Well-Posedness of a Gas-Disk Interaction System

by

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Abstract

This thesis concerns a gas-disk interaction system: the disk is immersed in a gas and acted on by a drag force and an external force. The evolution of the system is described by a coupled system of integro-differential equations. More specifically, we use a pure kinetic transport equation to model the gas and a Newton’s Second Law ODE to model the disk. The two are coupled via the drag force exerted on the disk by the gas and the boundary condition for the gas colliding with the disk.

Systems of this type have been extensively studied in the literature, both analytically and numerically. To the best of our knowledge, existing works focus on existence of near-equilibrium solutions and their long-time behaviour. However, uniqueness of solutions has not been investigated previously. In the first part of the thesis we will give the first rigorous proof of existence and uniqueness of solutions for general initial data and external forcing.

The most important physical feature of this system is its inherent recursivity: particles can collide with the disk time and time again. Recognizing this structure and introducing recursivity into the equations by the means of gas decomposition is the key to obtaining the well-posedness result.

In the second part of the thesis we will present a simple numerical method for computing the trajectory of the disk using the aforementioned gas decomposition. We will contrast it with methods used previously, and also use it to show that considering only one or two precollisions for the gas particles is sufficient to accurately compute the density distribution of the gas and the velocity of the disk.

Keywords: well-posedness, integro-differential equations, kinetic equations, gas-body interaction, drag force, system with memory.
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Chapter 1

Introduction

The focus of this thesis is the motion of a disk immersed in a gas and acted on by a drag force. This setup can be seen as a simple model for many different systems involving motion of a body through a medium; everyday examples of such systems are planes and cars.

To model the motion of the body we need a way to compute the drag force. There are empirical models that allow one to do so by assuming a power-law relationship between the velocity of the body and the drag force it is experiencing. However, such models neglect to incorporate the effect of the body on the gas: for instance, we would expect the gas in front of the body to get denser and thus exert more drag.

Capturing such phenomena requires modelling the dynamics of the medium as well as the motion of the body. To make the model more amenable to analysis, we will assume that gas does not interact with itself: the particles are simply moving with a constant velocity unless their path is obstructed by the disk. Such behaviour can be conveniently modelled with a simple kinetic equation. We will also use a Newton’s Second Law ODE to model the motion of the disk.

The existence of near-equilibrium solutions for this type of system with various boundary conditions has been extensively studied in [4, 2, 5, 6, 7, 8, 9, 10, 11]. However, the existence of solutions far from equilibrium and uniqueness of any solutions at all have been an open problem.
1.1 Problem Description

We consider a disk of unit mass, negligible thickness and radius $R$. It is moving through a gas under the action of an external force $F$ and the drag force $G$. We assume that the external force acts only in the horizontal direction, so the disk is moving only along the horizontal axis.

The gas surrounding the disk is represented by the phase space density function $f$, whose evolution away from the disk is described by the kinetic equation (1.1c). The full system is

$$\dot{p}(t) = F(\eta(t), t) + G[f], \quad p(0) = p_0, \quad (1.1a)$$
$$\dot{\eta}(t) = p(t), \quad \eta(0) = 0, \quad (1.1b)$$
$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = 0, \quad f(x, v, 0) = \phi_0(x, v). \quad (1.1c)$$

The boundary conditions for the gas and the form of the drag term $G$ will be derived in sections 1.3 and 1.4 respectively. Interactions between particles are ignored for the sake of simplicity, which makes the right hand side of (1.1c) zero.

We are going to use the following notations:

- $f(x, v, t)$ — distribution density of the gas,
- $f^\pm$ — distribution of incoming/outgoing gas,
- $\eta$ — position of the disk,
- $p$ — velocity of the disk,
- $\Omega$ — space occupied by gas,
- $\partial \Omega_{l,r}$ — left and right surfaces of the disk,
- $n_{l,r}$ — normal vectors to $\partial \Omega_{l,r}$,
- $\mathbb{V}$ — velocity space,
- $\mathbb{V}_\pm := \{ v \in \mathbb{V} \mid v_1 \gtrless p \}$ — velocities that are faster/slower than the disk.

We will distinguish the gas to the right of the disk from the gas on the left with sub- and super- scripts $r$ and $l$. Throughout the thesis superscripts $+$ and $-$ on the density functions denote the postcollisional and precollisional distributions respectively, understood as one-sided limits:

$$f^\pm(\eta(t), x_2, v, t) = \lim_{\epsilon \to 0^+} f(\eta(t) \pm \epsilon v_1, x_2, v, t \pm \epsilon). \quad (1.2)$$
We will use two different symmetry assumptions, which will allow us to work with fewer dimensions. The first one is that the disk is infinite, that is, a plane. In this case the spatial variables $x_2$ and $x_3$ can be ignored and the disk itself treated as a point. The velocity variables $v_2$ and $v_3$ will also be ignored, thus reducing the set of independent variables to $(x_1, v_1, t) \in (\mathbb{R}, \mathbb{R}, \mathbb{R}^+)$. We will refer to this as the “one-dimensional case”.

An alternative assumption is that the disk is “half-infinite”, that is, an infinite strip:

$$\text{disk}(t) = \{\eta(t)\} \times [-R, R] \times \mathbb{R}. \quad (1.3)$$

In this case we have $(x, v, t) = (x_1, x_2, v_1, v_2, t) \in (\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^+)$. This “two-dimensional” case does have the essential feature of higher dimensions—the importance of the transverse velocity—but lets us avoid some notational and technical difficulties arising in three dimensions.

Moreover, since the gas to the right of the disk behaves in the same way at the gas on the left, we will only work with the gas to the right. Note that if a particle had collided with the disk from the right, it could not have and never will collide from the left, and vice versa.

### 1.2 Assumptions

For an arbitrary density function $g(v_1, v_2)$, let $\hat{g}(v_1)$ be the partially integrated version:

$$\hat{g}(v_1) = \int_{\mathbb{R}} g(v_1, v_2) \, dv_2. \quad (1.4)$$

In other words, the “hat” denotes integration over all transverse velocities.

Throughout the thesis we let $T > 0$ be an arbitrary fixed time. The assumptions on the system are

(A0) Particles cannot penetrate the disk.

(A1) The initial distribution $\phi_0$ is a function of velocity only: $\phi_0(x, v) = \phi_0(v)$. 

(A2) The initial distribution $\phi_0$ is differentiable; furthermore, the function

$$h_{\phi_0} := \left(1 + |v|^2\right) \left(|\phi_0(v)| + |\nabla \phi_0(v)|\right)$$

is finite in the following norms:

$$\|h_{\phi_0}\|_{\infty} := \text{ess sup}_{v \in \mathbb{R}^2} |h_{\phi_0}(v)| < \infty,$$

$$\|h_{\phi_0}\|_{\infty,1} := \text{ess sup}_{v_1 \in \mathbb{R}} \int_{\mathbb{R}} |h_{\phi_0}(v_1, v_2)| \, dv_2 < \infty,$$

$$\|h_{\phi_0}\|_1 := \int_{\mathbb{R}^2} |h_{\phi_0}(v_1, v_2)| \, dv_1 \, dv_2 < \infty.$$

For any density function $g(v_1, v_2)$ we will write

$$\|g\| := \|h_g\|_{\infty} + \|h_g\|_{\infty,1} + \|h_g\|_1.$$

(A3) The external force $F(x_1, t)$ is Lipschitz in both variables, and there exist Lipschitz functions $F_-(t)$ and $F_+(t)$ such that for all $x_1$

$$F_-(t) < F(x_1, t) < F_+(t).$$

Without loss of generality, we assume $F_-(t) < 0 < F_+(t)$.

1.3 Boundary Condition for the Gas

We assume a diffusive boundary condition for the gas: the density distribution of the reflected gas has the form

$$f^+(\eta(t), x_2, v_1, v_2, t) = j(x_2, t) \phi(v_1 - p(t), v_2). \tag{1.5}$$

where $\phi$ is the known shape function and $j(x_2, t)$ is the scaling factor. The shape function $\phi$ is assumed to be positive and to satisfy assumption (A2). We make one more assumption regarding $\phi$ without losing generality:

(A4) We assume that $\phi$ is normalized:

$$\|\phi\| = 1.$$

The typical choice for both the initial distribution and the shape function is a Gaussian distribution.
The scaling factor $j(x_2, t)$ is determined from the conservation of mass via the following calculation:

$$\int_{V_+} (v_1 - p(t)) f_r^+ (\eta(t), x_2, v, t) \, dv = \int_{V_-} (p(t) - v_1) f_r^- (\eta(t), x_2, v, t) \, dv$$

(1.6)

$$j_r(x_2, t) \int_{V_+} (v_1 - p(t)) \phi(v_1 - p(t), v_2) \, dv = \int_{V_-} (p(t) - v_1) f_r^- (\eta(t), x_2, v, t) \, dv.$$

(1.7)

By definition of $V_+$, $\gamma_d$ does not actually depend on $p(t)$, and thus is a constant determined by $\phi$. Therefore the expression for the density flux is

$$j(x_2, t) = \frac{1}{\gamma_d} \int_{V_-} (p(t) - v_1) f^- (\eta(t), x_2, v, t) \, dv.$$

(1.8)

### 1.4 Derivation of the Drag Force

To derive the drag force we use Newton’s Second Law and Reynolds’ Transport Theorem; a similar derivation was presented in [2]. For this calculation we will omit $\pm$ on $f$. Since the disk moves only horizontally we have

$$F(\eta(t), t) = \frac{d}{dt} \left( \text{Horizontal} \right) \text{Momentum} = \dot{p}(t) + \frac{d}{dt} \int_{\Omega(t)} \int_V v_1 f(x, v, t) \, dv \, dx$$

(1.9a)

$$= \dot{p}(t) + \int_{\Omega(t)} \int_V v_1 \partial_t f \, dv \, dx + \int_{\partial \Omega(t)} \int_V \left( \mathbf{v}^b \cdot \mathbf{n} \right) v_1 f \, dv \, dx,$$

(1.9b)

where $\mathbf{v}^b$ is the velocity of the boundary. We will omit the temporal dependence of $\Omega$ for the rest of the calculation.

