Computation of Mountain Wave Clouds in a Moist Boussinesq Fluid Model

by

Hudson Edward Lynn

B.Sc., University of British Columbia, 2017

Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science

in the Department of Mathematics Faculty of Science

© Hudson Edward Lynn 2019
SIMON FRASER UNIVERSITY
Summer 2019

Copyright in this work rests with the author. Please ensure that any reproduction or re-use is done in accordance with the relevant national copyright legislation.
Approval

Name: Hudson Edward Lynn

Degree: Master of Science (Mathematics)

Title: Computation of Mountain Wave Clouds in a Moist Boussinesq Fluid Model

Examining Committee: Chair: Ben Adcock
Associate Professor

David Muraki
Senior Supervisor
Professor

Mary Catherine Kropinski
Supervisor
Professor

Ralf Wittenberg
Internal Examiner
Associate Professor

Date Defended: August 7, 2019
Abstract

This thesis presents computations of time-steady 2D clouds forming over mountain topography, with an interest in resolving the fine-scale physics at the cloud edge. The underlying model takes the incompressible Euler equations as its basis, and couples the atmospheric fluid flows to the physics of phase change in order to compute cloud-edge boundary locations in a vertically-stratified atmosphere. This coupling gives rise to a free-boundary problem for the cloud edge. This model has been employed here to successfully recover mountain cloud behaviour for a variety of atmospheric conditions, including computations of the well-known lens shape of the lenticular cloud.

The problem of time-steady, density-stratified flow over ground topography can be reduced to a 2D Helmholtz problem for the streamfunction due to Long’s theory. The domain of this PDE is a perturbed 2D half-space, and the streamfunction is specified by a bottom boundary that follows a localized mountain terrain. This thesis presents a new extension of Long’s theory to include cloudy air, where the derived Helmholtz equation now includes localized forcing associated with small regions of cloud.

The nature of the PDE domain makes the problem well-suited for a boundary method application. The scheme used in this work utilizes the method of fundamental solutions (MFS), a numerical approach related to the boundary integral equation method, which approximates the solutions to elliptic problems by finite sums of fundamental solutions of the PDE operator. The MFS is coupled with an iterative solver to resolve the free-boundary problem for the cloud geometry, and the numerical performance of this scheme is analyzed.

Keywords: Atmospheric Fluid Dynamics; Boussinesq; Cloud Edge Dynamics; Topographic Flow; Lenticular Clouds; Method of Fundamental Solutions
Acknowledgements

First and foremost I would like to thank my parents for all of their love and support. They have been there for me, unconditionally, at every level of my academic studies, and it certainly would not have been possible for me to make it this far without the care and dedication that they put in to raising all of their children.

I would like to extend my thanks and eternal gratitude to my supervisor, David Muraki. I am certain that the training and lessons he has provided will serve me for a long time to come. I would also like to thank my committee members, Professor Mary Catherine Kropinski and Professor Ralf Wittenberg for their support in the completion of my thesis.

Finally, and most importantly, I would like to thank Alicia. It is not an exaggeration to say that I would not have made it this far without her. Words cannot describe the debt I owe to her for the unwavering love and support she provides every day. Here’s to hoping I can pay it back some day.
# Table of Contents

A. Approval

B. Abstract

C. Acknowledgements

D. Table of Contents

E. List of Figures

1. Introduction

2. Model and Equations
   2.1 Marginal Cloud Model
      2.1.1 Thermodynamic Background
      2.1.2 Boussinesq Fluid Model
      2.1.3 Thermodynamic Advections
      2.1.4 Linearized Thermodynamics
      2.1.5 Stability in the Moist Atmosphere
   2.2 Topographic Flow within the Moist Boussinesq Model
      2.2.1 Scales for Mountain Flow
      2.2.2 Long’s Theory for Dry Air
      2.2.3 Boundary Conditions
      2.2.4 Long’s Theory Derivation for Cloud Model

3. Numerical Method
   3.1 Method of Fundamental Solutions
      3.1.1 Choice of Singular Points
      3.1.2 Helmholtz on Unit Disc
      3.1.3 General Closed Domains
      3.1.4 MFS for Infinite Domains
      3.1.5 MFS for Forced Problems
   3.2 MFS Applied to the Mountain Cloud Problem
      3.2.1 MFS for Dry Mountain Flow
      3.2.2 Iterative Cloud Corrections
4 Numerical Convergence and Scientific Insight 44
  4.1 Numerical Convergence ........................................ 44
    4.1.1 MFS Convergence ........................................ 44
    4.1.2 Convergence of Particular Solutions ..................... 47
    4.1.3 Convergence of Iterations ................................ 49
  4.2 Basic Mountain Clouds ....................................... 50
  4.3 Effects of Cloud on the Atmosphere .......................... 52
  4.4 Cloud-Edge Velocity ........................................ 54
  4.5 Rapidly Varying $r^\prime_\infty$ and Lenticular Clouds ....... 56
  4.6 Challenges for Computing in the Limit of Small $\sigma$ ....... 57

5 Conclusion 62

Bibliography 65
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Three examples of cloud edge phenomena</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>Image of a lenticular cloud</td>
<td>4</td>
</tr>
<tr>
<td>2.1</td>
<td>Moist air stability comparison</td>
<td>16</td>
</tr>
<tr>
<td>2.2</td>
<td>Moist air stability comparison</td>
<td>17</td>
</tr>
<tr>
<td>2.3</td>
<td>Flow past a mountain cartoon</td>
<td>18</td>
</tr>
<tr>
<td>2.4</td>
<td>A depiction of the linearized boundary for Long’s theory</td>
<td>22</td>
</tr>
<tr>
<td>3.1</td>
<td>MFS diagram for the unit circle</td>
<td>27</td>
</tr>
<tr>
<td>3.2</td>
<td>MFS set-up with exterior singularities</td>
<td>29</td>
</tr>
<tr>
<td>3.3</td>
<td>A crescent shaped domain inducing exterior singularities</td>
<td>31</td>
</tr>
<tr>
<td>3.4</td>
<td>A comparison of Green’s functions for the Helmholtz operator</td>
<td>36</td>
</tr>
<tr>
<td>3.5</td>
<td>Downstream decay of the Lyra Green’s function</td>
<td>37</td>
</tr>
<tr>
<td>3.6</td>
<td>Source point distribution used by [17] in (a), and by this project in (b)</td>
<td>38</td>
</tr>
<tr>
<td>3.7</td>
<td>Computational scheme cartoon</td>
<td>41</td>
</tr>
<tr>
<td>3.8</td>
<td>A figure of the cloud polygon used for numerical integration</td>
<td>43</td>
</tr>
<tr>
<td>4.1</td>
<td>Comparison of numerical conditioning for two source point depths</td>
<td>46</td>
</tr>
<tr>
<td>4.2</td>
<td>Estimated error convergence for MFS approximation to Long’s theory</td>
<td>47</td>
</tr>
<tr>
<td>4.3</td>
<td>Error plot for the computation of the particular solutions</td>
<td>48</td>
</tr>
<tr>
<td>4.4</td>
<td>Examples of two cases used to measure iterative rate of convergence</td>
<td>49</td>
</tr>
<tr>
<td>4.5</td>
<td>Relative corrections to $r'_l$ with increasing number of iterations</td>
<td>50</td>
</tr>
<tr>
<td>4.6</td>
<td>A standard wave cloud computation</td>
<td>51</td>
</tr>
<tr>
<td>4.7</td>
<td>Background atmosphere profiles</td>
<td>52</td>
</tr>
<tr>
<td>4.8</td>
<td>Two cases demonstrating the latent heating effect</td>
<td>53</td>
</tr>
<tr>
<td>4.9</td>
<td>Contours of corrections to $\theta'$</td>
<td>55</td>
</tr>
<tr>
<td>4.10</td>
<td>A vector plot showing computed cloud-edge velocities</td>
<td>56</td>
</tr>
<tr>
<td>4.11</td>
<td>A comparison between slowly and rapidly varying $r'_\infty$</td>
<td>58</td>
</tr>
<tr>
<td>4.12</td>
<td>Computation of a lenticular cloud</td>
<td>59</td>
</tr>
<tr>
<td>4.13</td>
<td>Depiction of the small $\sigma$ problem for the finite-difference solve</td>
<td>60</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

It is not uncommon to look up to the sky on a blustery day and observe clouds that are being blown about by the wind. This makes some intuitive sense: a cloud is simply a portion of the air in which liquid water is suspended, and thus it should react to the wind in a manner that is similar to the surrounding air. Therefore the study of cloud physics must overlap with the science of atmospheric fluid mechanics.

However, it is perhaps less obvious upon observation that to study clouds also requires the science of moist thermodynamics and phase change. The atmosphere is said to be at moist saturation when the maximum amount of water is present in vapour form, but none has condensed into the liquid state. Therefore, air that is at, or near, saturation is ripe for cloud formation upon small thermodynamic disturbances, such as a drop in temperature. Likewise, cloudy air can disappear by equal and opposite disturbances, such as a warming of the air. From this point of view, a cloud is simply a region of moist saturated air in which condensation has occurred and is surrounded by unsaturated (also known as clear) air.

It must then be the case that cloud phenomena exist in the overlap of fluid mechanics and thermodynamics. In fact, cloud events have been observed that require the intuition supplied by both of these sciences. For example, lenticular clouds which form over mountain tops typically do so in the presence of a strong wind. However, despite this high velocity flow the cloud appears essentially stationary above the topography, as opposed to being advected with the winds. This type of cloud presents one such example that requires the understanding of both atmospheric flow with the wind as well as the physics of phase change.

Cloud formation and edge dynamics are characteristically the study of a multi-fluid system. The atmosphere can largely be described by two components: dry air and water. In the absence of precipitation and ice crystals, the water component can either be water vapour, or condensed into suspended droplets. The mixture of dry air and water vapour is referred to as moist clear air. When some of that water vapour condenses into the liquid state, clouds form [3].

The study of cloud dynamics is largely focused on large-scale computations for applications such as weather forecasting. These types of computations cannot capture the fine
resolution needed to resolve cloud edges with high accuracy, and so fall short when attempting to explain more subtle phenomena of cloud geometry. Figure 1.1 shows three such examples of fine scale cloud features. Figure 1.1a is an image of an undulatus cloud forming in a thin altocumulus cloud layer, and Figure 1.1b includes an image of an asperitas cloud with its characteristic wave pattern in the cloud base. The wave structure present in both cases raises the question as to whether these phenomena can be caused by atmospheric gravity waves.

Figure 1.1: Examples of three fine scale cloud edge phenomena, all potentially attributable to gravity waves in the atmosphere. Indeed, the expansion of the hole-punch cloud in (c) has been explained in terms of a wave propagation phenomenon. Photos courtesy of the WMO International Cloud Atlas. (Photo credits: (a) Stanley Sharp 2015, (b) Gary McArthur 2004, (c) Tsz Cheung Lee 2015)

In particular, the “hole-punch cloud” pictured in Figure 1.1c, was the focus of a study carried out by Muraki et al. in 2016 [15]. This distinctive feature comes about when a thin cloud layer is punctured by a clear patch of air, usually caused by an aircraft passing through the layer. Surprisingly, however, this opening continues to expand long after the initialization. Though full physics models could reproduce the hole-punch cloud the computations did not suggest a simple mechanism to explain the continued expansion of the hole-punch.

The work by Muraki et al. was one of the early successes of a dynamical analysis that produces a nearly-linear set of model equations that couples the fluid motion to the thermodynamics of phase change. The extension this model brought to the standard set of fluid equations was a buoyancy term that depends on whether or not the surrounding air is clear or cloudy. This work was able to conclude that the hole-punch expansion can be explained by a gravity wave residing in the cloud layer causing the evaporation of the liquid water, therefore leading to the erosion of the cloud edge. The success of this study motivated the continued development of a more general model to explain the dynamics of cloud edge motion [14].

The underlying fluid equations in the model developed by Muraki et al. are those of a Boussinesq flow of disturbance winds in a stratified background atmosphere. One of the characteristics distinguishing a Boussinesq fluid model is a buoyancy response term that
comes from the advections of the stratified background state [2]. A particular scaling appropriate for the study of atmospheric phenomena supposes that the background quantities are slowly-varying with atmospheric depth, consistent with what has been referred to as the weak Boussinesq approximation [6].

Coupling this fluid model to the phase change dynamics needed for cloud formation is done through the buoyancy term, in a manner similar to that employed by Muraki et al. in the hole-punch study. The model assumes small disturbances from the moist saturated background state. The buoyancy force is represented as a disturbance temperature response in the equations, and the constitutive law for this temperature term depends on whether or not liquid water is present. This dependence on liquid water encompasses the inherent nonlinearity in the cloud model: the boundary of the cloudy air region depends on the solution itself, implying a free-boundary problem for the cloud edge.

One commonly used example to demonstrate wave clouds in the atmosphere is that of orographically generated clouds. It is well understood that a horizontal wind blowing over a surface topographic feature, such as a mountain ridge, can generate downstream gravity waves due to the buoyancy response. As the wind approaches the mountain peak it is displaced upward, and after it crosses this peak the stratified nature of the atmosphere drives the air back down. If the topography is an elongated ridge, the flow problem can be reduced by considering a vertical 2D cross-section of the mountain. The time steady case of such a mountain flow has been studied previously, and under the right conditions can be reduced to an elliptic solve known as Long’s theory [13].

When moist, nearly-saturated air is being advected over the mountain there is a chance for cloud formation. Displacing the air vertically drops the temperature relative to the background state. If this cooling is sufficient, the near-saturated water vapour may condense to form cloud. Figure 1.2 shows an image of one of these orographic wave clouds, classified as a lenticular cloud. A distinct feature present in this example is the oft-reported lens shape.

Numerically, the time-steady 2D flow over the mountain problem has been studied previously [17, 13]. In the context of Long’s theory the problem reduces to solving the Helmholtz equation on a semi-infinite domain. The bottom boundary of this domain follows the shape of the topography, usually characterized by a localized bump centered at the origin. Both the unbounded nature of the domain and the irregular bottom boundary have proved to make this a problem well-suited for a variety of numerical boundary methods.

In [13], Muraki employed a boundary integral equation (BIE) method to compute the flow of time-steady stratified flows over mountain ridges. That work showed the advantage of the boundary method approach over previously-applied Fourier methods to the same problem, particularly in the robustness of the BIE method to handling larger amplitude mountains and flow regimes far from the hydrostatic limit.
Figure 1.2: A lenticular cloud forming over a mountain. These clouds form when a layer of moisture in the atmosphere is advected over a mountain obstacle. The mountain displaces the air vertically upwards, causing the temperature to drop and the moisture condenses into liquid state, therefore forming the cloud. Photo courtesy of the WMO International Cloud Atlas. (Photo credit: Fabien Gillet 2008)

More recently, Masuda and Ishioka in [17] applied the Method of Fundamental Solutions (MFS), also known as the charge simulation method, to study the same problem. The MFS boundary method has been used to solve a variety of elliptic boundary value problems on irregular domains [4]. The method, which approximates PDE solutions using a linear combination of fundamental solutions, shares properties with both the method of particular solutions, as well as the BIE method insofar as the singularities of the fundamental solutions are placed selectively on a boundary exterior to the PDE domain, allowing for the interpretation of a discrete boundary integral formulation on an extended domain.

The MFS provides a high degree of flexibility of design in the selection of the singular point distribution, which often proves to be both an advantage and a numerical burden. Despite the lack of any general theory for minimizing error in a MFS approximation, it has regularly been observed that placing the singular points as distantly as possible leads to more accurate solutions, with the added benefit that the approximation can be evaluated up
to the boundary of the domain with high regularity. However, this very same practice tends to lead to ill-conditioned linear solves as well. Striking a balance between high accuracy and numerically well conditioned schemes tends to be the largest challenge in using a MFS approach [1, 4].

With a larger focus on the numerical performance of the method, Masuda and Ishioka showed that the MFS provided many advantages, mainly in terms of accuracy, over the BIE method for the mountain flow problem. It is this observation that provides the motivation to apply the MFS to the mountain cloud problem as well.

The goal of this thesis is to compute the location of clouds forming in the time-steady flow over a 2D mountain. In doing so, the science of cloud formation in the atmosphere can be studied. Particularly by quantitative comparison to the dry air flow over the mountain ridge, the effects of cloud presence in terms of heating and scattering can be assessed. Furthermore, the mountain cloud model is an extension of Long’s theory applied to the newly developed moist Boussinesq fluid system. Therefore analysis of the mountain cloud results will serve as some of the first validation of the moist Boussinesq model. Finally, the numerical considerations of the mountain cloud problem also present a learning opportunity. The proposed numerical scheme combines a flexible boundary method with the more traditional grid-based finite-difference and quadrature methods, and analysis of the method’s performance may provide insight into how such free-boundary problems could be solved in the future.

