Brauer-Severi varieties associated to twists of the Burkhardt quartic

by

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B.Sc., Simon Fraser University, 2019

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Mathematics
Faculty of Science

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Abstract

The Burkhardt quartic is a projective threefold which, geometrically, is birational to the moduli space of abelian surfaces with full level-3 structure. We study this moduli interpretation of the Burkhardt quartic in an arithmetic setting, over a general field k. As it turns out, some twists of the Burkhardt quartic have a nontrivial field-of-definition versus field-of moduli obstruction. Classically, if a twist has a k-rational point then the obstruction can be computed as the Brauer class of an associated conic. Using representation theory, we show how to compute the obstruction without assuming the existence of a k-rational point, giving rise to an associated 3-dimensional Brauer-Severi variety rather than a conic. This Brauer-Severi variety itself has a related moduli interpretation.

Keywords: full level-3 structures, field-of-definition versus field-of-moduli obstructions, Brauer-Severi varieties

Acknowledgements

I would like to express my sincere gratitude towards my supervisor Nils Bruin. My research project suits my mathematical interests and Dr. Bruin has provided, and continues to provide, invaluable guidance and support.

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Chapter 1

Introduction and statement of results

Let k be a field of characteristic different from 3. The Burkhardt quartic threefold is a hypersurface in $\mathbb{P}^4_k$ defined by the equation

$$\mathcal{B}\colon B(y_0, y_1, y_2, y_3, y_4) := y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

Our work focuses on a moduli space interpretation of the Burkhardt quartic threefold: over an algebraically closed field, the Burkhardt quartic parametrizes genus 2 curves with a full level-3 structure on their Jacobian. A genus 2 curve can be expressed as a double cover of $\mathbb{P}^1$, branched over 6 points. These 6 points determine the genus 2 curve. One of the realizations of the moduli interpretation of the Burkhardt presents this information in the form of the intersection of a nonsingular conic with a plane cubic. Such an intersection has degree 6 and over an algebraically closed field, a nonsingular conic is isomorphic to $\mathbb{P}^1$. In arithmetic settings however, where the base field k is not necessarily algebraically closed, conics may not be isomorphic to $\mathbb{P}^1$; a non-trivial base field extension may be required. This potentially poses an obstruction to the moduli interpretation over k. The conics obtained from k-rational points $\alpha \in \mathcal{B}(k)$ are isomorphic (see Section 6.1). For the standard model of the Burkhardt quartic given above, the conics are isomorphic to $\mathbb{P}^1$ over k, so the moduli interpretation of $\mathcal{B}$ over k holds without obstruction. The isomorphism class of a conic can be represented by an element of the Brauer group of the field k. In this thesis we consider *twists* of $\mathcal{B}$: threefolds over k that, over the algebraic closure of k, become isomorphic to $\mathcal{B}$. We show (Theorem 6.5) that such twists can again be expressed as quartic threefolds in $\mathbb{P}^4$. Such a twist $\mathcal{B}'$ has again a moduli interpretation: it provides data determining a genus 2 curve for which, if it exists over k, the Jacobian has an appropriately twisted full level-3 structure. We write $\mathrm{Ob}(\mathcal{B}'/k) \in \mathrm{Br}(k)$ for the corresponding obstruction.

One of the central results in this thesis is an explicit construction of a 3-dimensional Brauer-Severi variety S' in $\mathbb{P}^9$ whose class in $\mathrm{Br}(k)$ is $\mathrm{Ob}(\mathcal{B}'/k)$. The variety S' can be computed directly via the representation theory of the automorphism group of $\mathcal{B}'$. More precisely, we show the following.

Theorem 1.1. *Let $\mathcal{B}'$ be a twist of the Burkhardt quartic presented in $\mathbb{P}^4_k$, and $\Gamma' \subset \mathrm{PGL}_5(\overline{k})$ its automorphism group. Then there is a 3-dimensional Brauer-Severi variety S' such that the Brauer class of S' is exactly $\mathrm{Ob}(\mathcal{B}'/k)$. Moreover, the variety S' can be realized as an intersection of 20 quadrics in $\mathbb{P}^9_k$, and the span of these quadrics is invariant under the action of $(\bigwedge^2 \Gamma')^*$.*

The variety S' admits a degree 6 rational map to $\mathcal{B}'$. If $\mathrm{Ob}(\mathcal{B}'/k)$ is trivial, then S' is isomorphic to $\mathbb{P}^3_k$, so in that case $S'(k)$ and $\mathcal{B}'(k)$ lie dense in S' and $\mathcal{B}'$ respectively (provided that k is infinite). In particular, any unobstructed level-3 structure occurs over k. In fact, S' parametrizes genus 2 curves with a marked Weierstrass point. Such curves admit a model of the form $y^2 = f(x)$, with f a quintic polynomial. Hence, any unobstructed level-3 structure occurs over k for a curve of the form $y^2 = f(x)$, with $\deg(f) = 5$. We precisely formulate this last statement in the following theorem.

Theorem 1.2. *Let Σ' be a full level-3 structure over k, and let $\mathcal{B}'$ be the twist of $\mathcal{B}$ over k parametrizing genus 2 curves with full level-3 structure Σ' on their Jacobian. If $\mathrm{Ob}(\mathcal{B}'/k)$ is trivial, then there is a genus 2 curve $\mathcal{C}\colon y^2 = f(x)$, with f a degree 5 polynomial in $k[x]$, such that the group scheme $\mathrm{Jac}(\mathcal{C})[3]$ is isomorphic to Σ' over k.*

We also observe that for a rational point $\alpha \in \mathcal{B}'(k)$, we can construct a 1-dimensional Brauer-Severi variety representing $\mathrm{Ob}(\mathcal{B}'/k)$. Note that in the result above, we do not assume the existence of a rational point on $\mathcal{B}'$ for the construction of the 3-dimensional Brauer-Severi variety S'. From our construction, we know that S' has *period* dividing 2. It must have *index* 1,2, or 4,with index 1 meaning that $\mathrm{Ob}(\mathcal{B}'/k)$ is trivial. See Section 3.3 for the definitions of period and index. Over global and local fields we know that the index must equal the period, but in general this may not be the case. Since $\mathcal{B}'(k) \neq \emptyset$ implies that the index is at most 2, the following two questions arise:

Question 1. Are there any fields k and twists $\mathcal{B}'$ of the Burkhardt quartic such that $\mathrm{Ob}(\mathcal{B}'/k)$ has index 4?

Question 2. Is it the case that $\mathcal{B}'(k) = \emptyset$ implies that $\mathrm{Ob}(\mathcal{B}'/k)$ has index 4?

An affirmative answer to the latter question would imply that over a global field, any twist $\mathcal{B}'$ has rational points. Throughout this thesis, assume that k is a perfect field with $\mathrm{char}(k) \neq 2, 3$.

We now briefly describe the document layout. Chapters 2-5 contain necessary background material. Sections 1-2 of Chapter 6 discuss the relevant properties of the Burkhardt quartic, namely details on its automorphism group and its moduli interpretation. The rest of Chapter 6 is dedicated to our results. In particular, Sections 7-8 of Chapter 6 prove Theorem 1.1 and Theorem 1.2. Chapter 2 reviews the basics of Galois cohomology, concluding with Weil descent. Chapter 3 discusses the Brauer group of a field, first from the perspective of central simple algebras, and then from the perspective of Galois cohomology. The final section of Chapter 3 discusses the period-index problem for central simple algebras. Chapter 4 is about Brauer-Severi varieties, including a discussion on how to classify Brauer-Severi varieties by means of Galois cohomology, and a connection to the Brauer group. Chapter 5 deals with some theory of genus 2 curves. This theory includes a discussion of the Jacobian and the associated Kummer surface, a discussion of full level-3 structures, and a brief mention of different associated moduli spaces.

Chapter 2

Galois cohomology

We review some results from Galois cohomology, following [8].

2.1 Introduction

One may consider many different categories of objects over k, such as k-vector spaces, k-algebras, and k-varieties. More generally, consider the category of *finite dimensional vector spaces V over k equipped with a tensor Φ of type* (p, q). Objects in this category, called *k-objects*, are pairs (V, Φ) where Φ is an element of

$$\mathrm{Hom}_k(V^{\otimes q}, V^{\otimes p}).$$

A morphism between two objects (V, Φ) and (W, Ψ) is a k-linear map of vector spaces $f : V \to W$ such that

$$f^{\otimes p} \circ \Phi = \Psi \circ f^{\otimes q},$$

where $f^{\otimes p} : V^{\otimes p} \to V^{\otimes p}$ is given by $v_1 \otimes \cdots \otimes v_p \mapsto f(v_1) \otimes \cdots \otimes f(v_p)$.

Example 2.1. If $(p, q) = (0, 0)$, then the corresponding category is just the category of k-vector spaces, whereas if $(p, q) = (2, 1)$ then we get the category of (not necessarily associative) k-algebras.

Suppose we have an algebraic extension K/k. We may base change (V, Φ) to (V_K, Φ_K) by tensoring with K. This preserves the tensor type: if Φ is of type (p, q) then so is Φ_K.

Definition 2.2. We say that (V, Φ) and (W, Ψ) *become isomorphic* over K if (V_K, Φ_K) and (W_K, Ψ_K) are isomorphic over K. In this case, we say that (W, Ψ) is a (K/k)-*twisted form*, or *twisted form*, or simply *twist* of (V, Φ).

2.2 Classifying Twists by Galois cohomology

Given a k-object (V, Φ) and a Galois extension K/k, we can classify all twists up to k-isomorphism in the following, standard way (see Section 2.3 of [8]). First, note that

$\mathrm{Gal}(K/k)$ has an induced action on V_K, although not a K-linear one. If (W, Ψ) is another k-object and $f\colon V_K \to W_K$ is a K-object isomorphism, we set $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$, which is K-linear again. This defines an action of $\mathrm{Gal}(K/k)$ on isomorphisms $(V_K, \Phi_K) \to (W_K, \Psi_K)$. In particular, this makes $\mathrm{Aut}(V_K)$ into a $\mathrm{Gal}(K/k)$-set.

For the rest of this section, let G be a finite group and let A be a group with a G-action.

Definition 2.3. We say that a map $a\colon G \to A$ is a *1-cocycle* if a satisfies the following relation

$$a_{\sigma\tau} = a_\sigma \sigma(a_\tau) \quad \text{for all } \sigma, \tau \in G. \tag{2.1}$$

Two 1-cocycles a, b are called *cohomologous* if there exists an $m \in A$ such that

$$a_\sigma = m^{-1} b_\sigma \sigma(m) \quad \text{for all } \sigma \in G. \tag{2.2}$$

If a and b are cohomologous, we write $a \sim b$. This is readily checked to be an equivalence relation. We consider the equivalence classes.

Definition 2.4. The *first cohomology set of G with values in A* is defined to be

$$H^1(G, A) = \{\text{1-cocycles } a\colon G \to A\}/\sim .$$

In the set $H^1(G, A)$, the cohomology class of the trivial map $\sigma \mapsto 1_A$ forms a distinguished element. It is an example of an object in the following category.

Definition 2.5. The *category of pointed sets* consists of objects (X, x_0), where X is a set and $x_0 \in X$ is an element called the *base point*. A morphism between two objects (X, x_0) and (Y, y_0) in this category is a map $f\colon X \to Y$ such that $f(x_0) = y_0$. A *pointed set* is an object in this category.

The base point of $H^1(G, A)$ is the trivial map $\sigma \mapsto 1_A$. In general, $H^1(G, A)$ is only a pointed set with no (natural) group structure.

Proposition 2.6. *If A is abelian, and thus a $\mathbb{Z}[G]$-module, the set $H^1(G, A)$ forms a group under pointwise operations.*

Proof. If a, b are cocycles and c is defined by $c_\sigma = a_\sigma b_\sigma$ and A is commutative then c is a cocycle as well, because

$$c_{\sigma\tau} = a_{\sigma\tau} b_{\sigma\tau} = a_\sigma \sigma(a_\tau) b_\sigma \sigma(b_\tau) = a_\sigma b_\sigma \sigma(a_\tau b_\tau) = c_\sigma \sigma(c_\tau).$$

The trivial cocycle $\sigma \mapsto 1_A$ is a neutral element to this operation and for a cocycle a one can check that $\sigma \mapsto a_{\sigma^{-1}}$ is also a cocycle. This shows that the cocycles form a commutative group. The cocycles cohomologous to the trivial cocycle form a subgroup, so it follows that $H^1(G, A)$ indeed has a group structure. $\square$

Remark 2.7. Note that we really use that A is a commutative group. If A is not commutative, then $\sigma \mapsto a_\sigma b_\sigma$ need not be a cocycle if $\sigma \mapsto a_\sigma$ and $\sigma \mapsto b_\sigma$ are.

We now relate these notions back to the original problem of classifying the twisted forms of (V, Φ).

Lemma 2.8. *Let (V_K, Φ) and (W_K, Ψ_K) be the K-objects obtained from k-objects (V, Φ) and (W, Ψ) by tensoring with K. For a K-object isomorphism $f\colon V_K \to W_K$, the map $a_f\colon \mathrm{Gal}(K/k) \to \mathrm{Aut}_K\Phi$ defined by $\sigma \mapsto f^{-1} \circ \sigma(f)$ is a 1-cocycle. Furthermore, a_f and a_g are cohomologous if and only if $f^{-1} \circ g$ is induced by a k-automorphism of V.*

The proof of the above lemma is a fairly direct computation which we omit, see [8] for details. Write $\mathrm{TF}_{K/k}(V, \Phi)$ for the set of twisted (K/k)-forms of (V, Φ). Notice that $\mathrm{TF}_{K/k}(V, \Phi)$ is naturally a pointed set with the base point being the trivial twist (V, Φ). The lemma thus gives a morphism $\mathrm{TF}_{K/k}(V, \Phi) \to H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(\Phi))$ in the category of pointed sets. This brings us to the main result of the section.

Theorem 2.9 (see [8])**.** *The map $\mathrm{TF}_{K/k}(V, \Phi) \to H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(\Phi))$ described above is a bijection of pointed sets.*

Sketch of Proof. Let $[a] \in H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(\Phi))$ be a cohomology class, represented by a cocycle $a\colon \mathrm{Gal}(L/k) \to \mathrm{Aut}_K(\Phi)$. We construct a twisted form mapping to $[a]$. Define a *twisted action* of $\mathrm{Gal}(K/k)$ on (V_K, Φ_K) via $(\sigma, \alpha) \mapsto a_\sigma \sigma(\alpha)$. We then denote ${}_aV_K$ as the K-object on which the Galois group $\mathrm{Gal}(K/k)$ acts via the twisted action. Then taking the invariant part $W = ({}_aV_K)^{\mathrm{Gal}(K/k)}$ allows us (with additional work) to obtain a k-object (W, Ψ), which is a twisted form of (V, Φ). □

Remark 2.10. We can use Theorem 2.9 for affine k-varieties as well by considering their affine coordinate rings. These are k-algebras, so we can use Theorem 2.9 to consider their twists. We can then use that affine k-variety morphisms correspond exactly to morphisms between their coordinate rings in opposite direction via pullback.

