \) of \( S \) in one-to-one correspondence with \( T' \); so that the correspondence \( t(i,j,k) \rightarrow s(i,j,k) \) preserves the order on \( T' \). Since, multiplication by \( x \) provides a one-to-one order preserving correspondence such that

\[
t(i,j,k-1) + t(i,j,k) \quad (k \in P_{ij}(x))
\]

and

\[
t(i,j,k) + t(i,j,k-1) \quad (k \in N_{ij}(x));
\]

it follows that the correspondence

\[
s(i,j,k-1) + s(i,j,k) \quad (k \in P_{ij}(x))
\]

and

\[
s(i,j,k) + s(i,j,k-1) \quad (k \in N_{ij}(x))
\]

must also be one-to-one and order preserving.

Because \( H \) is 0-2 transitive on \( S \), it is also 0-n transitive; and thus there exist \( h(x) \in H \) such that,

\[
s(i,j,k-1)h(x) = x(i,j,k) \quad \text{for} \ k \in P_{ij}(x),
\]
and

\[ s(i,j,k)h(x) = s(i,j,k-1) \text{ for } k \in N_{ij}(x). \]

Since \( t = t(i,j,0) \) for each \((i,j)\), denote by \( s \) the elements \( s(i,j,0) \) for each \((i,j)\). Then for the substitution \( x \mapsto h(x) \), we have for each \((i,j) \in I \times J, \)

\[ s \Pi h(x_{ijk}) = s(i,j,n). \]

Because \( tw \neq t, \)

\[ t \neq t \forall \Pi x_{ijk} = \forall \Pi t(i,j,n). \]

Recalling that the set \( \{s(i,j,k)\} \) is in one-to-one order preserving correspondence with \( T' \),

\[ s \forall \Pi h(x_{ijk}) = \forall x \Pi h(x_{ijk}) \]

\[ = \forall s(i,j,n) \]

\[ \neq s. \]

Consequently, \( H \) does not satisfy \( w = e. \)

**Corollary 3.2.16.** If \( G \) is an 1-group which satisfies a law not satisfied by every 1-group, then \( G \) is normal valued.

**Proof:** Let \( G \) be an 1-group which is not normal valued and let \( w = e \) be a law not satisfied by every 1-group. Then by Theorem 2.4.13, \( G \) contains an 1-subgroup which acts 0-2 transitively on some totally
ordered set. Therefore, \( w = e \) is not satisfied by any variety containing \( G \).

**Corollary 3.2.17.** \( N \) is the largest proper variety of 1-groups.
CHAPTER IV

The Lattice $L$ of Varieties of Lattice Ordered Groups

In this chapter we will discuss the properties of the collection $L$ of all varieties of 1-groups. It will be shown that $L$ is a complete lattice; and that a multiplication on the elements of $L$ can be defined so that $L$ becomes a lattice ordered semigroup.

Some of the topics concerning the lattice ordered semigroup $L$ which will be discussed in this chapter are:

1) the idempotents of $L$;
2) the generation of varieties;
3) the factorization of varieties;

The material in this chapter may be found in Glass, Holland and McCleary [5], Martinez [11], [12], and Smith [14].

Section 1. Torsion Classes

A torsion class of 1-groups is a collection of 1-groups which is closed with respect to homomorphic images, convex 1-subgroups, and joins of convex 1-subgroups in the class. This idea of torsion class was first introduced and studied by Martinez [12].

An important consequence derived from this approach is the T-torsion radical of $G$ denoted by $T(G)$, which is defined as follows: for
an l-group $G$ and a torsion class $T$, $T(G)$ is the join of all convex l-subgroups of $G$ belonging to $T$. It should be clear to the reader that $T(G)$ is the largest convex l-subgroup of $G$ belonging to $T$, and hence, it is an l-ideal of $G$.

Several basic properties of the $T$-torsion radical of an l-group are found in the following proposition which is due to Martinez [12].

**Proposition 4.1.1.** Let $T$ be a torsion class and $G$ be an l-group.

1) If $A$ is a convex l-subgroup of $G$, then $T(A) = A \cap T(G)$.

2) If $f : G \to H$ is an l-homomorphism of $G$ onto $H$, then $f(T(G)) \subseteq T(H)$.

3) $T(T(G)) = T(G)$ i.e., $T(G)$ is closed.

4) If $\{ A_i \mid i \in I \}$ is a collection of convex l-subgroups of $G$, then $T(\bigvee A_i) = \bigvee T(A_i)$, and also $T(\bigwedge A_i) = \bigwedge T(A_i)$.

**Proof of (1).** The convex l-subgroup $A \cap T(G)$ of $T(G)$ is in $T$, and since $A \cap T(G)$ is also a convex l-subgroup of $A$, $A \cap T(G) \subseteq T(A)$.

Conversely, $T(A)$ is a convex l-subgroup of $A$, and hence, it is a convex l-subgroup of $G$. Since $T(A)$ is in $T$, $T(A) \subseteq A \cap T(G)$.

**Proof of (2).** Since $T$ is closed under l-homomorphic images, $f(T(G))$ is a convex l-subgroup of $H$ which is contained in $T$. Thus, $f(T(G)) \subseteq T(H)$.

The reader should note that if $h : G \to G$ is an l-automorphism of $G$, then $f(T(G)) \subseteq T(G)$, in other words $T(G)$ is a fully invariant l-ideal of $G$. 
Proof of (3). By (1), and the fact that $T(G)$ is a convex 1-subgroup of $G$, we have $T(T(G)) = T(G) \cap T(G) = T(G)$.

Proof of (4). $T(\lor A_i) \supseteq T(A_i)$ for all $i \in I$, thus,

$$T(\lor A_i) \supseteq \lor T(A_i).$$

For $\{A_i \mid i \in I\}$ a collection of convex 1-subgroups of $G$, $\lor A_i$ is also a convex 1-subgroup of $G$. Moreover, $T(A_i) \subseteq T(G)$, and thus $\lor T(A_i) \subseteq T(G)$. Consequently by (1),

$$T(\lor A_i) = (\lor A_i) \cap T(G)$$

$$= \lor (A_i \cap T(G))$$

$$= \lor T(A_i)$$

Since $T(\land A_i) \in T$ is a convex 1-subgroup of each $A_i$, we have $T(\land A_i) \subseteq \land T(A_i)$.

Conversely, $\land T(A_i)$ is a convex 1-subgroup of each $T(A_i)$, hence,

$\land T(A_i) \in T$. Also, $\land T(A_i)$ is a convex 1-subgroup of each $A_i$, so that $\land T(A_i) \subseteq T(\land A_i)$.

