A moneyness-adapting fee structure for guaranteed benefits embedded in variable annuities: Pricing and valuation

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Abstract

Guaranteed minimum death benefit (GMDB) and guaranteed minimum maturity benefit (GMMB) are two common guarantee riders embedded in variable annuities. To cover the financial risks incurring from the guarantees, fees are charged based on the underlying fund value, where a traditional approach funds the guarantees as a constant rate of fee over the period of the accumulation phase. This fee structure, however, potentially encourages surrendering when the options are out-of-money. To prevent the adverse incentives, Bernard et al. (2014) introduced a state-dependent fee, where fees are charged only when the guarantees are in-the-money or close to being in-the-money. This project proposes a moneyness-adapting fee structure, aiming to further reduce the insurer's reserve. Following the estimation of rate of fee charged for GMDB and/or GMMB under three pricing principles, the performances of three fee structures are compared with numerical illustrations, based on the measures of value-at-risk and conditional-tail-expectation.

Keywords: Variable annuities; Pricing principle; Guaranteed minimum death benefit; Guaranteed minimum maturity benefit

Dedication

To my beloved wife and parents, whose support and understanding are unconditional and endless.

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Chapter 1

Introduction

1.1 Overview of Variable Annuities

A variable annuity (VA) is typically an equity-linked annuity contract between an individual policyholder and an insurance company. It allows the policyholder to choose from a selection of investment options and offers life contingent benefits whose payments are linked to the account value of the policyholder's chosen investment portfolio, providing participation in the equity market.

VAs were introduced in the 1950s in the United States by the Teachers Insurance and Annuity Association of America-College Retirement Equities Fund (TIAA-CREF), whose earliest product was designed to generate incomes that reflect investment performance for retirement. Possessing a layer of insurance protection, the VA is also preferred due to its tax-deferrable feature. Figure 1.1 displays the sales amount of variable annuities sold in the U.S. from 1999 to the first quarter of 2019, as reported by the Life Insurance and Market Research Association (LIMRA)'s individual annuity sales survey (LIMRA (2019)). In the bar chart, the height of each bar (except for 2019) represents the total annual sales amount, where that of variable annuities is shown as the blue portion and that of fixed annuities is exhibited as the grey one. It can be observed from the figure that the sales amount of variable annuities are stably huge, averagely over 133 billion dollars annually. Although the proportion of the variable annuities sales to the total is slightly decreasing over time since 2011, the sales of VA still account for approximately 50% of the total sales on average.

For a typical variable annuity, a policyholder pays at time of issue to the insurer an single premium placed in a separate account. The premium is invested in the financial markets, for a couple of years, which is referred to as the accumulation phase. During the accumulation phase, the policyholder may make partial withdrawals, with or without penalties according to the contract provisions, or pay additional premiums to enhance the guaranteed amount (see the flexible-premium variable annuities (FPVA) in Bernard et al. (2017) as an example). Following the end of the accumulation phase is the annuitization

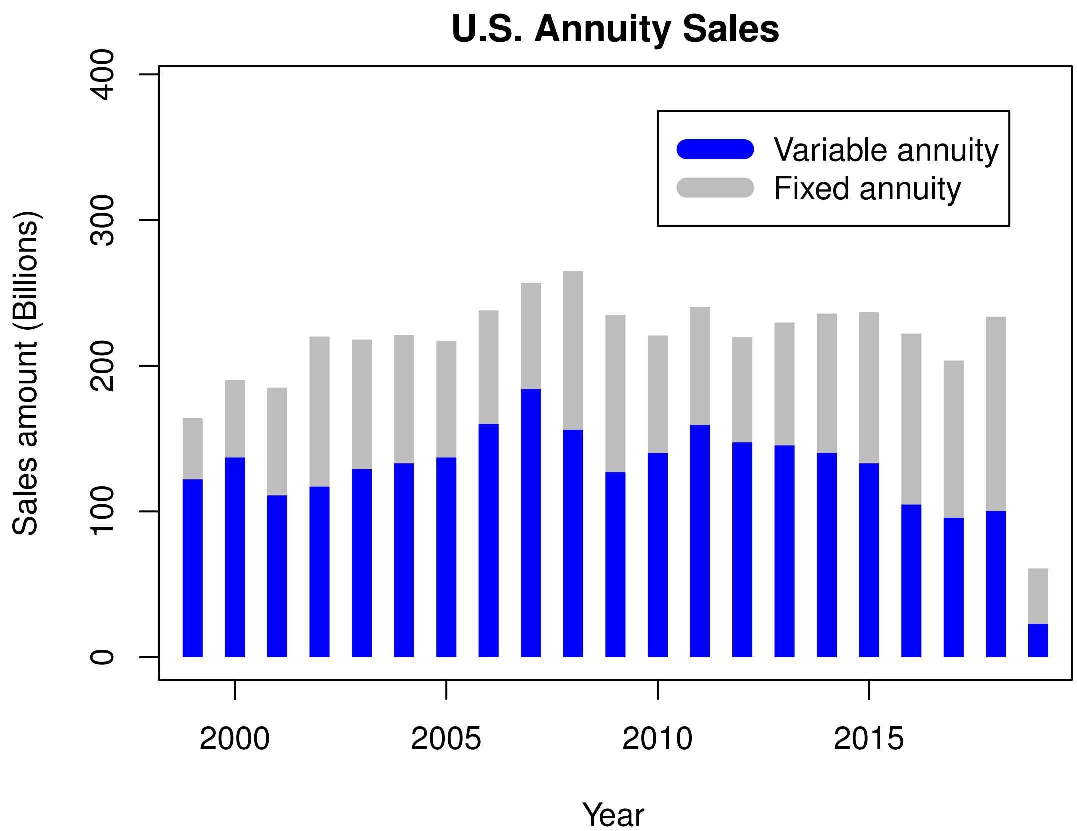

Figure 1.1: Variable annuity sales amount in the U.S. Data collected from LIMRA (2019).

phase, when the policyholder is allowed to either withdraw the entire account value as a lump sum payment or annuitize the amount.

1.2 Types of Guaranteed Minimum Benefits

Variable annuities are long-term investment products that are designed to offer financial protection against market risks and/or longevity risk, from which the protection can be provided by some guarantees embedded in VAs. As mentioned in Shen et al. (2016), the guarantees can generally be divided as two classes: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs), where GMLBs can be further categorized into four subclasses: the guaranteed minimum maturity benefits (GMMBs), the guaranteed minimum accumulation benefits (GMABs), the guaranteed minimum income benefits (GMIBs), and the guaranteed minimum withdrawal benefits (GMWBs).

Guaranteed Minimum Death Benefits (GMDBs)

A plain variable annuity without any guarantee riders would return to the policyholder's beneficiaries the account value at the time of death if the policyholder dies during the

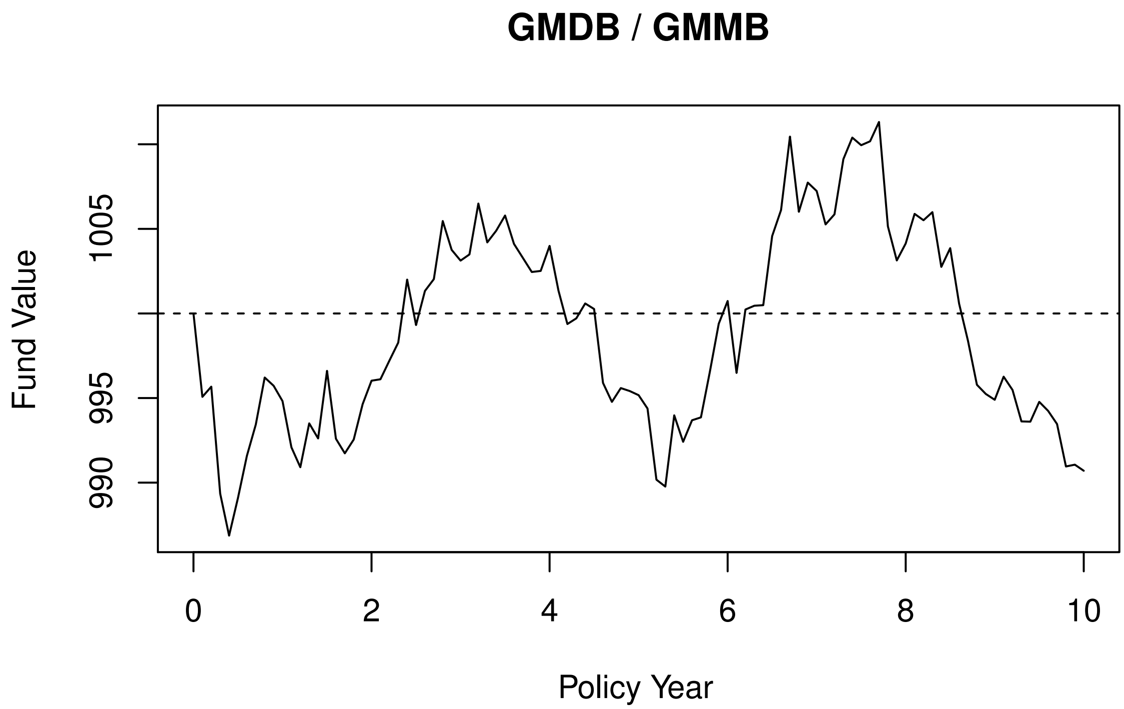

Figure 1.2: An illustrative fund value dynamics, initiating at 1,000, over time and an embedded GMDB and/or GMMB

accumulation phase. However, if a GMDB rider is embedded, a floor for the death benefit payment will be provided. Then the death benefit payment will be the greater of the account value and the guaranteed minimum death benefit amount at the time of death.

Suppose that a variable annuity, predetermined to be annuitized at the end of 10 years and embedded with a GMDB rider during the 10-year period, guarantees a return of the greater of the fund value and the guaranteed death benefit amount if the policyholder dies during the accumulation phase. Figure 1.2 gives an illustrative example for the fund dynamics with an initial deposit of 1,000 at time of issue of a variable annuity. For the case with a GMDB rider, the death benefit is guaranteed to be 1,000 if the fund value, at time of death, is below the guaranteed level; otherwise, the account value will be paid.

Guaranteed Minimum Maturity Benefits (GMMBs)

A GMMB rider provides a guaranteed minimum benefit at a predetermined time point to which the pocliyholder survives. Using the illustrative example in Figure 1.2 again, a maturity benefit of amount 1,000 will be paid to the policyholder at time 10 if the policyholder survives 10 years, and 0 otherwise.

1.3 Motivation

Existing literature regarding the pricing of guarantees embedded in variable annuities use the actuarial equivalence principle (AEP) to determine the premiums (i.e., letting the expected net loss at issue be zero.). However, the premiums can also be collected at a more conservative level rather than being determined using the actuarial equivalence principle. Therefore, in this project we propose the portfolio percentile principle (PPP) and the contract percentile principle (CPP) to determine the premiums for guarantees embedded in variable annuities.

Another common observation from the literature (e.g., Feng and Huang (2016)) regarding the pricing and valuation of guarantees is that a constant rate is charged for funding the guarantees, where the constant rate consists of the administration fee and the cost of guarantees. However, under this assumption, the deduction from the account is not reflecting the cost of the guarantees. To force the deduction only accounts for the cost of guarantees, a constant rate deducted continuously from the fund value will be used to only fund the guarantees in this project.

Moreover, traditionally the guarantees are funded during the accumulation phase by a deduction at a constant rate periodically, typically on a daily basis from the accumulative value in the separate account. However, this fee structure has some disadvantages introduced in Bernard et al. (2014), Delong (2014), and Zhou and Wu (2015). One issue under this fee structure is that policyholders are driven to lapse their policies when the account value is high, which makes the guarantees be out-of-money (when the fund value is beyond the guaranteed level). As a solution, Bernard et al. (2014) introduced a state-dependent fee structure so that the account value in the separate account is deducted only when the guarantees are in-the-money (when the fund value is below the guaranteed level) or close to being in-the money (when the fund value equals to the guaranteed level). In this project, we extend the idea and propose a moneyness-adapting fee structure, aiming to lower the reserving requirement while retaining the attractiveness of the guarantee features.

1.4 Outline

The remainder of this project is organized as follows. Chapter 2 provides the literature review on reserving methods for variable annuities embedded with guarantees and existing fee structures charged for the guarantees. In Chapter 3, we present the models for forecasting mortality rates, and the price dynamics for the underlying asset, as well as the fund value driven by the underlying asset under three fee structures charged for the guaranteed benefits. Additionally, three pricing principles determining the premium rate are studied, followed by the estimation of fees under each of the three pricing principles and the formulation of reserve calculations for the guaranteed benefits. In Chapter 4, we present the estimates of model parameters and introduce the simulation methodology for generating stochastic processes. Finally, the numerical results under a base case are illustrated, and a sensitivity analysis is conducted. Chapter 5 concludes this project.

Chapter 2

Literature Review

The literature on variable annuities has grown significantly during the past two decades. Providing thorough information about features of variable annuities, Bauer et al. (2008), Ledlie et al. (2008), and Kalberer and Ravindran (2009) can serve as good introductory articles.

To cover the variable annuity provider's mortality and expense risks, the administration expenses and the costs of guarantees, premiums that are calculated and deducted as a constant percentage of the underlying account value are charged periodically. Currently, a time-invariant fee structure is adopted (see, e.g., Grosen and Jørgensen (2002), Bacinello (2003), Siu (2005), Bernard and Lemieux (2008), and Bacinello et al. (2009, 2010)). However, Bernard et al. (2014) and Bernard and Moenig (2018) introduced a fee structure with a major drawback: since the embedded guaranteed minimum benefits resemble put options on the underlying account, the value of the guarantees decreases in the underlying account value. However, when the account value increases so that the guarantees lose the value, policyholders are still charged premiums under the current fee structure. Meanwhile, rational policyholders can benefit from surrendering their policies and reenter the financial market by purchasing an identical product. This is referred to the "lapse-and-reentry" strategy (or the commonly known "1035 exchange") mentioned in Moenig and Zhu (2018). When the lapse-and-reentry strategy is exercised, huge policy acquisition expenses will be accrued to variable annuity providers (see Bernard and Moenig (2018)). These huge costs, in combination with frequent policy lapses that give the providers less time to recover the up-front expenses, lead to a unsatisfactorily high fee associated with variable annuities (Pinquet et al. (2011) and Paris (2017)).

To retain the attractiveness of variable annuities to policyholders without reducing the insurers' profits, modifications of fee structure are proposed. Bernard and Moenig (2018) suggested a two-stage fee structure, where fees are reduced when the policy is still in force beyond the surrender schedule; Bernard et al. (2014) introduced state-dependent fees, where the policyholders are charged premiums only when the account value is below a predetermined level, and the fee structure is adopted with generalized assumptions in Delong (2014).

There are numerous articles about pricing of variable annuities with guaranteed minimum benefits (see, e.g., Milevsky and Posner (2001), Bauer et al. (2008), Dai et al. (2008), Bélanger et al. (2009), among many others). For pricing the guarantees, it is suggested to accurately quantify all major risk factors that affect the underlying fund dynamics (see Coleman et al. (2006), Du and Martin (2014), and Kling et al. (2011)). As one of the most popular asset price models, the geometric Brownian motion (GMB) under the Black-Scholes assumption was adopted to price the guarantees (see Bernard et al. (2014)). Platen and Rendek (2008) pointed out that the leptokurtic feature can be characterized by heavy tails in equity returns. While Christoffersen et al. (2009) modelled the asset price dynamics with time-varying volatilities, Van Haastrecht et al. (2010) adopted the stochastic volatility model when pricing guaranteed annuity options.

In addition to capturing the volatility of asset prices, the interest rate risk is another market risk factor that affects the pricing results. As the simplest assumption, a constant interest rate used for pricing can be found in Milevsky and Salisbury (2006), Bauer et al. (2008), Dai et al. (2008), Huang and Forsyth (2012), Huang and Kwok (2014), Chen and Forsyth (2008), while stochastic interest rate models, such as those in Vasicek (1977) and Cox et al. (1985), are implemented in Peng et al. (2012) and Bacinello et al. (2011). Kang and Ziveyi (2018) extended the assumptions made in Bernard et al. (2014) by incorporating both stochastic volatility (using Heston (1993)'s model) and stochastic interest rates (i.e., Hull and White (1990)).

In most literature for pricing the guaranteed minimum benefits, the mortality rates are deemed deterministic while the market risk is considered (e.g., Ballotta and Haberman (2006)). However, the underestimation of mortality risk can lead to huge losses for life insurer, so Liu et al. (2014) and Zhao and Mamon (2018) constructed a pricing model for guaranteed annuity options that incorporated the correlation between market and mortality risks, where mortality rates can be forecasted by models proposed in Lee and Carter (1992), Cairns et al. (2006), or Lin and Liu (2007).

The valuation of the guaranteed minimum benefits in the literature can be categorized as three approaches, including the partial differential equation (PDE) approach, the binomial tree approach, and Monte Carlo simulation. Shen and Xu (2005) conducted the valuation of equity-linked policies using the PDE approach under the GBM environment, whereas Feng and Huang (2016) gave a brief introduction to the U.S. statutory financial reporting and provided an approximation via a PDE method; Costabile et al. (2008) and Bacinello (2003) adopted a binomial tree approach to determining fair premium values for the guarantees. A unifying framework of valuation using Monte Carlo simulation is a common adoption when an analytical solution is not available (see Bacinello et al. (2011), and Piscopo and Haberman (2011)).

Chapter 3

Framework

In this chapter, we present the underlying asset price model, the corresponding fund value dynamics based on three fee structures for covering guaranteed benefits, the pricing principles, and the reserve calculations. Throughout the project, we consider a variable annuity which matures after n year since time of issue ($t = 0$). In addition, a guaranteed minimum death benefit, where the guaranteed level at time t is denoted by G_t^D and is payable at the time of death before the policy matures, and a guaranteed minimum maturity benefit, denoted by G^M, payable at time n if the policyholder survives to the maturity, are embedded in the variable annuity issued to a group of policyholders all aged x with homogeneous risk profiles.

3.1 Models for Risk Factors

For variable annuities, risk factors for pricing and valuation include but are not limited to the market risk (i.e., the actual investment performance of the underlying fund is worse than expected) and the mortality risk (i.e., the contingent benefits are paid more and earlier than expected). The following sections introduce mortality and equity models that are commonly adopted in the literature of pricing and valuation of variable annuities.

3.1.1 Mortality Model

Being the most widely adopted method for mortality fitting and prediction, Lee and Carter (1992) models the dynamics of the logarithm of the central death rate as a process driven by an age-specific constant and the speed of change at each age multiplied by an overall time trend of mortality rates:

$$\ln(m_{x,t}) = \alpha_x + \beta_x \cdot \kappa_t + \epsilon_{x,t}, \quad x = x_L, x_L + 1, \ldots, x_U, \quad t = t_L, t_L + 1, \ldots, t_U, \tag{3.1}$$

where $m_{x,t}$ represents the central death rate for an individual aged x in year t, α_x is the mean age-specific factor, κ_t is the time-varying index at time t, β_x is the age-specific reaction to the time-varying index, and $\epsilon_{x,t}$'s, the model errors not captured by the age-specific effects reflected in the model, are assumed independent and identically distributed.

The model is subject to two constraints: $\sum_{x=x_L}^{x_U} \beta_x = 1$ and $\sum_{t=t_L}^{t_U} \kappa_t = 0$. The model parameters can be estimated by the singular value decomposition. Alternatively, an approximation with the constraints gives the estimate of α_x,

$$\hat{\alpha}_x = \frac{1}{t_U - t_L + 1} \sum_{t=t_L}^{t_U} \ln(m_{x,t}), \tag{3.2}$$

for $x = x_L, x_L + 1, \ldots, x_U$, and the estimate of κ_t,

$$\hat{\kappa}_t = \sum_{x=x_L}^{x_U} \left[\ln(m_{x,t}) - \hat{\alpha}_x\right]. \tag{3.3}$$

Without the constant term being involved for each age x, the estimate of β_x can be obtained via regressing $[\ln(m_{x,t}) - \hat{\alpha}_x]$ on $\hat{\kappa}_t$ by

$$\hat{\beta}_x = \frac{\sum_{t=t_L}^{t_U} [\ln(m_{x,t}) - \hat{\alpha}_x] \cdot \hat{\kappa}_t}{\sum_{t=t_L}^{t_U} \hat{\kappa}_t^2} = \frac{\sum_{t=t_L}^{t_U} \ln(m_{x,t}) \cdot \hat{\kappa}_t}{\sum_{t=t_L}^{t_U} \hat{\kappa}_t^2}. \tag{3.4}$$

Generally, $\hat{\kappa}_t$ displays a decreasing time trend, indicating the mortality rates improved gradually over time. As suggested in Lee and Carter (1992), the time trend follows a random walk with drift θ: $\hat{\kappa}_t = \hat{\kappa}_{t-1} + \theta + e_t$, where the time trend errors $e_t, t = t_L + 1, t_L + 2, \ldots, t_U$, are assumed to be independent and identically distributed as $\mathcal{N}(0, \sigma_e^2)$, a normal random variable with mean 0 and variance σ_e^2.