The boundary term simplifies as follows:

$$I_b = \int_{\partial \Omega_l} \int_V \left( \mathbf{v}^b \cdot \mathbf{n}_l \right) v_1 f \, dv \, dx + \int_{\partial \Omega_r} \int_V \left( \mathbf{v}^b \cdot \mathbf{n}_r \right) v_1 f \, dv \, dx$$

$$= \int_{\partial \Omega_l} \int_V p(t) v_1 f \, dv \, dx - \int_{\partial \Omega_r} \int_V p(t) v_1 f \, dv \, dx.$$
We use (1.1c) to simplify the interior term:

\[
I_i = \int_{\Omega} \int_{V} v_1 \partial_t f \, dv \, dx = - \int_{\Omega} \int_{V} v_1 \cdot \nabla_x f \, dv \, dx = - \int_{\Omega} \int_{V} \nabla_x \cdot (v_1 v f) \, dx \, dv.
\]

Applying the divergence theorem to the right-hand side of the expression above yields

\[
I_i = - \int_{\Omega} \int_{\partial \Omega} n_{i} \cdot (v_1 v f) \, dv \, dx - \int_{V} \int_{\partial \Omega} n_{i} \cdot (v_1 v f) \, dv \, dx
\]

\[
= - \int_{\Omega} \int_{\partial \Omega} v_1^2 f \, dv \, dx + \int_{V} \int_{\partial \Omega} v_1^2 f \, dv \, dx.
\]

Hence the change of the horizontal momentum of the gas is

\[
I_i + I_b = \int_{\partial \Omega} \int_{V} v_1 (p(t) - v_1) f \, dv \, dx - \int_{\partial \Omega} \int_{V} v_1 (p(t) - v_1) f \, dv \, dx
\]

\[
= \int_{\partial \Omega} \int_{V} (v_1 - p(t))(p(t) - v_1) f \, dv \, dx - \int_{\partial \Omega} \int_{V} (v_1 - p(t))(p(t) - v_1) f \, dv \, dx
\]

\[
= - \int_{\partial \Omega} \int_{V} (p(t) - v_1)^2 f \, dv \, dx + \int_{\partial \Omega} \int_{V} (p(t) - v_1)^2 f \, dv \, dx
\]

(1.10)

where the second equality is valid because of conservation of mass (1.6):

\[
\int_{\partial \Omega} \int_{V} (p(t) - v_1) f \, dv \, dx = \int_{\partial \Omega} \int_{V} (p(t) - v_1) f \, dv \, dx - \int_{\partial \Omega} \int_{V} (v_1 - p(t)) f \, dv \, dx = 0.
\]

Using (1.10) in (1.9) lets us write down the ODE for the velocity of the disk:

\[
\dot{v}(t) = F(\eta(t), t) + \int_{\partial \Omega} \int_{V} (p(t) - v_1)^2 f \, dv \, dx - \int_{\partial \Omega} \int_{V} (p(t) - v_1)^2 f \, dv \, dx
\]

\[
= : F(\eta(t), t) + G^l(t; p) - G^r(t; p).
\]

Since the postcollisional distribution is given in terms of the precollisional distribution, the expressions for \(G^l\) and \(G^r\) can be simplified further. Using (1.5) we write

\[
\int_{\partial \Omega} \int_{V} (p(t) - v_1)^2 f \, dv \, dx = \int_{\partial \Omega} j_r(x_2, t) \int_{V} (p(t) - v_1)^2 \phi(v - p(t)) \, dv \, dx,
\]

\[
=: \beta_d
\]

where the quantity \(\beta_d\) can be evaluated explicitly.
Combining the calculation above with (1.8) we get

$$G^r(t; p) = \int_{\partial \Omega} \int_{\gamma^-} (p(t) - v_1)^2 + \frac{\beta_d}{\gamma_d} (p(t) - v_1) f \, d\mathbf{v} \, d\mathbf{x},$$

(1.11a)

$$G^l(t; p) = \int_{\partial \Omega} \int_{\gamma^+} (v_1 - p(t))^2 + \frac{\beta_d}{\gamma_d} (v_1 - p(t)) f \, d\mathbf{v} \, d\mathbf{x}.$$  

(1.11b)

### 1.5 Main Result

**Theorem 1.1.** Suppose the initial density $\phi_0$ and the shape function $\phi$ satisfy assumptions (A1) and (A2), and the external force $F$ satisfies assumption (A3). Let the initial velocity of the disk $p_0 \in \mathbb{R}$ and final time $T > 0$ be arbitrary. Then for the system (1.1) with the boundary condition given by (1.5, p.4) and the drag force given by (1.11, p.7) we have

(1.1.a) **Existence:** there exists at least one solution $(p, f)$ to the problem with

$$p \in W^{1,\infty}(0, T) \quad \text{and} \quad f \in L^\infty(\mathbb{R}_t^d; L^1(\mathbb{R}_v^d)).$$

(1.1.b) **Uniqueness:** the solution to the problem described above is unique.

(1.1.c) **Regularity:** any solution to the problem described above satisfies $p \in W^{2,\infty}(0, T)$.

The main difficulty in proving the theorem is possible recollisions of the gas particles with the disk, and the drag force they generate; we will refer to it as $G_{\text{rec}}$. Indeed, the physical system itself is inherently recursive: the distribution of the particles colliding with the disk for the $n$th time at time $t$ is determined by that of the particles colliding with the disk for the $(n - 1)$th time at some earlier time $s$. This in turn means that $G_{\text{rec}}$ depends on the entire history of the disk motion, rather than just its instantaneous velocity. To put the discussion of this recursion on a solid footing, we will decompose the gas into particles that have never collided with the disk, collided exactly once, exactly twice and so on.

The rest of the thesis is organized as follows: in chapter 2 we will define the aforementioned gas decomposition and introduce a change of variables: it turns out that identifying the particles by the time when they had last collided with the disk makes the density distributions much easier to work with. In chapter 3 we will prove existence of solutions by extracting a priori bound from a sub- and a super-solution. Chapter 4 is dedicated to showing uniqueness of solutions, and contains the most technical calculations of the thesis; improved regularity of solutions will follow from one of them. Finally, chapter 5 contains a description of a numerical method for solving the problem and a few computed examples.

The thesis extends the result obtained in [3], namely uniqueness in the one-dimensional case.
Chapter 2

Transformation of the Problem

In this chapter we are only working with the gas to the right of the disk.

2.1 Partition of the Gas

For the two-dimensional case we will denote the disk by \( D(t) \):

\[
D(t) = \{ \eta(t) \} \times [-R, R].
\]

In the one-dimensional case we simply have \( D(t) = \eta(t) \).

2.1.1 Partition of the Phase Space

Before proceeding to decompose the gas based on the number of precollisions the particles had, we need to formalize the notion of collision. For a given phase space point \((x, v; t)\) we let \( \omega_{(x,v,t)}(s) \) be the parametrization of the corresponding characteristic:

\[
\omega_{(x,v,t)}(s) = x - (t - s)v.
\]

We distinguish the particles that have collided with the disk in the past from the ones that have not by a condition on the corresponding phase space point:

**Definition 2.1 (Postcollisional Points).** A phase space point \((x, v; t)\) with \( x_1 > \eta(t) \) is said to be postcollisional if there exist times \( \bar{\tau} \) and \( \bar{s} \) such that \( 0 \leq \bar{\tau} < \bar{s} < t \) and

\[
\omega_{(x,v,t)}(\bar{s}) \in D(\bar{s}) \quad \text{and} \quad x_1 - (t - \bar{\tau})v_1 < \eta(\bar{\tau}).
\]
We extend the set of postcollisional points to include points \((x, v; t)\) that satisfy

\[ x \in D(t) \quad \text{and} \quad v_1 > p(t). \]

Note that \(\bar{s}\) is only needed to ensure that the particles did not pass over or under the disk; in the one-dimensional setting we only need to find \(\bar{\tau}\).

Let \(\chi_0(x, v, t)\) be the indicator function of the non-postcollisional part of the phase space:

\[
\chi_0(x, v, t) = \begin{cases} 
1 & \text{if } (x, v, t) \text{ is not postcollisional}, \\
0 & \text{if } (x, v, t) \text{ is postcollisional}.
\end{cases}
\]

### 2.1.2 Partition of the Density Function

We now introduce the idea of recursive scattering: let \(f_n(x, v, t)\) be the density functions of the particles that have collided with the disk exactly \(n\) times in the past. Away from the disk they satisfy the free transport equation (1.1c). For \(x \in D(t)\) we define \(f_n^\pm(x, v, t)\) in terms of the one-sided limits similar to those in (1.2):

\[
f_n^\pm(\eta(t), x_2, v, t) = \lim_{\epsilon \to 0^+} f_n(\eta(t) \pm \epsilon v_1, x_2, v, t) d\eta.
\]

The boundary conditions on the \(f_n\)s are similar to those for the full density function \(f\), with the exception that the collision with the disk now increments the sub-index. In particular, for \(x \in D(t)\), \(w_1 > p(t)\) and \(n \geq 0\) we define

\[
f_{n+1}^+(\eta(t), x_2, w, t) = j_n(x_2, t)\phi(w_1 - p(t), w_2),
\]

where

\[
j_n(x_2, t) = \frac{1}{\gamma d} \int_{\mathcal{V}_-} (p(t) - v_1) f_n(\eta(t), x_2, v, t) d\eta.
\]

For \(n \geq 1\) and \(x \notin D(t)\) we define \(f_n(x, v, t)\) as follows:

\[
f_n(x, v, t) = \begin{cases} 
\text{Solution to (1.1c, p.2) with BC given by (2.2)} & \text{for } (x, v, t) \notin \text{supp } \chi_0; \\
0 & \text{for } (x, v, t) \in \text{supp } \chi_0.
\end{cases}
\]

The same expression is also used if \(x \in D(t)\) and \(v_1 < p(t)\). For \(n = 0\) we simply have

\[f_0(x, v, t) = \chi_0(x, v, t)\phi_0(v).\]
We define \( f_{\text{rec}} \) to be the density function of all particles that have collided with the disk in the past:

\[
f_{\text{rec}}(x, v, t) = \sum_{n=1}^{\infty} f_n(x, v, t). \tag{2.4}
\]

The full density function is therefore explicitly constructed from the disk trajectory \( p \) and initial gas density \( \phi_0 \):

\[
f(x, v, t) = f_0(x, v, t) + f_{\text{rec}}(x, v, t). \tag{2.5}
\]

We will introduce explicit formulas for \( f_n \) and \( \chi_0 \) in subsection 2.2.5 (Change of Variables, p. 20) for \( x \in D(t) \), and section 3.4 (Construction of the Density Function, p.32) for \( x \notin D(t) \).