We now proceed to show that the 1-group variety $N$ is a torsion class. With this proposition we then reproduce a result by Holland [8] which states that every 1-group variety is a torsion class.

Proposition 4.1.2. The 1-group variety $N$ is a torsion class.

Proof: Since $N$ is a variety, it is closed with respect to taking 1-subgroups, 1-homomorphic images, and cardinal products of normal valued 1-groups. Thus, to prove $N$ is a torsion class it suffices to show that
$N$ is closed with respect to taking joins of normal valued convex l-subgroups from an l-group $G$. To this end, let $\{A_i \mid i \in I\}$ be a collection of normal valued convex l-subgroups of some l-group $G$.

We will show that for $0 \leq x, y \in \bigvee A_i$, the inequality $x + y \leq ny + nx$ is satisfied for some positive integer $n$.

First, however, we will concentrate on the special case of $0 \leq a \in A_i$ and $0 \leq b \in A_j$. Let $\tilde{a} = a - (a \wedge b)$ and $\tilde{b} = b - (a \wedge b)$. Then, $\tilde{a} \wedge \tilde{b} = 0$, $\tilde{a} + b = \tilde{b} + a$, and $a \wedge b$, $\tilde{a}$ and $\tilde{b}$ are all non-negative. Thus,

$$a + b = \tilde{a} + (a \wedge b) + b \leq 2(a \wedge b) + 2\tilde{a} + b \quad (\tilde{a}, a\wedge b \in A_i)$$

$$= 2(a \wedge b) + \tilde{a} + \tilde{b} + a$$

$$= 2(a \wedge b) + \tilde{b} + \tilde{a} + a$$

$$\leq 3b + 3a.$$

Utilizing this result, we may now show that for $0 \leq a, b \in \bigvee A_i$, $a + b \leq nb + na$. Since $0 \leq a, b \in \bigvee A_i$, by the Reiz Decomposition Theorem, $a$ and $b$ are finite sums ($\Sigma a_i$ and $\Sigma b_j$, respectively) of non-negative elements from $\bigcup A_i$. Thus,

$$a + b = \Sigma a_i + \Sigma b_j$$

$$\leq n \Sigma b_j + n \Sigma a_i,$$

$$= nb + na$$

for some positive integer $n$. Although this integer $n$ depends on $a$ and $b$ it is clear that condition (3) of Theorem 3.1.9 is satisfied by $\bigvee A_i$ and therefore that $\bigvee A_i$ is normal valued.
An 1-group $G$ is a **lex-extension** of a prime subgroup $C$ of $G$ if $0 < a \in G$ and $a \wedge b = 0$ for some $0 < b \in G$, then $a \in C$. Consequently, if $0 < g \in G \setminus C$ then $g > C$.

**Lemma 4.1.3.** Let $G$ be a subdirectly irreducible normal valued 1-group generated by $g_1, g_2, \ldots, g_n$. Then for some $1 \leq k \leq n$, $G = G(g_k)$.

**Proof:** Let $C$ be a value of some element in the minimum 1-ideal of $G$. Then $\{g_1, g_2, \ldots, g_n\} \not\subseteq C$.

Since the convex 1-subgroups of $G$, which contain a prime subgroup of $G$, are totally ordered under inclusion, the collection $\{K \supseteq C \mid K$ is a value of some $g_i\}$ is totally ordered. Thus, each $g_i$ has only one value which contains $C$. Moreover, since the number of generators of $G$ is finite, the set has a largest element which will be denoted by $M$, and we will say that $M$ is a value of $g_m$.

Since any convex 1-group of $G$ containing $M$ must contain all of the generators of $G$, the cover of $M$ is all of $G$. By the assumption that $G$ is a normal valued 1-group, and by Hölder's Theorem, we conclude that $G/M$ is 1-isomorphic to the real numbers.

We will now show that $G$ is a lex-extension of $M$. Let $0 < a, b \in G$ and let $a \wedge b = 0$. Since $\bigcap_{g \in G} -g + C + g$ is an 1-ideal of $G$ which does not contain the minimum 1-ideal of $G$, the 1-ideal $\bigwedge_{g \in G} -g + C + g$ is the identity element of $G$. Thus, there exists an element $g \in G$ such that $b \not\in -g + C + g$. However, since $-g + C + g$ is a prime subgroup of $G$ and $a \wedge b = 0$, we conclude that $a \in -g + C + g \subseteq -g + M + g = M$. Thus, $G$ is a lex-extension of $M$. 

79.
Finally, $g_m \not\in M$, and $G/M$ is an archimedean 0-group, and hence $G = G(g_m)$.

**Theorem 4.14.** Every variety $V$ of 1-groups is a torsion class.

**Proof:** Let $G$ be an 1-group, and let $\{A_i \mid i \in I\}$ be a collection of convex 1-subgroups of $G$ such that $A_i \in V \ (i \in I)$.

If $V$ is the variety consisting of all 1-groups then clearly $V A_i \in V$.

Thus, let $V \subseteq N$ and let $\omega(x_1, \ldots, x_n) = e$ be a law of $V$ not satisfied by every 1-group. Then $A_i \in N \ (i \in I)$. Thus $V A_i \in N$.

We wish to show that $\omega(x_1, \ldots, x_n) = e$ holds in $V A_i$.

For $h_1, h_2, \ldots, h_n \in V A_i$, each $h_i$ is a finite sum $\sum a_{ij}$ of elements $a_{ij} \in \bigcup A_i$. Let $H$ be the 1-subgroup of $V A_i$ generated by $\{a_{ij}\}$. Then, $H$ is a member of $N$.

Let $\bar{H}$ be a subdirectly irreducible factor of $H$. Then, $\bar{H}$ is also a member of $N$. Moreover, if $h \mapsto \bar{h}$ denotes the natural map of $H$ onto $\bar{H}$, then $\{\bar{a}_{ij}\}$ generates $\bar{H}$. Hence, by Lemma 4.1.3, we conclude that $\bar{H} = \bar{H}(\bar{a}_{ij})$ for some $i$ and some $j$.

The preimage $a_{ij}$ of $\bar{a}_{ij}$ is an element of $A_k$ for some $k \in I$. Thus, $A_k \cap H = \bar{H}$. From the assumption that $A_k$ satisfies $\omega(x_1, \ldots, x_n) = e$ we conclude that $A_k \cap H = \bar{H}$ satisfies $\omega(x_1, \ldots, x_n) = e$.

