Numerical Approximation Algorithms for Pension Funding

by

Yige (Vivian) Jin

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Declaration of Committee

Name: **Yige (Vivian) Jin**

Degree: **Master of Science**

Thesis title: **Numerical Approximation Algorithms for Pension Funding**

Committee: **Chair:** Joan Hu
Professor, Statistics and Actuarial Science

Jean-François Bégin
Co-Supervisor
Assistant Professor, Statistics and Actuarial Science

Barbara Sanders
Co-Supervisor
Associate Professor, Statistics and Actuarial Science

Yi Lu
Examiner
Professor, Statistics and Actuarial Science

Abstract

It is difficult to find closed-form optimal decisions in the context of pension plans. Therefore, we often need to rely on numerical algorithms to find approximate optimal decisions. In this report, we present two numerical algorithms that can be applied to solve optimal pension funding problems: the value function approximation and the grid value approximation. The value function approximation method applies to models with infinite time horizons and approximates the parameters of the value function by minimizing the difference between the true and approximate evaluations of the Hamilton–Jacobi–Bellman (HJB) equation. The grid value approximation method is used for models with finite time horizons. It works iteratively with backward and forward stages and approximates the optimal decisions directly without using the HJB equation. Numerical results are presented to compare approximate and true solutions for optimal contributions and share in risky assets for classic problems in the pension literature.

Keywords: Numerical Algorithm; Approximation; Optimal Pension Funding; Dynamic Programming.

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Chapter 1

Introduction

A pension plan is a retirement plan that requires the employer to set aside part of the paid salary and put it into some funds to cover the employee's future benefits after retirement. Most occupational pension plans in Canada can be separated into two main types: defined benefit (DB) plans and defined contribution (DC) plans. In defined benefit plans, the employee's retirement benefit is calculated by a specific pre-determined formula that is usually based on the year of service and annual salaries. Upon retirement, this amount of benefit is guaranteed to the employee, and the sponsor bears the risk if the money in the pool of funds is insufficient. Therefore, the sponsor needs to optimize their decisions to make sure the risk is minimized. For DC plans, on the other hand, this risk is transferred to the employee. After the contributions are made to the fund, its performance decides the amount the employee receives after retirement. As a consequence, the member is the one optimizing the decisions in the DC context to obtain the best possible risk-reward trade-off.

The literature on DC and DC-type plans has been active for a long time. Merton (1969) first discusses DC-type problems in his work which optimizes individual utility. The ideas of DC-type plans these days can be seen as extensions of Merton (1969). Because the retirement benefit depends on the performance of the assets, most of the studies focus on investigating the optimal asset allocation that maximizes employee's expected terminal utility (e.g., Battocchio and Menoncin, 2004; Cairns et al., 2006; Gao, 2008; Han and Hung, 2012). Some other studies also consider other aspects of the DC plans. Vigna and Haberman (2001) study optimal investment strategies that minimize the discounted future cost and also consider the investment risk and annuitisation risk. Yu et al. (2012) investigate a portfolio selection that matches the target liability so that it reduces the downside risk.

The objectives for DB-type plans are different than those of DC plans. Because the benefit for the DB plan is fixed, DB plans aim at securing the promised benefits instead of maximizing the employee's utility. For any DB plans, the sponsor has two targets to achieve this. First, the sponsor needs to keep the pension fund stable. Therefore, it is preferred if each contribution to the pension fund is close to the benefits that will be accrued during the period before the next contribution. The risk associated to this target is called contribution risk. Second, the sponsor wants the pension fund to meet its liabilities. This target leads to solvency risk. Haberman and Sung (1994), Josa-Fombellida and Rincón-Zapatero (2001), Josa-Fombellida and Rincón-Zapatero (2004), and Ngwira and Gerrard (2007) combine these two risks using a quadratic criterion and minimize them.

Most of the studies in both DC and DB contexts find optimal strategies through dynamic programming. To use dynamic programming, a well-educated guess of the value function needs to be made first. Then, based on this functional form, the optimal decisions can be solved using the Hamilton-Jacobi-Bellman (HJB) equation. This may take time if the HJB equation is complicated or if the value function is not mathematically convenient. To avoid the possible complicated process required to solve HJB equations, several numerical algorithms can approximate the optimal decisions.

The main objective of this report is to find optimal strategies of such retirement-related optimization problems via numerical algorithms. Taylor and Uhlig (1990) summarize several different numerical algorithms that can be used to solve nonlinear stochastic models, such as the value-function iteration (Christiano, 1990), the linear-quadratic approximation (Christiano, 1990), the parameterizing expectation (Den Haan and Marcet, 1990), and the least squares projection (Labadie, 1990).

This is not the first attempt to use numerical method to solve pension-related problems. Das and Sundaram (2002) apply a numerical algorithm that approximates the value function to solve DC-type problems. Changa et al. (2005) use Markov chain approximations to solve optimal DB-type pension funding problems. Cui et al. (2009) use a grid value approximation method to solve for the optimal individual consumption and portfolio choice problems, which are also DC-type problems.

Among these methods, the value function approximation and the grid value approximation methods are promising approaches that can be used to solve optimal decisions for both DC and DB-type pension funding problems. Therefore, we will examine the performances of these two methods in this report.

The value function approximation method applies to models with infinite time horizons. For this method, the first few steps are similar to those of solving dynamic programming problems in closed form. We first choose a functional form that approximates the value function and then derive the approximate HJB equation. Then, instead of solving this HJB equation explicitly, we obtain the value function parameters by finding the best parameter set that minimizes the differences between the approximate and true HJB equations. We then can use the approximate parameters to calculate the optimal controls of the problem.

While the value function approximation method applies to problems with infinite time horizons, the grid value approximation method can be used with finite time horizons. This method does not require a derivation of the HJB equation; we only need the objective function to perform this approximation. We first create a set of values of the post-control state variables, referred to as grid values. This set of values should cover all possible values of the state variables over time. This method then works iteratively and has two stages: the backward stage and the forward stage. In the backward stage, the method first finds the optimal controls for each grid value, as well as the values of corresponding pre-decision state variables at each time point. Then, given a specific path of the state variables, the forward stage finds the corresponding optimal controls.

The main results of this report are as follows. For problems with infinite time horizons, the value function approximation method provides approximate results that are very close to true ones. From the results of the robustness tests, the performance of the value function approximation method is very stable. For problems with finite time horizons, the approximations from the grid value approximation method get better as we consider times further into the future, and the overall performance is good. By examining the results of the robustness checks, we find that the performance of this method relies on the range of grid values we chose. That is, all the possible fund values need to be included in the range of the grid values. A smaller time step used for discretization improves the performance. Generally speaking, the two algorithms presented in this report are reliable for different parameter values and can be used to approximate the optimal decisions for retirement-related optimization problems.

This report is structured as follows. Chapter 2 introduces the three models that are used in this report. Then, the two numerical algorithms are discussed in Chapter 3. In Chapter 4, we apply the algorithms to the models. Robustness checks are presented in Chapter 5. Chapter 6 concludes.

Chapter 2

The Optimal Portfolio Choice Problems

This chapter introduces the three models used in this report: the Merton model, an optimal pension funding model with single risky asset, and an optimal pension funding model with multiple risky assets. The explicit solutions of the optimal decisions for each model are presented. The Merton model is used as a mean to illustrate the numerical algorithms described in the next chapter. The two pension funding models (with single risky asset and multiple risky assets) are our main models and will be used in the remaining chapters.

2.1 Merton's Model

The first model comes from Merton (1969). In his model, the investor seeks to find the optimal consumption and investment portfolio that maximize the lifetime utility of consumption. The investment portfolio consists of two assets, a risk-free asset and a risky asset. The risk-free asset has a non-stochastic constant continuously compounded rate of return r. The price of the risky asset $S(t)$ follows a geometric Brownian motion (gBm) with a constant drift μ and a volatility parameter σ:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{2.1}$$

where $W(t)$ is the value of a time-t standard Brownian motion. The investor's investment income depends on the performance of the two assets. At any time t, the investor can choose to invest a proportion $\pi(t)$ of wealth into the risky asset and $1 - \pi(t)$ into the risk-free asset. Meanwhile, the investor also consumes at rate $C(t)$. Thus, the stochastic differential equation (SDE) of wealth $F \equiv F(t)$ at time t is given by

$$dF(t) = [F(t)(r + \pi(t)(\mu - r)) - C(t)]dt + \sigma\pi(t)F(t)dW(t). \tag{2.2}$$

The objective function to be maximized is

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\beta t} U(C(t)) dt \right], \tag{2.3}$$

where $\mathbb{E}_0$ is the expectation conditional on the knowledge of $F(0)$, β is the investor's time preference factor, U is the utility of consumption, and the time horizon is assumed to be infinite. In this report, we only consider the constant relative risk aversion (CRRA) utility function:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \text{ and } \gamma \neq 1,$$

where γ is the risk aversion coefficient

At the initial time $t = 0$, the goal is to find the optimal decisions $\boldsymbol{x}^*(t) = \{C^*(t), \pi^*(t)\}$ for all time t. So, the optimized value of the objective function is defined as

$$V(F,t) = \max_{\{\boldsymbol{x}(s)\}_{s=t}^\infty} \mathbb{E}_t \left[\int_t^\infty e^{-\beta s} \frac{C(s)^{1-\gamma}}{1-\gamma} ds \right], \tag{2.4}$$

which is so called the value function. This can be written into an approximate recursive form:

$$V(F,t) \approx \max_{\boldsymbol{x}(t)} \mathbb{E}_t \left[\int_t^{t+h} e^{-\beta s} \frac{C(s)^{1-\gamma}}{1-\gamma} ds + V(F, t+h) \right], \tag{2.5}$$

where h is a small time interval. If this interval is sufficiently small, we have

$$\int_t^{t+h} e^{-\beta s} \frac{C(s)^{1-\gamma}}{1-\gamma} ds \approx e^{-\beta t} \frac{C(t)^{1-\gamma}}{1-\gamma} h,$$

$$V(F, t+h) \approx V(F,t) + V_t h + V_F \Delta F + \frac{1}{2} V_{FF} \sigma^2 \pi(t)^2 F(t)^2 h,$$

and

$$\Delta F \approx [F(t)(r + \pi(t)(\mu - r)) - C(t)]h + \sigma \pi(t) F(t) \Delta W, \tag{2.6}$$

where V_t is the first derivative of $V(F,t)$ with respect to t, V_F is the first derivative with respect to F, V_{FF} is the second derivative with respect to F, and $\Delta W = W(t+h) - W(t)$. Equation (2.6) follows Itō's lemma. Inputting the above equations into Equation (2.5), we obtain:

$$V(F,t) = \max_{\boldsymbol{x}(t)} \mathbb{E}_t \left[e^{-\beta t} \frac{C(t)^{1-\gamma}}{1-\gamma} h + V(F,t) + V_t h \right.$$

$$+V_F\left[(F(t)(r+\pi(t)(\mu-r))-C(t))h+\sigma\pi(t)F(t)\Delta W\right]$$
$$\left.+\frac{1}{2}V_{FF}\sigma^2\pi(t)^2F(t)^2h\right]. \qquad (2.7)$$

After removing $V(F,t)$ on both sides of Equation (2.7), we have zero on the left hand side (LHS). Also, since $W(t)$ is the time-t value of a standard Brownian motion, the expected value $\mathbb{E}_t[\Delta W(t)]$ is equal to zero. By dividing the right hand side (RHS) by h and by moving V_t to the LHS, we obtain

$$-V_t=\max_{C,\pi}\left\{e^{-\beta t}\frac{C^{1-\gamma}}{1-\gamma}+V_F\left[F(r+\pi(\mu-r))-C\right]+\frac{1}{2}V_{FF}\sigma^2\pi^2F^2\right\}, \qquad (2.8)$$

where C and π are abbreviated versions of $C(t)$ and $\pi(t)$. This is the stochastic Hamilton-Jacobi-Bellman (HJB) equation (Kirk, 2004). With an infinite time horizon, the above HJB equation can be simplified.

We define

$$\begin{aligned} J(F,t)&=e^{\beta t}V(F,t)\\ &=\max_{C,\pi}\mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{C(s)^{1-\gamma}}{1-\gamma}ds\right]\\ &=\max_{C,\pi}\mathbb{E}_t\left[\int_0^\infty e^{-\beta v}\frac{C(v+t)^{1-\gamma}}{1-\gamma}dv\right], \end{aligned} \qquad (2.9)$$

which depends on time only through C. Thus, we can write $J(F,t)$ as $J(F)$. The value function can be written as $V(F,t)=e^{-\beta t}J(F)$. By substituting $J(F)$ into Equation (2.8), we obtain

$$\beta e^{-\beta t}J(F)=\max_{C,\pi}\left\{e^{-\beta t}\frac{C^{1-\gamma}}{1-\gamma}+e^{-\beta t}J_F\left[F(r+\pi(\mu-r))-C\right]+\frac{1}{2}e^{-\beta t}J_{FF}\sigma^2\pi^2F^2\right\},$$

where J_F and J_{FF} are the first and second derivatives of J with respect to F. Dividing $e^{-\beta t}$ on both sides, we have the simplified HJB equation:

$$\beta J=\max_{C,\pi}\left\{\frac{C^{1-\gamma}}{1-\gamma}+\left[F(r+\pi(\mu-r))-C\right]J_F+\frac{1}{2}\sigma^2\pi^2F^2J_{FF}\right\}, \qquad (2.10)$$

where J is the abbreviated version of $J(F)$. We then follow the usual dynamic programming approach to solve for the optimal controls $C^*(t)$ and $\pi^*(t)$.

To find the maximum on the RHS, we choose the consumption C and the share in the risky asset π that satisfy the two following first order conditions (FOCs):

$$C^{-\gamma}-J_F=0,$$

$$F(\mu - r)J_F + \sigma^2 F^2 J_{FF}\pi = 0.$$

We solve the above system of equations and obtain:

$$C = J_F^{-\frac{1}{\gamma}}, \tag{2.11}$$

$$\pi = -\frac{(\mu - r)J_F}{\sigma^2 F J_{FF}}. \tag{2.12}$$

To find a solution to our value function, we can make a well-educated guess about the functional form of J based on Equation (2.9):

$$J(F) = v_1 F^{v_2}, \tag{2.13}$$

where v_1 and v_2 are constant coefficients. Thus, the first and second derivatives of J with respect to F are

$$J_F = v_1 v_2 F^{v_2 - 1}, \tag{2.14}$$

$$J_{FF} = v_1 v_2 (v_2 - 1) F^{v_2 - 2}. \tag{2.15}$$

Inputting these two derivatives back in Equations (2.11) and (2.12) leads to

$$C = (v_1 v_2 F^{v_2 - 1})^{-\frac{1}{\gamma}}, \tag{2.16}$$

$$\pi = -\frac{(\mu - r)}{\sigma^2} \frac{v_1 v_2 F^{v_2 - 1}}{F v_1 v_2 (v_2 - 1) F^{v_2 - 2}} = -\frac{(\mu - r)}{\sigma^2} \frac{1}{v_2 - 1}. \tag{2.17}$$

Now, we have that J_F, J_{FF}, C, and π are all expressed as functions of v_1 and v_2. We can solve the HJB equation by finding the values of v_1 and v_2. Specifically, we substitute Equations (2.14) to (2.17) in the HJB equation of (2.10) and get

$$0 = -\beta v_1 F^{v_2} + \frac{(v_1 v_2 F^{v_2 - 1})^{-\frac{1-\gamma}{\gamma}}}{1 - \gamma} + \left[F \left(r - \frac{(\mu - r)^2}{\sigma^2} \frac{1}{v_2 - 1} \right) - (v_1 v_2 F^{v_2 - 1})^{-\frac{1}{\gamma}} \right] v_1 v_2 F^{v_2 - 1}$$
$$+ \frac{1}{2} \sigma^2 \frac{(\mu - r)^2}{\sigma^4} \frac{1}{(v_2 - 1)^2} F^2 v_1 v_2 (v_2 - 1) F^{v_2 - 2}.$$

The above equation can be simplified by combining all the F terms that have the same exponents:

$$0 = \left(-\beta v_1 + r v_1 v_2 - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{1}{v_2 - 1} v_1 v_2 \right) F^{v_2} + \frac{\gamma}{1 - \gamma} (v_1 v_2)^{-\frac{1-\gamma}{\gamma}} (F^{v_2 - 1})^{-\frac{1-\gamma}{\gamma}}. \tag{2.18}$$

Because v_1 and v_2 should have non-zero values, it is impossible for the coefficient of $(F^{v_2 - 1})^{-\frac{1-\gamma}{\gamma}}$ to be equal to zero. So, Equation (2.18) will always hold if and only if the exponents of the F terms are the same and the summation of the coefficients of the two F

terms is equal to zero; that is,

$$v_2 = (v_2 - 1)\left(-\frac{1-\gamma}{\gamma}\right), \tag{2.19}$$

$$0 = \left(-\beta v_1 + r v_1 v_2 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1}{v_2 - 1}v_1 v_2 + \frac{\gamma}{1-\gamma}(v_1 v_2)^{-\frac{1-\gamma}{\gamma}}\right). \tag{2.20}$$

Then, we solve the system of equations above to get the values of v_1 and v_2:

$$v_1 = \left[\left(\beta - r(1-\gamma) - \frac{1-\gamma}{\gamma}\frac{(\mu - r)^2}{2\sigma^2}\right)\frac{1}{\gamma}\right]^{-\gamma}\frac{1}{1-\gamma},$$

$$v_2 = 1 - \gamma.$$

Finally, this gives us the optimal controls for our problem:

$$C^*(t) = \left(r + \frac{\beta - r}{\gamma} - \frac{1-\gamma}{\gamma^2}\frac{(\mu - r)^2}{2\sigma^2}\right)F(t), \tag{2.21}$$

$$\pi^*(t) = \frac{\mu - r}{\gamma\sigma^2}. \tag{2.22}$$

2.2 Optimal Pension Funding Problem with a Single Risky Asset

The models explained in the remaining parts of this chapter are the main focus of the report. They consider dynamic models of pension funding in the DB context (Josa-Fombellida and Rincón-Zapatero, 2001; Ngwira and Gerrard, 2007). We start with a simpler case for which the investment portfolio has only one risky and one risk-free asset. We consider both the infinite and finite time cases.

2.2.1 Infinite Time Horizon

We start with the infinite time horizon as it is simpler. We denote $F(t)$ the value of assets at time t and $C(t)$ the contribution rate at time t. The benefit paid out is denoted by P, the actuarial liability AL, and the normal cost for all participants of the plan NC. We assume that P, AL, and NC are all constant. The unfunded actuarial liability UAL is the excess of AL over F. The actuarial liability accumulates at a rate of $P - NC$ with a valuation rate of δ. Thus, the relationship between these three components satisfies the following equation:

$$\delta AL + NC - P = 0. \tag{2.23}$$

Same as Merton's model, the risk-free asset has a constant continuously compounded rate of return r. The value of the risky asset $S(t)$ follows Equation (2.1). The sponsor can choose to invest $\pi(t)$ of the wealth into the risky asset and $1 - \pi(t)$ in the risk-free asset.

So, the evolution of the pension fund depends on the assets performance, the contribution rate paid into the fund, and the benefits paid out:

$$dF(t) = [F(t)(r + \pi(t)(\mu - r)) + C(t) - P]dt + \sigma\pi(t)F(t)dW(t). \tag{2.24}$$

As discussed in the previous chapter, there are two main types of risks that need to be considered in the DB context. The first risk is the contribution risk, which is measured as the deviations of the contributions from the normal cost. The second one is the solvency risk, which is measured with the size of UAL. We want the combination of these two deviations to be as small as possible. Depending on the relative importance of these two risks, a weight factor $0 < \kappa \leq 1$ is given to one of the risks. As a consequence, the value function becomes

$$V(F,t) = \min_{\{C(s),\pi(s)\}_{s=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\beta s} \left(\kappa(C(s) - NC)^2 + (1-\kappa)(AL - F(s))^2 \right) ds \right], \tag{2.25}$$

where β is the sponsor's time preference factor (Josa-Fombellida and Rincón-Zapatero, 2001). Since the sponsor desires stable contributions and dislikes both overfunding and underfunding, we penalize negative and positive deviations equally by using the squared deviation of the contribution rate and fund value.