**Remark 2.1.** Since the particles are reflected by the disk randomly, two particles with different precollisional velocities can have the same velocity after the collision. Consequently, it is not possible to distinguish particles that have precollided different numbers of times: prior to the latest collision the particles could have had any velocity. For this reason we must allow different \( f_n \)’s, with \( n \geq 1 \), to have overlapping supports in the phase space.

Similarly to (2.2b), we define \( G_n \) to be the drag forces due to particles colliding with the disk for the \((n+1)\)th time and \( G_0 \) to be the drag forces due to first collisions:

\[
G_n(t; p) = \int_{\partial \Omega_r} \int_{\mathbb{R}^2} \left( (p(t) - v_1)^2 + \frac{\beta d}{2\gamma d}(p(t) - v_1) \right) f_n(\eta(t), x_2, v, t) \, dv \, dx_2, \tag{2.6a}
\]

\[
G_0(t; p) = \int_{\partial \Omega_r} \int_{\mathbb{R}^2} \left( (p(t) - v_1)^2 + \frac{\beta d}{2\gamma d}(p(t) - v_1) \right) \chi_0(\eta(t), x_2, v, t) \phi_0(v) \, dv \, dx_2. \tag{2.6b}
\]

We denote the total drag due to recollisions by \( G_{\text{rec}} \):

\[
G_{\text{rec}}(t; p) = \sum_{n=1}^{\infty} G_n(t; p); \quad \text{then} \quad G(t; p) = G_0(t; p) + G_{\text{rec}}(t; p). \tag{2.7}
\]

The expressions for the left side of the disk are similar.

### 2.2 Change of Variables

This section is dedicated to one of the key ideas of this thesis: changing the variable of integration in (2.2b) and (2.6a) from the velocity variable \( v \) to the time of precollision \( s \). To simplify the discussion, we consider the one-dimensional case and only talk about the gas colliding with the disk from the right. We also make a temporary assumption that there
exists a constant $M > 0$ such that

\[ |p(t)| + |\dot{p}(t)| \leq M \text{ for all } t \in [0,T]. \tag{2.8} \]

As we will see in section 3.2 (A Priori Estimates, p.24), such a bound need not be taken as an assumption; we are only stating it here to simplify the notation in what follows. As a last bit of preparation, we introduce a notation for the average velocity of the disk over the interval $[s,t]$:

\[ \langle p \rangle_{s,t} := \frac{1}{t-s} \int_s^t p(\tau) \, d\tau = \frac{\eta(t) - \eta(s)}{t-s}. \tag{2.9} \]

We can start building towards the change of variables. Suppose a particle colliding with the disk at time $t$ has previously collided with it at time $s$. Then this particle and the disk must have travelled the same distance between times $s$ and $t$, and hence have the same average velocity. Since the velocity of the particles does not change between collisions, this condition can be expressed as

\[ v = \langle p \rangle_{s,t}. \tag{2.10} \]

The change of variables will be based on this condition: we define the postcollisional velocity function $v(s,t)$ by

\[ v(s,t) := \frac{1}{t-s} \int_s^t p(\tau) \, d\tau. \tag{2.11} \]

A few of its basic properties are summarized in a lemma:

**Lemma 2.1 (Properties of $v(s,t)$).** Suppose $0 \leq s < t \leq T$ and $M$ is as defined by (2.8). Then

(2.1.a) $v(s,t)$ satisfies the bound $|v(s,t)| \leq M$;

(2.1.b) the derivatives of $v(s,t)$ are

\[ \frac{\partial v}{\partial s} = \frac{v(s,t) - p(s)}{t-s}, \quad \frac{\partial v}{\partial t} = \frac{p(t) - v(s,t)}{t-s}; \tag{2.12} \]

(2.1.c) if $\dot{p} \geq 0$ then $\partial_s v \geq 0$;

(2.1.d) the derivatives of $v$ satisfy the following estimates:

\[ \left| \frac{\partial v}{\partial s} \right| \leq \frac{1}{2} M, \quad \left| \frac{\partial v}{\partial t} \right| \leq \frac{1}{2} M. \]
Proof. Part (2.1.a) follows from assumption 2.8:

\[ |v(s, t)| = \left| \frac{1}{t-s} \int_s^t p(\tau) \, d\tau \right| \leq \frac{1}{t-s} \int_s^t M \, d\tau = M. \]

Part (2.1.b) follows from direct computations. For part (2.1.c) we write

\[ \frac{\partial v}{\partial s} = \frac{\partial}{\partial s} \left( \frac{1}{t-s} \int_s^t p(\tau) \, d\tau \right) = \frac{\partial}{\partial s} \int_0^1 p(s + \hat{\tau}(t-s)) \, d\hat{\tau} = \int_0^1 (1-\hat{\tau}) \hat{p}(\ldots) \, d\hat{\tau}, \]

and the sign equivalence follows. Part (2.1.d) is proven by taking the absolute value on both sides of the equality above:

\[ \left| \frac{\partial v}{\partial t} \right| = \left| \int_0^1 (1-\hat{\tau}) \hat{p}(\ldots) \, d\hat{\tau} \right| \leq \int_0^1 (1-\hat{\tau}) \| \hat{p} \|_\infty \, d\hat{\tau} \leq \frac{1}{2} M. \]

The estimate for \( \partial_t v \) is proved via a similar calculation. \( \square \)

### 2.2.1 The Monotone Case

To motivate and illustrate the change of variables idea we consider a simplified case: the disk is monotonically accelerating. With \( t \) fixed, we use the change of variables \( v \rightarrow s \) given by \( v = v(s, t) \). Since \( \hat{p} > 0 \), the function \( v(\cdot, t) \) is strictly increasing by Lemma 2.1.c (Properties of \( v(s, t), p.11 \)), and thus is a bijection between the precollisional velocities and the corresponding precollision times. This allows us to change the variable of integration in (2.6a) to obtain

\[ G_n(t; p) = \int_{-\infty}^{p(t)} \left[ (p(t) - v)^2 + \frac{\beta_1}{\gamma_1} (p(t) - v) \right] f_n^- (\eta(t), v, t) \, dv \]

\[ = \int_0^t \frac{\partial v}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_1}{\gamma_1} (p(t) - v(s, t)) \right] f_n^- (\eta(t), v(s, t), t) \, ds \]

\[ = \int_0^t \frac{\partial v}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_1}{\gamma_1} (p(t) - v(s, t)) \right] f_n^+ (\eta(s), v(s, t), s) \, ds, \]

where we can equate (2.14) to (2.15) because the distribution density does not change between collisions:

\[ f_n^- (\eta(t), v(s, t), t) = f_n^+ (\eta(s), v(s, t), s). \]
Similarly, (2.2b) becomes

$$j_n(s) = \frac{1}{\gamma_1} \int_0^s (p(s) - v(\tau, s)) f_n^+ (\eta(\tau), v(\tau, s), \tau) \, d\tau.$$  

One immediate advantage of the expression above over (2.2b) is that it leads to an explicit recursive relationship for \( \{f_n\} \). We can also write it as a recursion for \( \{j_n\} \) using (2.2a):

$$j_n(s) = \frac{1}{\gamma_1} \int_0^s (p(s) - v(\tau, s)) j_{n-1}(\tau) \phi(v(\tau, s) - p(\tau)) \, d\tau.$$  

The most difficult part of Theorem 1.1 is uniqueness: it requires a Lipschitz estimate for \( G_{\text{rec}} \), which in turn relies on Lipschitz estimates for \( \{j_n\} \). If both candidate solutions \( p \) and \( q \) were monotone, we could use the expression above to obtain such estimate inductively: all factors of the integrand satisfy Lipschitz and \( L^\infty \) estimates, so the standard approach of adding and subtracting “cross terms” would do the trick.

### 2.2.2 Postcollisional Velocities

Without the monotonicity assumption the change of variable is more difficult to set up.

Among all particles that are going to collide with the disk at time \( t \), we first need to identify those that have had a collision with the disk in past. Velocities of such particles will henceforth be called \textit{postcollisional}. Since every such velocity must satisfy condition (2.10, p.11), the range of the function \( v(\cdot, t) \) has an important role to play. We introduce the notation

$$\kappa(t) = \min_{s \in [0, t]} v(s, t);$$  

(2.16)

then postcollisional velocities can be characterized as follows:

**Proposition 2.1.** Suppose a particle with velocity \( v \) is colliding with the disk at time \( t \) and \( v \neq \kappa(t) \). Then it has collided with the disk in the past if and only if

$$\kappa(t) < v < p(t).$$  

(2.17)

**Proof.** Let \( \kappa(t) < v < p(t) \). As a consequence of assumption (2.8, p.11), \( v(s, t) \) is a continuous function of \( s \) for any \( t \), and thus must attain its minimum \( \kappa(t) \) at some \( s^* \in [0, t] \). Assume \( s^* < t \) and suppose to the contrary that the particle with velocity \( v \) has not collided with the disk in the past. Let \( \omega(s) \) be the position of the particle. Then

$$\omega(s) = \eta(t) - (t - s)v.$$  

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Since the particle is colliding with the disk from the right and could not have penetrated the disk by assumption (A0), it must have been to the right of the disk for all \( s \in [0, t) \), that is
\[
\omega(s) - \eta(s) \geq 0 \quad \forall s \in [0, t).
\]
However, this condition is violated at \( s = s^* \) since
\[
\omega(s^*) - \eta(s^*) = [\eta(t) - (t - s^*)v] - [\eta(t) - (t - s^*)\kappa(t)] = [\kappa(t) - v](t - s^*) < 0, \quad (2.18)
\]
which is a contradiction. If \( s^* = t \), then \( \kappa(t) = v(t, t) = p(t) \), which again violates (2.17).

The converse is an immediate consequence of (2.11, p. 11): while the range of \( v(\cdot, t) \) can be larger than \( (\kappa(t), p(t)) \), velocities in the interval
\[
[p(t), \max_{s \in [0, t]} v(s, t)]
\]
can not be colliding with the disk at time \( t \), and thus must be excluded.

Denote the set of all possible postcollisional velocities by \( \mathcal{V}_t \):
\[
\mathcal{V}_t = (\kappa(t), p(t)). \tag{2.19}
\]

### 2.2.3 Precollision Times

We now identify the times of the precollisions corresponding to velocities in \( \mathcal{V}_t \).