This interpretation of $H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(\Phi))$ immediately implies the following classical result.

Theorem 2.11 (Hilbert 90)**.** *The pointed set*

$$H^1(\mathrm{Gal}(K/k), \mathrm{GL}_n(K))$$

is trivial.

Proof. The set $H^1(\mathrm{Gal}(K/k), \mathrm{GL}_n(K))$ classifies the twisted forms of an n-dimensional k-vector space. Vector spaces, however, have no nontrivial twisted forms since they are entirely determined by their dimension, which is invariant under base field extension. □

Most varieties that we will be dealing with are projective or quasi-projective. The notion of twist for quasi-projective varieties X over k is analogous, and as in Theorem 2.9 we get a map $\mathrm{TF}_{K/k}(X) \to H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(X_K))$ in the category of pointed sets.

Theorem 2.12. *Let X be a quasiprojective variety. The map*

$$\mathrm{TF}_{K/k}(X) \to H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(X_K))$$

is a bijection of pointed sets.

Proof. See Theorem 4.5.2 of [12]. □

2.3 Higher cohomology

Before we define higher cohomology for a finite group G with values in a group A with a G-action, we identify the functor from which the cohomology theory is derived.

Definition 2.13. Let G be a finite group and A a group with a G-action. We define

$$H^0(G, A) = A^G,$$

where A^G is the G-invariant subset of A.

We now proceed to define the *second cohomology set of G with values in A*, which is denoted by $H^2(G, A)$. This is all we will need for this thesis. We say that a function $\xi\colon G \times G \to A$ is a *2-cocycle* if it satisfies the relation

$$\sigma(\xi(\tau, \rho)) \cdot \xi(\sigma\tau, \rho) = \xi(\sigma\tau, \rho) \cdot \xi(\sigma, \tau) \quad \text{for all } \sigma, \tau, \rho \in G. \tag{2.3}$$

We say that two 2-cocycles ξ, ξ' are *cohomologous*, and write $\xi \sim \xi'$ if there is a function $f : G \to A$ (any function, not necessarily a homomorphism) such that

$$\xi(\sigma, \tau) \cdot \xi'(\sigma, \tau)^{-1} = \sigma(f(\tau)) \cdot f(\sigma\tau)^{-1} \cdot f(\sigma) \quad \text{for all } \sigma, \tau \in G. \tag{2.4}$$

As for 1-cocycles, the relation $\sim$ is readily checked to be an equivalence. We define

$$H^2(G, A) = \{\text{2-cocycles } \xi : G \times G \to A\}/\sim .$$

This is naturally a pointed set, with base point being the trivial cocycle $(\sigma, \tau) \mapsto 1_A$.

Proposition 2.14. *If A is abelian, and thus a $\mathbb{Z}[G]$-module, the set $H^2(G, A)$ forms a group under pointwise operations.*

The proof is analogous to that of Proposition 2.6.

Under certain conditions, we get an associated long exact sequence of cohomology. By definition, an exact sequence of pointed sets is a sequence in which the kernel of each map equals the image of the previous one, the kernel being the subset of elements mapping to the base point (see Section 2.7 of [8]).

Theorem 2.15 (See Proposition 4.4.1 of [8])**.** *Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of groups with a G-action, where A is abelian and and contained in the centre of B. Then there is a long exact sequence of pointed sets

$$\begin{tikzcd} 0 \arrow[r] & H^0(G,A) \arrow[r] & H^0(G,B) \arrow[r] & H^0(G,C) \arrow[dll, out=-10, in=170] \\ & H^1(G,A) \arrow[r] & H^1(G,B) \arrow[r] & H^1(G,C) \arrow[dll, out=-10, in=170] \\ & H^2(G,A) \arrow[r] & H^2(G,B) \arrow[r] & H^2(G,C). \end{tikzcd}$$

Instead of proving this standard result, we just describe the connecting morphisms $C^G \to H^1(G,A)$ and $H^1(G,C) \to H^2(G,A)$. Suppose $n \in C^G$. To define an associated 1-cocycle, choose an element $m \in B^G$ which maps to n. then define a cocycle $a\colon G \to A$ by $a_\sigma = m \cdot \sigma(m)^{-1}$. Now let $a \in H^1(G,C)$ be a 1-cocyle. To associate a 2-cocycle, lift a to a map (not necessarily a cocycle) $\tilde{a}\colon G \to B$ and define a map $\xi\colon G \times G \to B$ by

$$\xi(\sigma,\tau) = \tilde{a}_\sigma \sigma(\tilde{a}_\tau)\tilde{a}_{\sigma\tau}^{-1}$$

which is checked to be a 2-cocycle taking values in A.

All the theory we have discussed in this section can be generalized to *profinite groups* (meaning projective limits of finite groups). We equip G with the profinite topology, A with the discrete topology, and insist that our cocycles are continuous maps.

Notation 2.16. Let $\overline{k}$ be an algebraic closure of k. Often we will be concerned with the case where $G = \mathrm{Gal}(\overline{k}/k)$. In this case, we often write

$$H^1(k,A) := H^1(\mathrm{Gal}(\overline{k}/k), A),$$

and similar for higher cohomology sets.

With this in mind, we can take K/k to be $\overline{k}/k$ in Section 2.2.

2.4 Weil descent

In this section, we present the some of the theory developed in Section I of [14]. To be consistent with the notation in [14], we use superscripts to denote a Galois action. Suppose K/k is a Galois extension, which we consider as a subfield of $\overline{k}$, and V is a (quasiprojective) variety defined over K. In this section, we are concerned with the following two, related problems.

(P) Determine if there exists a variety defined over k which is birationally equivalent to V over the extension K.

(P') Determine if there exists a variety defined over k which is biregularly equivalent to V over the extension K.

We first consider Problem (P). Suppose that there is a variety V_0 defined over k and a map f which is a birational correspondence between V_0 and V over K. Then for all $\sigma, \tau \in \mathrm{Gal}(K/k)$ the map $f_{\tau,\sigma} := f^\tau \circ (f^\sigma)^{-1}$, where $f^\sigma = \sigma(f) = \sigma \circ f \circ \sigma^{-1}$, is a birational correspondence between V^σ and V^τ. These maps satisfy the following properties

(i) $f_{\tau,\rho} = f_{\tau,\sigma} \circ f_{\sigma,\rho}$ for all $\rho, \sigma, \tau \in \mathrm{Gal}(K/k)$

(ii) $f_{\tau\omega,\sigma\omega} = (f_{\sigma,\tau})^\omega$ for all $\rho, \sigma, \tau \in \mathrm{Gal}(K/k)$ and $\omega \in \mathrm{Gal}(\overline{k}/k)$

The following theorem establishes the converse.

Theorem 2.17 (see Theorem 1 of [14]). *Suppose that for all pairs (σ, τ) of elements in $\mathrm{Gal}(K/k)$ there is a birational correspondence $f_{\tau,\sigma}$ between V^σ and V^τ such that conditions (i) and (ii) are satisfied. Then there is a variety V_0 defined over k and a birational correspondence f between V_0 and V over K such that $f_{\tau,\sigma} = f^\tau \circ (f^\sigma)^{-1}$. Moreover, such a pair (V_0, f) is unique up to birational transformation over k.*

The proof of Theorem 2.17 is largely by Galois theory, see [14] for details.

We now address Problem (P'). For details, see Section I of [14]. The strategy is to utilize the result we have already established. Suppose the conditions of Theorem 2.17 hold, so that we obtain a variety V_0 over k and a birational correspondence f between V_0 and V over K. If there exists a variety V_0' over k and a biregular correspondence f' between V_0' and V over K, then by the uniqueness statement in Theorem 2.17 we get a birational correspondence F between V_0 and V_0' over k such that $f' = f \circ F^{-1}$. In particular, $f \circ F^{-1}$ is biregular over K. The following result gives the necessary conditions for such a pair (V_0', F) to exist.

Theorem 2.18 (see Theorem 2 of [14]). *Suppose the conditions of Theorem 2.17 hold. Assume further that the birational correspondence $f^\sigma \circ f^{-1}$ between V and V^σ over K is* *biregular for all $\sigma \in \mathrm{Gal}(K/k)$. Then there exists a pair (V_0', F), with V_0' a variety over k and F a birational correspondence between V_0 and V_0' over k, such that the birational correspondence $f \circ F^{-1}$ between V_0' and V over K is biregular.*

The proof of Theorem 2.18 is quite involved, requiring several intermediate results. The important consequence is the following solution of Problem (P'), in analogy with Theorem 2.17.

Theorem 2.19 (see Theorem 3 of [14]). *Suppose that for all pairs (σ, τ) of elements in $\mathrm{Gal}(K/k)$ there is a biregular correspondence $f_{\tau,\sigma}$ between V^σ and V^τ such that conditions (i) and (ii) are satisfied. Then there is a variety V_0' defined over k and a biregular correspondence f' between V_0' and V over K such that $f_{\tau,\sigma} = f^\tau \circ (f^\sigma)^{-1}$. Moreover, such a pair (V_0', f') is unique up to biregular transformation over k.*

Chapter 3

The Brauer group

3.1 Central simple algebras

The Brauer group $\mathrm{Br}(k)$ of a field k plays an important role in this thesis. In order to define it, we need the following concept.

Definition 3.1. Let A be an associative, finite dimensional k-algebra with unity. If

(1) A is *simple*, which means that A has no nontrivial two-sided ideals, and

(2) A is *central* over k, meaning $Z(A) = k$ (the field k naturally embeds into A),

then we say that A is a *central-simple algebra* (CSA) over k.

A natural example of a CSA is the matrix algebra $M_n(k)$. This is the archetypal example of a CSA in the following sense.

Theorem 3.2 (see Corollary 2.2.6 of [8])**.** *All CSAs are twisted forms of $M_n(k)$ for some n.*

Remark 3.3. In particular, we see that the dimension of of central simple algebra must be a square n^2.

By Theorem 2.12, the class of all CSAs of dimension n^2 over k may naturally be viewed as $H^1(k, \mathrm{PGL}_n(\overline{k}))$, where we use the fact (see Corollary 2.4.2 of [8]) that $\mathrm{Aut}(M_n(k)) = \mathrm{PGL}_n(k)$. An important family of examples is formed by CSAs that are *quaternion algebras.*

Definition 3.4. Let $a, b \in k^*$. Define $(a, b)_k$ to be the following associative but noncommutative k-algebra

$$(a, b)_k := k\langle x, y\rangle/(x^2 - a, y^2 - b, xy + yx).$$

Any k-algebra of this form is called a *quaternion algebra.*

Remark 3.5. It can be checked that all quaternion algebras are CSAs, see Section 2.1 of [8] for details.

Remark 3.6. We are only interested in the quaternion algebra up to isomorphism. In particular, we may view the a, b modulo squares, so $(a, b)_k$ is defined for $a, b \in k^*/k^{*2}$.

Example 3.7. For any $a \in k^*$, the algebra $(a, 1)_k$ is isomorphic to the matrix algebra $M_2(k)$. A particular isomorphism $(a, 1)_k \to M_2(k)$ is given by mapping the generators x, y of $(a, b)_k$ as follows

$$x \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \ y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To check that this is an isomorphism, it suffices to verify the presentation equations $x^2 = a$, $y^2 = 1$, $xy = -yx$ for the associated matrices.

In general,we want to regard $M_n(k)$ as the "trivial" central-simple algebra because of Theorem 3.2, which shows that $M_n(k)$ is a natural "base point" for CSAs.

Definition 3.8. Two CSAs A and B over k are *Brauer equivalent* if there exists some $m, n \in \mathbb{Z}_+$ such that

$$A \otimes_k M_m(k) \cong B \otimes_k M_n(k).$$

In this case, we write $A \sim B$.

In particular, any two matrix algebras $M_m(k)$ and $M_n(k)$ are Brauer equivalent. The *Brauer group* of k, denoted $\mathrm{Br}(k)$, is the abelian group given by

$$\mathrm{Br}(k) = \{k\text{-algebras which are CSAs}\}/\sim$$

with operation $\otimes$. The class $[M_n(k)] \in \mathrm{Br}(k)$ acts as the identity element. The fact that this is a group requires some verification, see Section 2.4 of [8] for details.

The behaviour of quaternion algebras under tensor products is captured by the bimuliplicativity of the symbol $(a, b)_k$.

Proposition 3.9 (see Lemma 1.5.2 of [8])**.** *Let a, b, c be in k. Then there is an isomorphism*

$$(a, b)_k \otimes (a, c)_k \cong (a, bc)_k \otimes M_2(k)$$

or equivalently

$$[(a, b)_k] \cdot [(a, c)_k] = [(a, bc)_k]$$

in $\mathrm{Br}(k)$.

Which yields

Corollary 3.10. *Let a, b be in k. Then $[(a, b)_k] \cdot [(a, b)_k]$ is trivial.*

Proof. By Proposition 3.9, we get

$$(a,b)_k \otimes (a,b)_k \cong (a,b^2)_k \otimes (a,1)_k \cong (a,b^2)_k \otimes M_2(k) \cong (a,1)_k \otimes M_2(k) \cong M_2(k) \otimes M_2(k) \cong M_4(k),$$

giving $[(a,b)_k] \cdot [(a,b)_k] = [M_4(k)]$, as required. □

This shows that all quaternion algebras are in $\mathrm{Br}(k)[2]$, the 2-torsion part of the Brauer group of k. In fact, a partial converse holds.

Theorem 3.11 (Merkurjev). *The quaternion algebras $(a,b)_k$ generate all of* $\mathrm{Br}(k)[2]$.

Proof. See Chapter 8 of [8]. □

The Brauer class of a quaternion algebra $(a,b)_k$ has a natural geometric interpretation, namely it "corresponds to" the conic $L = \{ax^2 + by^2 = z^2\} \subset \mathbb{P}^2_k$, we will make this notion more precise in Chapter 4.

Theorem 3.12 (see Proposition 1.3.2 of [8]). *A conic $L\colon ax^2 + by^2 = z^2$ has a k-rational points if and only if the Brauer class of the quaternion algebra $(a,b)_k$ is trivial in the Brauer group.*

3.2 The cohomological Brauer group

In this section, we present an overview of a connection between Brauer groups and Galois cohomology, as applied to the theory of twisted forms, see [8] for details.

Definition 3.13. Let K/k be a Galois extension. A central-simple algebra A is *split* over K if $A_K := A \otimes_k K \cong M_n(K)$ for some n.

Remark 3.14. A k-algebra A is split over K if and only if the class of A_K is trivial in $\mathrm{Br}(K)$.

Definition 3.15. The *relative Brauer group* $\mathrm{Br}(K/k)$ is the subgroup of $\mathrm{Br}(k)$ consisting of CSAs A that *split* over K.

A refinement of Theorem 3.2 is the following result.

Theorem 3.16 (see Proposition 4.5.4 of [8]). *Let A be a CSA over k. Then there is a field extension K/k with the following properties:*

(1) There is an embedding $K \hookrightarrow A$ of k-algebras

(2) The algebra A is split over K.

Moreover, if K'/k is any other extension such that A is split over K', then $[K:k] \leq [K':k]$.