To solve this optimization problem, we follow similar steps to those discussed in the previous section. We first derive the infinite time horizon HJB equation:

$$\begin{aligned} \beta V = \min_{C,\pi} \Big\{ & \kappa(C - NC)^2 + (1-\kappa)(AL - F)^2 + V_F \left[F(r + \pi(\mu - r)) + C - P \right] \\ & + \frac{1}{2} V_{FF} \sigma^2 \pi^2 F^2 \Big\}, \end{aligned} \tag{2.26}$$

where C, π, F, and V are abbreviated versions of $C(t)$, $\pi(t)$, $F(t)$, and $V(F,t)$. The FOCs that minimize the RHS of Equation (2.26) are

$$C = NC - \frac{V_F}{2\kappa}, \tag{2.27}$$

$$\pi = -\frac{V_F}{FV_{FF}} \frac{\mu - r}{\sigma^2}. \tag{2.28}$$

By inputting Equations (2.27) and (2.28) in Equation (2.26), we obtain

$$\begin{aligned} \beta V = \; & \kappa \left(\frac{V_F}{2\kappa} \right)^2 + (1-\kappa)(AL - F)^2 + V_F \left[F \left(r - \frac{V_F}{FV_{FF}} \frac{(\mu - r)^2}{\sigma^2} \right) + NC - \frac{V_F}{2\kappa} - P \right] \\ & + \frac{1}{2} V_{FF} \sigma^2 \frac{V_F^2}{F^2 V_{FF}^2} \frac{(\mu - r)^2}{\sigma^4} F^2 \\ = \; & \frac{V_F^2}{4\kappa} + (1-\kappa)(AL - F)^2 \end{aligned}$$

$$+ V_F \left[rF - \frac{V_F}{V_{FF}} \frac{(\mu - r)^2}{\sigma^2} + NC - \frac{V_F}{2\kappa} - P \right] + \frac{1}{2} \frac{V_F^2}{V_{FF}} \frac{(\mu - r)^2}{\sigma^2}$$

$$= -\frac{V_F^2}{4\kappa} + (1-\kappa)(AL - F)^2 + V_F(rF - \delta AL) - \frac{1}{2} \frac{V_F^2}{V_{FF}} \frac{(\mu - r)^2}{\sigma^2}. \tag{2.29}$$

Then, to find a solution of the value function, we again need to make a well-educated guess about the functional form of V based on the objective function of Equation (2.25). In this case, we select the following:

$$V(F,t) = v_1 F^2 + v_2 AL^2 + v_3 F \cdot AL, \tag{2.30}$$

where v_1, v_2 and v_3 are constant coefficients. Thus, the first and second derivatives of V with respect to F are

$$V_F = 2v_1 F + v_3 AL, \tag{2.31}$$

$$V_{FF} = 2v_1, \tag{2.32}$$

respectively, and the FOCs become

$$C = NC - \frac{2v_1 F + v_3 AL}{2\kappa}, \tag{2.33}$$

$$\pi = -\frac{2v_1 F + v_3 AL}{2v_1 F} \frac{\mu - r}{\sigma^2}. \tag{2.34}$$

By substituting the two derivatives of Equations (2.31) and (2.32), the FOCs of Equations (2.33) and (2.34), and the value function of Equation (2.30) into Equation (2.29), we obtain

$$0 = -\beta \left(v_1 F^2 + v_2 AL^2 + v_3 F \cdot AL \right) - \frac{1}{4\kappa}(2v_1 F + v_3 AL)^2 + (1-\kappa)(AL - F)^2$$

$$+ (2v_1 F + v_3 AL)(rF - \delta AL) - \frac{1}{2} \frac{(2v_1 F + v_3 AL)^2}{2v_1} \frac{(\mu - r)^2}{\sigma^2}$$

$$= \left(-\beta v_1 - \frac{v_1^2}{\kappa} + (1-\kappa) + 2v_1 r - \frac{(\mu - r)^2}{\sigma^2} v_1 \right) F^2$$

$$+ \left(-\beta v_2 - \frac{v_3^2}{4\kappa} + (1-\kappa) - v_3 \delta - \frac{(\mu - r)^2}{\sigma^2} \frac{v_3^2}{4v_1} \right) AL^2$$

$$+ \left(-\beta v_3 - \frac{v_1 v_3}{\kappa} - 2(1-\kappa) - 2v_1 \delta + v_3 r - \frac{(\mu - r)^2}{\sigma^2} v_3 \right) F \cdot AL. \tag{2.35}$$

To make this equation hold for any value of F and AL, we need to set the coefficients in front of F^2, AL^2, and $F \cdot AL$ to zero. Thus, by solving the following system of equations:

$$-\beta v_1 - \frac{v_1^2}{\kappa} + (1-\kappa) + 2v_1 r - \frac{(\mu - r)^2}{\sigma^2} v_1 = 0, \tag{2.36}$$

$$-\beta v_2 - \frac{v_3^2}{4\kappa} + (1-\kappa) - v_3\delta - \frac{(\mu-r)^2}{\sigma^2}\frac{v_3^2}{4v_1} = 0, \tag{2.37}$$

$$-\beta v_3 - \frac{v_1 v_3}{\kappa} - 2(1-\kappa) - 2v_1\delta + v_3 r - \frac{(\mu-r)^2}{\sigma^2} v_3 = 0, \tag{2.38}$$

we can obtain the values for v_1, v_2, and v_3. Letting $\Delta = \kappa^2 \left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)^2 + 4\kappa(1-\kappa)$, the unique positive solution of Equation (2.36) is then

$$v_1 = \frac{\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)}{2}. \tag{2.39}$$

Inputting the above solution in Equation (2.38) gives the value of v_3:

$$v_3 = \frac{-4\kappa(1-\kappa) - 2\kappa\delta\sqrt{\Delta} + 2\kappa^2\delta\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)}{\sqrt{\Delta} + \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta\right)}. \tag{2.40}$$

Then, v_2 can be solved by inputting Equations (2.39) and (2.40) into Equation (2.37). Since our purpose is to find the optimal controls of the contribution rate $C^*(t)$ and the investment proportion $\pi^*(t)$, and these two controls do not require the value of v_2, we do not show the exact solution of v_2 here. Finally, from Equations (2.33) and (2.34), we find that the optimal controls for this pension funding problem are:

$$\begin{aligned} C^*(t) = NC &+ \frac{\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)}{2\kappa} \times \frac{1}{2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)\right)} \\ &\times \left\{\left[2(1-\kappa) + \delta\left(\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)\right)\right] AL \right. \\ &\left. - \left[2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)\right)\right] F\right\}, \end{aligned} \tag{2.41}$$

$$\pi^*(t) = \left(\frac{\left[2(1-\kappa) + \delta\left(\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)\right)\right] AL}{\left[2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\frac{(\mu-r)^2}{\sigma^2} + \beta - 2r\right)\right)\right] F} - 1\right)\frac{\mu - r}{\sigma^2}. \tag{2.42}$$

2.2.2 Finite Time Horizon

We now consider a more realistic case where the pension fund accumulates until time T. Under this assumption, we also want the deviation of the fund value from the actuarial liability at terminal time T to be as small as possible. So, a terminal condition is added to the objective function and the value function becomes

$$V(F,t) = \min_{\{C(s),\pi(s)\}_{s=t}^{T}} \mathbb{E}_t\left[\int_t^T e^{-\beta s}\left(\kappa(C(s) - NC)^2 + (1-\kappa)(AL - F(s))^2\right) ds\right.$$

$$+ \alpha e^{-\beta T}(F(T) - AL)^2\Big], \tag{2.43}$$

where $0 < \alpha \leq 1$ reflects the importance of the terminal condition (Ngwira and Gerrard, 2007). The pension fund and the value of the assets all follow the same dynamics as those used in the previous section. To find the optimal contribution rate $C^*(t)$ and the investment share $\pi^*(t)$, we again derive the HJB equation. Using the idea used to derive Equation (2.8), the HJB equation for this case is given by

$$\begin{aligned} -V_t = \min_{C,\pi} \Big\{ & e^{-\beta t} \left[\kappa(C - NC)^2 + (1-\kappa)(AL - F)^2\right] \\ & + V_F \left[F(r + \pi(\mu - r)) + C - P\right] + \frac{1}{2} V_{FF}\sigma^2\pi^2F^2 \Big\}, \end{aligned} \tag{2.44}$$

with the terminal condition

$$V(F,T) = \alpha e^{-\beta T}(F(T) - AL)^2, \tag{2.45}$$

where V_t is the first derivative of V with respect to t. The abbreviated notations from the previous section still apply here.

Unlike the infinite time case, our value function now depends on time t as well. The form of the terminal condition of Equation (2.45) suggests the following functional form for V:

$$V(F,t) = e^{-\beta t}L(t)\left(F^2 - 2Q(t)F + K(t)\right), \tag{2.46}$$

where $L(t), Q(t)$, and $K(t)$ are functions of time t that satisfy

$$\begin{aligned} L(T) &= \alpha, \\ Q(T) &= AL, \\ K(T) &= AL^2. \end{aligned}$$

The derivatives of V required for calculations are

$$V_t = -\beta V + e^{-\beta t}L_t\left(F^2 - 2Q(t)F + K(t)\right) + e^{-\beta t}L(t)\left(-2Q_tF + K_t\right), \tag{2.47}$$

$$V_F = 2e^{-\beta t}L(t)(F - Q(t)), \tag{2.48}$$

$$V_{FF} = 2e^{-\beta t}L(t), \tag{2.49}$$

where L_t, Q_t, and K_t are the first derivatives of the original functions with respect to t. The FOCs with respect to the contribution rate C and the share invested in the risky asset π

are

$$C = NC - \frac{V_F}{2\kappa e^{-\beta t}} = NC + \frac{L(t)}{\kappa}[Q(t) - F], \tag{2.50}$$

$$\pi = -\frac{(\mu - r)}{\sigma^2}\frac{V_F}{FV_{FF}} = \frac{\mu - r}{\sigma^2}\left[\frac{Q(t) - F}{F}\right]. \tag{2.51}$$

By substituting Equations (2.47) to (2.51) in Equation (2.44), we obtain

$$\begin{aligned}
0 = &(-\beta L(t) + L_t)\left(F^2 - 2Q(t)F + K(t)\right) + L(t)\left(-2Q_tF + K_t\right) \\
&+ 2rL(t)F^2 - 2rL(t)Q(t)F - \frac{(\mu - r)^2}{\sigma^2}L(t)\left(F^2 - 2Q(t)F + Q(t)^2\right) \\
&+ 2L(t)\left(F - Q(t)\right)(NC - P) - \frac{L(t)^2}{\kappa}\left(F^2 - 2Q(t)F + Q(t)^2\right) \\
&+ (1 - \kappa)\left(AL^2 - 2AL \cdot F + F^2\right).
\end{aligned} \tag{2.52}$$

The above equation is true under all circumstances if the terms independent of F and the coefficients in front of F^2 and F are all zero:

$$0 = L_t + \left(2r - \beta - \frac{(\mu - r)^2}{\sigma^2}\right)L(t) - \frac{1}{\kappa}L(t)^2 + (1 - \kappa), \tag{2.53}$$

$$0 = Q_t - \left(r + \frac{1 - \kappa}{L(t)}\right)Q(t) - (NC - P) + \frac{AL}{L(t)}(1 - \kappa), \tag{2.54}$$

$$\begin{aligned}
0 = K_t &+ \left(-\beta + \frac{L_t}{L(t)}\right)K(t) + (1 - \kappa)\frac{AL^2}{L(t)} - 2(NC - P)Q(t) \\
&- \left[\frac{1}{\kappa}L(t) + \frac{(\mu - r)^2}{\sigma^2}\right]Q(t)^2.
\end{aligned} \tag{2.55}$$

In order to find the optimal controls $C^*(t)$ and $\pi^*(t)$, we need to find the solutions of the system of ordinary differential equations (ODEs) above, yielding $L(t)$ and $Q(t)$. First, $L(t)$ can be obtained by solving the ODE of Equation (2.53), which can be rewritten as

$$L_t = -(1 - \kappa) - \left(2r - \beta - \frac{(\mu - r)^2}{\sigma^2}\right)L(t) + \frac{1}{\kappa}L(t)^2.$$

Indeed, this is a Riccati differential equation that has the following solution (Ngwira and Gerrard, 2007):

$$L(t) = \frac{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}}{\psi_1 - \psi_2 e^{\omega_3(T-t)}}, \tag{2.56}$$

where

$$\omega_1 = \frac{\kappa}{2}\left[2r - \beta - \frac{(\mu - r)^2}{\sigma^2} + \sqrt{\left[2r - \beta - \frac{(\mu - r)^2}{\sigma^2}\right]^2 + \frac{4(1-\kappa)}{\kappa}}\right], \tag{2.57}$$

$$\omega_2 = \frac{\kappa}{2}\left[2r - \beta - \frac{(\mu - r)^2}{\sigma^2} - \sqrt{\left[2r - \beta - \frac{(\mu - r)^2}{\sigma^2}\right]^2 + \frac{4(1-\kappa)}{\kappa}}\right], \tag{2.58}$$

$$\omega_3 = \frac{\omega_1 - \omega_2}{\kappa}, \tag{2.59}$$

$$\psi_1 = \alpha - \omega_1, \psi_2 = \alpha - \omega_2. \tag{2.60}$$

Then, we solve the ODE associated with $Q(t)$. By following the general steps of solving linear differential equations with variable coefficients, we have

$$\begin{aligned} Q(t) = AL - & \frac{\kappa(r-\delta)AL}{\left(\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r - \omega_1)(\kappa r - \omega_2)} \\ & \times \Big\{(\kappa r - \omega_2)\omega_2\psi_1 - (\kappa r - \omega_1)\omega_1\psi_2 e^{\omega_3(T-t)} \\ & -[(\kappa r - \omega_2)\omega_2\psi_1 - (\kappa r - \omega_1)\omega_1\psi_2]e^{-(r-\frac{\omega_1}{\kappa})(T-t)}\Big\}. \end{aligned} \tag{2.61}$$

The detailed steps of this derivation are given in Appendix A. Using $L(t)$ and $Q(t)$, we can finally find our optimal controls for the finite time horizon version of our problem:

$$C^*(t) = NC + \frac{L(t)}{\kappa}[Q(t) - F], \tag{2.62}$$

$$\pi^*(t) = \frac{\mu - r}{\sigma^2}\left[\frac{Q(t) - F}{F}\right]. \tag{2.63}$$

2.3 Optimal Pension Funding Problem with Multiple Risky Assets

We now consider more general cases where multiple risky assets are available.

2.3.1 Infinite Time Horizon

Similar to the single risky asset case, we have a risk-free asset with a constant compounded rate of return r. Instead of having only one risky asset, however, the portfolio manager now has access to n risky assets. The values of these risky assets are given by the following stochastic differential equations:

$$dS^i(t) = \mu_i S^i(t)dt + S^i(t)\sum_{j=1}^{n}\sigma_{ij}dW_j(t), \tag{2.64}$$

where $S^i(t)$ is the value of the i-th risky asset at time t, μ_i is the average rate of return for the i-th asset, and σ_{ij} is the i-th row, j-th column element of the Cholesky decomposition of the asset covariance matrix. The vector $\boldsymbol{W}(t) = (W_1(t), \ldots, W_n(t))^\top$ is the time-t value of an n-dimensional standard Brownian motion. At any time t, the sponsor can choose to invest a proportion $\pi_i(t)$ of the plan's asset into the i-th risky asset. The remaining proportion $1 - \sum_{i=1}^n \pi_i(t)$ of the asset is invested in the risk-free asset. So, the evolution of the pension fund now becomes

$$dF(t) = \left[rF(t) + \sum_{i=1}^{n} \pi_i(t)(\mu_i - r)F(t) + C(t) - P \right] dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\pi_i(t)F(t)dW_j(t). \tag{2.65}$$

As we are still dealing with an infinite time horizon in this design, we select the value function of Equation (2.25). We let the matrix $\boldsymbol{\sigma} = (\sigma_{ij})$, for $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}$, be the Cholesky decomposition of the covariance matrix. $\boldsymbol{\Pi}(t) = (\pi_1(t), \ldots, \pi_n(t))^\top$ is defined as the column vector of the share invested in n risky assets, and the $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^\top$ is the column vector of expected returns on the n risky assets. The HJB equation considering multiple risky assets can thus be written as

$$\beta V = \min_{C, \boldsymbol{\Pi}} \Big\{ \kappa(C - NC)^2 + (1-\kappa)(AL - F)^2 + V_F \left[rF + \boldsymbol{\Pi}^\top(\boldsymbol{\mu} - r\mathbf{1})F + C - P \right] + \frac{1}{2} V_{FF} \boldsymbol{\Pi}^\top \boldsymbol{\Sigma} \boldsymbol{\Pi} F^2 \Big\}, \tag{2.66}$$

where $\mathbf{1}$ denotes the column vector of size n, with the elements of the vector being 1, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$ is the covariance matrix, and $\boldsymbol{\Pi}$ is the abbreviated version of $\boldsymbol{\Pi}(t)$. As usual, we solve this HJB equation by first finding the FOCs:

$$C = NC - \frac{1}{2\kappa} V_F, \tag{2.67}$$

$$\boldsymbol{\Pi} = -\frac{V_F}{F V_F F} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}). \tag{2.68}$$

Since the objective function is the same as that presented in Section 2.2.1, we choose Equation (2.30) as our guess for the functional form of V. As a consequence, our first and second derivatives of V with respect to F follow Equations (2.31) and (2.32). By substituting the two derivatives to Equations (2.67) and (2.68), the FOCs become

$$C = NC - \frac{1}{2\kappa}(2v_1 F + v_3 AL), \tag{2.69}$$

$$\boldsymbol{\Pi} = -\frac{2v_1 F + v_3 AL}{2v_1 F} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}). \tag{2.70}$$

Then, we substitute Equations (2.69) and (2.70) in the HJB equation of Equation (2.66), which gives

$$\begin{aligned}
0 = &\left(-\beta v_1 - \frac{v_1^2}{\kappa} + (1-\kappa) + 2v_1 r - \boldsymbol{\theta}^\top \boldsymbol{\theta} v_1\right) F^2 \\
&+ \left(-\beta v_2 - \frac{v_3^2}{4\kappa} + (1-\kappa) - v_3\delta - \boldsymbol{\theta}^\top \boldsymbol{\theta} \frac{v_3^2}{4v_1}\right) AL^2 \\
&+ \left(-\beta v_3 - \frac{v_1 v_3}{\kappa} - 2(1-\kappa) - 2v_1\delta + v_3 r - \boldsymbol{\theta}^\top \boldsymbol{\theta} v_3\right) F \cdot AL,
\end{aligned}$$

where $\boldsymbol{\theta} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$ is a vector containing the market prices of risk. Using an idea similar to that used in Section 2.2.1, we solve for v_1, v_2, and v_3 by setting the coefficients in front of F^2, AL^2, and $F \cdot AL$ to zero. Because the value of v_2 is not used, we only provide the equations for v_1 and v_3:

$$v_1 = \frac{\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)}{2}, \tag{2.71}$$

$$v_3 = \frac{-4\kappa(1-\kappa) - 2\kappa\delta\sqrt{\Delta} + 2\kappa^2\delta\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)}{\sqrt{\Delta} + \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta\right)}, \tag{2.72}$$

where $\Delta = \kappa^2\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)^2 + 4\kappa(1-\kappa)$. If we compare Equations (2.71) and (2.72) to Equations (2.39) and (2.40), the only difference is that we need to substitute $\frac{(\mu-r)^2}{\sigma^2}$ by $\boldsymbol{\theta}^\top \boldsymbol{\theta}$, which is a general expression for the squared price of risk in the multiple asset context. Thus, our optimal controls $C^*(t)$ and $\boldsymbol{\Pi}^*(t)$ for the problem indeed look similar to Equations (2.41) and (2.42):

$$\begin{aligned}
C^*(t) = NC + &\frac{\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)}{2\kappa} \times \frac{1}{2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)\right)} \\
&\times \left\{\left[2(1-\kappa) + \delta\left(\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)\right)\right] AL \right. \\
&\qquad \left. - \left[2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)\right)\right] F\right\},
\end{aligned} \tag{2.73}$$

$$\boldsymbol{\Pi}^*(t) = \left(\frac{\left[2(1-\kappa) + \delta\left(\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)\right)\right] AL}{\left[2(1-\kappa) + r\left(\sqrt{\Delta} - \kappa\left(\boldsymbol{\theta}^\top \boldsymbol{\theta} + \beta - 2r\right)\right)\right] F} - 1\right) \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}). \tag{2.74}$$

2.3.2 Finite Time Horizon

The last case we want to introduce is the model with multiple risky assets and a finite time horizon, which is also the most complicated one investigated in this report. Because we are still considering multiple risky assets, the values of n risky assets follow Equation (2.64), and the fund value follows Equation (2.65). As discussed in Section 2.2.2, we need a terminal condition under the finite time case. We therefore use the value function presented in Equation (2.43). Our HJB equation thus looks very similar to Equation (2.44), but takes multiple risky assets into account:

$$-V_t = \min_{C,\mathbf{\Pi}} \Big\{ e^{-\beta t} \left[\kappa \left(C - NC \right)^2 + (1-\kappa)(AL - F)^2 \right] \\ + V_F \Big[rF + \mathbf{\Pi}^\top(\boldsymbol{\mu} - r\mathbf{1})F + C - P \Big] + \frac{1}{2} V_{FF} \mathbf{\Pi}^\top \mathbf{\Sigma} \mathbf{\Pi} F^2 \Big\}, \tag{2.75}$$

with the terminal condition of Equation (2.45). Again, because the objective function is the same as that given in Section 2.2.2, Equation (2.46) is selected as our well-educated guess for the functional form of V. Consequently, we have that our V_t, V_F, and V_{FF} follow Equations (2.47) to (2.49). Then, our FOCs needed to solve for the optimal controls, $C^*(t)$ and $\mathbf{\Pi}^*(t)$, are given by the following equations:

$$C = NC - \frac{V_F}{2\kappa e^{-\beta t}} = NC + \frac{L(t)}{\kappa}[Q(t) - F], \tag{2.76}$$

$$\mathbf{\Pi} = -\frac{V_F}{F V_{FF}} \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = \left[\frac{Q(t) - F}{F} \right] \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}). \tag{2.77}$$

If we input Equations (2.47) to (2.49) and Equations (2.76) to (2.77) into Equation (2.75), we obtain

$$\begin{aligned} 0 = & \left(-\beta L(t) + L_t + 2rL(t) - \boldsymbol{\theta}^\top \boldsymbol{\theta} L(t) - \frac{L(t)^2}{\kappa} + (1-\kappa) \right) F^2 \\ & + \Big(-2rL(t)Q(t) + 2\boldsymbol{\theta}^\top \boldsymbol{\theta} L(t)Q(t) + \frac{2}{\kappa} L(t)^2 Q(t) - 2\delta ALL(t) - 2(1-\kappa)AL \\ & \quad + 2\beta L(t)Q(t) - 2Q(t)L_t - 2L(t)Q_t \Big) F \\ & + \Big(-\boldsymbol{\theta}^\top \boldsymbol{\theta} L(t)Q(t)^2 - \frac{1}{\kappa} L(t)^2 Q(t)^2 + 2\delta ALQ(t)L(t) + (1-\kappa)AL^2 - \beta L(t)K(t) \\ & \quad + L_t K(t) + L(t)K_t \Big). \end{aligned}$$

As discussed above, the above equation holds if and only if the term independent of F and the coefficients in front of F^2 and F are all equal to zero. So, we have

$$0 = L_t + \left(2r - \beta - \boldsymbol{\theta}^\top \boldsymbol{\theta} \right) L(t) - \frac{1}{\kappa} L(t)^2 + (1-\kappa), \tag{2.78}$$

$$0 = \left[\left(r - \beta - \boldsymbol{\theta}^\top \boldsymbol{\theta} \right) L(t) - \frac{1}{\kappa} L(t)^2 + L_t \right] Q(t) \\ - (NC - P)L(t) + (1-\kappa)AL + L(t)Q_t, \tag{2.79}$$

$$0 = K_t + \left(-\beta + \frac{L_t}{L(t)} \right) K(t) + (1-\kappa)\frac{AL^2}{L(t)} \\ - 2(NC - P)Q(t) - \left[\frac{1}{\kappa} L(t) + \boldsymbol{\theta}^\top \boldsymbol{\theta} \right] Q(t)^2, \tag{2.80}$$

where δAL is substituted by $-(NC-P)$ according to Equation (2.23). The coefficient in front of $Q(t)$ in Equation (2.79) can be written as

$$\left[\left(2r-\beta-\boldsymbol{\theta}^\top\boldsymbol{\theta}\right)L(t)-\frac{1}{\kappa}L(t)^2+L_t+(1-\kappa)\right]-rL(t)-(1-\kappa),$$

where the part inside the square brackets follows Equation (2.78) and is equal to zero. As a consequence, Equation (2.79) can be expressed as

$$0=\left[0-rL(t)-(1-\kappa)\right]Q(t)-(NC-P)L(t)+(1-\kappa)AL+L(t)Q_t,$$

or equivalently,

$$0=Q_t-\left(r+\frac{1-\kappa}{L(t)}\right)Q(t)-(NC-P)+\frac{AL}{L(t)}(1-\kappa). \tag{2.81}$$

By comparing Equations (2.78) and (2.81) to Equations (2.53) and (2.54), we again find that the only difference between them is the terms of $\frac{(\mu-r)^2}{\sigma^2}$ and $\boldsymbol{\theta}^\top\boldsymbol{\theta}$ in the equations. Thus, we can obtain the solutions for $L(t)$ and $Q(t)$ from Equations (2.56) and (2.61) by replacing $\frac{(\mu-r)^2}{\sigma^2}$ with $\boldsymbol{\theta}^\top\boldsymbol{\theta}$ inside ω_1 and ω_2 of Equations (2.57) and (2.58):

$$\omega_1=\frac{\kappa}{2}\left[2r-\beta-\boldsymbol{\theta}^\top\boldsymbol{\theta}+\sqrt{\left[2r-\beta-\boldsymbol{\theta}^\top\boldsymbol{\theta}\right]^2+\frac{4(1-\kappa)}{\kappa}}\right],$$

$$\omega_2=\frac{\kappa}{2}\left[2r-\beta-\boldsymbol{\theta}^\top\boldsymbol{\theta}-\sqrt{\left[2r-\beta-\boldsymbol{\theta}^\top\boldsymbol{\theta}\right]^2+\frac{4(1-\kappa)}{\kappa}}\right].$$

Finally, our optimal controls for the finite time horizon with multiple risky assets are:

$$C^*(t)=NC+\frac{L(t)}{\kappa}[Q(t)-F], \tag{2.82}$$

$$\boldsymbol{\Pi}^*(t)=\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-r\mathbf{1})\left[\frac{Q(t)-F}{F}\right]. \tag{2.83}$$

Chapter 3

Numerical Algorithms

In this chapter, we present two different numerical algorithms that can be used to approximately solve the optimization problems discussed in the previous chapter. The first algorithm is called value function approximation, and it applies to problems with infinite time horizons. The second one is called grid value approximation, and it works well for the problems with finite time horizons. These two algorithms are introduced using Merton's model in this chapter as an example of the methods. We will apply these two algorithms to the remaining models in the next chapter.