The rest of this thesis will be laid out as follows: in Chapter 2 a review of the underlying moist Boussinesq model will be carried out, highlighting the essential assumptions required. In that same chapter, the derivation of Long’s theory for the case of dry air will be presented, as well as the equivalent derivation consistent with a moist saturated atmosphere. Chapter 3 will begin by reviewing the MFS in some generality, as well as explain the numerical considerations of the mountain cloud problem. Chapter 4 presents the basic numerical convergence results, as well as conclusions pertaining to the physics of the time-steady mountain cloud problem. Finally, Chapter 5 includes some concluding remarks on the project.
Chapter 2

Model and Equations

The physics of the flow model are an extension of the inviscid Euler equations in which the primary driving force comes from the pressure gradient. When such a flow is used to model vertically-stratified fluids (such as the atmosphere) a second main force comes into play, known as the buoyancy force. In a vertically-stratified medium fluid parcels resist upward displacement, and so any fluid that is forced upwards (for example by the presence of a mountain obstacle) will feel a strong vertical restoring force to move back down. This buoyancy force is important for modeling the atmosphere, where standard background states are vertically-stratified in thermodynamic quantities such as density and temperature. Incorporation of these vertical variations can lead to the so-called Boussinesq approximation, in which the change in density or temperature scales out of the equations everywhere except the buoyancy response.

Adding moisture to this vertically-stratified flow sets the scene for modeling cloud dynamics. When the air is nearly-saturated with water vapour, upward displacement causes a drop in temperature, which can lead to condensation and therefore cloud formation. The atmosphere then has two components: clear air (a mixture of dry air and water vapour) and cloudy air (clear air with suspended liquid water droplets), making the cloudy atmosphere a two fluid system. The presence of liquid water in an air parcel increases its temperature relative to its surroundings, and similar to the example of vertical displacement the parcel feels an upward driving force. Therefore the moist air model differs from that of dry air in that the buoyancy response driving vertical motion now depends on two factors: the vertical stratification and the presence of liquid water.

In this context the equations of the cloud model are those of Boussinesq flow coupled with moist thermodynamics. The model assumes small perturbations from a vertically-stratified background atmosphere that can lead to phase change processes causing the condensation of water vapour into suspended liquid water droplets that are also known as clouds [14]. The coupling occurs through a buoyancy response that depends on whether or not the surrounding air is clear or cloudy.
Assumptions on the thermodynamic state of the model are also required. The atmosphere is assumed to be both adiabatic and reversible. For the dynamics to be adiabatic means that there is no net heat or mass transfer into or out of the system. An important consequence of this is that the clouds that are captured by the equations are non-precipitating. Furthermore, in an adiabatic atmosphere the thermodynamic quantities of potential temperature \( \theta \) and total water mixing ratio \( r_T \) are perfectly advected with the flow. Potential temperature is a thermodynamic quantity closely related to the entropy of the atmosphere, and is useful here due to this conservation property. The total water mixing ratio \( r_T \), usually expressed in units of g/kg, indicates the ratio of water in the atmosphere to dry air. Likewise, the mixing ratios of water vapour \( r_v \) and liquid water \( r_l \) indicate the amount of water present in each phase, and the magnitude of all three mixing ratios is typically small.

A further assumption is that the background atmosphere is dry stable. For the atmosphere to be stable means that only small vertical displacements of air are permitted. In a strongly unstable atmosphere vertically displaced air can continue to rise, causing strong updrafts. Consequences of an unstable atmosphere can be highly turbulent winds and thunder storms. In the model this stability requirement is controlled by the stratification of the atmosphere that leads to the buoyancy force counteracting vertical displacement of air parcels. A consequence of this condition is that the potential temperature of the background atmosphere must not decrease with height.

As is explained at the end of this chapter, the time-steady 2D mountain cloud problem can be reduced to a constant coefficient elliptic solve. The goal of this chapter is to explain the underlying model from which this elliptic equation is derived, and then to understand the conditions under which this elliptic simplification holds within said model. In Section 2.1 the basic set of equations governing the system will be introduced, and in the following subsections various comments on the assumptions and ideas contained within this model will be explained. The model as presented here is the work of Muraki et al. in [14]. Then Section 2.2 will review the basics of Long’s theory for stratified flows of dry air over mountain obstacles before applying a similar derivation to the moist Boussinesq model in order to derive the set of equations for the time-steady mountain cloud problem.

### 2.1 Marginal Cloud Model

The governing equations for the marginal cloud model are an extension of the standard Boussinesq equations for dry flow. In particular, the physics of moist thermodynamics and phase change in the atmosphere have been incorporated into the dynamical system, giving rise to the following set of equations for 2D disturbance velocity \( \mathbf{u}' = (u', w') \), pressure \( p' \),
temperature $T'$, potential temperature $\theta'$ and total water mixing ratio $r'_T$:

\[
\nabla' \cdot \vec{u}' = 0 \tag{2.1} \\
\frac{Du'}{Dt'} = -\frac{\partial p'}{\partial x'} \tag{2.2} \\
\sigma^2 \frac{Dw'}{Dt'} = -\frac{\partial p'}{\partial z'} + T' \tag{2.3} \\
\frac{D\theta'}{Dt'} + \tilde{N}^2 w' = 0 \tag{2.4} \\
\frac{Dr'_T}{Dt'} - \frac{\bar{\Gamma}}{T^2} w' = 0 \tag{2.5}
\]

where $\sigma$ is the hydrostatic parameter, $\tilde{N}(z')$ represents the dry Brunt-Väisälä frequency that controls stratification of the background atmosphere (discussed in Section 2.1.1), $\bar{T}(z')$ is the background temperature profile, and $\bar{\Gamma}(z')$ is the temperature lapse rate. The bar variables represent background quantities, and they are prescribed functions described in Section 2.1.1. Primed and bar variables have been scaled to be $O(1)$. In the above, equations (2.1)-(2.3) represent the equations of Boussinesq fluid flow. The buoyancy response is represented as a temperature disturbance $T'$, indicating that warming the air causes an upward force on $w'$. Equations (2.4) and (2.5) are simply advection equations for potential temperature and the total water mixing ratio.

The above system is not closed, as it is short one equation. Indeed, a constitutive law for disturbance temperature $T'$ is required, representative of the fact that the buoyancy response in the atmosphere couples the vertically-stratified flow to the thermodynamics. This equation, that determines $T'$ in terms of dynamical variables $\theta'$ and the liquid water mixing ratio $r'_l$, which indicates whether the surrounding air is clear ($r'_l = 0$) or cloudy ($r'_l > 0$), is given as

\[
T'(\theta', r'_l) = \theta' + \Lambda \frac{\bar{r}^*}{T} r'_l. \tag{2.6}
\]

The constant $\Lambda$ is an $O(1)$ number whose value depends on the thermodynamic reference state. This constant represents the amount of energy, known as latent heat, released into the atmosphere by the condensation of water vapour into liquid water, hence explains why $r'_l > 0$ implies a temperature increase. The profile $\bar{r}^*(z')$ is the background state of the total water mixing ratio, where the star denotes saturation conditions.

Of course, an equation for $r'_l$ is also required, which is given in terms of the liquid water functional $l'(\theta', r'_T)$:

\[
l'(\theta', r'_T) = \frac{\bar{T}}{\Lambda \bar{r}^* + \bar{T}^2} (-\theta' + \bar{T} r'_T) \tag{2.7}
\]
where \( r'_l \) is then defined as the positive part of \( l' \):

\[
\begin{align*}
  r'_l &= \begin{cases} 
l' & \text{for } l' \geq 0, \\
 0 & \text{for } l' < 0.
\end{cases}
\end{align*}
\] (2.8)

It is this equation that captures the inherent free-boundary in the cloud model: the location of the cloud edge must be solved for along with the other physical variables. Equation (2.7) also indicates the main two processes of cloud formation: both the cooling of moist air \((\theta' < 0)\) or a local increase in total water \((r'_T > 0)\) can cause \( l' > 0 \), implying cloud formation. The local increase of water potentially forming cloud is a consequence of the fact that the background atmosphere is saturated; the maximum amount of water vapour is already present for the current temperature and pressure, and so any additional water must condense into liquid form.

The model presented in this thesis appears essentially unchanged from the work of Muraki et al. in [14]. However, there are two exceptions: first, the model presented here does not assume \( \sigma = 1 \) as is done in [14]. The second difference concerns the latent heat parameter \( \Lambda \). In general \( \Lambda \) is defined as dependent on the thermodynamic state as

\[
\Lambda(T,p) = \left( \frac{l_v}{R_v T} \right)^2 \frac{R_v r^*(T,p)}{c_{p,d}}
\] (2.9)

where \( l_v \) represents the latent heat energy, \( R_v, R_d \) and \( c_{p,d} \) are thermodynamic constants defined in the following sections, and \( r^* \) is the saturation mixing ratio. In this thesis, \( \Lambda \) is taken as a constant by fixing \( T \) and \( r^* \) to reference value \( T_0 \) and \( r_0(T_0,p_0) \). In contrast, Muraki et al. define \( \bar{\Lambda}(\bar{T},\bar{p}) \) as a non-constant coefficient in the equations. In practice this difference only amounts to a notational change.

Various aspects and assumptions of this model will be discussed in the following sections. Starting in Section 2.1.1 the background atmosphere will be established. In Section 2.1.2 the underlying assumptions of the Boussinesq model are discussed. Section 2.1.3 outlines how the total advective nature of potential temperature and total water mixing ratio complete the set of dynamical equations. Finally, section 2.1.4 reviews the linearization of the thermodynamic equations in order to determine the set of free scales in the model.

### 2.1.1 Thermodynamic Background

An explanation of the background state of the atmosphere is required in order to understand the non-constant coefficients (denoted by bars) in (2.1)-(2.7). The background atmosphere is assumed to be a function of the vertical coordinate \( z' \) only, which is consistent with the dominant behaviour observed in atmospheric measurements. Furthermore, the background is also assumed to be time steady. To this end, total thermodynamic variables are defined
in terms of this background as

\[
\phi(x', z')/\phi_0 = \bar{\phi}(z')(1 + \tilde{\phi}(x', z'))
\] (2.10)

where the bar denotes a background quantity and the tilde a disturbance quantity whose magnitude can be assumed to be small. Furthermore, the tilde variables are unscaled versions of the prime variables in the set of dynamic equations. In the above, \( \phi \) can be any of \( T, p \) or \( \theta \). In general, \( \phi_0 \) represents a reference value chosen at some height \( z_0 \), and so all background quantities are equal to 1 at this reference height.

The standard approach in atmospheric sciences is to define the atmosphere in terms of lapse rates that dictate how the thermodynamic variables change with height. To this end, the thermodynamic background quantities have been shown to be governed by the following equations:

\[
\frac{d \log \bar{\theta}}{dz'} = \epsilon \bar{N}^2(z')
\] (2.11)

\[
\frac{d \log \bar{p}}{dz'} = -\epsilon \frac{c_{p,d}}{R_m} \frac{1}{\bar{T}}
\] (2.12)

\[
-\frac{d \bar{T}}{dz'} = \epsilon \bar{\Gamma}
\] (2.13)

where \( c_{p,d} \) is the specific heat capacity of dry air, and \( R_m \) is the effective gas constant of moist air, a weighted average of the gas constants for dry air \( R_d \) and for water vapour \( R_v \). Specific heat capacity is a measure of the energy required to raise the temperature of a substance (in this case dry air), and the gas constant is the constant of proportionality coming from the ideal gas law of a substance.

Expression (2.11) is the dimensionless definition of the clear air Brunt-Väisälä frequency \( \bar{N}(z') \), and it sets the stratification of the entire background atmosphere. The \( \epsilon \) parameter in this expression is given by

\[
\epsilon = \frac{H}{\bar{H}_0} \frac{R_d}{c_{p,d}}
\] (2.14)

where the so-called pressure height scale \( \bar{H}_0 = R_d T_0 / g \) has been introduced. The value of \( H/\bar{H}_0 \) can be assumed to be small in accordance with the Boussinesq scaling, and therefore \( \epsilon \) encodes the slowly varying nature of the background atmosphere. Equation (2.12) is simply the dimensionless form of the hydrostatic relation.

The final expression (2.13) defines the temperature lapse rate \( \bar{\Gamma} \), which is connected to the rest of the background atmosphere through

\[
\frac{1 - \bar{\Gamma}(z')}{\bar{T}} = \bar{N}^2(z')
\] (2.15)
so that specifying $\bar{N}(z')$ and the thermodynamic reference state closes the system (2.11)-(2.13), therefore specifying the entire background state. Equation (2.11) provides important information about how $\bar{N}$ controls the stability of the background atmosphere. The assumption that the background atmosphere is dry stable implies that $\bar{\theta}$ is non-decreasing in height, and so (2.11) dictates that $\bar{N}(z') \geq 0$. Furthermore, (2.15) indicates that whenever $\bar{N}(z') \geq 1$, $\bar{T}$ increases with height. This is known as an inversion layer in the atmosphere, and is distinct from the dominant behaviour of $\bar{T}$ decreasing with height (hence why it is colder on top of a mountain then at its base).

In addition to the thermodynamic variables discussed above are the moisture variables $r_T, r_v, r_l$ which are the mixing ratios of total water, water vapour, and liquid water in the atmosphere. They obey the relationship $r_T = r_v + r_l$, as all water in the model atmosphere is assumed to be in either liquid or vapour form. A high representative value for total water would be $r_T = 28 g/kg$, or $r_T = 0.028$ at about 300K and 1000 hPa.

The reference state for the moisture variables is assumed to be critically saturated. For the atmosphere to be at saturation means that for a given temperature and pressure, the maximum amount of water vapour is present in the air without any of it condensing to form liquid water. To this end, the reference value $r_0$ is defined as

$$r_0 = \frac{R_d}{R_v} \frac{e^* (T_0)}{p_0}$$

where $e^* (T_0)$ denotes the saturation vapour pressure. Furthermore, the background state of the moisture variables is defined in terms of the saturation mixing ratio $\bar{r}^*(z')$, which is given by

$$\frac{d \log \bar{r}^*}{dz'} = -\epsilon \left( \frac{l_v}{R_v T_0} \right) \frac{\bar{T}}{T^2}$$

where the latent heat constant $l_v$ accounts for the energy released into the atmosphere by water condensing into liquid.

Using the reference value $r_0$ and the background saturation mixing ratio $\bar{r}^*$, the complete definitions of the moisture variables in the atmosphere are

$$r_T/r_0 = \bar{r}^*(z')(1 + \tilde{r}_T)$$
$$r_v/r_0 = \bar{r}^*(z')(1 + \tilde{r}_v)$$
$$r_l/r_0 = \bar{r}^*(z') \tilde{r}_l$$

where all three quantities use the same reference and background values. An important consequence of the critically saturated background is that $r_l = 0$ in the zero-disturbance state, indicating that cloudy air (regions of $r_l > 0$) is entirely a product of atmospheric disturbances. Furthermore, the above definitions preserve the total water property in the disturbance variables, i.e $\tilde{r}_T = \tilde{r}_v + \tilde{r}_l$. 11
2.1.2 Boussinesq Fluid Model

The assumptions leading to the Boussinesq approximation are essential to understanding how the size of the fluid velocities sets the scales of the thermodynamic disturbances. To that end, the underlying fluid model for the system is that of inviscid, irrotational, and incompressible flow. Indeed, equations (2.1)-(2.3) represent the standard Boussinesq equations:

\[
\nabla' \cdot \vec{u}' = 0 \quad (2.21)
\]
\[
\frac{D\vec{u}'}{Dt'} = -\frac{\partial p'}{\partial x'} \quad (2.22)
\]
\[
\sigma^2 \frac{Dw'}{Dt'} = -\frac{\partial p'}{\partial z'} + T' \quad (2.23)
\]

where the Boussinesq nature is captured by the fact that variations in disturbance temperature \( T' \) have scaled out of every term except the buoyancy force in (2.23). This buoyancy response implies that warming the air drives it upwards, and cooling forces it back down.

The space, time, and velocity variables have all been non-dimensionalized as

\[
x = Lx', \quad z = Hz', \quad t = \tau t', \quad u = Uu', \quad w = \sigma Uw'
\]

(2.24)

where \( \sigma = H/L \) is the hydrostatic parameter, included so that the scaled velocities remain divergence free with respect to the scaled distances.