Therefore, since $H$ is a subdirect product of subdirectly irreducible factors, it satisfies $\omega(x_1, \ldots, x_n) = e$. Thus $\omega(h_1, \ldots, h_n) = 0$ and so $V A_i$ satisfies $\omega(x_1, \ldots, x_n) = e$ as required.
Corollary 4.1.5. For an 1-group $G$ and a variety of 1-groups $U$, there exists a unique 1-ideal $U(G)$ of $G$ such that $U(G) \in U$ and $U(G)$ contains every convex 1-subgroup of $G$ which is a member of $U$.

The reader is now aware of the fact that $U(G)$ is the $U$-torsion radical of $G$. Consequently, Proposition 4.1.1 is applicable to varieties of 1-groups.

With this background we now proceed to Section 2, where we consider the varietal product of 1-group varieties.

Section 2. Varietal products

Since the intersection of a class of 1-group varieties is again a variety; it is natural to define a partial order, on the class of all 1-group varieties $L$, by using set containment. For all $U$ and $V$ in $L$ define

$$V \leq U \text{ if and only if } V \subseteq U.$$ 

The partial order $\leq$ on $L$ becomes a lattice order, if one defines for $V_i \in L$ ($i \in I$),

$$\bigwedge_{i \in I} V_i = \bigcap_{i \in I} V_i,$$

and

$$\bigvee_{i \in I} V_i = \bigvee \{ U \in L \mid U \geq V_i, i \in I \}.$$ 

Since $L$ contains both a largest element and a smallest element, the definitions of $\vee$ and $\wedge$ on $L$ suffice to make $L$ a complete lattice.
One may also define a multiplication on $L$ as follows: for $U$ and $V$ members of $L$, $G \in UV$ if and only if $G$ contains an 1-ideal $H$ such that $H \in U$ and $G/H \in V$. Or, $G \in UV$ if and only if $G/\langle U(G) \rangle \in V$.

**Proposition 4.2.1.** If $U$ and $V$ are members of $L$, then $UV$ is an element of $L$.

**Proof:** It will be shown that $UV$ satisfies the definition of a variety.

Let $G \in UV$, then $G$ contains an 1-ideal $H$ such that $H \in U$ and $G/H \in V$. For an 1-subgroup $K$ of $G$, $H \land K$ is an 1-ideal of $K$, and an 1-subgroup of $H$. Also $K/(H \land K)$ is 1-isomorphic to $HK/H$, and $HK/H$ is an 1-subgroup of $G/H$. Consequently, $H \land K \in U$ and $K/(H \land K) \in V$, since $U$ and $V$ are varieties. Thus $K \in UV$.

Next it will be shown that the cardinal product of elements from $UV$ is in $UV$.

Let $\{G(i) \mid i \in I\}$ be elements of $UV$. Then for each $i \in I$, there exists an 1-ideal $H(i)$ of $G(i)$, such that, $H(i) \in U$ and $G(i)/H(i) \in V$. Consequently, $\prod H(i) \in U$ and $\prod G(i)/H(i) \in V$. Whence, $\prod H(i) \in U$ and $\prod G(i)/\prod H(i) \in V$. Thus, $\prod G(i) \in UV$.

Finally, it is shown that the 1-homomorphic image of an element $G$ in $UV$ is in $UV$.

Let $G \in UV$, then $G$ contains an 1-ideal $H$ such that $H \in U$ and $G/H \in V$. Let $f$ be an 1-homomorphism of $G$ onto $G'$, then $G/\ker(f)$ is 1-isomorphic to $G'$, the 1-homomorphic image of $H$, $f(H)$,
in $G'$ is an 1-ideal of $G'$. The quotient group $G'/f(H)$ is an 1-homomorphic image of $G/H$ by Theorem 1.1.13. Thus, $f(H) \in U$ and $G'/f(H) \in V$. Therefore $G' \in UV$.

**Proposition 4.2.2.** Let $U$, $V$ and $W$ be elements of $L$, then $U(VW) = (UV)W$. In other words, multiplication is associative.

**Proof:** Let $G \in U(W)$, then there exists an 1-ideal $H$ of $G$ such that $H \in U$ and $G/H \in VW$. Since $G/H \in VW$, $G/H$ contains an 1-ideal $K/H$ such that $K/H \in V$ and $(G/H)/(K/H)$ which is 1-isomorphic to $G/K \in W$.

Now, $H \in U$ and $K/H \in V$ implies that $K \in (UV)$. But, since $G/K \in W$; it must be that $G \in (UV)W$. Therefore $U(W) \subseteq (UV)W$.

Conversely, let $G \in (UV)W$ and let $A$ be the $UV$-torsion radical of $G$. Then $G/A \in W$. Let $B$ be the $U$-torsion radical of $A$, then $A/B \in V$.

Since torsion radicals are fully invariant, $B$ is an 1-ideal of $G$. Thus, $B \in U$, $A/B \in V$ and since $(G/B)/(A/B)$ is 1-isomorphic to $G/A \in W$, we also have $(G/B)/(A/B) \in W$. Therefore, $G \in U(W)$.

**Proposition 4.2.3.** For elements $U$, $V$ and $W$ of $L$, if $U \leq V$ then $UW \leq VW$ and $WU \leq WV$.

**Proof:** Assume $U \leq V$ and $G \in UW$. Then, there exists an 1-ideal $H$ of $G$ such that $H \in U$ and $G/H \in W$. Since $U \leq V$, $H \in V$ and thus, $G \in VW$. Therefore, $UW \subseteq VW$.

Similarly, it may be shown that $WU \subseteq VW$. 

By Proposition 4.2.1, 4.2.2, and 4.2.3, \( L \) is a lattice, and a partially ordered semigroup. The semigroup \( (L,\cdot) \) has an identity, namely \( E \), since \( EV = V \) and \( VE = V \) for all \( V \in L \).

For \( V \) and element of \( L \), \( V^n (n \in \mathbb{N}) \) will denote the product of \( V \) with itself \( n \) times. It is called the \( n \)th power of \( V \).

If \( V^2 = V \), then \( V \) is called an idempotent of \( (L,\cdot) \). It will be shown that the only idempotents of \( (L,\cdot) \) are \( E, N \) and \( L \).

**Lemma 4.2.4.** Let \( (H,A) \) and \( (G,B) \) be 1-permutation groups. Then, \( 1\text{-var}\{(H,B)\text{wr}(G,A)\} = 1\text{-var}\{(H,B)\text{wr}(G,A)\} \).