3.1 Value Function Approximation

3.1.1 Introduction of the Algorithm

This method solves the optimization problem by approximating the value function (Das and Sundaram, 2002). We first choose a functional form that approximates the value function based on the functional form of the objective function and derive the initial HJB equation based on this approximation. Here, we denote the approximate value function by $\hat{V}(\boldsymbol{\nu}) \equiv \hat{V}(F, t; \boldsymbol{\nu})$, where $\boldsymbol{\nu} \in \mathbb{R}^p$ is a set of parameters $\{\nu_1, \ldots, \nu_p\}$ that defines the approximate value function. Based on $\hat{V}(\boldsymbol{\nu})$, we can derive the FOCs of the controls and thus the controls with respect to $\boldsymbol{\nu}$, i.e., $\tilde{\boldsymbol{x}}(\boldsymbol{\nu})$. Then, we apply $\hat{V}(\boldsymbol{\nu})$ and the controls $\tilde{\boldsymbol{x}}(\boldsymbol{\nu})$ to the initial HJB equation. If the functional form of $\hat{V}(\boldsymbol{\nu})$ is the same as the functional form of the true value function, and if the values of the parameters $\boldsymbol{\nu}$ used are the correct values, this approximate HJB equation should be the same as the true one and hold for any values of the state variables F. Instead, if the values of $\boldsymbol{\nu}$ are different from the true values, the approximate HJB equation would deviate from the true HJB equation. The purpose of this method is to obtain the best approximation parameter set $\boldsymbol{\nu}$ that minimizes the deviation between the true HJB equation and the HJB equation derived from the approximate value function.

For instance, suppose we want to find the optimal controls of Merton's model under the CRRA utility of Equation (2.4). The approximate value function $\hat{V}(\boldsymbol{\nu})$ can be written as

$\hat{V}(\boldsymbol{\nu}) = e^{-\beta t}\hat{J}(\boldsymbol{\nu})$. Instead of using the approximate value function $\hat{V}$, we use the transformed version of the approximate value function, $\hat{J}$. Based on the functional form of the objective function, $\hat{J}(\boldsymbol{\nu})$ should be same as Equation (2.13):

$$\hat{J}(\boldsymbol{\nu}) = \nu_1 F^{\nu_2}, \tag{3.1}$$

where $\boldsymbol{\nu} = \{\nu_1, \nu_2\}$. Using Equation (3.1), we derive the first and second derivatives of $\hat{J}(\boldsymbol{\nu})$ with respect to F: $\hat{J}_F$ and $\hat{J}_{FF}$. Furthermore, the approximate optimal controls $\tilde{\boldsymbol{x}}(\nu) = \{\tilde{C}(\boldsymbol{\nu}), \tilde{\pi}(\boldsymbol{\nu})\}$ follow Equations (2.16) and (2.17):

$$\tilde{C}(\boldsymbol{\nu}) = \left(\nu_1\nu_2F^{\nu_2-1}\right)^{-\frac{1}{\gamma}}, \tag{3.2}$$

$$\tilde{\pi}(\boldsymbol{\nu}) = -\frac{\mu - r}{\sigma^2}\frac{1}{\nu_2 - 1}, \tag{3.3}$$

where $\tilde{C}(\boldsymbol{\nu})$ and $\tilde{\pi}(\boldsymbol{\nu})$ are both functions with respect to $\boldsymbol{\nu}$. The initial true HJB equation follows Equation (2.10). The initial HJB equation based on this approximation is, on the other hand,

$$\beta\hat{J}(\boldsymbol{\nu}) \approx \max_{\boldsymbol{x}(\nu)}\left\{\frac{\tilde{C}(\boldsymbol{\nu})^{1-\gamma}}{1-\gamma} + [F(r + \tilde{\pi}(\boldsymbol{\nu})(\mu - r)) - \tilde{C}(\boldsymbol{\nu})]\hat{J}_F + \frac{1}{2}\sigma^2\tilde{\pi}^2(\boldsymbol{\nu})F^2\hat{J}_{FF}\right\}.$$

By moving $\beta\hat{J}(\boldsymbol{\nu})$ to the RHS of the equation, we obtain

$$0 \approx \max_{\boldsymbol{x}(\nu)}\left\{\frac{\tilde{C}(\boldsymbol{\nu})^{1-\gamma}}{1-\gamma} + [F(r + \tilde{\pi}(\boldsymbol{\nu})(\mu - r)) - \tilde{C}(\boldsymbol{\nu})]\hat{J}_F + \frac{1}{2}\sigma^2\tilde{\pi}^2(\boldsymbol{\nu})F^2\hat{J}_{FF}\right\} - \beta\hat{J}(\boldsymbol{\nu}), \tag{3.4}$$

where the LHS and RHS of Equation (3.4) are equal only if the functional form of $\hat{J}(\boldsymbol{\nu})$ is consistent with the true one, and the values used for ν_1 and ν_2 are the correct values.

Next, using $\hat{J}_F$, $\hat{J}_{FF}$, and inputting Equations (3.1) to (3.3) into Equation (3.4) leads to

$$0 \approx \left(-\beta\nu_1 + r\nu_1\nu_2 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1}{\nu_2 - 1}\nu_1\nu_2\right)F^{\nu_2} + \frac{\gamma}{1-\gamma}(\nu_1\nu_2)^{-\frac{1-\gamma}{\gamma}}(F^{\nu_2-1})^{-\frac{1-\gamma}{\gamma}}, \tag{3.5}$$

which should be true for any value of F, and should be strictly equal to zero if $J = \hat{J}$. Because this is normally not the case in most applications, we define

$$M = \left(-\beta\nu_1 + r\nu_1\nu_2 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1}{\nu_2 - 1}\nu_1\nu_2\right)F^{\nu_2} + \frac{\gamma}{1-\gamma}(\nu_1\nu_2)^{-\frac{1-\gamma}{\gamma}}(F^{\nu_2-1})^{-\frac{1-\gamma}{\gamma}}.$$

as being the error between the approximate and true HJB equations. The idea of this algorithm is then to find the best approximate values of ν_1 and ν_2 that make this error M

as close to zero as possible for any given value of F. Assume we have N different values of F such that each value of F is indexed by i, and we want the total errors to be minimal. As a consequence, one way to find parameters is to solve

$$\min_{\boldsymbol{\nu}} \sum_{i=1}^{N} M_i^2, \tag{3.6}$$

where

$$M_i = \left(-\beta\nu_1 + r\nu_1\nu_2 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1}{\nu_2 - 1}\nu_1\nu_2 \right) F_i^{\nu_2} + \frac{\gamma}{1-\gamma}(\nu_1\nu_2)^{-\frac{1-\gamma}{\gamma}} (F_i^{\nu_2 - 1})^{-\frac{1-\gamma}{\gamma}}.$$

As discussed above, we use M_i^2 here to penalize both the negative and positive deviations, so that the total error from the approximation is as close to zero as possible.

To make this algorithm work, we need to pick a set of values for F. We can choose any set of values that is reasonable. If we have an initial fund value $F_0 = 20$, a reasonable set of F values would have a lower bound of 0 and an upper bound of 40. By having the set of F and the optimization function of Equation (3.6), we can use any programming language to perform the optimization.

3.1.2 Optimization Method

The optimization method we use for this algorithm is a limited-memory version of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method (L-BFGS) as explained in Byrd et al. (1994) and Byrd et al. (1995). This method allows us to solve large nonlinear optimization problems with bounds on the variables. The L-BFGS method starts with the user choosing an initial estimate of the solution, $\boldsymbol{\nu}_0$, and the method then tries to find a better solution iteratively by making small changes to the existing solution. At the beginning of each iteration, the method calculates an estimate of the Hessian matrix by using the gradient function, the current estimate of the solution $\boldsymbol{\nu}$, and the value of the optimized function. Then, the estimated Hessian matrix is used to find the next approximated solution. The iterations stop either when the value difference of the optimized function between two iterations is less than the threshold or when the maximum number of iterations is reached. For this report, we set the threshold to be 10^{-8} and allow the method to have a maximum of 10,000 iterations.

3.2 Grid Value Approximation

The grid value approximation is an algorithm that works for finite time problems (Carroll, 2006; Cui et al., 2009). This algorithm combines backward and forward dynamic programming. In the backward step, we first construct a set of grid values. We then find the optimal controls at time $t \in \{0, h, 2h, \ldots, T-h, T\}$ for all grids and create a matrix for each control variable. Then, in the forward step, given a value of the state variable, we search for the optimal controls from these precomputed matrices. We again use Merton's model as an example to introduce this algorithm.

3.2.1 Derivations of Approximation

Since this algorithm works only with finite time horizons, we need to change Merton's optimization problem. The objective function of Equation (2.3) thus becomes

$$\mathbb{E}_0\left[\int_0^T e^{-\beta t}U(C(t))dt\right], \tag{3.7}$$

where the utility follows the CRRA utility function. Even though we change the time horizon to be finite, the value function of the problem still follows Equation (2.4). In order to apply this algorithm, we first need to discretize the SDE of wealth of Equation (2.2) and the value function of Equation (2.4). The discretized wealth SDE of Equation (2.2) is

$$\begin{aligned} F(t+h)-F(t) &= \left[F(t)(r+\pi(t)(\mu-r))-C(t)\right]h+\sigma\pi(t)F(t)(W(t+h)-W(t)) \\ &= F(t)\left[(r+\pi(t)(\mu-r))h+\sigma\pi(t)\sqrt{h}Z(t+h)\right]-C(t)h, \end{aligned}$$

where $W(t+h)-W(t)=\sqrt{h}Z(t+h)$ and $Z(t+h)$ is the value of a standard normal random variable. The discrete evolution of the wealth is thus:

$$F(t+h)=F(t)\left[1+(r+\pi(t)(\mu-r))h+\sigma\pi(t)\sqrt{h}Z(t+h)\right]-C(t)h. \tag{3.8}$$

This Euler-based discretization assumes that we first invest $F(t)$ at time t. Then, at time $t+h$, we consume an amount of $C(t)h$. If we use this equation as our recursive formula for the wealth, we have three unknown variables (i.e., $F(t), \pi(t)$, and $C(t)$) in the backward step, making it challenging to apply the algorithm. Therefore, to streamline the algorithm, we assume that the consumption is made at the beginning of the time period and the remaining wealth is invested into the portfolio at time t. If we let the time step h be small enough, these two situations are almost equivalent. As a consequence, our recursive formula becomes

$$\begin{aligned} \tilde{F}(t+h) &= \left(\tilde{F}(t)-\tilde{C}(t)h\right)\left[1+(r+\tilde{\pi}(t)(\mu-r))h+\sigma\tilde{\pi}(t)\sqrt{h}Z(t+h)\right] \\ &= \left(\tilde{F}(t)-\tilde{C}(t)h\right)\left[1+rh-\tilde{\pi}(t)rh+\tilde{\pi}(t)\left(\mu h+\sigma\sqrt{h}Z(t+h)\right)\right] \\ &= \left(\tilde{F}(t)-\tilde{C}(t)h\right)\left[1+rh-\tilde{\pi}(t)(1+rh)+\tilde{\pi}(t)\left(1+\mu h+\sigma\sqrt{h}Z(t+h)\right)\right], \end{aligned}$$

where $\tilde{F}(t)$ is the approximate discrete wealth evaluated at time t, $\tilde{\pi}(t)$ is the approximate discrete share in the risky asset, $\tilde{C}(t)$ is the discrete consumption evaluated at time t, and the terminal optimal consumption is $\tilde{C}^*(T) = \tilde{F}(T)$. Because the risk-free asset follows

$$dB(t) = rB(t)dt,$$

we have the Euler's discretization of the evolution as

$$B(t+h) = (1+rh)B(t),$$

and the exact solution of $B(t+h)$ as

$$B(t+h) = e^{rh}B(t).$$

Similarly, because the risky asset follows Equation (2.1), Euler's discretization of $S(t+h)$ is

$$S(t+h) = \left(1 + \mu h + \sigma\sqrt{h}Z(t+h)\right)S(t),$$

and the exact solution of the SDE is

$$S(t+h) = e^{\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}Z(t+h)}S(t).$$

Thus, $1 + rh$ can be seen as Euler's discretization of e^{rh}, and $1 + \mu h + \sigma\sqrt{h}Z(t+h)$ as Euler's discretization of $e^{\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}Z(t+h)}$. Consequently, the approximate wealth $\tilde{F}(t+h)$ becomes

$$\tilde{F}(t+h) = \left(\tilde{F}(t) - \tilde{C}(t)h\right)\left[R^0_{t+h} + \tilde{\pi}(t)\left(R_{t+h} - R^0_{t+h}\right)\right], \tag{3.9}$$

where $R^0_{t+h} = e^{rh}$ and $R_{t+h} = e^{\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}Z_{t+h}}$. Then, we define $\boldsymbol{a}(t)$ to be a set that contains possible values of $\tilde{F}(t) - \tilde{C}(t)h$, called the grids. Any value in this set $a \in \boldsymbol{a}(t)$ is called a grid value. The discrete recursive value function is the Euler discretization of Equation (2.4) with integration from t to T and with the state variable $\tilde{F}$:

$$\tilde{V}(\tilde{F}, t) = \max_{\tilde{C}(t), \tilde{\pi}(t)} \mathbb{E}_t\left[e^{-\beta t}U(\tilde{C}(t))h + \tilde{V}(\tilde{F}, t+h)\right]. \tag{3.10}$$

To find the optimal controls $\tilde{C}^*(t)$ and $\tilde{\pi}^*(t)$, we first find the FOCs using Equation (3.10). Even though $\tilde{V}(\tilde{F}, t)$ is not a direct function of $\tilde{\pi}(t)$, $\tilde{\pi}(t)$ affects the value of $\tilde{V}(\tilde{F}, t)$ through $\tilde{F}(t+h)$. The FOC of $\tilde{\pi}(t)$ is thus

$$\tilde{V}_{\tilde{\pi}}(t) = 0 + \mathbb{E}_t\left[\frac{\partial \tilde{V}(\tilde{F}, t+h)}{\partial \tilde{\pi}(t)}\right]$$

$$
\begin{aligned}
&= \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{\pi}(t)}\right] \\
&= \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\left(\tilde{F}(t)-\tilde{C}(t)h\right)\left(R_{t+h}-R^0_{t+h}\right)\right] = 0,
\end{aligned}
$$

where $\tilde{V}_{\tilde{\pi}}(t)$ is the partial derivative of $\tilde{V}(\tilde{F},t)$ with respect to $\tilde{\pi}(t)$ and $\tilde{V}_{\tilde{F}}(t+h)$ is the partial derivative of $\tilde{V}(\tilde{F},t+h)$ with respect to $\tilde{F}(t+h)$. Because R_{t+h} is the only random variable in the above expectation and $\tilde{F}(t)-\tilde{C}(t)h$ is known at time t, we obtain

$$
0 = \mathbb{E}_t\left[\hat{V}_{\tilde{F}}(t+h)\left(R_{t+h}-R^0_{t+h}\right)\right]. \tag{3.11}
$$

The FOC for $\tilde{C}(t)$ is therefore

$$
\tilde{V}_{\tilde{C}}(t) = \mathbb{E}_t\left[e^{-\beta t}hU_{\tilde{C}}(t)\right] + \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\right] = 0, \tag{3.12}
$$

where $\tilde{V}_{\tilde{C}}(t)$ is the partial derivative of $\tilde{V}(\tilde{F},t)$ with respect to $\tilde{C}(t)$ and $U_{\tilde{C}}(t)$ is the derivative of $U(c)$ with respect to c evaluated at $\tilde{C}(t)$.

We need the expression of $\tilde{V}_{\tilde{F}}(t+h)$ to solve Equations (3.11) and (3.12). We start with the partial derivative at time $t, \tilde{V}_{\tilde{F}}(t)$. By applying the envelope theorem and using Equations (3.9) and (3.10), the partial derivative is calculated as

$$
\begin{aligned}
\tilde{V}_{\tilde{F}}(t) &= 0 + \mathbb{E}_t\left[\frac{\partial \tilde{V}(\tilde{F},t+h)}{\partial \tilde{F}(t)}\right] \\
&= \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\left(R^0_{t+h}+\tilde{\pi}(t)\left(R_{t+h}-R^0_{t+h}\right)\right)\right] \\
&= -\mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\frac{1}{h}\right]
\end{aligned} \tag{3.13}
$$

We need to satisfy

$$
\mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\right] = -\mathbb{E}_t\left[e^{-\beta t}U_{\tilde{C}}(t)h\right] \tag{3.14}
$$

so that the FOC of Equation (3.12) holds. As a consequence, Equation (3.13) can be written as

$$
\begin{aligned}
\tilde{V}_{\tilde{F}}(t) &= \mathbb{E}_t\left[e^{-\beta t}U_{\tilde{C}}(t)\right] \\
&= e^{-\beta t}\mathbb{E}_t\left[U_{\tilde{C}}(t)\right],
\end{aligned} \tag{3.15}
$$

Since $\tilde{C}(t)$ is deterministic and known at time t, Equation (3.15) becomes

$$
\tilde{V}_{\tilde{F}}(t) = e^{-\beta t}U_{\tilde{C}}(t).
$$

Then, we can replace $\tilde{V}_{\tilde{F}}(t+h)$ by $e^{-\beta(t+h)}U_{\tilde{C}}(t+h)$ in the two FOCs and obtain:

$$
\begin{aligned}
0 &= e^{-\beta(t+h)}\mathbb{E}_t\left[U_{\tilde{C}}(t+h)\left(R_{t+h}-R^0_{t+h}\right)\right], \\
0 &= e^{-\beta t}hU_{\tilde{C}}(t) - h\mathbb{E}_t\left[e^{-\beta(t+h)}U_{\tilde{C}}(t+h)\left[R^0_{t+h}+\tilde{\pi}(t)\left(R_{t+h}-R^0_{t+h}\right)\right]\right].
\end{aligned}
$$

Because h is constant, we can further simplify the above equations to

$$
\begin{aligned}
0 &= e^{-\beta(t+h)}\mathbb{E}_t\left[U_{\tilde{C}}(t+h)\left(R_{t+h}-R^0_{t+h}\right)\right], \\
U_{\tilde{C}}(t) &= \mathbb{E}_t\left[e^{-\beta h}U_{\tilde{C}}(t+h)\left[R^0_{t+h}+\tilde{\pi}(t)\left(R_{t+h}-R^0_{t+h}\right)\right]\right].
\end{aligned}
$$

From the above equations, we know that the values of $\tilde{C}(t)$ and $\tilde{\pi}(t)$ both depend on $\tilde{C}(t+h)$, which is affected by $\tilde{F}(t+h)$. Because the value of $\tilde{F}(t+h)$ is determined by some grid value a, the optimal controls $\tilde{\pi}^*(t)$ and $\tilde{C}^*(t)$ are affected by the grid value a as well. Therefore, to solve for $\tilde{\pi}^*(t)$ and $\tilde{C}^*(t)$, we simply need to obtain the solutions of these grid value dependent equations:

$$
0 = e^{-\beta(t+h)}\mathbb{E}_t\left[U_{\tilde{C}^*}(t+h)[a]\left(R_{t+h}-R^0_{t+h}\right)\right], \tag{3.16}
$$

$$
\tilde{C}^*(t)[a] = I_U\left(e^{-\beta h}\mathbb{E}_t\left[U_{\tilde{C}^*}(t+h)[a]\left(R^0_{t+h}+\tilde{\pi}^*(t)[a]\left(R_{t+h}-R^0_{t+h}\right)\right)\right]\right), \tag{3.17}
$$

where $\tilde{\pi}^*(t)[a]$ is the optimal value of $\tilde{\pi}(t)$ given grid value a applicable at time t, $\tilde{C}^*(t)[a]$ is the time-t optimal consumption given grid value a, $U_{\tilde{C}^*}(t+h)[a]$ is the derivative of $U(c)$ with respect to c evaluated at $\tilde{C}^*(t+h)[a]$, $\tilde{C}^*(t+h)[a]$ is time-$(t+h)$ optimal consumption given a, and $I_U(\cdot)$ is the inverse function of $U_{\tilde{C}}(t)$. Using any root finding techniques, Equation (3.16) gives the optimal share $\tilde{\pi}^*(t)[a]$ for a specific a. Then, Equation (3.17) gives the corresponding optimal consumption level $\tilde{C}^*(t)[a]$.