Under these scales, the material derivative operator can be replaced by a non-dimensional equivalent:

\[
\frac{D}{Dt} = \frac{1}{\tau} \left( \frac{\partial}{\partial t'} + \mu (\vec{u}' \cdot \nabla') \right) \equiv \frac{1}{\tau} \frac{D}{Dt'} \quad (2.25)
\]

where \( \mu = U\tau/L \) is the nonlinearity parameter. In the limit \( \mu \ll 1 \), all advections are linear.

There are three main classes of assumptions under which this moist Boussinesq flow model is derived. First are the assumptions on the state of the background atmosphere, which are that it is hydrostatic (as explained in Section 2.1.1) as well as that the total atmosphere obeys the ideal gas law

\[
p = \rho R_m T
\]

(2.26)

where \( R_m \) is the effective gas constant of the moist atmosphere, discussed below. This ideal gas relation effectively allows for the elimination of density \( \rho \) from the equations in favour of \( p \) and \( T \).

Second is the so-called small mixing ratio (SMR) approximation. Using that the representative values of \( r_T \) are \( o(1) \) small, many terms in the equations can be replaced by their dry air equivalents. For example, the effective moist air gas constant, \( R_m \), is given by the expression

\[
R_m = \frac{R_d + r_v R_v}{1 + r_T}
\]

(2.27)
but under the SMR approximation, it can be approximated as $R_m = R_d$. The evidence of this SMR approximation in the Boussinesq fluid equations is seen by the fact there is no distinction between dry air (where there is no water at all) and clear air (where there is simply no liquid water, but quite possibly water vapour is present). Without this SMR approximation there would be non-constant coefficients on the pressure gradient terms of the form $R_m/R_d$, which would only differ from 1 by a small amount.

The final set of assumptions concerns the size of scales for the disturbance temperature $\tilde{T}$ and pressure $\tilde{p}$. It has been shown that the correct scales required to recover the Boussinesq flow model are

\[
\tilde{T}, \tilde{\theta} = O \left( \frac{U H_0}{\tau R_d T_0 \sigma} \right) \tag{2.28}
\]

\[
\tilde{p} = O \left( \frac{H H_0 U H_0}{\tau R_d T_0 \sigma} \right) \tag{2.29}
\]

which are both assumed to be $o(1)$. Such an assumption allows for linearization of a number of terms in the fluid equations. Scaling $\tilde{T}, \tilde{\theta}$ and $\tilde{p}$ based on the above gives rise to the $O(1)$ variables $T', \theta'$ and $p'$. It was shown in [14] that the time scale $\tau$ can be determined independently based on the coupling of the fluid mechanics and thermodynamics, and therefore the statement that both disturbance scales are small becomes a limit on the magnitude of the disturbance winds, characterized by $U \ll 550$ m/s. Furthermore, it should be noted that (2.28) also sets the scale on $\tilde{\theta}$, since the constitutive law (2.6) implies $\theta' = T'$ in the case of clear air ($r_l' = 0$).

### 2.1.3 Thermodynamic Advections

The equations for $\theta'$ and $r_T'$ ((2.4) and (2.5)) are a direct result of the fact the total potential temperature and total water mixing ratio are advected quantities in the adiabatic atmosphere:

\[
\frac{D \log(\theta)}{D t} = 0 \tag{2.30}
\]

\[
\frac{D \log(r_T)}{D t} = 0 \tag{2.31}
\]

where the logarithmic form is used so that replacing $\theta$ and $r_T$ by their multiplicative disturbance forms

\[
\frac{\theta}{T_0} = \tilde{\theta}(z')(1 + \tilde{\theta}) \tag{2.32}
\]

\[
\frac{r_T}{r_0} = \tilde{r}^*(z')(1 + \tilde{r}_T) \tag{2.33}
\]
gives the following set of equations for the disturbance quantities

\[
\frac{D \log(1 + \tilde{\theta})}{Dt'} + \mu \frac{d \log \tilde{\theta}}{dz'} w' = 0 \tag{2.34}
\]

\[
\frac{D \log(1 + \tilde{r}_T)}{Dt'} + \mu \frac{d \log \tilde{r}_T}{dz'} w' = 0. \tag{2.35}
\]

Since the lapse rates for the background atmosphere have already been set by (2.11) and (2.17), the natural scales of disturbance potential temperature and total water mixing ratio become

\[
\tilde{\theta} = O(\epsilon \mu) \tag{2.36}
\]

\[
\tilde{r}_T = O \left( \frac{l_v}{R_v T_0} \epsilon \mu \right) \tag{2.37}
\]

where the assumed smallness of both scales allows for linearization that leads to (2.4) and (2.5). As was explained in Section 2.1.1 the scale on \( \tilde{\theta} \) must also be set by the Boussinesq model. Matching scales means that the magnitude of \( \tilde{r}_T \) is larger by a factor of \( l_v/(R_v T_0) \) compared to the other disturbance variables. This further limits the magnitude of the disturbance winds under which the above assumptions are valid, characterized by \( U \ll 25 \text{ m/s} \).

### 2.1.4 Linearized Thermodynamics

The final piece of information needed to determine the minimum set of scales required in the moist Boussinesq model is a thermodynamic connection between the disturbance variables. The fact that (2.28) and (2.36) are both valid scales for \( \tilde{\theta} \) was already defended in Section 2.1.2 in terms of the constitutive law (2.6), and the origin of this argument will be outlined here. As was explained in [14], the equation for the total entropy of the moist air system, under the SMR assumption and the background state described in Section 2.1.1, implies the following relationship between the disturbance variables:

\[
\log(1 + \tilde{\theta}) = \log(1 + \tilde{T}) - \frac{R_d}{c_{p,d}} \log(1 + \tilde{p}) - \left( \frac{l_v}{c_{p,d} T_0} r_0 \right) \frac{\tilde{r}_T}{\tilde{T}} \tilde{r}_l. \tag{2.38}
\]

The scale of \( \tilde{p} \) is smaller than that of \( \tilde{T} \) by a factor of \( H/H_0 \), and so the pressure term in the above can be neglected. Furthermore, (2.38) must hold even when \( \tilde{r}_l = 0 \). The Boussinesq scaling for clear air already dictates that the scales of \( \tilde{\theta} \) and \( \tilde{T} \) must match, and the dominant balance of (2.38) indicates that this result holds even in a cloudy atmosphere. Matching the Boussinesq and thermodynamic scales implies that

\[
\epsilon \mu = \frac{U H_0}{\tau R_d T_0 \sigma} \tag{2.39}
\]
which is really a statement on how the various free scales must relate in order for the thermodynamics of phase change to properly couple to the Boussinesq fluid model.

Indeed, rewriting (2.39) shows that it is an expression for the time scale, \( \tau \), given by

\[
\frac{1}{\tau} = \frac{g \sigma}{\sqrt{c_{p,d} T_0}}.
\] (2.40)

Under these scaling arguments it can be seen that the linearized version of (2.38), when the pressure term is dropped, exactly accounts for the constitutive law (2.6). Furthermore, it was shown in [14] that a second linearized thermodynamic relation

\[
T' = \tilde{T}(r_T' - r_l')
\] (2.41)

can be derived from an SMR approximation for the vapour pressure using the Clausius-Clapeyron equation to write saturation vapour pressure \( e^* \) as solely a function of \( T \). The expressions (2.6) and (2.41) together explain the origin of the liquid water functional, \( l' \).

Careful tracking would indicate that the required smallness of scales can be set by the disturbance wind velocity \( U \), as well as by the height and length scales \( H \) and \( L \). Exactly what these scales are varies from problem to problem. The forms of these scales relevant to the mountain flow problem will be discussed in Section 2.2.1.

### 2.1.5 Stability in the Moist Atmosphere

The stability requirement in dry air, where the buoyancy response is simply given by \( T' = \theta' \), is that the buoyancy stratification coefficient \( \tilde{N}^2 \) must be positive, so that vertical displacement of air parcels is resisted. Recall that in Section 2.1.1, the criterion \( \tilde{N}(z') \geq 0 \) was derived based on the fact that \( \tilde{\theta}(z') \) must increase with height. However, in cloudy air the buoyancy stratification coefficient is no longer given by the dry-air Brunt-Väisälä frequency \( \tilde{N} \), and so a new stability criterion must be derived.

It was shown by Muraki et al. that the advective derivative of the buoyancy response \( T' \) in cloudy air is given by the expression

\[
\frac{DT'}{Dt'} + \frac{1}{T} \left( \frac{1}{1 + \Lambda r^*/T^2} - \tilde{\Gamma} \right) w' = 0
\] (2.42)

implying that the condition of a positive stratification coefficient is met whenever

\[
\tilde{\Gamma}(z') \leq \frac{1}{1 + \Lambda r^*/T^2} \equiv \tilde{\Gamma}_c(z')
\] (2.43)
where $\bar{\Gamma}_c$ is the cloudy air temperature lapse rate. The latent heat parameter $\Lambda$ is defined by equation (2.9) where $T$ and $r^*$ are taken as reference values $T_0$ and $r_0$:

$$\Lambda = \left( \frac{t_v}{R_v T_0} \right)^2 \frac{R_v}{c_p d} r_0$$ \hspace{1cm} (2.44)$$

and so the only values determining $\bar{\Gamma}$ and $\bar{\Gamma}_c$ are $\bar{N}$ and the reference state $(T_0, p_0)$. When condition (2.43) holds for all $z'$ the atmosphere is said to be absolutely stable. When there are layers in the atmosphere for which (2.43) does not hold, the atmosphere is said to be conditionally unstable. The choices of $\bar{N}$ consistent with absolute stability are determined by the choice of the thermodynamic reference state. For example, using the simplest possible case of $\bar{N}$ being a constant, Figure 2.1 compares the choice of $\bar{N} = 0.3$ to $\bar{N} = 0.6$, both using a reference state of $T_0 = 273$ K and $p_0 = 1000$ hPa. Both examples have conditionally unstable layers below $z' \approx 3$ and $z' \approx 0.5$ respectively. However, it can also be observed that as the atmosphere becomes more dry stable ($\bar{N}$ increases) it also becomes more moist stable, since the conditionally unstable layer in the $\bar{N} = 0.6$ example is shallower than in the $\bar{N} = 0.3$ example.

![Figure 2.1: Comparison of the stability criterion $\bar{\Gamma} \leq \bar{\Gamma}_c$ for two choices of constant $\bar{N}$, using a fixed reference state of $T_0 = 273$ K and $p_0 = 1000$ hPa. These examples are conditionally unstable below $z' \approx 3$ (for $\bar{N} = 0.3$) and $z' \approx 0.5$ (for $\bar{N} = 0.6$). However, increasing $\bar{N}$ does increase the stability of the atmosphere, which can be seen by the fact that the unstable layer is shallower in the $\bar{N} = 0.6$ example.](image-url)

Furthermore, Figure 2.2 shows how the choice of $\bar{N}$ depends on the reference state. Both cases in this figure use $\bar{N} = 0.8$, but the first has a cooler reference temperature of $T_0 = 273$ K (consistent with a typical winter day), and the second uses $T_0 = 293$ K (a typical summer temperature). Both cases use $p_0 = 1000$ hPa. Notice how the cooler reference state is absolutely stable in the entire atmosphere, whereas the warmer reference state has
a conditionally unstable layer below $z' \approx 1$. The notion that warming the atmosphere introduces unstable layers is consistent with the observation of more extreme weather phenomena occurring in warm tropical climates.

Understanding the simple case of $\bar{N}$ being constant is important for the mountain flow applications studied in this thesis. One of the key assumptions in Long’s theory for 2D mountain flow is that the stratification coefficient $\bar{N}$ is constant in the whole atmosphere. Therefore absolute stability in the atmosphere can be confirmed by simply referring to plots like those in Figures 2.1 and 2.2.

Figure 2.2: Comparison of the stability criterion $\bar{\Gamma} \leq \bar{\Gamma}_c$ for two choices of $T_0$, holding $\bar{N} = 0.8$ constant, and fixing $p_0 = 1000$ hPa. These examples show how the choice of $\bar{N}$ leading to absolute stability is controlled by the reference state $(T_0, p_0)$. For the cooler temperature $T_0 = 273$ K the atmosphere is absolutely stable. However, picking a warmer reference state introduces a conditionally unstable layer below $z' \approx 1$.

### 2.2 Topographic Flow within the Moist Boussinesq Model

Atmospheric flow over ground topography can be described by an incompressible, inviscid, and Boussinesq flow theory. Figure 2.3 depicts the standard set up: far upstream of the mountain the wind is constant and horizontal, with velocity $U_\infty$. The bottom boundary of the problem follows the ground topography, given by a known function $h(x)$, and the domain is infinite in the horizontal and positive vertical directions. The solid black lines are the streamlines of the flow.

#### 2.2.1 Scales for Mountain Flow

The mountain flow problem introduces two scales not present in the standard marginal cloud model. In this problem there is assumed to be a uniform horizontal upstream wind
Figure 2.3: A cartoon depicting flow over a mountain. The horizontal upstream wind is assumed to move with constant velocity $U_\infty$. The shaded grey region represents a mountain, and the solid black lines are the total streamlines of the flow.

with velocity $U_\infty$ that is being blown over a topographic feature (e.g. a mountain) of height $H_m$, whose shape is given by a function $h(x)$, so that $\max |h(x)| = H_m$.

Because the total wind is now $U_\infty \hat{x} + \vec{u}$ the material derivative becomes

$$ \frac{D}{Dt} = \frac{1}{\tau} \left( \frac{\partial}{\partial t'} + U_\infty \frac{\partial}{L} \frac{\partial}{\partial x'} + \mu \vec{u}' \cdot \nabla' \right). $$ (2.45)

In the case of a linear and time-steady flow, the horizontal advective term must be $O(1)$, therefore giving the scale relationship

$$ \frac{1}{\tau} = \frac{U_\infty}{L} $$ (2.46)

which, using the time scale $\tau$ implied by (2.40), allows for determination of the height scale $H$ (since $\sigma = H/L$) as

$$ H = \sqrt{\frac{c_{p,d} T_0}{g}} U_\infty. $$ (2.47)

Furthermore, the flow must move parallel to the bottom boundary defined by $h(x)$. That is, the component of the total velocity that is normal to $h(x)$ must vanish on the bottom boundary. This is equivalent to the kinematic boundary condition

$$ -(U_\infty + u) \frac{\partial h}{\partial x} + w = 0. $$ (2.48)
The scaling argument that $U_{\infty} \tau / L = 1$ and $\mu \ll 1$ implies that $U \ll U_{\infty}$. Therefore, the dominant behaviour of the bottom boundary condition is governed by

$$\frac{U_{\infty} H_m}{L} \frac{\partial h'}{\partial x'} = \sigma U w' \tag{2.49}$$

where $h = H_m h'$. The balance of the left and right hand sides gives a second scaling relation, which is simply that

$$U = \frac{H_m}{H} U_{\infty}. \tag{2.50}$$

Combining the velocity scale (2.50) with the time scale (2.46) implies that the nonlinearity parameter $\mu$ is set as $\mu = H_m / H$, effectively limiting the height of the mountain for linear flow regimes.

The length scale $L$ is simply set by the characteristic width of the mountain coming from $h'(x')$. Knowing $H$ and $L$ in turn sets the hydrostatic parameter $\sigma$.

### 2.2.2 Long’s Theory for Dry Air

The 2D time-steady flow of a vertically-stratified atmosphere over a mountain obstacle has been studied previously, and in the case of a constant horizontal wind Long’s theory provides a simple linear equation to solve for the streamfunction $\psi$ [9]. It turns out that this result holds in both the cases of linear and nonlinear flows, and is one of the amazing quirks that makes Long’s theory a powerful tool for gravity wave problems.

Starting from equations (2.1)-(2.5), an equivalent set of equations can be stated for incompressible, inviscid Boussinesq flow of dry air. In this case $r' = 0$ everywhere, and so (2.6) states that $T' = \theta'$ everywhere. Therefore, a closed system of dynamical equations describing the flow can be stated as

$$\nabla \cdot \vec{u} = 0 \tag{2.51}$$

$$\frac{D u'}{Dt'} = -\frac{\partial p'}{\partial x'} \tag{2.52}$$

$$\sigma^2 \frac{D w'}{Dt'} = -\frac{\partial p'}{\partial z'} + \theta' \tag{2.53}$$

$$\frac{D \theta'}{Dt'} + \bar{N}^2 w' = 0 \tag{2.54}$$

where equation (2.5) is no longer needed for the dry system.