The model error $\epsilon_{x,t}$ and the time trend error e_t are assumed independent. The conditional joint distribution of $\epsilon_{x,t}$ and e_t is given by

$$\begin{bmatrix} \epsilon_{x,t} \\ e_t \end{bmatrix} \Bigg| \mathcal{F}_{t-1} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix} \right), \tag{3.5}$$

where $\mathcal{F}_t$ stands for the information up to time t and σ_x^2 is the variance of $\epsilon_{x,t}$. Then the drift parameter θ, the variance of the time trend error σ_e^2, and the variance of the model error σ_x^2 can be estimated by

$$\hat{\theta} = \frac{1}{t_U - t_L} \sum_{t=t_L+1}^{t_U} (\hat{\kappa}_t - \hat{\kappa}_{t-1}) = \frac{\hat{\kappa}_{t_U} - \hat{\kappa}_{t_L}}{t_U - t_L}, \tag{3.6}$$

$$\hat{\sigma}_e^2 = \frac{1}{t_U - t_L - 1} \sum_{t=t_L+1}^{t_U} \left(\hat{\kappa}_t - \hat{\kappa}_{t-1} - \hat{\theta}\right)^2, \tag{3.7}$$

and

$$\hat{\sigma}_x^2 = \frac{1}{t_U - t_L - 1} \sum_{t=t_L}^{t_U} \left[\ln(m_{x,t}) - \hat{\alpha}_x - \hat{\beta}_x \cdot \hat{\kappa}_t \right]^2, \tag{3.8}$$

respectively.

3.1.2 Underlying Asset Price Model

Let $\mathcal{S}_T = \{S_t : 0 \leq t \leq T\}$, $T \geq 0$, be the price dynamics of the underlying asset over the time period $[0, T]$. The underlying asset can be a specific stock or a mutual fund, or an index such as the Standard & Poor's 500 Index. Assuming the asset price follows a geometric Brownian motion (GBM) and a management fee is charged continuously as a constant rate d of the underlying asset price, the dynamics of the asset price is given, under the physical measure, by

$$dS_t = (\mu - d) \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t, \quad 0 \leq t \leq T \tag{3.9}$$

where μ is the expected annual growth rate of the underlying asset, σ is the constant volatility, and $\{W_t : t \geq 0\}$ is the standard Brownian motion under the physical measure.

As an illustrative textbook exercise, Shreve (2004) applied the Itô-Doeblin formula and provided the solution to this stochastic differential equation by

$$S_T = S_t \cdot \exp\left\{ \left(\mu - d - \frac{\sigma^2}{2} \right) \cdot (T - t) + \sigma \cdot (W_T - W_t) \right\}, \quad 0 \leq t \leq T, \tag{3.10}$$

where, according to the normality of the standard Brownian motion, $(W_T - W_t)$ is normally distributed with mean zero and standard deviation $\sqrt{T - t}$.

3.2 Fee Structure

In this section, we discuss three types of fee structures for funding the guarantees embedded in variable annuities: (1) a traditional approach that is similar to the management fee of mutual funds, which charges the policyholder a constant rate of the underlying fund; (2) a state-dependent fee structure that premiums are collected only when the fund lies below a predetermined barrier; and (3) the proposed approach in which the premiums are calculated based on a moneyness-adapting amount, attempting to better track the loss incurring from the issuance of protection.

3.2.1 Charge Proportional to the Fund (F1)

As a traditional charging method, policyholders of the guaranteed benefits are charged continuously by a constant rate of the fund value at time of charge. Given an initial investment amount F_0 from the policyholder, a constant rate of fee c, $0 \leq c < \infty$, and the underlying asset price dynamics $\mathcal{S}_T = \{S_t : 0 \leq t \leq T\}, 0 \leq T \leq n$, the corresponding fund value F_t

associated to a variable annuity at time t, $0 \le t \le n$, can be determined by

$$dF_t(c;\mathcal{S}_t) \;=\; F_t(c;\mathcal{S}_t)\,\frac{dS_t}{S_t} - c\,F_t(c;\mathcal{S}_t)\,dt, \quad F_0(c;\mathcal{S}_0) = F_0. \tag{3.11}$$

If the GBM model is adopted for the underlying asset price, we have the fund value dynamics to be formulated by

$$dF_t(c;\mathcal{S}_t) \;\overset{(3.9)}{=}\; F_t(c;\mathcal{S}_t)\left[((\mu-d)dt + \sigma dW_t) - c\,dt\right], \quad 0 \le t \le n. \tag{3.12}$$

Possessing a similar dynamics as the underlying asset does, the fund value at time t can be solved by

$$F_t(c;\mathcal{S}_t) = F_0 \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right)t + \sigma W_t - c\,t\right\}, \quad 0 \le t \le n. \tag{3.13}$$

3.2.2 Charge Proportional to the Fund until Fund Exceeds a Barrier (F2)

Bernard et al. (2014) proposed an alternative fee structure aiming to prevent policyholders from surrendering by setting a predetermined barrier $B(>0)$, where the policyholders are charged premiums continuously when the guarantees are in-the-money (i.e., $F_t(c;\mathcal{S}_t) < B$) or at-the-money (i.e., $F_t(c;\mathcal{S}_t) = B$) since policyholders of guarantees tend to surrender their policies when the guarantees are out-of-money (i.e., $F_t(c;\mathcal{S}_t) > B$). Then the premium is charged as a constant proportion of the fund value at time of charge. Adopting the design of this fee structure, the dynamics of the funds is given by, for $0 \le t \le n$,

$$\begin{aligned} dF_t(c;\mathcal{S}_t) =& F_t(c;\mathcal{S}_t)\,\frac{dS_t}{S_t} - c \cdot F_t(c;\mathcal{S}_t) \cdot 1_{\{F_t(c;\mathcal{S}_t) \le B\}} dt \\ \overset{(3.9)}{=}& F_t(c;\mathcal{S}_t)\left[((\mu-d)dt + \sigma dW_t) - c \cdot 1_{\{F_t(c;\mathcal{S}_t) \le B\}} dt\right], \quad F_0(c,\mathcal{S}_0) = F_0. \end{aligned} \tag{3.14}$$

Applying the Itô's lemma to $d\ln F_t(c;\mathcal{S}_t)$ and then integrating, we obtain the following representation of (3.14) as

$$F_t(c;\mathcal{S}_t) = F_0 \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right)t + \sigma W_t - c\int_0^t 1_{\{F_s(c;\mathcal{S}_s) \le B\}} ds\right\}, \quad 0 \le t \le n, \tag{3.15}$$

where $\int_0^t 1_{\{F_s(c;\mathcal{S}_s) \le B\}} ds$ represents the total occupation time in the interval $(0,t)$ during which the fund value is below the barrier B.

3.2.3 Charge Proportional to the Deficiency of Fund to Guaranteed Level (F3)

Based on the fee structure introduced by Bernard et al. (2014), we propose an extension: choose a reference fund level $H(>0)$ and define the deficiency of fund at time t by

$[H - F_t(c;\mathcal{S}_t)]_+$, $0 \leq t \leq n$, where $[x]_+ = \max(x, 0)$. Then the fee, used for funding the guarantee and charged at time t, is calculated at the rate of the multiplication of a constant c and the deficiency at time t when the guarantee is in-the-money or close to being in-the-money. Under this fee structure, where the base amount for determining the premium can better mimic the moneyness of the guarantee, the fund value dynamics is given by, for $0 \leq t \leq n$, $F_0(c, \mathcal{S}_0) = F_0$,

$$
\begin{aligned}
dF_t(c;\mathcal{S}_t) =& F_t(c;\mathcal{S}_t)\frac{dS_t}{S_t} - c\,[H - F_t(c;\mathcal{S}_t)]_+ \, 1_{\{F_t(c;\mathcal{S}_t)\leq B\}} dt \\
=& \begin{cases} F_t(c;\mathcal{S}_t)\frac{dS_t}{S_t} - c\,[H - F_t(c;\,\mathcal{S}_t)]\, 1_{\{F_t(c;\mathcal{S}_t)\leq B\}} dt, & H \geq B \\ F_t(c;\mathcal{S}_t)\frac{dS_t}{S_t} - c\,[H - F_t(c;\mathcal{S}_t)]\, 1_{\{F_t(c;\mathcal{S}_t)\leq H\}} dt, & H < B \end{cases} \\
\overset{(3.9)}{=}& \begin{cases} F_t(c;\mathcal{S}_t)\left\{((\mu-d)dt + \sigma dW_t) - c\left[\frac{H}{F_t(c;\,\mathcal{S}_t)} - 1\right] 1_{\{F_t(c;\mathcal{S}_t)\leq B\}} dt\right\}, & H \geq B \\ F_t(c;\mathcal{S}_t)\left\{((\mu-d)dt + \sigma dW_t) - c\left[\frac{H}{F_t(c;\,\mathcal{S}_t)} - 1\right] 1_{\{F_t(c;\mathcal{S}_t)\leq H\}} dt\right\}, & H < B \end{cases} \\
=& F_t(c;\mathcal{S}_t)\left\{((\mu-d)dt + \sigma dW_t) - c\left[\frac{H}{F_t(c;\mathcal{S}_t)} - 1\right] 1_{\{F_t(c;\mathcal{S}_t)\leq (H\wedge B)\}} dt\right\},
\end{aligned}
\tag{3.16}
$$

where $x \wedge y$ denotes the smaller of x and y. Similarly, applying the Itô's lemma to $d \ln F_t(c;\mathcal{S}_t)$ and then integrating obtains the following representation of (3.16) as

$$
F_t(c;\mathcal{S}_t) = F_0 \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right) t + \sigma W_t - c\int_0^t \left[\frac{H}{F_s(c;\mathcal{S}_s)} - 1\right] 1_{\{F_s(c;\mathcal{S}_s)\leq (H\wedge B)\}} ds\right\}. \tag{3.17}
$$

Similarly, we can define the total occupation time in the interval $(0, t)$ during which the premium is paid by $\int_0^t 1_{\{F_s(c;\mathcal{S}_s)\leq (H\wedge B)\}} ds$.

As a brief conclusion, the fund value dynamics under the three fee structures can be summarized by (3.12) – (3.17), for $j = 1, 2, 3$, as follows:

$$
dF_t^j(c_j;\mathcal{S}_t) = F_t^j(c_j;\mathcal{S}_t) \cdot \left[((\mu - d)t + \sigma W_t) - c_j \cdot X_t^j(c_j;\mathcal{S}_t) \cdot 1_t^j(c_j;\mathcal{S}_t) dt\right], \tag{3.18}
$$

with the representation

$$
F_t^j(c_j;\mathcal{S}_t) = F_0 \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right) t + \sigma W_t - c_j \int_0^t X_s^j(c_j;\mathcal{S}_s) \cdot 1_s^j(c_j;\mathcal{S}_s) ds\right\}, \quad 0 \leq t \leq n, \tag{3.19}
$$

where

$$
X_t^j(c_j;\mathcal{S}_t) = \begin{cases} 1, & j = 1, 2, \\ \frac{H}{F_t^j(c_j;\mathcal{S}_t)} - 1, & j = 3, \end{cases}
$$

and

$$1_t^j(c_j;\mathcal{S}_t) = \begin{cases} 1, & j=1, \\ 1_{\left\{F_t^j(c_j;\mathcal{S}_t)\le B\right\}}, & j=2, \\ 1_{\left\{F_t^j(c_j;\mathcal{S}_t)\le (H\wedge B)\right\}}, & j=3. \end{cases}$$

Here j indicates that the j-th fee structure is adopted, where $j=1$ stands for the traditional charge method, $j=2$ is the state-dependent approach of Bernard et al. (2014), and $j=3$ is the proposed method of Section 3.2.3. In addition, c_j, F_t^j, 1_t^j, and X_t^j represent the fee rate, the fund value, the trigger for paying premium based on the fund value level at time of charge, and the ratio of the charge base amount to the current fund value, respectively.

3.3 Pricing

A number of articles regarding the pricing of the embedded guarantees in variable annuities adopted the no-arbitrage pricing approach which relies on the fundamental theorem of asset pricing. Under the theorem, it is implied that the price of a financial derivatives in a complete market is given by the discounted expected value of its future payoff under a unique risk-neutral measure whose existence requires that the market is arbitrage-free. However, we observe from the life insurance market that insurance products with the same risk may have been priced differently which can lead to potential arbitrage. In addition, mortality-linked products are not traded in the financial market, so, even if the conversion of the mortality rates under the physical measure to those under the risk-neutral measure can be achieved through transforms (e.g., Wang's transform by Wang (2000)), the market price of mortality risk is still inaccessible. Therefore, in this project we conduct the pricing under the physical measure, where the future payoffs are discounted by a subjectively constant force of interest that reflects the yield on asset backing up the liability (see Feng and Huang (2016)).

We consider the market risk as the main driver of the uncertainty and project the mortality rates by the best estimates when determining the premiums. Suppose the j-th fee structure is selected under a level force of interest for discounting, r_p, over the duration of the contract, and a constant rate of fee, c_j, to be solved.

3.3.1 Loss Function

Let T_x be the time to death of an x-year-old policyholder at the beginning of the year T_U+1 of the variable annuity embedded with a GMDB and a GMMB, whose guaranteed minimum benefit payments at time t, $0\le t\le n$, and at the time of maturity are G_t^D and G^M, respectively. Then we can write the insurer's loss function at time of issue for this individual policy, given the stock price $\mathcal{S}_T = \{S_t : 0 \le t \le T\}, 0 \le T \le n$, by

$$L_0(c_j) = \begin{cases} e^{-r_p T_x} \cdot \left[G^D_{T_x} - F^j_{T_x}(c_j; \mathcal{S}_{T_x})\right]_+ \\ \quad - \int_0^{T_x} e^{-r_p t} \cdot \left[c_j \cdot F^j_t(c_j; \mathcal{S}_t) \cdot X^j_t(c_j; \mathcal{S}_t) \cdot 1^j_t(c_j; \mathcal{S}_t)\right] dt, \quad 0 \le T_x < n, \\ e^{-r_p n} \cdot \left[G^M - F^j_n(c_j; \mathcal{S}_n)\right]_+ \\ \quad - \int_0^{n} e^{-r_p t} \cdot \left[c_j \cdot F^j_t(c_j; \mathcal{S}_t) \cdot X^j_t(c_j; \mathcal{S}_t) \cdot 1^j_t(c_j; \mathcal{S}_t)\right] dt, \quad T_x \ge n. \end{cases} \tag{3.20}$$

The first component in the case $0 \le T_x < n$ above stands for the actuarial present value of the excess payment of the guaranteed death benefit to the fund value at time of death by the maturity, whereas the first component in the case $T_x \ge n$ is that of the guaranteed maturity benefit if the policyholder survives to the maturity; the second components in both cases represents the actuarial present value of the premium collected continuously until the earlier of the maturity of the policy and the time of death.

3.3.2 Pricing Principles

In this subsection, we propose three pricing principles: actuarial equivalence principle, portfolio percentile principle, and contract percentile principle. Assume that the deaths among the policyholders in the portfolio, given an asset price scenario, are independent. We are able to estimate the rate of fee charged for covering the guaranteed benefits by simulating the asset price for N_a times, where the k-th asset price scenario over the time period $[0, T]$ is denoted by $\mathcal{S}^k_T = \left\{S^k_t : 0 \le t \le T\right\}, 0 \le T \le n, k = 1, 2, \ldots, N_a$, and each pair of two simulated scenarios are mutually independent.

Actuarial Equivalence Principle (AEP)

Under the equivalence principle, a fair premium under the j-th fee structure can be determined by zeroing the expected loss at time of issue per policy, i.e., by solving the c_j such that

$$\mathbb{E}\left[L_0(c_j)\right] = 0. \tag{3.21}$$

Applying the law of total expectation we obtain

$$\mathbb{E}\left[L_0(c_j)\right] = \mathbb{E}\left[\mathbb{E}\left[L_0(c_j) | \mathcal{S}_n\right]\right], \tag{3.22}$$

where

$$
\begin{aligned}
&\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right] \\
&= \int_0^n e^{-r_p t}\cdot\left[G_t^D - F_t^j(c_j;\mathcal{S}_t)\right]_+ \cdot {}_t\hat{p}_{x,\,t_U+1}\cdot\hat{\mu}_{x+t,\,t_U+1+t}dt \\
&\quad - \int_0^n\left\{\int_0^t e^{-r_p s}\left[c_j\cdot F_s^j(c_j;\mathcal{S}_s)\cdot X_s^j(c_j;\mathcal{S}_s)\cdot 1_s^j(c_j;\mathcal{S}_s)\right]ds\right\}\cdot {}_t\hat{p}_{x,\,t_U+1}\cdot\hat{\mu}_{x+t,\,t_U+1+t}dt \\
&\quad + \; e^{-r_p n}\cdot\left[G^M - F_n^j(c_j;\mathcal{S}_n)\right]_+ \cdot {}_n\hat{p}_{x,\,t_U+1} \\
&\quad - \left\{\int_0^n e^{-r_p s}\left[c_j\cdot F_s^j(c_j;\mathcal{S}_s)\cdot X_s^j(c_j;\mathcal{S}_s)\cdot 1_s^j(c_j;\mathcal{S}_s)\right]ds\right\}\cdot {}_n\hat{p}_{x,\,t_U+1},
\end{aligned}
\tag{3.23}
$$

and, for a policyholder aged x in the beginning of year t_U+1, ${}_t\hat{p}_{x,t_U+1}$ and $\hat{\mu}_{x+t,t_U+1+t}$ stand for the projected probability of surviving to time t and the projected force of mortality at time t, respectively. With the help of integration by parts, we have

$$
\begin{aligned}
\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right] \;=\; & \int_0^n e^{-r_p t}\cdot\left[G_t^D - F_t^j(c_j;\mathcal{S}_t)\right]_+ \cdot {}_t\hat{p}_{x,t_U+1}\cdot\hat{\mu}_{x+t,t_U+1+t}dt \\
& + \; e^{-r_p n}\cdot\left[G^M - F_n^j(c_j;\mathcal{S}_n)\right]_+ \cdot {}_n\hat{p}_{x,\,t_U+1} \\
& - \; \int_0^n e^{-r_p t}\cdot\left[c_j\cdot F_t^j(c_j;\mathcal{S}_t)\cdot X_t^j(c_j;\mathcal{S}_t)\cdot 1_t^j(c_j;\mathcal{S}_t)\right]\cdot {}_t\hat{p}_{x,\,t_U+1}dt.
\end{aligned}
\tag{3.24}
$$

The fact that $F_t^j(c_j;\mathcal{S}_t) > 0$, where $0 \le c_j < \infty$, leads to the following boundaries: for $0 \le t \le n < \infty$,

$$
\begin{aligned}
& 0 < e^{-r_p t} \le 1, \\
& 0 \le {}_t\hat{p}_{x,\,t_U+1} \le 1, \\
& 0 \le [G_t^D - F_t^j(c;\mathcal{S}_t)]_+ < \max_{0\le t\le n} G_t^D < \infty, \\
& 0 \le [G^M - F_t^j(c_j;\mathcal{S}_n)]_+ < G^M < \infty, \\
& 0 < F_t^j(c_j;\mathcal{S}_t)X_t^j(c_j;\mathcal{S}_t) = F_t^j(c_j;\mathcal{S}_t) < \infty, \quad j=1,2, \\
& 0 < F_t^3(c_3;\mathcal{S}_t)X_t^3(c_3;\mathcal{S}_t) = \left[H - F_t^3(c_3;\mathcal{S}_t)\right]_+ < H < \infty, \quad \text{and} \\
& 1_t^j(c_j;\mathcal{S}_t) \in \{0,1\}.
\end{aligned}
\tag{3.25}
$$

Therefore,

$$
\begin{aligned}
& \int_0^n e^{-r_p t}\cdot\left[G_t^D - F_t^j(c_j;\mathcal{S}_t)\right]_+ \cdot {}_t\hat{p}_{x,\,t_U+1}\cdot\hat{\mu}_{x+t,\,t_U+1+t}dt \\
& \qquad < \max_{0\le t\le n} G_t^D \cdot \int_0^n {}_t\hat{p}_{x,\,t_U+1}\cdot\hat{\mu}_{x+t,\,t_U+1+t}dt \\
& \qquad = \max_{0\le t\le n} G_t^D \cdot (1 - {}_n\hat{p}_{x,\,t_U+1}) < \infty,
\end{aligned}
\tag{3.26}
$$

$$
e^{-r_p n}\cdot\left[G^M - F_n^j(c_j;\mathcal{S}_n)\right]_+ \cdot {}_n\hat{p}_{x,\,t_U+1} < G^M < \infty, \tag{3.27}
$$

and

$$c_3 \cdot \int_0^n e^{-r_p t} \cdot \left[F_t^3(c_3;\mathcal{S}_t) \cdot X_t^3(c_3;\mathcal{S}_t)\right] \cdot 1_t^3(c_3;\mathcal{S}_t) \cdot {}_t\hat{p}_{x,\,t_U+1} dt \ < \frac{c_3 H}{r_p}\left(1-e^{-r_p n}\right) < \infty, \tag{3.28}$$

which conclude that $\mathbb{E}\left[L_0(c_j)\right] < \infty$ and that $\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right] < \infty$. Then, by the law of large numbers,

$$\frac{1}{N_a}\sum_{k=1}^{N_a}\mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n^k\right] \xrightarrow{P} \mathbb{E}\left[\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]\right] = \mathbb{E}\left[L_0(c_j)\right],$$

where we denote $X_n \xrightarrow{P} X$ if X_n converges in probability to X and say X_n is a consistent estimator of X if $X_n \xrightarrow{P} X$. Consequently, $\hat{\mathbb{E}}\left[\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]\right] := \frac{1}{N_a}\sum_{l=1}^{N_a}\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]$ is a consistent estimator of $\mathbb{E}\left[L_0(c_j)\right]$, and the rate of fee under the actuarial equivalence principle can be estimated by

$$\hat{c}_j^E = \inf\left\{c_j : \hat{\mathbb{E}}\left[\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]\right] = 0\right\}. \tag{3.29}$$