In particular, this implies

$$\mathrm{Br}(k) = \bigcup_{K/k \text{ Galois}} \mathrm{Br}(K/k).$$

Remark 3.17. For quaternion algebras, the theorem follows from Example 3.7. If $A = (a,b)_k$, then A splits over $k(\sqrt{b})$ and also over $k(\sqrt{a})$.

The Brauer group can be characterized in terms of a Galois-cohomological object.

Theorem 3.18 (see Corollary 2.7.9 of [7])**.** *Let K/k be a Galois extension. Then we have a natural isomorphism of abelian groups $H^2(\mathrm{Gal}(K/k), K^*) \cong \mathrm{Br}(K/k)$.*

Sketch of Proof. There is an explicit construction. Let $\xi \in H^2(\mathrm{Gal}(K/k), K^*)$ be a cocycle. We need to associate to ξ a CSA over k, say A_ξ. Since we need that $K \hookrightarrow A_\xi$, we choose a K-basis $\{u_\sigma\}_{\sigma \in \mathrm{Gal}(K/k)}$ for A_ξ labeled by the elements of $\mathrm{Gal}(K/k)$. In other words

$$A_\xi = \bigoplus_{\sigma \in K} K u_\sigma.$$

By viewing K as a k-vector space, we see that A_ξ is also a k-vector space.

We define multiplication on A_ξ by the presentation

$$u_\sigma \beta = \sigma(\beta) u_\sigma, \quad u_\tau u_\sigma = \xi(\tau,\sigma) u_{\tau\sigma} \quad \text{for } \beta \in K.$$

By using the 2-cocycle condition (2.3), one can check that this multiplication is associative. Of course one needs to also verify that this multiplication is well-defined with respect to the cohomologous relation on 2-cocycles.

As to why A_ξ is a CSA, and why the map $H^2(\mathrm{Gal}(K/k), K^*) \to \mathrm{Br}(K/k)$ given by $\xi \mapsto A_\xi$ is an isomorphism, see [7]. $\square$

Recall Notation 2.16. In view of the previous result, the next theorem is perhaps expected, but it does require more theory that we omit.

Theorem 3.19 (see Theorem 4.4.7 of [8])**.** *Let k be a field. Then there is a natural isomorphism of abelian groups*

$$H^2(k, \overline{k}^*) \cong \mathrm{Br}(k).$$

We, however, do not need the entirety of $\mathrm{Br}(k)$. Since we are mainly focused on quaternion algebras, we need to consider only $\mathrm{Br}(k)[2]$. For this purpose, let $\mu_2 = \{\pm 1\} \subset \overline{k}^*$. Then μ_2 is naturally a $\mathrm{Gal}(\overline{k}/k)$-module (a trivial one), and there is a short exact sequence

$$0 \longrightarrow \mu_2 \longrightarrow \overline{k}^* \xrightarrow{\cdot 2} \overline{k}^* \longrightarrow 0 \tag{3.1}$$

of $\mathrm{Gal}(\overline{k}/k)$-modules. This is known as the "Kummer sequence". By Theorem 2.15 this gives a long exact sequence of cohomology, a segment of which is

$$H^1(k, \overline{k}^*) \longrightarrow H^2(k, \mu_2) \longrightarrow H^2(k, \overline{k}^*) \xrightarrow{\cdot 2} H^2(k, \overline{k}^*).$$

Now, by Theorem 2.11 with $n = 1$, we have $H^1(k, \overline{k}^*) = \{0\}$. Thus the 2-torsion of $H^2(k, \overline{k}^*) \cong \mathrm{Br}(k)$ is precisely $H^2(k, \mu_2)$, giving the following result.

Theorem 3.20. *There is a natural isomorphism of abelian groups*

$$H^2(k, \mu_2) \cong \mathrm{Br}(k)[2].$$

Corollary 3.21. *Let $A, B \in \mathrm{Br}(k)[2]$. Suppose for all Galois extensions K/k the class $A \otimes K$ is trivial in $\mathrm{Br}(K)[2]$ if and only if $B \otimes K$ is trivial in $\mathrm{Br}(K)[2]$. Then $A = B \in \mathrm{Br}(k)[2]$.*

Proof. By Theorem 3.20, A and B can be represented by 2-cocycles ξ_A and ξ_B in $H^2(k, \mu_2)$. Viewing $\mathrm{Gal}(\overline{k}/k)$ as a projective limit of Galois groups $\mathrm{Gal}(K/k)$, we see that ξ_A is determined by the extensions K/k for which ξ_A is trivial on $\mathrm{Gal}(K/k)$, and similar for ξ_B. $\square$

Another consequence of taking cohomology of the exact sequence (3.1) and applying Theorem 2.11 is the following natural isomorphism

$$H^1(k, \mu_2) \cong k^*/k^{*2}, \tag{3.2}$$

and more generally

$$H^1(k, \mu_n) \cong k^*/k^{*n}, \tag{3.3}$$

which follows directly from looking at the initial segment of the relevant cohomology sequence cohomology sequence.

3.3 Index and period

We discuss two important invariants of CSAs, namely *period* and *index*. Recall Theorem 3.16, which ensures that the following definition makes sense.

Definition 3.22. Let A be a central-simple k-algebra. Then the *index* of A, written $\mathrm{ind}(A)$, is the greatest common divisor of the degrees of finite separable extensions K/k that split A.

Remark 3.23. In Section 4.5 of [8], the main reference for this section, the definition of $\mathrm{ind}(A)$ is based on a theorem of Wedderburn. The theorem states that every central-simple algebra A is isomorphic to $M_r(D)$ for some $r > 0$ and D a division algebra. The index of A is defined to be the degree of D, and the definition above is Proposition 4.5.8.

Notice that $\mathrm{ind}(A)$ only depends on the class $[A]$ in $\mathrm{Br}(k)$, so we can consider the *index of a Brauer class*. To define period,we use the following result.

Theorem 3.24. *Let K/k be a Galois extension of degree n. Then every element of the relative Brauer group $\mathrm{Br}(K/k)$ has order dividing n. Consequently, $\mathrm{Br}(k)$ is torsion.*

Definition 3.25. Let A be a central-simple k-algebra. Then the *period* of A, written $\mathrm{per}(A)$, is the order of $[A]$ in the Brauer group $\mathrm{Br}(k)$.

The relationship between these two invariants is given by the following theorem.

Theorem 3.26 (Brauer)**.** *Let A be a central-simple k-algebra. Then*

(a) The period $\mathrm{per}(A)$ divides the index $\mathrm{ind}(A)$.

(b) The period $\mathrm{per}(A)$ and the index $\mathrm{ind}(A)$ have the same prime factors.

In general, however, $\mathrm{per}(A)$ need not equal $\mathrm{ind}(A)$. This is the famous *period-index problem*. For $k = \mathbb{Q}$ or any number field, or any completion of the aforementioned, the problem has been resolved by Albert–Brauer–Hasse–Noether.

Theorem 3.27 (see Section 18.4 of [11])**.** *Let k be a local or global field. Then for all classes $A \in \mathrm{Br}(k)$, the period and index of A coincide.*

Recalling Theorem 3.11, we have the following stronger statement about $\mathrm{Br}(k)[2]$ in these cases.

Proposition 3.28. *Let k be a local or global field. Then every class $A \in \mathrm{Br}(k)[2]$ can be represented by a quaternion algebra.*

Proof. See the discussion in [15] and [16]. □

The interplay between period and index will be an important feature of our discussion of geometric Brauer equivalence in Section 4.3.

Chapter 4

Brauer-Severi Varieties

We follow Chapter 5 of [8].

4.1 Basic Properties

Definition 4.1. A *Brauer-Severi variety* is a projective variety X over a field k such that the base extension $X_K := X \times_k K$ is isomorphic to $\mathbb{P}_K^{n-1}$ (where $n - 1 = \dim X$) for some finite field extension K/k, i.e. X is a twisted form of $\mathbb{P}^{n-1}$. The field K is said to *split* X.

Proposition 4.2. *A k-variety X is a Brauer-Severi variety if and only if $X_{\bar{k}} \cong \mathbb{P}_{\bar{k}}^{n-1}$ for some n.*

Proof. Necessity is clear, and sufficiency follows from the fact that a morphism $X_{\bar{k}} \to \mathbb{P}_{\bar{k}}^{n-1}$ is defined by finitely many polynomials. The coefficients of these polynomials give a finite extension K/k. □

The easiest nontrivial example of a Brauer-Severi variety is a conic. By Theorem 3.12, K splits a conic if and only if the conic has a K-rational point. This property generalizes to all Brauer-Severi varieties.

Theorem 4.3 (Châtelet)**.** *Let X be a Brauer-Severi variety of dimension $n - 1$ over k. The following are equivalent.*

(1) X is isomorphic to $\mathbb{P}_k^{n-1}$ over k

(2) X has a k-rational point.

4.2 Classification by Galois cohomology

By Theorem 2.12, we can assign to each twist X of $\mathbb{P}_k^{n-1}$ a class in

$$H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(\mathbb{P}_k^{n-1})) = H^1(\mathrm{Gal}(K/k), \mathrm{PGL}_n(K)),$$

where K is a Galois splitting field of X. Let $G = \mathrm{Gal}(K/k)$. If Y, Z are k-varieties, then we get a natural action on the K-morphisms $\phi\colon Y_K \to Z_K$ given by $\sigma(\phi) = \sigma \circ \phi \circ \sigma^{-1}$. Given an isomorphism $\phi\colon \mathbb{P}^{n-1}_K \to X_K$, for $\sigma \in G$ we define an automorphism $a_\sigma \in \mathrm{Aut}_K(\mathbb{P}^{n-1}_K) = \mathrm{PGL}_n(K)$ by

$$a_\sigma = \phi^{-1} \circ \sigma(\phi).$$

This construction is identical to the one in Lemma 2.8. Direct computation shows that $a\colon G \to \mathrm{Aut}_K(\mathbb{P}^{n-1}_K)$ is a 1-cocycle. Furthermore, choosing a different isomorphism $\mathbb{P}^{n-1}_K \to X_K$ gives a cocycle cohomologous to a. Thus we have assigned to X a class

$$[a] \in H^1(G, \mathrm{Aut}_K(\mathbb{P}^{n-1}_K)).$$

Fixing an isomorphism $\mathrm{Aut}_K(\mathbb{P}^{n-1}_K) \cong \mathrm{PGL}_n(K)$ then gives a corresponding cocycle $[a] \in H^1(G, \mathrm{PGL}_n(K))$. By Theorem 3.2, this yields the following result.

Theorem 4.4. *The association $X \mapsto [a]$ described above induces a base point preserving bijection between isomorphism classes of Brauer-Severi k-varieties of dimension $n-1$ and isomorphism classes of central simple k-algebras of dimension n^2.*

Given a central simple k-algebra A, we shall refer to the corresponding Brauer-Severi variety (or rather its isomorphism class) as the *Brauer-Severi variety associated to A*.

The exact sequence

$$1 \longrightarrow K^* \longrightarrow \mathrm{GL}_n(K) \longrightarrow \mathrm{PGL}_n(K) \longrightarrow 1.$$

yields a corresponding long exact sequence of Galois-cohomology, a segment of which is

$$H^1(G, \mathrm{GL}_n(K)) \longrightarrow H^1(G, \mathrm{PGL}_n(K)) \longrightarrow H^2(G, K^*) \cong \mathrm{Br}(K/k).$$

The second map is injective since $H^1(G, \mathrm{GL}_n(K)) = 0$ by Hilbert 90 (Theorem 2.11). Thus we can assign to each X a distinct class $[X]$ in the Brauer group $\mathrm{Br}(K/k)$

Theorem 4.5 (see Theorem 5.2.2 of [8])**.** *Let X be a Brauer-Severi k-variety of dimension $n-1$ over k, and let d be the period of X (the order of $[X] \in \mathrm{Br}(K/k)$. Then there exists a projective embedding*

$$\rho_K\colon X \hookrightarrow \mathbb{P}^{N-1}_k, \;\; N = \binom{n+d-1}{d}$$

such that $\rho_K\colon X_K \hookrightarrow \mathbb{P}^{N-1}_K$ is isomorphic to the d-tuple embedding.

4.3 Geometric Brauer equivalence

In the previous section, we saw that Brauer-Severi varieties are naturally identified with central simple k-algebras (see Theorem 4.4). One may wonder what "Brauer-equivalence" of Brauer-Severi varieties may mean geometrically. In this section, we present some well-known results in that direction. Before we do so, we need the notion of a *twisted-linear* subvariety.

Definition 4.6. Let X be a Brauer-Severi variety over k of dimension $n-1$. We say that a closed subvariety $Y \subset X$ defined over k is a *twisted-linear* subvariety of X if Y is a Brauer-Severi variety, and moreover under a $\overline{k}$-isomorphism $X_{\overline{k}} \to \mathbb{P}^{n-1}_{\overline{k}}$ the subvariety $Y_{\overline{k}}$ is mapped to a linear subvariety of $\mathbb{P}^{n-1}_{\overline{k}}$.

Lemma 4.7. *The above notion is well-defined.*

Proof. Any automorphism of $\mathbb{P}^{n-1}_{\overline{k}}$ sends linear subvarieties to linear subvarieties, hence different isomorphisms from $X_{\overline{k}}$ to $\mathbb{P}^{n-1}_{\overline{k}}$ define the same twisted-linear subvarieties of X. $\square$

The following two results may be found in Section 5.3 of [8].

Theorem 4.8. *Let X be a Brauer-Severi k-variety, and Y a twisted linear subvariety of X. Then X and Y have the same class in* $\mathrm{Br}(k)$.

Very important for us will be the following result, which concerns the *minimal* twisted-linear subvarieties of a Brauer-Severi variety, meaning the twisted-linear subvarieties which contain no smaller twisted-linear subvarieties.

Theorem 4.9. *Let X be a Brauer-Severi variety, and A the associated CSA. Then the minimal twisted-linear subvarieties of X all have dimension d, satisfying*

$$d = \mathrm{ind}(A) - 1.$$

Chapter 5

Curves of genus 2

This chapter contains some background on genus 2 curves. The material in the first 3 sections may be found in the first few chapters of [3].

5.1 Nonsingular models

Suppose k is a field with char$k \neq 2$. We shall be concerned with genus 2 curves that have an affine model of the form

$$\mathcal{C}\colon y^2 = f(x) := f_0 + f_1 x + \cdots + f_6 x^6$$

where $f(x) \in k[x]$ is squarefree. Every genus 2 curve is birationally equivalent over k to a curve of this type. A model of this form is not complete, but if we try to complete it by replacing x, y with $x/z, y/z^3$ then we get a singularity at $z = 0$. Instead, for our purposes we consider the weighted projective model

$$\mathcal{C}\colon y^2 = f(x, z) := f_0 z^6 + f_1 z^5 x + \cdots + f_6 x^6.$$

Viewing $V(f) \subset \mathbb{P}^1$, we take the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ given by $(x : z) \mapsto (x^3 : x^2 z : x z^2 : z^3)$. This yields a complete, nonsingular model for $\mathcal{C}$ in $\mathbb{P}^4$ with coordinates x_0, x_1, x_2, x_3 and y given by

$$y^2 = Q_4,\ Q_1 = Q_2 = Q_3 = 0$$

where

$$\begin{aligned}
Q_1 &= x_0 x_2 - x_1^2 \\
Q_2 &= x_0 x_3 - x_1 x_2 \\
Q_3 &= x_1 x_3 - x_2^2 \\
Q_4 &= f_0 x_0^2 + f_1 x_0 x_1 + f_2 x_1^2 + f_3 x_1 x_2 + f_4 x_2^2 + f_5 x_2 x_3 + f_6 x_3^2.
\end{aligned}$$

Notice that

$$L\colon Q_1 = Q_2 = Q_3 = 0$$

is the image of the Veronese embedding.