Derivation of the Long’s theory equation is fairly straightforward. First, the disturbance streamfunction $\psi'$ is defined so that

$$u' = \psi'_z, \quad w' = -\psi'_x \tag{2.55}$$
which is made possible by the incompressibility condition. In the case of time-steady flow, the material derivative $\frac{D}{Dt}$ can be written

$$\frac{D}{Dt} = (1 + \mu u') \frac{\partial}{\partial x'} + \mu w' \frac{\partial}{\partial z'}$$  \hspace{1cm} (2.56)$$

where the scale $\mu = H_m/H$ controls the nonlinearity of the advective terms. Pressure can be eliminated from the equations by converting to a streamfunction-vorticity representation. Taking the $z'$ derivative of (2.52) and the $x'$ derivative of (2.53), and then subtracting, the system is rewritten

$$\frac{\partial \eta'}{\partial x'} - \mu J(\psi', \eta') = -\frac{\partial \theta'}{\partial x'}$$ \hspace{1cm} (2.57)

$$\frac{\partial \theta'}{\partial x'} - \mu J(\psi', \theta') = \bar{N}^2 \frac{\partial \psi'}{\partial x'}$$ \hspace{1cm} (2.58)

where $\eta' = u'_z - \sigma^2 w'_{x'}$ is the $\hat{y}$-component of vorticity, and $J(f, g)$ is the Jacobian defined as

$$J(f, g) = f_x g_z - f_z g_x.$$  \hspace{1cm} (2.60)$$

The linearized case is easy: when $\mu = \frac{H_m}{H} \ll 1$ the Jacobian terms can be dropped and (2.57) and (2.58) can be integrated in $x'$, since the disturbances are zero far upstream, to give

$$\eta' = -\theta'$$ \hspace{1cm} (2.61)

$$\theta' = \bar{N}^2 \psi'.$$ \hspace{1cm} (2.62)$$

These two algebraic equations then allow for (2.59) to be rewritten as the following Helmholtz equation

$$\sigma^2 \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial z'^2} + \bar{N}^2 \psi' = 0$$ \hspace{1cm} (2.63)$$

which is the standard linear equation quoted in reference to Long’s theory.

It is indeed true that the exact same equation still holds in the nonlinear case. Observe that, in the case when $\bar{N}$ is constant in $z'$, (2.58) can be rewritten as

$$J(z' + \mu \psi', \bar{N}^2 z' + \mu \theta') = 0$$ \hspace{1cm} (2.64)$$

and since the Jacobian of these quantities is zero, it must be that one can be written as a function of the other. In particular

$$\bar{N}^2 z' + \mu \theta' = F_1(z' + \mu \psi')$$ \hspace{1cm} (2.65)$$
for some function $F_1$. Since both $ψ'$ and $θ'$ are assumed to be zero far upstream, the following equation must hold:

$$F_1(z') = \bar{N}^2 z'$$

(2.66)

which directly gives the functional form of $F_1$. Using this form of $F_1$, (2.65) implies that $θ' = \bar{N}^2 ψ'$, even in the fully nonlinear case.

Using this relation to replace $−θ'x'$ by $−\bar{N}^2 ψ'x'$ in (2.57) gives the exact same equation as (2.58), but with $−η$ replacing $θ$. The conclusion is therefore that

$$\bar{N}^2 z - \mu η' = F_2(z + \mu ψ')$$

(2.67)

where again it can be concluded that $F_2(z') = \bar{N}^2 z'$. Therefore it must be that $η' = −\bar{N}^2 ψ'$ which exactly recovers the Helmholtz equation (2.63) from (2.59).

### 2.2.3 Boundary Conditions

The bottom topographic condition that goes with the Helmholtz equation (2.63) is rather straightforward, and follows from the kinematic condition that the flow must follow the topography on the surface. Without neglecting any terms, and using the $U$ scale (2.50), the bottom boundary condition can be expressed as

$$\mu u' \frac{dh'}{dx'} - w' = -\frac{dh'}{dx'}$$

(2.68)

which is simply the scaled version of (2.48). Representing the left hand side in terms of the stream function gives the expression:

$$\mu \frac{∂ψ'}{∂z'} \frac{dh'}{dx'} + \frac{∂ψ'}{∂x'} = \frac{d}{dx'} (ψ'(x', \mu h'(x')))$$

(2.69)

and since both $ψ'$ and $h'$ vanish far upstream, the boundary condition can be integrated in $x'$ to give the expression

$$ψ'(x', \mu h'(x')) = −h'(x')$$

(2.70)

for the bottom boundary condition. By not having made any assumptions about the size of the nonlinearity parameter $µ = H_m/H$ this boundary condition holds for both linear and nonlinear flows. That being said, for small amplitude mountains when $µ \ll 1$ this boundary condition can be approximated by

$$ψ'(x', 0) = −h'(x')$$

(2.71)

which converts the PDE domain into the $z' > 0$ half-space. Figure 2.4 depicts the comparison between the linear and nonlinear boundary conditions. Though this simplification is not necessary, it does provide a numerical shortcut when solving problems with a homogeneous
bottom boundary condition, as will be the case in the mountain cloud problem. For these problems, an odd-symmetric variant of the free-space Green’s function can be constructed which is exactly zero on the $z’ = 0$ axis, and therefore captures the boundary condition exactly. See Section 3.2.2 for more details.

Figure 2.4: A visual comparison of the linear and nonlinear boundary conditions for the flow over the mountain problem. The black dashed line indicates the topographic feature, a Gaussian $h(x’)$ of height 0.1. The blue line following this is the boundary along which the nonlinear condition enforces $\psi’ = -h’$. The red line indicates the linear boundary $z’ = 0$.

However, as it stands right now (2.63) with the boundary condition (2.70) is not uniquely solvable. Furthermore, apart from the hydrostatic parameter $\sigma$, there is nothing in (2.63) or the boundary condition (2.70) that stops the solution $\psi’$ from being symmetric in $x’$ and $z’$. In particular, this would allow for both upstream and downstream waves equally, which is uncharacteristic of uniform flow past a mountain obstacle.

Therefore, a far-field condition is required, which both makes the equation well posed, as well as encodes the upstream/downstream asymmetry of the wave propagation. A variety of such conditions have been posed previously. For example, a Fourier approach can be taken to ensure that radiating waves have an upward group velocity only [16]. More directly, this condition can be represented as

$$|\psi’(x’, z’)| = o(1/\sqrt{r}), \quad (x’ < 0, r = \sqrt{x'^2 + z'^2} \to \infty)$$

(2.72)

and this form leads to a derivation of a free-space Green’s function for the mountain flow problem that has the correct upstream and radiative behaviour [11].

2.2.4 Long’s Theory Derivation for Cloud Model

To incorporate the effects of a moist-saturated atmosphere into the mountain flow problem, the linear Long’s theory derivation that was applied to the dry Boussinesq system (2.51)-(2.54) can now be applied to the marginal cloud model, (2.1)-(2.7). Using the same
streamfunction-vorticity formulation, the dynamical equations are written

\[
\begin{align*}
\frac{\partial \eta'}{\partial x'} - \mu J(\psi', \eta') &= -\frac{\partial T'}{\partial x'} \quad (2.73) \\
\frac{\partial \theta'}{\partial x'} - \mu J(\psi', \theta') &= \bar{N}^2 \frac{\partial \psi'}{\partial x'} \quad (2.74) \\
\frac{\partial r_T'}{\partial x'} - \mu J(\psi', r_T') &= -\frac{\bar{\Gamma}}{T^2} \frac{\partial \psi'}{\partial x'} \quad (2.75) \\
\eta' &= \sigma^2 \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial z'^2}. \quad (2.76)
\end{align*}
\]

Since this is the linear case, where all \( \mu \) terms are dropped, expressions (2.73)-(2.75) can be integrated in \( x' \). Just as in the dry case, \( \psi' \), \( T' \), and \( \theta' \) will be zero upstream. However, \( r_T'(x', z') \) will be given an upstream profile \( r^2_{\infty}(z') \) which depends only on \( z' \), and which is everywhere negative. The reason for this choice is to slightly subsaturate the atmosphere at all heights. The profile of \( r^2_{\infty} \) will be a design parameter in the model, allowing for control of the upstream relative humidity profile.

The integrated equations, where (2.76) is being used to eliminate \( \eta' \) everywhere, are

\[
\begin{align*}
\sigma^2 \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial z'^2} &= -T' \quad (2.77) \\
\theta' &= \bar{N}^2 \psi' \quad (2.78) \\
r_T' - r^2_{\infty} &= -\frac{\bar{\Gamma}}{T^2} \psi'. \quad (2.79)
\end{align*}
\]

In order to eliminate \( T' \) from the equations, as was done in the dry case, the constitutive law (2.6) will be used. Replacing the \( \theta' \) in that expression with \( \bar{N}^2 \psi' \) then gives the cloud equivalent of (2.63)

\[
\sigma^2 \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial z'^2} + \bar{N}^2 \psi' = -\Lambda \bar{r}^*_T r_T'. \quad (2.80)
\]

Of course (2.80) requires the expression for the liquid water functional (2.7) in order to determine the right hand side. Using (2.78) and (2.79) this expression can be represented entirely in terms of \( \psi' \) and known thermodynamic functions as

\[
l'(x', z') = \frac{1}{\Lambda \bar{r}^*_T + T^2} \left( -\psi' + T^2 r^2_{\infty}(z') \right). \quad (2.81)
\]

The Long’s theory equivalent for moist air is the set of two linear equations, (2.80) and (2.81). The system has an inherent nonlinearity due to the free-boundary solve required for the cloud-edge. However, this set of linear equations does not hold in the limit of a nonlinear flow regime, as it is essential to the derivation that the \( O(\mu) \) terms may be neglected. This assumption is valid as long as \( \mu \ll 1 \), with one possible exception. Specifically, since the upstream water \( r^2_{\infty}(z') \) is a design parameter of the problem it is possible to select a profile...
which violates the assumption that the nonlinear term $\mu J(\psi', r'_T)$ may be neglected. The following subsection will explore this case further.

**Nonlinearities Introduced by Rapidly-Varying $r'_\infty$**

The linear Long’s theory derivation for the cloud model assumes that all $\mu$ terms can be neglected in (2.73)-(2.75). However, it is possible to select a profile for $r'_\infty(z')$ that varies rapidly enough to violate this assumption. To understand the consequences of this, the nonlinear advection equation of $r'_T$ (2.75) must be analyzed.

Following the steps of the clear air derivation, note that (2.74) still implies $\theta' = \bar{N}^2 \psi'$ holds regardless of the size of $\mu$. The expression (2.75) can then be rewritten as

$$J \left( z' + \mu \psi, \log \bar{r}^*(z') + \frac{l_v}{R_v T_0} \epsilon \mu r'_T \right) = 0 \tag{2.82}$$

where the lapse rate of the saturation mixing ratio (2.17) has been used. Note that the second term in the Jacobian is the linearized form of $\log \bar{r}^*(z')(1 + \bar{r}_T)$, where the linearization happens about $\bar{r}_T$ being small. Therefore (2.82) is equivalent to the statement that the total water mixing ratio is advected with the flow.

Just as was done in the dry air flow for $\theta'$ and $\eta'$, (2.82) implies

$$\log \bar{r}^*(z') + \frac{l_v}{R_v T_0} \epsilon \mu r'_T = F(z' + \mu \psi') \tag{2.83}$$

and using the upstream values of $\psi'$ and $r'_T$ gives the following definition of $F$:

$$F(z') = \log \bar{r}^*(z') + \frac{l_v}{R_v T_0} \epsilon \mu r'_\infty(z'). \tag{2.84}$$

Therefore it can be concluded that the correct nonlinear relationship between $r'_T$ and $\psi'$ is

$$r'_T = \frac{1}{\mu} \frac{R_v T_0}{\epsilon l_v} (\log \bar{r}^*(z' + \mu \psi') - \log \bar{r}^*(z')) + r'_\infty(z' + \mu \psi'). \tag{2.85}$$

In the case when $\mu$ is small, since $\bar{r}^*$ is a slowly-varying quantity, the term $\log \bar{r}^*(z' + \mu \psi')$ can be expanded about $\mu \psi' = 0$ to give the simplified expression

$$r'_T = -\frac{\bar{\Gamma}}{T^2} \psi' + r'_\infty(z' + \mu \psi') \tag{2.86}$$

where the neglected terms are $O(\mu^2)$. Note that this expression almost exactly matches (2.79), albeit with $r'_\infty$ being evaluated along streamlines $z' + \mu \psi'$. The conclusion is that when $\mu$ is small, dropping all the nonlinear terms in (2.73)-(2.75) is only strictly valid when
\( r'_{\infty} \) is slowly-varying enough such that

\[ r'_{\infty}(z' + \mu \psi') \approx r'_{\infty}(z'). \tag{2.87} \]

However, this need not be the case always, as \( r'_{\infty} \) is a prescribed profile in the model. The vertical scale on which \( r'_{\infty} \) must vary such that (2.87) is not valid can be deduced. In particular, by expanding \( r'_{\infty}(z' + \mu \psi') \) about \( \mu = 0 \) implies that

\[
r'_{\infty}(z' + \mu \psi') - r'_{\infty}(z') = \mu \psi' \frac{dr'_{\infty}(z')}{dz'} + O(\mu^2) \tag{2.88}
\]

where the first term cannot be neglected if \( \frac{dr'_{\infty}}{dz} = O(1/\mu) \). When this is the case, (2.73) and (2.76) can still be used to imply that (2.80) still holds (at least for \( \mu \ll 1 \)), but with the new expression for the liquid water functional:

\[
l'(x', z') = \frac{1}{\Lambda r^* + T^2} (-\psi' + T^2 r'_{\infty}(z' + \mu \psi')). \tag{2.89}
\]
Chapter 3

Numerical Method

The Helmholtz-type equation to be solved for the disturbance streamfunction $\psi'(x,z)$, per the Long’s theory derivation applied to the cloud model, is

$$\sigma^2 \psi''_{xx} + \psi''_{zz} + \bar{N}^2 \psi' = -\Lambda \bar{r}\bar{T}r_l' \quad (3.1)$$

$$\psi'(x,\mu h'(x')) = -h'(x') \quad (3.2)$$

where the right hand side term is defined by the free-boundary of the cloud edge, $r_l' = \max(l',0)$ where

$$l'(x',z') = \frac{1}{\Lambda r^* + T_2}(-\psi + T^2 r_{l\infty}')$$

and $r_{l\infty}'$ is evaluated at $z'$ when it is not rapidly varying, and $z' + \mu \psi'$ when it is.

The nature of this system of equations makes the numerical implementation of the mountain cloud problem of interest. The semi-infinite domain of the PDE, together with the irregular bottom boundary make the problem well suited for a numerical boundary method. As such, Section 3.1 will give a brief introduction to the Method of Fundamental Solutions, a boundary method which has been shown to work well for the dry air flow problem. To extend the method to capture mountain clouds an iterative scheme will be adopted to resolve the free-boundary that is the cloud edge. The iterative method that solves for the cloud corrections to Long’s theory will be discussed in Section 3.2.2. At each iterative step a forced Helmholtz equation will need to be solved, and the grid based finite-difference/quadrature scheme used for this step is discussed in Section 3.2.2 as well.

3.1 Method of Fundamental Solutions

The Method of Fundamental Solutions (MFS) has previously been shown to be an accurate and efficient boundary method to solve the mountain flow problem [17]. Provided one knows a fundamental solution for their operator, the MFS has proven effective for general elliptic
boundary value problems. Furthermore, the MFS provides many of the same advantages as other boundary methods, such as a boundary integral method [4].

In 2D, a fundamental solution $G(x, z)$ for an elliptic operator $\mathcal{L}$ can be defined as satisfying

$$\mathcal{L}G(x, z) = \delta(x, z)$$

(3.3)

where $\delta$ denotes the Dirac delta-function. Then the MFS approximation to the following elliptic boundary value problem with Dirichlet conditions

$$\mathcal{L}u(x, z) = 0, \quad \text{in } \Omega$$
$$u = g(x, z), \quad \text{on } \partial\Omega$$

(3.4)\hspace{1cm}(3.5)

is simply given by the finite sum

$$u_N(x, z) = \sum_{j=1}^{N} q_j G(x - \xi_j, z - \eta_j).$$

(3.6)

The singular points $\xi_j = \{\xi_j, \eta_j\}_{j=1}^{N}$ (also known as source points) are selected to lie outside the PDE domain $\Omega \cup \partial\Omega$, in contrast to a boundary integral formulation where the analogous singular points would lie on the boundary domain $\partial\Omega$. Figure 3.1 shows an example for a simple circular domain.