**Definition 2.2 (Precollision Time).** Suppose a particle with velocity \( v \in \mathcal{V}_t \) collides with the disk at time \( t \). Then the time \( s \) is a corresponding **precollision time** if \( (v, s) \) satisfies (2.10, p. 11) and the particle has been ahead of the disk for all \( \tau \in (s, t) \), that is,
\[
\eta(t) - (t - \tau)v \geq \eta(\tau) \quad \forall \tau \in (s, t). \quad (2.20)
\]

Note that this condition is a mathematical formulation of Assumption (A0).

Let \( \mathcal{N}_t \) be the set of all possible precollision times. To construct a bijection between \( \mathcal{V}_t \) and \( \mathcal{N}_t \) we need a more explicit characterization of the latter. To this end, we first rewrite (2.20) as
\[
v \leq \langle p \rangle_{\tau, t} \quad \forall \tau \in (s, t), \tag{2.21}
\]
which in turn is equivalent to
\[ v \leq \min_{\tau \in [s,t]} \langle p \rangle_{\tau,t} \leq \langle p \rangle_{s,t}. \] (2.22)

Combining condition above with (2.10, p. 11) gives
\[ \langle p \rangle_{s,t} = v \leq \min_{\tau \in [s,t]} \langle p \rangle_{\tau,t} \leq \langle p \rangle_{s,t}, \]
so we must conclude that (2.20) is equivalent to
\[ v = \min_{\tau \in [s,t]} \langle p \rangle_{\tau,t} = \langle p \rangle_{s,t}. \] (2.23)

Motivated the equality above, we define the modified postcollisional velocity \( \underline{v}(s,t) \) as follows:
\[ \underline{v}(s,t) := \min_{\tau \in [s,t]} \langle p \rangle_{\tau,t}. \] (2.24)

We can now say that \( s \) is a precollision time corresponding to \( v \) if and only if \( v = v(s,t) = \underline{v}(s,t) \). Consequently, we define
\[ \mathcal{N}_t := \left\{ s \in [0,t] \mid v(s,t) = \underline{v}(s,t) \right\}. \] (2.25)

Note the function \( \underline{v}(\cdot,t) \) is monotonically, but not necessarily strictly, increasing. It can be thought of as the tightest monotonically increasing lower envelope for \( v(s,t) \); the notation \( \underline{v} \) has been chosen to reflect that. We give an example of \( \mathcal{N}_t \) and \( \underline{v} \) in Figure 2.1 to help intuitive understanding of their properties.

**Notation.** For a given \( t \), we will use \( \overline{v}(\mathcal{A},t) \) to denote the image of the set \( \mathcal{A} \) under the map \( \overline{v}(\cdot,t) \) and \( \overline{v}^{-1}(\mathcal{B},t) \) to denote the pre-image of the set \( \mathcal{B} \) under the map \( \overline{v}(\cdot,t) \). Note that the inversion is only performed in the first variable with the second variable \( t \) fixed. The same notation will be used for \( v(\cdot,\cdot) \).

We now establish an important property of \( \mathcal{N}_t \). A large part of the analysis is essentially Riesz’s rising sun lemma [12] with a sign change.
Lemma 2.2. Fix $t \in [0, T]$ and let $\mathcal{N}_t$ be the set defined in (2.25). Then $\mathcal{N}_t^c = [0, t] \setminus \mathcal{N}_t$ is open in $[0, t]$.

Proof. Note that since $\underline{v}(t, t) = v(t, t) = p(t)$, we have $t \in \mathcal{N}_t$. We write $\mathcal{N}_t^c$ as

$$\mathcal{N}_t^c = \{ s \in [0, t) \mid v(s, t) > \underline{v}(s, t) \} = \{ s \in [0, t) \mid v(s, t) > v(\tau, t) \text{ for some } \tau \in (s, t) \} = \bigcup_{\tau \in (0, t]} \mathcal{O}_\tau,$$

where $\mathcal{O}_\tau = [0, \tau) \cap \overline{\{-v^{-1}(v(\tau, t), \infty), t\}}$. Since $\underline{v}(\cdot, t)$ is continuous, the pre-image $\mathcal{O}_\tau$ is open in the subspace topology on $[0, t]$. Therefore $\mathcal{N}_t^c$ is an open subset of $[0, t]$. \hfill \Box

2.2.4 The Bijection

Lemma 2.3 (Properties of $\underline{v}$). Let $\underline{v}(s, t)$ be the modified velocity defined in (2.24). Then

(2.3.a) $\underline{v}(\cdot, t) : [0, t] \to \mathcal{V}_t$ is continuous.

(2.3.b) Let $(a, b)$ be a maximal connected subset of $\mathcal{N}_t^c$. Then for all $s \in (a, b)$ we have

$$\underline{v}(a, t) = \underline{v}(s, t) = \underline{v}(b, t).$$

Consequently, $\partial_s \underline{v}(s, t) = 0$ on $\mathcal{N}_t^c$.

(2.3.c) Restricting $\underline{v}$ to $\mathcal{N}_t$ does not change its range: $\underline{v}([0, t], t) = \underline{v}(\mathcal{N}_t, t)$.

(2.3.d) For any fixed $t \in [0, T]$, $\underline{v}(s, t)$ is Lipschitz in $s$ with $|\partial_s \underline{v}(s, t)| \leq M/2$ for almost every $s \in [0, t]$.

(2.3.e) For any fixed $s \in [0, t]$, $\underline{v}(s, t)$ is Lipschitz in $t$ with $|\partial_t \underline{v}(s, t)| \leq M/2$ for almost every $t \in [0, T]$. 

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(2.3.f) Let $\mathcal{L}(A)$ be the Lebesgue measure of $A$ and define

$$D_t := \left\{ s \in [0, t] \mid \frac{\partial v(s, t)}{\partial s} \text{ exists} \right\}.$$  

Then $\mathcal{L}(v(D_t', t)) = \mathcal{L}(v(D_t, t)) = 0.$

(2.3.g) For all $s \in \mathcal{N}_t \cap D_t$ we have $\partial_s v(s, t) = \partial_s v(s, t)$.

Proof. (2.3.a) Let $\epsilon > 0$ be given. By Lemma 2.1.d (Properties of $v(s, t)$, p.11) the function $v(\cdot, t)$ is uniformly continuous on $[0, t]$, and thus there exists $\delta > 0$ such that

$$|v(s, t) - v(s', t)| < \epsilon \quad \text{whenever} \quad |s - s'| < \delta. \quad (2.26)$$

Without loss of generality, suppose $0 \leq s - s' < \delta$. Then by definition of $v(\cdot, t)$ we have

$$0 \leq v(s, t) - v(s', t) = v(s, t) - \min\{ \min_{\tau \in [s', s]} v(\tau, t), \ v(s, t) \}$$

$$\quad = v(s, t) - \min\{ v(\hat{\tau}, t), v(s, t) \} \quad (2.27)$$

for some $\hat{\tau} \in [s', s]$. If $v(\hat{\tau}, t) < v(s, t)$ we have

$$v(s, t) - v(s', t) = v(s, t) - v(\hat{\tau}, t) \leq v(s, t) - v(\hat{\tau}, t) < \epsilon$$

by (2.26). If $v(\hat{\tau}, t) \geq v(s, t)$, then the right-hand side of (2.27) vanishes.

(2.3.b) Since $v(\cdot, t)$ is non-decreasing we have $v(s, t) \leq v(b, t)$. Suppose to the contrary that $v(s, t) < v(b, t)$. Then there must exist $\tau \in [s, b]$ such that $v(s, t) = v(\tau, t)$, which in turn implies that $v(\tau, t) = v(\tau, t)$. But then $\tau \in \mathcal{N}_t$ by the definition of $\mathcal{N}_t$, which is a contradiction since $\tau \in (a, b) \subseteq \mathcal{N}_t$. The equality $v(a, t) = v(s, t)$ follows from continuity of $v(\cdot, t)$.

(2.3.c) Take $s \in \mathcal{N}_t^c$ and let $(a, b)$ be the largest connected subset of $\mathcal{N}_t^c$ containing it. By part (2.3.b), we have $v(s, t) = v(b, t)$. Thus $v(s, t) \in v(\mathcal{N}_t, t)$ since $b \in \mathcal{N}_t$.

(2.3.d) For $s \in \mathcal{N}_t^c$ let $(a_s, b_s)$ be the largest connected subset of $\mathcal{N}_t^c$ containing $s$. In other words, the lower limit $a_s$ is the largest time less than $s$ such that $v(a_s, t) = v(a_s, t)$. Similarly, the upper limit $b_s$ is the smallest time greater than $s$ such that $v(b_s, t) = v(b_s, t)$. For $s \in \mathcal{N}_t$ we simply let $a_s = b_s = s$. Then in both cases we have

$$v(a_s, t) = v(a_s, t) = v(s, t) = v(b_s, t) = v(b_s, t).$$

Let $\tau, \tau' \in [0, t]$ and assume, without loss of generality, that $\tau \leq \tau'$. If $(\tau, \tau') \subseteq \mathcal{N}_t^c$ then $v(\tau', t) - v(\tau, t) = 0$ by part (2.3.b). Otherwise, by Lemma 2.1.d (Properties of $v(s, t)$, p.11)
we have
\[ |v'(t) - v(t)| = |v(a_{\tau'}, t) - v(b_{\tau}, t)| \leq \frac{M}{2} (a_{\tau'} - b_{\tau}) \leq \frac{M}{2} (\tau' - \tau). \]

Thus the function \( v(s,t) \) is Lipschitz in \( s \). Therefore, \( \partial_s v(s,t) \) exists for almost all \( t \) and \( |\partial_s v(s,t)| \leq M/2 \).

(2.3.e) Let \( \tau' > t \) and fix \( s \in [0, t] \). Since \( v(s,t) \) is continuous, \( v(s,t) = v(\tau, t) \) for some \( \tau \in [s, t] \). We have
\[ v(s, \tau') \leq v(\tau, t) \leq v(\tau, \tau') + \frac{M}{2} |\tau' - t| \implies v(s, \tau') - v(s, t) \leq \frac{M}{2} |\tau' - t|. \]

On the other hand, for all \( \tau \in [0, t] \) we have
\[ v(\tau, \tau') \geq v(\tau, t) - \frac{M}{2} |\tau' - t| \implies v(s, \tau') \geq v(s, t) - \frac{M}{2} |\tau' - t|. \]

Thus the function \( v(s,t) \) is Lipschitz in \( t \). Therefore, \( \partial_t v(s,t) \) exists for almost all \( t \) and \( |\partial_t v(s,t)| \leq M/2 \).