Portfolio Percentile Principle (PPP)

Select an acceptable confidence level $q,\ 0 < q < 1$. Let $L_0^l(c_j)$ denote the loss of the l-th policy in the portfolio of size l_0 under the j-th fee structure. Due to the independence of deaths among the policyholders, we have that, given a stock price scenario $\mathcal{S}_n$, $L_0^l(c_j)|\mathcal{S}_n, l = 1, 2, \ldots, l_0$, are independent and identically distributed as $L_0(c_j)|\mathcal{S}_n$. Under the portfolio percentile principle, a premium can be determined at a level such that the probability of incurring a positive total loss of the whole portfolio, $\sum_{l=1}^{l_0} L_0^l(c_j)$, is at most q. That is, the insurer charges the policyholders a corresponding rate of fee $c_j^P(q)$ for the guarantees such that

$$c_j^P(q) = \inf\left\{c_j : \mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] \le q\right\}, \quad j = 1, 2, 3. \tag{3.30}$$

To approximate the probability of a positive total loss, we begin with applying the law of total expectation:

$$\begin{aligned}
\mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] &= \mathbb{E}\left[1_{\left\{\sum_{l=1}^{l_0} L_0^l(c_j)>0\right\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_{\left\{\sum_{l=1}^{l_0} L_0^l(c_j)>0\right\}}\middle|\mathcal{S}_n\right]\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > 0\middle|\mathcal{S}_n\right]\right].
\end{aligned} \tag{3.31}$$

Applying the central limit theorem (CLT), the assumed independence of deaths among the policyholders in the portfolio leads to the the following convergence in distribution,

$$\frac{\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - l_0 \cdot \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \operatorname{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}} = \frac{\sqrt{l_0}\left(\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]\right)}{\sqrt{\operatorname{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}} \xrightarrow{D} \mathcal{N}(0,1), \tag{3.32}$$

where we denote $X_n \xrightarrow{D} X$ if X_n converges in distribution to X. In (3.32),

$$\operatorname{Var}\left[L_0(c_j)|\mathcal{S}_n\right] = \mathbb{E}\left[L_0^2(c_j)\middle|\mathcal{S}_n\right] - \left(\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]\right)^2, \tag{3.33}$$

$\mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]$ is given in (3.24), and the second moment of the loss can be derived similarly to (3.24) by

$$\begin{aligned}
&\mathbb{E}\left[L_0^2(c_j)\middle|\mathcal{S}_n\right] \\
=\ &\int_0^n e^{-2r_p t} \cdot \left\{\left[G_t^D - F_t^j(c_j;\mathcal{S}_t)\right]_+\right\}^2 \cdot {}_t\hat{p}_{x,\,t_U+1} \cdot \hat{\mu}_{x+t,\,t_U+1+t}dt \\
&+ \int_0^n \left\{\int_0^t e^{-r_p s}\left[c_j \cdot F_s^j(c_j;\mathcal{S}_s) \cdot X_s^j(c_j;\mathcal{S}_s) \cdot 1_s^j(c_j;\mathcal{S}_s)\right] ds\right\}^2 \cdot {}_t\hat{p}_{x,\,t_U+1} \cdot \hat{\mu}_{x+t,\,t_U+1+t}dt \\
&- 2\int_0^n e^{-r_p t} \cdot \left[G_t^D - F_t^j(c_j;\mathcal{S}_t)\right]_+ \cdot \left[\int_0^t e^{-r_p s} \cdot c_j \cdot F_s^j(c_j;\mathcal{S}_s) \cdot X_s^j(c_j;\mathcal{S}_s) \cdot 1_s^j(c_j;\mathcal{S}_s)ds\right] \\
&\qquad \cdot {}_t\hat{p}_{x,\,t_U+1} \cdot \hat{\mu}_{x+t,\,t_U+1+t}dt \\
&+ e^{-2r_p n} \cdot \left\{\left[G^M - F_n^j(c_j;\mathcal{S}_n)\right]_+\right\}^2 \cdot {}_n\hat{p}_{x,\,t_U+1} \\
&+ \left\{\int_0^n e^{-r_p s}\left[c_j \cdot F_s^j(c_j;\mathcal{S}_s) \cdot X_s^j(c_j;\mathcal{S}_s) \cdot 1_s^j(c_j;\mathcal{S}_s)\right] ds\right\}^2 \cdot {}_n\hat{p}_{x,\,t_U+1}\cdot \\
&- 2e^{-r_p n}\left[G^M - F_n^j(c_j;\mathcal{S}_n)\right]_+ \left[\int_0^n e^{-r_p s} \cdot c_j \cdot F_s^j(c_j;\mathcal{S}_s) \cdot X_s^j(c_j;\mathcal{S}_s) \cdot 1_s^j(c_j;\mathcal{S}_s)ds\right] \\
&\qquad \cdot {}_n\hat{p}_{x,\,t_U+1}\cdot
\end{aligned} \tag{3.34}$$

Through the definition of convergence in distribution, we have

$$\lim_{l_0\to\infty} F_{\frac{\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - l_0\cdot\mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n\right]}{\sqrt{l_0\cdot\operatorname{Var}\left[L_0(c_j)\middle|\mathcal{S}_n\right]}}}(x) = \Phi(x), \quad x \in \mathbb{R}, \tag{3.35}$$

where F_X stands for the cumulative distribution function of a random variable X, and Φ represents that of a standard normal random variable. Therefore, the conditional probability that the total loss is greater than a selected level a, $a \in \mathbb{R}$, is

$$\begin{aligned}\mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n\right] &= \mathbb{P}\left[\frac{\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - l_0 \cdot \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}} > \frac{a - l_0 \cdot \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}}\right] \\ &= 1 - F_{\frac{\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - l_0 \cdot \mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)\middle|\mathcal{S}_n\right]}}}\left(\frac{a - l_0 \cdot \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}}\right) \\ &\to 1 - \Phi\left(\frac{a - l_0 \cdot \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]}}\right) \quad \text{as } l_0 \to \infty. \end{aligned} \tag{3.36}$$

Applying the law of large numbers, by (3.31) we have

$$\begin{aligned}\hat{\mathbb{P}}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > a\right] &:= \frac{1}{N_a}\sum_{k=1}^{N_a} \mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n^k\right] \\ &\xrightarrow{P} \mathbb{E}\left[\mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n\right]\right] = \mathbb{P}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > a\right]. \end{aligned} \tag{3.37}$$

As a result, $\hat{\mathbb{P}}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right]$ is a consistent estimator of the probability of generating a positive total loss when l_0 is large, and the rate of fee can be estimated by

$$\begin{aligned}\hat{c}_j^P(q) &:= \inf\left\{c_j : \hat{\mathbb{P}}\left[\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] \le q\right\} \\ &= \inf\left\{c_j : \frac{1}{N_a}\sum_{k=1}^{N_a}\left[1 - \Phi\left(\frac{-l_0 \cdot \mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n^k\right]}{\sqrt{l_0 \cdot \mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n^k\right]}}\right)\right] \le q\right\} \\ &= \inf\left\{c_j : \frac{1}{N_a}\sum_{k=1}^{N_a}\left[\Phi\left(\frac{\sqrt{l_0} \cdot \mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n^k\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n^k\right]}}\right)\right] \le q\right\}. \end{aligned} \tag{3.38}$$

Alternatively, a rate of fee applying the portfolio percentile principle can be determined at the level so that the probability of generating a positive average loss from the portfolio, $\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j)$, is at most q. That is, this alternative rate of fee $c_j^{\bar{P}}(q)$ covering the guarantee is charged so that

$$c_j^{\bar{P}}(q) = \inf\left\{c_j : \mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] \le q\right\}. \tag{3.39}$$

To approximate the average loss, we start from using the law of total expectation similar to (3.31), and obtain

$$\mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] = \mathbb{E}\left[\mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > 0 \middle| \mathcal{S}_n\right]\right]. \tag{3.40}$$

Then, by the central limit theorem, we have, for a given $\mathcal{S}_n$,

$$\frac{\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]/l_0}} \xrightarrow{D} \mathcal{N}(0,1), \tag{3.41}$$

which leads to the asymptotic conditional probability,

$$\begin{aligned}
\mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n\right] &= \mathbb{P}\left[\frac{\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]/l_0}} > \frac{a - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]/l_0}}\right] \\
&= 1 - F_{\frac{\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j)|\mathcal{S}_n - \mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)\middle|\mathcal{S}_n\right]/l_0}}}\left(\frac{a - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]/l_0}}\right) \\
&\to 1 - \Phi\left(\frac{a - \mathbb{E}\left[L_0(c_j)|\mathcal{S}_n\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n\right]/l_0}}\right) \quad \text{as } l_0 \to \infty.
\end{aligned} \tag{3.42}$$

Applying the law of large numbers again, we have

$$\begin{aligned}
\hat{\mathbb{P}}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > a\right] &:= \frac{1}{N_a}\sum_{k=1}^{N_a} \mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n^k\right] \\
&\xrightarrow{P} \mathbb{E}\left[\mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > a \middle| \mathcal{S}_n\right]\right] = \mathbb{P}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > a\right].
\end{aligned} \tag{3.43}$$

Consequently, $\hat{\mathbb{P}}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right]$ is a consistent estimator of the probability of generating a positive average loss when l_0 is large, and the alternative rate of fee can be estimated by

$$\begin{aligned}
\hat{c}_j^{\bar{P}}(q) :=\; & \inf\left\{c_j : \hat{\mathbb{P}}\left[\frac{1}{l_0}\sum_{l=1}^{l_0} L_0^l(c_j) > 0\right] \le q\right\} \\
=\; & \inf\left\{c_j : \frac{1}{N_a}\sum_{k=1}^{N_a}\left[1-\Phi\left(\frac{-\mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n^k\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n^k\right]/l_0}}\right)\right] \le q\right\} \\
=\; & \inf\left\{c_j : \frac{1}{N_a}\sum_{k=1}^{N_a}\left[\Phi\left(\frac{\sqrt{l_0}\cdot\mathbb{E}\left[L_0(c_j)\middle|\mathcal{S}_n^k\right]}{\sqrt{\mathrm{Var}\left[L_0(c_j)|\mathcal{S}_n^k\right]}}\right)\right] \le q\right\} =\; \hat{c}_j^P(q).
\end{aligned} \tag{3.44}$$

These two estimated rates of fee, $\hat{c}_j^{\bar{P}}(q)$ and $\hat{c}_j^P(q)$, under the portfolio percentile principle are identical, but this does not hold if the least total/average loss from the portfolio, a in (3.36) and (3.42), is not set to zero.

Contract Percentile Principle (CPP)

Select an acceptable insolvency probability $q,\ 0 < q < 1$. By approaching the percentile of the loss function at issue, we can solve the corresponding rate of fee $c_j^C(q)$ by zeroing the $100(1-q)$-th percentile of the loss at issue from a single policy, i.e.,

$$c_j^C(q) = \inf\left\{c_j : \xi_{L_0(c_j)}(1-q) = 0\right\}, \tag{3.45}$$

where $\xi_X(1-q),\ 0 < q < 1$, stands for the $100(1-q)$-th percentile of a continuous random variable X, whose probability density function at x is denoted as $f_X(x)$.

Using the central limit theorem, Serfling (2009) proposed an asymptotic distribution of the $100(1-q)$-th sample percentile of a continuous random variable X, $\hat{\xi}_X(1-q)$, where X has a non-zero density at $\xi_X(1-q)$, by

$$\sqrt{n}\left(\hat{\xi}_X(1-q) - \xi_X(1-q)\right) \xrightarrow{D} \mathcal{N}\left(0, \frac{q(1-q)}{\left[f_X(\xi_X(1-q))\right]^2}\right),$$

or

$$\hat{\xi}_X(1-q) \xrightarrow{D} \mathcal{N}\left(\xi_X(1-q), \frac{q(1-q)}{n\left[f_X(\xi_X(1-q))\right]^2}\right), \tag{3.46}$$

and proved that $\hat{\xi}_X(1-q)$ is a consistent estimator of $\xi_X(1-q)$. Then the rate of fee under the contract percentile principle can be estimated by solving

$$\hat{c}_j^C(q) = \inf\left\{c_j : \hat{\xi}_{L_0(c_j)}(1-q) = 0\right\}. \tag{3.47}$$

3.4 Valuation

For traditional life insurance products, a formula-based static approach is usually used for reserving. However, life insurance policies nowadays have been sharing co-movements in value with the capital market. The complexity, arising from product feature of equity-linked life insurance policies, drives the needs for a new reserving method by taking market risks into consideration. Established by the American Academy of Actuaries (AAA) in 2008, the Variable Annuity Commissioner's Annuity Reserve Valuation Method (VA-CARVM), known as the Actuarial Guideline XLIII (AG-43), was adopted by the National Association of Insurance Commissioners (NAIC) as a principle-based approach for reserving of variable annuities with guaranteed benefits.

3.4.1 Actuarial Guideline XLIII (VA–CARVM)

Considering the so-called stochastic scenario method introduced in AG-43, the underlying fund performance should be simulated as individual scenarios on a representative single policy before determining the reserve. Denote ${}_t\tilde{p}^k_{x,\,t_U+1}$ and $\tilde{\mu}^k_{x+t,\,t_U+1+t}$ be the probability of surviving t years for the policyholder aged x in year t_U+1 and the associate force of mortality at time t (or at age $x+t$ in year t_U+1+t) in the k-th scenario used in valuation. Assuming r_v, a level force of interest, is used for discounting in valuation, the present value of accumulated deficiency (PVAD) up to time t, $PVAD^k_t$, of the guaranteed minimum benefit(s) embedded in a variable annuity under the k-th scenario is defined by, for $0 \le t \le n$,

- if only GMDB is embedded in the variable annuity,

$$\begin{aligned} PVAD^k_t := & \int_0^t e^{-r_v s} \cdot \left[G^D_s - F^j_s(c_j;\mathcal{S}^k_s)\right]_+ \cdot {}_s\tilde{p}^k_{x,\,t_U+1} \cdot \tilde{\mu}^k_{x+s,\,t_U+1+s} ds \\ & - \int_0^t e^{-r_v s} \cdot \left[c_j \cdot X^j_s(c_j;\mathcal{S}^k_s) \cdot 1^j_s(c_j;\mathcal{S}^k_s) \cdot F^j_s(c_j;\mathcal{S}^k_s)\right] \cdot {}_s\tilde{p}^k_{x,\,t_U+1} ds, \end{aligned} \tag{3.48}$$

  where the first component represents the present value of losses from providing the guaranteed minimum death benefit payment on death during $[0,t]$, and the second one is the present value of premiums collected up to time t;

- if only GMMB is embedded in the variable annuity,

$$\begin{aligned} PVAD^k_t := & \, e^{-r_v n} \cdot \left[G^M - F^j_n(c_j;\mathcal{S}^k_n)\right]_+ \cdot {}_n\tilde{p}^k_{x,\,t_U+1} \cdot 1_{\{t=n\}} \\ & - \int_0^t e^{-r_v s} \cdot \left[c_j \cdot X^j_s(c_j;\mathcal{S}^k_s) \cdot 1^j_s(c_j;\mathcal{S}^k_s) \cdot F^j_s(c_j;\mathcal{S}^k_s)\right] \cdot {}_s\tilde{p}^k_{x,\,t_U+1} ds, \end{aligned} \tag{3.49}$$

where the first term arises from the present value of the guaranteed minimum maturity benefit payment if the policyholder survives to the end of n years when the fund value is lower than the guaranteed level; and

- if both GMDB and GMMB are embedded in the variable annuity,

$$
\begin{aligned}
PVAD_t^k := & \int_0^t e^{-r_v s} \cdot \left[G_s^D - F_s^j(c_j; \mathcal{S}_s^k)\right]_+ \cdot {}_s\tilde{p}_{x,\,t_U+1}^k \tilde{\mu}_{x+s,\,t_U+1+s}^k ds \\
& + \; e^{-r_v n} \cdot \left[G^M - F_n^j(c_j; \mathcal{S}_n^k)\right]_+ \cdot {}_n\tilde{p}_{x,\,t_U+1}^k \cdot 1_{\{t=n\}} \\
& - \; \int_0^t e^{-r_v s} \cdot \left[c_j \cdot X_s^j(c_j; \mathcal{S}_s^k) \cdot 1_s^j(c_j; \mathcal{S}_s^k) \cdot F_s^j(c_j; \mathcal{S}_s^k)\right] \cdot {}_s\tilde{p}_{x,\,t_U+1}^k ds.
\end{aligned}
\tag{3.50}
$$

The greatest present value of accumulated deficiencies (GPVAD) over the period $[0, t]$ is given by the supremum of all PVAD up to time t:

$$
\mathcal{M}_t^k \; := \; \sup_{0 \le s \le t} \left\{PVAD_s^k\right\}, 0 \le t \le n. \tag{3.51}
$$

For computational convenience in practice, the projection time points of PVAD can be selected to be a series of time points $\{t_0, t_1, \ldots, t_{N_p}\}$, where $0 = t_0 < t_1 < \ldots < t_{N_p} = n$ and N_p represents the number of time points selected. An illustrative projection on a quarterly basis was demonstrated in Feng and Huang (2016). Then the GPVAD is defined by

$$
\mathcal{M}_{t_m}^k := \max_{l=0,1,\ldots,m} \left\{PVAD_{t_l}^k\right\}, \quad m = 0, 1, \ldots, N_p. \tag{3.52}
$$

A collection of GPVAD, $\mathcal{M}_t^k,\ 0 \le t \le n,\ k = 1, 2, \ldots, N_a$, can be simulated if the single scenario procedure is repeated for thousands of randomly generated scenarios, where we denote $\mathcal{M}_t^k \sim \mathcal{M}_t$.

To quantify the risk that the insurer takes for providing coverage, a common measure for the worst $100(1-\alpha)\%$ scenarios is the percentile, which is usually known as the value-at-risk (VaR) in finance. The $\alpha\%$-VaR of a loss random variable X is defined by, for $0 < \alpha < 1$,

$$
\mathrm{VaR}_\alpha(X) := \; \inf\left\{x \in \mathbb{R} : F_X(x) > \alpha\right\}. \tag{3.53}
$$

Once VaR is reached, one may be interested in the average loss for the worst $100(1-\alpha)\%$ scenario, which can be formulated by the conditional-tail-expectation as

$$
\mathrm{CTE}_\alpha(X) := \; \mathbb{E}\left[X | X > \mathrm{VaR}_\alpha(X)\right]. \tag{3.54}
$$

The CTE amount in the AG-43 is defined as $\mathrm{CTE}_{70\%}(\mathcal{M}_n)$, the 70%-CTE of the GPVAD over the n-year period (the accumulation phase) of a variable annuity, and the AG-43 reserve is determined by the greatest between the CTE amount and a standard scenario amount, which usually serves a deterministic floor for the AG-43 reserve. Since the standard scenario amount contains no stochastic terms, this project focuses only on the stochastic component of the reserve (i.e., the CTE amount).

Chapter 4

Numerical Results

This chapter calculates the CTE amounts in AG-43 reserve using Monte Carlo simulation for a portfolio of variable annuities, embedded with guaranteed minimum death and/or maturity benefits, issued to 10,000 x-year-old male policyholders with homogeneous risk profiles at the beginning of year 2017, based on the assumptions and fee estimates determined by the three pricing principles described in Chapter 3. The simulations are run with 10,000 sample paths for both pricing and valuation, on a weekly basis.

This chapter is organized as follows. In Section 4.1, the data used for the estimation of model parameters are illustrated. Section 4.2 introduces the discretization of the stochastic processes for simulation. The following section demonstrates the fees, the occupation times, the actuarial present values of the premium payments, the value-at-risks, and the estimated CTE amounts in the AG-43 reserve under a representative base case. In the last section, a sensitive analysis is conducted to study the effect of parameter value changes on the AG-43 reserve.

4.1 Estimation for Mortality Model Parameters

This section introduces the mortality data used for estimation and illustrates the estimates of the model parameters in Chapter 3. We fit the model to the U.S. male mortality rates, from the Human Mortality Database, for an age-year window $[25, 84] \times [1955, 2016]$, and exhibit the estimates of the model parameters in Figure 4.1. The estimated model parameters can be used to project future male mortality rates starting in year 2017.

4.2 Simulation Methodology

Sections 4.2.1 and 4.2.2 introduce how we can simulate the stochastic mortality rates, the underlying asset prices, and the corresponding fund values that are formulated in Chapter 3.