![Figure 3.1: A basic MFS set-up. The problem domain $\Omega$ (shaded in red) is a circle of radius $r$, and the source points (in blue) are placed on the edge of a larger circle of radius $R > r$.](image-url)
The coefficients $q_j$ are solved for using a least-squares fit to the boundary condition on a set of points $\{x_k, z_k\}_{k=1}^M \subset \partial \Omega$. That is, the $q_j$’s solve the following set of equations:

$$ g(x_k, z_k) = \sum_{j=1}^{N} q_j G(x_k - \xi_j, z_k - \eta_j), \quad k = 1, \ldots, M. \quad (3.7) $$

In the case when $N = M$, this is simply a square solve. The singular points $\xi_j$ can be interpreted as lying on a boundary $\partial \Omega_R$ that encloses the PDE boundary $\partial \Omega$. In this way, the MFS approximation can be seen as a discrete layer-potential approximation on the larger domain enclosed by the boundary $\partial \Omega_R$ [4].

The clear advantage that the MFS shares with a boundary integral equation method is the need to discretize only the boundary of the domain. This can save on computational cost for large domains that are closed, or as is the case in the mountain flow problem where both the domain and the boundary are infinite.

The MFS also provides benefits over a traditional BIE method. Since the singular points are chosen to lie outside the PDE domain, the approximation can be evaluated up to the boundary with high regularity. Furthermore, there is flexibility in the choice of fundamental solution used in the approximation (3.6). The most common choice for problems on closed domains is that $G(\mathbf{x})$ is simply the free-space Green’s function, but in the case of infinite domains it is often useful to use a modified version of the Green’s function to capture the correct far-field behaviour [8].

The main drawback of the MFS is the ill-conditioning of the linear system (3.7). It is well known that for many PDEs and domains, spectral accuracy can be achieved using the MFS, but usually at the cost of ill-conditioned linear solves [4]. The main choice affecting both the accuracy and the conditioning of the approximation is the distribution of the singular points outside the domain $\Omega$.

### 3.1.1 Choice of Singular Points

The most important design choice in a MFS approximation, in terms of controlling error, is the distribution of the singular points $\xi_j$. This distribution generally includes both the number of singular points $N$, as well as their locations.

There is no general theory known for the optimal choice of singular points given an elliptic operator $\mathcal{L}$, domain $\Omega$, and boundary data $g(\mathbf{x})$ [4] [1]. However, error bounds have been proven for the geometry of $\Omega$ being a disk and the source points lying on a larger circle, as is the set-up in Figure 3.1, for both the Laplacian and Helmholtz operators. Reinterpreting these results for more complex problems and domains to make empirical choices of source point distributions has been shown to work well.
For instance, in the case when $\mathcal{L}$ is the Laplacian, and the domain is as in Figure 3.1, the error in the approximation can be shown to be

$$||u_N - u||_\Omega = O((r/R)^N)$$

(3.8)

where $N$ is the number of source points [7]. The conclusion of such a result is that not only can increasing the number of source points decrease the error, but so can placing the source points farther from the boundary $\partial\Omega$.

3.1.2 Helmholtz on Unit Disc

For the case when $\mathcal{L}$ is the Helmholtz operator, Barnett and Betcke have shown similar error bounds as (3.8) for the same domain and source point distribution, in the case of $r = 1$ [1]. The error bounds are slightly more general, as they attempt to incorporate some information about the regularity of the solution.

The general set-up can be seen in Figure 3.2. Their results assume that the boundary data, $g$, can be analytically continued up to a distance $\rho$. That is, the analytically continued boundary data reaches its first singularity at a point distance $\rho$ from the origin. The location of this singularity, relative to the source point distance $R$, controls the convergence rate of the MFS for this problem.

Figure 3.2: The MFS set-up, now including the possibility of singularities outside the domain. This figure shows two cases: in the first case the singularity in the continuation of the boundary data is at the point $Q$, a distance $\rho$ from the origin, and $\rho > R$. In the other case, the singularity is at the point $Q'$, where $\rho' < R$. The comparison of $\rho$ to $R$ becomes important when considering the numerical conditioning of the MFS.
Using this distance \( \rho \), Barnett and Betcke show that the error on the boundary, \( t = \|u_N - g\|_{\partial \Omega} \), is controlled by the following two cases:

\[
t = \begin{cases} 
O(\rho^{-N/2}), & \text{for } \rho < R^2, \\
O(R^{-N}), & \text{for } \rho > R^2. 
\end{cases}
\]  

(3.9)

It was concluded by Barnett and Betcke to be sufficient to bound the boundary error, as it has been shown that the interior error can in turn be bounded by a domain-dependent constant multiplying \( t \). Note that the condition is not really comparing \( \rho \) (a distance) to \( R^2 \) (a squared distance), as this would be a mismatch of dimensions. In reality, the condition is whether or not \( r\rho \) is larger or smaller than \( R^2 \), where \( r = 1 \) is simply the unit radius of the disk.

There are two main conclusions from these error bounds: In the case of \( g \) being relatively regular \((\rho > R^2)\), the error is controlled by the distance of the charge points, \( R \), and agrees with the bound (3.8). In the other case, of relatively irregular boundary data \((\rho < R^2)\), the error is instead controlled by the distance of the singularity, \( \rho \). This second case turns out to be important not only in the case of irregular boundary data, but also in the case of more complex domains, as discussed in Section 3.1.3.

A statement about the ill-conditioning of the solve, again limited to the unit disc, was also proven in [1]. The conclusion was that in the case when \( \rho > R \) the MFS coefficient vector \( q \) remains bounded as \( N \to \infty \). That is, \( |q| < C \) as \( N \to \infty \), where \( C \) is independent of \( N \). However, in the other case, when \( \rho < R \), the coefficient vector sees exponential growth:

\[
|q| \geq C\sqrt{N} \left( \frac{R}{\rho} \right)^{N/2}
\]  

(3.10)

where \( C \) is again independent of \( N \). This result highlights the other main observation regarding source point distribution in the MFS: taking \( R \) arbitrarily large can lead to exponentially large coefficients and ill-conditioned systems.

Though the possible exponential growth of the coefficients with \( N \) does not contradict the error decaying with \( N \) in an analytic sense, it can lead to computational limitations. In particular, the roundoff errors associated with the linear solve for \( q \) are typically of size \( \epsilon_{\text{mach}}|q| \). Therefore it is generally possible, and indeed has been shown in some cases, that the minimum measurable error is orders of magnitude higher than \( \epsilon_{\text{mach}} \) due to the presence of roundoff errors related to exponentially large coefficients [1].

### 3.1.3 General Closed Domains

Many of the same general principles from the unit disk case, in terms of both distance and number of source points, have been observed to hold in the case of more general domains. The error bounds conjectured for the Helmholtz operator on general domains by Barnett
and Betcke depend on the analytic continuation of a solution $u$, denoted $\tilde{u}$. This continuation is defined such that if $u$ satisfies $Lu = 0$ in $\Omega$, then $L\tilde{u} = 0$ in any larger domain $\tilde{\Omega} \supset \Omega$ and $\tilde{u} = u$ on $\partial \Omega$. Singularities in this continuation $\tilde{u}$ have been shown to arise both from singularities in the analytic continuation of the boundary data as well as singularities induced by the shape of the domain $\Omega$.

In particular if the domain $\Omega$ is non-convex, as is the case in Figure 3.3 for a crescent shaped domain, singularities can be present in the continued solutions due entirely to the shape of the boundary $\partial \Omega$, regardless of the regularity of $g$.

![Figure 3.3](image.png)

**Figure 3.3:** A figure showing a crescent shaped domain and the location of a singularity induced by the boundary shape denoted by the star. The blue dots represent a choice of source points based on a conformal mapping principle.

In short, a boundary curve $\partial \Omega$ can be shown to have a Schwarz function $G(z)$, where $z \in \mathbb{C}$. The interpretation of such a function is that for any point $z_1$ close to the boundary, the point $z_2$ defined by

$$z_2 = \overline{G(z_1)}$$

is the reflection of $z_1$ on $\partial \Omega$, where the bar denotes a complex conjugate. It has been shown that in most cases, any singularities of $G$ lying outside $\Omega$ will also be singularities of the analytic continuation of $u$ [12].

Using the example of the crescent domain, any $(x, y)$ point on this boundary can be parameterized by a function $Z(s) = x(s) + iy(s)$ where

$$Z(s) = e^{is} - \frac{a}{e^{is} + b}.$$  \hspace{1cm} (3.12)

The Schwarz function of this crescent domain can be shown to have an exterior singularity at the point $z = \bar{b}^{-1}$, and this point is labeled by a star in Figure 3.3. Given that the case of
circular domains has proven error bounds for one intuitive choice of source points, it seems reasonable to search for a class of source point distributions for which similar results may hold for more complex domains.

That being said, error bounds for the case of more general domains were conjectured by Barnett and Betcke [1]. These error bounds depend on source point distributions that are based on conformal mappings \( \Psi \) of the boundary \( \partial \Omega \) to the unit circle. Specifically, if \( w \in \mathbb{C} \) is a point on the unit circle, then \( \Psi(w) \) would be the corresponding point on \( \partial \Omega \). The source points can then be defined as \( \xi_j = \Psi(\text{Re}^{2\pi ij/N}) \), and the singularity can be characterized by its conformal distance \( \rho \). The conformal charge points for the crescent domain are plotted in blue in Figure 3.3.

Then, based on the unit disk results, it was conjectured that the boundary error \( t \) can be bounded as

\[
t = \begin{cases} 
O(\rho^{-N/2}), & \text{for } \rho < R^2, \\
O(R^{-N}), & \text{for } \rho > R^2.
\end{cases}
\]  

(3.13)

Though it has been shown for computational examples that this error bound is accurate, it has also been shown that placing charge points according to a conformal mapping can be quite limiting. For example, in the case of the crescent domain in Figure 3.3, the conformal distance of the singularity is only \( \rho = 1.1 \), meaning that the maximum radius of the source points that is advisable, \( R = \sqrt{\rho} \), is not much larger than 1.

The authors of [1] have suggested an alternate approach to selecting the source points for general domains. The exact details are not relevant for this work, but the general idea is to choose source points that cluster densely near the singularity at distance \( \rho \), and are placed more sparsely and far from the boundary away from the singularity. This adaptive approach has been shown to work exceptionally well, in some cases converging \( \sim 5 \) times faster than the conformal mapping approach.

The main drawback of the adaptive approach is that it is entirely empirical in nature. There are no theorems or conjectures that suppose its convergence rate. Furthermore, the exact selection of points relies on the selection of a number of parameters that can only be found by sweeping through a range of choices. Lastly, the method does rely on knowledge of the location of the singularity at distance \( \rho \), so that the points can be clustered near it.

This covers the basics of applying the MFS to homogeneous Helmholtz equations on closed domains. These basic principles will be useful in designing an MFS approach to the mountain cloud problem. However, there are two other features of the mountain cloud equations, namely that the domain is infinite and there is a non-zero right hand side, that will require special attention. As such, a review of how the MFS can be applied to problems with these features will be presented in the next two subsections.
3.1.4 MFS for Infinite Domains

The MFS has also been applied to elliptic problems where $\Omega$ is infinite, such as exterior problems. For such problems there is still a boundary condition on $\partial \Omega$, and this boundary may be finite (in the case of some exterior problems) or infinite (as it is in the mountain flow problem). In such problems a far-field condition is required to uniquely solve the PDE. Such conditions might include far-field decay, boundedness, or a radiation boundary condition [8, 17].

The standard choice for implementing the MFS for such infinite domain problems is to find the fundamental solution that already captures the correct far-field behaviour, meaning that the coefficients $q_j$ can still be solved for by fitting the boundary data on $\partial \Omega$. For example, in the case when $\Omega$ is the exterior of the unit circle, and the decay condition is $u \to 0$ as $|x| \to \infty$, a valid fundamental solution would satisfy

\begin{align}
\mathcal{L}G &= \delta(x), \quad \text{in } \Omega \\
G(x) &\to 0, \text{as } |x| \to \infty.
\end{align}

Provided that such a fundamental solution has been found, a distribution of source points still needs to be selected. Much like the case of finite domains, there is no general theory for selecting the source point distribution for infinite domains. For domains that are exterior to some closed boundary curve $\partial \Omega$, it has been shown that the simple choice of choosing the $\xi_j$ source points to lie on a circle that is contained by $\partial \Omega$ works well [8].

At least for the case of the mountain flow problem, one selection of source points has been shown to work quite well, and seems reminiscent of the adaptive scheme mentioned above, insofar as the source points are selected at a shallow depth near the topographic peak (therefore being quite close to the boundary) and to gradually move deeper the farther they move from the peak [17]. Furthermore, the source points are spaced quite closely when they are placed near the boundary, and spread out as they move farther up- and downstream. Further discussion of the source point distribution in the mountain flow problem can be found in Section 3.2.1.

3.1.5 MFS for Forced Problems

The MFS relies on the fact that the set of fundamental solutions $G(x - \xi_j)$ satisfies the homogeneous equation $\mathcal{L}G = 0$ in the domain $\Omega$. In the case of a forced elliptic PDE, such as:

\begin{align}
\mathcal{L}u &= f(x, z), \quad \text{in } \Omega \\
u &= g(x, z), \quad \text{on } \partial \Omega
\end{align}
other numerical techniques must be paired with the MFS [4]. A common class of numerical methods for these problems rely on decomposing the solution \( u \) into a particular and a homogeneous part, \( u = u_P + u_H \), where the particular solution is chosen to solve the equation

\[
\mathcal{L}u_P = f(x, z) \quad \text{in } \Omega
\]  

(3.18)

without regard for the Dirichlet boundary condition. Then the homogeneous solution is found by uniquely solving the homogeneous PDE below:

\[
\mathcal{L}u_H = 0 \quad \text{in } \Omega
\]

(3.19)

\[
u_H = g(x, z) - u_P(x, z) \quad \text{on } \partial \Omega.
\]

(3.20)

The modified BC above ensures that the sum \( u_H + u_P \) satisfies the original BC. The homogeneous problem can of course be approximated using the MFS. This means that the main numerical addition to the problem is in computing the particular solution \( u_P \).

The particular solution to (3.18) is not unique, and there is a selection of approximation methods that have been applied to the problem previously [4]. For this work, the analytical form selected for the particular solution is that of convolution with the free-space Green’s function:

\[
u_P(x, z) = \int_{\Omega} f(x', z')G(x - x', z - z')dx'dz'.
\]

(3.21)

For some cases, the above integral can be evaluated directly using numerical quadrature, particularly when the forcing term \( f \) is known exactly. This will not be the case in the mountain cloud problem, and so an alternate approach is detailed in Section 3.2.2.

Other approximation methods have been studied for the computation of \( u_P(x, z) \). In the case of \( \mathcal{L} \) being the Laplacian radial basis functions have been employed [4]. Use of this type of method relies on knowledge of a set of basis functions \( u_i \) and \( f_i \) for the operator \( \mathcal{L} \) such that \( \mathcal{L}u_i = f_i \). Furthermore, in the case of an infinite domain, the \( u_i \) must also satisfy the correct far-field conditions, making the use of such a method non-trivial for the mountain flow problem.

### 3.2 MFS Applied to the Mountain Cloud Problem

The PDE for the mountain cloud problem has both a non-zero right hand side, as well a free-boundary problem for the cloud edge. The general approach relies on first computing the Long’s theory solution for dry air, and then iteratively solving the free-boundary problem for the correction to this solution due to the presence of cloudy air. Therefore, Section 3.2.1 will outline how the MFS is used to solve the homogeneous Long’s theory equation. Then Section 3.2.2 describes the iterative approach taken to computing the cloud corrections.
3.2.1 MFS for Dry Mountain Flow

One motivator for choosing the MFS as the basis of the computational method is that it has previously been shown to work well for the dry air mountain flow problem. Under the right conditions, the MFS has even been shown to produce smaller errors than a traditional BIE method [17]. In order for this application to work, knowledge of both the free-space Green’s function as well as a valid source point distribution are required. A review of these points follows in the next two sections.