3.2.2 Expectation Computation

Before applying a root finding technique, we need to calculate the expectations in Equations (3.16) and (3.17) first. Using Equation (3.16) as an example, the expectation can be written as

$$
\begin{aligned}
&\mathbb{E}_t\left[\left(\tilde{C}^*(t+h)[a]\right)^{-\gamma}\left(e^{\left(\mu-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}Z_{t+h}}-e^{rh}\right)\right] \\
&=\int_{-\infty}^{\infty}\left(\tilde{C}^*(t+h)[a]\right)^{-\gamma}\left(e^{\left(\mu-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}z}-e^{rh}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz.
\end{aligned} \tag{3.18}
$$

If we let $x=\frac{z}{\sqrt{2}}$, the above equation becomes

$$
\int_{-\infty}^{\infty}\left(\tilde{C}^*(t+h)[a]\right)^{-\gamma}\left(e^{\left(\mu-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}\sqrt{2}x}-e^{rh}\right)\frac{1}{\sqrt{\pi}}e^{-x^2}\,dx, \tag{3.19}
$$

which has the form of $\int_{-\infty}^{\infty} e^{-x^2} f(x)dx$. We can therefore use the Gauss-Hermite quadrature to approximate the integral:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where x_i are the roots of the physicists' version of the Hermite polynomial $H_n(x)$, n is the number of sample points used, and w_i are weights given by[1]

$$w_i = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(x_i)]^2}.$$

Here, $H_{n-1}(x)$ is the Hermite polynomial of order $n-1$ evaluated at x and is given by

$$H_{n-1}(x) = (-1)^{n-1} e^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}.$$

We can write the expectation at any time t in a form similar to Equation (3.19) and thus use the Gauss-Hermite quadrature.

3.2.3 Backward and Forward Steps

From Equations (3.16) and (3.17), we can now perform the backward and forward steps of the algorithm. The first step is to construct the grids at time t, $\boldsymbol{a}(t)$, which contains J possible values of the post-consumption wealth, $\tilde{F}(t) - \tilde{C}(t)h$. Then, in the backward step, we need to find the optimal decisions at time $t \in \{0, h, 2h, \dots, T-h, T\}$ for each grid value $a \in \boldsymbol{a}(t)$. We know that at time T, $\tilde{C}^*(T) = \tilde{F}(T)$. So, it is easy to find the optimal decisions at time $T-h$. At time $t = T-h$ and given the grid value $a \in \boldsymbol{a}(t)$, Equation (3.16) can be written as

$$\begin{aligned}
0 &= \mathbb{E}_t \left[U_{\tilde{C}^*}(t+h)[a] \left(R_{t+h} - R^0_{t+h} \right) \right] \\
&= \mathbb{E}_{T-h} \left[U_{\tilde{C}^*}(T)[a] \left(R_T - R^0_T \right) \right] \\
&= \mathbb{E}_{T-h} \left[\left(\tilde{F}(T) \right)^{-\gamma} \left(R_T - R^0_T \right) \right] \\
&= \mathbb{E}_{T-h} \left[\left(\left(\tilde{F}(T-h) - \tilde{C}(T-h)h \right) \left[R^0_T + \tilde{\pi}^*(T-h)[a] \left(R_T - R^0_T \right) \right] \right)^{-\gamma} \left(R_T - R^0_T \right) \right] \\
&= \mathbb{E}_{T-h} \left[\left(a \left[R^0_T + \tilde{\pi}^*(T-h)[a] \left(R_T - R^0_T \right) \right] \right)^{-\gamma} \left(R_T - R^0_T \right) \right], \qquad (3.20)
\end{aligned}$$

using Equation (3.9). By root finding techniques, we can solve for the optimal portfolio weight $\tilde{\pi}^*(T-h)[a]$ for each $a \in \boldsymbol{a}(T-h)$. The optimal consumption can then be obtained directly from Equation (3.17).

[1] See Abramowitz and Stegun (1972) for more details.

From Equation (3.20), we see that the optimal consumption at time $T-h$ depends on the grid value $a \in \boldsymbol{a}(T-h)$. If we change the value of a, we have a new value of $\tilde{\pi}^*(T-h)[a]$ and thus a new $\tilde{C}^*(T-h)[a]$, leading to $\tilde{F}^*(T-h)[a] = a + \tilde{C}^*(T-h)[a]h$. So, for each given grid value a applicable at time $T-h$, we have unique values of $\tilde{\pi}^*(T-h)[a], \tilde{C}^*(T-h)[a]$, and $\tilde{F}^*(T-h)[a]$. Then, at time $T-2h$, we have

$$0 = e^{-\beta(T-h)}\mathbb{E}_{T-2h}\left[(\tilde{C}^*(T-h)[a])^{-\gamma}\left(R_{T-h} - R^0_{T-h}\right)\right], \tag{3.21}$$

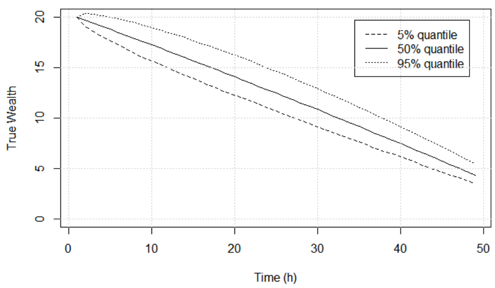
where the value of $\tilde{C}^*(T-h)[a]$ depends on $\tilde{F}^*(T-h)[a]$. Given a grid value a applicable at time $T-2h$, the root finding technique tries different candidate values of $\tilde{\pi}(T-2h)[a]$. For each candidate value of $\tilde{\pi}(T-2h)[a]$ tried, a new value of wealth $\tilde{F}(T-h)[a]$ is produced using Equation (3.9), where $a = \tilde{F}(T-2h) - \tilde{C}(T-2h)h$. From the previous calculation at time $t = T-h$, we already have J groups of values for $\tilde{F}^*(T-h)$ and $\tilde{C}^*(T-h)$. Therefore, all the new values of $\tilde{F}(T-h)[a]$ can be expressed as the linear interpolation of two nearest values of $\tilde{F}^*(T-h)$ we already have. By doing this, we can also obtain the corresponding $\tilde{C}(T-h)[a]$ using the similar linear interpolation of the two values of the $\tilde{C}^*(T-h)$ that are paired with the two values of $\tilde{F}^*(T-h)$. We then will have the pair of $\left(\tilde{F}^*_{new}(T-h)[a], \tilde{C}^*_{new}(T-h)[a]\right)$ for each $\tilde{\pi}(T-2h)[a]$ used. Among all these pairs, we finally can find the optimal $\tilde{\pi}^*(T-2h)[a]$ that satisfies Equation (3.21). After obtaining the optimal $\tilde{\pi}^*(T-2h)[a]$, we again use the linear interpolation to find the corresponding pair of wealth and consumption at time $T-h$ and solve for $\tilde{C}^*(T-2h)[a]$ using Equation (3.17). We finally repeat this procedure till time $t = 0$ is reached.

After the backward pass is done, we have J groups of optimal decisions as well as the state variables for each time t. Then, we start the forward part of our algorithm. The purpose of the forward pass is to produce the optimal controls given a specific path of the state variables over time. By reference to the J grid values $\boldsymbol{a}(0)$ and their corresponding state variables $\tilde{F}^*(0)[a]$, we can again express the initial wealth $F(0)$ as a linear interpolation between the two nearest values of $\tilde{F}^*(0)[a]$ from the grid. We can then locate the two values of $\tilde{C}^*(0)[a]$ and the two values of $\tilde{\pi}^*(0)[a]$ associated with the same grid values. We therefore obtain the final optimal decisions at time zero; that is, $\tilde{C}^*(0)$ and $\tilde{\pi}^*(0)$. By relying on the optimal decisions at time zero and wealth $F(0)$, we can thus compute the time-h wealth from Equation (3.9). By repeating this procedure till time $T-h$, we can find all the needed optimal decisions.

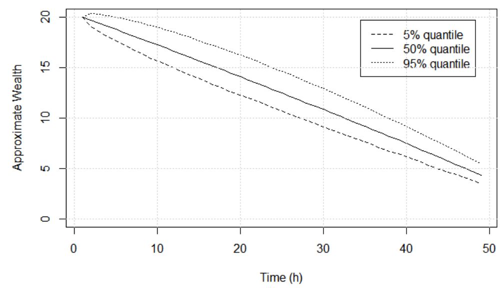
3.3 Results

To test if these two methods work for the simplest case, we set the model parameters to $r = 0.02, \mu = 0.05, \sigma = 0.15$, and $\gamma = 3$. The time preference factor β is equal to the risk-free rate r. The grid we used for the value function approximation is from 0.1 to 40 with increments of 0.05. We choose $\boldsymbol{\nu}_0 = (-30{,}000, -2)$ as our starting points, with a lower

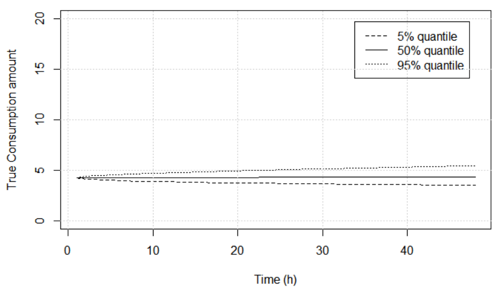
Figure 3.1: **Distributions of Approximate Solution Based on the Grid Value Approximation Method and True Solution with Merton's Model.**



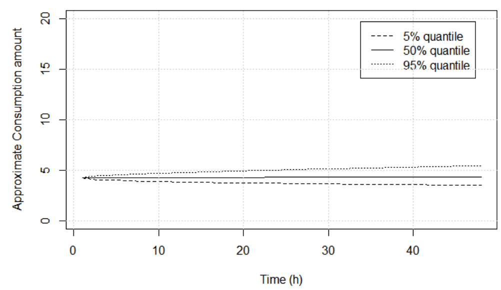
(a) Distribution of True Fund Value



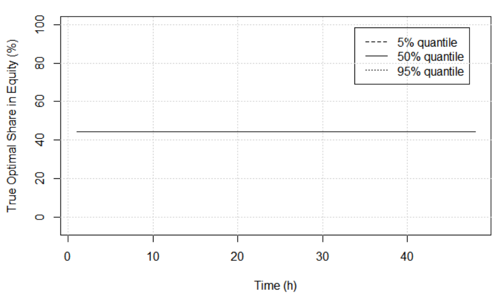
(b) Distribution of Approximate Fund Value



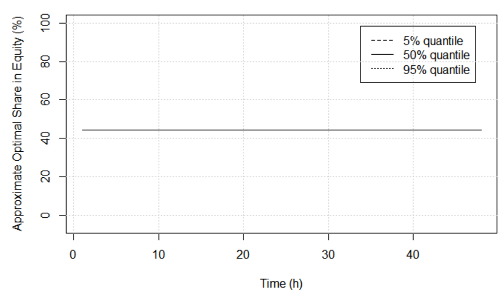
(c) Distribution of True Optimal Contribution



(d) Distribution of Approximate Optimal Contribution



(e) Distribution of True Optimal Share



(f) Distribution of Approximate Optimal Share

bound of $(-100{,}000, -10)$ and an upper bound of $(-0.01, -0.01)$ in the optimization. The approximate and true value function parameters are presented in Table 3.1. The relative differences, $|\frac{\nu_i - v_i}{v_i}| \times 100\%$, between the approximate and true values are 0.022% and 0.005% respectively. Because these differences are less than 0.1%, we consider that this method works well for Merton's model.

For the grid value approximation, we use a monthly time step of $h = 1/12$ and a time horizon of $T = 4$ years. The set of grids used has a minimum value of 0.01 and a maximum of 40. The initial wealth is set to $F(0) = 20$. Even though the method works with one path, we set the number of simulation runs in the forward pass to $N = 5{,}000$ to

	Approximate		True	
Parameters	ν_1	ν_2	v_1	v_2
Values	−34,239.344443	−1.999894	−34,231.780616	−2.000000

Table 3.1: **Optimal Value Function Parameters of Merton's Model Based on Value Function Approximation.**

evaluate the performance of the method. The results are given in Figure 3.1. Solutions of true contribution amount and share in the risky asset can be found in Merton (1969). We compare different quantiles between the true and approximate solutions. We notice that the differences between the true and approximate solutions are almost nil. We can thus conclude that the grid value approximation works well for Merton's model, too.

Chapter 4

Applications of the Numerical Algorithms

In this chapter, we apply the two algorithms discussed in Chapter 3 to the optimal pension funding problems of Chapter 2. In each section, we first show the equations and steps required to apply the method. Then, we follow with numerical results for each model.

4.1 Single Risky Asset

We start with the models that only have one risky asset, which are discussed in Section 2.2.

4.1.1 Infinite Time Horizon

For optimal pension funding problems with infinite time horizon, we use the value function approximation introduced in Section 3.1. The problem under study is that of Equation (2.25). The approximate value function $\hat{V}(\boldsymbol{\nu})$ has a functional form similar to that of Equation (2.30):

$$\hat{V}(\boldsymbol{\nu}) = \nu_1 F^2 + \nu_2 AL^2 + \nu_3 F \cdot AL, \tag{4.1}$$

where $\boldsymbol{\nu} = \{\nu_1, \nu_2, \nu_3\}$. Using Equation (4.1), the first and second derivatives of $\hat{V}$ with respect to F are

$$\hat{V}_F(\boldsymbol{\nu}) = 2\nu_1 F + \nu_3 AL, \tag{4.2}$$

$$\hat{V}_{FF}(\boldsymbol{\nu}) = 2\nu_1, \tag{4.3}$$

respectively. The FOCs for the two controls, $\tilde{C}(\boldsymbol{\nu})$ and $\tilde{\pi}(\boldsymbol{\nu})$, follow the same functional forms as of Equations (2.33) and (2.34):

$$\tilde{C}(\boldsymbol{\nu}) = NC - \frac{2\nu_1 F + \nu_3 AL}{2\kappa}, \tag{4.4}$$

$$\tilde{\pi}(\boldsymbol{\nu}) = -\frac{2\nu_1 F + \nu_3 AL}{2\nu_1 F}\frac{\mu - r}{\sigma^2}. \tag{4.5}$$

Therefore, the initial approximate HJB equation follows

$$\begin{aligned} 0 \approx & \left(-\beta\nu_1 - \frac{\nu_1^2}{\kappa} + (1-\kappa) + 2\nu_1 r - \frac{(\mu-r)^2}{\sigma^2}\nu_1\right)F^2 \\ & + \left(-\beta\nu_2 - \frac{\nu_3^2}{4\kappa} + (1-\kappa) - \nu_3\delta - \frac{(\mu-r)^2}{\sigma^2}\frac{\nu_3^2}{4\nu_1}\right)AL^2 \\ & + \left(-\beta\nu_3 - \frac{\nu_1\nu_3}{\kappa} - 2(1-\kappa) - 2\nu_1\delta + \nu_3 r - \frac{(\mu-r)^2}{\sigma^2}\nu_3\right)F\cdot AL. \end{aligned} \tag{4.6}$$

According to Section 3.1.1, the error between the approximate and true HJB equations for any given value of F_i can be set to

$$\begin{aligned} M_i = & \left(-\beta\nu_1 - \frac{\nu_1^2}{\kappa} + (1-\kappa) + 2\nu_1 r - \frac{(\mu-r)^2}{\sigma^2}\nu_1\right)F_i^2 \\ & + \left(-\beta\nu_2 - \frac{\nu_3^2}{4\kappa} + (1-\kappa) - \nu_3\delta - \frac{(\mu-r)^2}{\sigma^2}\frac{\nu_3^2}{4\nu_1}\right)AL^2 \\ & + \left(-\beta\nu_3 - \frac{\nu_1\nu_3}{\kappa} - 2(1-\kappa) - 2\nu_1\delta + \nu_3 r - \frac{(\mu-r)^2}{\sigma^2}\nu_3\right)F_i\cdot AL, \end{aligned} \tag{4.7}$$

and the minimization problem that needs to be solved to is given in Equation (3.6).

To solve the problem, we need to first set the grid over which the minimization equation is computed. As the initial fund value is equal to $F(0) = 20$, we choose the grid to have a lower bound of 0 and an upper bound of 40, where the upper bound refers to a total return of 100%. The increment of the grid is set to 0.05, and we use a total of 801 points in the minimization problem. We set the risk-free rate to $r = 0.05$, the rate of return for the single risky asset to $\mu = 0.1$, and the volatility to $\sigma = 0.15$ (see, e.g., Ngwira and Gerrard, 2007). The valuation rate δ and the time preference factor β are both assumed to be equal to r. The weight of the contribution risk κ is set to 0.8.

We first approximate the values $\boldsymbol{\nu}$ and compare them with the true values of v_1, v_2, and v_3. After testing different initial values of the optimization and comparing the results, we choose $\boldsymbol{\nu}_0 = (0.4, 0.4, -0.5)$ as our starting point. We set the lower and upper bounds for the three parameters to -1 and 3, respectively. The estimated approximate and true values solved are reported in Table 4.1 for the problem under study. To examine the performance of the approximation, we calculate the relative differences, $|\frac{\nu_i - v_i}{v_i}| \times 100\%$, between the approximate and true values of the parameters. The relative differences are all less than 0.01%. Therefore, we expect the approximate and true values of the optimal controls to be very close.

	Approximate			True		
Parameters	ν_1	ν_2	ν_3	v_1	v_2	v_3
Values	0.376298	0.376306	-0.752597	0.376302	0.376302	-0.752604

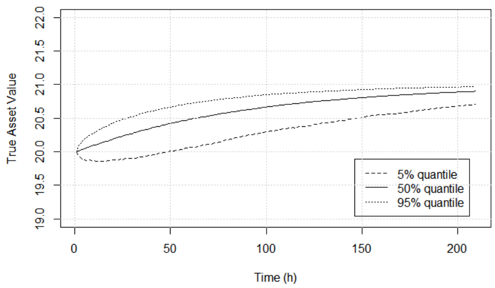
Table 4.1: **Optimal Value Function Parameters of Optimal Pension Funding Model Based on Value Function Approximation Method with Single Risky Asset.**

To evaluate the performance of the approximate value function in producing optimal controls, we perform a simulation study. We generate N simulated paths of future risky asset returns and, along each path, compute the evolution of the fund value and the corresponding optimal controls. We then compare the resulting distributions of the true and approximate optimal controls as well as the fund value. We use a total of $N = 5{,}000$ simulation paths to perform the comparison. Because it is impossible for us to test the infinite time horizon, we assume a terminal time T of 4 years. The time step is set to one week; that is, $h = 1/52$. We start the simulation at time $t = 0$ with an initial fund level $F(0) = 20$. The actuarial liability AL has a deterministic value of 21. The function used to calculate the approximate and true optimal controls follow Equations (4.4) and (4.5), respectively, where the approximate controls use estimated values of ν_1, ν_2, and ν_3, while the true ones use v_1, v_2, and v_3 in Table 4.1. The normal cost NC is calculated using Equation (2.23) with a benefit of $P = 2$. To perform the simulation, we use the discrete evolution of the fund value of Equation (2.24). For consistency, Equation (2.24) is discretized using the idea discussed in Section 3.2.1:

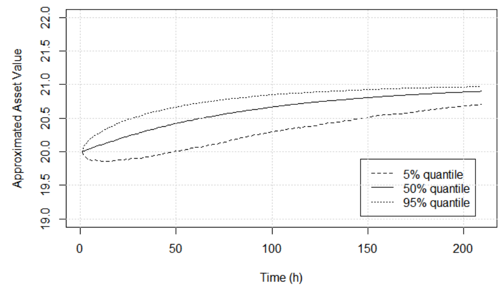
$$\tilde{F}(t+h) = \left(\tilde{F}(t) + \tilde{C}(t)h\right)\left[R^0_{t+h} + \tilde{\pi}(t)\left(R_{t+h} - R^0_{t+h}\right)\right] - P \cdot h. \tag{4.8}$$

The 5^{th}, 50^{th}, and 95^{th} quantiles of the true and approximate fund value, optimal contribution, and optimal share are presented in Figure 4.1. Because the initial unfunded actuarial liability is quite large, we need to make a large contribution to the plan and invest a moderate share of wealth into the risky asset. As a consequence, the unfunded actuarial liability decreases. The fund value keeps increasing as t increases and reaches the target level eventually, which leads to a decrease in the UAL. The UAL decrease then leads to a decrease in the optimal contributions and share in the risky asset. As time passes, the different percentiles of both approximate and true optimal contributions start to converge to their long run level. We expect that all these percentiles will eventually reach the target level; that is, the normal cost. Similarly, different percentiles of the optimal share in the single risky asset tend to go to zero as t increases. If we compare the distributions between the approximate and true solutions, we find that the differences are almost nil, which meets our expectations. Thus, we conclude that this method works quite well with the optimal pension funding problem under the given assumptions.