For a stochastic process whose closed-form solution is not available, a common method is to approach the continuous process by partitioning the time interval of the accumulation

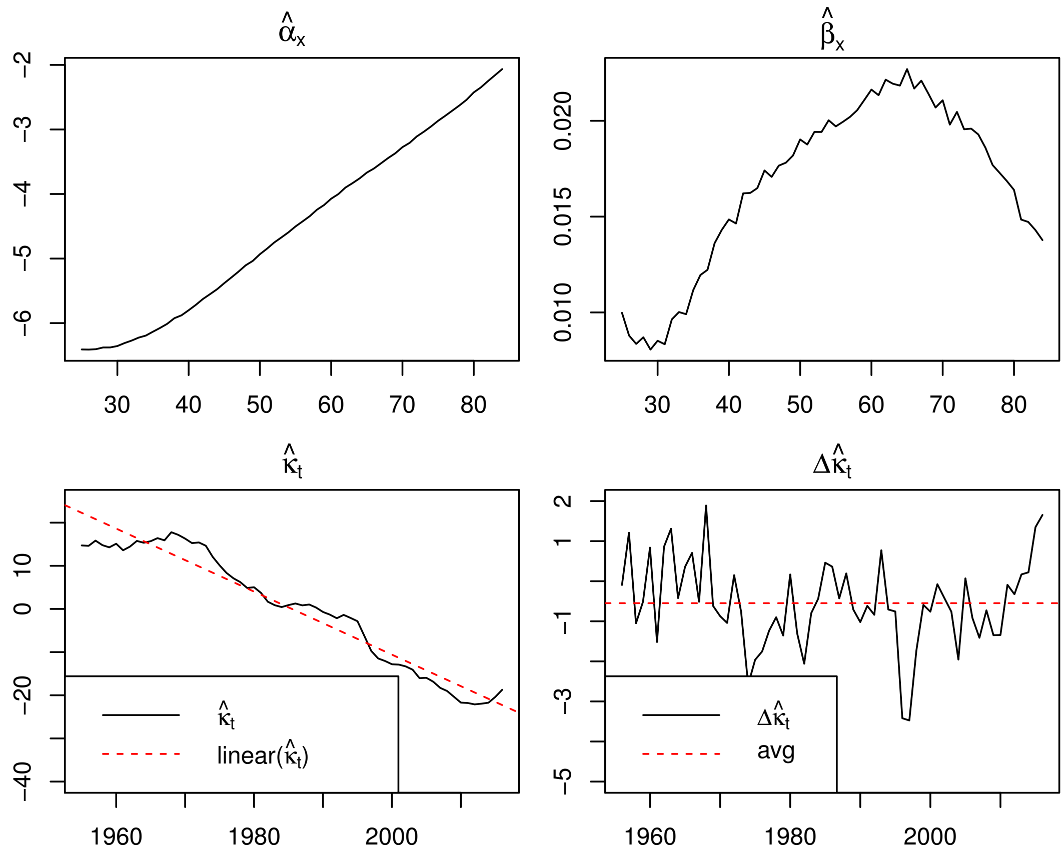

Figure 4.1: $\hat{\alpha}_x, \hat{\beta}_x$ versus x, and $\hat{\kappa}_t$ versus t for U.S. males

phase into infinitesimal subintervals of equal time length. We select a large integer m for approximation and label a series of time points $\left\{t_i = \frac{i}{m} : i = 0, 1, \ldots, nm\right\}$ as the time of projection for pricing and valuation. With this discretization, we provide approximations for mortality projections and simulations, underlying asset prices, corresponding fund values, and the CTE amounts in AG-43 reserve.

4.2.1 Generate Mortality Rates

The process of the logarithm of the central death rate under the Lee-Carter (1992) model is given by (3.1). First, we estimate the model parameters. Then, we project the logarithm of the central mortality rates used in pricing for the years ranging from $t_U + 1$ to $t_U + n$ by

$$\ln(\hat{m}_{x,t_U+t}) = \hat{\alpha}_x + \hat{\beta}_x(\hat{\kappa}_{t_U} + t \cdot \hat{\theta}), \tag{4.1}$$

where $\hat{\alpha}_x$, $\hat{\beta}_x$, $\hat{\kappa}_{t_U}$, and $\hat{\theta}$ are given in Section 3.1.1.

Since the model errors and the time trend errors are assumed independent, we can simulate in the k-th scenario the mortality rates used in valuation for the years ranging from $t_U + 1$ to $t_U + n$ by simulating ϵ_{x,t_U+t}'s from $\mathcal{N}(0, \hat{\sigma}_x^2)$ and e_{t_U+t}'s from $\mathcal{N}(0, \hat{\sigma}_e^2)$,

$t = 1, 2, \ldots, n$, where $\hat{\sigma}_x^2$ and $\hat{\sigma}_e^2$ are given in (3.8) and (3.7). Specifically, the logarithm of central death rate for age x in year $t_U + t$ can be simulated by

$$\ln(\tilde{m}^k_{x,\, t_U+t}) = \hat{\alpha}_x + \hat{\beta}_x(\hat{\kappa}_{t_U} + t \cdot \hat{\theta} + \sum_{i=1}^{t} e^k_{t_U+i}) + \epsilon^k_{x,\, t_U+t}, \tag{4.2}$$

where $\tilde{m}^k_{x,\, t_U+t}$, $e^k_{t_U+i}$, and $\epsilon^k_{x,\, t_U+t}$ are the realizations of $\tilde{m}_{x,\, t_U+t}$, e_{t_U+i}, and $\epsilon_{x,\, t_U+t}$, respectively, in the k-th scenario.

Since the mortality rates used for fitting are usually given in integer ages and years, and so are the projected mortality rates, we are required to simulate mortality rates for fractional ages/years for pricing and valuation. It is common to assume locally constant forces of mortality during each fractional age and year, i.e., $\hat{\mu}_{x+s,\, t+s} = \hat{\mu}_{x,\, t}$ and $\tilde{\mu}_{x+s,\, t+s} = \tilde{\mu}_{x,\, t}$ for integer x's and t's, and $0 \le s < 1$. Under this assumption, it can be shown that $\hat{p}_{x,\, t} = e^{-\hat{\mu}_{x,\, t}} = e^{-\hat{m}_{x,\, t}}$ and $\tilde{p}_{x,\, t} = e^{-\tilde{\mu}_{x,\, t}} = e^{-\tilde{m}_{x,\, t}}$. From the selected time points $\left\{t_i = \frac{i}{m} : i = 0, 1, \ldots, nm\right\}$, we have $t_{lm} = l$, an integer. Assume $t_i \in [l, l+1)$ for $l = 0, 1, \ldots, n-1$ and $i = lm, lm+1, \ldots, (l+1)m - 1$. The projected survival probability, ${}_{t_i}\hat{p}_{x,\, t_U+1}$, can be formulated by

$$\begin{aligned} {}_{t_i}\hat{p}_{x,\, t_U+1} &= {}_{l}\hat{p}_{x,\, t_U+1} \cdot {}_{t_i - l}\hat{p}_{x+l,\, t_U+1+l} \\ &= \left[\prod_{j=0}^{l-1} \hat{p}_{x+j,\, t_U+1+j}\right] \cdot e^{-\int_0^{t_i - l} \hat{\mu}_{x+l+s,\, t_U+1+l+s} ds} \\ &= \left[\prod_{j=0}^{l-1} e^{-\hat{\mu}_{x+j,\, t_U+1+j}}\right] \cdot e^{-\hat{\mu}_{x+l,\, t_U+1+l}(t_i - l)} \\ &= \left[\prod_{j=0}^{l-1} e^{-\hat{m}_{x+j,\, t_U+1+j}}\right] \cdot e^{-\hat{m}_{x+l,\, t_U+1+l}(t_i - l)}, \end{aligned} \tag{4.3}$$

and, similarly, the simulated survival probability in the k-th scenario, ${}_{t_i}\tilde{p}^k_{x,\, t_U+1}$, can be formulated as

$${}_{t_i}\tilde{p}^k_{x,\, t_U+1} = \left[\prod_{j=0}^{l-1} e^{-\tilde{m}^k_{x+j,\, t_U+1+j}}\right] \cdot e^{-\tilde{m}^k_{x+l,\, t_U+1+l}(t_i - l)}. \tag{4.4}$$

4.2.2 Generate Underlying Asset Prices and Fund Value

We model the underlying asset price dynamics as a GBM as presented in (3.9), whose solution is given in (3.10). Due to the normality, independence and stationary properties of the standard Brownian motion, we have $\Delta(W_{t_i}) := W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, \frac{1}{m})$ for all $i = 1, 2, \ldots, nm$. Then, given the initial underlying asset price $S_0 > 0$, the underlying asset price can be simulated in the k-th scenario by the following recursive formula,

$$S_{t_i}^k = S_{t_{i-1}}^k \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right) \cdot \frac{1}{m} + \sigma \cdot \Delta(W_{t_i}^k)\right\}, \quad i = 1, 2, \ldots, nm, \tag{4.5}$$

where $\Delta(W_{t_i}^k)$ is the realization of $\Delta(W_{t_i})$ in the k-th scenario.

According to (3.19), the corresponding simulated fund value under the j-th fee structure can be obtained by, for $i = 1, 2, \ldots, nm$,

$$\begin{aligned} F_{t_i}^j(c_j; \mathcal{S}_{t_i}^k) =& F_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) \\ & \cdot \exp\left\{\left(\mu - d - \frac{\sigma^2}{2}\right)\frac{1}{m} + \sigma\Delta(W_{t_i}^k) - c \cdot X_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) \cdot 1_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k)\right\}, \end{aligned} \tag{4.6}$$

where

$$X_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) = \begin{cases} 1, & j = 1, 2, \\ \frac{H}{F_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k)} - 1, & j = 3, \end{cases}$$

and

$$1_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) = \begin{cases} 1, & j = 1, \\ 1_{\left\{F_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) \le B\right\}}, & j = 2, \\ 1_{\left\{F_{t_{i-1}}^j(c_j; \mathcal{S}_{t_{i-1}}^k) \le (H \wedge B)\right\}}, & j = 3, \end{cases}$$

stand for the ratio of the charge base amount to the fund value at time t_{i-1} and the trigger indicator for premium payment based on the fund value at time t_{i-1}, respectively.

4.2.3 Estimation of the CTE Amount for the AG-43 Reserve

The stochastic processes of mortality rates, the underlying asset price, and the fund value are calculated on a weekly basis ($m = 52$). The $PVAD_{t_i}^k$ in (3.48)–(3.50) can be approximated, for $i = 1, 2, \ldots, nm$, by

$$\begin{aligned} PVAD_{t_i}^k \approx & \sum_{h=1}^{i} e^{-r_v t_h} \cdot \left[G_{t_h}^D - F_{t_h}^j(c_j; \mathcal{S}_{t_h}^k)\right]_+ \cdot {}_{t_h}\tilde{p}_{x,\, t_U+1}^k \cdot \tilde{\mu}_{x+t_h,\, t_U+1+t_h}^k \cdot \frac{1}{m} \\ & - \sum_{h=1}^{i} e^{-r_v t_{h-1}} \cdot \left[c_j \cdot X_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot 1_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot F_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k)\right] \\ & \qquad \cdot {}_{t_{h-1}}\tilde{p}_{x,\, t_U+1}^k \cdot \frac{1}{m} \end{aligned} \tag{4.7}$$

if only GMDB is embedded;

$$
\begin{aligned}
PVAD_{t_i}^k \approx\ & e^{-r_v n} \cdot \Big[G^M - F_n^j(c_j; \mathcal{S}_n^k)\Big]_+ \cdot {}_n\tilde{p}_{x,\,t_U+1}^k \cdot 1_{\{t_i=n\}} \\
& - \sum_{h=1}^{i} e^{-r_v t_{h-1}} \cdot \Big[c_j \cdot X_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot 1_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot F_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k)\Big] \\
& \qquad \cdot {}_{t_{h-1}}\tilde{p}_{x,\,t_U+1}^k \cdot \frac{1}{m}
\end{aligned} \tag{4.8}
$$

if only GMMB is embedded; and

$$
\begin{aligned}
PVAD_{t_i}^k \approx\ & \sum_{h=1}^{i} e^{-r_v t_h} \cdot \Big[G_{t_h}^D - F_{t_h}^j(c_j; \mathcal{S}_{t_h}^k)\Big]_+ \cdot {}_{t_h}\tilde{p}_{x,\,t_U+1}^k \cdot \tilde{\mu}_{x+t_h,\,t_U+1+t_h}^k \cdot \frac{1}{m} \\
& + e^{-r_v n} \cdot \Big[G^M - F_n^j(c_j; \mathcal{S}_n^k)\Big]_+ \cdot {}_n\tilde{p}_{x,\,t_U+1}^k \cdot 1_{\{t_i=n\}} \\
& - \sum_{h=1}^{i} e^{-r_v t_{h-1}} \cdot \Big[c_j \cdot X_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot 1_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k) \cdot F_{t_{h-1}}^j(c_j; \mathcal{S}_{t_{h-1}}^k)\Big] \\
& \qquad \cdot {}_{t_{h-1}}\tilde{p}_{x,\,t_U+1}^k \cdot \frac{1}{m}
\end{aligned} \tag{4.9}
$$

if both GMDB and GMMB are embedded, respectively. By the order statistics of $\mathcal{M}_n^k$'s (defined in (3.52)) with respect to $k, k = 1, \cdots, N_a$, the $100(1-\alpha)\%$-CTE amount for the AG-43 reserve can be estimated by averaging the top $100\alpha\%$ greatest simulated $\mathcal{M}_n^k$'s.

4.3 Base Case

This section begins with the assumptions on the parameter values for the stochastic models and the guarantee features. After that, we present the comparison of the estimated fees among three fee structures and three pricing principles, the estimated PVADs over time, and the estimated CTE amounts in AG-43 reserve versus the confidence level.

4.3.1 Assumptions

The values of all parameters that are used in the base case are summarized in Table 4.3.1. We consider three kinds of guarantee features: a GMDB (a variable annuity with GMDB only), a GMMB (a variable annuity with GMMB only), and a GMDMB (a variable annuity with both GMDB and GMMB). We assume that, if applicable, both the guaranteed minimum death benefit over the accumulation phase and the guaranteed minimum maturity benefit are set to the initial investment amount, F_0.

4.3.2 Rate of Fee

As the estimates for the rate of fee charge under three principles are formulated, we compare the estimated rate of fee between fee structures. Figures 4.2–4.4 display the estimated rates

Table 4.1: Values of parameters

	Parameter	Value
Mortality fitting		
$[x_L, x_U]$	Age fitting span	[25, 84]
$[t_L, t_U]$	Year fitting span	[1955, 2016]
Underlying asset price model		
μ	Expected annual growth rate of return	0.03
d	Rate of management fee	0.005
σ	Underlying asset volatility	0.3
Product feature		
x	Policy issue age	55
$t_U + 1$	Policy issue year	2017
n	Length of accumulative phase period	10 (years)
G_t^D	Base amount of the guaranteed minimum death benefit (if applicable)	1,000
G^M	Base amount of the guaranteed minimum maturity benefit (if applicable)	1,000
B	Barrier that triggers a fee charge under the fee structures F2 and F3	1,000
H	Reference fund value for the moneyness-adapting base amount under the fee structure F3	1,000
F_0	Initial investment amount in a variable annuity	1,000
Pricing		
l_0	Initial number of policyholders in the portfolio	10,000
m	Number of projection time points in one year in pricing	52
N_a	Number of simulations for the underlying asset prices in pricing	10,000
r_p	Constant force of interest for discounting in pricing	0.03
q	Confidence level: the largest probability of generating a positive total (individual) loss under the PPP (CPP)	[1%, 99%][1]
Valuation		
m	Number of projection time points in one year in valuation	52
N_a	Number of simulations for the underlying asset prices in valuation	10,000
r_v	Constant force of interest for discounting in valuation	0.03
α	Confidence level for calculating VaR and CTE	0.70

[1] q changes by 1%, ranging from 1% to 99%

of fee, under the confidence level q, for the GMDB, GMMB, and GMDMB, respectively, defined in the base case. In each plot, the dotted lines show the estimated rates for adopting the fee structure F1, while the dashed and the solid ones are for F2 and F3, respectively. The left panels collect the results when the fees are determined under the actuarial equivalence principle (AEP); the middle panels show those under the portfolio percentile principle (PPP); the right ones portrait those under the contract percentile principle (CPP).

We observe that, since the confidence level q is not considered under the actuarial equivalence principle, the estimated rates of fee determined under this pricing principle are leveled in the plots. Using the AEP, the fee determined under the fee structure F3 is the highest, while that under F1 results in the lowest and that under the F2 is in the middle. This is reasonable because there exists a trigger for the premium charge under F2 and F3, i.e., the premiums are only paid when the fund value is below the barrier B, while F1 charges the policyholders over the whole period of the accumulation phase (see Section 4.3.3 for more details).

Furthermore, it is intuitive that the fees under the PPP and CPP are generally decreasing in q, where q represents the probability of incurring a positive loss. To reach a better fund adequacy and to create the profitability of issuing the guarantee, the insurers usually choose small q for conservative premiums. We observe from the figures that, when a small q is selected for the fee calculation ($q < 0.10$ for GMDB, $q < 0.37$ for GMMB, and $q < 0.42$ for GMDMB in the base case), the rate of fee is the smallest if the fee structure F3 is adopted, which makes the fee structure F3 more attractive to the policyholders.

On the other hand, although the rates of fee using F3 under the PPP and CPP for GMDB and a moderate q (say, 0.5) are significantly higher than those using F1 and F2, it is less likely for a rational guarantee provider to accept such a large possibility of generating a positive loss during the pricing stage. Moreover, when q is smaller than 0.38, the rates of fee using the PPP and CPP for GMMB and GMDMB are lower for the fee structure F3 than those found with the fee structures F1 and F2. Therefore, if the attractiveness of a certain fee structure is measured based on the level of the rate of fee, the fee structure F3 is most likely favoured by the policyholders.

An interesting observation in Figure 4.4 is that there are drops in the estimated rates of fee at q around 0.62 for GMDMB under the PPP and CPP. Figure 4.3 shows that, when q is larger than 0.60, there are no solutions $\hat{c}_j^P(q) \geq 0$ and $\hat{c}_j^C(q) \geq 0$ for GMMB under the PPP and CPP, respectively. That is, even if no rates of fee are charged for GMMB, the probabilities of generating a positive loss from the whole portfolio and a single policy are still smaller than q. Furthermore, according to the mortality forecast, the probability of surviving a 10-year period for (55) in 2017 is around 90.49%, which infers that the payment of the guaranteed minimum maturity benefit accounts for most of the loss from GMDMB. Therefore, if the confidence level q is larger than 0.6, the estimated rates of fee for GMDMB resemble those for GMDB, which are less than 1% in the base case.

4.3.3 Occupation Time

Figures 4.5–4.7 demonstrate the occupation times if premiums are determined by the AEP, PPP, and CPP using the fee structures F1, F2, and F3 for GMDB, GMMB, and GMDMB, respectively, in the base case.