5.2 The Jacobian $\mathrm{Jac}(\mathcal{C})$ and the associated Kummer surface

The *Jacobian variety of* $\mathcal{C}$, denoted $\mathrm{Jac}(\mathcal{C})$ is an abelian surface that represents $\mathrm{Pic}^0(\mathcal{C})$. In Chapter 2 of [3] there is an explicit construction of the Jacobian. Geometrically, $\mathrm{Jac}(\mathcal{C})$ is obtained by considering the symmetric square $\mathcal{C}^{(2)}$ and blowing down the line corresponding to the canonical class $[\kappa] \in \mathrm{Pic}^2(\mathcal{C})$. The result is an intersection of 72 quadrics in $\mathbb{P}^{15}$. This description is cumbersome to work with. Instead, it is useful to consider a much simpler object, the *Kummer surface* $\mathcal{K}_{\mathcal{C}}$ of $\mathcal{C}$, which contains a lot of the relevant information of $\mathrm{Jac}(\mathcal{C})$. The Kummer surface is given by $\mathcal{K}_{\mathcal{C}} = \mathrm{Jac}(\mathcal{C})/\{\pm 1\}$, and has a model in $\mathbb{P}^3$ as a quartic surface. Over $\overline{k}$, the surface $\mathcal{K}_{\mathcal{C}}$ has 16 nodal singularities, which are the image of $\mathrm{Jac}(\mathcal{C})[2]$.

5.3 The dual Kummer surface

The projective dual $\mathcal{K}_{\mathcal{C}}^*$ of $\mathcal{K}_{\mathcal{C}}$ will be important for describing the moduli interpretation of the Burkhardt quartic. Classically, if k is algebraically closed, then it turns out that $\mathcal{K}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}}^*$ are isomorphic. The 16 nodes of $\mathcal{K}_{\mathcal{C}}$ are dual to 16 planes, called *tropes*, each of which meets $\mathcal{K}_{\mathcal{C}}^*$ in a conic containing 6 nodes.

We now follow Section 6 of [1]. We use the quadrics Q_i in Section 5.1 defining $\mathcal{C}$ to give a description of $\mathcal{K}_{\mathcal{C}}^*$. First, consider

$$\mathcal{W}_{\mathcal{C}} = V\left(\det\left(\frac{\partial Q_i}{\partial x_j}\right)_{0\le i,j\le 4}\right) \subset \mathbb{P}^3,$$

which is classically known as a *Weddle surface*. Now, let $M(Q_i) \in k^{4\times 4}$ be the 4×4 symmetric matrix associated to the quadratic form Q_i. A model for the dual Kummer surface of $\mathcal{C}$ is given by

$$\mathcal{K}_{\mathcal{C}}^* = V\left(\det\left(\sum_{i=1}^{4} \eta_i M(Q_i)\right)\right)$$

with coordinates $(\eta_1, ..., \eta_4)$. The surfaces $\mathcal{W}$ and $\mathcal{K}_{\mathcal{C}}^*$ are birational, the map being given by the relation

$$\frac{\partial}{\partial x_j}\left(\mathbf{x}\left(\sum_{i=1}^{4}\eta_i M(Q_i)\right)\mathbf{x}^T\right)\Big|_P = 0 \quad \text{for } j = 1, ..., 4.$$

for $P \in \mathcal{W}_C$. The composition $\mathbb{P}^1 \to \mathcal{W}_C \to \mathcal{K}_{\mathcal{C}}^*$, where $\mathbb{P}^1 \to \mathcal{W}_C \subset \mathbb{P}^3$ comes from the Veronese embedding, is given by $(x : z) \mapsto (z^2 : -xz : x^2 : 0)$. The image turns out to be

$$\eta_4 = \eta_1\eta_3 - \eta_2^2 = 0,$$

5.4 Geometric Kummer surfaces and the obstruction

We generally try to follow the terminology from [1], which is not entirely standard but convenient for our particular application. Suppose we are given the surface $\mathcal{K}_{\mathcal{C}}$. In this section, we discuss how to reconstruct the curve $\mathcal{C}$ if k is algebraically closed, and what goes wrong if k is a general field.

Definition 5.1. A *geometric Kummer surface* is a quartic surface in $\mathbb{P}^3$ with 16 nodal singularities.

As stated in section 5.2, a Kummer surface $\mathcal{K}_{\mathcal{C}}$ coming from a genus 2 curve $\mathcal{C}$ is a geometric Kummer surface, the 16 nodes are the image of $\mathrm{Jac}(\mathcal{C})[2]$. Note that the image of the identity element of $\mathrm{Jac}(\mathcal{C})$ maps to a node of $\mathcal{K}_{\mathcal{C}}$, so $\mathcal{K}_{\mathcal{C}}$ comes with a distinguished node. This motivates the following definition.

Definition 5.2. A *Kummer surface over k* is a geometric Kummer surface defined over k, with a marked node.

As mentioned in Section 5.3, a node of $\mathcal{K}_{\mathcal{C}}$ corresponds to a trope of $\mathcal{K}_{\mathcal{C}}^*$, which is a plane in $\mathbb{P}^3$ that contains 6 nodes of $\mathcal{K}_{\mathcal{C}}^*$.

Definition 5.3. A *dual Kummer surface over k* is a geometric Kummer surface with a marked trope.

We now come to the main point of the section. Suppose we are given a Kummer surface $\mathcal{K}$ over k. This gives rise to a dual Kummer surface $\mathcal{K}^*$ over k, i.e. a geometric Kummer surface with a marked trope over k. The marked trope contains 6 nodes of $\mathcal{K}^*$, determining a conic L. If L has a k-rational point, which is guaranteed if k is algebraically closed, then $L \cong \mathbb{P}^1$. Choosing a k-isomorphism $L \cong \mathbb{P}^1$, the 6 nodes determine a degree 6 polynomial $f(x) \in k[x]$ with those 6 roots up to scaling. This detail is important, and we will expound on it in Section 5.6. The important result for this section is the following.

Theorem 5.4 (see Section B.5.1 of [9])**.** *Let $\mathcal{C}$ be the genus 2 curve $y^2 = f(x)$, where $f(x)$ is obtained from a Kummer surface $\mathcal{K}$ over k as described above. Then $\mathcal{K}_{\mathcal{C}} = \mathcal{K}$.*

If the conic L does not have k-rational points, then we have an obstruction to constructing an associated genus 2 curve.

Definition 5.5. Let $\mathcal{K}$ be a Kummer surface over k, and L the associated conic on $\mathcal{K}^*$. We define the *obstruction of* $\mathcal{K}$ as the Brauer class of the conic L. We denote this obstruction by $\mathrm{Ob}(\mathcal{K}) \in \mathrm{Br}(k)[2]$.

A consequence of the previous theorem is the following characterization of $\mathrm{Ob}(\mathcal{K})$, see Section B.5.1 of [9] for details.

Proposition 5.6. *Let $\mathcal{K}$ be a Kummer surface over k. Then $\mathrm{Ob}(\mathcal{K})$ is trivial if and only if $\mathcal{K}$ is the Kummer surface of a genus 2 curve $\mathcal{C}$ defined over k, i.e., $\mathcal{K} = J/\langle -1 \rangle$ for some Jacobian variety J defined over k.*

In summary, we see that 6 points on a conic defined over k uniquely determine a Kummer surface which, as it turns out, is defined over k. Thus a Kummer surface may be specified by a cubic curve intersecting a conic in 6 points, this is how we will obtain Kummer surfaces from points on the Burkhardt quartic in Section 6.1. The 6 points also determine a corresponding genus 2 curve over k up to quadratic twist (where the quadratic twists come from different scalings of the polynomial $f(x) \in k[x]$) if and only if the conic has a k-rational point. We state this in the following proposition.

Proposition 5.7 (see Chapter 4 of [3])**.** *Let L be a conic defined over k, and let $\mathcal{D}$ be a cubic curve defined over k such that $\mathcal{D}$ intersects L in 6 distinct (geometric) points. The pair $(L, \mathcal{D})$ determines a Kummer surface over k. Furthermore, the pair $(L, \mathcal{D})$ determines a genus 2 curve over k (up to quadratic twist) if and only if L has a k-rational point.*

5.5 Full level-3 structures

We are interested in curves $\mathcal{C}$ of genus 2 together with a certain structure on the 3-torsion subgroup of $\mathrm{Jac}(\mathcal{C})$. A well-known result is the following.

Theorem 5.8 (see Section 1.2 of [9])**.** *There is an isomorphism of groups*

$$\mathrm{Jac}(\mathcal{C})[3] \cong (\mathbb{Z}/3\mathbb{Z})^4.$$

We are, however, interested in $\mathrm{Jac}(\mathcal{C})[3]$ as a variety over k. As such, it has some Galois structure. For motivation, we briefly recall how to describe the 3-torsion on an elliptic curve. Let

$$\mathcal{E} \colon y^2 = x^3 + ax + b$$

be an elliptic curve over k. The 3-torsion points on $\mathcal{E}$ are exactly those points $(x, y) \in \mathcal{E}$ such that

$$3x^4 + 6ax^2 + 12bx - a^2 = 0.$$

The equation defining $\mathcal{E}$ and the equation above define an octic algebra over $\mathbb{Q}$. After some manipulation (see e.g. [2]) this algebra can be expressed as

$$\mathbb{Q}[z]/(F(z)) \quad \text{where} \quad F(z) = z^8 + 18az^4 + 108bz^2 - 27a^2,$$

thus giving an easy description of the action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathcal{E}[3]$. As abstract groups $\mathcal{E}[3] \cong (\mathbb{Z}/3\mathbb{Z})^2$. The Weil-pairing on $\mathcal{E}$ restricts to a symplectic form on $\mathcal{E}[3]$. The standard symplectic form on $(\mathbb{Z}/3\mathbb{Z})^2$ is given by the following matrix

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We want to regard the space $\mathcal{E}[3]$ as a symplectic space with respect to the Weil pairing. Thus the automorphism group of $\mathcal{E}[3]$ is isomorphic to the symplectic group

$$\mathrm{Sp}_2(\mathbb{F}_3) = \{P \in \mathrm{GL}_2(\mathbb{F}_3) \mid PWP^T = W\}$$

which preserves the symplectic pairing on $(\mathbb{Z}/3\mathbb{Z})^2$. It happens to be the case that $\mathrm{Sp}_2(k) = \mathrm{SL}_2(k)$, but for higher dimensional matrix groups, this is no longer the case. If we fix a symplectic isomorphism $\mathcal{E}[3] \cong (\mathbb{Z}/3\mathbb{Z})^2$, and if we assume that k contains the cube roots of unity (we will expand on this detail later in the section) we get an induced Galois representation $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Sp}_2(\mathbb{F}_3)$. For another elliptic curve $\mathcal{E}'$ over k, an isomorphism $\mathcal{E}[3] \cong \mathcal{E}'[3]$ is then an isomorphism of groups which is compatible with the Galois representation structure. If we fix $\mathcal{E}$, then the moduli space of pairs $(\mathcal{E}', \iota)$, where $\iota \colon \mathcal{E}[3] \cong \mathcal{E}'[3]$ is an isomorphism, has an explicit description as the complement of 4 geometric points in $\mathbb{P}^1$. This result is Theorem 1 of [2]. With all of this in mind, we now move on to the 3-torsion structure of genus 2 curves, and the resulting structure on the associated Kummer surface.

The following definition is from Section 7 of [1].

Definition 5.9. A *full level-3 structure*, for us, is a group scheme Σ that over $\overline{k}$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$, and is equipped with a symplectic pairing $\Sigma \times \Sigma \to \mu_3$. An *abelian surface A with full level-3 structure* is a principally polarized abelian surface A together with an embedding $\Sigma \to A$, such that the bilinear pairing on Σ is compatible with the Weil pairing on $A[3]$.

For background on principally polarized abelian surfaces see [5]. Jacobians of genus 2 curves are principally polarized abelian surfaces, and these are the only examples that we will be concerned with in this thesis.

Remark 5.10. A full level-3 structure on an abelian surface A is thus specified by a choice of symplectic isomorphism

$$A[3] \cong (\mathbb{Z}/3\mathbb{Z})^4,$$

where the symplectic form on $(\mathbb{Z}/3\mathbb{Z})^4$ is given by the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The automorphism group of a full level-3 structure, in the sense of symplectic spaces, is $\mathrm{Sp}_4(\mathbb{F}_3)$, which is given by

$$\mathrm{Sp}_4(\mathbb{F}_3) = \{P \in \mathrm{GL}_4(\mathbb{F}_3) \mid PMP^T = M\} \subset \mathrm{GL}_4(\mathbb{F}_3).$$

This is a group of order 51840. Geometrically, there is only one full level-3 structure available.

Theorem 5.11 (see [9])**.** *Over $\overline{k}$, every full level-3 structure is isomorphic to*

$$\Sigma = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$$

where the pairing $\Sigma \times \Sigma \to \mu_3$ is given by viewing $(\mu_3)^2$ as the Cartier dual of $(\mathbb{Z}/3\mathbb{Z})^2$.

Over non-algebraically closed fields k, abelian surfaces may have non-isomorphic full level-3 structures. The points of $A[3]$ may not all be k-rational points, thus giving a nontrivial action of $\mathrm{Gal}(\overline{k}/k)$ on $\Sigma \cong A[3]$. The Weil pairing on $A[3]$ is $\mathrm{Gal}(\overline{k}/k)$-covariant, giving that the Weil pairing is preserved by $\mathrm{Gal}(\overline{k}/k)$ up to an automorphism of the target μ_3. Thus we get a Galois representation

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{GSp}_4(\mathbb{F}_3) := \{P \in \mathrm{GL}_4(\mathbb{F}_3) \mid PMP^T = \pm M\} \cong \mathrm{Sp}_4(\mathbb{F}_3) \rtimes C_2,$$

which restricts to a representation $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Sp}_4(\mathbb{F}_3)$ if k contains the cube roots of unity.

Remark 5.12. Let Σ be a full level-3 structure over k. Regardless of whether or not k contains the cube roots of unity, the associated Galois representation $\mathrm{Gal}(\overline{k}/k) \to \mathrm{GSp}_4(\mathbb{F}_3)$ makes $\mathrm{Sp}_4(\mathbb{F}_3)$ into a group with an action of $\mathrm{Gal}(\overline{k}/k)$. Thus we can make sense of the object $H^1(k, \mathrm{Sp}_4(\mathbb{F}_3))$, which by Theorem 2.12 classifies the twists of Σ over k. Take $\Sigma = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$. The cohomology set $H^1(k, \mathrm{Sp}_4(\mathbb{F}_3))$ classifies all other full level-3 structures over k, as by Theorem 5.11 they are all twisted forms of Σ.