(2.3.f) Since \( v(s,t) \) is Lipschitz in \( s \), it is almost everywhere differentiable and thus \( \mathcal{L}(\mathcal{D}_s) = 0 \). Since \( v(\cdot, t) \) is absolutely continuous, it possesses the Lusin property: \( \mathcal{L}(v(\mathcal{D}_t, t)) = 0 \). The same argument holds for \( v(\cdot, t) \).

(2.3.g) Let \( s \in \mathcal{N}_t \cap \mathcal{D}_t \). Then \( \partial_t v(s,t) \) is given by the definition of the classical derivative. Therefore,
\[ \frac{\partial v(s,t)}{\partial s} = \lim_{\tau \to s^+} \frac{v(\tau, t) - v(s, t)}{\tau - s} = \lim_{\tau \to s^+} \frac{v(\tau, t) - v(s, t)}{\tau - s} \leq \lim_{\tau \to s^+} \frac{v(\tau, t) - v(s, t)}{\tau - s} = \frac{\partial v(s,t)}{\partial t}, \]
\[ \frac{\partial v(s,t)}{\partial s} = \lim_{\tau \to s^-} \frac{v(s, t) - v(\tau, t)}{s - \tau} = \lim_{\tau \to s^-} \frac{v(s, t) - v(\tau, t)}{s - \tau} \leq \lim_{\tau \to s^-} \frac{v(s, t) - v(\tau, t)}{s - \tau} = \frac{\partial v(s,t)}{\partial t}. \]

It now follows that \( \partial_s v(s,t) = \partial_t v(s,t) \).

\[ \square \]

From Lemma 2.3.c (Properties of \( v \), p.16) it follows that the map \( v(\cdot, t) : \mathcal{N}_t \rightarrow \mathcal{V}_t \) is a surjection. However, it is not necessarily an injection, so further restriction is required. To show that the restriction we are about to make is of no consequence we will need the following lemma from [12] (page 77):

**Lemma 2.4** (Measure of an Image). Let \( I \subseteq \mathbb{R} \) be an interval and let \( u : I \rightarrow \mathbb{R} \). Assume that there exists a set \( E \subseteq I \) (not necessarily measurable) and \( M \geq 0 \) such that \( u \) is differentiable for all \( t \in E \), with
\[ |\dot{u}(t)| \leq M \quad \text{for all } t \in E. \]
Transformation of the Problem

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Then $L_\circ(u(E)) \leq M L_\circ(E)$, where $L_\circ$ denotes the Lebesgue outer measure.

We are now ready to make the restriction and create a bijection.

**Theorem 2.2 (Bijection).** For any fixed $t \in [0, T]$, let $v(\cdot, t)$ be the function defined in (2.24). Let

$$
\Phi_t := \left\{ s \in [0, t] \mid \frac{\partial v(s, t)}{\partial s} > 0 \right\} \quad \text{and} \quad W_t := v(\Phi_t, t). \tag{2.28}
$$

Then $v(s, t) = v(s, t)$ for all $s \in \Phi_t$, the mapping $v(\cdot, t) : \Phi_t \to W_t$ is a bijection and is strictly increasing, and $W_t$ contains almost all postcollisional velocities, that is, $L(V_t \setminus W_t) = 0$.

**Proof.** From Lemma 2.3.b (Properties of $v$, p.16) we know that $\partial_s v(s, t) \equiv 0$ for all $s \in N_t^c$, so it must be the case that $\Phi_t \subseteq N_t$. Furthermore, since $v(\cdot, t)$ is a monotonically increasing function on the interval $[0, t]$, its restriction to $\Phi_t$ is strictly increasing and thus is a bijection between its domain and range.

Let $I = [0, t]$ and

$$
\Phi_t^0 := \left\{ s \in [0, t] \mid \frac{\partial v(s, t)}{\partial s} = 0 \right\},
$$

then $I = \Phi_t \cup \Phi_t^0 \cup D_t^c$. Since $V_t = v(I, t)$ and $W_t = v(\Phi_t, t)$ we have

$$
V_t = W_t \cup v(\Phi_t^0, t) \cup v(D_t^c, t).
$$

Intersecting both sides with $W_t^c$ gives

$$
V_t \setminus W_t = \left( v(\Phi_t^0, t) \cup v(D_t^c, t) \right) \cap W_t^c \subseteq v(\Phi_t^0, t) \cup v(D_t^c, t),
$$

and thus by monotonicity and subadditivity of the Lebesgue outer measure we have

$$
L_\circ(V_t \setminus W_t) \leq L_\circ(v(\Phi_t^0, t) \cup v(D_t^c, t)) \leq L_\circ(v(\Phi_t^0, t)) + L_\circ(v(D_t^c, t)). \tag{2.29}
$$

Choosing $u(\cdot) = v(\cdot, t)$, $E = \Phi_t^0$ and $M = 0$ in Lemma 2.4 (Measure of an Image, p.18) gives

$$
L_\circ(v(\Phi_t^0, t)) = 0.
$$

The second term in (2.29) vanishes by Lemma 2.3.f (Properties of $v$, p.17), so we have $L_\circ(V_t \setminus W_t) = 0$. Hence $V_t \setminus W_t$ is measurable with $L(V_t \setminus W_t) = 0$ and almost all postcollisional velocities are in $W_t$. 

\[\square\]
Remark 2.2 (Another way to write $\partial_s v$). Let $\chi(s; \Phi_t)$ be the indicator function for $\Phi_t$. Since $\partial_s v(s,t) = 0$ on $\Phi^c_t$, we can write

$$\frac{\partial v(s,t)}{\partial s} = \frac{\partial v(s,t)}{\partial s} \chi(s; \Phi_t).$$

Furthermore, since $\Phi_t \subseteq \mathcal{N}_t$, Lemma 2.3.g (p.17) allows us to write

$$\frac{\partial v(s,t)}{\partial s} = \frac{\partial v(s,t)}{\partial s} \chi(s; \Phi_t),$$

which turns out to be convenient for numerical computations.

Remark 2.3. We have not yet discussed the relationship between the velocity of the particle that had precollided with the disk at time $s$ and the velocity of the disk itself at time $s$; one would expect the particle to be moving faster in that case. Indeed, combining (2.20) with (2.10) yields

$$\eta(s) + (\tau - s)v \geq \eta(\tau) \quad \forall \tau \in (s, t),$$

which can in turn be rewritten as $v \geq \langle p \rangle_{s, \tau}$ for all $\tau \in (s, t)$. Letting $\tau \to s^+$ gives $v \geq \langle p \rangle_{s, s} = p(s)$, so the particle is, at least, not slower than the disk. The case $v = p(s)$ is the grazing collision, which produces no drag force. Since

$$\frac{\partial v(s,t)}{\partial s} = \frac{v(s,t) - p(s)}{t - s},$$

all velocities that have had a grazing precollision and their corresponding (non-unique!) precollision times are collected in $\mathcal{W}^c_t$ and $\Phi^0_t$ respectively. Since $\mathcal{L}(v(\Phi^0_t, t)) = 0$, particles that have had a grazing precollision have no effect on the dynamics of the disk, and thus can be safely excluded.

### 2.2.5 Change of Variables

We now make a change of variables in (2.6a) for the one-dimensional case. By (2.19, p.14) and Theorem 2.2 (Bijection, p.19), for $n \geq 1$ we have

$$G_n(t; p) = \int_{V_t} \left[ (p(t) - v)^2 + \frac{\beta_1}{\gamma_1} (p(t) - v) \right] f_n^-(\eta(t), v, t) dv$$

$$= \int_{V_t} \left[ (p(t) - v)^2 + \frac{\beta_1}{\gamma_1} (p(t) - v) \right] f_n^-(\eta(t), v, t) dv$$

$$= \int_{\Phi_t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_1}{\gamma_1} (p(t) - v(s,t)) \right] f_n^-(\eta(t), v(s,t), t) ds.$$
Furthermore, since \( \partial_s^2 v(s, t) \) vanishes on \( \Phi_c \), we can write

\[
G_n(t; p) = \int_0^t \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_1}{\gamma_1} (p(t) - v(s, t)) \right] f_n^{-}(\eta(t), v(s, t), t) \, ds.
\]

To compute the drag due to first collisions we integrate over the rest of the velocities:

\[
G_0(t; p) = \int_{-\infty}^{g(t)} \left[ (p(t) - v)^2 + \frac{\beta_1}{\gamma_1} (p(t) - v) \right] \phi_0(v) \, dv
\]

(the domain of integration is obtained by subtracting \( \mathcal{V}_t \) from the original domain \( \mathcal{V}_- \)).

We now proceed to the two-dimensional case. We can interpret the time of precollision as the time the particle would have precollided if the disk was infinite. To distinguish the actual precollisions we need to impose a condition on the transverse velocities:

\[
x_2 - \frac{R}{t-s} \leq v_2 \leq \frac{x_2 + R}{t-s}.
\]

For convenience, we define

\[
a_{\pm} = \frac{x_2 \pm R}{t-s}.
\]

Then the expression for the drag forces due to recollisions can be written as follows:

\[
G_n(t; p) = \int_{-R}^R \int_0^t \int_{a_-}^{a_+} \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s, t)) \right] \times
\]

\[
f_n^{-}(\eta(t), x_2, v(s, t), v_2, t) \, dv_2 \, ds \, dx_2.
\]  

To obtain the expression for \( G_0 \) we again integrate over the complement of \( \mathcal{V}_t \times R \), which now consists of three parts:

\[
G_{00}(t; p) = \int_{-R}^R \int_{-\infty}^{\infty} \int_{R} \left[ (p(t) - v_1)^2 + \frac{\beta_1}{\gamma_1} (p(t) - v_1) \right] \phi_0(v_1, v_2) \, dv_2 \, dv_1 \, dx_2,
\]

\[
G_{0+}(t; p) = \int_{-R}^R \int_0^t \int_{a_+}^{\infty} \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s, t)) \right] \times
\]

\[
\phi_0(v(s, t), v_2) \, dv_2 \, ds \, dx_2,
\]
We can now write
\[
G_{0^-}(t; p) = \int_{-R}^{R} \int_{0}^{t} \int_{-\infty}^{a^-} \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s, t)) \right] \times \phi_0(v(s, t), v_2) \, dv_2 \, ds \, dx_2, \tag{2.32c}
\]

\[
G_0(t; p) = G_{00}(t; p) + G_{0^+}(t; p) + G_{0^-}(t; p). \tag{2.32d}
\]