Under the fee structure F1, policyholders are charged premiums over the whole period of the accumulation phase; regardless of the moneyness of the guarantees, the occupation time always equals the length of the accumulation phase. On the contrary, the occupation time under the fee structures F2 and F3, which are formulated in Sections 3.2.2 and 3.2.3, is the total time when the fund value is below a certain level, i.e., B for F2 and $B \wedge H$ for F3. From the figures, we see that the policyholders for GMDB are charged premiums for only about half of the length of the accumulation phase if F2 or F3 is adopted, while the

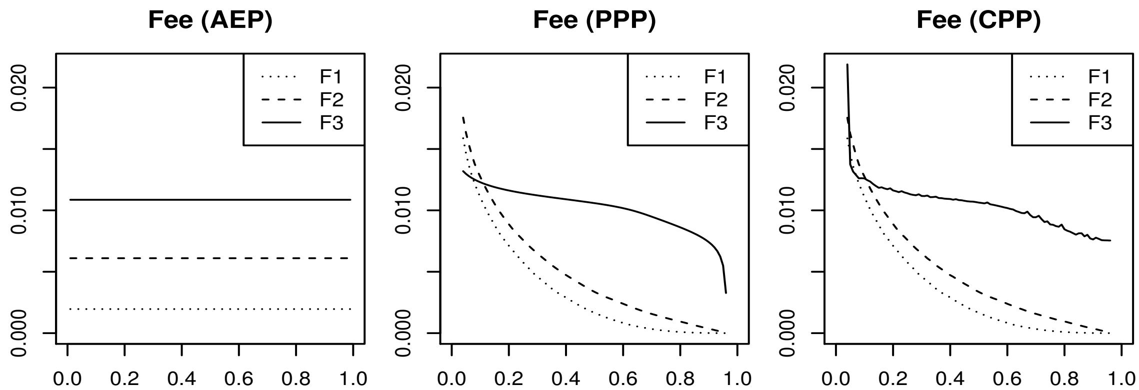

Figure 4.2: Estimated rates of fee charge for GMDB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

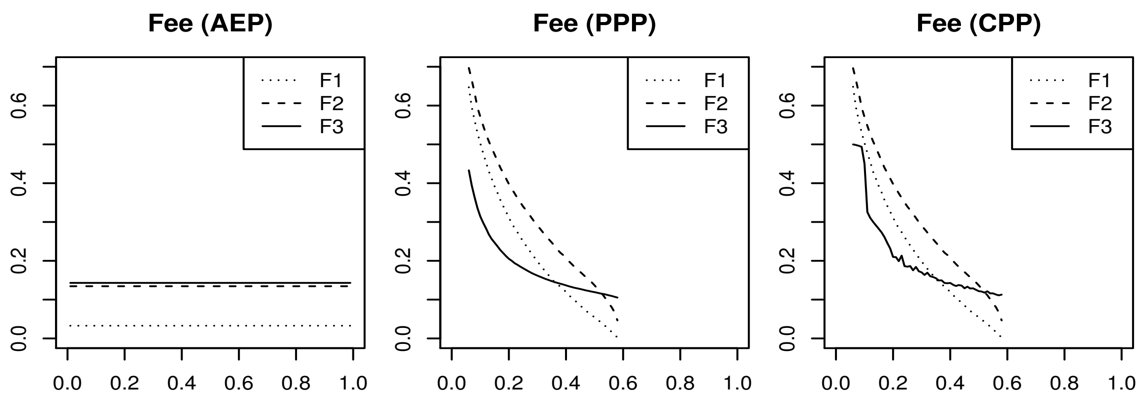

Figure 4.3: Estimated rates of fee charge for GMMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

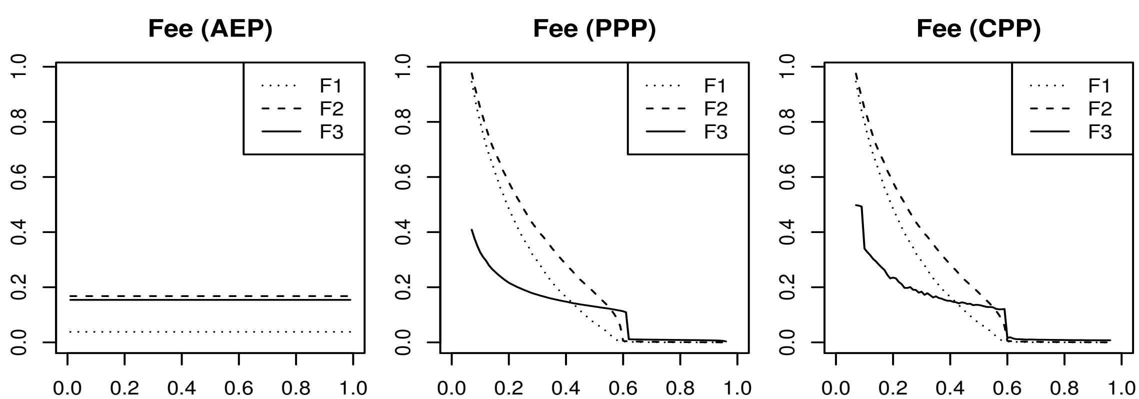

Figure 4.4: Estimated rates of fee charge for GMDMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

premiums are charged over the whole period if F1 is adopted. Figures 4.6 and 4.7 show that the occupation times for GMMB and GMDMB decrease in q and approach the length of the accumulation phase when the fee structure F2 and a small q are selected, while those under the fee structure F3 are still only about half, making the fee structure F3 more attractive to policyholders.

4.3.4 Actuarial Present Value of Premium Payments

From the perspective of policyholders, contracts providing identical protections yet asking for lower premium payments are more favored. This section compares the actuarial present values of the premium payments under the three fee structures using the three pricing principles, whose results are displayed in Figures 4.8–4.10.

From the figures, we obtain results similar to those in Section 4.3.2: adopting the fee structure F3 results in the lowest actuarial present values of premium payments, comparing to those charged under the fee structures F1 and F2 when pricing with the PPP and CPP, and selecting a small q for GMDB and a q value between around 0.15 and 0.5 for GMMB and GMDMB.

4.3.5 Value-at-Risk

The value-at-risk of the loss helps quantify the risk. The 70%-VaR's (or the 70%-percentiles) of the GPVAD for GMDB, GMMB, and GMDMB under the three fee structures and the three pricing principles are provided in Figures 4.11–4.13. From the figures for the GMDB using the PPP and CPP, we observe that the VaR under the fee structure F3, although not the lowest, is slightly higher than those under the other two fee structures when q is small. Moreover, for a q value beyond about 0.37, the fee structure F3 does generate a significantly smaller VaR. In addition, the left panels in Figures 4.12 and 4.13 show that the VaRs under the AEP for GMMB and GMDMB are the lowest when the fee structure F3 is adopted. However, the VaRs for GMMB and GMDBM with the fee structure F3 under the PPP and CPP are not satisfactory when the confidence level q is smaller than 0.30.

4.3.6 CTE Amount in the AG-43 Reserve

The 70%-CTE amount is one of the components for calculating the AG-43 reserve. This determines the statutory reserving requirement for the insurers who issue variable annuities embedded with guarantees. The 70%-CTE amounts for GMDB, GMMB, and GMDMB, under three fee structures and three fee structures are illustrated in Figures 4.14–4.16. We observe in Figure 4.14 that, using any of the three pricing principles, the 70%-CTE under the fee structure F3 is lower than those under the fee structures F1 and F2. Even though the rate of fee charged with the AEP for GMMB (GMDMB) using the fee structure F3 is close to (smaller than) that charged using the fee structure F2, the CTE amount for GMMB (GMDMB) under the fee structure F3 is the lowest, compared to those under the fee structures F1 and F2. However, when using the PPP and CPP, the CTE amounts under

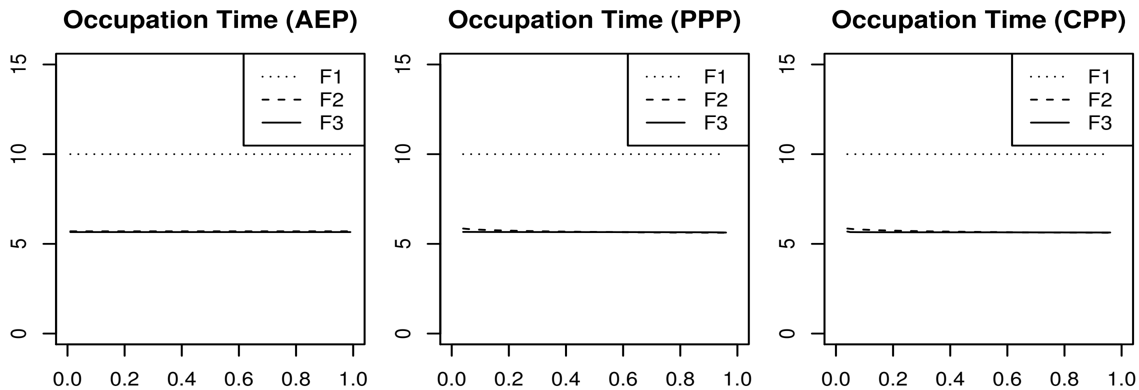

Figure 4.5: Estimated occupation times for GMDB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

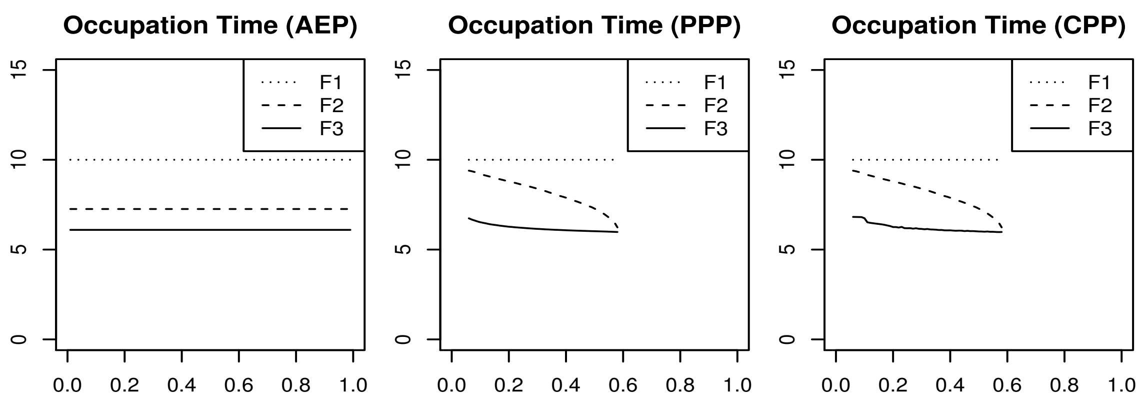

Figure 4.6: Estimated occupation times for GMMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

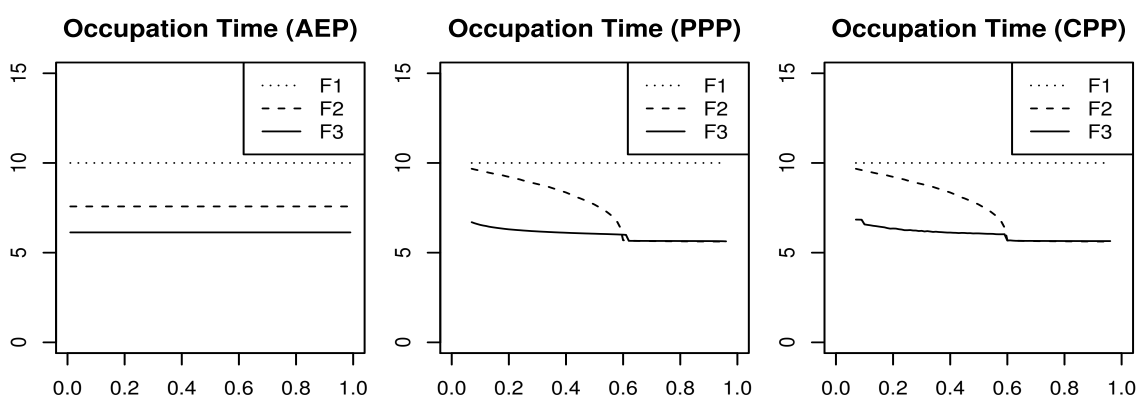

Figure 4.7: Estimated occupation times for GMDMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

the fee structure F3 are higher than those under the fee structures F1 and F2 for a confidence level q smaller than 0.3. Therefore, if the performance of a fee structure is measured by its corresponding CTE amount, the fee structure F3 outperforms the other two fee structures F1 and F2 for GMDB under all of the three pricing principles and for GMMB and GMDMB under the AEP, but it is not guaranteed for GMMB and GMDMB under the PPP and CPP when q is small.

4.4 Sensitivity Analysis

This section analyzes the sensitivities of the rate of fee, the 70%-VaR of the loss, and the 70%-CTE of GMDB to some key parameters that may significantly affect the pricing and valuation results, including the volatility of the underlying asset σ, the barrier for the premium payments B, the rate of management fee d for holding the underlying asset, and the policy issue age x.

In the following subsections, each group of plots consists of 9 subplots. Each subplot shows the fee, the 70%-VaR, and the 70%-CTE for three different values of a key parameter, where the solid lines display the results under the base case, the dotted lines exhibit those under a lower parameter value, and the dashed lines demonstrate those for a higher parameter value. Moreover, the upper panels display the results when adopting the fee structure F1, the middle panels show those when using the fee structure F2, and lower panels gives the results under the fee structure F3; the left panels demonstrate the results using the AEP, whereas the middle and the right panels illustrate those using the PPP and CPP, respectively.

4.4.1 Effect of Changes in the Underlying Asset Volatility

Since the policyholders are usually allowed to choose from a selection of investment options, and the performance of the selected vehicle has a sound impact on the guaranteed minimum benefit payments, the occupation time (for the fee structures F2 and F3), and the total premium payments. Therefore, we conduct a sensitivity analysis to study the impacts on the rate of fee, the VaR, and the CTE amounts caused by a change in the volatility of the selected underlying asset.

Figure 4.17 plots the rates of fee at three different values of σ under the three fee structures and the three pricing principles. It shows that the rate of fee is increasing with the volatility of the underlying asset. Since the guarantee pays off only when the fund value is below the guaranteed level at the time of death and/or the time of maturity, and the guarantee is set to at-the-money at time of issue, an increase in the volatility increases the probability of making a positive benefit payment. Furthermore, comparing to the results under the fee structures F1 and F2, the rate of fee is less sensitive to the volatility of the underlying asset when the fee structure F3 is adopted.

Figure 4.18 plots the 70%-VaR's at three different values of σ. It shows that the 70%-VaR is increasing with the volatility under the three fee structures and the three pricing principles. In addition, we observe that the VaR is less sensitive to the volatility when the PPP or the CPP is adopted for a small q. The low sensitivity to the volatility can also be found when the fee structure F3 is adopted under the AEP and for all values of q. A similar conclusion can also be reached for the sensitivity of the 70%-CTE, whose results are displayed in Figure 4.19.

4.4.2 Effect of Changes in the Barrier for Premium Payments

The barrier for premium payments determines when the policyholders are charged premiums if the fee structures F2 and F3 are adopted. A higher barrier B leads to a longer occupation time, so it is intuitive that the rate of fee is decreasing with the barrier, which can be observed from the lower two panels in Figure 4.20. Also, since a barrier is not adopted under the fee structure F1, there is no effect on the rate of fee for different values of the barrier.

Figures 4.21 and 4.22 demonstrate the 70%-VaR's and 70%-CTE under three different levels of the barrier. From the left panels in these two figures, we observe that the VaR and CTE are stable when the AEP is used for pricing. In addition, the VaR and CTE are also insensitive to the barrier level change when the fee structure F3 is adopted for small q's.

4.4.3 Effect of Changes in the Management Fee

The level of the management fee in the underlying asset affects the price dynamics. The management fee is one of the components in the drift term of the price, so the fund value is deducted faster for a larger management fee. Thus, the excess of the guaranteed minimum benefit payment and the rates of fee are expected to be larger when the rate of management fee is raised, and so are the VaR's and the CTE's. Although these expectations can be found in Figures 4.23–4.25, there is no significant impact on the results for a 0.5% change in the rate of management fee under the three fee structures and the three pricing principles.

4.4.4 Effect of Changes in the Policy Issue Age

The last sensitivity test compares the results when the GMDB is issued to policyholders with three different ages (35, 55, and 75) at the time of issue. The mortality rates projected for policyholder aged 75 during the period of the accumulation phase are expected to be larger, leading the GMDB issuer to a higher probability of paying death benefits and collecting less premiums. This accounts for the highest rates of fee, displayed in Figure 4.26, when the GMDB is issue to (75), compared to the results for (55) and (35). In addition, for (75) we observe that, under the PPP and CPP when the confidence level q is smaller than 0.20, the rate of fee charged under the fee structure F3 is significantly smaller than those under the fee structures F1 and F2.

Figures 4.27 and 4.28 display that the VaRs and the CTE amounts are less sensitive to the policy issue age for almost all confidence level q's under the fee structure F3, compared to those under the fee structures F1 and F2. Under the AEP, the CTE amounts required for issuing a GMDB to (75) when using the fee structures F1 and F2 are 144 and 91, respectively, while the CTE amount is only 20 when the fee structure F3 is adopted. The smaller CTE amounts under the fee structure F3 can also be verified for the three issue ages when the PPP and CPP are used, which provides further evidence to support the attractiveness of the proposed fee structure F3.

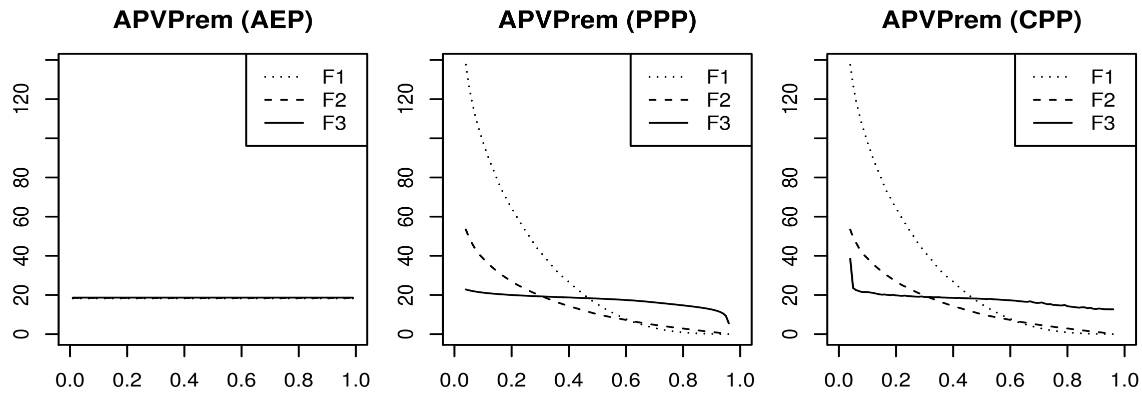

Figure 4.8: Estimated actuarial present values of the premium payments for GMDB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

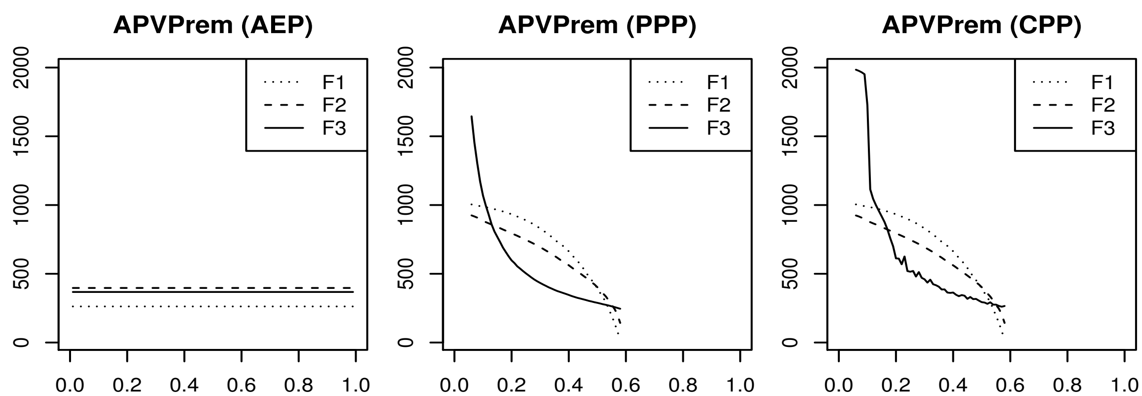

Figure 4.9: Estimated actuarial present values of the premium payments for GMMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

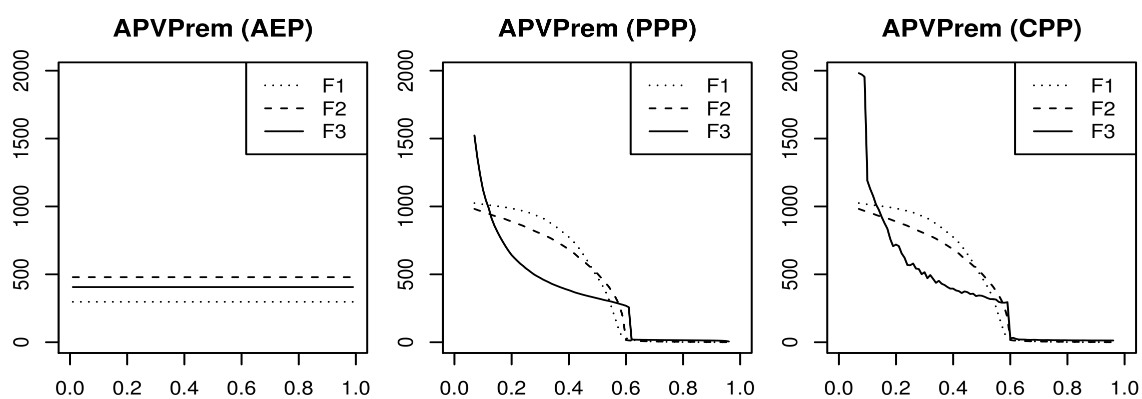

Figure 4.10: Estimated actuarial present values of the premium payments for GMDMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

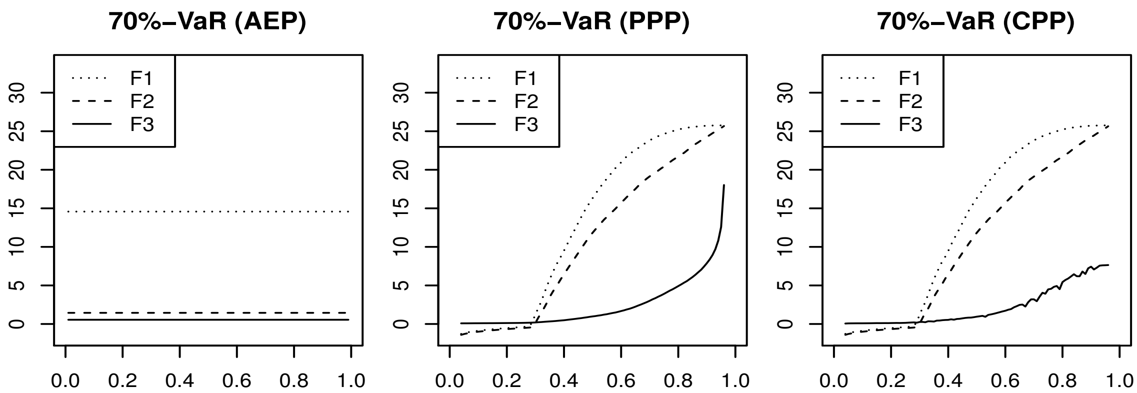

Figure 4.11: Estimated value-at-risks of the loss for GMDB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