Remark 5.13. If k does not contain a cube root of unity, then the image of an automorphism $\sigma \in \mathrm{Gal}(\overline{k}/k)$ which interchanges ζ_3 (a primitive cube root of unity) and ζ_3^2 is the following matrix

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{F}_3),$$

which doesn't preserve the Weil pairing but inverts it.

In the case where A is the Jacobian of a genus 2 curve, a full level-3 structure on A naturally gives rise to the notion of a full level-3 structure for the associated Kummer surface. We write $\mathbb{P}\Sigma = \Sigma/\langle -1 \rangle$, meaning $\mathbb{P}\Sigma$ is Σ modulo the negation automorphism. This is a degree 41 separated 0-dimensional scheme. If we identify $\Sigma(\overline{k})$ with the additive group of the vector space $\mathbb{F}_3^4$, we see that $\mathbb{P}\Sigma(\overline{k})$ corresponds to the identity element and the 40 order three subgroups of $\Sigma(\overline{k})$. Let $A = \mathrm{Jac}(\mathcal{C})$ and suppose that we have a full level-3 structure $\Sigma \to A[3]$. If we compose with the map $A \to A/\langle -1 \rangle = \mathcal{K}_{\mathcal{C}}$, we get an injective morphism $\mathbb{P}\Sigma \to \mathcal{K}_{\mathcal{C}}$. In other words, $\mathbb{P}\Sigma$ fits into the commutative diagram

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[d] & \mathrm{Jac}(\mathcal{C}) \arrow[d] \\ \mathbb{P}\Sigma \arrow[r] & \mathcal{K}_{\mathcal{C}}. \end{tikzcd}$$

We call $\mathbb{P}\Sigma \to \mathcal{K}$ a *full level-3 structure on a Kummer surface*. We call $\mathbb{P}\Sigma$ itself a *Kummer full level-3 structure*.

Remark 5.14. One can describe a level-3 structure on a Kummer surface $\mathcal{K}_{\mathcal{C}}$ over k in a different way as well. Note that for the multiplication-by-3 map on $\mathrm{Jac}(\mathcal{C})$, we have that $3(-D) = -(3D)$, so multiplication-by-3 descends to a morphism $\mathcal{K}_{\mathcal{C}} \to \mathcal{K}_{\mathcal{C}}$. A full level-3 structure on $\mathcal{K}_{\mathcal{C}}$ is a labelling of the points of the fibre over the marked node of $\mathcal{K}_{\mathcal{C}}$.

By a slight abuse of notation, we will denote a Kummer full level-3 structure by $\mathbb{P}\Sigma$, with the understanding that a corresponding lift Σ may not exist over k (it is only guaranteed to exist over $\overline{k}$).

Remark 5.15. Similarly to the case of a full level-3 structure over k on an abelian surface, a full level-3 structure $\mathbb{P}\Sigma$ on a Kummer surface induces a Galois representation

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{PGSp}_4(\mathbb{F}_3)$$

which restricts to a representation

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{PSp}_4(\mathbb{F}_3)$$

if and only if k contains a cube root of unity. Either way, this makes $\mathrm{PSp}_4(\mathbb{F}_3)$ into a group with $\mathrm{Gal}(\overline{k}/k)$-action. The relationship between $\mathrm{Sp}_4(\mathbb{F}_3)$ and $\mathrm{PSp}_4(\mathbb{F}_3)$ is given by the following exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Sp}_4(\mathbb{F}_3) \longrightarrow \mathrm{PSp}_4(\mathbb{F}_3) \longrightarrow 1. \tag{5.1}$$

Analogously to Remark 5.12, we take the standard full level-3 structure $\Sigma = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$, and the corresponding Kummer full level-3 structure $\mathbb{P}\Sigma$. Taking Galois cohomology of the exact sequence (5.1) gives the following long exact sequence of cohomology

$$H^1(k, \mu_2) \longrightarrow H^1(k, \mathrm{Sp}_4(\mathbb{F}_3)) \longrightarrow H^1(k, \mathrm{PSp}_4(\mathbb{F}_3)) \longrightarrow H^2(k, \mu_2). \tag{5.2}$$

The cohomology set $H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$ classifies all Kummer full level-3 structures over k, as by Theorem 5.11 they are all twisted forms of $\mathbb{P}\Sigma$.

Let $\mathbb{P}\Sigma$ be any Kummer full level-3 structure over k. Remark 5.15 allows us to view $\mathbb{P}\Sigma$ as a cocycle $\xi \in H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$. In view of the exact sequence 5.2, define $\mathrm{Ob}(\mathbb{P}\Sigma) \in \mathrm{Br}(k)[2]$ as the image of ξ in $H^2(k, \mu_2) \cong \mathrm{Br}(k)[2]$. Notice that the exactness of (5.2) implies that $\mathrm{Ob}(\mathbb{P}\Sigma)$ is trivial if and only if ξ lifts to a cocycle which takes values in $\mathrm{Sp}_4(\mathbb{F}_3)$. In particular, we get the following interpretation of $\mathrm{Ob}(\mathbb{P}\Sigma)$.

Proposition 5.16. *Let $\mathbb{P}\Sigma$ be a Kummer full level-3 structure over k. Then* $\mathrm{Ob}(\mathbb{P}\Sigma)$ *is trivial if and only if there exists a full level-3 structure Σ' defined over k such that* $\mathbb{P}\Sigma' = \mathbb{P}\Sigma$.

5.6 Moduli of full level-3 structures

Let $\mathcal{K}$ be a Kummer surface over k with full level-3 structure $\mathbb{P}\Sigma$. We have seen two seemingly different obstructions, namely $\mathrm{Ob}(\mathcal{K})$ and $\mathrm{Ob}(\mathbb{P}\Sigma)$. We prove that these are in fact the same.

Theorem 5.17. *Let $\mathcal{K}$ be a Kummer surface over k with full level-3 structure $\mathbb{P}\Sigma$. Then* $\mathrm{Ob}(\mathcal{K}) = \mathrm{Ob}(\mathbb{P}\Sigma)$.

In other words, the obstruction to $\mathcal{K}$ being the quotient of a Jacobian variety defined over k is completely determined by the full level-3 structure on $\mathcal{K}$. To prove this theorem we need a general result about abelian varieties.

Theorem 5.18. *Every automorphism of A acts faithfully on* $A[3]$.

Proof. see Proposition 17.5 of [10]. □

Proof of Theorem 5.17. By Corollary 3.21, it suffices to show that if $\mathrm{Ob}(\mathbb{P}\Sigma)$ is trivial then so is $\mathrm{Ob}(\mathcal{K})$; the converse holds because $\mathbb{P}\Sigma$ is a subscheme of $\mathcal{K}$. By Proposition 5.16, the class $\mathrm{Ob}(\mathbb{P}\Sigma)$ is trivial if and only if there is a corresponding full level-3 structure Σ over k, so assume such a Σ exists. By the discussion in Section 5.4, we can specify $\mathcal{K}$ by a conic L over k with 6 marked points. Let K/k be a quadratic extension such that L has a K-rational point, and write $\mathrm{Gal}(K/k) = \langle \sigma \rangle$. Then there is an abelian surface A_K, defined over K, with full level-3 structure Σ such that $\mathcal{K} = A_K/\langle -1 \rangle$. Also, since $\mathcal{K}$ is completely determined by L with 6 marked points (Proposition 5.7), we must have $\mathcal{K} = A_K^\sigma/\langle -1 \rangle$. We are then guaranteed a K-isomorphism $\phi_\sigma \colon A_K \to A_K^\sigma$ which preserves Σ and makes the diagram

$$\begin{tikzcd} A_K \arrow[rr, dotted, "\phi_\sigma"] \arrow[dr] & & A_K^\sigma \arrow[dl] \\ & \mathcal{K} & \end{tikzcd}$$

commute. By Theorem 5.18, a K-isomorphism $A_K \to A_K^\sigma$ of abelian surfaces which preserves Σ must be unique. The uniqueness guarantees that conditions (i) and (ii) in Section 2.4 are satisfied. Therefore, by Theorem 2.18, there is a model for A_K over k. $\square$

For a full level-3 structure Σ over k, let $\mathcal{M}_2(\Sigma)$ be the moduli space of abelian surfaces $A = \mathrm{Jac}(\mathcal{C})$ with full level-3 structure Σ. For an explicit description of this space, see Theorem 2 of [2]. The following result is a direct consequence of Theorem 5.17.

Theorem 5.19. *Let Σ be a full level-3 structure over k. Then the k-rational points of $\mathcal{M}_2(\Sigma)$ correspond to abelian surfaces defined over k with full level-3 structure Σ, i.e. there is no field-of-definition versus field-of-moduli obstruction.*

We also have the associated moduli space $\mathrm{Kum}(\mathbb{P}\Sigma)$ of Kummer surfaces with full level-3 structure $\mathbb{P}\Sigma$ (see Section 5.3.3 of [9]). There is a natural map $\mathcal{M}_2(\Sigma) \to \mathrm{Kum}(\mathbb{P}\Sigma)$. Conversely, however, by Proposition 5.7 a point of $\mathrm{Kum}(\mathbb{P}\Sigma)$ only determines a genus 2 curve $\mathcal{C}$ up to quadratic twist. For $d \in k^*/k^{*2}$, denote $\mathcal{C}^{(d)}$ to be the corresponding quadratic twist. This gives rise to the notion of a *quadratic twist of the full-level 3 structure* Σ, denoted $\Sigma^{(d)}$, which comes from the geometric isomorphism $\mathrm{Jac}(\mathcal{C}) \cong_{\overline{k}} \mathrm{Jac}(\mathcal{C}^{(d)})$. Now, notice that $\mathbb{P}\Sigma = \mathbb{P}\Sigma^{(d)}$, as $\mathcal{C}$ and $\mathcal{C}^{(d)}$ certainly have the same Kummer surface over k. We thus get a k-isomorphism

$$\mathcal{M}_2(\Sigma) \cong \mathcal{M}_2(\Sigma^{(d)})$$

induced by the diagram

$$\begin{tikzcd} \mathcal{M}_2(\Sigma) \arrow[rr, dotted] \arrow[dr] & & \mathcal{M}_2(\Sigma^{(d)}) \\ & \mathrm{Kum}(\mathbb{P}\Sigma) \arrow[ur] & \end{tikzcd}$$

where the map $\mathrm{Kum}(\mathbb{P}\Sigma) \to \mathcal{M}_2(\Sigma^{(d)})$ is given by an appropriate lift of a Kummer surface over k to a Jacobian over k which lies above it (recall Section 5.4).

Remark 5.20. The fact that $\mathbb{P}\Sigma$ determines Σ up to quadratic twist can be seen at the level of cohomology. By (3.2), we have $H^1(k, \mu_2) \cong k^*/k^{*2}$. The map $H^1(k, \mu_2) \to H^1(k, \mathrm{Sp}_4(\mathbb{F}_3))$ in (5.2) is injective, and the image gives a distinguished subset: the cocycles corresponding to quadratic twists. The exactness of (5.2) implies that $H^1(k, \mu_2)$ is the kernel of $H^1(k, \mathrm{Sp}_4(\mathbb{F}_3)) \to H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$.

Chapter 6

The Burkhardt quartic threefold

6.1 The moduli interpretation

The theory in this section is found in [1]. A particular model for the Burkhardt quartic in $\mathbb{P}_k^4$ over a field k is given by

$$\mathcal{B}\colon B(y_0, y_1, y_2, y_3, y_4) := y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

Over $\overline{k}$, a Zariski-open part of $\mathcal{B}$ parametrizes genus 2 curves with full level-3 structure on their Jacobian. This open part is given by $\mathcal{B}\backslash\mathrm{He}(\mathcal{B})$, where

$$\mathrm{He}(\mathcal{B}) = V\left(\det\left(\frac{\partial^2 B}{\partial y_i \partial y_j}\right)_{0\le i,j\le 4}\right)$$

is the Hessian of $\mathcal{B}$. We describe how to explicitly obtain a genus 2 curve from a given point $\alpha = (\alpha_0 : \cdots : \alpha_4) \in (\mathcal{B}\backslash\mathrm{He}(\mathcal{B}))(\overline{k})$. We define the polars $P_\alpha^{(1)}, P_\alpha^{(2)}, P_\alpha^{(3)}$ of $\mathcal{B}$ at α. These are classical objects in algebraic geometry (see Section 1.1 of [6]) , and in a geometric sense they are “tangent hypersurfaces” to the point α on $\mathcal{B}$. The polars of $\mathcal{B}$ at α are given by the following equations:

$$\begin{aligned}
P_\alpha^{(1)}\colon\ & (4y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3)\alpha_0 + (3y_0y_1^2 + 3y_2y_3y_4)\alpha_1 + (3y_0y_2^2 + 3y_1y_3y_4)\alpha_2 \\
& + (3y_0y_3^2 + 3y_1y_2y_4)\alpha_3 + (3y_0y_4^2 + 3y_1y_2y_3)\alpha_4 = 0, \\
P_\alpha^{(2)}\colon\ & 2\alpha_0^2y_0^2 + \alpha_1^2y_0y_1 + \alpha_2^2y_0y_2 + \alpha_3^2y_0y_3 + \alpha_4^2y_0y_4 + \alpha_0\alpha_1y_1^2 + \alpha_3\alpha_4y_1y_2 + \alpha_2\alpha_4y_1y_3 \\
& + \alpha_2\alpha_3y_1y_4 + \alpha_0\alpha_2y_2^2 + \alpha_1\alpha_4y_2y_3 + \alpha_1\alpha_3y_2y_4 + \alpha_0\alpha_3y_3^2 + \alpha_1\alpha_2y_3y_4 + \alpha_0\alpha_4y_4^2 = 0, \\
P_\alpha^{(3)}\colon\ & (4\alpha_0^3 + \alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3)y_0 + (3\alpha_0\alpha_1^2 + 3\alpha_2\alpha_3\alpha_4)y_1 + (3\alpha_0\alpha_2^2 + 3\alpha_1\alpha_3\alpha_4)y_2 \\
& + (3\alpha_0\alpha_3^2 + 3\alpha_1\alpha_2\alpha_4)y_3 + (3\alpha_0\alpha_4^2 + 3\alpha_1\alpha_2\alpha_3)y_4 = 0.
\end{aligned}$$

Note that $P_\alpha^{(3)}$ is simply the tangent space of $\mathcal{B}$ at α. A further geometric observation is that $P_\alpha^{(2)} \cap P_\alpha^{(3)}$ is a cone in 3-dimensional space (the linear equation for $P_\alpha^{(3)}$ allows us to eliminate a variable) with vertex α . Now, let π_α be a projection from α onto any chosen hyperplane not containing α. From the above construction, we obtain a plane conic $L_{\mathcal{B},\alpha}$ given by

$$L_{\mathcal{B},\alpha} = \pi_\alpha(P_\alpha^{(2)} \cap P_\alpha^{(3)}).$$

We then mark 6 points on $L_{\mathcal{B},\alpha}$, which are distinct by the discussion in Section 6 of [4], via projecting the third polar:

$$\pi_\alpha(P_\alpha^{(1)} \cap P_\alpha^{(2)} \cap P_\alpha^{(3)}).$$

This is the same as intersecting $L_{\mathcal{B},\alpha}$ with the cubic curve $\pi_\alpha(P_\alpha^{(1)} \cap P_\alpha^{(3)})$. By Proposition 5.7, this already determines a dual Kummer surface over k. In fact, the following is true.