Green’s Function

To employ the MFS for the mountain flow problem, an appropriate fundamental solution needs to be found for the PDE

$$\sigma^2 \psi_{xx} + \psi_{zz} + \bar{N}^2 \psi = 0. \quad (3.22)$$

Since the domain is infinite, the fundamental solution must also satisfy the far-field radiation condition consistent with Long’s Theory. Luckily, the free-space Green’s function for this problem has already been derived by Lyra [10] when $\bar{N} = \sigma = 1$:

$$G(x, z) = -\frac{1}{4} Y_0(r) - \frac{1}{\pi} \sum_{j=0}^{\infty} J_{2j+1}(r) \frac{\cos(2j+1)\phi}{2j+1}. \quad (3.23)$$

This Green’s function has two parts, as illustrated in Figure 3.4. This figure also includes a semi-circular mountain, shaded in grey, for visualization purposes, as well as to mask the singularity at the origin. Plotted in Figure 3.4b is the $-\frac{1}{4} Y_0(r)$ Bessel function, a traditional free-space Green’s function for the Helmholtz equation that radiates in all directions symmetrically from the semi-circular mountain. The solid black lines are contours of $z - \frac{1}{4} Y_0(r)$ and represent the streamlines that would result from the traditional Hankel Green’s function. Notice how the waves emanate out from the mountain both up- and downstream equally. However, the mountain flow problem has a clear upstream/downstream asymmetry. Therefore, the non-singular Bessel series portion is added to cancel upstream radiating waves, giving the expression in (3.23). The effect of this can be seen plotted in the total Green’s function in Figure 3.4a. In particular, the leading order behaviour of $Y_0(r)$ is $O(1/\sqrt{r})$ on the upstream side of the domain. The Bessel series cancels this leading order term, so that $G(x, z) = o(1/\sqrt{r})$ upstream, as is required by the far-field boundary condition of the Long’s theory PDE.

With this Green’s function in hand, a valid approximation of the solution $\psi$ is:

$$\psi_N(x, z) = \sum_{j=1}^{N} q_j G_z(x - \xi_j, z - \eta_j) \quad (3.24)$$
where \( q_j \)'s are the coefficients, and \( (\xi_j, \eta_j) \) represent the singular points, to be discussed in the next subsection. Notice that this approximation uses the \( z \)-derivative of \( G(x, z) \) as opposed to the Green’s function itself. This choice takes advantage of the flexibility of the MFS. All that is required of the fundamental solution is that it satisfies the PDE (with delta-function forcing) and the far-field conditions, which both \( G \) and its derivatives do.

The choice of using \( G_z \) has to do with the downstream behaviour of the solution near \( z = 0 \). The boundary condition

\[
\psi(x, h(x)) = -h(x)
\]

implies that \( \psi \to 0 \) as \( r \to \infty \) near \( z = 0 \). The behaviour of various fundamental solutions near \( z = 0 \) has been plotted in Figure 3.5. It turns out that while \( G \) decays like \( r^{-1/2} \) near \( z = 0 \), the \( z \)-derivative \( G_z(x, z) \) decays like \( r^{-3/2} \) [17]. This extra factor of \( r^{-1} \) is enough to justify using the derivative in place of the original Green’s function. The odd-symmetric Green’s function \( G_{sym} \), defined as

\[
G_{sym}(x - x_0, z - z_0) = G(x - x_0, z - z_0) - G(x - x_0, -z - z_0)
\]

is included for comparison as it will be important when computing the particular solutions in Section 3.2.2. Specifically, since \( G_{sym} = 0 \) on \( z = 0 \) it will be useful for computing the solutions with homogeneous boundary conditions using the linear boundary condition, since \( G_{sym} \) is exactly the Green’s function for \( z' > 0 \) half-space.

**Source Point Distribution**

The distribution in [17] is claimed to be an empirical choice not based on any particular methodology. That being said, the form shares qualitative similarity with the adaptive method suggested by [1]. The source points are determined by the collocation points on the
Figure 3.5: Downstream decay comparison of three fundamental solutions for the mountain flow Helmholtz problem. Figure (a) shows contour plots of $G_z$ and $G_{sym}$, where it should be noted that near $z = 0$ both display stronger decay than $G$ itself (contrast to Figure 3.4a). An offset of $x_0 = (0, 1/2)$ has been included in these plots. The downstream decay is highlighted in (b) where a vertical slice at $z = 0.1$ has been plotted for all three versions of the Lyra’s Green’s function.

boundary, $(x_j, h(x_j))$ via

$$\xi_j = x_j + \frac{\alpha s_j h_x(x_j)}{\sqrt{1 + h_x(x_j)^2}} \quad (3.27)$$

$$\eta_j = h(x_j) - \frac{\alpha s_j}{\sqrt{1 + h_x(x_j)^2}} \quad (3.28)$$

where $\alpha$ is a tuning parameter, and $s_j$ is the spacing of the points $(x_j, h(x_j))$ along the curve $h(x)$, given by the expression:

$$s_j = D_2 - \frac{D_2 - D_1}{\cosh^2(j/M)}, \quad j = -M, -M + 1, \ldots, M, \quad (3.29)$$

so that $s_j$ nearly equals $D_1$ near the origin ($m = 0$) and nearly equals $D_2$ at the farthest points. In this expression $D_1$ and $D_2$ are constants that are chosen to minimize the error of the MFS approximation.

The interpretation of such a distribution is that the source points are placed normal to the boundary topography, at a depth of $\alpha s_j$. The distribution is plotted in Figure 3.6a. It can be observed in this figure that the distribution behaves like the adaptive method suggested in [1]: the source points are clustered near the boundary close to the topographic
peak (where there may be a boundary induced singularity) and are placed farther away up- and downstream.

![Graph showing source point distribution](image)

(a) Distribution used in [17]

(b) Distribution used for this work

Figure 3.6: Source point distribution used by [17] in (a), and by this project in (b).

Though this distribution of source points has been shown to work well for the cases studied by [17], there are some limitations when applied to the mountain cloud problem. First, there is a large number of parameters that needed to be tuned to achieve the reported accuracy. It is not clear that this choice of parameters is robust to a number of physical cases, such as $\sigma \neq 1$ or $\bar{N} \neq 1$, or drastically different topographies. Second, as demonstrated in Figure 3.6a, the source points extend into the $z > 0$ half-plane. This will become an issue when computing the cloud corrections, as the method employed requires the solution to be extended to the positive half-plane. If there are singularities located there, then the extended solution will be singular, and the method will not work.

In lieu of this distribution, a significantly simpler choice has been made, namely, to simply place all the singular points on a shifted axis at depth $-R$ from the $z = 0$ axis. Specifically,

$$\xi_j = x_j$$ \hspace{1cm} (3.30)

$$\eta_j = -R$$ \hspace{1cm} (3.31)

The collocation points $(x_j, h(x_j))$ are simply selected with equal spacing $d_x$ in the horizontal. This distribution is also shown in Figure 3.6b. The inspiration for this choice comes from the case of a linearized boundary condition; when $H_m/H \ll 1$, the bottom topographic condition can be replaced by

$$\psi'(x', 0) = -h'(x')$$ \hspace{1cm} (3.32)

in which case the PDE domain really is the positive half-space. In this linearized case, the choice of a shifted axis is influenced by the unit circle domain, where simply choosing a
source point distribution identical to the boundary, but lying outside the domain, is found to be suitable.

This simple choice, which is used in both the linear and nonlinear boundary condition cases, has a few advantages over more sophisticated choices. First, it ensures that any approximate solutions are non-singular in the positive half-plane, as is required for the cloud correction solve. Secondly, with only one control parameter (the depth \( R \)) the method is easily implemented and tuned.

The main disadvantage would appear to be a loss of accuracy. As has already been mentioned, the distribution used in [17] bears a qualitative resemblance to the adaptive scheme proposed in [1], which was shown to be incredibly accurate, even compared to the conformal mapping distribution. Conversely, the choice of a shifted axis would be more comparable to solving the crescent domain in Figure 3.3 using a circular distribution of source points that contain the singularity. This would be a case in which convergence is limited by the singularity distance, not the source point distance.

However, these accuracy limitations are not a large concern for this method. As is explained in Section 3.2.2, the computation of the particular solution uses a finite-difference solve, which is only 2nd-order accurate. Therefore the accuracy of the entire approximation is limited by the finite-difference step, and therefore the MFS approximations need not be of exceptional accuracy, only comparable to the 2nd-order results.

### 3.2.2 Iterative Cloud Corrections

To extend from Long’s theory and solve (3.1), the total solution \( \psi' \) will be decomposed as

\[
\psi' = \psi'_L + \psi'_c
\]  

(3.33)

where \( \psi'_L \) is the Long’s theory solution solving

\[
\sigma^2(\psi'_L)_{xx} + (\psi'_L)_{zz} + \bar{N}^2 \psi'_L = 0
\]  

(3.34)

\[
\psi'_L(x', \mu h'(x')) = -h'(x')
\]  

(3.35)

and the cloud corrections portion \( \psi'_c \) is defined to solve

\[
\sigma^2(\psi'_c)_{xx} + (\psi'_c)_{zz} + \bar{N}^2 \psi'_c = -\Lambda \frac{\bar{F}^*}{T} r''_l
\]  

(3.36)

\[
\psi'_c(x', \mu h'(x')) = 0.
\]  

(3.37)

This corrective approach captures the fact that while \( \psi'_L \) is \( O(1) \), the corrections \( \psi'_c \) may be small, depending mostly on the size and location of the clouds forming in the atmosphere. The method also separates the two main driving causes of the flow: the presence of the mountain, captured by \( \psi'_L \), and the presence of the cloud, captured by \( \psi'_c \).
To account for the nonlinearity of the free-boundary solve in (3.36), an iterative approach will be taken. An approximation to (3.34) can be found using the MFS with the singular point distribution given by (3.30) and (3.31). Then this solution can be used to generate an initial guess for the location of the cloud edge, defined by the curve $l'_1 = 0$, where

$$l'_1 = \frac{1}{\Lambda r^* + T^2}(\bar{T}^2 r'_\infty - \psi'_L).$$  \hfill (3.38)

The first iteration of $\psi'_c$, denoted $\psi'_1$, can then be computed as the solution to

$$\sigma^2(\psi'_1)_{xx} + (\psi'_1)_{zz} + \bar{N}^2 \psi'_1 = -\Lambda \bar{r}^* r'_{l,1}$$

$$\psi'_1(x', h'(x')) = 0$$  \hfill (3.39)

with $r'_{l,1} = \max\{l'_1, 0\}$. Then a second guess for the cloud edge $l'_2$ can be computed using (3.38) but replacing $\psi'_L$ by $\psi'_L + \psi'_1$.

In general, at the $j$th iteration $\psi'_c \approx \psi'_j$, where $\psi'_j$ satisfies the forced Helmholtz equation

$$\sigma^2(\psi'_j)_{xx} + (\psi'_j)_{zz} + \bar{N}^2 \psi'_j = -\Lambda \bar{r}^* r'_{l,j}$$

$$\psi'_j(x', h'(x')) = 0$$  \hfill (3.40)

with $r'_{l,j} = \max\{l'_j, 0\}$ and the $j$th liquid water functional $l'_j$ being defined as

$$l'_j = \frac{1}{\Lambda r^* + T^2}(\bar{T}^2 r'_\infty - (\psi'_L + \psi'_{j-1})).$$  \hfill (3.41)

The solution to the forced Helmholtz equation (3.41) can be approximated by computing a particular solution of the form

$$\psi'_{P,j}(x', z') = \int \int_{l'_j > 0} -\Lambda \bar{r}^* r'_{l,j}(x, z)G(x' - x, z' - z)dx\,dz$$  \hfill (3.42)

as well as a homogeneous solution $\psi'_{H,j}$ with bottom boundary condition $\psi'_{H,j}(x, h(x)) = -\psi'_{P,j}(x, h(x))$. The homogeneous solution is solved using the MFS again.

**Computation of Particular Solution**

Though the particular solutions $\psi'_{P,j}$ are given by the expression (3.44), they are not computed by direct evaluation of this integral everywhere in the domain. This is due to the fact that the kernel, $G(x' - x, z' - z)$ is singular whenever $(x', z')$ lie within the cloud.

However, a key point in the integral form is that the integration only needs to be carried out over the cloudy region, $l'_j > 0$, since this is where the forcing is localized. The strategy employed for this project is pictured in Figure 3.7. The integral (3.44) is evaluated
everywhere outside of a rectangular box (denoted by green x’s) containing the cloudy region, and then using the boundary values on this box carry out a finite-difference solve of the Helmholtz equation to fill in the interior. In the most extreme cases it is possible to take the box to be the entire computational domain, therefore carrying out the quadrature only to determine the boundary values. In this case the finite-difference solve is carried out over the whole computational domain, including the mountain. This is why it is important that the singular points lie entirely outside.

Figure 3.7: Cartoon depicting the scheme used for computation of the cloud corrections to Long’s theory. The grey shaded region denotes the cloud location, and the green x’s denote the boundary of the box on which the finite-difference solve is carried out. The purple points represent the discretized cloud boundary that is integrated over in computing the particular solution. Lastly, the blue stars are the singular points used in the MFS, and the red points show the discretized bottom boundary.

However, this strategy is not robust in general. This is because the Helmholtz equation (3.36) can have a homogeneous solution $\phi$ satisfying

$$\sigma^2 \phi_{xx} + \phi_{zz} + \tilde{N}^2 \phi = 0$$

$$\phi \bigg|_{\partial \Omega_C} = 0$$

$$\phi \bigg|_{\partial \Omega_C} = 0 \quad (3.45)$$

$$\phi \bigg|_{\partial \Omega_C} = 0 \quad (3.46)$$
where \( \Omega_C = [-L_x, L_x] \times [0, L_z] \) is the rectangle representing the computational domain. Such a solution \( \phi \) exists whenever

\[
N^2 = \left( \frac{n\pi}{L_z} \right)^2 + \left( \frac{m\pi}{2L_x/\sigma} \right)^2, \quad n, m = 1, 2, 3, \ldots
\]  

which then becomes a restriction on the size of the computational domain. This is because whenever \( N^2 \) is close to one of the resonance wave numbers given above, the finite-difference solve can become corrupted by the homogeneous solutions. Numerically this is a statement on the ill-conditioning of the finite-difference matrix, and is connected to issues of solving the Helmholtz equation numerically for high wavenumber. Reducing the size of the finite-difference solve to a smaller box only containing the cloudy region alleviates this problem somewhat.

The integral (3.44) is approximated by integrating only within the cloudy region \( l_j \geq 0 \). The boundary of this cloudy region is approximated by a 'cloud polygon', pictured as purple points in Figure 3.7. This polygon is found in the following way: a bilinear interpolant of \( l_j(x, z) \), labeled \( L_j(x, z) \) is computed using the values of \( l_j \) on the equispaced grid. Then the following set of 1D root-finding problems can be solved using a standard root-finding algorithm:

\[
L_j(x_i, z) = 0, \quad i = 1, \ldots, N_x \tag{3.48}
\]

\[
L_j(x, z_k) = 0, \quad k = 1, \ldots, N_z. \tag{3.49}
\]

This approach will locate a series of points lying on the cloud edge that also intersect with grid-lines. Then these points form the vertices of a polygon approximating the cloud shape. In practice, these points are equivalent to those found using Matlab's contour function, and so this function is used to locate all approximate points on the cloud edge. Replacing the integrand in (3.44) by its own bilinear interpolant, the integral can be approximated using Matlab’s quad2d routine, and integrating only inside the regions of the cloud polygon. See Figure 3.8 for a visual representation.

In reality, when computing (3.44), the free-space Green’s function \( G(x, z) \) is not used, but instead replaced by its odd symmetric counterpart \( G_{sym}(x - x_0) \), defined as

\[
G_{sym}(x - x_0, z - z_0) = G(x - x_0, z - z_0) - G(x - x_0, -z - z_0). \tag{3.50}
\]

This modified Green’s function has the property that \( G_{sym}(x, 0) = 0 \), and so it exactly captures the correct boundary condition in the case of the linearized bottom boundary. However, using the symmetric Green’s function is in fact always a sound choice, even for the nonlinear boundary condition. This is because \( G_{sym}(x, z) \) shares the favourable downstream decay property of \( G_z(x, z) \), as can be seen in Figure 3.5. Therefore, computing par-
Figure 3.8: The approximate location of the cloud edge is found via interpolation. The grey shaded region above represents the true location of the cloud, and the red dots indicate the the approximate cloud edge locations found on the grid-lines (plotted in black). The red dots form a polygon, inside which the numerical quadrature routine integrates a bilinear interpolant of the integrand in (3.44).

ticular solutions using $G_{sym}$ and homogeneous solutions using $G_z(x, z)$ ensures a consistent downstream decay near $z = 0$. 

Chapter 4

Numerical Convergence and Scientific Insight

The mountain cloud problem provides an interesting learning opportunity both in terms of the numerical analysis and the scientific results. From a numerical perspective the scheme uses the MFS as its basis, with a simplified source point distribution that must be validated in terms of accuracy. Furthermore, this is the first such computation of the moist Boussinesq extension to Long’s theory, meaning that the iterative scheme must be validated in terms of its ability to efficiently and accurately resolve cloud-edges. From a scientific point of view, since the mountain cloud results will represent some of the first results to come out of the moist Boussinesq model it is important to take stock of what physical insights can be obtained. Analyzing the results in terms of both how cloud presence might affect atmospheric flows, and checking certain steady-state edge phenomena will serve as an initial validation of the model.