*Proof:* Since \( (H,A)\text{wr}(G,B) \) is an 1-subgroup of \( (H,A)\text{wr}(G,B) \),

\[
1\text{-var}\{(H,A)\text{wr}(G,B)\} \leq 1\text{-var}\{(H,A)\text{wr}(G,B)\}.
\]

Conversely, suppose \( w(x) = e \) is a law which fails in \( (H,A)\text{wr}(G,B) \), and let \( x \mapsto f \) be a substitution in \( (H,A)\text{wr}(G,B) \) for which

\[
(a,b) = \alpha \neq \alpha f(\cdot) = \forall \alpha \prod_{ij\in I\times J} f_{ijk},
\]

for some \( \alpha = (a,b) \in A^+ \times B \).

For \( (\hat{h}_{ij}, \tilde{h}_{ij}) = \prod_{K}\hat{f}_{ijk} ((i,j) \in I \times J) \), \( \alpha(\hat{h}_{ij}, \tilde{h}_{ij}) \neq \alpha \) for a finite subset of \( I \times J \). Therefore, \( a_{\hat{h}_{ij}}(\cdot) \neq a \) for a finite subset of \( I \times J \). For all \( c \in B \setminus \{b\} \cup \{b\tilde{h}_{ij}\} \), let \( \hat{h}_{ijc} \) be the identity. Then, as required, \( w(x) = e \) fails in \( (H,A)\text{wr}(G,B) \), and

\[
1\text{-var}\{(H,B)\text{wr}(G,A)\} \subseteq 1\text{-var}(H,B)\text{wr}(G,A)).
\]

**Lemma 4.2.5.** Let \( (H,B) \) be an 1-permutation group in the 1-group variety \( \mathcal{U} \), and let \( (G,A) \) be a transitive 1-permutation group in the 1-group variety \( \mathcal{V} \). Then, \( W = (H,B)\text{wr}(G,A) \in \mathcal{U}\mathcal{V} \).
Proof: The natural 1-ideal $\prod_{\alpha \in A} H^\alpha$ of $W$ is an element of $U$ and $W/\prod_{\alpha \in A} H^\alpha$ is 1-isomorphic to $G$ which is a member of $V$. Therefore, $W \in UV$.

Corollary 4.2.6. For any positive integer $n$, $\text{Wr}^n Z \in A^n$ and $\text{Wr}^n r \in A^n$.

Lemma 4.2.7. For an 1-group $G$, an 1-ideal $H$ of $G$, and a value $M$ in $G$; if $M$ does not contain $H$, then $M \cap H$ is a value in $H$.

Proof: We will show that $M \cap H$ is a value in $H$ by showing that $M^* \cap H$ covers $M \cap H$, where $M^*$ is the cover of $M$ in $G$.

First, if $H \cap (M^* \setminus M) = \emptyset$ then $M \supset H$ which contradicts one of our assumptions. Thus $H \cap M^* \supset H \cap M$.

If $C$ is a proper convex 1-subgroup of $H$, then $
\{x \in G \mid |x| \land |h| \in C \text{ for all } h \in H\}\text{ is a convex 1-subgroup of } G$, which we will denote by $C'$.

Clearly, $C \subseteq C' \cap H$. Conversely, if $C' \cap H \nsubseteq C$, then choose $0 < x \in (C' \cap H) \setminus C$. Since $x \in C'$, and $x \in H$ we have $|x| \land |t| \in C$ for all $t \in H$ implies that $x = |x| \land |x| \in C$, which is a contradiction. Consequently, $C = C' \cap H$.

Moreover, if $B$ and $C$ are convex 1-subgroups of $H$ such that $B \subseteq C$ then $C' \cap H = C \supset B = B' \cap H$. Therefore $C' \supset B'$.

Applying the previous two paragraphs to the lemma establishes that $H \cap M$ is a value of $H$ and $M^* \cap H$ is its cover.
Theorem 4.2.8. The l-group variety \( N \) is an idempotent in \((L,*)\).

Proof: It suffices to show that for \( G \in N^2 \), each regular subgroup \( M \) of \( G \) is normal valued.

Let \( G \in N^2 \), then \( G \) contains an l-ideal \( H \) such that \( H \in N \) and \( G/H \in N \).

Let \( M \) be a regular subgroup of \( G \). If \( M \supseteq H \), then \( M/H \) is a regular subgroup of \( G/H \). Thus, \( M/H \) is normal in its cover \( (M/H)^* = M^*/H \) (where \( M^* \) is the cover of \( M \) in \( G \)). Therefore, \( M \) is normal in \( M^* \).

If, however, \( M \nsubseteq H \), then by Lemma 4.1.2 \( M \cap H \) is a regular subgroup of \( H \). Hence, \( M \cap H \) is normal in its cover \( (M \cap H)^* = M^* \cap H \).

Observe that by Theorem 1.1.16

\[
\frac{(M^* \cap H)}{(M \cap H)} \text{ is l-isomorphic to } M + \frac{(M^* \cap H)}{M},
\]

and

\[
M + \frac{(M^* \cap H)}{M} = M^*.
\]

Therefore, \( M \) is normal in \( M^* \).

Theorem 4.2.9. \( N = \bigvee_{n=1}^{\infty} A^n \).

Proof: Since \( A \) is contained in \( N \) and \( N \) is an idempotent, by an induction argument we obtain the conclusion that \( A^n \subseteq N \) for all positive integers.

Thus, \( \bigvee_{n=1}^{\infty} A^n \subseteq N \).
Conversely, let $w(x) = \forall_{IJK} x_{ijk} = e$ be a law which is not satisfied by some subdirectly irreducible element $G$ in $N$. Since $G$ is an element of $N$, $G$ may be taken to be an $1$-subgroup of the wreath product of copies of the real number, (by Theorem 2.4.5), say $W \{ R(r), r \in \Gamma \} = (W, A)$. Hence, there exists a substitution $x \mapsto g$ and an element $\lambda$ in $A$, such that, $\lambda \neq \lambda w(g) = \forall_{IJK} g_{ijk}$.  

We will show that $w(x) = e$ is not satisfied by $W \{ R(\delta) \mid \delta \in \Delta \}$, where $\Delta$ is a finite subset of $\Gamma$. 

Let $B = \{ \lambda \} \cup \{ \lambda w_{ijk}^k(g) \mid i \in I, j \in J, k \in K \}$. Then, $B$ is a finite subset of $A$; and for each pair of distinct points $(a, b)$ in $B$, there exists a unique $r \in \Gamma$, such that, $a \equiv^r b$. Let $\Lambda$ be this finite subset of points of $\Gamma$. 

Let $\tilde{\Lambda} = \Pi_{\delta \in \Delta} R(\delta)$ and let $a \mapsto \bar{a}$ denote the projection of $A$ onto $\tilde{\Lambda}$. 