Since the density does not change between collisions, we have
\[
f_n^-(\eta(t), x_2, v(s, t), v_2, t) = f_n^+(\eta(s), x_2 - (t - s)v_2, v(s, t), v_2, s)
\]
\[= j_{n-1}(x_2 - (t - s)v_2, s)\phi(v(s, t) - p(s), v_2). \tag{2.33}
\]

We will write
\[
f_n(x_2, v_2, s, t) := f_n^-(\eta(t), x_2, v(s, t), v_2, t).
\]

We now derive the recurrence relation for \(f_n\)'s in two dimensions: for \(n \geq 1\) we have
\[
j_n(z, s) = \frac{1}{\gamma_2} \int_{V_1} \int_{ \mathbb{R} } (p(s) - v_1)f_n^-(\eta(s), z, v_1, v_2, s) \, dv_2 \, dv_1
\]
\[= \frac{1}{\gamma_2} \int_{-\infty}^{\infty} \int_{-\infty}^{z} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau. \tag{2.34}
\]

The case \(n = 0\) is similar to (2.32):
\[
j_{00}(s) = \frac{1}{\gamma_2} \int_{-\infty}^{\infty} \int_{ \mathbb{R} } (p(s) - v_1)\phi_0(v_1, v_2) \, dv_2 \, dv_1, \tag{2.35a}
\]
\[
j_{0^+}(z, s) = \frac{1}{\gamma_2} \int_{0}^{\infty} \int_{z}^{\infty} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \, dv_2 \, d\tau, \tag{2.35b}
\]
\[
j_{0^-}(z, s) = \frac{1}{\gamma_2} \int_{0}^{\infty} \int_{-\infty}^{z} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \, dv_2 \, d\tau, \tag{2.35c}
\]
\[
j_{0\perp}(z, s) = j_{0^+}(z, s) + j_{0^-}(z, s), \quad j_0(z, s) = j_{00}(z, s) + j_{0\perp}(z, s). \tag{2.35d}
\]

We can now write
\[
f_{n+1}(x_2, v_2, s, t) = j_n(x_2 - (t - s)v_2, s)\phi(v(s, t) - p(s), v_2) \tag{2.36}
\]
for all \(n \geq 0\). This recurrence relation is the key for most estimates in the thesis.
Chapter 3

Existence

In this chapter we will work toward proving Theorem 1.1.a (Existence, p.7). We begin by establishing an important property of the physical system:

3.1 Galilean Invariance

For some of the calculations in this chapter it would be convenient to assume that $p(t)$ is always positive. To do that we would have to change the frame of reference, which requires the system to be invariant under Galilean transformations. And indeed, it should be: otherwise we would not have been able to describe its dynamics using Newton’s Second Law. To show the invariance explicitly we let $u \in \mathbb{R}$ be an arbitrary velocity shift and define the corresponding shifted variables as follows:

\begin{align*}
\tilde{v} &= v + u, \quad \tilde{x} = x + tu, \quad (3.1) \\
\tilde{p}(t) &= p(t) + u, \quad \tilde{\eta}(t) = \eta(t) + tu, \quad (3.2) \\
\tilde{f}(\tilde{x}, \tilde{v}, t) &= f(x, v, t), \quad (3.3) \\
\tilde{F}(\tilde{\eta}(t), t) &= F(\eta(t), t). \quad (3.4)
\end{align*}

We can now specifically state what we are trying to prove:

Theorem 3.1 (Galilean Invariance). Suppose $(\eta(t), p(t), f(x, v, t))$ satisfies (1.1, p. 2), then

\begin{align*}
\dot{\tilde{\eta}}(t) &= \tilde{p}(t), \quad \tilde{\eta}(0) = 0, \quad (3.5b) \\
\dot{\tilde{f}}(\tilde{x}, \tilde{v}, t) + \tilde{v} \partial_{\tilde{x}} \tilde{f}(\tilde{x}, \tilde{v}, t) &= 0, \quad \tilde{f}(\tilde{x}, \tilde{v}, 0) = \phi_{0}(\tilde{x}, \tilde{v} - u), \quad (3.5c)
\end{align*}

\begin{align*}
\dot{\tilde{p}}(t) &= \tilde{F}(\tilde{\eta}(t), t) + G[\tilde{f}], \quad \tilde{p}(0) = p_{0} + u, \quad (3.5a)
\end{align*}
The boundary condition for $f$ and the expression for the drag force are given by (1.5, p.4) and (1.11, p.7) respectively.

**Proof.** Differentiating both sides of definition (3.3) with respect to $t$ and $x$ yields, respectively,

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial t} \tilde{f} + \frac{\partial}{\partial t} \tilde{f} f = \frac{\partial}{\partial t} \tilde{f} + u \frac{\partial}{\partial x} \tilde{f};$$

substituting the above into (1.1c, p.2) gives (3.5c). To verify that the boundary condition (1.5, p.4) is satisfied we show that the fluxes (1.8, p.5) corresponding to $f$ and $\tilde{f}$ are equal:

$$\gamma_1 j_r(t) = \int_{-\infty}^{p(t)} (\tilde{p}(t) - \tilde{v}) \tilde{f}^{-} (\tilde{\eta}(t), \tilde{v}, t) \, d\tilde{\eta} = \int_{-\infty}^{p(t)+u} (p(t) - v) f^{-} (\eta(t), v, t) \, d\eta = \gamma_1 j_r(t).$$

Equality of the drag forces can be shown in the same way; together with the definition of $\tilde{F}$ this shows that equation (3.5a) holds and thus finishes the proof.

Note that since assumptions (A2) and (A3) are translation invariant, the theorem above will let us choose a convenient frame of reference later on.

### 3.2 A Priori Estimates

#### 3.2.1 Envelope for Solutions

We will now construct an envelope—a sub- and a super-solution to (1.1a). We will work with the one-dimensional case. Using the gas decomposition established in subsection 2.1.2 (*Partition of the Density Function, p.9*) we write the equation for the disk velocity as follows:

$$\dot{p}(t) = F(\eta(t), t) + G^t_0(t; p) + G^l_{rec}(t; p) - G^r_0(t; p) - G^r_{rec}(t; p).$$

The simplest solutions to the equation above are monotone, so we will construct a subsolution by altering the equation above in a way that readily leads to monotonically decreasing solution. Specifically, we will pretend that there is no gas to the left of the disk, so that $G^l_0(t; p) = G^l_{rec}(t; p) = 0$. The subsolution is then obtained by solving the problem

$$\dot{p}_-(t) = F_-(t) - G^r_0(t; p-) - G^r_{rec}(t; p-), \quad p_-(0) = p_0,$$

where $G^r_0$ and $G^r_{rec}$ are as defined in (2.6b, p.10) and (2.7, p.10) respectively, and $F_-$ is as defined by assumption (A3).
**Existence**

**Lemma 3.1.** There exists a unique solution to (3.7).

**Proof.** Since the right hand side of (3.7) is always negative, \( p_- \) is monotonically decreasing. This in turn implies that no recollisions from the right can occur, and thus \( G_{\text{rec}}^r(t;p_-) = 0 \). This leaves us with the following ODE:

\[
\dot{p}_-(t) = F_-(t) - G_0^r(p_-),
\]

where

\[
G_0^r(p_-) = \int_{-\infty}^{p_-} \left[ (p_- - v)^2 + \frac{\beta_1}{\gamma_1} (p_- - v) \right] \phi_0(v) \, dv.
\]

Note that we no longer need \( \chi_0 \) from (2.6b, p.10): all particles are colliding for the first time. To prove global existence of solutions to (3.7) we will show that \( G_0^r(p_-) \) is also uniformly bounded: since \( G_0^r(p_-) \) is continuous, it is bounded for \( p_- \in [-1, p_0] \). For \( p_- < -1 \) we have

\[
G_0^r(p_-) \leq \int_{-\infty}^{-1} \left[ v^2 + \frac{\beta_1}{\gamma_1} |v| \right] \phi_0(v) \, dv \lesssim \| \phi_0 \|,
\]

so \( G_0^r \) is indeed uniformly bounded. Since \( F_-(t) \) is continuous by assumption (A3), it is uniformly bounded for all \( t \in [0, T] \), which now means that the right hand side of (3.7) is uniformly bounded. Moreover, \( G_0^r \) is Lipschitz continuous, so there exists a unique solution to (3.7) for \( t \in [0, T] \).

The supersolution \( p_+ \) can be constructed by pretending that there is no gas to the right of the disk and repeating the argument above.

Next we will show that \( p_- \) and \( p_+ \) indeed form an envelope for possible solutions of the full problem:

**Theorem 3.2.** Suppose \( p \) is a solution of (3.6, p. 24). Then \( p_+(t) \geq p(t) \geq p_-(t) \) for all \( t \in [0, T] \).

**Proof.** We first show that the desired conclusion holds locally by comparing \( \dot{p}(0) \) with \( \dot{p}_-(0) \). Since all particles colliding with the disk at time 0 are doing so for the first time, the recollisional terms drop out:

\[
\dot{p}(0) = F(0, 0) + G_0^l(p_0) - G_0^r(p_0) > F_-(0) - G_0^r(p_0) = \dot{p}_-(0),
\]

where we have used assumption (A3) and the fact that \( G_0^l \geq 0 \). Together with \( p(0) = p_-(0) \), the calculation above implies that \( p(t) \geq p_-(t) \) for all sufficiently small \( t \). The upper bound can be obtained in the same way.
The next step is to extend the result to the interval $[0, T]$. To this end, suppose $p(t) \geq p_-(t)$ for all $t \in [0, t_1]$ and $p(t_1) = p_-(t_1)$; we will show that no such $t_1 \leq T$ can exist. Since $p_-$ is decreasing, $p(t_1) = p_-(t_1)$ implies that $p$ attains its global minimum over $[0, t_1]$ at $t_1$, and therefore $G^r_{\text{rec}}(t_1; p) = 0$. We now have

$$\dot{p}(t_1) = F(\eta(t_1), t_1) + G^l_0(t_1; p) + G^l_{\text{rec}}(t_1; p) - G^r_0(t_1; p) - G^r_{\text{rec}}(t_1; p)$$

$$> F_-(t_1) - G^r_0(t_1; p) = F_-(t_1) - G^r_0(t_1; p_-) = \dot{p}_-(t_1),$$

which in turn implies that $p(t) \geq p_-(t)$ for all $t$. The upper bound can be obtained in the same way.