4.1 Numerical Convergence

4.1.1 MFS Convergence

As was explained in Chapter 3 both the rate of convergence and ill-conditioning of the MFS approximation are controlled by the source point distribution. Because of the simplified source point distribution of a shifted axis, described by

\[ \xi_j = x_j, \quad j = 1, \ldots, N \]
\[ \eta_j = -R, \quad j = 1, \ldots, N \]

(4.1)

(4.2)

there is only one control parameter, the depth \( R \) (not counting the number of points, \( N \)). To set \( R \), a technique will be borrowed from [17]. In that study, where the MFS was also applied to the mountain flow problem, they had two characteristic depths for source points,
called $D_1$ and $D_2$. The depth $D_1$ dominated near the origin, while $D_2$ dominated far up and downstream. See Figure 3.6 for an illustration.

Importantly, in the parameter sweep conducted in that study it was found that a near optimal choice for the shallower depth was one that depended on the number of source points as $D_1 = 8/N$. It was not explained why this choice should work, but intuitively it seems to strike a balance between the fact that MFS solutions tend to become more accurate (at least in an analytic sense) as $R$ and $N$ increase, but also that the linear solves become more ill-conditioned, leading to potentially severe round-off error. Therefore decreasing $R$ as $N$ grows appears to strike the right balance to preserve both convergence as well as numerical stability.

Indeed, if a uniform depth of $R$ is used, independent of $N$, then the condition number $\kappa$ for the linear solve has been observed to grow to the order of $1/\epsilon_{\text{mach}}$ for relatively small values of $N$. Adopting the choice of an adaptive depth, similar to that of [17]:

$$R(N) = \frac{2}{N}$$

leads to significantly better conditioning, as well as bounded coefficient norms. The value of 2 in the numerator was found by simply testing a selection of numbers near 8 (based on the study in [17]). In reality it was found to only make a small difference to the measured errors when other factors were used.

To illustrate the rate of convergence for the MFS, a test problem of dry air flow over mountain topography will be used. The mountain height is set at $H_m = 100$ m, with the upstream wind being set as $U_\infty = 20$ m/s and a Brunt-Väisälä frequency being set as $\tilde{N} = 0.8$. This implies a height scale of $H = 1069$ m, and for a non-hydrostatic atmosphere the length scale $L$ is set as the same, so that $\sigma = 1$. The nonlinearity parameter $H_m/H \approx 0.1$ is small enough to justify using the linear bottom boundary condition. The thermodynamic reference state is taken as $T_0 = 273$ K and $p_0 = 1000$ hPa at a height $z_0 = 0$. For the sake of comparison the computations are repeated using both the adaptive choice of $R$ in (4.3) and a constant choice of $R = 0.2$.

The comparison between the two choices in terms of condition number $\kappa$ and coefficient norm

$$|q| = \sqrt{\sum_{j=1}^{N} |q_j|}$$

is shown in Figure 4.1. It becomes clear that the constant choice of $R = 0.2$ works well for a moderate number of points $N$, but tends to lead to poorly conditioned solves for larger systems. In contrast, the adaptive choice produces both bounded condition numbers and coefficient norms.

The error associated with these source point distributions is estimated by comparison to a fine grid solution. Specifically, an approximate solution $\psi'_{\text{fine}}$ is computed using $N_{\text{max}} =$
Figure 4.1: Comparison of the non-adaptive choice of source point depth, $R = 0.2$ versus the adaptive choice given by equation (4.3). Figure (a) shows how the condition numbers behave as $N$ grows. Figure (b) shows how the coefficient norms behave. In both plots red circles are the adaptive scheme and blue stars are non-adaptive.

6 · $2^7$ source points and collocation points (the factor of 6 comes from the horizontal width of the domain, where the height is 3). Approximate solutions $\psi'_N$ are computed for $N = 6 · 2^3, ..., 6 · 2^6$, and the relative error for each is estimated as

$$E_N = \frac{\max_{j,k} |\psi'_N(x_j, z_k) - \psi_{\text{fine}}(x_j, z_k)|}{\max_{j,k} |\psi_{\text{fine}}(x_j, z_k)|} \quad (4.5)$$

where the $(x_j, z_k)$ are points that lie on both the coarse and fine grids. The error plot is included in Figure 4.2. The fixed choice of $R$, plotted in blue, appears to produce smaller error estimates for all $N$ tested. However, it should be noted that the gap between the fixed scheme and the adaptive scheme (plotted in red) is shrinking as $N$ grows. It seems possible that for larger values of $N$, the numerical ill-conditioning of the fixed scheme would limit the minimal achievable error.

Furthermore, the rate at which the error decreases using the adaptive scheme is comparable to that computed in [17]. For mountains of height $H_m/H = O(1)$, they estimate errors on the order of $E_N = 10^{-3}$ for $N = 2^8$ source points. Compared to the smallest error estimated here, which is also $E_N \sim 10^{-3}$ using $N = O(2^8)$ source points, the comparisons are fairly strong. Therefore it would appear that at least for the case of the linear boundary condition the simplified choice of source points does not lead to a significant reduction in accuracy.
Figure 4.2: Estimated errors for the MFS computation of dry air flow over mountain topography. The errors are estimated by comparison to an approximate solution that uses \( N = 6 \cdot 2^7 \). Red circles use the adaptive scheme \( R = 2/N \), and blue stars use the constant scheme \( R = 0.2 \).

4.1.2 Convergence of Particular Solutions

The particular solution at the \( j \)th iteration is computed to have the form

\[
\psi_{j,P} = \int \int_{l_j' \geq 0} f_i(x', z') G_{sym}(x - x', z - z') dx' dz'
\]  

(4.6)

where the integration only needs to be carried out over the cloudy region \( l_j' \geq 0 \) since the forcing vanishes everywhere else. The integration formula is used to compute \( \psi_{j,P} \) on the exterior of a rectangular box containing the cloud, and a standard finite-difference scheme based on a 5-point Laplacian stencil is used to solve for the interior values.

Both the integration scheme and finite-difference scheme work out to be 2nd-order accurate. To test the convergence rate, the exact same experimental set-up as in Section 4.1.1 was used. The finest grid solution, which uses uniform grid spacings \( d_x = d_z = 1/128 \), is used to generate the grid values for \( l_1'(x, z) \) that are used in the forcing term for all resolutions. This is done to remove the influence of the MFS from each finite-difference solve.
The finest grid overlaps with all coarser grids, and so no interpolation is needed to generate the forcing term for each solve. The errors are estimated according to (4.5), but with $\psi_{1,P}'$ replacing $\psi'$. For these error estimates the quadrature formula is used only on the boundaries with a finite-difference everywhere in the interior.

Figure 4.3: Estimated error convergence for the 2nd-order accurate computation of the particular solutions. The blue circles represent the error estimates of the combined quadrature/finite-difference scheme, and the black reference line represents exact 2nd-order accurate convergence. Here $N$ is the number of points taken in the horizontal direction.

Figure 4.3 displays the computed error estimates, along with a line representing the idealized 2nd-order accurate convergence. The estimated rate of convergence coming from the best fit slope is 2.09 which explains the agreement between the measured errors in red and the idealized error.

It can be noted that the estimated relative errors of the particular solution coming from the quadrature/finite-difference scheme are comparable to the errors coming from the MFS approximation for $\psi_L'$, with both producing $E_N \approx 10^{-3}$ at the finest resolution measured. It is important to compare relative errors because the corrections $\psi_1'$ tend to be an order of magnitude smaller than the Long’s theory solution. This agreement of the errors serves to justify the simplified choice of source points in the MFS, as was explained in Chapter 3.
4.1.3 Convergence of Iterations

The tolerance on the iterative corrections is measured in terms of the relative change of the liquid water mixing ratio. The relative change can be defined as

\[ e_j = \frac{\max(r_{l,j} - r_{l,j-1})}{\max(r_{l,j-1})}. \]  

(4.7)

The relative corrections could be defined in terms of \( \psi' \) with little effect on the overall number of iterations typically required. However, using the liquid water mixing ratio instead ensures that the cloud-edge is being computed accurately.

Figure 4.4: Examples of two different cases used to measure the convergence of the iterative corrections. Each run uses the same physical set-up, with the exception of the peak of \( r'_{\infty} \) being adjusted to produce a smaller cloud in (a) and a larger cloud in (b).

The primary impact on the speed of the convergence seems to be the size of the resultant cloud. Figure 4.4 shows examples of two runs, each set up identically to the basic example in the previous section, with the only difference being that the upstream profile \( r'_{\infty}(z') \) has been adjusted in (b) to be both wider and have a larger peak value, thus generating a larger cloud. Specifically:

\[ r'_{\infty}(z') = \begin{cases} 0.7 \exp(-(z' - 1)^2/0.2^2) - 1, & \text{in (a),} \\ 0.8 \exp(-(z' - 1)^2/0.4^2) - 1, & \text{in (b).} \end{cases} \]  

(4.8)

Both runs were set to converge within a tolerance of \( e_j < 10^{-10} \).

Figure 4.5 shows the measured corrections at each iteration. It is clear from the relative slopes of each plot that the larger cloud exhibits a slower rate of convergence. Furthermore, the fact that the larger cloud also starts with a higher relative correction is indicative of
the fact that, relative to a Long’s theory initial solution, a larger cloud must have a larger effect on the atmosphere.

Figure 4.5: Plot of the relative corrections to $r'_1$ at each iteration, for two cases of differing cloud size. The blue dots represent the error computed for the small cloud in Figure 4.4a, and the red dots are the corrections for the large cloud in Figure 4.4b.

### 4.2 Basic Mountain Clouds

An illustrative example of a standard mountain cloud will be used to highlight some of the basic features before discussing a more diverse set of examples.

To understand where mountain wave clouds might form within a flow, it is instructive to refer to (2.81), but replacing $\psi'$ by the equivalent $(1/N^2)\theta'$. This gives the expression

$$l'(x',z') = \frac{1}{\Lambda r^* + T^2} \left( -\frac{1}{N^2} \theta' + \bar{T}^2 r'_\infty(z') \right)$$

where $r'_\infty(z')$ is the upstream profile of $r'_T$. Since $r'_\infty$ is a prescribed function, the only dependence of $l'$ on the solution is through $\theta'$. Indeed, $r'_\infty < 0$ for all $z'$, and so cloudy regions (where $l' > 0$) can only form when $\theta'$ is suitably negative. In other words, in the
subsaturated atmosphere sufficient cooling through $\theta'$ is required in order to condense water vapour into cloud.

Figure 4.6: A standard wave cloud forming over a Gaussian topography of height $H_m = 100$ m. The upstream water profile, $r'^\infty(z')$ is plotted in (a), and it can be observed that the humidity peak lines up with the location of the cloud in (b). In (b) black lines indicate total streamlines, and blue contours are $\theta'$ indicating regions of cooling (dashed) and warming (solid), with the zero contour bolded.

This intuition is supported by Figure 4.6. This basic run uses an upstream wind of $U^\infty = 20$ m/s blowing over a mountain of height $H_m = 100$ m. The Brunt-Väisälä frequency is set at a constant $\bar{N} = 0.8$, and the reference temperature and pressure on the surface are $T_0 = 273$ K and $p_0 = 1000$ hPa. These values are consistent with a height scale of $H = 1069$ m, indicating that the flow is suitably linear at $\mu = 0.094$. The length scale $L$ is set equal to the height scale $H$, so that the flow is non-hydrostatic at $\sigma = 1$. Plots of the background thermodynamic profiles are included in Figure 4.7.

Both the mountain profile $h'(x')$ and the upstream water profile $r'^\infty(z')$ are Gaussians given by

$$h'(x') = e^{-x'^2}$$
$$r'^\infty = 0.7e^{(z-1)^2/(0.3)^2} - 1$$

where the $r'^\infty$ profile is plotted in Figure 4.6a. This profile peaks at $z = 1$ at a value $r'^\infty = -0.3$, which translates to a relative humidity of 97.9%. For comparison, the relative
humidity away from the peak is about 93%, and a fully-saturated atmosphere would have a relative humidity of 100%.

The dashed contours in Figure 4.6b indicate negative values of $\theta'$, where the atmosphere is being cooled relative to the background state. As such, a cloud (shaded in grey) forms in the region of the humidity peak of $z' = 1$. The cloud is top/bottom asymmetric about the peak location of $z' = 1$, representing the fact that cooling is stronger closer to the mountain surface, causing cloud to form more readily on the lower half. The upstream/downstream asymmetry is representative of the fact that the downstream gravity waves cause alternating warming and cooling patches to form, where the first of the warm patches is indicated by the solid blue contours. Upstream, where there is negligible wave propagation, only cooling can occur.

### 4.3 Effects of Cloud on the Atmosphere

When water vapour condenses into liquid form, therefore forming cloud, energy in the form of latent heat is released. In the iterative computations this is represented by the fact that the corrections tend to warm the atmosphere in the cloudy region relative to the Long’s theory solution. In other words, in the absence of the heating effect, Long’s theory tends to predict atmospheres that are slightly cooler than those that are consistent with the presence of cloud.
This warming in the vicinity of the cloud can be directly observed in the iterative convergence of the solutions. Figure 4.8 shows the results of a computation using a nearly identical set-up as in Section 4.2, except with the upstream water profile replaced by

\[
r'_{\infty}(z') = 0.8e^{-\left(z' - 1.0\right)^2/(0.5)^2} - 1
\] (4.12)

which has both a more saturated peak value of \(-0.2\), and a wider profile. The other change is to increase the constant value of \(\bar{N}\) from 0.8 to 0.85 in order to preserve cloudy air stability in both reference states studied. The resultant cloud location can be seen in Figure 4.8 for different choices of \(T_0\), along with a plot of the \(r'_{\infty}\) profile.

Figure 4.8: The effect of latent heating on the atmosphere due to cloud presence. Both figures depict the same set-up, with blue contours indicating converged \(\theta'\) and green contours indicating the Long’s theory \(\theta'\). The fact that blue contours are contracted from the green ones indicates warming in the vicinity of the cloud and cooling downstream. In (b) the larger value of \(T_0\) causes the latent heat effect to be stronger, which explains why the blue contours are slightly further contracted than in (c).

The reason for testing two different values of \(T_0\) is to probe the effect of latent heating. The latent heat parameter \(\Lambda\) increases with \(T_0\), therefore emphasizing the impact of latent heating on the atmosphere. The example of \(T_0 = 20^\circ\text{C}\) is consistent with a typical summer temperature and has a value of \(\Lambda = 2.28\), and \(T_0 = 0^\circ\text{C}\) would be winter with a value of \(\Lambda = 0.69\). Figure 4.8 supports the notion that the warming effect is due to latent heating, as the warmer \(T_0\) sees a larger contraction of the iterated contours compared to the Long’s theory solution.

That being said, the difference between the two cases is slight, and furthermore the change from the Long’s theory solution is not large in either case. The lack of distinction between the two cases may be attributed to the fact that the model deals with multiplicative disturbances. This means that the results being analyzed are only disturbances relative to the background state, and so Figure 4.8 indicates that the relative change to the atmosphere

53
due to the latent heat release is only weakly dependent on the thermodynamic reference state itself.

The largest deviations in both of the examples occur near the centre of the cloud, at height \( z' \approx 1 \). This implies that the strongest warming effects are fairly local to the humidity peak. Furthermore, comparing the blue and green contours in 4.8, the disturbances resulting from both the cloud and the mountain (in blue) look similar to the disturbances that would result from flow over a slightly smaller mountain.

This is further supported by Figure 4.9, which shows the corrections to the \( \theta' \) field due to the cloud, computed as \( \bar{N}^2 \psi'_c \). As expected, there is a significant warm patch sitting right on top of the cloudy region, due to the latent heating effect. Slightly downstream of the cloud is a cooling region, indicated by the dashed contours. This wave pattern of successive heating and cooling due to the cloud is the same as the disturbance pattern produced by the mountain, albeit with the signs reversed and at a considerably smaller magnitude. This is further highlighted by the velocity vectors plotted in Figure 4.9, which represent the disturbance velocities coming from \( \psi'_c \) alone. Near the tail of the cloud there is a downward acceleration on the flow. Then due to the vertically stratified nature of the atmosphere, there is a buoyancy response on this displaced air that drives it back up near the head of the cloud. Furthermore, these correction velocities point against the direction of the Long’s theory disturbances, which can be seen by comparing the vector directions to the solid black streamlines. In this sense, the cloud corrections to the Long’s theory solution are counteracting the disturbances caused by the mountain, but to a relatively small degree.