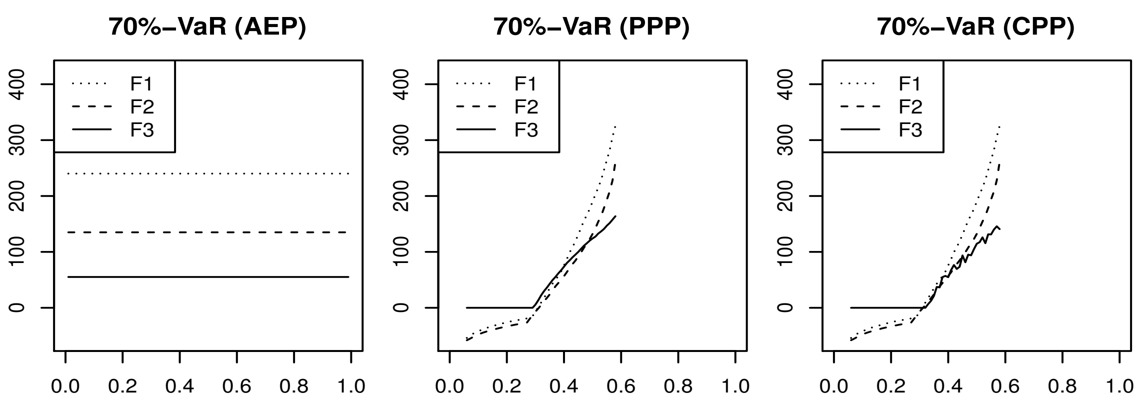

Figure 4.12: Estimated value-at-risks of the loss for GMMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

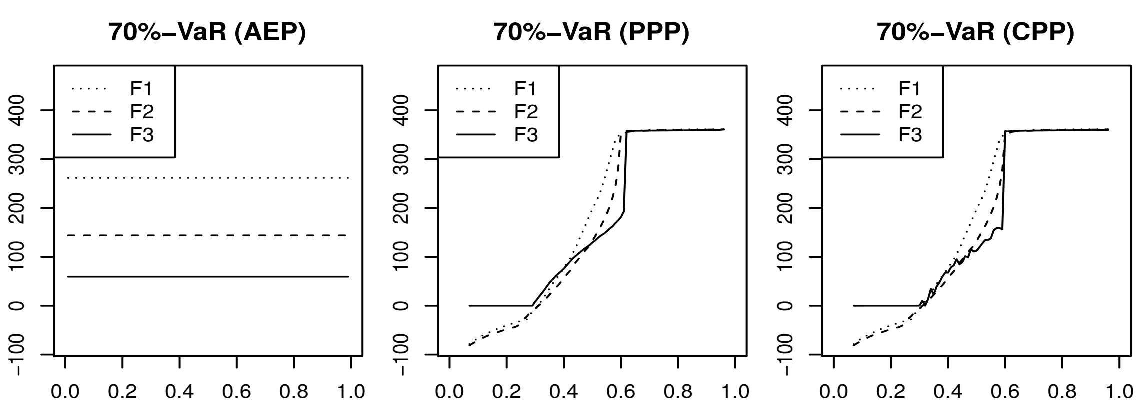

Figure 4.13: Estimated value-at-risks of the loss for GMDMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

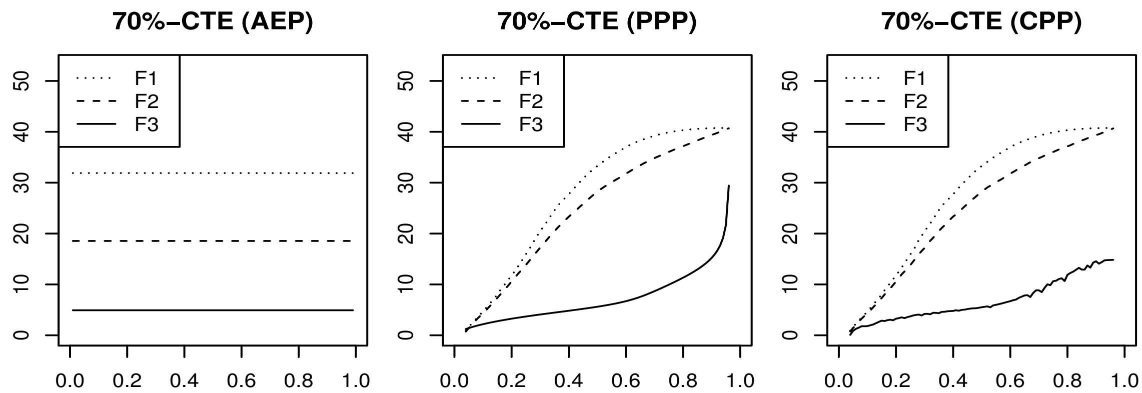

Figure 4.14: Estimated CTE amounts for GMDB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

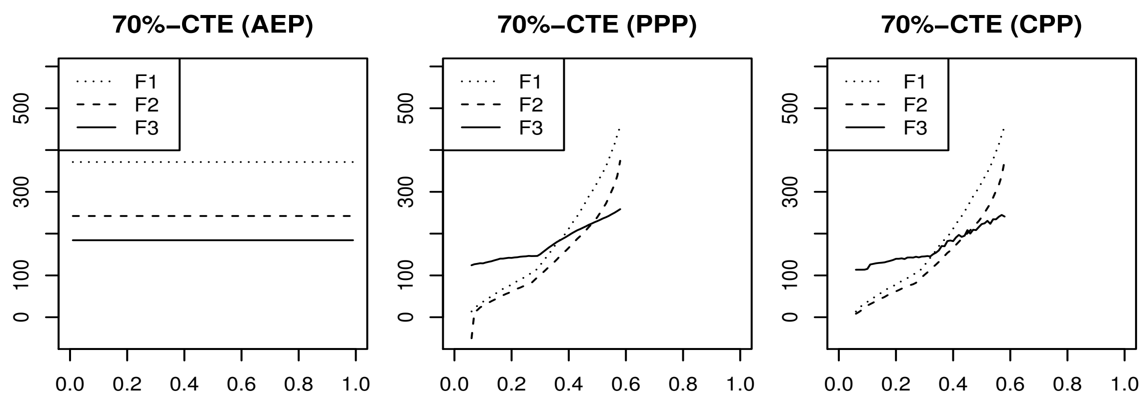

Figure 4.15: Estimated CTE amounts for GMMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

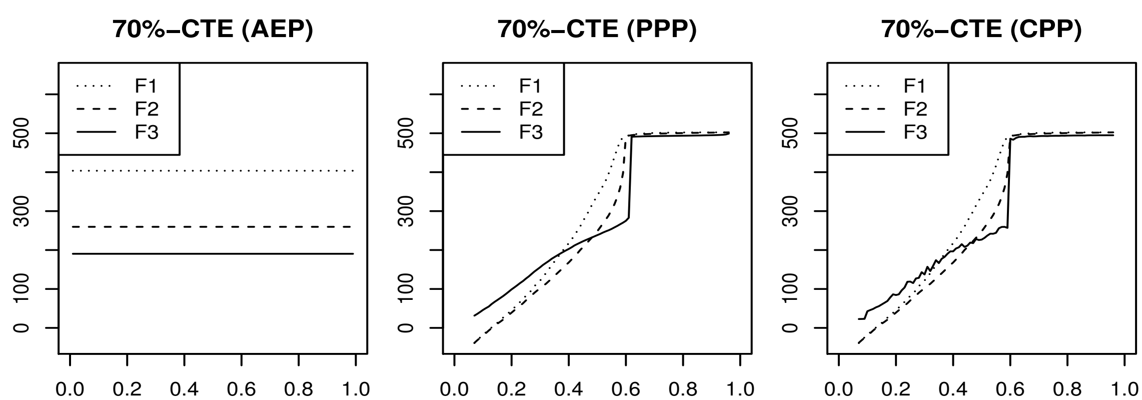

Figure 4.16: Estimated CTE amounts for GMDMB, under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q

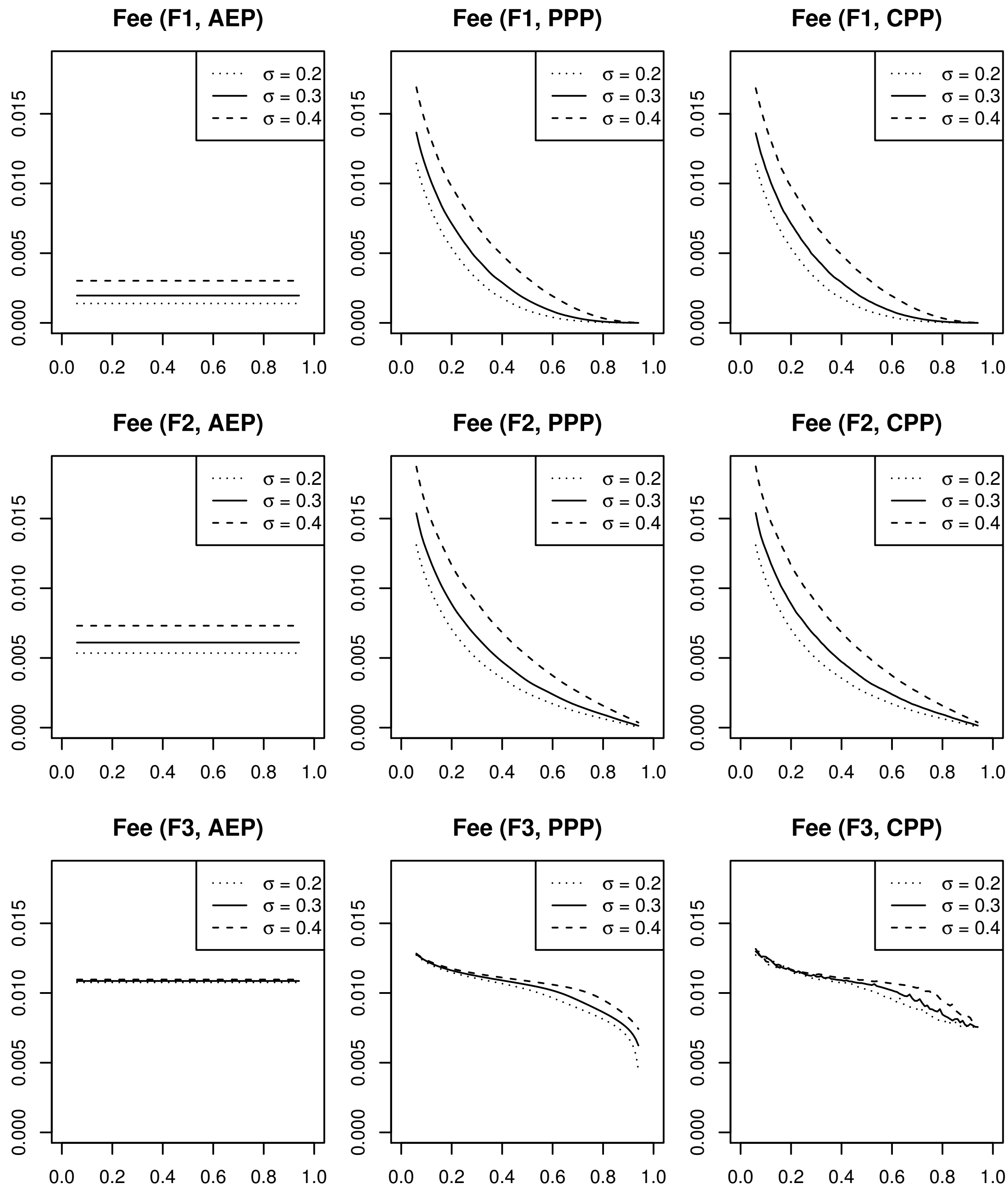

Figure 4.17: Estimated rates of fee charge for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the volatility of the underlying asset

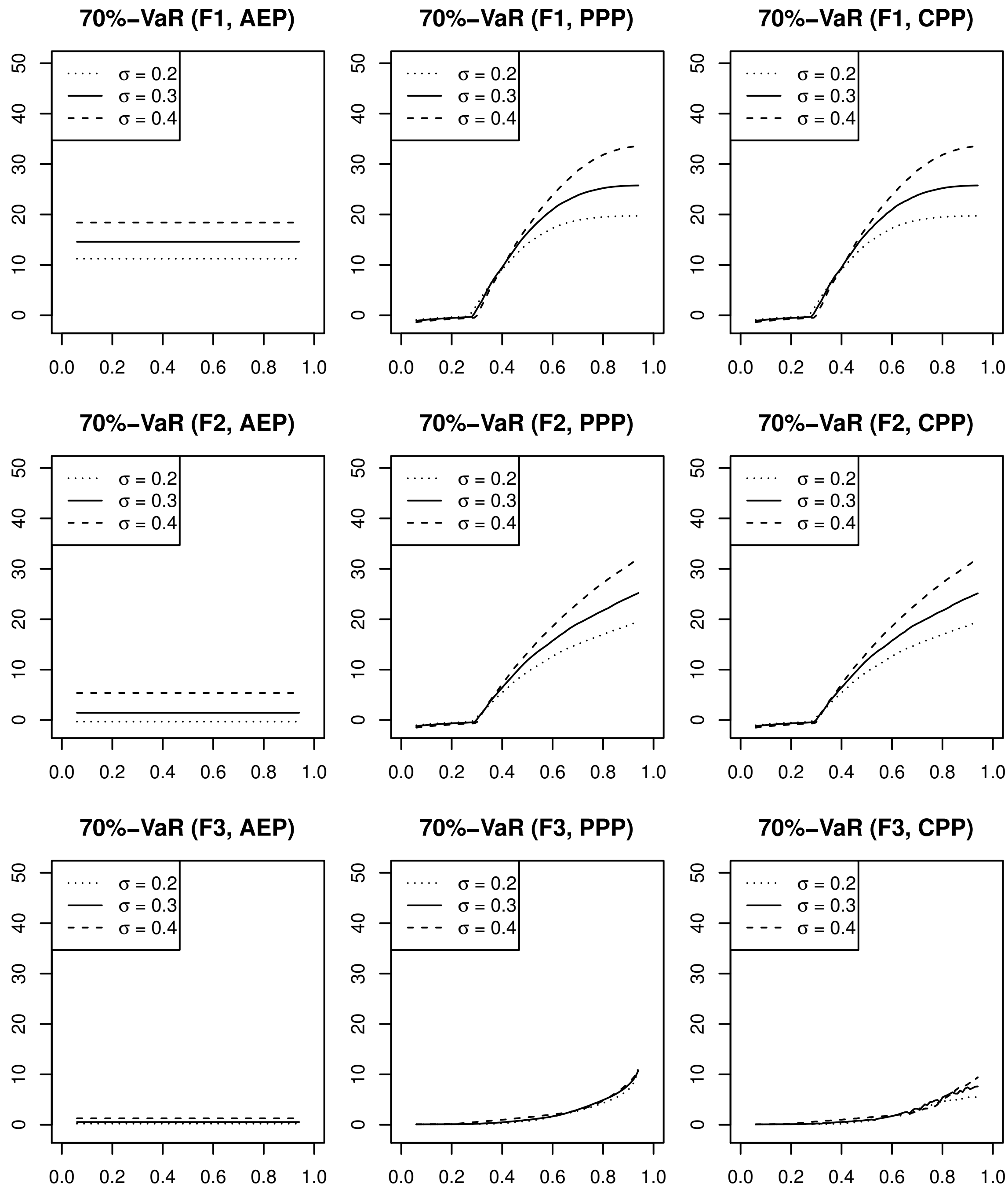

Figure 4.18: Estimated 70%-VaR's for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the volatility of the underlying asset

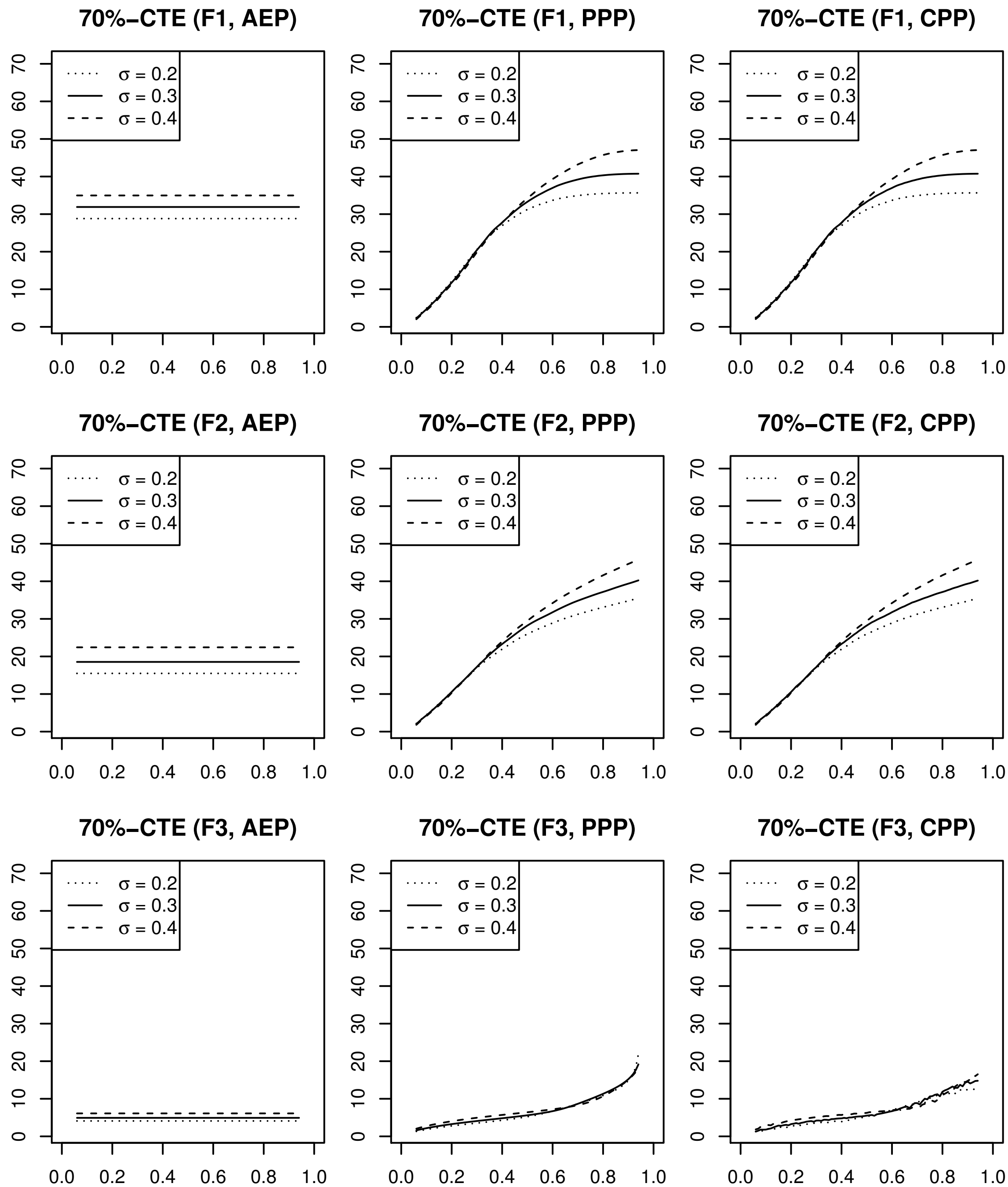

Figure 4.19: Estimated 70%-CTE amounts for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the volatility of the underlying asset

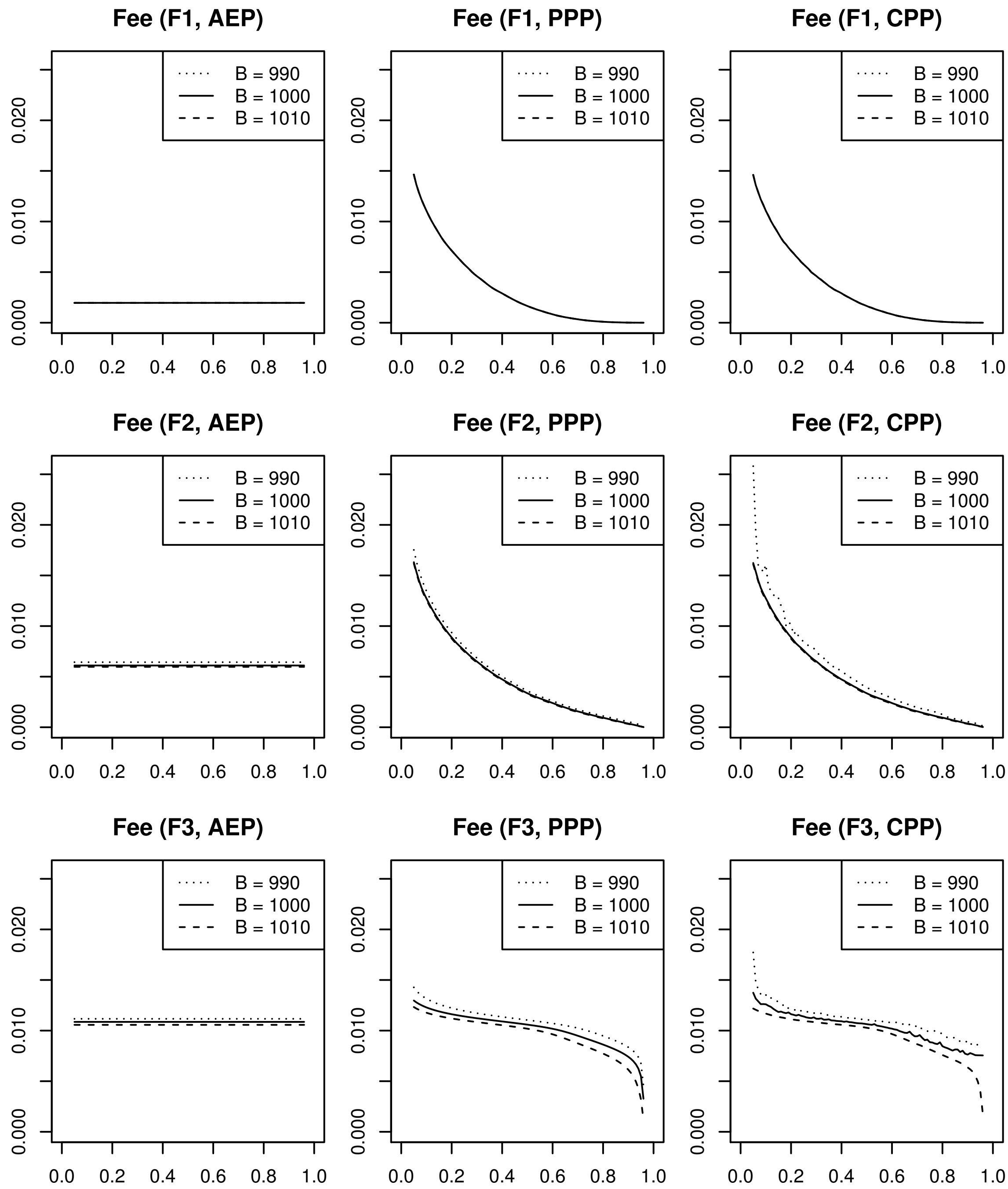

Figure 4.20: Estimated rates of fee charge for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the barrier for premium payments