Since $p_-(t)$ and $p_+(t)$ are both continuous, we can obtain uniform a priori bounds for $p$: for all $t \in [0, T]$ we have

$$M_+ := \max_{t \in [0, T]} p_+(t) \geq p_+(t) \geq p(t) \geq p_-(t) \geq \min_{t \in [0, T]} p_- (t) =: M_-,$$

so $p(t)$ is uniformly bounded above and below. By Theorem 3.1 (Galilean Invariance, p.23) we can assume that $M_- = 0$ and that the disk velocity $p(t)$ is always positive, which in turn implies that $\kappa > 0$.

Since the properties of $\phi_0$ stated in (A2) are translation invariant, we are not losing any generality by translating the velocity.

### 3.2.2 Bound for $\dot{p}$

The next step for us is bounding the drag force in terms of $p$ (but not $\dot{p}$). We begin with a technical lemma:

**Lemma 3.2 (Bound for $p(s)$).** For all $0 \leq s < t$ and all $m \geq 1$ we have

$$\frac{\partial v(s, t)}{\partial s} (p(s))^m \leq \frac{\partial v(s, t)}{\partial s} (v(s, t))^m.$$

The bound above is not immediate since we do not always have $p(s) \leq v(s, t)$.

**Proof.** Suppose $s \in \Phi_t$ (2.28, p.19), then

$$\frac{\partial v(s, t)}{\partial s} = \frac{v(s, t) - p(s)}{t - s} > 0$$

and hence

$$v(s, t) = v(s, t) > p(s),$$

so

$$\frac{\partial v(s, t)}{\partial s} (p(s))^m \leq \frac{\partial v(s, t)}{\partial s} (v(s, t))^m = \frac{\partial v(s, t)}{\partial s} (v(s, t))^m.$$
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If \( s \in \Phi_0^t \), then \( \partial_s g(s,t) = 0 \) by definition (2.28, p. 19) and the inequality is satisfied automatically. The set \( D^c_t \) is too small to be an issue.

We now use the lemma to prove another technical-looking bound. However, we are actually getting close to recursively bounding \( j_n \): compare the left-hand side of the inequality below to definition (2.34, p. 22).

**Lemma 3.3.** For all \( t \geq 0 \) and all \( m \geq 1 \) we have

\[
\int_0^t \frac{\partial v(s,t)}{\partial s} (p(t) - v(s,t))(v(s,t))^{m-1} \, ds \leq \frac{(p(t))^{m+1}}{m(m+1)}.
\]

**Proof.** Denote the quantity to be bounded by \( I(m) \). Using Lemma 3.2 (*Bound for \( p(s) \)) we calculate

\[
I(m) \leq \int_0^t \frac{\partial v(s,t)}{\partial s} (p(t) - v(s,t))(v(s,t))^{m-1} \, ds
\]

\[
= p(t) \int_0^t \frac{\partial v(s,t)}{\partial s} (v(s,t))^{m-1} \, ds - \int_0^t \frac{\partial v(s,t)}{\partial s} (v(s,t))^m \, ds
\]

\[
= \frac{1}{m} \left[ (p(t))^{m+1} - p(t)(\kappa(t))^m \right] - \frac{1}{m+1} \left[ (p(t))^{m+1} - (\kappa(t))^{m+1} \right]
\]

\[
= \left( \frac{1}{m} - \frac{1}{m+1} \right) (p(t))^{m+1} + (\kappa(t))^m \left( \frac{\kappa(t)}{m+1} - \frac{p(t)}{m} \right) \leq \frac{(p(t))^{m+1}}{m(m+1)}. \quad \Box
\]

We define

\[
\lambda_n(s) = \frac{\| \phi_0 \|}{\gamma_2^{n+1}} \left( \frac{p(s)^{2n}}{(2n)!} + \frac{p(s)^{2n+1}}{(2n+1)!} \right); \quad (3.9)
\]

we can now use the lemma above to obtain a bound for \( f_n \):

**Theorem 3.3 (Bound for \( f_n \)).** For all \( x_2 \in [-R,R] \), \( s \geq 0 \) and \( n \geq 0 \) we have

\[
\hat{f}_{n+1}(x_2, s, t) \leq \lambda_n(s),
\]

where \( \hat{f} \) is defined by (1.4, p. 3).
Proof. Let \( z \in [-R, R] \) and \( s \geq 0 \). Using equality (2.35, p.22) we can bound \( j_{0\perp} \) and \( j_{00} \) as follows:

\[
\gamma_2 j_{0\perp}(z, s) \leq \int_0^s \int_R \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \, dv_2 \, d\tau
\]

\[
= \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \hat{\phi}_0(v(\tau, s)) \, d\tau = \int_{\mathcal{E}(s)} (p(s) - v) \hat{\phi}_0(v) \, dv,
\]

\[
\gamma_2 j_{00}(s) = \int_{-\infty}^s \int_R (p(s) - v_1) \phi_0(v_1, v_2) \, dv_2 \, dv_1 = \int_{-\infty}^s (p(s) - v) \hat{\phi}_0(v) \, dv.
\]

Combining the two gives

\[
j_0(z, s) \leq \frac{1}{\gamma_2} \int_{-\infty}^s (p(s) - v) \hat{\phi}_0(v) \, dv \leq \frac{\|\hat{\phi}_0\|_{L^1(v, dv)} + \|\hat{\phi}_0\|_{L^p(s)}}{\gamma_2}
\]

\[
\leq \frac{\|\hat{\phi}_0\|}{\gamma_2} (1 + p(s)) = \lambda_0(s).
\]

Combining the estimate above with equality (2.36, p.22) and assumption ((A4), p.4) gives

\[
f_1(x_2, v_2, s, t) \leq \lambda_0(s) \phi(v(s, t) - p(s), v_2)
\]

\[
\hat{f}_1(x_2, s, t) \leq \lambda_0(s) \hat{\phi}(v(s, t) - p(s)) \leq \lambda_0(s).
\]

We now use equality (2.34, p.22) and Lemma 3.3 (p.27) to inductively obtain the bound for \( n \geq 1 \):

\[
\gamma_2 j_n(z, s) \leq \int_0^s \int_R \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau
\]

\[
= \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \hat{f}_n(z, \tau, s) \, d\tau
\]

\[
\text{(induction hypothesis)} \leq \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \lambda_{n-1}(\tau) \, d\tau \leq \gamma_2 \lambda_n(s).
\]

Finally, we have

\[
f_{n+1}(x_2, v_2, s, t) = j_n(x_2 - (t - s)v_2, s) \phi(v(s, t) - p(s), v_2) \leq \lambda_n(s) \phi(v(s, t) - p(s), v_2)
\]

\[
\hat{f}_{n+1}(x_2, s, t) \leq \lambda_n(s) \hat{\phi}(v(s, t) - p(s)) \leq \lambda_n(s).
\]

We can now bound \( G_{\text{rec}} \) using the result above:
Theorem 3.4 (Bound for $G_{rec}$). There exists a constant $C_0$, possibly depending on $\|\phi_0\|$, $p_0$ and $F(x,t)$, such that for all $t \geq 0$ we have

$$G_{rec} \leq C_0(p(t)^2 + 1) \left( e^{p(t)/\sqrt{\gamma_2}} - 1 \right).$$

Proof. Using equality (2.31, p. 21) and positivity of $f_n$ we get

$$G_n(t;p) = \int_{-R}^{R} \int_{0}^{t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s,t)) \right] \int_{x_2-R}^{x_2+R} f_n(x_2, v_2, s, t) \, dv_2 \, ds \, dx_2$$

$$\leq \int_{-R}^{R} \int_{0}^{t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s,t)) \right] f_n(x_2, s, t) \, ds \, dx_2.$$ 

Next we use Theorem 3.3 (Bound for $f_n$, p. 27) and Lemma 3.2 (Bound for $p(s)$, p. 26):

$$G_n(t;p) \leq 2R \int_{0}^{t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s,t)) \right] \lambda_{n-1}(s) \, ds$$

$$= \frac{2R\|\phi_0\|}{\gamma_2^n} \int_{0}^{t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s,t)) \right] \left( \frac{p(s)^{2n-2}}{(2n-2)!} + \frac{p(s)^{2n-1}}{(2n-1)!} \right) \, ds$$

$$\leq \frac{2R\|\phi_0\|}{\gamma_2^n} \int_{0}^{t} \frac{\partial v(s,t)}{\partial s} \left[ (p(t) - v(s,t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s,t)) \right] \left( \frac{v(s,t)^{2n-2}}{(2n-2)!} + \frac{v(s,t)^{2n-1}}{(2n-1)!} \right) \, ds$$

$$\leq 4Rp(t)^2 \gamma_2 \lambda_n(t) + 2R\beta_2 \lambda_n(t) = 2R\lambda_n(t) \left( 2p(t)^2 \gamma_2 + \beta_2 \right).$$

Now we add it all up:

$$G_{rec}(t;p) = \sum_{n=1}^{\infty} G_n(t;p) \leq 2R \left( 2p(t)^2 \gamma_2 + \beta_2 \right) \sum_{n=1}^{\infty} \lambda_n(t) \leq C_0(p(t)^2 + 1) \left( e^{p(t)/\sqrt{\gamma_2}} - 1 \right),$$

(3.12)

and the proof is complete. \qed

This result together with (3.8, p. 26) lets us bound the derivative of a solution of (1.1a, p. 2) uniformly:

**Theorem 3.5 (Uniform Bound for $\dot{p}$).** There exists a constant $M$, possibly depending on $\|\phi_0\|$, $p_0$, $F(x,t)$ and $T$, such that for any $p$ that solves (1.1a, p. 2) we have

$$|p(t)| + |\dot{p}(t)| \leq M \quad \text{for all} \quad t \in [0, T].$$

(3.13)
Proof. First, we use assumption (A3) to uniformly bound the external forcing:

$$|F(\eta(t), t)| \leq \|F_-\|_{\infty} + \|F_+\|_{\infty}.$$  

We also have a uniform bound on the drag force due to first collisions:

$$G^{r}_0(t; p) = \int_{-R}^{R} \int_{\mathcal{V}_-} \left[ (p(t) - v_1)^2 + \frac{\beta_2}{\gamma_2}(p(t) - v_1) \right] \chi_0(\eta(t), x_2, v, t) \phi_0(v) \, dv \, dx_2$$

$$\leq 2R \left( M_+^2 + \frac{\beta_d}{\gamma_d} M_+ \right) \|\phi_0\|;$$

the drag force term $G^{r}_0$ can be bounded in a similar way.