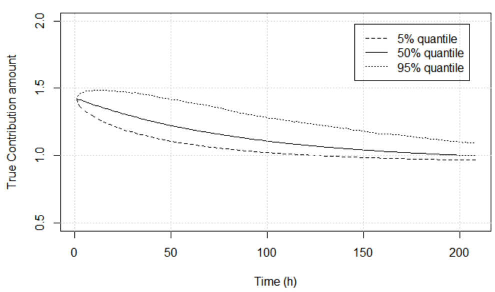
Figure 4.1: **Distributions of Approximate Solution Based on the Value Function Approximation Method and True Solution of Optimal Pension Funding Model with Single Risky Asset.**



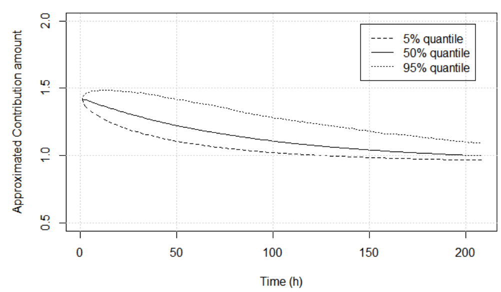
(a) Distribution of True Fund Value



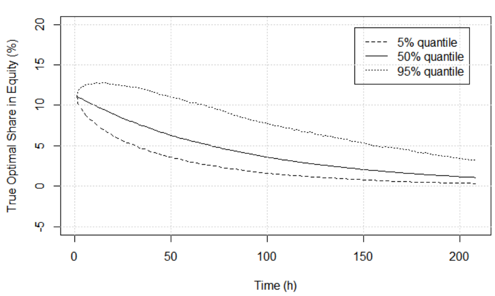
(b) Distribution of Approximate Fund Value



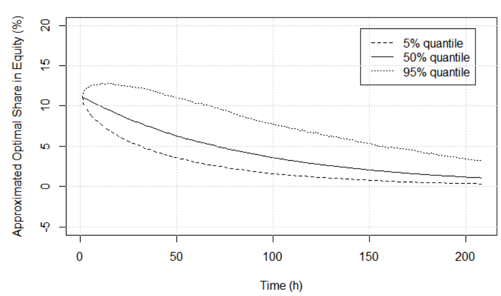
(c) Distribution of True Optimal Contribution



(d) Distribution of Approximate Optimal Contribution



(e) Distribution of True Optimal Share



(f) Distribution of Approximate Optimal Share

4.1.2 Finite Time Horizon

For the same pension funding problem discussed above but with a finite time horizon, we use the grid value method to solve for the optimal controls. The model is the one presented in Section 2.2.2. As discussed in Section 3.2.1, we first need to discretize the value function and the wealth evolution equation. The discrete evolution of the fund value follows Equation (4.8) and can be obtained by following similar steps to those done in the derivation of Equation (3.9). The initial value function follows Equation (2.43) and can be written in a recursive form:

$$\begin{aligned} &V(F,t) \\ &= \min_{C(t),\pi(t)} \mathbb{E}_t \left[\int_t^{t+h} e^{-\beta s} \left(\kappa(C(s) - NC)^2 + (1-\kappa)(AL - F(s))^2 \right) ds + V(F, t+h) \right]. \end{aligned} \tag{4.9}$$

We discretize the above equation to get

$$\begin{aligned} &\tilde{V}(\tilde{F},t) \\ &= \min_{\tilde{C}(t),\tilde{\pi}(t)} \mathbb{E}_t \left[e^{-\beta t} \left(\kappa(\tilde{C}(t) - NC)^2 + (1-\kappa)(AL - \tilde{F}(t))^2 \right) h + \tilde{V}(\tilde{F}, t+h) \right], \end{aligned} \tag{4.10}$$

which is based on the discrete fund value $\tilde{F}$ and the discrete controls $\tilde{C}$ and $\tilde{\pi}$. The terminal condition satisfies

$$\tilde{V}(\tilde{F},T) = \alpha e^{-\beta T} (\tilde{F}(T) - AL)^2.$$

Because $\tilde{F}(T)$ is determined by $\tilde{C}(T-h)$ and $\tilde{\pi}(T-h)$ and is known at time T, the last decision is made at time $T-h$. We therefore start the backward pass at time $T-h$, and due to the existence of the terminal condition, we have unique optimal controls at $t = T-h$. From Equation (4.10), the value function at $t = T-h$ is

$$\begin{aligned} &\tilde{V}(\tilde{F}, T-h) \\ &= \min_{\tilde{C}(T-h),\tilde{\pi}(T-h)} \Bigg\{ \mathbb{E}_{T-h} \Bigg[e^{-\beta(T-h)} \Bigg(\kappa \left(\tilde{C}(T-h) - NC \right)^2 \\ &\qquad + (1-\kappa) \left(AL - \tilde{F}(T-h) \right)^2 \Bigg) h \Bigg] + \mathbb{E}_{T-h} \left[\tilde{V}(\tilde{F}, T) \right] \Bigg\}. \end{aligned} \tag{4.11}$$

The optimal controls $\tilde{C}^*(T-h)$ and $\tilde{\pi}^*(T-h)$ can be obtained by solving the above equation. We first give the FOC for $\tilde{\pi}(T-h)$ using Equations (4.8) and (4.11):

$$\begin{aligned} &\tilde{V}_{\tilde{\pi}}(T-h) \\ &= \mathbb{E}_{T-h} \left[2\alpha e^{-\beta T} (\tilde{F}(T) - AL) \frac{\partial \tilde{F}(T)}{\partial \tilde{\pi}(T-h)} \right] \\ &= \mathbb{E}_{T-h} \left[2\alpha e^{-\beta T} (\tilde{F}(T) - AL) \left(\tilde{F}(T-h) + \tilde{C}(T-h)h \right) \left(R_T - R_T^0 \right) \right] \\ &= 2\alpha e^{-\beta T} \mathbb{E}_{T-h} \left[\left(a \left[R_T^0 + \tilde{\pi}(T-h) \left(R_T - R_T^0 \right) \right] - P \cdot h - AL \right) a \left(R_T - R_T^0 \right) \right] \\ &= 0, \end{aligned} \tag{4.12}$$

where $a \in \boldsymbol{a}(T-h)$ is the grid value used to represent the post-contribution fund value, $\tilde{F}(T-h) + \tilde{C}(T-h)h$. Because a is not random, Equation (4.12) can be simplified:

$$0 = \mathbb{E}_{T-h}\left[\left(a\left[R_T^0 + \tilde{\pi}(T-h)\left(R_T - R_T^0\right)\right] - P \cdot h - AL\right)\left(R_T - R_T^0\right)\right]. \tag{4.13}$$

As the only unknown variable in Equation (4.13) is $\tilde{\pi}(T-h)$ and the only random variable is R_T, $\tilde{\pi}(T-h)$ can be solved directly without using a root finding technique:

$$\tilde{\pi}^*(T-h)[a] = \frac{P \cdot h + AL - aR_T^0}{a} \frac{\mathbb{E}_{T-h}\left[R_T - R_T^0\right]}{\mathbb{E}_{T-h}\left[\left(R_T - R_T^0\right)^2\right]}, \tag{4.14}$$

where $\tilde{\pi}^*(T-h)[a]$ is the optimal value of $\tilde{\pi}(T-h)$ given the value of a.

Next, we state the FOC for the contribution amount at time $T-h$, $\tilde{C}(T-h)$. Using Equation (4.10), we obtain the FOC for $\tilde{C}(T-h)$:

$$\begin{aligned} &\tilde{V}_{\tilde{C}}(\tilde{F}, T-h) \\ =& \mathbb{E}_{T-h}\left[2he^{-\beta(T-h)}\kappa\left(\tilde{C}(T-h) - NC\right) + 2\alpha e^{-\beta T}\left(\tilde{F}(T) - AL\right)\frac{\partial \tilde{F}(T)}{\partial \tilde{C}(T-h)}\right] \\ =& 0. \end{aligned} \tag{4.15}$$

Similar to the optimal share, Equation (4.15) can be solved directly as well:

$$\begin{aligned} &\tilde{C}^*(T-h)[a] \\ =& NC - \frac{\alpha}{\kappa}e^{-\beta h}\mathbb{E}_{T-h}\left[\left(a\left[R_T^0 + \tilde{\pi}^*(T-h)[a]\left(R_T - R_T^0\right)\right] - P \cdot h - AL\right)\right. \\ &\left. \times\left(R_T^0 + \tilde{\pi}^*(T-h)[a]\left(R_T - R_T^0\right)\right)\right], \end{aligned} \tag{4.16}$$

where $\tilde{C}^*(T-h)[a]$ is the optimal contribution given $a \in \boldsymbol{a}(T-h)$.

After obtaining the specific equations for the optimal controls at time $T-h$, we can now find the general solution for time $t \in \{0, h, \dots, T-2h\}$. We use Equation (4.10) to solve for the optimal controls $\tilde{C}^*(t)$ and $\tilde{\pi}^*(t)$. The FOCs for $\tilde{C}(t)$ and $\tilde{\pi}(t)$ are:

$$\begin{aligned} \tilde{V}_{\tilde{\pi}}(t) &= \mathbb{E}_t\left[\frac{\partial \tilde{V}(\tilde{F}, t+h)}{\partial \tilde{\pi}(t)}\right] = \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{\pi}(t)}\right] \\ &= \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)(\tilde{F}(t) + \tilde{C}(t)h)\left(R_{t+h} - R_{t+h}^0\right)\right] = 0, \end{aligned} \tag{4.17}$$

$$\tilde{V}_{\tilde{C}}(t) = \mathbb{E}_t\left[e^{-\beta t}2h\kappa\left(\tilde{C}(t) - NC\right)\right] + \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\right] = 0. \tag{4.18}$$

Similar to Merton's model, we need the expression of $\tilde{V}_{\tilde{F}}(t+h)$ to solve Equations (4.17) and (4.18). We start with the partial derivative at time $t, \tilde{V}_{\tilde{F}}(t)$:

$$\begin{aligned}\tilde{V}_{\tilde{F}}(t) &= \mathbb{E}_t\left[-2he^{-\beta t}(1-\kappa)(AL-\tilde{F}(t))\right] + \mathbb{E}_t\left[\frac{\partial \tilde{V}(\tilde{F}, t+h)}{\partial \tilde{F}(t)}\right] \\ &= \mathbb{E}_t\left[-2he^{-\beta t}(1-\kappa)(AL-\tilde{F}(t))\right] + \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t)\left(R^0_{t+h} + \tilde{\pi}(t)\left(R_{t+h} - R^0_{t+h}\right)\right)\right] \\ &= \mathbb{E}_t\left[-2he^{-\beta t}(1-\kappa)(AL-\tilde{F}(t))\right] + \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\frac{1}{h}\right]. \end{aligned} \tag{4.19}$$

Equation (4.18) needs to satisfy

$$\mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial \tilde{F}(t+h)}{\partial \tilde{C}(t)}\right] = -\mathbb{E}_t\left[e^{-\beta t}2h\kappa\left(\tilde{C}(t) - NC\right)\right]. \tag{4.20}$$

As a consequence, Equation (4.19) can be written as

$$\tilde{V}_{\tilde{F}}(t) = \mathbb{E}_t\left[-2he^{-\beta t}(1-\kappa)(AL-\tilde{F}(t))\right] - \frac{1}{h}\mathbb{E}_t\left[e^{-\beta t}2h\kappa\left(\tilde{C}(t) - NC\right)\right]. \tag{4.21}$$

Because there is no randomness in $\tilde{C}(t)$ and $\tilde{F}(t)$ at time t, we can remove the expectation in Equation (4.21). We thus obtain

$$\tilde{V}_{\tilde{F}}(t+h) = -2he^{-\beta(t+h)}(1-\kappa)\left(AL - \tilde{F}(t+h)\right) - 2e^{-\beta(t+h)}\kappa\left(\tilde{C}(t+h) - NC\right). \tag{4.22}$$

By inputting Equation (4.22) into Equation (4.17), we obtain the equation needed to find $\tilde{\pi}^*(t)$:

$$\begin{aligned} & -2e^{-\beta(t+h)}\mathbb{E}_t\left[a\left(R_{t+h} - R^0_{t+h}\right)\left(h(1-\kappa)\left(AL - \tilde{F}(t+h)\right) + \kappa\left(\tilde{C}(t+h) - NC\right)\right)\right] \\ =& \mathbb{E}_t\left[\left(h(1-\kappa)\left[AL - \left(a\left(R^0_{t+h} + \tilde{\pi}(t)\left(R_{t+h} - R^0_{t+h}\right)\right) - P\cdot h\right)\right]\right.\right. \\ & \qquad \left.\left. + \kappa\left[\tilde{C}(t+h)[a] - NC\right]\right)\left(R_{t+h} - R^0_{t+h}\right)\right] \\ =& 0, \end{aligned} \tag{4.23}$$

where a is a value of $\tilde{F}(t) + \tilde{C}(t)h$ and $\tilde{C}(t+h)[a]$ is the contribution amount at time $t+h$ given a at time t. By substituting Equation (4.22) into Equation (4.18), we get the solution for $\tilde{C}^*(t)$:

$$\begin{aligned} & \tilde{C}^*(t)[a] \\ =& NC + \frac{e^{-\beta h}}{\kappa}\mathbb{E}_t\left[\left(h(1-\kappa)\left[AL + P\cdot h - a\left(R^0_{t+h} + \tilde{\pi}^*(t)[a]\left(R_{t+h} - R^0_{t+h}\right)\right)\right]\right.\right. \\ & \qquad \left.\left. + \kappa\left(\tilde{C}^*(t+h)[a] - NC\right)\right)\left(R^0_{t+h} + \tilde{\pi}^*(t)[a]\left(R_{t+h} - R^0_{t+h}\right)\right)\right]. \end{aligned} \tag{4.24}$$

Because there is no closed-form solution for the expectations in Equations (4.23) and (4.24), we need to use numerical integration discussed in Section 3.2.2 to calculate the expectations. Using any root finding technique, Equation (4.23) gives the optimal share $\tilde{\pi}^*(t)[a]$ for any given value of a. Then, Equation (4.24) gives the corresponding optimal contribution amount $\tilde{C}^*(t)[a]$.

To investigate the performance of this approximate algorithm, we compare the approximate numerical results to the true solutions through a simulation study. The choice of grid values depends on the values of $F(0)$ and AL. Because the fund value is supposed to reach the actuarial liability, we expect the lower and upper bounds of the fund value to be close to the initial fund value and the actuarial liability, respectively. If the initial fund value is 20 and the actuarial liability is 21, $\boldsymbol{a}(t)$ is then assumed to have a minimum value equals to 19 and maximum value equals to 22. The increment of the grid is set to be 0.005 and thus we have a total of 601 possible values. All other parameters follow the values given in Section 4.1.1; that is, $r = 0.05, \mu = 0.1, \sigma = 0.15, \delta = \beta = r$, and $\kappa = 0.8$. The rate of benefit paid out is $P = 2$, and the value of the normal cost is calculated using Equation (2.22). Furthermore, the parameter dealing with the importance of the terminal condition, α, is set to 0.2.

We generate 5,000 scenarios with a weekly time step of $h = 1/52$ for 4 years. The values for the explicit optimal controls are obtained from Equations (2.62) and (2.63) with F replaced by its discrete approximation $\hat{F}(t)$, where

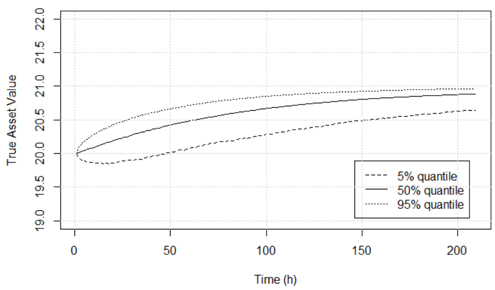
$$\hat{F}(t+h) = \left(\hat{F}(t) + C(t)h\right)\left[R^0_{t+h} + \pi(t)\left(R_{t+h} - R^0_{t+h}\right)\right] - P \cdot h.$$

The 5^{th}, 50^{th}, and 95^{th} quantiles of both the true and approximate solutions are presented in Figure 4.2. For each panel, the three quantiles tend to converge to a certain target level as time passes by. If we compare the true and approximate solutions, the distributions are very similar.

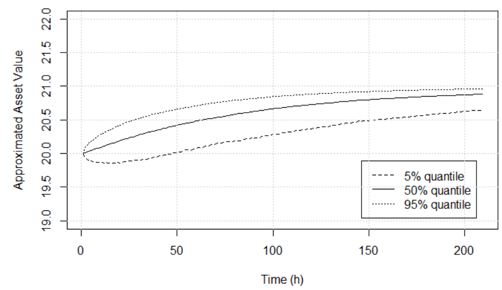
To further explore how the approximation performs, we plot the time-t empirical cumulative distribution functions (ECDF) for the true and approximate optimal controls as well as the fund values in the same plot. The results are given in Figures 4.3.

At any time t, the true optimal contribution amount and true optimal share in the risky asset are determined by reference to the optimal controls derived in Section 2.2.2, which depend on $F(t)$. However, in the approximate solutions, since we use discrete time, the approximate optimal share in the risky asset is determined after the approximate optimal contribution amount is made, which is based on the fund value after contribution, $F(t) + C(t)$. Since the sponsor contributes more to the pension plan when UAL is high, the difference between the fund value before and after the contribution is large at the beginning of the simulation horizon. As a consequence, the difference between the ECDFs of the true and approximate optimal shares in the risky asset is larger for the initial steps. This difference diminishes as time passes. After two years (i.e, 104 weeks), the ECDFs are almost the same for both controls. We can thus conclude that the grid value approximation works well under the given assumptions.

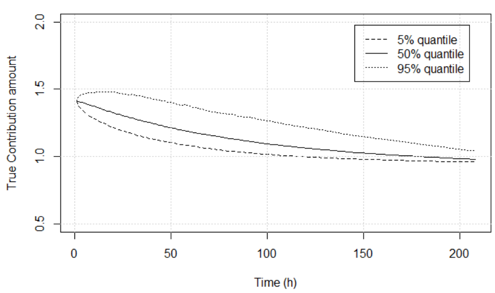
Figure 4.2: **Distributions of Approximate Solution Based on the Grid Value Approximation Method and True Solution of Optimal Pension Funding Model with Single Risky Asset.**



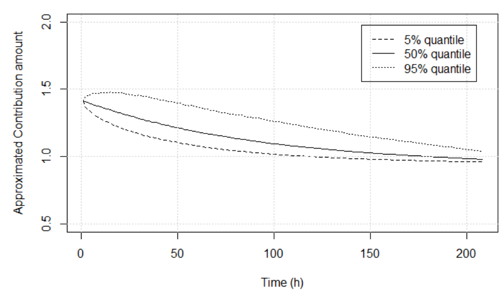
(a) Distribution of True Fund Value



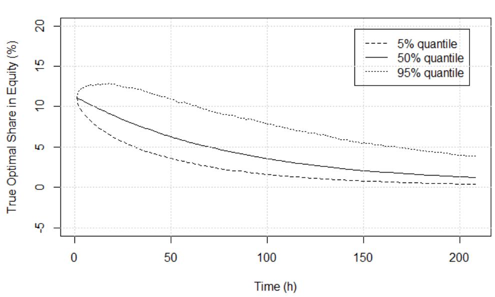
(b) Distribution of Approximate Fund Value



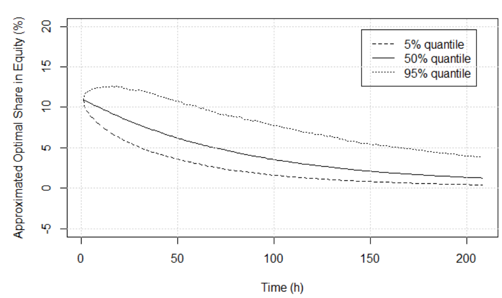
(c) Distribution of True Optimal Contribution



(d) Distribution of Approximate Optimal Contribution

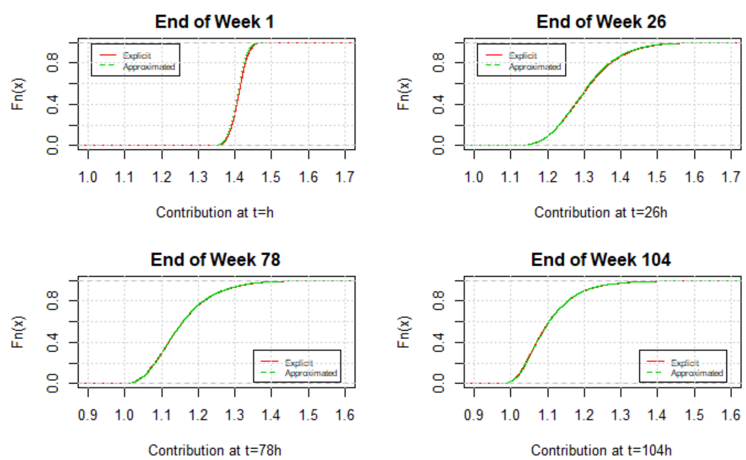


(e) Distribution of True Optimal Share

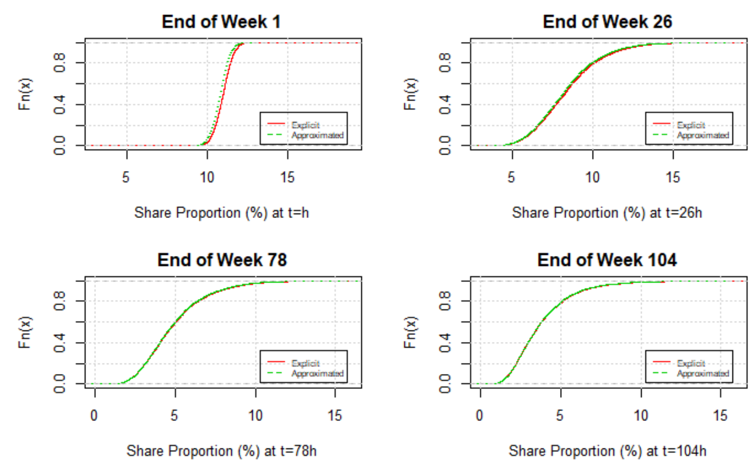


(f) Distribution of Approximate Optimal Share

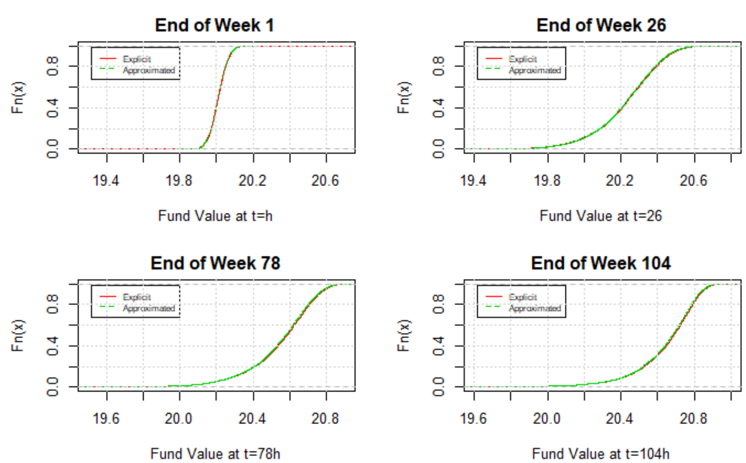
Figure 4.3: **Comparison Between Empirical Cumulative Distribution Functions of True and Approximate Optimal Controls and State Variables Based on the Grid Value Approximation Method with Single Risky Asset for Different Horizons.**



(a) Comparison Between True and Approximate Contribution



(b) Comparison Between True and Approximate Share



(c) Comparison Between True and Approximate Fund Value

4.2 Multiple Risky Assets

We now apply the same two algorithms to more general cases with multiple risky assets discussed in Section 2.3. To simplify the simulation process, we consider two different risky assets.