### 4.4 Cloud-Edge Velocity

Accurately computing edge motion is one of the main goals of the moist Boussinesq fluid model. Many cloud phenomena, such as the holepunch cloud discussed in Chapter 1, typically require an understanding of how the cloud-edge motion depends on both advection by the wind and phase change processes. Being able to capture fine scale cloud-edge dynamics within the nearly-linear moist Boussinesq system is an attractive potential benefit of the model.

It was shown in [14] that within the moist Boussinesq model, the velocity of a point \( \vec{x}_c \) on the cloud-edge can be computed as

\[
\frac{d\vec{x}_c}{dt'} = 1 + \mu \vec{u}_c' - \left( \frac{Dl'}{Dt'} \right) \frac{\nabla' l'}{\left| \nabla' l' \right|^2} \bigg|_c
\]

where the term \( 1 + \mu \vec{u}_c' \) represents advection by the upstream and disturbance winds, and the final term encompasses phase change effects. The \( c \) indicates that all values must be taken on the cloud-edge.
Figure 4.9: Plot of corrections to potential temperature due to the presence of cloud, $\theta' - \theta_{\text{dry}}'$. The solid blue contours emanating from the cloud indicate a warming effect that can be explained by the release of latent heat. The red velocity vectors represent $\vec{u}_c'$, the corrections made to the dry velocity field $\vec{u}_\text{dry}'$ due to the presence of cloud. The largest of these arrows corresponds to a relative value of $|\vec{u}_c'|/|\vec{u}_\text{dry}'| = 5.8 \times 10^{-3}$.

Notice that even in the case of a time steady problem the right hand side need not be zero. However, the computed velocities must be consistent with the overall cloud-edge remaining stationary. Therefore there can be nonzero $d\vec{x}_c/dt$ on the cloud-edge, but it must point perpendicular to the outward normal in order to maintain the time steady cloud shape, and so it is only the normal component of $d\vec{x}_c/dt$ that dictates cloud motion.

Figure 4.10 shows a vector plot of the computed velocities for the cloud in Section 4.2. As expected, the velocity vectors point entirely parallel to the cloud-edge. Though the time-steady mountain cloud is essentially as simple a case as possible for which to compute edge velocities, it is still promising to see that results match what is expected physically.

The perhaps somewhat counterintuitive result that there can be nonzero velocity on the edge of the cloud without the actual cloud body moving is exactly what makes the study of cloud dynamics so fascinating. In fact, mountain clouds such as these represent a relatively simple example, and the effect can even be observed in many time-lapse videos of
the phenomena. This processes can be understood physically by picturing a moist air parcel moving with the flow: as the parcel is lifted into the humidity peak it cools, causing the water vapour to condense at the cloud tail. This cloudy air then continues to be advected along a streamline until it reaches the rear edge of the cloud, where the air is driven back down and warms, causing the liquid water to evaporate. In this idealized example of a constant horizontal wind, the continual condensation at the tail and evaporation at the head results in a steady state in which the visible cloud remains entirely stationary.

4.5 Rapidly Varying $r'_\infty$ and Lenticular Clouds

Lenticular clouds have potentially the most distinct edge shape of all topographically generated clouds. They are typically thin, smooth clouds with an apparent concavity that makes them look like a large lens sitting aloft of the mountain. See Figure 4.12b for one such example. The thinness of such clouds would seem to indicate that they could form if a narrow enough band of moist air was advected over the mountain top. In the context of the moist Boussinesq fluid model, a critical feature of such a narrow band of moist air is the effect it has on the upstream profile $r'_\infty(z')$.

As explained in Section 2.2.4, a important scale for the mountain cloud problem is the size of $dr'_\infty/dz'$. In particular, if $r'_\infty(z')$ is relatively slowly varying (compared to a value of $1/\mu$), it is sufficient to evaluate $r'_\infty$ as a function of $z'$ only. However, if

$$
\frac{dr'_\infty}{dz'} = O(1/\mu)
$$

(4.14)
it is in fact necessary to incorporate nonlinear effects by evaluating the upstream water profile on streamlines, \( r'_\infty(z' + \mu \psi') \).

This phenomenon is showcased in Figure 4.11 (a) and (b), where both use an upstream water profile consistent with

\[
r'_\infty(z') = 0.3 e^{-(z-1)^2/c_w^2} - 1
\]  

(4.15)

where \( c_w \) represents the cloud width parameter. Furthermore, both figures use the nonlinear evaluation of \( r'_\infty \) to compute the cloud-edge. In (a), \( c_w = 0.3 \), and so the liquid water profile is identical to the computations in Section 4.2. Due to the relatively wide nature of this Gaussian profile, Figure 4.11a is nearly indistinguishable from the linear computation in Figure 4.6 (a).

In contrast, the nonlinear evaluation of \( r'_\infty \) presents a large effect in the case when the profile is rapidly varying. Figure 4.11b supports this conclusion. In this computation the width parameter was taken to be \( c_w = \mu \). The shaded grey region indicates the cloud resulting from the nonlinear effects of \( r'_\infty \), and the red dashed contour indicates the cloud-edge that would result from the purely linear computation. The key difference is that the nonlinear computation generates a cloud that conforms to the streamlines of the flow, accounting for the fact that when \( r'_\infty \) is a rapidly varying profile, its nonlinear advections become important.

This incorporation of nonlinear effects turns out to be essential in reproducing the oft observed lens shape of lenticular clouds. Figure 4.12 demonstrates the results of a computation using the upstream water profile

\[
r'_\infty(z') = 0.7 e^{-(z-0.3)^2/(0.2\mu)^2} - 1
\]  

(4.16)

with a mountain of height \( H_m = 200 \) m. All other parameters are the same as in previous examples, so that \( \mu = 0.187 \). The humidity peak has been lowered in order to accentuate the concave shape of the cloud. Figure 4.12 (a) shows the computed cloud shape with a distinct concavity. Compare this to the image in Figure 4.12 (b), depicting such a cloud in reality. Even though the computation is only 2D, the distinct lenticular shape is still captured.

4.6 Challenges for Computing in the Limit of Small \( \sigma \)

One of the goals of this thesis work was to make a comparison to preexisting, full-physics computations of 2D mountain wave clouds. One such study that was of particular interest was the computation of clouds forming on the mountain surface in the hydrostatic limit \( \sigma \rightarrow 0 \), computations of which were carried out by Grabowski and Smolarkiewicz [5]. In this study, the implied height scale of \( H = 534 \) m and length scale of \( L = 25 \) km dictate a
Figure 4.11: A comparison of the slowly and rapidly varying $r'_\infty$. In (a), $r'_\infty$ is slowly varying with respect to the scale $1/\mu$, and so the cloud-edge is indistinguishable from the linear case. However, in (b), $r'_\infty$ varies on the order of $1/\mu$, and so it is essential to incorporate the nonlinear evaluation of $r'_\infty$. The cloud-edge computed by the linear theory is outlined in a red dash for comparison.

The hydrostatic parameter of $\sigma = 0.02$. Furthermore, the implied non-dimensional Brunt-Väisälä frequency used was $\bar{N} = 0.8$.
Recall from Section 3.2.2 that the finite-xsdifference solve for the particular solutions becomes numerically ill-conditioned whenever $\hat{N}^2$ is near one of the wavenumbers

$$k_{n,m}^2 = \left(\frac{n \pi}{L_z}\right)^2 + \left(\frac{m \pi}{2L_x/\sigma}\right)^2 \quad (4.17)$$

for integer values of $n$ and $m$, where $L_z$ and $2L_x$ can be interpreted as the height and width of the box containing the cloud. For the surface cloud in the scaled model, characteristic values of the box size would be $L_z = 5$ and $2L_x = 10$. The consequence of the relatively large box and small $\sigma$ is that the resonance wave numbers leading to the ill-conditioned Helmholtz solve become fairly densely packed near $\hat{N}$.

This is showcased in Figure 4.13, where the resonance wavenumbers $k_{1,m}^2$ have been plotted against $m$ for four values of $\sigma$, using the characteristic values of $L_z$ and $2L_x$ given above. Note that $k_{n,m} > 1$ for all $n \geq 2$ due to the height of the box, independently of $\sigma$, and so it suffices to only consider the $n = 1$ wavenumbers. The main observation to be made from this plot is that for larger values of $\sigma$ there are relatively few resonance wavenumbers $k_{1,m}^2 \in [0,1]$, and so it is easy enough to avoid the resonance issue by tuning the size of the cloud box, or by changing $\hat{N}$ (if possible). However, as $\sigma \to 0$ it becomes impossible to tune the computation such that $\hat{N}^2$ isn’t nearby a resonance value. In particular when $\sigma = 0.02$ the closest value of $k_{1,m}$ to $\hat{N} = 0.8$ is easily within $10^{-3}$, and the next nearest values are not far off.

This numerical ill-conditioning makes the current method poorly suited for computations in the hydrostatic limit of $\sigma \to 0$. There are a few potential modifications to the method that could improve robustness. The most straightforward would be to simply find a suitable quadrature scheme to handle the singular integrand in the particular solution, therefore
Figure 4.13: Depiction of the small $\sigma$ problem for the finite-difference solve. Notice that for $\sigma = 1$ there are only two resonance wave numbers in $[0, 1]$, making it relatively easy to set up a computation that avoids the ill-conditioning issue. However, as $\sigma$ approaches 0 the resonance wave numbers become more densely packed, implying that any finite-difference solves carried out on this domain are at significant risk of corruption by the homogeneous solution.

eliminating the need for a finite-difference solve entirely. However, no such scheme has been found for implementation in this project.

A second idea involves decomposing the cloud box into many smaller boxes. This naturally comes from the notion that the cloud itself can be thought of as the sum of many smaller, overlapping clouds. Therefore, instead of computing on one large box containing the whole cloud, compute on a series of smaller, overlapping length 1 boxes each containing a portion of the whole cloud. In the example discussed above, instead of performing one finite-difference solve on a box of height 5 and width 10, one could carry out $\sim 50$ finite-difference solves on overlapping boxes.
A final suggestion to improve on the performance of the finite-difference solve would be to decompose the particular solution into a singular and non-singular integral as

$$\psi'_P(x') = \iint_{l'^+ > 0} f_l(x) (G(x - x') + \frac{1}{2\pi} \log |x - x'|) dx - \iint_{l'^+ > 0} f_l(x) \frac{1}{2\pi} \log |x - x'| dx \quad (4.18)$$

where the factor of $\frac{1}{2\pi} \log |x - x'|$ has been introduced to cancel the singular behaviour of the free-space Green’s function. The first term is a non-singular integral and can be evaluated anywhere in the computational domain. The second term is the convolution form of the solution to the Poisson problem

$$\sigma^2 \psi'_{xx} + \psi'_{zz} = f_l \quad (4.19)$$

since $\frac{1}{2\pi} \log(r)$ is the free-space Green’s function for the Laplace operator. Therefore, one could easily compute the solution to the above using a standard finite-difference scheme on any reasonable sized domain, as the Poisson problem does not have the same ill-conditioning issue as the forced Helmholtz problem.
Chapter 5

Conclusion

The focus of this research project was on computing 2D mountain wave clouds using a recently developed moist Boussinesq fluid model. Doing so also serves as one of the first validations that the moist Boussinesq model captures accurate cloud-edge dynamics. The mountain cloud problem provides a relatively simple test case because it is time steady and the general results regarding cloud shape are well known in the atmospheric science community. However, the problem still provides a relatively high degree of computational complexity, due to the the semi-infinite domain and irregular bottom boundary.

The work presented in this thesis represents a combination of a variety of mathematical disciplines. The underlying fluid equations require understanding not only of the standard inviscid Euler equations, but also of the extension due to the Boussinesq approximation. Independently of this, some knowledge on moist thermodynamics was also required. Further still, the detailed scaling arguments that lead to the dynamic coupling of the fluid flow and thermodynamics are essential in order to complete the marginal cloud model. Original to this work was the derivation of the free-boundary Helmholtz problem for the mountain cloud problem, an extension of the pre-existing Long’s theory for stratified flows.

The numerical implementation of the model was also a nontrivial task. Though individual elements of the mountain cloud problem have been dealt with independently before, there appear to be few numerical problems that combine all of the semi-infinite domain, irregular bottom boundary, and iterative solve required to resolve the cloud-edge as a free-boundary problem. Though the scheme presented here may not be perfect it seems to achieve satisfactory accuracy in reasonable time for a fairly robust series of cases. At the very least the numerical implementation can serve as a benchmark for any potential future work on this project.

To speak to the numerical results of the thesis more specifically, the MFS was shown to be a suitable method for this problem as it achieves a high enough accuracy to accurately capture the physics of the edge dynamics. Furthermore, it was shown that at least for the case of low-amplitude mountains a simplified choice of the source point distribution could be used, compared to what has been suggested in the literature previously. However this
freedom of choice also represents one drawback of the MFS: with no general theory for optimal source point distributions, the user has to simply make an educated guess to be validated by error estimates. However, the educated guess used in this project, of a simple shifted axis at a finite depth, was shown to moderate the other drawback of the MFS: the ill-conditioning of the associated linear solves.

The choice of the shifted axis was also necessary when considering the second part of the numerical scheme. To compute particular solutions to the forced Helmholtz problems at every iterative step a finite-difference scheme was used, where the boundary values were computed using a familiar convolution form of the particular solution. There were two main advantages of using this finite-difference/quadrature scheme. First, the finite-difference solve is significantly faster than simply computing the quadrature formula at all points on the domain. Second, and more significantly, it allowed for avoidance of computing singular integrals numerically, which though not impossible would have been difficult considering the forcing terms were only known numerically.

The main drawback to this scheme is associated with the challenges arising from solving the Helmholtz equation via a finite-difference scheme. In particular, depending on the size of the computational domain there may exist resonances for which a homogeneous solution to the Helmholtz equations exists, with homogeneous boundary conditions. The existence, or near-existence, of such solutions restricts the size of possible domains. This can be somewhat alleviated by simply using the finite-difference solve on a small box containing the cloud, and then using the quadrature formula on the exterior. However, there are still some cloud cases that may fall outside even this scheme, depending on the cloud size.

Despite these limitations, the numerical implementation of the moist Boussinesq model still proves to provide physically relevant results. The basic cloud shapes produced by the simple 2D solver bear a resemblance to the edge shapes observed in reality. Furthermore, the presence of the upstream water profile $r'_\infty$ allowed for tuning of the atmosphere in order to capture specific cloud cases.

The effect of the cloud on the surrounding atmosphere could also be observed. In particular, the latent heating effect was shown to cause an appreciable amount of warming the atmosphere. Such an effect is consistent with what would be expected from the phase change processes that produce clouds. Furthermore, the overall effects of the corrections to the Long’s theory solution due to cloud are to lessen the magnitude of the disturbances caused by the mountain, albeit by only a small amount.

The cloud-edge velocities were also computed, and this provides an important benchmark for the moist Boussinesq model. One of the motivations for this linearized cloud model is to be able to accurately capture edge dynamics, particularly motion that is sometimes counterintuitive based on the underlying flow. Though the edge motion of the time-steady lenticular cloud is fairly basic, it is still essential that the model can accurately capture this
behaviour. It was shown that indeed, the computed edge velocities are consistent with a stationary cloud-edge.

The fact that the concave lenticular cloud shapes live within this model is also promising. Furthermore, it is a satisfying feature that the only additional nonlinearity that needs to be accounted for is in the evaluation of the upstream water profile, and that largely the conditions under which these clouds form in the model is still consistent with a linear flow regime.

Of course, there are still further directions in which this work could be taken. Work is already being undertaken to apply the moist Boussinesq model to more general, time-dependent problems, but even within the scope of the time-steady mountain clouds there are results nearly in reach. In particular, work by Grabowski and Smolarkiewicz [5] also includes computations of mountain clouds, specifically those that form directly on the surface of the mountain itself.

It is not the surface interaction of the cloud that makes this problem numerically intractable for the current project, but rather its size. The physical set-up in the above cited work was consistent with producing clouds whose horizontal width was on the order of tens of kilometers. The issue with this comes back to the limitations of the finite-difference scheme for computing the particular solutions. Even if the cloud is bounded by a small box, therefore limiting the domain of the finite-difference solve, there still proves to be sufficient corruption by homogeneous solutions to make the computation infeasible. The best course of action to take to overcome this limitation would likely be to replace the combined finite-difference/quadrature scheme by a purely quadrature based approach. Doing so would allow for the quadrature to be carried on any size domain, therefore making the scheme more robust to larger clouds. Being able to compare to a published computation would serve as strong validation for the mountain cloud computations carried on in this thesis work.
Bibliography