Theorem 6.1 (see [1])**.** *The space $\mathcal{B}\backslash\mathrm{He}(\mathcal{B})$ is the moduli space of dual Kummer surfaces with full level-3 structure. Moreover, over k there is no field-of-definition versus field-of-moduli obstruction i.e. k-rational points of $\mathcal{B}\backslash\mathrm{He}(\mathcal{B})$ correspond to dual Kummer surfaces over k. This holds for any twist $\mathcal{B}'$ of $\mathcal{B}$ over k if $\mathcal{B}'$ is a quartic threefold in $\mathbb{P}^4$.*

The authors of [1] do not explicitly deal with the last statement concerning twists, but this statement will become apparent once we describe the rest of the construction.

Remark 6.2. As we shall show later (see Theorem 6.5), every twist $\mathcal{B}'$ of $\mathcal{B}$ can be realized as a quartic threefold in $\mathbb{P}^4$, so Theorem 6.1 holds for all twists of $\mathcal{B}$.

Proceeding as in the discussion before Proposition 5.7, if there is a k-rational point on $L_{\mathcal{B},\alpha}$, then $L_{\mathcal{B},\alpha} \cong \mathbb{P}^1_k$ via stereographic projection (so in particular $(L_{\mathcal{B},\alpha})_{\bar{k}} \cong \mathbb{P}^1_{\bar{k}}$). This gives us 6 points on $\mathbb{P}^1$. Taking f to be a polynomial with those 6 points as roots gives a corresponding curve $\mathcal{C}_\alpha\colon y^2 = f(x)$ up to quadratic twist, exactly as in Proposition 5.7.

Notice that the above construction works for any twist $\mathcal{B}'$ of $\mathcal{B}$, and thus for each point $\alpha \in \mathcal{B}'\backslash\mathrm{He}(\mathcal{B}')$ we get a corresponding conic $L_{\mathcal{B}',\alpha}$.

Theorem 6.3. *Let $\mathcal{B}'$ be a twist of $\mathcal{B}$. The isomorphism class of the conic $L_{\mathcal{B}',\alpha}$ does not depend on the choice of $\alpha \in \mathcal{B}'\backslash\mathrm{He}(\mathcal{B}')$.*

We postpone the proof of this statement until the end of the section. In essence, it follows quite directly from Theorem 5.19.

We now describe how to mark the full level-3 structure on $\mathrm{Jac}(\mathcal{C}_\alpha)$, following [1]. It turns out that the intersection of the Burkhardt quartic and its Hessian, the locus where the above moduli interpretation breaks down, is a union of 40 planes. We write

$$\mathcal{B} \cap \mathrm{He}(\mathcal{B}) = \bigcup_{i=1}^{40} J_i.$$

The J_i are classically referred to as j-planes. This 40 is not an accident: notice that the vector space $\mathbb{F}_3^4$, which is the 3-torsion part of the Jacobian of a genus 2 curve, has $\frac{3^4-1}{2} = 40$ cyclic subgroups (each of which is a point on the corresponding Kummer surface). We can take the *enveloping cone* $\mathrm{EC}_\alpha(P_\alpha^{(1)})$ of the first polar with respect to α (see Section 1.1 of [6] for the definition). It turns out that $\pi_\alpha(\mathrm{EC}_\alpha(P_\alpha^{(1)}))$ is a model for the dual Kummer surface $(\mathcal{K}_{\mathcal{C}_\alpha})^*$, and $\pi_\alpha(J_i)$ is tangent to this $(\mathcal{K}_{\mathcal{C}_\alpha})^*$. Thus J_i corresponds to a point of $\mathcal{K}_{\mathcal{C}_\alpha}$. This point on $\mathcal{K}_{\mathcal{C}_\alpha}$ lifts to 3-torsion points on $\mathrm{Jac}(\mathcal{C}_\alpha)$, see Proposition 2.6 of [1] for details.

Proof of Theorem 6.3. By Definition 5.5, the Brauer class of the conic $L_{\mathcal{B}',\alpha}$ is $\mathrm{Ob}(\mathcal{K}_{\mathcal{C}_\alpha})$. The discussion above shows that the Kummer surface $\mathcal{K}_{\mathcal{C}_\alpha}$ comes with a full level-3 structure $\mathbb{P}\Sigma$. By Theorem 5.17

$$\mathrm{Ob}(\mathcal{K}_{\mathcal{C}_\alpha}) = \mathrm{Ob}(\mathbb{P}\Sigma),$$

where $\mathbb{P}\Sigma$ is constant by the moduli interpretation of $\mathcal{B}'$. Therefore the Brauer class of $L_{\mathcal{B}',\alpha}$ does not depend on α, by Section 4.2 this means the conics $L_{\mathcal{B}',\alpha}$ are isomorphic. $\square$

6.2 The Automorphism group of the Burkhardt

By Section 3 of [1], there is a projective representation $\mathrm{PSp}_4(\mathbb{F}_3) \cong \Gamma \subseteq \mathrm{PGL}_5(\overline{k})$ where Γ is generated by

$$-\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1&2&2&2&2\\1&-1&-1&2&-1\\1&-1&-1&-1&2\\1&-1&2&-1&-1\\1&2&-1&-1&-1\end{pmatrix}, -\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&\zeta_3^{-1}&0\\0&0&\zeta_3&0&0\\0&0&0&0&1\end{pmatrix}.$$

The group Γ acts on $\mathcal{B}$ by right multiplication on the row vector $(y_0, ..., y_4)$.

In Section 6.3, we will discuss the representation theory of $\mathrm{Sp}_4(\mathbb{F}_3)$ and its connection to the representation theory of $\mathrm{PSp}_4(\mathbb{F}_3)$. In particular, we will see a proof of the following statement.

Proposition 6.4. *If we view the above matrices as elements of* $\mathrm{GL}_5(\overline{k})$*, then they generate a subgroup of* $\mathrm{GL}_5(\overline{k})$ *which is isomorphic to* Γ*.*

In other words, the above matrices actually give a representation $\mathrm{PSp}_4(\mathbb{F}_3) \to \mathrm{GL}_5(\overline{k})$, which then induces the projective representation $\rho_1\colon \mathrm{PSp}_4(\mathbb{F}_3) \xrightarrow{\sim} \Gamma \subseteq \mathrm{PGL}_5(\overline{k})$.

6.3 Representation theory of $\mathrm{Sp}_4(\mathbb{F}_3)$

For this section, denote $\zeta = \zeta_3$. The group $\mathrm{Sp}_4(\mathbb{F}_3)$ has 34 conjugacy classes. Table 6.1 shows character values for 12 of the 34 conjugacy classes C. The representations of $\mathrm{Sp}_4(\mathbb{F}_3)$ are listed up to dimension 30. The table was generated in Magma via `CharacterTable(Sp(4,3));`.

$\lvert C\rvert$:	1	1	90	40	40	240	480	540	540	540	5184	40
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_4	4	-4	0	$1+3\zeta$	$-2-3\zeta$	-2	1	-2	0	2	-1	$2+3\zeta$
$\overline{\chi_4}$	4	-4	0	$-2-3\zeta$	$1+3\zeta$	-2	1	-2	0	-2	-1	$-1-3\zeta$
χ_5	5	5	-3	$2+3\zeta$	$-1-3\zeta$	-1	2	1	1	1	0	$-1-3\zeta$
$\overline{\chi_5}$	5	5	-3	$-1-3\zeta$	$2+3\zeta$	-1	2	1	1	1	0	$2+3\zeta$
χ_6	6	6	-2	-3	-3	3	0	2	2	2	1	-3
χ_{10}	10	10	2	$-5-3\zeta$	$-2+3\zeta$	1	1	2	-2	2	0	$-2+3\zeta$
$\overline{\chi_{10}}$	10	10	2	$-2+3\zeta$	$-5-3\zeta$	1	1	2	-2	2	0	$-5-3\zeta$
χ_{15}	15	15	-1	6	6	3	0	3	-1	3	0	6
χ'_{15}	15	15	7	-3	-3	0	3	-1	3	-1	0	-3
χ_{20}	20	-20	0	-7	-7	2	2	-2	0	2	0	7
χ'_{20}	20	20	4	2	2	5	-1	0	4	0	0	2
χ''_{20}	20	-20	0	$5-3\zeta$	$8+3\zeta$	2	2	-2	0	2	0	$-8-3\zeta$
$\overline{\chi''_{20}}$	20	-20	0	$8+3\zeta$	$5-3\zeta$	2	2	-2	0	2	0	$-5+3\zeta$
χ'''_{20}	20	-20	0	$5+6\zeta$	$-1-6\zeta$	-4	-1	-2	0	2	0	$1+6\zeta$
$\overline{\chi'''_{20}}$	20	-20	0	$-1-6\zeta$	$5+6\zeta$	-4	-1	-2	0	2	0	$-5-6\zeta$
χ_{24}	24	24	8	6	6	0	3	0	0	0	-1	6
χ_{30}	30	30	-10	3	3	3	3	-2	2	-2	0	3
χ'_{30}	30	30	6	$3+9\zeta$	$-6-9\zeta$	-3	0	2	2	2	0	$-6-9\zeta$
$\overline{\chi'_{30}}$	30	30	6	$-6-9\zeta$	$3+9\zeta$	-3	0	2	2	2	0	$3+9\zeta$

Table 6.1: Character table of $\mathrm{Sp}_4(\mathbb{F}_3)$ for 12 of 34 conjugacy classes C, up to dimension 30

For a character χ in the table, $\overline{\chi}$ denotes the conjugate character. In terms of representations, if χ corresponds to the representation V_χ, then $\overline{\chi}$ corresponds to the dual $V_{\overline{\chi}} = V_\chi^*$.

Column 3 of Table 6.1 gives the character values on $-1 \in \mathrm{Sp}_4(\mathbb{F}_3)$. By the exact sequence (5.1), this allows us to connect the representation theory of $\mathrm{Sp}_4(\mathbb{F}_3)$ to the representation theory of $\mathrm{PSp}_4(\mathbb{F}_3)$, namely

$$\chi(-1) = \begin{cases} \chi(1) & \text{if } V_\chi \text{ is a representation of } \mathrm{PSp}_4(\mathbb{F}_3) \\ -\chi(1) & \text{otherwise.} \end{cases}$$

The only normal subgroup of $\mathrm{Sp}_4(\mathbb{F}_3)$ is μ_2. Therefore, every non-trivial representation of $\mathrm{Sp}_4(\mathbb{F}_3)$ is either faithful, or is a faithful representation of $\mathrm{PSp}_4(\mathbb{F}_3)$. For instance, we see that the only two 5-dimensional representations of $\mathrm{Sp}_4(\mathbb{F}_3)$ are also representations of $\mathrm{PSp}_4(\mathbb{F}_3)$, which implies Proposition 6.4.

6.4 Twists of the Burkhardt quartic

As we will see in the following result, Proposition 6.4 implies that all twists of the Burkhardt quartic “fit" in $\mathbb{P}^4_k$.

Theorem 6.5. *Let $\mathcal{B}'$ be a twist of the Burkhardt quartic over k. Then $\mathcal{B}'$ can be embedded into $\mathbb{P}^4_k$ such that $\mathcal{B}$ and $\mathcal{B}'$ become isomorphic over $\overline{k}$ via a linear transformation of the ambient $\mathbb{P}^4$.*

Proof. Recall that by Theorem 2.12, twists of $\mathcal{B}$ are naturally in bijection with $H^1(k, \Gamma)$ and twists of $\mathbb{P}^4_k$ are naturally in bijection with $H^1(k, \mathrm{PGL}_5(\overline{k}))$. The natural map $H^1(k, \Gamma) \to H^1(k, \mathrm{PGL}_5(\overline{k}))$ shows that a twist of $\mathcal{B} \subset \mathbb{P}^4_k$ gives rise to a twist of the ambient $\mathbb{P}^4_k$. Now, Since Γ lifts to $\mathrm{GL}_5(\overline{k})$ by Proposition 6.4, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{GL}_5(\overline{k}) \arrow[rr] & & \mathrm{PGL}_5(\overline{k}) \\
& \Gamma \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The diagram implies that the map $H^1(k, \Gamma) \to H^1(k, \mathrm{PGL}_5(\overline{k}))$ factors through $H^1(k, \mathrm{GL}_5(\overline{k}))$, which is trivial by Hilbert 90. The above shows that any cocycle corresponding to a twist of $\mathcal{B}$ is trivial in $H^1(k, \mathrm{PGL}_5(\overline{k}))$, hence the corresponding twist of the ambient $\mathbb{P}^4_k$ is trivial. $\square$

6.5 The Field-of-definition versus field-of-moduli obstruction

Let $\mathcal{B}'$ be a twist of $\mathcal{B}$ over k. By Theorem 6.5, we may assume that $\mathcal{B}'$ is a quartic threefold in $\mathbb{P}^4_k$. If we are given a k-rational point $\alpha \in (\mathcal{B}' \backslash \mathrm{He}(\mathcal{B}'))(k)$, we can perform the construction in Section 6.1 to obtain a conic $L_{\mathcal{B}',\alpha}$ with 6 marked points. By Proposition 5.7, this determines an abelian surface over k if and only if $L_{\mathcal{B}',\alpha}$ has k-rational points. Hence there is a potential field-of-definition versus field-of-moduli obstruction: the point α may not necessarily correspond to an abelian surface with full level-3 structure defined over k. We are only guaranteed a Kummer surface $\mathcal{K}_{\mathcal{C}_\alpha}$ with full level-3 structure $\mathbb{P}\Sigma$ over k, where $\mathbb{P}\Sigma$ is independent of α (it is the $\mathbb{P}\Sigma$ such that $\mathcal{B}'$ is birational to $\mathrm{Kum}(\mathbb{P}\Sigma)$ over k).

Definition 6.6. With the above setup, we define the *obstruction associated to $\mathcal{B}'$* by $\mathrm{Ob}(\mathcal{B}') := \mathrm{Ob}(\mathbb{P}\Sigma) \in \mathrm{Br}(k)[2]$.

Recalling Theorem 5.17, we have the equality

$$\mathrm{Ob}(\mathcal{B}') = \mathrm{Ob}(\mathcal{K}_{\mathcal{C}_\alpha}) = \mathrm{Ob}(\mathbb{P}\Sigma).$$

All of the above are the Brauer class of the conic $L_{\mathcal{B}',\alpha}$, Theorem 6.3 shows that this class is independent of α. It turns out that $\mathrm{Ob}(\mathcal{B})$ is trivial.