For each $g \in \{ g_{ijk} \mid i \in I, j \in J, k \in K \}$ there exists 

$\tilde{g} = (\tilde{g}_{l,a}) \in W \{ R(\delta) \mid \delta \in \Delta \}$ defined by 

$$\tilde{g}_{l,a} = \begin{cases} 
g_{l,a} & \text{if } a \in B \text{ and there exists } b \in B \text{ such that } ag \equiv_1 b \text{ in } \tilde{A} \\
 &  \\
e & \text{otherwise.} 
\end{cases}$$

The permutation $\tilde{g}_{l,a}$ is well defined since if $a$ and $c \in B$ such that $\bar{a} \equiv_\delta \bar{c}$ in $\tilde{A}$, then by the way in which we choose the points of $\Lambda$, $a \equiv_\delta c$ in $A$. Consequently, $g_{l,a} = g_{l,c}$. 

We wish to show that the map $x_{ijk} \mapsto g_{ijk}$ is a substitution.
To do so, it suffices to show that if \( x_{ijk} \) and \( x_{i',j',k'} \) are inverses, then \( g_{i+j} \) and \( g_{i',j',k'} \) are inverses.

If \( x_{ijk} \) and \( x_{i',j',k'} \) are inverses, then \( g_{ijk} \) and \( g_{i',j',k'} \) are inverses. Hence, if \((g_{ijk})_{\delta} = (g_{ijk})_{\delta, a}\), then there exists an element \( b \in B \) such that \( b = \delta \) \( g_{ijk} \) in \( \bar{A} \). Consequently, \((g_{ijk})_{\delta, a} = (g_{ijk})_{\delta, \bar{a}} = (g_{i',j',k'})_{\delta, a}\) as required.

We will now show that \( w(x) = e \) fails in \( \text{Wr}(R(\delta)/\delta \in \Delta) \).

Let \( g \in \{g_{ijk} \mid i \in I, j \in J\} \), and let \( a = \lambda w_{ij}(g) \in B \) then, for \( \delta \in \Delta \)

\[
(a \bar{g})_{\delta} = \bar{a} \bar{g}_{\delta, \bar{a}} = a g_{\delta, a} = (ag)_{\delta} = (\bar{a} \bar{g})_{\delta} .
\]

Hence, for each pair \((i, j) \in I \times J\), \( \lambda \Pi g_{ijk} = \lambda \Pi \bar{g}_{i',j',k'} \).

Since \( a \rightarrow \bar{a} \) is one-to-one and an order preserving correspondence from \( B \) to \( \bar{B} \);

\[
\forall \lambda \Pi g_{ijk} = \forall \lambda \Pi \bar{g}_{i',j',k'} \neq \lambda .
\]

Thus, \( w(x) = e \) fails in \( \text{Wr}(R(\delta)/\delta \in \Delta) \).

Consequently, by Corollary 4.1.7, \( w(x) = e \) fails in \( \bar{A}^n \), where \( n \) is the cardinality of \( \Delta \); and so \( w(x) = e \) fails in \( \bigvee_{n=1}^\infty \bar{A}^n \).

**Corollary 4.2.10.** If \( V \) is any proper variety of 1-groups, then the powers of \( V \) generate \( N \).

Upon noting that both \( L \) and \( E \) are idempotents, we have the following result:
Theorem 4.2.11. The only idempotents of \((L, *)\) are \(E, N\) and \(L\).

Section 3. The Generation of Varieties

In the first section of this chapter it was shown that if \((H,A)\) and \((G,B)\) are subdirectly irreducible elements of the varieties \(U\) and \(V\) respectively, then \((H,A)\wr (G,B)\) is an element of \(UV\). In this section we will extend this result to determine how particular elements from \(U\) and \(V\) generate \(UV\).

Lemma 4.3.1. Let \((G,A)\) be a transitive 1-permutation group, and \(G \in UV\). The orbits of \(U(G)\) determine a convex congruence \(C\) on \((G,A)\), and the lazy subgroup for \(C\) is \(U(G)\) itself. Let \((G/U(G), A/C)\) denote the induced transitive 1-permutation group; and let \((\bar{U}(G)_{\alpha}, \bar{a})\) denote the transitive 1-permutation group obtained by restricting \(U(G)\) to the \(C\)-class \(\bar{a}\). Then, there is an 1-embedding \(\phi\) of \((G,A)\) onto \((\bar{W},B) = (\bar{U}(G)_{\bar{\alpha}}, \bar{a})\) \((G/U(G), A/C)\).

Proof: In light of Theorem 2.3.6, it suffices to show that the lazy subgroup of \(C\), \(L(C)\), is \(U(G)\). To demonstrate this point, we will need to introduce some notation first.

Let \(\bar{A}\) denote the Dedekind completion of \(A\). Let \(F_x(U(G)) = \{\bar{a} \in \bar{A} | \bar{a}g = \bar{a} \text{ for all } g \in U(G)\}\), and \(H = \{g \in G | \bar{a}g = \bar{a} \text{ for all } \bar{a} \in F_x(U(G))\}\). Finally, let \(\text{Conv}_A(aH) = \{b \in A | c \leq b \leq d \text{ for some } c, d \in aH\}\).
We would like to establish that $\text{Conv}_A(aH) \subseteq \text{Conv}_A(\text{all}(G))$ for all $a \in H$. To this end, suppose that $\text{Conv}_A(aH) \nsubseteq \text{Conv}_A(\text{all}(G))$, and without loss of generality let $\bar{b} = \sup(\text{Conv}_A(a(G))) \in \text{Conv}_A(aH) \subseteq A$.

Then, there exists an $h \in H$ such that $a \leq \bar{b} \leq ah$. However, $\bar{b} \in \text{Fx}(U(G))$ and $h \in H$, so that $ah \leq \bar{b}h = \bar{b} < ah$. Thus, $\text{Conv}_A(aH) \subseteq \text{Conv}_A(\text{all}(G))$ for all $a \in A$.

Now we will show that $H \subseteq U(G)$. To this end let $a \in A$ and $1 \leq g \in H$. Then, from the previous paragraph there exist $1 \leq f_a \in U(G)$ such that $ag \leq af_a$. Thus, $(g \land f_a)^{-1} \in G_a$. Also, $1 \leq g \land f_a \leq f_a \in U(G)$, and hence by convexity, $g \land f_a \in U(G)$.

Moreover, since $g \land f_a \leq g$ for all $a \in A$, $\forall (g \land f_a) \leq g$. Conversely, for each $b \in A$, $b[ \lor (g \land f_a)] \geq b(g \land f_b) = bh$. Thus $g = \lor (g \land f_a) \in U(G)$ since $U(G)$ is closed.

The reader may now convince himself that $H = L(C)$, and consequently $L(C) = U(G)$.