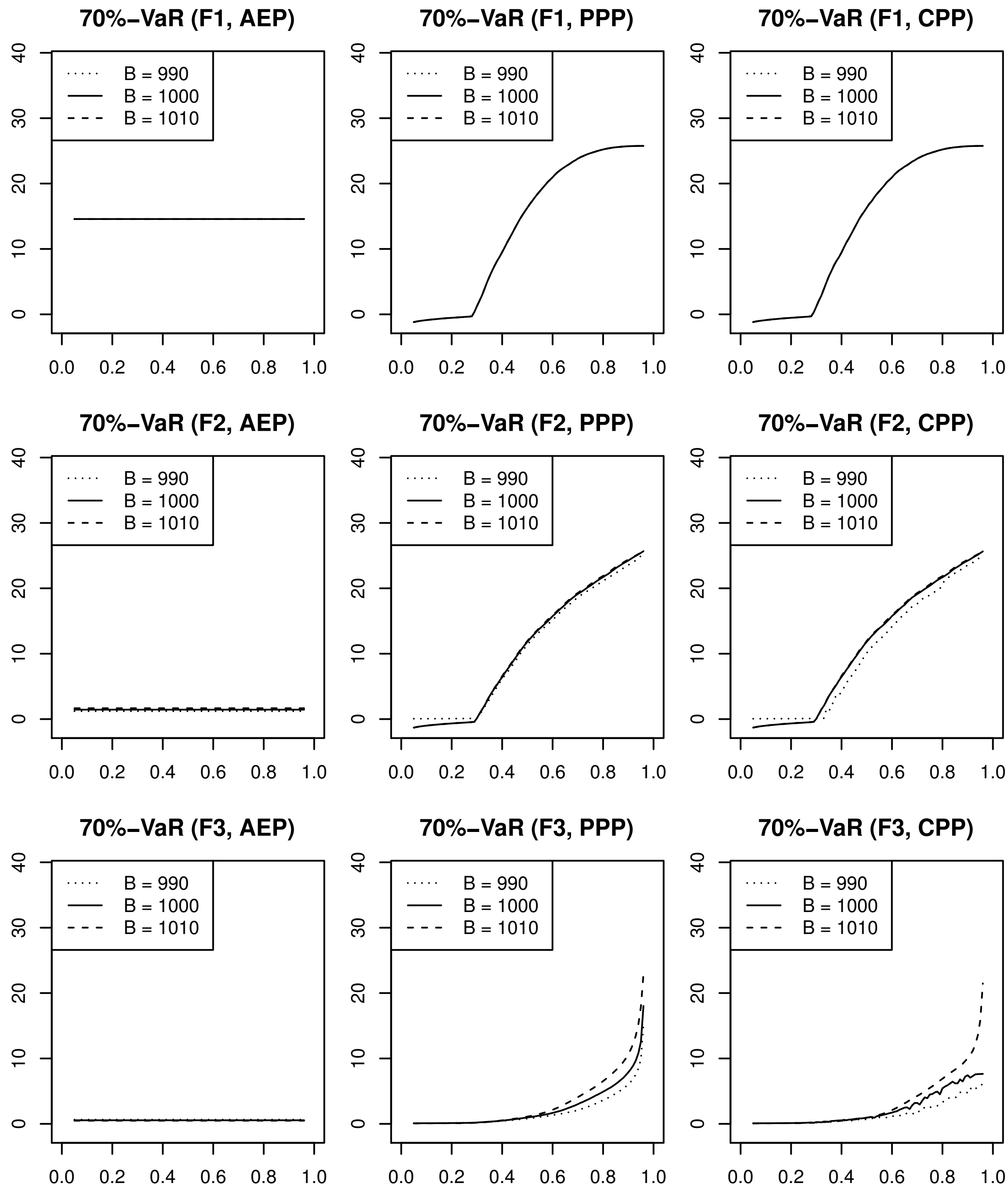

Figure 4.21: Estimated 70%-VaR's for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the barrier for premium payments

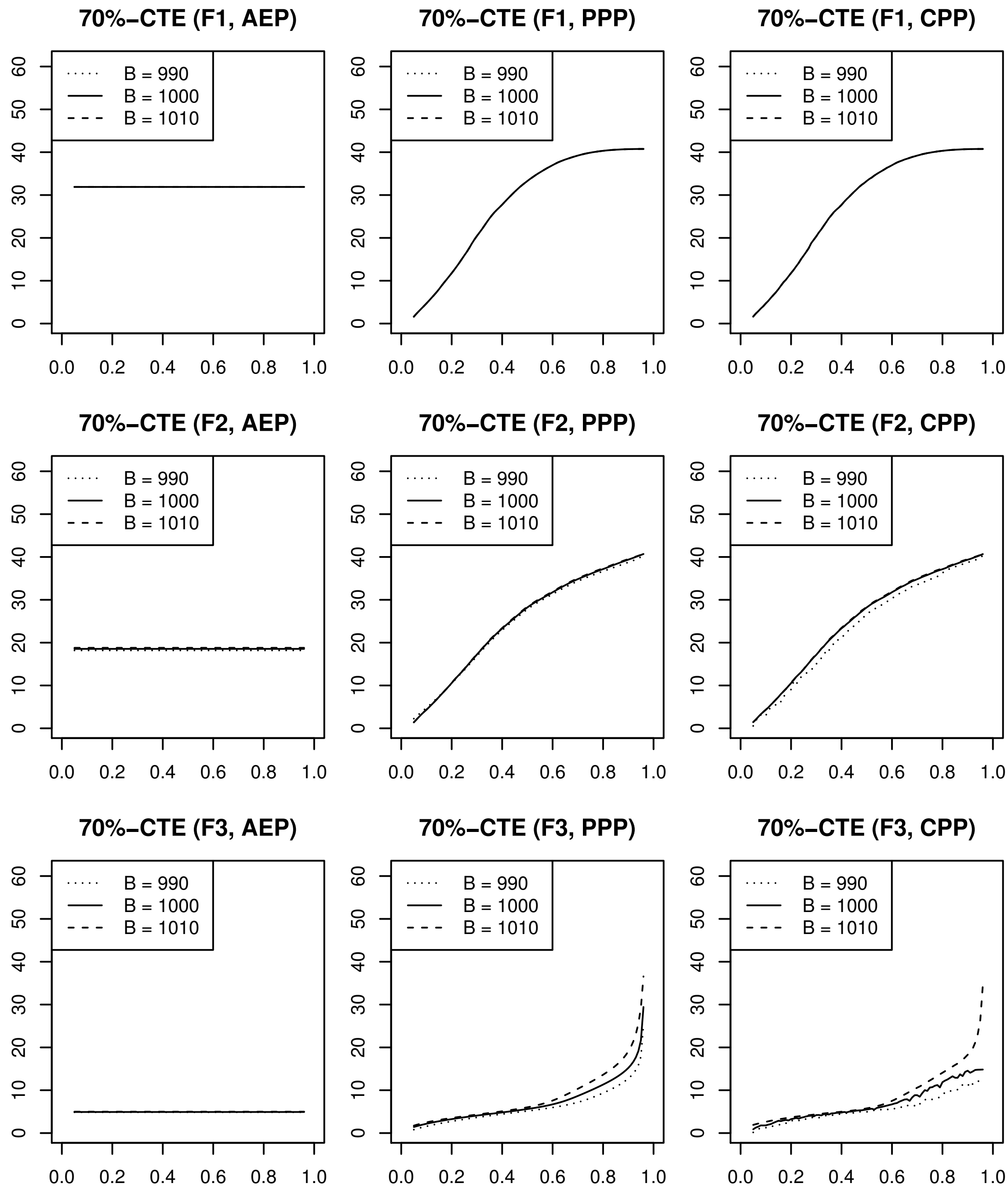

Figure 4.22: Estimated 70%-CTE amounts for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the barrier for premium payments

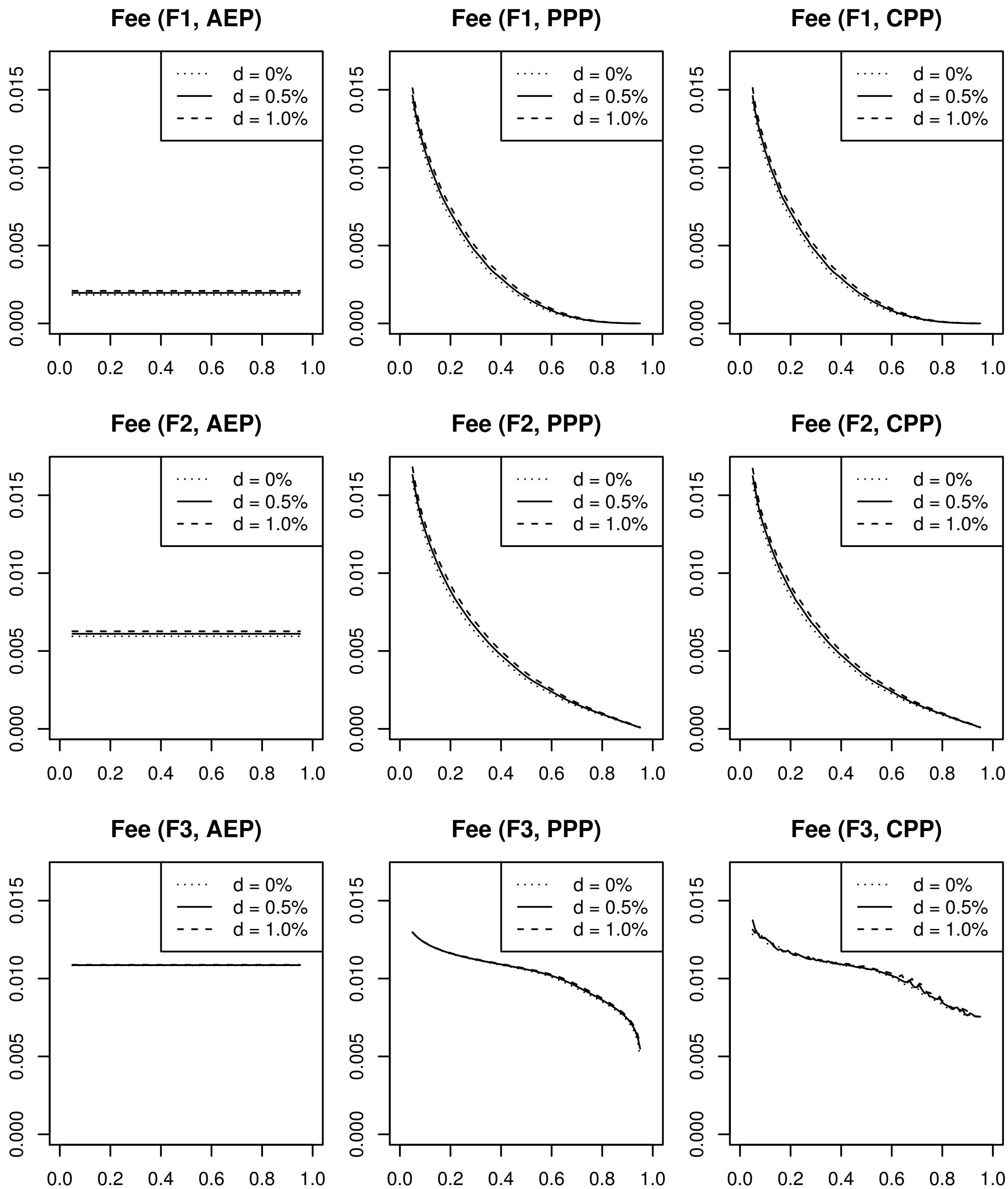

Figure 4.23: Estimated rates of fee charge for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the management fee

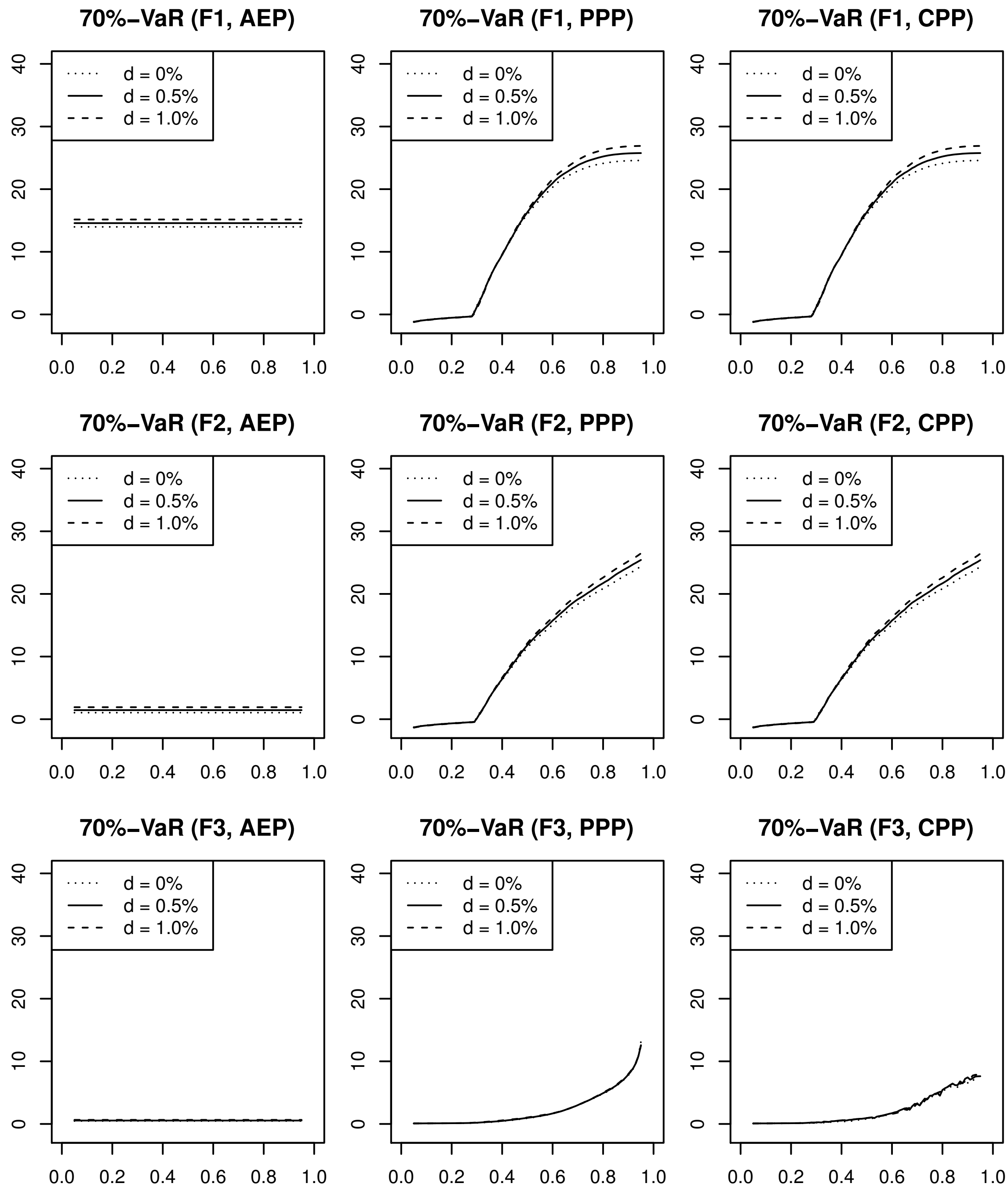

Figure 4.24: Estimated 70%-VaR's for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the management fee

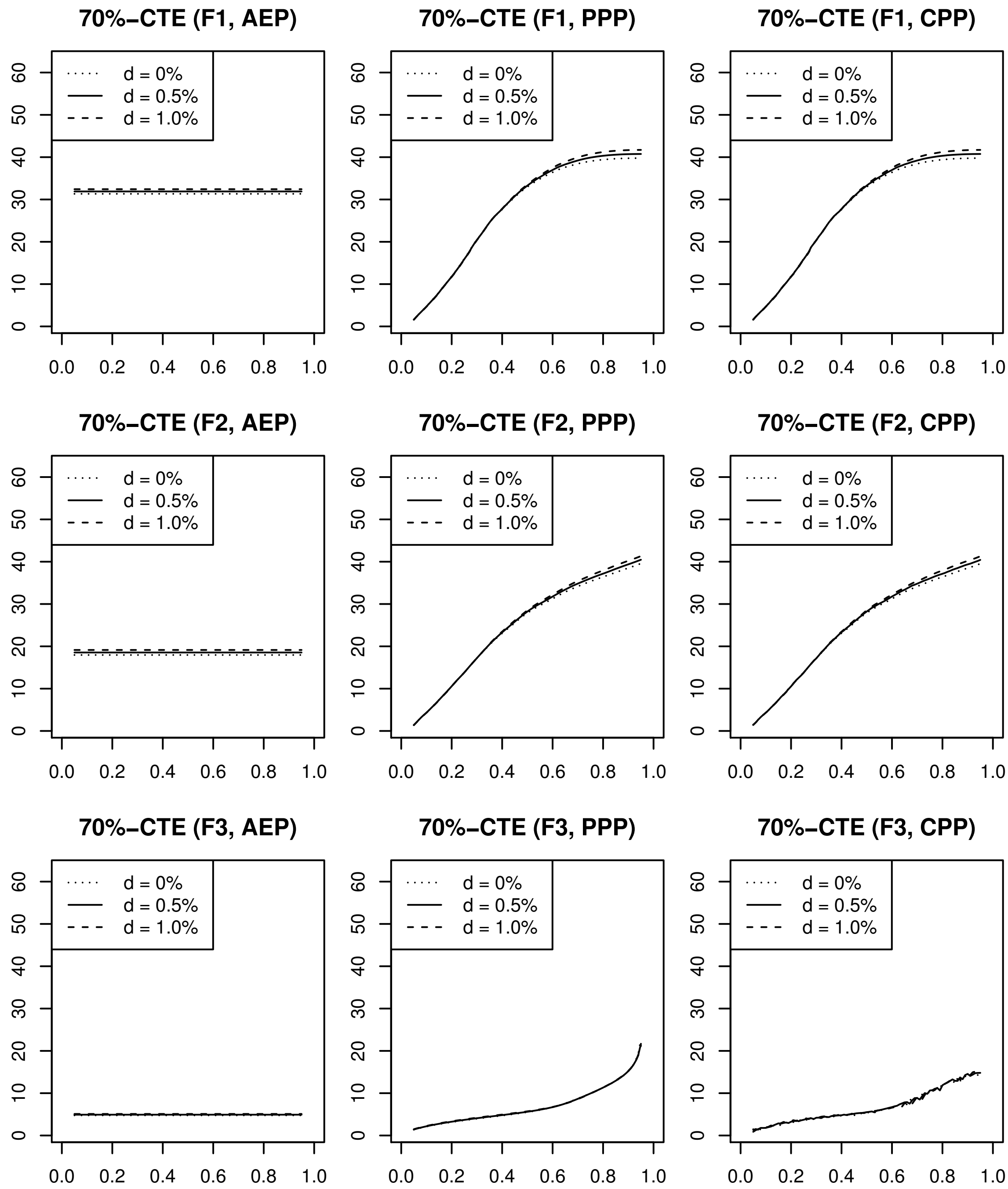

Figure 4.25: Estimated 70%-CTE amounts for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different values of the management fee