4.2.1 Infinite Time Horizon

Since we only change the number of assets in the portfolio, the approximate value function $\hat{V}(\boldsymbol{\nu})$ keeps the same functional form as Equation (4.1). Therefore, the first and second derivatives of $\hat{V}$ with respect to F follow Equations (4.2) and (4.3). The FOC for the contribution, $\tilde{C}(\boldsymbol{\nu})$, still follows Equation (4.4), while the FOC for optimal shares, $\tilde{\boldsymbol{\Pi}}(\boldsymbol{\nu})$, is now given by Equation (2.70):

$$\tilde{\boldsymbol{\Pi}}(\boldsymbol{\nu}) = -\frac{2\nu_1 F + \nu_3 AL}{2\nu_1 F}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}), \tag{4.25}$$

where $\tilde{\boldsymbol{\Pi}}(\boldsymbol{\nu}) = (\tilde{\pi}_1(\nu), \tilde{\pi}_2(\nu))^\top$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^\top$, and

$$\begin{aligned}\boldsymbol{\Sigma} &= \boldsymbol{\sigma}\boldsymbol{\sigma}^\top \\ &= \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}^\top \\ &= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad 0 < \rho < 1. \end{aligned} \tag{4.26}$$

The initial approximate HJB equation thus becomes

$$\begin{aligned} 0 \approx &\left(-\beta\nu_1 - \frac{\nu_1^2}{\kappa} + (1-\kappa) + 2\nu_1 r - \boldsymbol{\theta}^\top\boldsymbol{\theta}\nu_1\right) F^2 \\ &+ \left(-\beta\nu_2 - \frac{\nu_3^2}{4\kappa} + (1-\kappa) - \nu_3\delta - \boldsymbol{\theta}^\top\boldsymbol{\theta}\frac{\nu_3^2}{4\nu_1}\right) AL^2 \\ &+ \left(-\beta\nu_3 - \frac{\nu_1\nu_3}{\kappa} - 2(1-\kappa) - 2\nu_1\delta + \nu_3 r - \boldsymbol{\theta}^\top\boldsymbol{\theta}\nu_3\right) F \cdot AL, \end{aligned} \tag{4.27}$$

where

$$\boldsymbol{\theta} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - \begin{pmatrix} r \\ r \end{pmatrix}\right]. \tag{4.28}$$

Therefore, the error between the approximate and true HJB equations for any given value of F_i is

$$M_i = \left(-\beta\nu_1 - \frac{\nu_1^2}{\kappa} + (1-\kappa) + 2\nu_1 r - \boldsymbol{\theta}^\top\boldsymbol{\theta}\nu_1\right)F_i^2$$
$$+ \left(-\beta\nu_2 - \frac{\nu_3^2}{4\kappa} + (1-\kappa) - \nu_3\delta - \boldsymbol{\theta}^\top\boldsymbol{\theta}\frac{\nu_3^2}{4\nu_1}\right)AL^2$$
$$+ \left(-\beta\nu_3 - \frac{\nu_1\nu_3}{\kappa} - 2(1-\kappa) - 2\nu_1\delta + \nu_3 r - \boldsymbol{\theta}^\top\boldsymbol{\theta}\nu_3\right)F_i \cdot AL. \tag{4.29}$$

Then, we solve the minimization problem of Equation (3.6) to find $\boldsymbol{\nu}$.

We use the grid given in Section 4.1.1. The risk-free rate is again $r = 0.05$. In addition to the single risky asset used in the model of Section 4.1.1, we add a new risky asset with a higher return of $\mu_2 = 0.15$ and a higher volatility of $\sigma_2 = 0.25$. Thus, we have $\boldsymbol{\mu} = (0.1, 0.15)^\top$ and

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.15 & 0 \\ 0.25\rho & 0.25\sqrt{1-\rho^2} \end{pmatrix}, \tag{4.30}$$

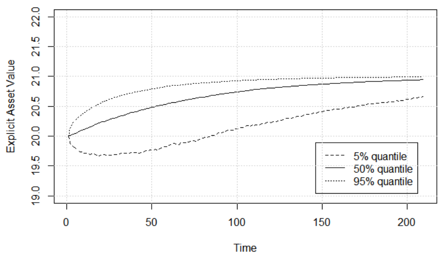
where the correlation coefficient is set to $\rho = 0.85$. The values of δ, β, and κ used the same as those in Section 4.1.1. Since we only add one additional risky asset, we expect the parameter values of the value function to be close to those of the single risky asset case. We therefore use the initial starting point, $\boldsymbol{\nu}_0 = (0.4, 0.4, -0.5)$, and the lower and upper bounds same as those in Section 4.1.1. The approximate and true parameter values are presented in Table 4.2.

	Approximate			True		
Parameters	ν_1	ν_2	ν_3	v_1	v_2	v_3
Values	0.358448	0.358337	−0.716678	0.358356	0.358356	−0.716711

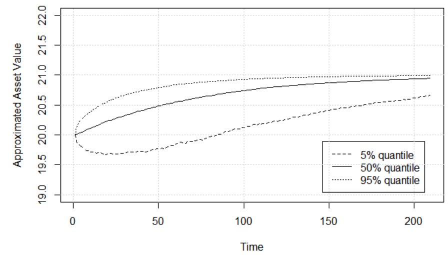
Table 4.2: **Optimal Value Function Parameters of Optimal Pension Funding Model Based on Value Function Approximation Method with Two Risky Assets.**

Similar to the single risky asset case, we obtain approximate parameter values that are very close to the true values. The relative differences are all less than 0.01%. We then again generate 5,000 scenarios to simulate possible situations. By doing that, we can compare the approximate and true distributions of the optimal controls and the state variables. The time horizon is assumed to be $T = 4$ years. We again use weekly time steps of $h = 1/52$. Values for $F(0), AL$, and P are those introduced in Section 4.1.1. The function used to calculate the approximate and true optimal controls now follow Equations (4.4) and (4.25), where the approximate controls use estimated values of ν_1, ν_2, and ν_3, and the true ones are given

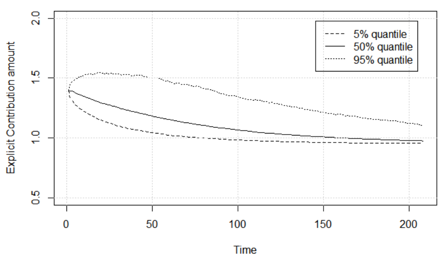
Figure 4.4: **Distributions of Approximate Solution Based on the Value Function Approximation Method and True Solution of Optimal Pension Funding Model with Two Risky Assets.**



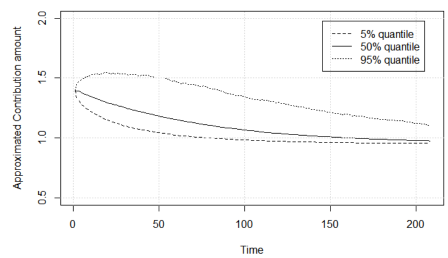
(a) Distribution of True Fund Value



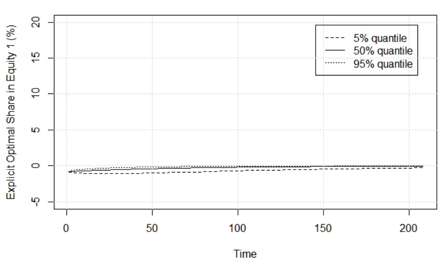
(b) Distribution of Approximate Fund Value



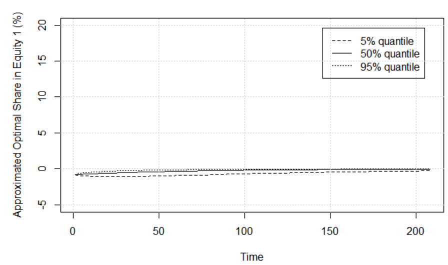
(c) Distribution of True Optimal Contribution



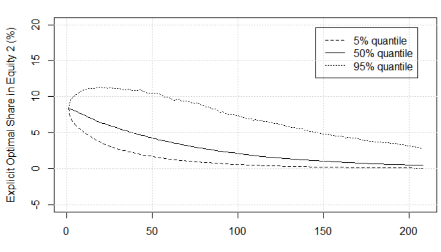
(d) Distribution of Approximate Optimal Contribution



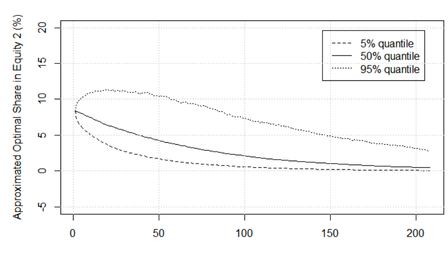
(e) Distribution of True Optimal Share in Risky Asset 1



(f) Distribution of Approximate Optimal Share in Risky Asset 1



(g) Distribution of True Optimal Share in Risky Asset 2



(h) Distribution of Approximate Optimal Share in Risky Asset 2 in Table 4.2. The evolution of approximate fund value follows

$$\tilde{F}(t+h) = \left(\tilde{F}(t) + \tilde{C}(t)h\right)\left[R^0_{t+h} + \tilde{\mathbf{\Pi}}^\top(t)\left(\mathbf{R}_{t+h} - R^0_{t+h}\mathbf{1}\right)\right] - P \cdot h, \tag{4.31}$$

and the true fund value follows

$$\hat{F}(t+h) = \left(\hat{F}(t) + C(t)h\right)\left[R^0_{t+h} + \mathbf{\Pi}^\top(t)\left(\mathbf{R}_{t+h} - R^0_{t+h}\mathbf{1}\right)\right] - P \cdot h,$$

where $\mathbf{R}_{t+h} = \left(R^1_{t+h}, R^2_{t+h}\right)^\top$ is the column vector of the risky asset returns and

$$R^i_{t+h} = e^{\left(\mu_i - \frac{\sum_{j=1}^2 \sigma_{ij}^2}{2}\right)h + \sqrt{h}\sum_{j=1}^2 \sigma_{ij} Z_j}, \tag{4.32}$$

where Z_1 and Z_2 are two independent standard normal random variables. Figure 4.4 shows the distributions of the optimal controls and fund value. It should not come as a surprise that the patterns of the distributions in Figure 4.4 are all similar to those of Figure 4.1. Furthermore, the differences between the distributions of the approximate and true solutions are very small, meaning that the approximation works.

4.2.2 Finite Time Horizon

The model for the finite time case is the one presented in Section 2.3.2. Since we only add assets to the portfolio, the value function has the same functional form as the one used for single risky asset case. The initial value function therefore follows Equation (4.9) and can be written in a recursive form:

$$\begin{aligned} &V(F,t) \\ &= \min_{C(t),\mathbf{\Pi}(t)} \mathbb{E}_t\left[\int_t^{t+h} e^{-\beta s}\left(\kappa(C(s)-NC)^2 + (1-\kappa)(AL - F(s))^2\right)ds + V(F,t+h)\right]. \end{aligned} \tag{4.33}$$

Thus, the discrete recursive form we used for the algorithm is

$$\begin{aligned} &\tilde{V}(\tilde{F},t) \\ &= \min_{\tilde{C}(t),\tilde{\mathbf{\Pi}}(t)} \mathbb{E}_t\left[e^{-\beta t}\left(\kappa(\tilde{C}(t)-NC)^2 + (1-\kappa)(AL - \tilde{F}(t))^2\right)h + \tilde{V}(\tilde{F},t+h)\right], \end{aligned} \tag{4.34}$$

where $\tilde{\mathbf{\Pi}}(t) = (\tilde{\pi}_1(t), \tilde{\pi}_2(t))^\top$ is the column vector of discrete shares in the two risky assets, and $\tilde{F}$ follows Equation (4.31). Since we now only have two risky assets, we can expand Equation (4.31)

$$\tilde{F}(t+h) = \left(\tilde{F}(t) + \tilde{C}(t)h\right)\left[R^0_{t+h} + \tilde{\pi}_1(t)\left(R^1_{t+h} - R^0_{t+h}\right) + \tilde{\pi}_2(t)\left(R^2_{t+h} - R^0_{t+h}\right)\right] - P \cdot h,$$

where R^1_{t+h} and R^2_{t+h} follow Equation (4.32). The terminal condition is again

$$\tilde{V}(\tilde{F}, T) = \alpha e^{-\beta T}(\tilde{F}(T) - AL)^2.$$

Similar to the single risky asset case, we first need to find the unique optimal controls at $t = T - h$. The recursive form of the value function at $t = T - h, \tilde{V}(\tilde{F}, T - h)$, follows Equation (4.34). We first solve the FOCs for $\tilde{\pi}_1(T-h)$ and $\tilde{\pi}_2(T-h)$, respectively. The FOC of $\tilde{\pi}_1(T-h)$ is

$$\begin{aligned}
&\tilde{V}_{\tilde{\pi}_1}(T-h) \\
&= \mathbb{E}_{T-h}\left[2\alpha e^{-\beta T}(\tilde{F}(T) - AL)\frac{\partial \tilde{F}(T)}{\partial \tilde{\pi}_1(T-h)}\right] \\
&= \mathbb{E}_{T-h}\left[2\alpha e^{-\beta T}(\tilde{F}(T) - AL)\left(\tilde{F}(T-h) + \tilde{C}(T-h)h\right)\left(R^1_T - R^0_T\right)\right] \\
&= 2\alpha e^{-\beta T}\mathbb{E}_{T-h}\left[\left(a\left[R^0_T + \tilde{\mathbf{\Pi}}^\top(T-h)\left(\mathbf{R}_T - R^0_T\mathbf{1}\right)\right] - P\cdot h - AL\right)a\left(R^1_T - R^0_T\right)\right] \\
&= \mathbb{E}_{T-h}\left[\left(a\left[R^0_T + \tilde{\mathbf{\Pi}}^\top(T-h)\left(\mathbf{R}_T - R^0_T\mathbf{1}\right)\right] - P\cdot h - AL\right)\left(R^1_T - R^0_T\right)\right] \\
&= 0.
\end{aligned} \tag{4.35}$$

Similarly, the FOC of $\tilde{\pi}_2(T-h)$ is

$$0 = \mathbb{E}_{T-h}\left[\left(a\left[R^0_T + \tilde{\mathbf{\Pi}}^\top(T-h)\left(\mathbf{R}_T - R^0_T\mathbf{1}\right)\right] - P\cdot h - AL\right)\left(R^2_T - R^0_T\right)\right]. \tag{4.36}$$

Equations (4.35) and (4.36) can be written in the form of

$$\begin{aligned}
&\left(\frac{P\cdot h + AL}{a} - R^0_T\right)\mathbb{E}_{T-h}\left[R^1_T - R^0_T\right] \\
&= \tilde{\pi}_1(T-h)\mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)^2\right] + \tilde{\pi}_2(T-h)\mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)\left(R^2_T - R^0_T\right)\right],
\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}
&\left(\frac{P\cdot h + AL}{a} - R^0_T\right)\mathbb{E}_{T-h}\left[R^2_T - R^0_T\right] \\
&= \tilde{\pi}_2(T-h)\mathbb{E}_{T-h}\left[\left(R^2_T - R^0_T\right)^2\right] + \tilde{\pi}_1(T-h)\mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)\left(R^2_T - R^0_T\right)\right].
\end{aligned} \tag{4.38}$$

We can combine the above two equations into one equation using a matrix representation:

$$\begin{aligned}
&\left(\frac{P\cdot h + AL}{a} - R^0_T\right)\begin{pmatrix}\mathbb{E}_{T-h}\left[R^1_T - R^0_T\right] \\ \mathbb{E}_{T-h}\left[R^2_T - R^0_T\right]\end{pmatrix} \\
&= \begin{pmatrix}\mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)^2\right] & \mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)\left(R^2_T - R^0_T\right)\right] \\ \mathbb{E}_{T-h}\left[\left(R^1_T - R^0_T\right)\left(R^2_T - R^0_T\right)\right] & \mathbb{E}_{T-h}\left[\left(R^2_T - R^0_T\right)^2\right]\end{pmatrix}\tilde{\mathbf{\Pi}}(T-h).
\end{aligned} \tag{4.39}$$

Let

$$\mathbf{D}(T) = \begin{pmatrix} \mathbb{E}_{T-h}\left[\left(R_T^1 - R_T^0\right)^2\right] & \mathbb{E}_{T-h}\left[\left(R_T^1 - R_T^0\right)\left(R_T^2 - R_T^0\right)\right] \\ \mathbb{E}_{T-h}\left[\left(R_T^1 - R_T^0\right)\left(R_T^2 - R_T^0\right)\right] & \mathbb{E}_{T-h}\left[\left(R_T^2 - R_T^0\right)^2\right] \end{pmatrix}.$$

Then, the optimal value of $\tilde{\mathbf{\Pi}}(T-h)$ given a can be solved by using the following equation:

$$\tilde{\mathbf{\Pi}}^*(T-h)[a] = \left(\frac{P \cdot h + AL}{a} - R_T^0\right)\mathbf{D}(T)^{-1}\begin{pmatrix} \mathbb{E}_{T-h}\left[R_T^1 - R_T^0\right] \\ \mathbb{E}_{T-h}\left[R_T^2 - R_T^0\right] \end{pmatrix}. \tag{4.40}$$

Next, we solve the FOC for the contribution amount, $\tilde{C}(T-h)$. The FOC of $\tilde{C}(T-h)$ follows Equation (4.15) and can be solved directly as well:

$$\begin{aligned} &\tilde{C}^*(T-h)[a] \\ =&NC - \frac{\alpha}{\kappa}e^{-\beta h}\mathbb{E}_{T-h}\left[\left(a\left[R_T^0 + \tilde{\mathbf{\Pi}}^{*\top}(T-h)[a]\left(\mathbf{R}_T - R_T^0\mathbf{1}\right)\right] - P \cdot h - AL\right)\right. \\ &\left.\times\left(R_T^0 + \tilde{\mathbf{\Pi}}^{*\top}(T-h)[a]\left(\mathbf{R}_T - R_T^0\mathbf{1}\right)\right)\right]. \end{aligned} \tag{4.41}$$

Starting from $t = T-2h$, we have general solutions for the optimal controls. Using the idea employed to derive Equation (4.40), we first find the FOCs of the contribution and the two shares separately:

$$\tilde{V}_{\tilde{\pi}_1}(t) = \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)(\tilde{F}(t) + \tilde{C}(t)h)\left(R_{t+h}^1 - R_{t+h}^0\right)\right] = 0, \tag{4.42}$$

$$\tilde{V}_{\tilde{\pi}_2}(t) = \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)(\tilde{F}(t) + \tilde{C}(t)h)\left(R_{t+h}^2 - R_{t+h}^0\right)\right] = 0, \tag{4.43}$$

$$\tilde{V}_{\tilde{C}}(t) = \mathbb{E}_t\left[e^{-\beta t}2h\kappa\left(\tilde{C}(t) - NC\right)\right] + \mathbb{E}_t\left[\tilde{V}_{\tilde{F}}(t+h)\frac{\partial\tilde{F}(t+h)}{\partial\tilde{C}(t)}\right] = 0. \tag{4.44}$$

Applying Equation (4.22) to Equations (4.42) and (4.43), we then obtain

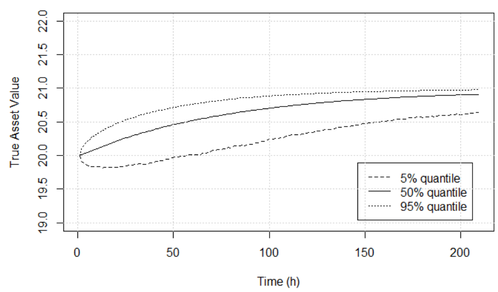
$$\begin{aligned} 0 =& \mathbb{E}_t\left[\left(R_{t+h}^1 - R_{t+h}^0\right)\left(h(1-\kappa)\left(AL - \tilde{F}(t+h)\right) + \kappa\left(\tilde{C}(t+h) - NC\right)\right)\right], \\ 0 =& \mathbb{E}_t\left[\left(R_{t+h}^2 - R_{t+h}^0\right)\left(h(1-\kappa)\left(AL - \tilde{F}(t+h)\right) + \kappa\left(\tilde{C}(t+h) - NC\right)\right)\right]. \end{aligned}$$

These two equations can be combined:

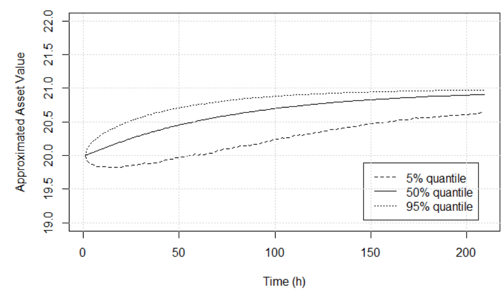
$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \mathbb{E}\left[\begin{pmatrix} R_{t+h}^1 - R_{t+h}^0 \\ R_{t+h}^2 - R_{t+h}^0 \end{pmatrix}\left(h(1-\kappa)\left(AL - \tilde{F}(t+h)\right) + \kappa\left(\tilde{C}(t+h) - NC\right)\right)\right] \\ &= \mathbb{E}\left[\mathbf{d}(t+h)\left(h(1-\kappa)\left(AL - \left(a\left(R_{t+h}^0 + \tilde{\mathbf{\Pi}}^\top(t)\mathbf{d}(t+h)\right) - P \cdot h\right)\right)\right.\right. \\ &\qquad\left.\left. + \kappa\left(\tilde{C}(t+h)[a] - NC\right)\right)\right], \end{aligned} \tag{4.45}$$

where

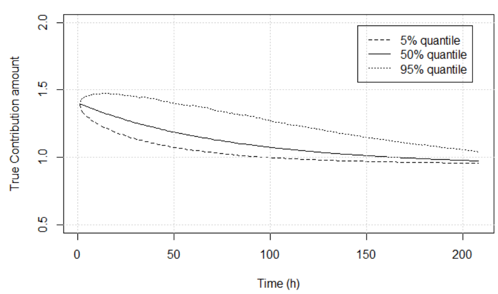
Figure 4.5: **Distributions of Approximate Solution Based on the Grid Value Function Approximation Method and True Solution of the Optimal Pension Funding Model with Two Risky Assets.**