Definition: A family $\{(G_i, \Omega_i) \mid i \in I\}$ of 1-permutation groups is said to mimic a variety $V$ if and only if the following two conditions are satisfied:

1. $G_i \in V$, for all $i \in I$;
2. for any transitive 1-permutation group $(H, \Lambda)$ with $H \in V$, for any $\lambda \in \Lambda$, any finite set of words $\{w_p(x)\}$ and any substitution $x \rightarrow h$ in $(H, \Lambda)$ there exist elements $i \in I$, $\alpha \in \Omega_i$ and a substitution $x \rightarrow g$ in $G_i$ such that $\lambda w_p(h) < \lambda w_q(h)$ if and only if $\alpha w_p(q) < \alpha w_q(g)$.
Theorem 4.3.2. If $U = 1\text{-var} \{ (U_s, \Gamma_s) \mid s \in S \}$ and $\{ (G_t, \Omega_t) \mid t \in T \}$ mimics $V$, then $1\text{-var} \{ (U_s, \Gamma_s) \text{ wr } (G_t, \Omega_t) \mid s \in S, t \in T \} = UV$.

Proof: By lemma 4.2.5, $1\text{-var} \{ (U_s, \Gamma_s) \text{ wr } (G_t, \Omega_t) \}$ is contained in $UV$.

Conversely, suppose $w(x) = e$ is a law which fails in a subdirectly irreducible member $F$ of $V$. Then, the $1$-group $F$ has a representing subgroup, and so $F$ may be viewed as a transitive $1$-permutation group $(F, \Lambda)$. Applying lemma 4.3.1, we embed $(F, \Lambda)$ in $(H, \Lambda)$, where $(H, \Lambda)$ is

$$(U(F)^B, \Lambda) \text{ wr } (F/U(F), \Lambda)$$

and we indentify the two sets denoted by $\Lambda$. Thus, $w(x) = e$ fails in $(H, \Lambda)$. If $w(x) = e$ fails in $V$ then $w(x) = e$ fails in some $(G_t, \Omega_t)$, since $\{ (G_t, \Omega_t) \}$ mimics $V$. Thus, $w(x) = e$ fails in any $(U_s, \Gamma_s) \text{ wr } (G_t, \Omega_t)$. Therefore, throughout the remainder of the proof we will assume that $w(x) = e$ holds in $V$, and in particular $F/U(F)$, but fails in $(H, \Lambda)$.

For convenience we will write $w_{ij}^k(h) = h_{ij2}h_{ij2} \cdots h_{ijk}$, if $x \rightarrow h$ is a substitution. Consequently, since there exists a $(\lambda, \lambda) \in \Lambda$ such that $\lambda \neq \lambda$ w(h) = $\forall \lambda \{ h_{ij} \} \rightarrow$ (k-1)(h), $\lambda w_{ij}^k(h)$.

Consider the set of words $\{ w_{ij}^k(x) \} \cup \{ e \} \cup \{ w_{ij}^k(x) \}$. Since $(G_t, \Omega_t)$ mimics $V$, there exists $(G, \Omega) \in \{ (G_t, \Omega_t) \}$ and
a substitution \( x \leftrightarrow g \in G \) such that for some \( \alpha \in \Omega \)

\[
\lambda_p^\alpha (h) < \lambda_q^\alpha (h) \quad \text{if and only if} \quad \alpha w_p (g) < \alpha w_q (g),
\]

for all \( w_p (x) \) and \( w_q (x) \) in our collection of words. Consequently this forces the following:

(1) \( \lambda_{w_{ij}} (h) < \lambda_{w_{mn}} (h) \quad \text{if and only if} \quad \alpha w_{ij} (g) < \alpha w_{mn} (g) \)

(2) \( \lambda_{w_{ij}} (h) < \lambda \quad \text{if and only if} \quad \alpha w_{ij} (g) < \alpha \)

(3) \( \lambda_{w_{ij}} (h) > \lambda \quad \text{if and only if} \quad \alpha w_{ij} (g) > \alpha \)

(4) \( \lambda_{w_{ij}}^k (h) = \lambda_{w_{mn}}^S (h) \quad \text{if and only if} \quad \alpha w_{ij}^k (g) = \alpha w_{mn}^S (g) \).

We would now like to show that there exists \( U \in U \), such that the word \( w(x) = e \) fails in the wreath product \( (U, \Gamma) \) wr \( (G, \Omega) \).

To this end, let \( I = \{ i \mid \lambda_{w_{ij}} (h) = \lambda \} \) and \( J_i = \{ j \mid \lambda_{w_{ij}} (h) = \lambda \} \) for all \( i \in I \). Then call \( \prod_i \lambda_{w_{ij}} x_{ijk} = w'(x) \). It is easily shown that \( \lambda w'(h) = \lambda w(h) \).

We will now construct a new word which we will then show fails in \( U(F)^\beta \). Let \( w''(y) = \prod_i y_{ijk} \), where \( y_{ijk} \) replaces \( x_{ijk} \) in \( w'(x) \) as follows:

(a) \( y_{ijk} = y_{mns} \quad \text{if and only if} \quad x_{ijk} = x_{mns} , \) and

\[
\lambda_{w_{ij}}^k (h) = \lambda_{w_{mn}}^S (h)
\]

(b) \( y_{ijk} \leftrightarrow y_{mns} \quad \text{if and only if} \quad x_{ijk} \leftrightarrow x_{mns} , \) and

\[
\lambda_{w_{ij}}^k (h) = \lambda_{w_{mn}}^S (h)
\]

(c) \( y_{ijk} = e \quad \text{if and only if} \quad x_{ijk} = e \)

Now \( w''(y) = e \) fails in \( U(F)^\beta \), since the substitution given
by \( y_{ijk} \leftrightarrow h_{ijk}(\lambda_{wij}^{(k - 1)}(h)) \) takes inverses to inverses. More precisely if \( y_{ijk} \) and \( y_{mns} \) are inverses, then \( h_{ijk}(\lambda_{wij}^{(k - 1)}(h)) \) and \( h_{mns}(\lambda_{wmn}^{(s - 1)}(h)) \) are inverses. Thus, since

\[
\lambda \neq \lambda w(h) = \lambda w'(h) = \forall \land (\lambda h_{ijk}(\lambda_{wij}^{(k - 1)}(h)), \lambda_{wij}^{(k - 1)}(h))
\]

we have \( \forall \land (\lambda h_{ijk}(\lambda_{wij}^{(k - 1)}(h)) \neq e \), and so \( w''(y) \neq e \).