Example 6.7. Take $k = \mathbb{Q}$. There is another classical model of the Burkhardt quartic, usually presented in $\mathbb{P}^5_{\mathbb{Q}}$ by the equations

$$\mathcal{B}' \colon \sigma_1(x_0, ..., x_5) = \sigma_4(x_0, ..., x_5) = 0,$$

where the σ_i are the elementary symmetric polynomials. Eliminating the variable x_5 in the linear equation $\sigma_1 = 0$ gives a model for $\mathcal{B}'$ in $\mathbb{P}^4_{\mathbb{Q}}$. On this model, there are $\mathbb{Q}$-rational points away from $\mathrm{He}(\mathcal{B}')$. Choosing a point $\alpha \in (\mathcal{B}' \backslash \mathrm{He}(\mathcal{B}'))(\mathbb{Q})$ such as

$$\alpha = (1 : -1/4 : -4 : -1/4 : 1)$$

gives a conic $L_{\mathcal{B}',\alpha}$ which does not have $\mathbb{Q}$-rational points. In fact, it doesn't even have $\mathbb{R}$-rational points, and doesn't have $\mathbb{Q}_3$-rational points. Using the Hasse-Minkowski theorem (see Theorem 8 in Chapter IV of [13]), one can show that $\mathrm{Ob}(\mathcal{B}') = [(-1, -3)_{\mathbb{Q}}] \in \mathrm{Br}(k)[2]$.

6.6 The Witting configuration and the Maschke $\mathbb{P}^3$

For historical context, in the 1890s Klein and his students studied two different representations of $\mathrm{PSp}_4(\mathbb{F}_3)$. Burkhardt studied the action on $\mathbb{P}^4$ which we encountered in Section 6.2, while Maschke studied a certain action on $\mathbb{P}^3$ which we will discuss in this section. Both of these projective representations come from $\mathrm{PSp}_4(\mathbb{F}_3)$ acting on a $\mathbb{P}^8$ whose coordinates are certain *theta functions*, where the decomposition into Burkhardt's and Maschke's representations is determined by decomposing the theta functions in even and odd functions. We direct the interested reader to Chapters 3 and 4 of [9] for details. In this section, we use results from Chapters 4 and 5 of [9], which contain significantly more detail than what we present in this thesis. There is a projective representation $\rho_2 \colon \mathrm{PSp}_4(\mathbb{F}_3) \to \mathrm{PGL}_4(\overline{k})$. Denote the image of ρ_2 by $\Gamma_4 \subset \mathrm{PGL}_4(\overline{k})$. The significance of this representation is that it presents $\mathrm{PSp}_4(\mathbb{F}_3)$ as the automorphism group of $\mathbb{P}^3$ together with extra structure. This extra structure can be described as 40 marked points W_{40} sitting inside what is classically referred to as the *Witting configuration.* The Witting configuration consists of 40 planes, each of which contain 21 lines that form the *extended Hesse pencil*, "the arrangement of 21 lines which are the 12 lines of the Hesse pencil together with the 9 lines joining the corners of the four triangles" (see Section 4.3.1 of [9]). Each plane in the Witting configuration contains 12 points of W_{40}, and conversely through each point of W_{40} pass 12 of the planes. Each point of W_{40} is associated to a unique plane in the configuration, see Section 5.3.1 of [9] for details.

Remark 6.8. As stated above, the Witting configuration contains 40 planes and 40 points such that each point lies on 12 planes and each plane contains 12 points. This gives an incidence structure which corresponds to the following incidence structure in $\mathbb{P}(\mathbb{F}_3^4)$ (notice $\#\mathbb{P}(\mathbb{F}_3^4) = 40$). Endow $\mathbb{F}_3^4$ with the structure of a symplectic space (as in Section 5.5); one can check that each point in $\mathbb{P}(\mathbb{F}_3^4)$ (i.e. a 1-dimensional subspace of $\mathbb{F}_3^4$) pairs trivially with exactly 12 other points. By definition, $\mathrm{Sp}_4(\mathbb{F}_3)$ preserves the symplectic pairing on $\mathbb{F}_3^4$, and hence $\mathrm{Sp}_4(\mathbb{F}_3)$ preserves the incidence structure in $\mathbb{P}(\mathbb{F}_3^4)$. Thus one can derive explicitly the action of $\mathrm{PSp}_4(\mathbb{F}_3)$ on the Witting configuration.

The 40 points W_{40} of the Witting configuration are the base locus of 5 quartics in $\mathbb{P}^3$ with coordinates $z_1, ..., z_4$:

$$W_{40} = V(U_1) \cap V(U_2) \cap V(U_3) \cap V(U_4) \cap V(U_5)$$

where

$$\begin{aligned}
U_1 &= 3z_1z_2z_3z_4 \\
U_2 &= -z_1(z_2^3 + z_3^3 + z_4^3) \\
U_3 &= z_2(z_1^3 + z_3^3 - z_4^3) \\
U_4 &= z_3(z_1^3 - z_2^3 + z_4^3) \\
U_5 &= z_4(z_1^3 + z_2^3 - z_3^3).
\end{aligned}$$

Remark 6.9. This a slight rescaling of the quartics found on page 358 of [4], the quartics given in [9] contain a minor typographical error.

The 40 planes of the Witting configuration are given by

$$\begin{aligned}
z_i &= 0 \quad \text{(4 planes)} \\
(z_2^3 + z_3^3 + z_4^3)^3 - 27z_2^3z_3^3z_4^3 &= 0 \quad \text{(9 planes)} \\
(z_1^3 + z_2^3 + z_4^3)^3 - 27z_1^3z_2^3z_4^3 &= 0 \quad \text{(9 planes)} \\
(z_1^3 + z_2^3 + z_3^3)^3 - 27z_1^3z_2^3z_3^3 &= 0 \quad \text{(9 planes)} \\
(z_1^3 + z_3^3 + z_4^3)^3 - 27z_1^3z_3^3z_4^3 &= 0 \quad \text{(9 planes)}
\end{aligned}$$

Remark 6.10. These equations are found on page 151 of [9] (there the coordinates are $z_0, ..., z_3$ rather than $z_1, ..., z_4$).

The $\mathbb{P}^3$ together with the Witting configuration is called the *Maschke* $\mathbb{P}^3$. The quartics $U_1, ..., U_5$ above yield a unirational $6:1$ parametrization $\mathbb{P}^3 \to \mathcal{B}$ which is compatible with the $\mathrm{PSp}_4(\mathbb{F}_3)$ action, meaning that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{P}^3 \arrow[r, "\rho_2(g)"] \arrow[d] & \mathbb{P}^3 \arrow[d] \\
\mathcal{B} \arrow[r, "\rho_1(g)"] & \mathcal{B}
\end{tikzcd} \tag{6.1}$$

for every $g \in \mathrm{PSp}_4(\mathbb{F}_3)$.

We now give, explicitly, the action of $\mathrm{PSp}_4(\mathbb{F}_3)$ on the Witting configuration (thus a special case of Remark 6.8).

Theorem 6.11. *Let*

$$\Gamma_4 = \langle H_1, H_2, H_3 \rangle \subset \mathrm{PGL}_4(\overline{k})$$

where

$$H_1 = \frac{1}{3}\begin{pmatrix} 3\zeta_3 & 0 & 0 & 0 \\ 0 & (\zeta_3+2) & (\zeta_3+2) & (-\zeta_3+1) \\ 0 & (\zeta_3+2) & (\zeta_3-1) & (-\zeta_3-2) \\ 0 & (-\zeta_3+1) & (-\zeta_3-2) & (\zeta_3+2) \end{pmatrix}$$
$$H_2 = \frac{1}{3}\begin{pmatrix} (2\zeta_3+1) & (-2\zeta_3-1) & 0 & (\zeta_3+2) \\ (\zeta_3-1) & (2\zeta_3+1) & 0 & (-\zeta_3+1) \\ 0 & 0 & (3\zeta_3+3) & 0 \\ (\zeta_3+2) & (2\zeta_3+1) & 0 & (2\zeta_3+1) \end{pmatrix}$$
$$H_3 = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group Γ_4 preserves the Witting configuration, and is isomorphic to $\mathrm{PSp}_4(\mathbb{F}_3)$*.*

Proof. It is a direct verification that Γ_4 preserves the Witting configuration (it suffices to check this for the generators H_1, H_2, H_3). Thus $\Gamma_4 \leq \mathrm{PSp}_4(\mathbb{F}_3)$. One can check with the help of a computer algebra system like Magma that the above matrices generate a projective group of order $25920 = 51840/2$. $\square$

Remark 6.12. If we view the matrices in Theorem 6.11 as being in $\mathrm{GL}_4(\overline{k})$, we get a faithful representation $\mathrm{Sp}_4(\mathbb{F}_3) \to \mathrm{GL}_4(\overline{k})$ which is entry $\overline{\chi_4}$ in Table 6.1.

The Maschke $\mathbb{P}^3$ has a modular interpretation.

Theorem 6.13 (see Section 5.4 of [9])**.** *The quasiprojective variety $\mathbb{P}^3 \backslash W_{40}$ is the moduli space of Jacobians of genus 2 curves with full level-3 structure together with a marked Weierstrass point. The 6:1 unirational parametrization $\mathbb{P}^3 \to \mathcal{B}$ given by $U_1, ..., U_5$ corresponds to forgetting the marked Weierstrass point.*

6.7 The Brauer-Severi variety associated to a twist of the Burkhardt quartic

We now get to defining the Brauer-Severi variety associated to twists of the Burkhardt quartic, as promised in the title of this thesis. We have seen that the obstruction $\mathrm{Ob}(\mathcal{K}_{\mathcal{C}_\alpha})$ for a point $\alpha \in (\mathcal{B}' \backslash \mathrm{He}(\mathcal{B}'))(k)$, where $\mathcal{K}_{\mathcal{C}_\alpha}$ is the Kummer surface that realizes the moduli interpretation of $\mathcal{B}'$ as given in Theorem 6.1, is independent of α. This gave rise to the definition of $\mathrm{Ob}(\mathcal{B}')$ in Definition 6.6. We also saw that for any Kummer full level-3 structure $\mathbb{P}\Sigma$, the moduli space $\mathrm{Kum}(\mathbb{P}\Sigma)$ admits a model as a twist of the Burkhardt quartic, and that $\mathrm{Ob}(\mathcal{B}')$ really is just $\mathrm{Ob}(\mathbb{P}\Sigma)$, the obstruction for $\mathbb{P}\Sigma$ to be lifted to a full level-3 structure.

We also saw that given a point $\alpha \in (\mathcal{B}' \backslash \mathrm{He}(\mathcal{B}'))(k)$, we can represent $\mathrm{Ob}(\mathcal{B}')$ by the class of a conic. In this section we determine a Brauer-Severi variety representing $\mathrm{Ob}(\mathcal{B}')$ without assuming the existence of a point α in $(\mathcal{B}' \backslash \mathrm{He}(\mathcal{B}'))(k)$. This Brauer-Severi variety is a twist of the Maschke $\mathbb{P}^3$ (meaning that it is a twisted form of $\mathbb{P}^3$ that contains a twisted Witting configuration).

To motivate the construction, we first examine the Maschke $\mathbb{P}^3$ without any twisting. As we saw in Section 6.6, we have a projective representation Γ_4 of $\mathrm{PSp}_4(\mathbb{F}_3)$ acting on the Maschke $\mathbb{P}^3$. Recall that by Remark 6.12, Γ_4 gives a faithful representation of $\mathrm{Sp}_4(\mathbb{F}_3) \to \mathrm{GL}_4(\overline{k})$. Denote the image of $\mathrm{Sp}_4(\mathbb{F}_3)$ in $\mathrm{GL}_4(\overline{k})$ by $\widetilde{\Gamma}_4$, in other words, $\widetilde{\Gamma}_4$ is just the group generated by the matrices H_1, H_2, H_3 in Theorem 6.11 when viewed in $\mathrm{GL}_4(\overline{k})$ instead of $\mathrm{PGL}_4(\overline{k})$. Let V_4 be the natural $\widetilde{\Gamma}_4$-module. For this section, it is useful to recall the following, well-known formulas from character theory:

$$\begin{aligned}
\chi_{V^*} &= \overline{\chi_V} \\
\chi_{\mathrm{Sym}^2 V}(g) &= \frac{1}{2}((\chi_V(g))^2 + \chi_V(g^2)) \\
\chi_{\bigwedge^2 V}(g) &= \frac{1}{2}((\chi_V(g))^2 - \chi_V(g^2)).
\end{aligned}$$

Using Veronese embedding, we can view a $\mathbb{P}^3$ inside $\mathbb{P}^9$. Denote the image by $\nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$. The Veronese embedding is given by the degree 2 monomials in the coordinates of $\mathbb{P}^3$, so there is a corresponding action of $\widetilde{\Gamma}_4$ on $\overline{k}^{10}$ given by the representation $W := \mathrm{Sym}^2 V_4$. The space W is a faithful representation of $\mathrm{PSp}_4(\mathbb{F}_3)$, it is entry $\overline{\chi_{10}}$ in Table 6.1. Using character theory, we can recover W directly from the representation $\Gamma \subset \mathrm{GL}_5(\overline{k})$ of $\mathrm{PSp}_4(\mathbb{F}_3)$ in Section 6.2, where Γ is the automorphism group of $\mathcal{B}$.

Proposition 6.14. *Let V_5 be the natural Γ-module. There is an isomorphism of representations*

$$\mathrm{Sym}^2 V_4 \cong (\bigwedge^2 V_5)^*$$

which is compatible with the action of $\mathrm{Gal}(\overline{k}/k)$.

Remark 6.15. If we choose coordinates, then the isomorphism can be described by a matrix $T \in k^{10\times 10}$, which we have explicitly computed. Thus we get an isomorphism between the associated groups $\operatorname{Sym}^2 \Gamma_4$ and $(\bigwedge^2 \Gamma)^*$, given by conjugation with T. The important thing for us is that T is defined over k, thus ensuring that the isomorphism is compatible the $\operatorname{Gal}(\overline{k}/k)$-structure.

As is well known, $\nu_2(\mathbb{P}^3)$ is defined by 20 quadratic equations in $\mathbb{P}^9$, hence giving a 20-dimensional subspace of $\operatorname{Sym}^2 W^*$. We can obtain this 20-dimensional representation by decomposing the representation $\operatorname{Sym}^2 W^*$ into irreducible representations of $\operatorname{PSp}_4(\mathbb{F}_3)$.

Proposition 6.16. *The representation* $\operatorname{Sym}^2 W^*$ *decomposes into*

$$\operatorname{Sym}^2 W^* = V_5 \oplus V_{20} \oplus V_{30}$$

where V_i *is an irreducible representation of* $\operatorname{PSp}_4(\mathbb{F}_3)$ *of dimension* i*.*

Proof. Using character theory, we find that the character of $\operatorname{Sym}^2 W^*$ decomposes into $\chi_5 + \chi'_{20} + \chi'_{30}$ in the notation of Table 6.1. □

Remark 6.17. In Table 6.1 we see that there is only one faithful representation of $\operatorname{PSp}_4(\mathbb{F}_3)$ of dimension 20. So V_{20} is uniquely determined by its dimension.