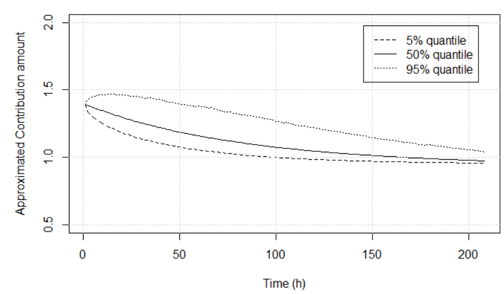
(a) Distribution of True Fund Value



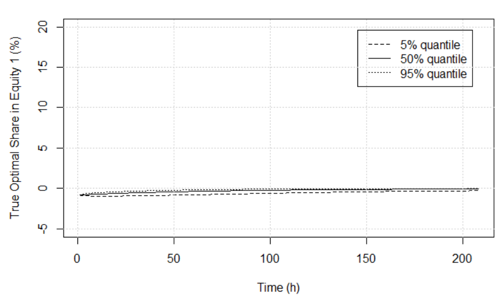
(b) Distribution of Approximate Fund Value



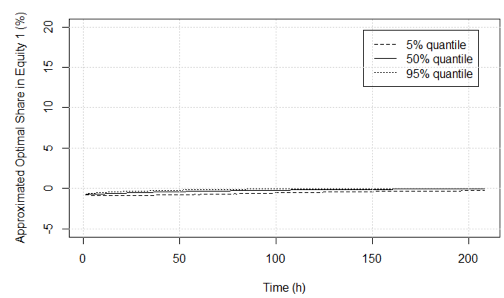
(c) Distribution of True Optimal Contribution



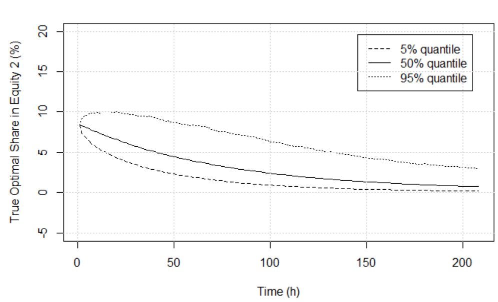
(d) Distribution of Approximate Optimal Contribution



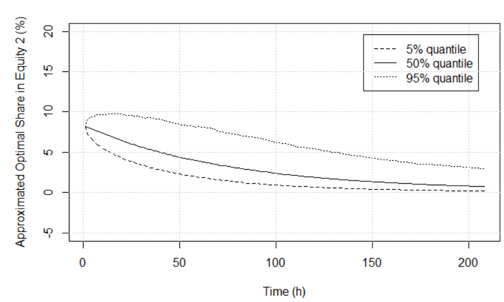
(e) Distribution of True Optimal Share in Risky Asset 1



(f) Distribution of Approximate Optimal Share in Risky Asset 1

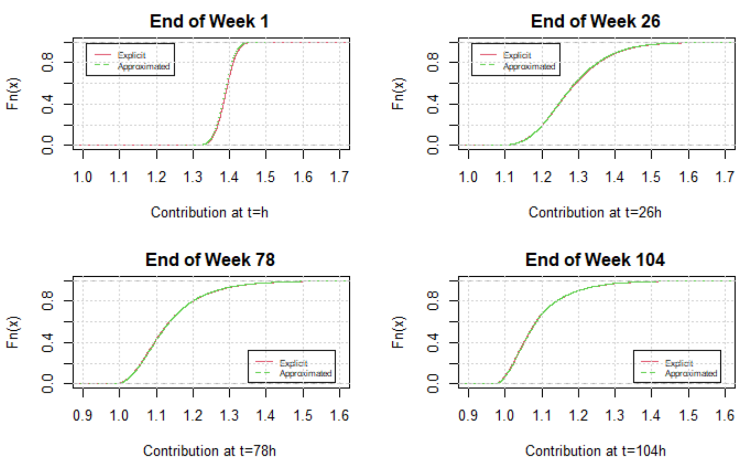


(g) Distribution of True Optimal Share in Risky Asset 2

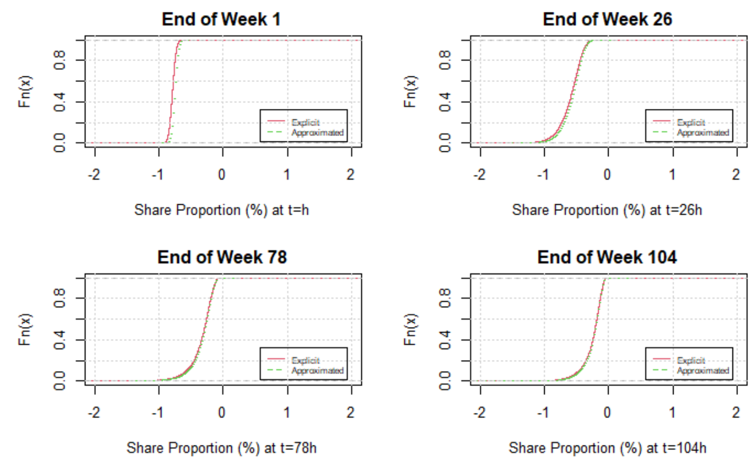


(h) Distribution of Approximate Optimal Share in Risky Asset 2

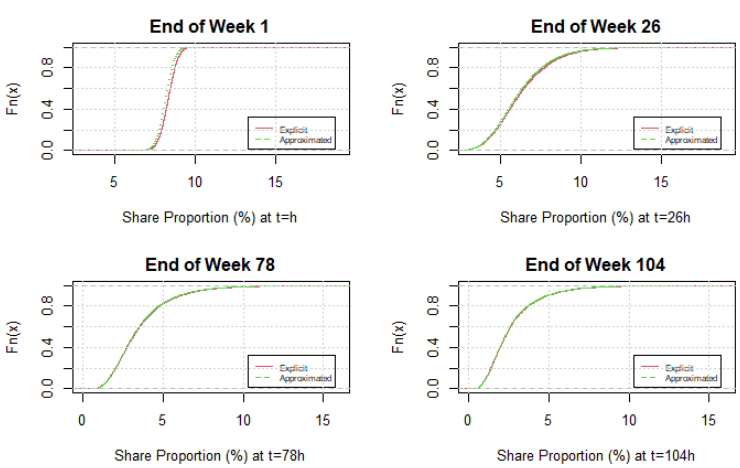
Figure 4.6: **Comparison Between Empirical Cumulative Distribution Functions of True and Approximate Optimal Controls and State Variables Based on the Grid Vale Approximation Method with Two Risky Assets for Different Horizons.**



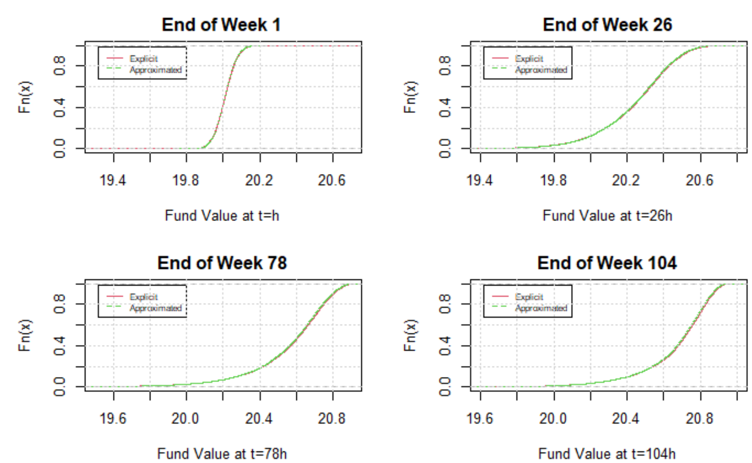
(a) Comparison Between True and Approximate Contribution



(b) Comparison Between True and Approximate Share in Risky Asset 1



(c) Comparison Between True and Approximate Share in Risky Asset 2



(d) Comparison Between True and Approximate Fund Value

$$\mathbf{d}(t+h) = \begin{pmatrix} R^1_{t+h} - R^0_{t+h} \\ R^2_{t+h} - R^0_{t+h} \end{pmatrix}.$$

The optimal shares in the two risky assets, $\tilde{\mathbf{\Pi}}^*(t)$ can therefore be found using Equation (4.45). Then, we apply Equation (4.22) to Equation (4.44) to find the solution of $\tilde{C}^*(t)$:

$$\begin{aligned} \tilde{C}^*(t)[a] =& NC + \frac{e^{-\beta h}}{\kappa} \times \mathbb{E}_t \Big[\Big(R^0_{t+h} + \tilde{\mathbf{\Pi}}^{*\top}(t)[a]\mathbf{d}(t+h) \Big) \\ & \times \Big[h(1-\kappa) \Big(AL + P \cdot h - a \Big(R^0_{t+h} + \tilde{\mathbf{\Pi}}^{*\top}(t)[a]\mathbf{d}(t+h) \Big) \Big) \\ & + \kappa \Big(\tilde{C}^*(t+h)[a] - NC \Big) \Big] \Big]. \end{aligned} \tag{4.46}$$

We can then use Equations (4.45) and (4.46) to find the approximate numerical results and compare them with the true solutions. The initial fund value $F(0)$, the actuarial liability AL, and the choice of grid values are similar to those given in Section 4.1.2. Using the parameter values discussed in Section 4.2.1, Figure 4.5 shows the 5^{th}, 50^{th}, and 95^{th} quantiles of both true and approximate solutions.

Overall, the different percentiles in each panel tend to reach the target level. The distributions of the true and approximate solutions are almost identical. Figure 4.6 reports the ECDF plots that examine the performance of the algorithm at different times. The algorithm performs poorly for the assets' shares at the beginning, but improves as time passes by. When we reach the end of Year 2, the ECDFs are almost the same. For the contribution amount, the approximation works well for any time t. As a consequence, the true and approximate fund values are very close all the times. We can thus conclude that the grid value approximation also works well for the multiple risky assets case, generally speaking.

Chapter 5

Robustness Checks

In this chapter, we perform robustness checks on the two algorithms. We apply a series of tests to assess whether the two algorithms are robust to other parameter choices. Due to the long computing time required to run the multiple-asset cases, we only focus on robustness tests for single-asset cases.

5.1 Parameter Sets

Other than the base case that has been discussed in the previous chapters, 30 additional parameter sets are tested. We divide these sets into several different groups.

5.1.1 Weight of the Contribution Risk

We first assess the impact of the importance of the contribution risk on the performance of the algorithms. In the base case, we use a value of $\kappa = 0.8$, which puts a large weight on the contribution risk. We therefore add two tests where the importance of contribution risk is moderate at $\kappa = 0.5$, and low at $\kappa = 0.2$.

5.1.2 Market Price of Risk

By investigating the true optimal solutions, we find that the share in the risky asset is directly impacted by the market price of risk (i.e., Sharpe ratio). Therefore, instead of testing the risk-free return r, the risky return μ, and volatility σ separately, we assess robustness tests based on different market prices of risks. Other than the market price of risks of 0.3333 in the base case, we test two sets: market price of risks of 0.2 and 0.5.

5.1.3 Time Preference Factor, Valuation Rate, and Risk-Free Return

The third group of scenarios we consider is related to the time preference factor β, the valuation rate δ, and the risk-free return r. Because the relationship between the values of δ and r affects the target level of the contribution amount, we test the robustness of the two algorithms under different target levels by changing the values of these two parameters.

Moreover, δ and β are two parameters that control the discount level and both affect the optimal solutions. Therefore, it is better to test these two parameters together. Thus, we run robustness tests for different combinations of these three parameters. That is, for each specific relationship between the values of δ and r ($\delta > r, \delta = r$, or $\delta < r$), we test the performance of two algorithms with different value combinations of δ and β.

5.1.4 Combined Effects

The above group is tested given a specific market price of risk. Because the value of r has an impact on the market price of risk, we need to combine the above two types of groups together as well. Hence, we apply the third group of sets introduced in Section 5.1.3 to each market price of risks mentioned in Section 5.1.2.

5.2 Results

This section presents the performance of the two algorithms for the 30 different parameter sets mentioned above. We use the root-mean-square error (RMSE) and the normalized root-mean-square error (NRMSE) to examine the performance. The steps for calculating RMSE and NRMSE are as follows:

1. Calculate the squared errors computed from the true and approximate solutions from each time point along in each of the $N = 5{,}000$ simulated paths.

2. Sum all squared errors to find the total square error.

3. Compute the mean of the total square error.

4. Compute the square root of the total mean square error to obtain the RMSE.

5. For each time point, calculate the mean of the N true solutions.

6. Find the time-t variance of the N true solutions using the mean of N paths, $t \in [0, T-h]$ for C^* and π^*, and $t \in [h, T]$ for F.

7. Compute the mean of all the variances to find the average variance and its square root to find the standard deviation.

8. Divide the RMSE by this standard deviation to find the NRMSE.

We use the parameter values introduced in Chapter 4 as the reference, which is Set 0 in Tables 5.1 and 5.2. Sets 1 and 2 belong to the first group which tests the weight of the contribution risk. Starting from Set 3, the value of κ is back to 0.8. Sets 3 and 4 are related to the second group that examines the Sharpe ratio. The next eight sets are part of the third group, which uses a Sharpe ratio of 0.3333. The fourth group assesses combined effects and includes the remaining 18 sets, with Sets 13 to 21 using a Sharpe ratio of 0.2 and the last nine sets a Sharpe ratio of 0.5. Table 5.1 shows the performance of value function approximation, and Table 5.2 gives the results using grid value approximation.

From Table 5.1, we find that the RMSEs for all sets are very small. Even though the NRMSE values are higher than the RMSE values, all of them are still below 0.1%. We therefore conclude that value function approximation has a stable performance for different parameter values. The RMSE values from Table 5.2 are much higher than those in Table 5.1, which is not a surprise. The approximation errors in value function approximation mainly come from the approximation errors in the value function parameters, which lead to approximation errors in the optimal controls and fund values.

The approximation errors in grid value approximation come from multiple sources. First, there are discretization errors when converting the continuous-time problem to a discrete-time problem. Second, during the backward stage, the root finding technique produces approximation errors for each optimal share. Since the contribution amount depends on the optimal share, the approximation errors in the optimal share also cause errors in the contribution amount, which in turn lead to errors in the fund value. These approximation errors also accumulate. Finally, in the forward stage, the linear interpolation creates errors as well.

Although the values of the RMSEs for grid value approximation are higher, most of the RMSEs are still below 0.4% for C^* and π^* and below 0.8% for F. Overall, grid value approximation performs quite well. There are three sets that yield quite high RMSEs for both C^* and π^*: Sets 13 to 15. By using the parameter values from these three sets and a time horizon of 4 years, the range of the values of $F^*(t) + C^*(t)$ is quite wide, which goes beyond the range of the set of grid values we chose. A higher valuation rate δ than the risk-free return r seems to cause this. As a consequence of the narrow grid values, many of the approximation optimal controls are computed based on linear interpolation (or extrapolation), which increases the approximation errors. If we increase the range of the grid values, the RMSE values decrease. The value of the time preference factor has a small impact on the accuracy of the approximation. From Equations (4.23) and (4.24), we find that as κ decreases, additional weight is put on the part that may cause approximation errors, which leads to higher RMSEs in Sets 1 and 2. This can be improved by choosing a smaller time interval h.

If we focus on the NRMSE, many sets have values larger than 4%. Because the RMSE values for most of them are less than 4%, the high NRMSE values are caused by small standard deviations. These high values can be lowered by further decreasing the RMSE values, which can be done by choosing a smaller time step h. The remaining several large NRMSE values are due to high RMSE values, which can be improved by choosing a wider range of grid values, as discussed above.

Set	Parameter Values						RMSE			NRMSE		
	r	μ	σ	δ	κ	β	C^*	π^*	F^*	C^*	π^*	F^*
0	0.05	0.1	0.15	0.05	0.8	0.05	0.001%	0.001%	0.005%	0.018%	0.033%	0.031%
1	0.05	0.1	0.15	0.05	**0.5**	0.05	0.001%	0.000%	0.004%	0.016%	0.027%	0.046%
2	0.05	0.1	0.15	0.05	**0.2**	0.05	0.004%	0.000%	0.009%	0.047%	0.064%	0.200%
3	**0.02**	**0.08**	**0.12**	**0.02**	0.8	**0.02**	0.001%	0.001%	0.007%	0.014%	0.027%	0.029%
4	**0.04**	**0.09**	**0.25**	**0.04**	0.8	**0.04**	0.001%	0.000%	0.004%	0.028%	0.048%	0.040%
5	0.05	0.1	0.15	**0.06**	0.8	**0.08**	0.001%	0.001%	0.006%	0.011%	0.027%	0.021%
6	0.05	0.1	0.15	**0.07**	0.8	**0.06**	0.001%	0.001%	0.005%	0.005%	0.019%	0.012%
7	0.05	0.1	0.15	**0.08**	0.8	**0.08**	0.001%	0.001%	0.006%	0.005%	0.021%	0.013%
8	0.05	0.1	0.15	**0.03**	0.8	**0.04**	0.000%	0.000%	0.001%	0.007%	0.006%	0.010%
9	0.05	0.1	0.15	**0.04**	0.8	**0.02**	0.002%	0.000%	0.005%	0.042%	0.052%	0.066%
10	0.05	0.1	0.15	**0.01**	0.8	**0.01**	0.002%	0.000%	0.005%	0.013%	0.008%	0.017%
11	0.05	0.1	0.15	**0.05**	0.8	**0.07**	0.001%	0.001%	0.005%	0.018%	0.033%	0.030%
12	0.05	0.1	0.15	**0.05**	0.8	**0.03**	0.001%	0.001%	0.005%	0.017%	0.031%	0.030%

Table 5.1: **Root-Mean-Square Error and Normalized Root-Mean-Square Error between True and Approximate Solutions for Different Groups using Value Function Approximation.**

Set	Parameter Values						RMSE			NRMSE		
	r	μ	σ	δ	κ	β	C^*	π^*	F^*	C^*	π^*	F^*
13	**0.02**	**0.08**	**0.12**	**0.06**	0.8	**0.08**	0.001%	0.002%	0.006%	0.002%	0.011%	0.008%
14	**0.02**	**0.08**	**0.12**	**0.07**	0.8	**0.06**	0.001%	0.002%	0.008%	0.003%	0.017%	0.011%
15	**0.02**	**0.08**	**0.12**	**0.08**	0.8	**0.08**	0.002%	0.008%	0.011%	0.004%	0.025%	0.010%
16	**0.05**	**0.15**	**0.2**	**0.03**	0.8	**0.04**	0.002%	0.000%	0.006%	0.029%	0.027%	0.042%
17	**0.05**	**0.15**	**0.2**	**0.04**	0.8	**0.02**	0.001%	0.001%	0.006%	0.027%	0.037%	0.048%
18	**0.05**	**0.15**	**0.2**	**0.01**	0.8	**0.01**	0.002%	0.000%	0.006%	0.013%	0.007%	0.015%
19	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.07**	0.001%	0.001%	0.007%	0.014%	0.029%	0.030%
20	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.01**	0.001%	0.001%	0.006%	0.011%	0.023%	0.025%
21	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.04**	0.001%	0.001%	0.007%	0.014%	0.027%	0.029%
22	**0.04**	**0.08**	**0.2**	**0.06**	0.8	**0.08**	0.002%	0.001%	0.007%	0.019%	0.041%	0.028%
23	**0.04**	**0.08**	**0.2**	**0.07**	0.8	**0.06**	0.001%	0.001%	0.005%	0.009%	0.030%	0.015%
24	**0.04**	**0.08**	**0.2**	**0.08**	0.8	**0.08**	0.002%	0.001%	0.006%	0.009%	0.032%	0.015%
25	**0.05**	**0.1**	**0.25**	**0.03**	0.8	**0.06**	0.001%	0.000%	0.004%	0.044%	0.037%	0.060%
26	**0.05**	**0.1**	**0.25**	**0.04**	0.8	**0.02**	0.001%	0.000%	0.004%	0.057%	0.070%	0.083%
27	**0.05**	**0.1**	**0.25**	**0.01**	0.8	**0.01**	0.000%	0.000%	0.000%	0.002%	0.003%	0.003%
28	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.07**	0.001%	0.000%	0.004%	0.025%	0.046%	0.036%
29	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.01**	0.002%	0.000%	0.007%	0.047%	0.050%	0.068%
30	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.04**	0.001%	0.000%	0.004%	0.028%	0.048%	0.043%

Table 5.1: **(Continued) Root-Mean-Square Error and Normalized Root-Mean-Square Error between True and Approximate Solutions for Different Groups using Value Function Approximation.**

Set	**Parameter Values**						**RMSE**			**NRMSE**		
	r	μ	σ	δ	κ	β	C^*	π^*	F^*	C^*	π^*	F^*
0	0.05	0.1	0.15	0.05	0.8	0.05	0.162%	0.081%	0.406%	2.283%	4.372%	2.447%
1	0.05	0.1	0.15	0.05	**0.5**	0.05	0.338%	0.086%	0.401%	4.168%	9.235%	4.760%
2	0.05	0.1	0.15	0.05	**0.2**	0.05	0.808%	0.103%	0.430%	9.615%	21.735%	10.076%
3	**0.02**	**0.08**	**0.12**	**0.02**	0.8	**0.02**	0.166%	0.180%	0.451%	1.999%	3.847%	2.039%
4	**0.04**	**0.09**	**0.25**	**0.04**	0.8	**0.04**	0.141%	0.027%	0.332%	3.054%	6.459%	3.226%
5	0.05	0.1	0.15	**0.06**	0.8	**0.08**	0.192%	0.107%	0.492%	1.812%	3.596%	1.896%
6	0.05	0.1	0.15	**0.07**	0.8	**0.06**	0.236%	0.140%	0.584%	1.656%	3.458%	1.680%
7	0.05	0.1	0.15	**0.08**	0.8	**0.08**	0.283%	0.201%	0.686%	1.586%	3.760%	1.540%
8	0.05	0.1	0.15	**0.03**	0.8	**0.04**	0.150%	0.048%	0.259%	5.474%	6.282%	3.532%
9	0.05	0.1	0.15	**0.04**	0.8	**0.02**	0.146%	0.059%	0.325%	3.787%	6.283%	3.785%
10	0.05	0.1	0.15	**0.01**	0.8	**0.01**	0.214%	0.069%	0.226%	2.417%	3.085%	1.002%
11	0.05	0.1	0.15	**0.05**	0.8	**0.07**	0.160%	0.081%	0.406%	2.264%	4.288%	2.411%
12	0.05	0.1	0.15	**0.05**	0.8	**0.03**	0.164%	0.081%	0.406%	2.304%	4.458%	2.485%