Thus, there exists a substitution \( y \leftrightarrow u \) in some \( U \) such that \( e \neq w''(u) \). Combining these results, we will define a substitution in \( (U, \Gamma) \ wr (G, \Omega) \) by \( x_{ijk} \leftrightarrow (b_{ijk}, g_{ijk}) = m_{ijk} \) where \( b_{ijk}: \Omega \rightarrow U \) is defined by

(i) whenever \( x_{ijk} \leftrightarrow x_{mns} \) and \( w_{mn}(x) \) is involved in \( w'(x) \),

\[
(\alpha w_{mn}^{s}(g)) = u_{mns}^{-1},
\]

(ii) whenever \( x_{ijk} = x_{mns} \) and \( w_{mn}(x) \) is involved in \( w'(x) \),

\[
(\alpha w_{mn}^{s}(g)) = u_{mns},
\]

(iii) all the other components of \( b_{ijk}(\beta) = e \).

Summing up, in \( (U, \Gamma) \ wr (G, \Omega) \), \( w(m) = (b, e) \) where by (ii)

\[
b(\alpha) = \bigvee \bigwedge b_{ijk}(\alpha w_{ij}^{(k - 1)}(g)) \]
\[
= \bigvee \bigwedge u_{ijk}
\]
\[
= w''(y) \neq e.
\]

Thus, \( w(x) = e \) fails in \( (U, \Gamma) \ wr (G, \Omega) \) as required.
Corollary 4.3.3. Let \( \text{l-var} \{ (U_s, \Gamma_s) \} = U \) and \( \{ (G_t, \Omega_t) \} \) be the collection of all transitive \( l \)-permutation groups in \( V \).

Then \( \text{l-var} \{ (U_s, \Gamma_s) \wr (G_t, \Omega_t) \} = UV \).

Now that we know how a product of varieties is generated, it is of interest to study when a subset of a variety mimics the variety. Also, it is advantageous to study some examples. The next theorem gives an important example of an \( l \)-group which mimics a variety.

Theorem 4.3.4. The regular representation \( (\mathbb{Z}, \mathbb{Z}) \) of the \( l \)-group of integers mimics \( A \).

Proof: Let \( (H, \Lambda) \) be a transitive abelian \( l \)-permutation group. Since \( (H, \Lambda) \) is transitive and abelian, \( H \) is an \( o \)-group, and so \( (H, \Lambda) \) is the right regular representation \( (H, H) \).

Since \( H \) is an abelian \( o \)-group, \( H \) may be embedded in a Hahn group \( V(\Gamma, R) \). In this case the Hahn group is the lexicographic product \( \prod_{\delta} R_\delta \) of copies of the ordered real numbers \( R \), where \( \Gamma \) is a totally ordered set.

Let \( \{ w_p(x) \} \) be a finite set of words. For each word \( w_p(x) \) let \( x \leftrightarrow s \) denote a substitution in \( \prod_{\delta} R_\delta \). By an argument similar to that in theorem 4.2.9 we may now assume that the index set \( \Gamma \) is finite, say \( \{ 1, 2, \ldots, m \} \).

Let \( D \) be the set of all positive differences \( w_p(s) - w_q(s) \). A typical element of \( D \) will be \( d = (d_1, d_2, \ldots, d_m) \). Let \( 1/c = \min \{ |d_i| \mid d_i \neq 0 \} \). Let \( k_1 = 1 \), and for \( 2 \leq i \leq m \),
let \( k_i > \max | d_1k_1c + d_2k_2c + \cdots + d_{i-1}k_{i-1}c | \).

We will now show that there exists an \( \mathbb{1} \)-homomorphism \( \Theta: V(\Gamma, R) \rightarrow R \), where \( R \) denotes the real numbers. Also the reader should note that the \( \sigma \) - permutation groups involved in this discussion are right regular representations, and thus we ignore the sets that the groups act upon.

Define \( \Theta: V(\Gamma, R) \rightarrow R \) by \( \Theta(r_1, \ldots, r_n) = \sum r_i k_i c \).

Clearly, \( \Theta \) is a group homomorphism. If \( w_p(s) = (p_1, \ldots, p_m) \), \( w_q(s) = (q_1, \ldots, q_m) \), and \( w_p(s) > w_q(s) \), then \( (p_1 - q_1, \ldots, p_m - q_m) > 0 \). Thus if \( p_j - q_j \) is the first non-zero entry from the right (the order is the lexicographic order) then \( p_j - q_j > 0 \). Therefore,

\[
(p_j - q_j)k_j c > d_j c | d_1k_1c + \cdots + d_{j-1}k_{j-1}c |
\]

\[
> | d_1k_1c + \cdots + d_{j-1}k_{j-1}c |
\]

\[
> 0,
\]

where \( d_i = p_i - q_i \). Thus, \( \Theta \) preserves the order of the elements from \( \{ w_p(s) \} \).

Now we would like to show that for a finite set of words \( \{ w_p(x) \} \) and a substitution \( x \rightarrow r \in R \), there is a substitution \( x \rightarrow z \in Z \) (the integers) such that \( w_p(r) < w_q(r) \) if and only if \( w_p(z) < w_q(z) \). Let \( x \rightarrow r \) be a substitution in \( R \).

Since the number of words is finite, the number of images under the substitution \( \rightarrow \) is also finite, say \( a_1, \ldots, a_n \). The subgroup \( S \) of \( R \) generated by these elements is a free abelian
As before, let \( D \) be the set of positive differences \( w_p(r) - w_q(r) \). Each \( d \in D \) has the form \( d = \sum f_i(d)\alpha_i > 0 \) with each \( f_i(d) \) an integer. Thus, \( (\alpha_1, \ldots, \alpha_n) \) is a solution to the system of inequalities \( \sum f_i(d)z_i > 0 \). By continuity, this system must have a rational solution and thus an integer solution. That is, \( \sum f_i(d)z_i > 0 \). Let \( \phi : S \rightarrow Z \) be the homomorphism defined by \( \phi (\sum m_i\alpha_i) = \sum m_i\alpha_i \). Then, \( w_p(r) > w_q(r) \) if and only if \( w_p(\phi(r)) > w_q(\phi(r)) \), as required.

**Theorem 4.3.5**  
For each positive integer \( n \),  
\[ 1\text{-var} (\text{Wr}^n Z) = A^n, \quad \text{and} \quad 1\text{-var} (\text{Wr}^\infty Z) = 1\text{-var} (\text{Wr}^\infty Z) = N. \]

**Proof:** The theorem has been proved for \( n = 1 \). For a positive integer \( n > 1 \), \( \text{Wr}^n Z = (\text{Wr}^n Z) \text{Wr} Z \). Thus, by Theorem 4.3.2 and by induction, \( 1\text{-var} (\text{Wr}^n Z) = A^n \).