Remark 6.18. We are reusing the notation V_5. This is because the two representations we have denoted by V_5 are in fact the same. The V_5 in Proposition 6.16 corresponds to the 5-dimensional invariant subspace $\langle U_1, ..., U_5\rangle$ of $\operatorname{Sym}^2 W^*$. As we have discussed, these $U_1, ..., U_5$ can be viewed as the coordinates of $\mathcal{B}$, hence agreeing with the representation V_5 coming from Γ.

We are now ready to define the Brauer-Severi variety associated to a a twist of the Burkhardt quartic which, as we shall see, measures the obstruction. Let $\mathcal{B}'$ be a twist of $\mathcal{B}$. The *Brauer-Severi variety* S' *associated to* $\mathcal{B}'$ is constructed as follows. Let $\Gamma' \cong \operatorname{PSp}_4(\mathbb{F}_3)$ be the automorphism group of $\mathcal{B}'$. By Theorem 6.5, we may assume $\Gamma' \subset \operatorname{PGL}_5(\overline{k})$. Let V'_5 be the natural Γ'-module, and let $W' = (\bigwedge^2 V'_5)^*$. Then by Proposition 6.14 and Proposition 6.16, we get a decomposition $\operatorname{Sym}^2(W')^* = V'_5 \oplus V'_{20} \oplus V'_{30}$ into irreducible representations.

Definition 6.19. Let $\mathcal{B}'$ be a twist of $\mathcal{B}$. With the notation above, we define S' to be the variety in $\mathbb{P}^9$ defined by the 20 quadratic equations corresponding to V'_{20}.

Over $\overline{k}$ the representations V_{20} and V'_{20} are isomorphic. Therefore S' is a linear twist of $\nu_2(\mathbb{P}^3)$, and hence a Brauer-Severi variety.

Remark 6.20. Not all Brauer-Severi 3-folds in $\mathbb{P}^9$ arise in this way. We are twisting through the group $\operatorname{PSp}_4(\mathbb{F}_3)$. By the discussion in Section 6.6, this means that we get exactly those Brauer-Severi 3-folds S' in $\mathbb{P}^9$ that contain a twisted Witting configuration which is $\mathrm{Gal}(\overline{k}/k)$-invariant. We can thus regard the S' as twists of the quasiprojective k-variety $\mathbb{P}^3_k \backslash \{\text{Witting configuration}\}$. Therefore by Theorem 2.12, these S' are classified by $H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$, where the Galois action is given by the representation Γ_4 of $\mathrm{PSp}_4(\mathbb{F}_3)$.

Example 6.21. Recall the symmetric model $\mathcal{B}'$ of $\mathcal{B}$ in Example 6.7. Using the linear relation $\sigma_1(x_0, ..., x_5) = 0$, we eliminate the variable x_5, thus presenting $\mathcal{B}'$ in $\mathbb{P}^4$ with coordinates $x_0, ..., x_4$. The model $\mathcal{B}'$ is a nontrivial twist of $\mathcal{B}$ over $\mathbb{Q}$, but there is an isomorphism $\mathcal{B} \to \mathcal{B}'$ over $\mathbb{Q}(\zeta_3)$ given by

$$(x_0, ..., x_4) = (y_0, ..., y_4) \begin{pmatrix} 1 & -1 & -1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\zeta_3 \\ 0 & 0 & 0 & -1 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 & 0 & 0. \end{pmatrix}$$

Conjugating Γ by the above matrix gives the automorphism group of Γ' of $\mathcal{B}'$ as a subgroup of $\mathrm{PGL}_5(\overline{\mathbb{Q}})$ which is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We may then compute the group $\mathrm{Sym}^2(\bigwedge^2 \Gamma')^*$, which gives a matrix group which naturally acts on a 55-dimensional space V'_{55}. We can then decompose V'_{55} into the $V'_5 \oplus V'_{20} \oplus V'_{30}$ described above. The V'_{20} can then be viewed as the space of the following 20 quadrics in $\mathbb{P}^9$ with coordinates $u_1, ..., u_{10}$ (see Appendix A for the full list of quadrics)

(1) $u_1u_8 - u_1u_9 - u_2u_7 + u_2u + 10 - u_3u_5 + u_3u_7 + u_4u_6 - u_4u_8 - u_5u_{10} - u_6u_9$

(2) $u_3^2 - u_3u_6 - u_3u_8 + u_3u_{10} + u_6^2 - u_6u_8 + u_6u_{10} + u_8^2 + u_8u_{10} + u_{10}^2,$

$$\vdots$$

(20) $u_1u_{10} - u_2u_{10} - u_3u_4 + u_3u_7 - u_3u_{10} + u_4u_8 + u_4u_{10} + u_5u_{10} - u_6u_7 + u_6u_9 - u_6u_{10} + u_7u_{10} - u_8u_9 - u_8u_{10} + u_9u_{10} - 2u_{10}^2.$

Let $S' \subset \mathbb{P}^9$ be the variety defined by these quadrics. Then S' is the Brauer-Severi variety associated to the twist $\mathcal{B}'$ of $\mathcal{B}$. Further, the representation V'_5 of Γ' gives a 6:1 rational map $S' \to \mathcal{B}'$, which we can compute explicitly as

$$\begin{aligned} x_0 &= u_1u_8 - u_2u_6 + u_3u_5 \\ x_1 &= u_1u_9 - u_2u_7 + u_4u_5 \\ x_2 &= u_1u_{10} - u_3u_7 + u_4u_6 \\ x_3 &= u_2u_{10} - u_3u_9 + u_4u_8 \\ x_4 &= u_5u_{10} - u_6u_9 + u_7u_8. \end{aligned}$$

Theorem 6.22. *Let $\mathcal{B}'$ be a twist of $\mathcal{B}$ over k, and S' the associated Brauer-Severi variety. Then the obstruction associated to $\mathcal{B}'$ is trivial if and only if S' has a k-rational point.*

We will prove a stronger statement in Section 6.8 (see Theorem 6.23).

6.8 Proof of Theorem 1.1 and Theorem 1.2, and another result

Let $\mathcal{B}'$ be a twist of the Burkhardt quartic. Recall that the obstruction associated to $\mathcal{B}'$ (Definition 6.6) is an element of $\mathrm{Br}(k)[2] \cong H^2(k, \mu_2)$. From Definition 6.19, we have a Brauer-Severi variety S' of dimension 3. By the discussion in Section 4.2, we know that S' corresponds to an element in $\mathrm{Br}(k)$. We show that this is in fact the obstruction.

Theorem 6.23. *The Brauer class of S' is the obstruction associated to $\mathcal{B}'$, i.e. $[S'] = \mathrm{Ob}(\mathcal{B}')$.*

Proof. View $\mathrm{Sp}_4(\mathbb{F}_3)$ as a group with $\mathrm{Gal}(\overline{k}/k)$-action by considering the representation Γ_4 in Section 6.6. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Sp}_4(\mathbb{F}_3) & \longrightarrow & \mathrm{PSp}_4(\mathbb{F}_3) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \mathrm{GL}_4(\overline{k}) & \longrightarrow & \mathrm{PGL}_4(\overline{k}) & \longrightarrow & 1. \end{array}$$

This gives the following commutative diagram of Galois cohomology

$$\begin{array}{ccc} H^1(k, \mathrm{PSp}_4(\mathbb{F}_3)) & \longrightarrow & H^2(k, \mu_2) \cong \mathrm{Br}(k)[2] \\ \downarrow & & \downarrow \\ H^1(k, \mathrm{PGL}_4(\overline{k})) & \longrightarrow & H^2(k, \bar{k}^*) \cong \mathrm{Br}(k). \end{array}$$

By Remark 6.20, the group $H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$ classifies all the Brauer-Severi varieties S' arising from twists $\mathcal{B}'$ of the Burkhardt quartic. Now, recall the representation of $\mathrm{PSp}_4(\mathbb{F}_3)$ in Section 6.2, which represents $\mathrm{PSp}_4(\mathbb{F}_3)$ as the automorphism group Γ of $\mathcal{B}$. Because diagram (6.1) commutes, we can identify $H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$ with $H^1(k, \Gamma)$. Thus the images of $H^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$ and $H^1(k, \Gamma)$ in $H^2(k, \mu_2)$ must be the same. □

This proves Theorem 1.1.

Proof of Theorem 1.2. Suppose $\mathrm{Ob}(\mathcal{B}')$ is trivial. By Theorem 6.23, this implies that S' is isomorphic to $\mathbb{P}^3$ over k. In particular, S' has a k-rational point. The moduli interpretation of S' (recall Theorem 6.13) gives that this k-rational point corresponds to a curve $\mathcal{C}\colon y^2 = f(x)$ which satisfies the conditions in the theorem. □

Recall that in Section 6.1, we originally obtained the obstruction associated to $\mathcal{B}'$ as the Brauer class of a conic $L_{\mathcal{B}',\alpha}$, which is a twist of $\mathbb{P}^1$. The variety S' is, however, a twist of $\mathbb{P}^3$. To conclude this section, we show that if k is a local or global field, then $L_{\mathcal{B}',\alpha}$ is isomorphic to a subvariety of S'.

Theorem 6.24. *Let k be a local or global field. Then S' has a k-subvariety that is a conic isomorphic to $L_{\mathcal{B}',\alpha}$ over k.*

Proof. One can check that lines in $\mathbb{P}^3_k$ are mapped to conics on $\nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$ via the Veronese embedding. Since S' is a twist of $\nu_2(\mathbb{P}^3)$, its twisted-linear k-subvarieties of dimension 1 (if they exist) are thus conics. By Theorem 6.23, the period of S' is 2. Since k is a local or global field, Theorem 3.27 gives that the index of S' is also 2. Theorem 4.9 then guarantees the existence of a conic L' on S' that is defined over k. We are done by Theorem 4.8. $\square$

Remark 6.25. The twisted linear subvarieties (the conics) over k on S' are parametrized by a twisted Grassmanian $\mathrm{Gr}(2,4)$ over k. The representation of $\mathrm{PSp}_4(\mathbb{F}_3)$ acting on $\mathrm{Gr}(2,4)$ is, by definition, $\bigwedge^2 \Gamma_4$.

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Appendix A

Quadrics Example

The following are the quadrics in Example 6.21.

```
u1*u8 - u1*u9 - u2*u7 + u2*u10 - u3*u5 + u3*u7 + u4*u6 - u4*u8 - u5*u10
- u6*u9,

u3^2 - u3*u6 - u3*u8 + u3*u10 + u6^2 - u6*u8 + u6*u10 + u8^2 + u8*u10
+ u10^2,

u4^2 - u4*u7 - u4*u9 - u4*u10 + u7^2 - u7*u9 - u7*u10 + u9^2 - u9*u10
+ u10^2,

u1*u2 - u2^2 + u2*u5 - u3*u8 + u3*u9 - u3*u10 + u4*u8 - u4*u9 + u4*u10
- u6*u7  - u8*u10 + u9*u10 - u10^2,

u1*u2 + u3*u10 - u4*u10 + u5^2 - u5*u6 - u5*u7 + u5*u8 + u5*u9 + u6*u7
- u6*u9 - u7*u8 + u8*u9 + u10^2,

u1*u3 + u2*u9 - u4*u9 - u5*u6 + u5*u7 - u5*u10 + u6^2 - u6*u7 - u6*u8
+ u6*u10 + u7*u8 - u8*u10 + u9^2,

u2*u6 - u2*u8 + u3*u5 - u3*u7 + u3*u8 + u3*u10 - u4*u6 - u4*u10 - u5*u6
+ u6*u7 - u8^2 + u8*u9 - u9*u10 + u10^2,

u1^2 - u1*u3 + u1*u5 + u1*u6 + u1*u10 - u2*u10 - u4*u6 + u4*u7 + u4*u8
+ u5^2 - u5*u7 + u5*u9 + u5*u10 + u6*u8 - u7*u8,

u1*u2 + u1*u8 + u1*u10 - u2^2 + u2*u5 - u2*u8 - u2*u10 - u3*u5 + u3*u7
- u3*u9 - u4*u9 - u5*u6 - u8^2 + u8*u9 - u8*u10,

u1*u3 - u1*u9 + u1*u10 - u2*u7 + u3*u7 + u3*u10 - u4*u10 - u5*u6 + u6^2
- u6*u8- u6*u9 + u6*u10 - u7*u8 + u8*u9 + u10^2,
```

```
u1*u3 - u2*u6 + u2*u7 + u2*u8 - u3*u5 - u3*u8 + u4*u5 - u5*u10 + u6^2
- u6*u7 -u6*u8 + u6*u10 + u7*u8 + u8^2 + u9*u10,

u1*u3 - u2*u7 + u2*u8 + u2*u10 + u3*u7 - u3*u8 - u4*u5 + u4*u6 - u4*u8
+ u6^2 -u6*u7 - u6*u8 + u6*u10 + u8^2 + u9*u10,

u1*u4 - u2*u6 + u2*u9 - u2*u10 - u3*u5 + u3*u7 - u3*u9 - u4^2 + u4*u6
+ u4*u7 +u4*u10 - u6*u7 - u8*u10 + u9*u10 - u10^2,

u1*u6 + u2*u7 + u3*u4 - u3*u6 - u3*u7 + u4*u5 - u4*u6 - u5*u8 - u5*u10
+ u6^2 +u6*u10 + u7*u8 + u8*u10 - u9*u10 + u10^2,

u1*u7 + u2*u6 + u3*u4 + u3*u5 - u3*u7 - u4*u6 - u4*u7 - u5*u9 + u5*u10
+ u6*u9+ u7^2 - u7*u10 + u8*u10 - u9*u10 + u10^2,

u1*u4 + u1*u5 + u2*u3 - u2*u5 + u3*u10 - u4*u10 + u5^2 - u5*u7 + u5*u8
+ u5*u10+ u7^2 - u7*u8 - u7*u10 + u8^2 + u8*u10 + u10^2,

u1*u3 - u2*u6 - u3*u5 + u3*u7 - u3*u9 + u3*u10 + u4*u6 - u4*u8 - u4*u10
- u5*u6- u5*u10 + u6^2 - u6*u8 - u6*u9 + u6*u10 + u8*u9 + u10^2,

u1*u9 - u2*u4 + u3*u7 - u3*u9 - u3*u10 - u4*u5 + u4*u6 - u4*u8 + u4*u10
- u5*u9+ u5*u10 - u6*u7 + u6*u9 + u7*u9 - u9^2 + u9*u10 - u10^2,

u1*u10 - u2*u3 + u2*u6 - u2*u10 + u3*u5 - u3*u10 - u4*u6 + u4*u8 + u4*u10
-u5*u8 + u5*u10 - u6*u7 + u6*u8 + u6*u9 - u8^2 - u8*u10 - u10^2,

u1*u10 - u2*u10 - u3*u4 + u3*u7 - u3*u10 + u4*u8 + u4*u10 + u5*u10 - u6*u7
+u6*u9 - u6*u10 + u7*u10 - u8*u9 - u8*u10 + u9*u10 - 2*u10^2
```