Table 5.2: **Root-Mean-Square Error and Normalized Root-Mean-Square Error between True and Approximate Solutions for Different Groups using Grid Value Approximation.**

Set	Parameter Values						RMSE			NRMSE		
	r	μ	σ	δ	κ	β	C^*	π^*	F^*	C^*	π^*	F^*
13	**0.02**	**0.08**	**0.12**	**0.06**	0.8	**0.08**	0.587%	2.124%	1.781%	2.377%	12.261%	2.492%
14	**0.02**	**0.08**	**0.12**	**0.07**	0.8	**0.06**	0.849%	3.480%	2.581%	2.913%	16.576%	3.099%
15	**0.02**	**0.08**	**0.12**	**0.08**	0.8	**0.08**	1.226%	5.994%	3.817%	3.695%	23.322%	3.955%
16	**0.05**	**0.15**	**0.2**	**0.03**	0.8	**0.04**	0.157%	0.073%	0.345%	4.131%	5.752%	3.167%
17	**0.05**	**0.15**	**0.2**	**0.04**	0.8	**0.02**	0.159%	0.088%	0.425%	3.160%	5.610%	3.372%
18	**0.05**	**0.15**	**0.2**	**0.01**	0.8	**0.01**	0.251%	0.111%	0.397%	2.322%	3.321%	1.310%
19	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.07**	0.178%	0.145%	0.509%	2.089%	3.977%	2.217%
20	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.01**	0.184%	0.146%	0.507%	2.128%	4.131%	2.287%
21	**0.04**	**0.12**	**0.16**	**0.04**	0.8	**0.04**	0.181%	0.146%	0.508%	2.107%	4.052%	2.252%
22	**0.04**	**0.08**	**0.2**	**0.06**	0.8	**0.08**	0.195%	0.052%	0.472%	2.082%	4.482%	2.112%
23	**0.04**	**0.08**	**0.2**	**0.07**	0.8	**0.06**	0.236%	0.064%	0.549%	1.991%	4.302%	1.960%
24	**0.04**	**0.08**	**0.2**	**0.08**	0.8	**0.08**	0.278%	0.075%	0.623%	1.955%	4.020%	1.810%
25	**0.05**	**0.1**	**0.25**	**0.03**	0.8	**0.06**	0.148%	0.016%	0.228%	8.611%	9.289%	5.043%
26	**0.05**	**0.1**	**0.25**	**0.04**	0.8	**0.02**	0.141%	0.020%	0.285%	5.674%	9.559%	5.498%
27	**0.05**	**0.1**	**0.25**	**0.01**	0.8	**0.01**	0.202%	0.024%	0.178%	3.426%	4.571%	1.237%
28	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.07**	0.136%	0.059%	0.325%	2.964%	5.615%	3.086%
29	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.01**	0.141%	0.060%	0.327%	3.053%	6.023%	3.251%
30	**0.04**	**0.06**	**0.1**	**0.04**	0.8	**0.04**	0.138%	0.060%	0.325%	3.006%	5.815%	3.166%

Table 5.2: **(Continued) Root-Mean-Square Error and Normalized Root-Mean-Square Error between True and Approximate Solutions for Different Groups using Grid Value Approximation.**

Chapter 6

Conclusion

In the pension context, sponsor or employee's optimal strategies could be determined by solving dynamic programming problems. These optimal decisions may be difficult to obtain in closed-form if the value function cannot be found or the HJB equation is hard to solve. Therefore, numerical algorithms can be used to approximately solve these problems.

In this report, we presented two different numerical algorithms, the value function approximation and the grid value approximation. We then applied them to retirement related-problems. The value function approximation can be applied to problems with infinite time horizons, and the grid value approximation to those with finite time horizons. We used Merton's model as an example of a DC-type problem and the model from Josa-Fombellida and Rincón-Zapatero (2001) as an example of a DB-type problem. We consider both single and multiple risky assets cases. The approximation results are compared with the true solutions using different quantiles and the ECDFs at different time. The comparisons indicated that the two algorithms perform well. In the robustness tests, we examined the performances of two algorithms by using different parameter values. These combinations of parameter values represented different choices of the assets, different market performances, and different sponsor preferences. By comparing the RMSEs and NRMSEs, we concluded that the two approximation methods can be used as alternative ways to find the optimal decisions of similar problems, in general. Despite the fact that we know the true solutions for these models, normally for more complex models, we would not. The good performance of the two methods gives some guidance about solving optimal controls for more complex models.

There are some limitations in our models and assumptions. First, we assume that the price of the assets follows a gBm. This makes the problem simpler but not representative of the behavior of our markets. Further works can consider more complicated asset return models; for example, by adding a jump term to the model. Second, we keep the normal cost and the benefit paid out in our models constant, which may not be the case in real life. It would be relevant to consider numerical algorithms used in this study to models that account for the evolution of the normal cost and the benefit. The third limitation is related to the objective function; we assume a quadratic criterion when combining the two risks in

DB plans. It would be interesting to consider whether the two algorithms work for other types of objective functions (e.g., absolute criterion).

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Appendix A

A.1 Detailed Solution of $Q(t)$

By Equation (2.54), $Q(t)$ can be written as

$$Q_t = \left(r + \frac{1-\kappa}{L(t)}\right) Q(t) + \left[(NC - P) - \frac{AL}{L(t)}(1-\kappa)\right]. \tag{A.1}$$

Let $y(t) = r + \frac{1-\kappa}{L(t)}$ and $b(t) = (NC - P) - \frac{AL}{L(t)}(1-\kappa)$. Equation (A.1) therefore has a form of $Q_t = y(t)Q(t) + b(t)$, which is a linear differential equation that allows for the following solution:

$$Q(t) = l \cdot e^{Y(t)} + e^{Y(t)} \int b(t)e^{-Y(t)}\, dt, \tag{A.2}$$

where $l = e^k$ is a constant and $Y(t) = \int y(t)\, dt$. First, we solve for $Y(t)$:

$$\begin{aligned} Y(t) &= \int r + \frac{1-\kappa}{L(t)}\, dt \\ &= \int r + \frac{\left(\psi_1 - \psi_2 e^{\omega_3(T-t)}\right)(1-\kappa)}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}}\, dt \\ &= \int r dt + (1-\kappa)\int \frac{\psi_1 - \psi_2 e^{\omega_3(T-t)}}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}}\, dt. \end{aligned}$$

The first integral is trivial and we mainly focus on the second integral. We multiply both the nominator and denominator by $e^{\omega_3 t}$ and the integral becomes

$$\int \frac{\psi_1 e^{\omega_3 t} - \psi_2 e^{\omega_3 T}}{\omega_2\psi_1 e^{\omega_3 t} - \omega_1\psi_2 e^{\omega_3 T}}\, dt. \tag{A.3}$$

To solve Equation (A.3), several substitutions are needed. First, let $\nu = -\omega_3 t$:

$$-\frac{1}{\omega_3}\int \frac{\psi_2 e^{\nu+\omega_3 T} - \psi_1}{\omega_1\psi_2 e^{\nu+\omega_3 T} - \omega_2\psi_1} d\nu.$$

Then, we let $u = -\nu - \omega_3 T$:

$$\begin{aligned}
&-\int \frac{\psi_1 e^u - \psi_2}{\omega_2\psi_1 e^u - \omega_1\psi_2} du \\
=&-\int \frac{\frac{\psi_1}{\omega_2\psi_1}\left(\omega_2\psi_1 e^u - \omega_1\psi_2\right) + \frac{\omega_1\psi_2\psi_1}{\omega_2\psi_1} - \psi_2}{\omega_2\psi_1 e^u - \omega_1\psi_2} du \\
=&-\int \frac{1}{\omega_2} + \frac{\frac{\omega_1\psi_2}{\omega_2} - \psi_2}{\omega_2\psi_1 e^u - \omega_1\psi_2} du \\
=&-\left[\left(\frac{\omega_1\psi_2}{\omega_2} - \psi_2\right)\int \frac{1}{\omega_2\psi_1 e^u - \omega_1\psi_2} du + \int \frac{1}{\omega_2} du\right]. \qquad \text{(A.4)}
\end{aligned}$$

After applying two substitutions, Equation (A.4) looks much simpler than Equation (A.3) does. To solve Equation (A.4), two final substitutions are required. Let $z = e^u$ and $p = \omega_2\psi_1 - \frac{\omega_1\psi_2}{z}$, so our equation can be further simplified and solved:

$$\begin{aligned}
&-\left[\left(\frac{\omega_1\psi_2}{\omega_2} - \psi_2\right)\frac{1}{\omega_1\psi_2}\ln p + \int \frac{1}{\omega_2} du\right] \\
=&-\left[\left(\frac{\omega_1\psi_2}{\omega_2} - \psi_2\right)\frac{1}{\omega_1\psi_2}\ln\left(\omega_2\psi_1 - \frac{\omega_1\psi_2}{z}\right) + \frac{u}{\omega_2}\right] \\
=&-\left[\left(\frac{\omega_1\psi_2}{\omega_2} - \psi_2\right)\frac{1}{\omega_1\psi_2}\ln\frac{\omega_2\psi_1 e^u - \omega_1\psi_2}{e^u} + \frac{u}{\omega_2}\right] \\
=&-\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right)\ln\frac{\omega_2\psi_1 e^u - \omega_1\psi_2}{e^u} - \frac{u}{\omega_2}. \qquad \text{(A.5)}
\end{aligned}$$

Now, we have solved the integral of Equation (A.3) with two substitutions. To obtain the final solution, we need to undo all the substitutions. First, we undo the substitution of $u = -\nu - \omega_3 T$:

$$-\frac{\omega_1 - \omega_2}{\omega_1\omega_2}\ln\left(\frac{\omega_2\psi_1 e^{-\nu-\omega_3 T} - \omega_1\psi_2}{e^{-\nu-\omega_3 T}}\right) - \frac{-\nu - \omega_3 T}{\omega_2}.$$

Then, we undo the second substitution of $\nu = -\omega_3 t$:

$$\begin{aligned}
&-\frac{1}{\omega_3}\left[-\frac{\omega_1 - \omega_2}{\omega_1\omega_2}\ln\left(\frac{\omega_2\psi_1 e^{\omega_3 t-\omega_3 T} - \omega_1\psi_2}{e^{\omega_3 t-\omega_3 T}}\right) - \frac{\omega_3 t - \omega_3 T}{\omega_2}\right] \\
=&\frac{t-T}{\omega_2} + \frac{\omega_1 - \omega_2}{\omega_1\omega_2\omega_3}\ln\frac{\omega_2\psi_1 e^{-\omega_3(T-t)} - \omega_1\psi_2}{e^{-\omega_3(T-t)}}. \qquad \text{(A.6)}
\end{aligned}$$

The expression denoted by (A.6) is our final solution of Equation (A.3). Using this solution, $Y(t)$ becomes

$$Y(t) = (1-\kappa)\left(\frac{t-T}{\omega_2} + \frac{\omega_1 - \omega_2}{\omega_1\omega_2\omega_3}\ln\frac{\omega_2\psi_1 e^{\omega_3(t-T)} - \omega_1\psi_2}{e^{\omega_3(t-T)}}\right) + rt.$$

After finding $Y(t)$, we can compute $e^{Y(t)}$:

$$e^{Y(t)} = e^{(1-\kappa)\left(\frac{t-T}{\omega_2} + \frac{\omega_1-\omega_2}{\omega_1\omega_2\omega_3}\ln\frac{\omega_2\psi_1 e^{\omega_3(t-T)} - \omega_1\psi_2}{e^{\omega_3(t-T)}}\right) + rt}$$

$$= \left(e^{\ln \frac{\omega_2\psi_1 e^{\omega_3(t-T)} - \omega_1\psi_2}{e^{\omega_3(t-T)}}} \right)^{(1-\kappa)\frac{\omega_1-\omega_2}{\omega_1\omega_2\omega_3}} \cdot e^{\frac{1-\kappa}{\omega_2}(t-T)+rt}$$

$$= \left(\frac{\omega_2\psi_1 e^{\omega_3(t-T)} - \omega_1\psi_2}{e^{\omega_3(t-T)}} \right)^{(1-\kappa)\frac{\omega_1-\omega_2}{\omega_1\omega_2\frac{\omega_1-\omega_2}{\kappa}}} \cdot e^{\frac{1-\kappa}{\omega_2}(t-T)+rt}$$

$$= \left(\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)} \right)^{\frac{\kappa(1-\kappa)}{\omega_1\omega_2}} \cdot e^{\frac{1-\kappa}{\omega_2}(t-T)+rt}$$

$$= \frac{1}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} e^{\frac{1-\kappa}{\omega_2}(t-T)+rt}. \tag{A.7}$$

By having $e^{Y(t)}$, it is easy to obtain $e^{-Y(t)}$. Thus, we are able to find the solution of $\int b(t)e^{-Y(t)}\,dt$:

$$\begin{aligned}
&\int b(t)e^{-Y(t)}\,dt \\
=&\int \left[(NC-P) - \frac{\left(\psi_1 - \psi_2 e^{\omega_3(T-t)}\right)(1-\kappa)AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \right] \left(\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}\right) e^{-\frac{1-\kappa}{\omega_2}(t-T)-rt}\,dt \\
=&\int (NC-P)\left(\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}\right) e^{-\frac{1-\kappa}{\omega_2}(t-T)-rt}\,dt \\
&\quad - \int (1-\kappa)AL\left(\psi_1 - \psi_2 e^{\omega_3(T-t)}\right) e^{-\frac{1-\kappa}{\omega_2}(t-T)-rt}\,dt \\
=&(NC-P)\left[\frac{\omega_2\psi_1}{-\frac{1-\kappa}{\omega_2} - r} e^{\left(-\frac{1-\kappa}{\omega_2}-r\right)t + \frac{1-\kappa}{\omega_2}T} - \frac{\omega_1\psi_2}{-\omega_3 - \frac{1-\kappa}{\omega_2} - r} e^{\left(-\omega_3 - \frac{1-\kappa}{\omega_2}-r\right)t + \left(\omega_3 + \frac{1-\kappa}{\omega_2}\right)T} \right] \\
&\quad - (1-\kappa)AL\left[\frac{\psi_1}{-\frac{1-\kappa}{\omega_2} - r} e^{\left(-\frac{1-\kappa}{\omega_2}-r\right)t + \frac{1-\kappa}{\omega_2}T} - \frac{\psi_2}{-\omega_3 - \frac{1-\kappa}{\omega_2} - r} e^{\left(-\omega_3 - \frac{1-\kappa}{\omega_2}-r\right)t + \left(\omega_3 + \frac{1-\kappa}{\omega_2}\right)T} \right] \\
=&e^{\frac{1-\kappa}{\omega_2}(T-t)-rt}\left[(NC-P)\frac{\omega_2^2\psi_1}{-(1-\kappa) - r\omega_2} - (1-\kappa)AL\frac{\omega_2\psi_1}{-(1-\kappa) - r\omega_2} \right] \\
&\quad - e^{\left(\frac{1-\kappa}{\omega_2}\right)(T-t)-rt} e^{\omega_3(T-t)} \left[(NC-P)\frac{\omega_1\omega_2\psi_2}{-\omega_2\omega_3 - (1-\kappa) - r\omega_2} \right. \\
&\qquad\qquad \left. -(1-\kappa)AL\frac{\omega_2\psi_2}{-\omega_2\omega_3 - (1-\kappa) - r\omega_2} \right] \\
=&e^{\frac{1-\kappa}{\omega_2}(T-t)-rt}\left[\frac{AL\omega_2\psi_1(\delta\omega_2 + (1-\kappa))}{(1-\kappa) + r\omega_2} - \frac{\omega_2\psi_2 AL(\delta\omega_1 + (1-\kappa))}{\omega_2\omega_3 + (1-\kappa) + r\omega_2} e^{\omega_3(T-t)} \right].
\end{aligned}$$

As a consequence, the solution of $e^{Y(t)} \int b(t)e^{-Y(t)}dt$ is

$$\frac{AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \left[\frac{\omega_2\psi_1(\delta\omega_2 + (1-\kappa))}{(1-\kappa) + r\omega_2} - \frac{\omega_2\psi_2(\delta\omega_1 + (1-\kappa))}{\omega_2\omega_3 + (1-\kappa) + r\omega_2} e^{\omega_3(T-t)} \right]. \tag{A.8}$$

Since we have $\omega_1\omega_2 = -\kappa(1-\kappa)$ and $\omega_3 = \frac{\omega_1-\omega_2}{\kappa}$ from Equations (2.57) to (2.59), Equation (A.8) can be written as

$$\frac{AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \left[\frac{\kappa\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\kappa\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2} e^{\omega_3(T-t)} \right].$$

The second to last step is to determine the constant l in Equation (A.2). From Section 2.2.2, we know that $Q(T) = AL$, which means that

$$\begin{aligned} AL =& l \cdot \frac{1}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-T)}} e^{\frac{1-\kappa}{\omega_2}(T-T)+rT} \\ &+ \frac{AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-T)}} \left[\frac{\kappa\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\kappa\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2} e^{\omega_3(T-T)}\right] \\ =& l \cdot \frac{1}{\omega_2\psi_1 - \omega_1\psi_2} e^{rT} \\ &+ \frac{AL}{\omega_2\psi_1 - \omega_1\psi_2} \left[\frac{\kappa\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\kappa\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2}\right]. \end{aligned} \tag{A.9}$$

Using Equation (A.9), we can find the constant l:

$$l = ALe^{-rT} \left\{(\omega_2\psi_1 - \omega_1\psi_2) - \kappa \left[\frac{\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2}\right]\right\}. \tag{A.10}$$

Finally, we can obtain the equation for $Q(t)$ by applying Equations (A.7), (A.8) and (A.10) to Equation (A.2):

$$\begin{aligned} Q(t) =& \frac{ALe^{-rT} e^{\frac{1-\kappa}{\omega_2}(t-T)+rt}}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \\ &\times \left\{(\omega_2\psi_1 - \omega_1\psi_2) - \kappa \left[\frac{\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2}\right]\right\} \\ &+ \frac{AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \left[\frac{\kappa\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\kappa\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2} e^{\omega_3(T-t)}\right] \\ =& \frac{AL}{\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}} \\ &\times \left\{ e^{\left(\frac{1-\kappa}{\omega_2}+r\right)(t-T)} \left[(\omega_2\psi_1 - \omega_1\psi_2) - \kappa \left(\frac{\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2}\right)\right] \right. \\ &\left. + \left[\frac{\kappa\psi_1(\delta\omega_2 + (1-\kappa))}{\kappa r - \omega_1} - \frac{\kappa\psi_2(\delta\omega_1 + (1-\kappa))}{\kappa r - \omega_2} e^{\omega_3(T-t)}\right]\right\}. \end{aligned} \tag{A.11}$$

The above equation can be simplified. First, from Equations (2.57) to (2.60), we have

$$\begin{aligned} \psi_1\omega_2 - \psi_2\omega_1 &= \alpha(\omega_2 - \omega_1) \\ \frac{1-\kappa}{\omega_2} &= -\frac{\omega_1}{\kappa} \\ \psi_2\omega_1^2 - \psi_1\omega_2^2 &= \alpha(\omega_1^2 - \omega_2^2) - \omega_1\omega_2(\omega_1 - \omega_2) \\ \psi_1 - \psi_2 &= \omega_2 - \omega_1. \end{aligned}$$

By applying these four equivalences to Equation (A.11), we can finally obtain

$$\begin{aligned} Q(t) =& \frac{AL}{\left(\omega_2\psi_1 - \omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r - \omega_1)(\kappa r - \omega_2)} \\ &\times \left\{ e^{-\left(r - \frac{\omega_1}{\kappa}\right)(T-t)} (r-\delta)\kappa[(\kappa r - \omega_2)\omega_2\psi_1 - (\kappa r - \omega_1)\omega_1\psi_2] \right. \end{aligned}$$

$$
\begin{aligned}
&+\kappa\delta\left[(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2 e^{\omega_3(T-t)}\right] \\
&-\omega_1\omega_2\left[(\kappa r-\omega_2)\psi_1-(\kappa r-\omega_1)\psi_2 e^{\omega_3(T-t)}\right]\Big\} \\
=&\frac{-AL\kappa(r-\delta)}{\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r-\omega_1)(\kappa r-\omega_2)} \\
&\times\Big\{(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2 e^{\omega_3(T-t)} \\
&-[(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2]e^{-\left(r-\frac{\omega_1}{\kappa}\right)(T-t)}\Big\} \\
&+\frac{AL}{\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r-\omega_1)(\kappa r-\omega_2)} \\
&\times\Big\{\kappa r\left[(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2 e^{\omega_3(T-t)}\right] \\
&-\omega_1\omega_2\left[(\kappa r-\omega_2)\psi_1-(\kappa r-\omega_1)\psi_2 e^{\omega_3(T-t)}\right]\Big\} \\
=&\frac{-AL\kappa(r-\delta)}{\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r-\omega_1)(\kappa r-\omega_2)} \\
&\times\Big\{(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2 e^{\omega_3(T-t)} \\
&-[(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2]e^{-\left(r-\frac{\omega_1}{\kappa}\right)(T-t)}\Big\} \\
&+\frac{AL}{\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r-\omega_1)(\kappa r-\omega_2)} \\
&\times(\kappa r-\omega_2)(\kappa r-\omega_1)\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right) \\
=&AL-\frac{\kappa(r-\delta)AL}{\left(\omega_2\psi_1-\omega_1\psi_2 e^{\omega_3(T-t)}\right)(\kappa r-\omega_1)(\kappa r-\omega_2)} \\
&\times\Big\{(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2 e^{\omega_3(T-t)} \\
&-[(\kappa r-\omega_2)\omega_2\psi_1-(\kappa r-\omega_1)\omega_1\psi_2]e^{-\left(r-\frac{\omega_1}{\kappa}\right)(T-t)}\Big\}.
\end{aligned}
$$