Now since any \( G \in N \) is 1-isomorphic to a wreath product of the real numbers \( R \) by Theorem 2.4.5 we have \( 1\text{-var} (\text{Wr}^\infty Z) \leq N \).

Conversely, since \( N = \forall A^n \) and \( 1\text{-var} (\text{Wr}^n Z) = A^n \) is a subset of \( 1\text{-var} (\text{Wr}^\infty Z) \) for all positive integers \( n \), we have \( N \leq 1\text{-var} (\text{Wr} Z) \).

Similarly, it can be shown that \( 1\text{-var} (\text{Wr}^{-\infty} Z) = N. \)

**Corollary 4.3.6.** The varieties \( N \) and \( L \) are the only 1-group varieties closed under taking wreath products.
Section 4. The Factorization of 1-group Varieties.

In this section we will give a partial converse to proposition 4.2.3.

Lemma 4.4.1. Let $H$ be an 1-group and $(G,\Omega)$ a transitive 1-permutation group. Then every 1-ideal $N$ of $(H,H) \text{ wr } (G,\Omega)$ is related by inclusion to $\Sigma H^\alpha$. Moreover, if $N \subseteq \Sigma H^\alpha$ then $N = \Sigma (N \cap H^\alpha)$.

Proof: Let $N$ be an 1-ideal of the wreath product, and suppose that $N \nsubseteq \Sigma H^\alpha$. Then there exists $(h,g) \in N$ such that $g > e$ (the identity of $G$). Hence, there exists $\alpha \in \Omega$ such that $\alpha g > \alpha$. By convexity, $H^\alpha \subseteq N$. Let $\beta \in \Omega$, then there exists $f \in G$ such that $\alpha f = \beta$, and by normality $(e,f)^{-1}(h,g)(e,f) \in N$. Moreover, $\beta f^{-1}gf > \beta$ and so $H^\beta \subseteq N$ for all $\beta \in \Omega$ as required.

Lemma 4.4.2. Let $U_i, V_i$ ($i = 1, 2$) be 1-group varieties with $U_i V_i \subseteq U_2 V_2$. If $U_1 \nsubseteq U_2$ then $A V_1 \subseteq V_2$; and if $V_1 \nsubseteq V_2$ then $U_1 A \subseteq U_2$.

Proof: Let $(G,\Omega)$ be a transitive 1-permutation group with $G \in V_1$. Let $H$ be an 1-group in $U_1 \setminus U_2$. Then $(H,H) \text{ wr } (G,\Omega)$ is an element in $U_1 V_1 \subseteq U_2 V_2$.

Since $H \nsubseteq U_2$, $U_2( (H,H) \text{ wr } (G,\Omega) ) = F \subseteq \Sigma H^\alpha$. Thus, $F \subseteq \Sigma H^\alpha$, so that $( (H,H) \text{ wr } (G,\Omega) ) / F \in V_2$. Hence, $(Z,Z) \text{ wr } (G,\Omega) \subseteq ( (H,H) \text{ wr } (G,\Omega) ) / F \in V_2$, and so $V_2 \subseteq 1\text{-var} \{ (Z,Z) \text{ wr } (G,\Omega) \mid G \in V_1 \} = AV_1$, by corollary 4.3.3.
Corollary 4.4.3. Let $U, V,$ and $W$ be $1$-group varieties and $V \neq N, L$. If $UV \subseteq VW$ then $U \subseteq W$; and if $VL \subseteq VW$ then $U \subseteq W$.

If $UV = VW$ then $U = W$; and if $VL = VW$ then $U = W$.

Proof: Since by corollary 4.3.6, $N$ and $L$ are the only $1$-group varieties closed under taking wreath products, $AV \downarrow V$ and $VA \downarrow V$ whenever $V$ is neither $N$ nor $L$. 
APPENDIX

In the first chapter of this thesis some of the results concerning the properties of 1-groups and 1-subgroups were presented without proof. We expound those ideas now.

**Theorem 1.** If $G$ is a 1-group and $x, y, a, b, \in G$ then

\[ x + (a \lor b) + y = (x + a + y) \lor (x + b + y) \]

and

\[ x + (a \land b) + y = (x + a + y) \land (x + b + y). \]

**Proof:** We will show that

\[ x + (a \lor b) + y = (x + a + y) \lor (x + b + y). \]

Since $a \lor b \geq a$ and $b$, we have $x + (a \lor b) + y$ is greater than or equal to $x + a + y$ and $x + b + y$. Thus

\[ x + (a \lor b) + y \geq (x + a + y) \lor (x + b + y). \]

Assume that for some $z \in G$,

\[ z \geq (x + a + y) \lor (x + b + y). \]

Then, $z \geq x + a + y$ and $x + b + y$, and so $-x + z - y \geq a$ and $b$.

Consequently, $-x + z - y \geq a \lor b$, and therefore $z \geq x + (a \lor b) + y$.

We have shown that $x + (a \lor b) + y$ is the least upper bound of $x + a + y$ and $x + b + y$.

The proof of the dual is similar.
Theorem 2. If \( G \) is a \( \ast \)-group and \( a, b \in G \), then

\[-(a \lor b) = (-a) \land (-b) \quad \text{and} \quad -(a \land b) = (-a) \lor (-b).\]

**Proof:** We will show that \(-a \lor b \geq -(a \land b)\). Since \( a, b \leq a \lor b \), we have \(-a, -b \geq -(a \lor b)\). Thus, \((-a) \land (-b) \geq -(a \lor b)\).

If \( z \in G \) and \( z \geq (-a) \land (-b) \), then \(-z \geq a \) and \( b \). Thus, \(-z \geq a \lor b \), or \( z \leq -(a \lor b)\).

We have shown that \(-a \lor b\) is the greatest lower bound of \(-a\) and \(-b\).

Theorem 3. For a \( \ast \)-group \( G \), \( a \in G \), and a positive integer \( n \),

\[n(a \land 0) = na \land (n-1)a \land \ldots \land a \land 0 .\]

**Proof:** (By Induction) The result holds for \( n=1 \). Assume that the result is true for \( k-1 \). Then,

\[k(a \land 0) = (a \land 0) + (k-1)(a \land 0) \]

\[= (a \land (k-1)(a \land 0)) \land (0 + (k-1)(a \land 0)) \]

\[= (ka \land (k-1)a \land \ldots \land a \land ((k-1)(a \land 0)) \land (ka \land (k-1)a \land \ldots \land a \land 0).\]

The reader may also observe that the result holds for the dual.
BIBLIOGRAPHY