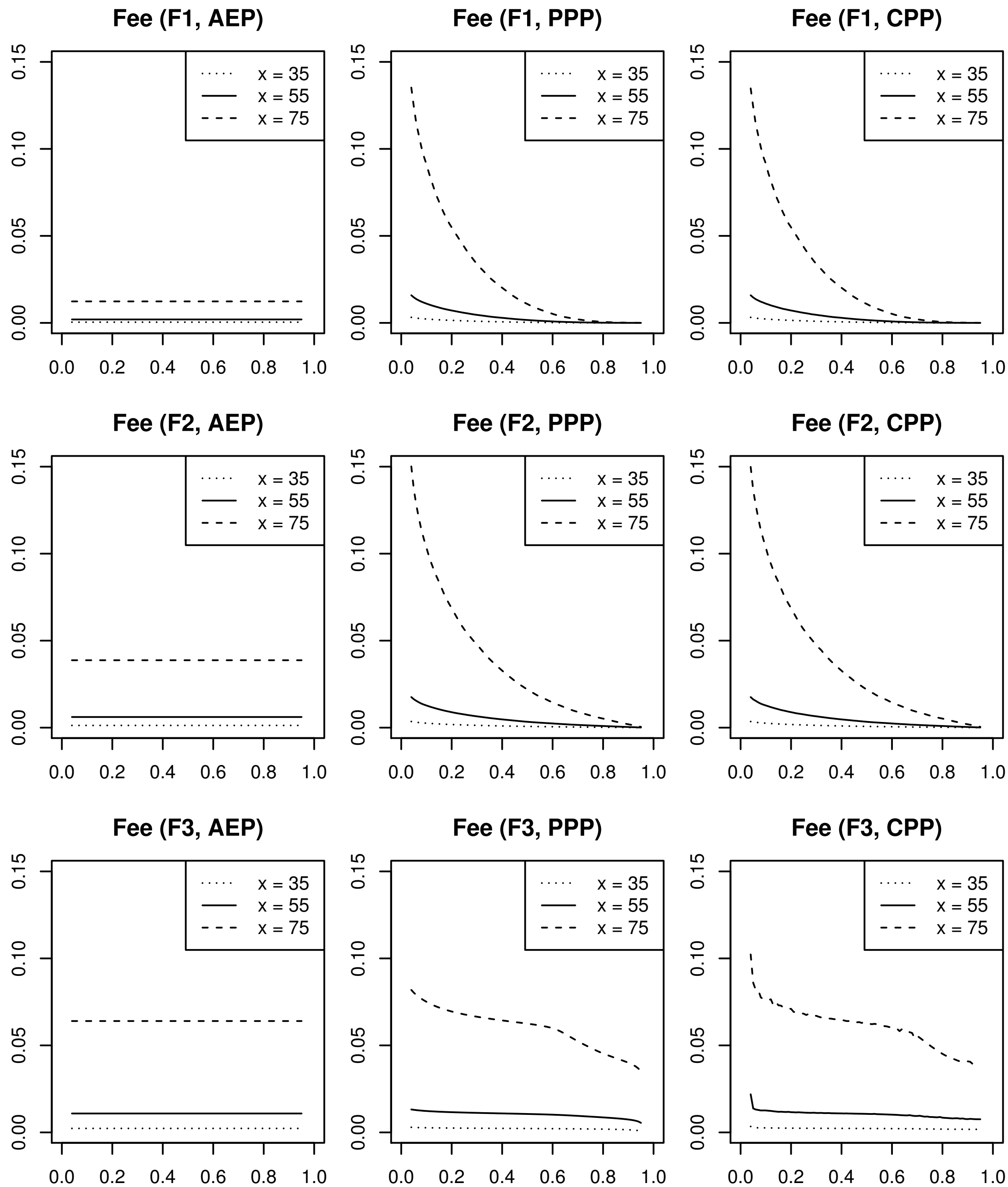

Figure 4.26: Estimated rates of fee charge for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different policy issue ages

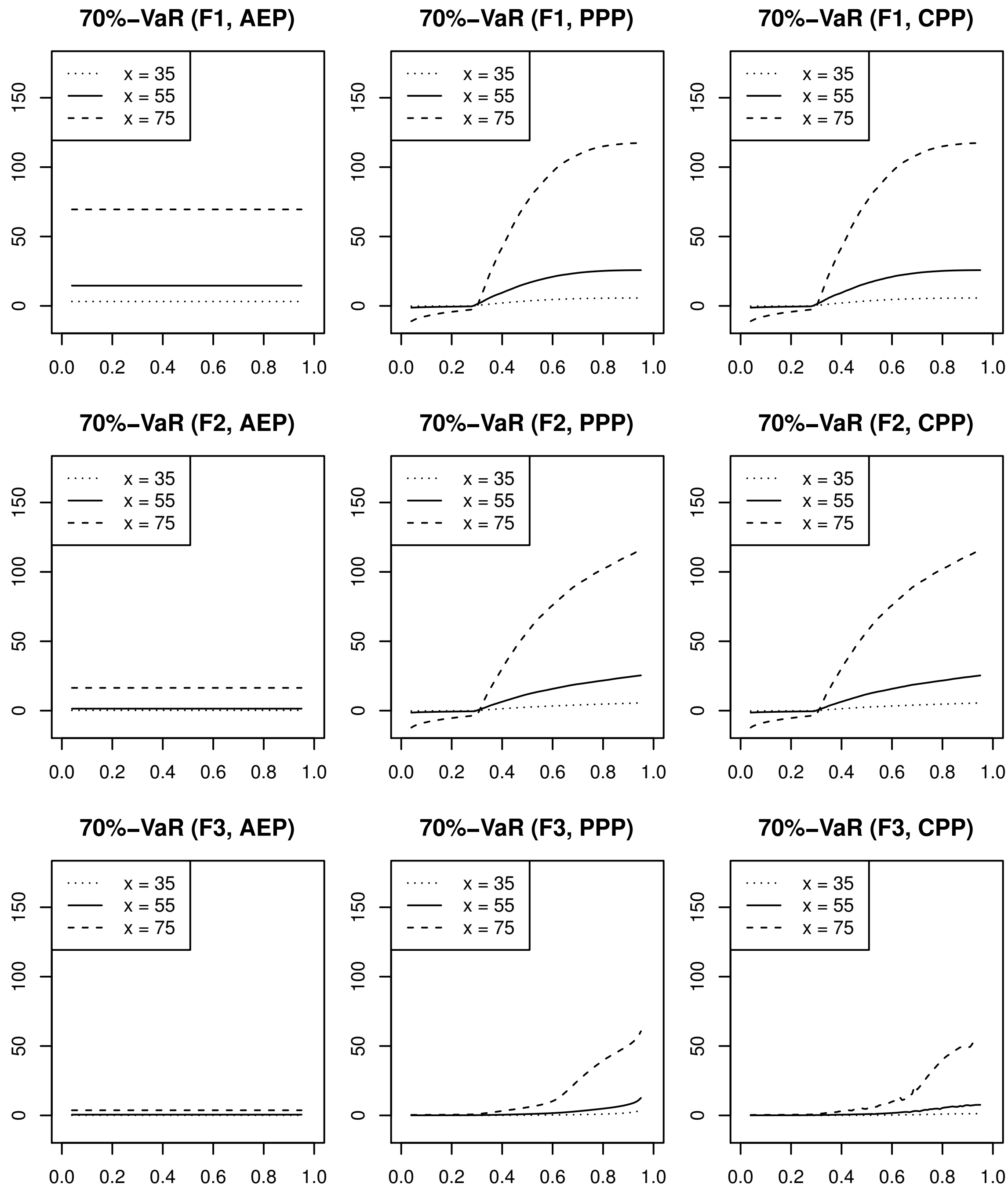

Figure 4.27: Estimated 70%-VaR's for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different policy issue ages

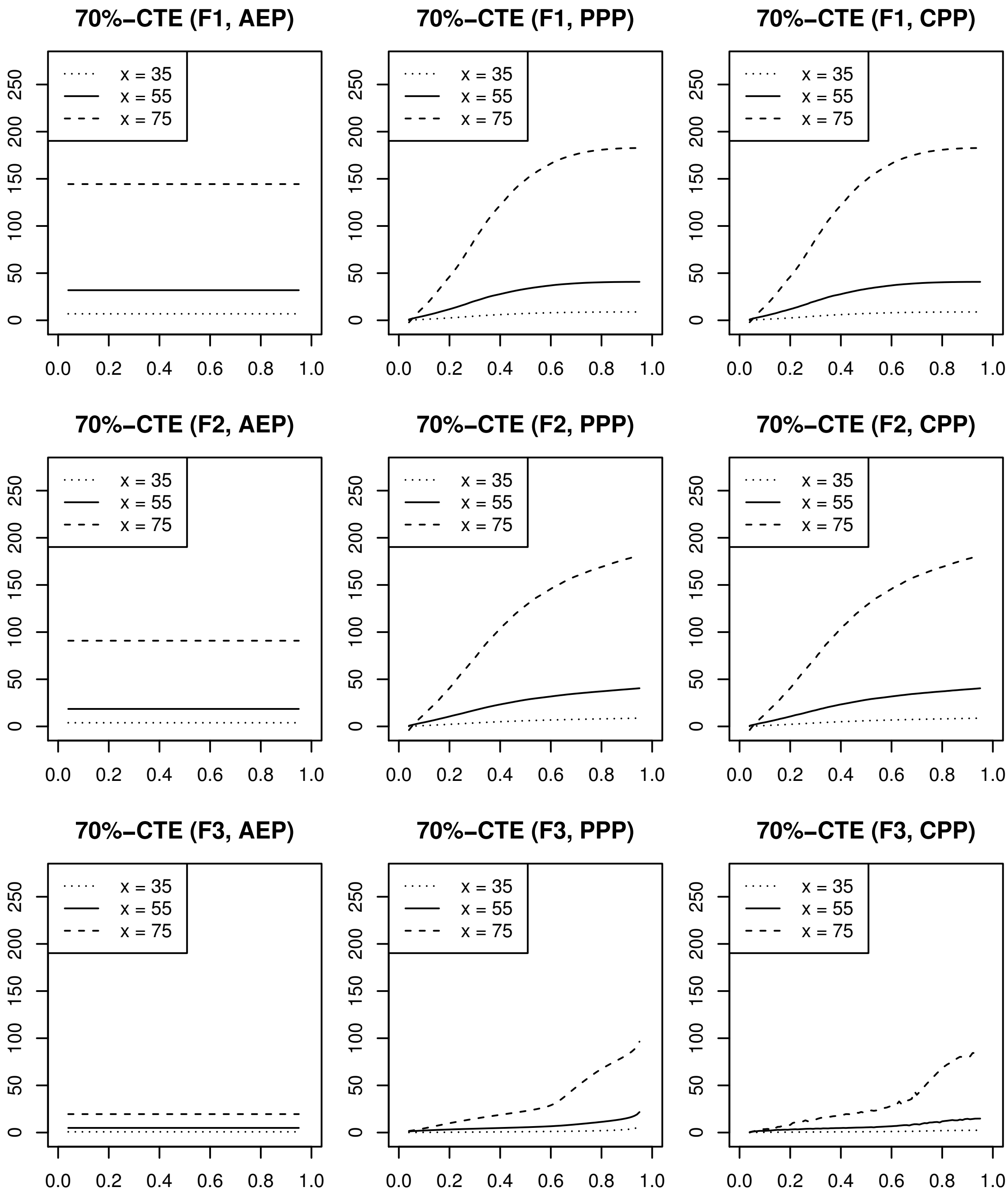

Figure 4.28: Estimated 70%-CTE amounts for GMDB, determined under the fee structures F1, F2, and F3 using the AEP, PPP, and CPP, versus the confidence level q for three different policy issue ages

Chapter 5

Conclusion

This project studies the guaranteed minimum benefits embedded in variable annuities, which are long-term investments that provide payments for retirement. Issued by life insurers, the guarantees protect policyholders from mortality risk and market risk for a predetermined period. We introduce three fee structures for funding the guarantees, including the one currently adopted in practice, the one suggested in Bernard et al. (2014), and a new method which is proposed in the project. Three pricing principles, the actuarial equivalence principle, the portfolio percentile principle, and the contract percentile principle, are adopted to determine the level of fee rate. In the pricing framework, we model the dynamics of the underlying asset price as the geometric Brownian motion and adopt the Lee and Carter (1992) model for mortality forecasting. To compare the performances under the three fee structures, we formulate the CTE amount defined in the AG-43, the U.S. statutory reserving method for variable annuities, as a measure of risk. Under each pricing principle, we plot the estimated rates of fee, occupation times, actuarial present values of premium payments, 70%-VaRs, and 70%-CTEs for each fee structure, and demonstrate the advantage of our proposed fee structure.

The model is illustrated with numerical results and a sensitivity analysis. We start with the estimation for the model parameters and generate the stochastic equity prices and the survival probabilities for every time point of projection. Then, we approximate the simulated fund value dynamics and the estimate of the CTE amounts in AG-43 reserve. For the base case and for three guarantees, GMDB, GMMB, and GMDMB, we compare the estimates for the rates of fee and the CTE amounts in AG-43 reserve under the three fee structures and the three pricing principles. From the numerical results, we observe that the estimated rates of fee, occupation times, APV of premium payments, VaRs, and CTE amounts under the fee structure F3 are more satisfactory than those under the other two fee structures when the PPP and CPP are adopted and the confidential level q is small.

For the future work, we will extend our model to include stochastic interest rates and stochastic volatilities, so that the term structure and the heavy tails of equity returns can be captured in the framework, as suggested in Kang and Ziveyi (2018).

Bibliography

Bacinello, A. R. (2003). Pricing guaranteed life insurance participating policies with annual premiums and surrender option. *North American Actuarial Journal*, 7(3):1–17.

Bacinello, A. R., Biffis, E., and Millossovich, P. (2009). Pricing life insurance contracts with early exercise features. *Journal of Computational and Applied Mathematics*, 233(1):27–35.

Bacinello, A. R., Biffis, E., and Millossovich, P. (2010). Regression-based algorithms for life insurance contracts with surrender guarantees. *Quantitative Finance*, 10(9):1077–1090.

Bacinello, A. R., Millossovich, P., Olivieri, A., and Pitacco, E. (2011). Variable annuities: A unifying valuation approach. *Insurance: Mathematics and Economics*, 49(3):285–297.

Ballotta, L. and Haberman, S. (2006). The fair valuation problem of guaranteed annuity options: The stochastic mortality environment case. *Insurance: Mathematics and Economics*, 38(1):195–214.

Bauer, D., Kling, A., and Russ, J. (2008). A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bulletin*, 38(2):621–651.

Bélanger, A. C., Forsyth, P. A., and Labahn, G. (2009). Valuing the guaranteed minimum death benefit clause with partial withdrawals. *Applied Mathematical Finance*, 16(6):451–496.

Bernard, C., Cui, Z., and Vanduffel, S. (2017). Impact of flexible periodic premiums on variable annuity guarantees. *North American Actuarial Journal*, 21(1):63–86.

Bernard, C., Hardy, M., and MacKay, A. (2014). State-dependent fees for variable annuity guarantees. *ASTIN Bulletin*, 44(3):559–585.

Bernard, C. and Lemieux, C. (2008). Fast simulation of equity-linked life insurance contracts with a surrender option. In *2008 Winter Simulation Conference*, 444–452. IEEE.

Bernard, C. and Moenig, T. (2018). Where less is more: Reducing variable annuity fees to benefit policyholder and insurer. *Journal of Risk and Insurance*. (Early view).

Cairns, A. J., Blake, D., and Dowd, K. (2006). A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration. *Journal of Risk and Insurance*, 73(4):687–718.

Chen, Z. and Forsyth, P. A. (2008). A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (GMWB). *Numerische Mathematik*, 109(4):535–569.

Christoffersen, P., Heston, S., and Jacobs, K. (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science*, 55(12):1914–1932.

Coleman, T. F., Li, Y., and Patron, M.-C. (2006). Hedging guarantees in variable annuities under both equity and interest rate risks. *Insurance: Mathematics and Economics*, 38(2):215–228.

Costabile, M., Massabó, I., and Russo, E. (2008). A binomial model for valuing equity-linked policies embedding surrender options. *Insurance: Mathematics and Economics*, 42(3):873–886.

Cox, J., Ingersoll, J., and Ross, S. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Dai, M., Kuen Kwok, Y., and Zong, J. (2008). Guaranteed minimum withdrawal benefit in variable annuities. *Mathematical Finance*, 18(4):595–611.

Delong, Ł. (2014). Pricing and hedging of variable annuities with state-dependent fees. *Insurance: Mathematics and Economics*, 58(1):24–33.

Du, D. F. and Martin, C. (2014). https://www.bostonfed.org/-/media/Documents/bankinfo/publications/variable-annuities.pdf. Accessed 14 June 2019.

Feng, R. and Huang, H. (2016). Statutory financial reporting for variable annuity guaranteed death benefits: Market practice, mathematical modeling and computation. *Insurance: Mathematics and Economics*, 67:54–64.

Grosen, A. and Jørgensen, P. L. (2002). Life insurance liabilities at market value: An analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of Risk and Insurance*, 69(1):63–91.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343.

Huang, Y. and Forsyth, P. (2012). Analysis of a penalty method for pricing a guaranteed minimum withdrawal benefit (GMWB). *IMA Journal of Numerical Analysis*, 32(1):320–351.

Huang, Y. T. and Kwok, Y. K. (2014). Analysis of optimal dynamic withdrawal policies in withdrawal guarantee products. *Journal of Economic Dynamics and Control*, 45:19–43.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *Review of Financial Studies*, 3(4):573–592.

Kalberer, T. and Ravindran, K. (2009). *Variable Annuities: A global perspective.* Risk Books, London.

Kang, B. and Ziveyi, J. (2018). Optimal surrender of guaranteed minimum maturity benefits under stochastic volatility and interest rates. *Insurance: Mathematics and Economics*, 79:43–56.

Kling, A., Ruez, F., and Russ, J. (2011). The impact of stochastic volatility on pricing, hedging, and hedge efficiency of withdrawal benefit guarantees in variable annuities. *ASTIN Bulletin*, 41(2):511–545.

Ledlie, M. C., Corry, D. P., Finkelstein, G. S., Ritchie, A. J., Su, K., and Wilson, D. (2008). Variable annuities. *British Actuarial Journal*, 14(2):327–389.

Lee, R. D. and Carter, L. R. (1992). Modeling and forecasting US mortality. *Journal of the American Statistical Association*, 87(419):659–671.

LIMRA (2019). US individual annuity sales. https://www.limra.com/en/newsroom/fact-tank. Accessed 28 June 2019.

Lin, X. S. and Liu, X. (2007). Markov aging process and phase-type law of mortality. *North American Actuarial Journal*, 11(4):92–109.

Liu, X., Mamon, R., and Gao, H. (2014). A generalized pricing framework addressing correlated mortality and interest risks: a change of probability measure approach. *Stochastics An International Journal of Probability and Stochastic Processes*, 86(4):594–608.

Milevsky, M. A. and Posner, S. E. (2001). The titanic option: Valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *Journal of Risk and Insurance*, 68(1):93–128.

Milevsky, M. A. and Salisbury, T. S. (2006). Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, 38(1):21–38.

Moenig, T. and Zhu, N. (2018). Lapse-and-reentry in variable annuities. *Journal of Risk and Insurance*, 85(4):911–938.

Paris, T. (2017). Behavioral analytics for annuities. https://www.soa.org/globalassets/assets/files/e-business/pd/events/2017/las/pd-2017-05-las-session-082.pdf. Accessed 3 July 2019.

Peng, J., Leung, K. S., and Kwok, Y. K. (2012). Pricing guaranteed minimum withdrawal benefits under stochastic interest rates. *Quantitative Finance*, 12(6):933–941.

Pinquet, J., Guillén, M., and Ayuso, M. (2011). Commitment and lapse behavior in long-term insurance: A case study. *Journal of Risk and Insurance*, 78(4):983–1002.

Piscopo, G. and Haberman, S. (2011). The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. *North American Actuarial Journal*, 15(1):59–76.

Platen, E. and Rendek, R. (2008). Empirical evidence on student-t log-returns of diversified world stock indices. *Journal of Statistical Theory and Practice*, 2(2):233–251.

Serfling, R. J. (2009). *Approximation theorems of mathematical statistics*. Volume 162. John Wiley & Sons, New York.

Shen, W. and Xu, H. (2005). The valuation of unit-linked policies with or without surrender options. *Insurance: Mathematics and Economics*, 36(1):79–92.

Shen, Y., Sherris, M., and Ziveyi, J. (2016). Valuation of guaranteed minimum maturity benefits in variable annuities with surrender options. *Insurance: Mathematics and Economics*, 69:127–137.

Shreve, S. E. (2004). *Stochastic calculus for finance II: Continuous-time models*. Volume 11. Springer Science & Business Media, New York.

Siu, T. K. (2005). Fair valuation of participating policies with surrender options and regime switching. *Insurance: Mathematics and Economics*, 37(3):533–552.

Van Haastrecht, A., Plat, R., and Pelsser, A. (2010). Valuation of guaranteed annuity options using a stochastic volatility model for equity prices. *Insurance: Mathematics and Economics*, 47(3):266–277.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.

Wang, S. S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of risk and insurance*, 67(1):15–36.

Zhao, Y. and Mamon, R. (2018). An efficient algorithm for the valuation of a guaranteed annuity option with correlated financial and mortality risks. *Insurance: Mathematics and Economics*, 78:1–12.

Zhou, J. and Wu, L. (2015). The time of deducting fees for variable annuities under the state-dependent fee structure. *Insurance: Mathematics and Economics*, 61:125–134.