The combination of the two estimates above with Theorem 3.4 (Bound for $G_{rec}$, p.28) lets us bound $\dot{p}(t)$:

$$|\dot{p}(t)| = \left| F(\eta(t), t) + G^{r}_0(t; p) + G^{d}_{rec}(t; p) - G^{r}_{0}(t; p) - G^{r}_{rec}(t; p) \right|$$

$$\leq \|F_-\|_{\infty} + \|F_+\|_{\infty} + 4R \left( M_+^2 + \frac{\beta_2}{\gamma_2} M_+ \right) \|\phi_0\| + 2C_0(M_+^2 + 1)e^{M_+^2/\sqrt{2}}. \quad \text{(3.14)}$$

We finish the proof by explicitly defining $M$:

$$M := M' + M_+.$$

\[\square\]
3.3 Fixed Point Iteration

We are now ready to prove Theorem 1.1.a (Existence, p.7). Let

\[ \Gamma := \left\{ p \in W^{1,\infty}(0,T) \mid p(0) = p_0 \text{ and } \|\dot{p}\|_{\infty} \leq M+1 \right\}. \]

**Proof.** We first observe that the space \( \Gamma \) is compact and convex in \( C([0,T]) \) by the Arzelà-Ascoli Theorem [13]. Let \( \eta_p(t) = \int_0^t p(s) \, ds \) and define mappings \( \Psi : \Gamma \to L^\infty(0,T) \) and \( \Sigma : L^\infty(0,T) \to L^\infty(0,T) \) as follows:

\[
\Psi p = F(\eta_p(t), t) + G_0^t(t; p) - G_0^t(t; p) + G_{\text{rec}}^t(t; p) - G_{\text{rec}}^t(t; p),
\]

\[
\Sigma q = \begin{cases} 
M + 1 & \text{if } q > M + 1, \\
q & \text{if } |q| \leq M + 1, \\
-M - 1 & \text{if } q < -M - 1.
\end{cases}
\]

Simply put, \( \Sigma \) retracts \( q \) onto the interval \([-M-1, M+1]\) to force the map into the set we started with; since we know the solution must live in this set anyway, it does not ruin the iteration.

Consider the mapping

\[ \overline{\Psi} : p \mapsto p_0 + \int_0^t (\Sigma \Psi p)(s) \, ds. \]

Since \( \Sigma \) is continuous and \( \Psi \) is Lipschitz continuous by Theorem 4.7 (Lipschitz Bound for \( G_{\text{rec}}, \) p.57), the mapping \( \overline{\Psi} \) is also continuous. Furthermore, it maps \( \Gamma \) to itself by construction, and thus has a fixed point by the Schauder Fixed Point Theorem [14].

Let \( \tilde{p} \) be that fixed point. Since \( |\dot{\tilde{p}}(0)| < M \) by Theorem 3.5 (Uniform Bound for \( \dot{p}, \) p.29) and \( \dot{p} \) is continuous by Theorem 1.1.c (Regularity, p.7), there exists a short interval \( I_\epsilon = [0, \epsilon] \) such that \( \dot{\tilde{p}}(t) < M \) for all \( t \in I_\epsilon \). Therefore, if we temporarily replace \( T \) with \( \epsilon \) in definition of \( \Gamma \), we would obtain \( \tilde{p} \) which is invariant under \( \Sigma \), and thus is a solution to (1.1,p.2).

We extend this result to the original domain \([0,T]\) as follows: suppose to the contrary that there exists \( t_1 \in [\epsilon, T] \) such that \( \dot{\tilde{p}} < M + 1 \) for all \( t \in [0, t_1) \) and \( \dot{\tilde{p}}(t_1) = M + 1 \). Then for \( t \in [0, t_1) \), \( \tilde{p} \) is invariant under \( \Sigma \) and thus is a solution to the original problem. But then by Theorem 3.5 we must have

\[ \dot{\tilde{p}} < M \text{ for all } t \in [0, t_1), \]

which is the a contradiction: \( \dot{p} \) must be continuous by Theorem 1.1.c (Regularity, p.7). \( \square \)
3.4 Construction of the Density Function

We will now conclude the proof of Theorem 1.1.a (Existence, p.7) by constructing the density function. For \( x \in D(t) \) it is defined by (2.33, p.22), so we only consider \( x \notin D(t) \).

At first we will construct \( f \) only for \( x_2 \in [-R, R] \), and then extend to the rest of the phase space. We begin by extending Definition 2.2 (Precollision Time, p.14) to \( x \notin D(t) \):

**Definition 3.1 (Precollision Time for \( x \notin D(t) \)).** Let the phase space point \((x, v; t)\) with \( x_1 > \eta(t) \) and \( x_2 \in [-R, R] \) be given. A time \( s(x, v, t) \) is said to be the corresponding precollision time if it satisfies (2.20, p.14) and

\[
x - (t - s(x, v, t))v \in D(s(x, v, t)).
\]

We can show that the precollision time defined in this way is unique for almost all \((x, v, t)\) by slightly modifying Theorem 2.2 (Bijection, p.19): we would define the precollisional velocity function

\[
v(s, t, x) := \frac{x - \eta(s)}{t - s}.
\]

Keeping \( x = x_1 \) fixed, we can repeat arguments in subsections 2.2.2 (Postcollisional Velocities, p.13) and 2.2.3 (Precollision Times, p.14) and see that for all \( x \) and \( t \), almost all \( v \) can indeed be identified with a unique precollision time. We can now use this Definition 3.1 to write \( \chi_0 \) with \( x_2 \in [-R, R] \) as follows:

\[
\chi_0(x, v, t) = \begin{cases} 
0 & \text{if there exists } s \in (0, t) \text{ satisfying Definition 3.1}, \\
1 & \text{otherwise}.
\end{cases}
\]

This, in turn, uniquely specifies \( f_0 \) in (2.5, p.10). For \( f_n \) with \( n \geq 1 \) and \((x, v, t) \notin \text{supp} \chi_0 \), we recursively solve the transport equation with boundary condition given by (2.2a, p.9) to obtain

\[
f_n(x, v, t) = j_{n-1}(x_2 - (t - s(x, v, t))v_2, s(...))\phi(v_1 - p(s(...)), v_2), \quad (3.15)
\]

which now completely specifies the full density distribution \( f(x, v, t) \) for \( x_2 \in [-R, R] \). We now extend it further: if \( x_2 > R \) and \( v_2 < 0 \), or \( x_2 < -R \) and \( v_2 > 0 \), we have

\[
f(x, v, t) = \phi_0(v).
\]
Existence

Construction of the Density Function

If \( x_2 > R \) and \( v_2 > 0 \) we define \( s' = t - (x_2 - R)/v_2 \) and set

\[
f(x, v, t) := f(x - (t - s')v, v, s'),
\]

where the right-hand side is defined by (3.15). The case with \( x_2 < R \) and \( v_2 < 0 \) is similar.

We now show that \( f \in L^\infty(R^d; L^1(R^d)) \). Using the bound for \( j_n \) obtained in (3.11, p.28) we write

\[
f_{\text{rec}}(x, v, t) = \sum_{n=1}^{\infty} n^{-1} \left( x_2 - (t - s(x, v, t))v_2, s(...) \right) \phi(v_1 - p(s(...)), v_2)
\]

By (3.13, p.29) and a calculation similar to (3.12, p.29) we can obtain

\[
f_{\text{rec}}(x, v, t) \lesssim \phi(v_1 - p(s(x, v, t)), v_2),
\]

where the implied constant depends on \( \|\phi_0\|, \gamma_2 \) and \( M \). We now have

\[
\int_{R} \int_{R} |f_{\text{rec}}(x, v, t)| \, dv \, dv_1 \lesssim \int_{R} \int_{R} \phi(v_1 - p(s(x, v, t)), v_2) \, dv_2 \, dv_1
\]

\[
= \int_{|v_1| < 2M} \phi(v_1 - p(\ldots)) \, dv_1 + \int_{|v_1| \geq 2M} \phi(v_1 - p(\ldots)) \, dv_1
\]

Since \( \hat{\phi} \in L^\infty \) by assumption (A2), we can bound \( I_1 \) as follows:

\[
I_1 = \int_{|v_1| < 2M} \phi(v_1 - p(s(x, v, t))) \, dv_1 \leq 2M\|\hat{\phi}\|_\infty < 2M\|\phi\| = 2M.
\] (3.16)

To bound \( I_2 \) we use the weighted norm from assumption (A2):

\[
\hat{\phi}(v_1 - p(s(x, v, t))) \leq \frac{1}{1 + (v_1 - p(s(x, v, t)))^2}.
\]
Since $|p(\cdot)| \leq M$, the quantity on the right-hand side is integrable over $|v_1| \geq 2M$; we can now write
\[
\int_{|v_1| \geq 2M} \hat{\varphi}(v_1 - p(\cdot)) \, dv_1 \leq \int_{|v_1| \geq 2M} \frac{1}{1 + (v_1 - p(\cdot))^2} \, dv_1 \lesssim 1.
\]
Combining the bound above with (3.16) we get
\[
\|f_{\text{rec}}\|_{L^1(dv)} = \int_{\mathbb{R}} \int_{\mathbb{R}} |f_{\text{rec}}(x, v, t)| \, dv_2 \, dv_1 \lesssim 1,
\]
where the implied constant depends on $M$. Since this bound holds for all $x$, we can conclude that $f_{\text{rec}}$ indeed belongs to $L^\infty(\mathbb{R}_x^d, L^1(\mathbb{R}_v^d))$. Since $f_0$ belongs to the same space automatically, we have shown that the full density function $f$ does too. The construction is now complete.

From the construction we see that there is a unique $f$ corresponding to the disk trajectory $p$, and thus uniqueness of $p$ would imply uniqueness of the whole solution.
Chapter 4

Uniqueness

This chapter is dedicated to proving Theorem 1.1.b (*Uniqueness, p.7*). We will also obtain a proof of Theorem 1.1.c (*Regularity, p.7*) by a small alteration of one of the steps towards uniqueness.

4.1 Preliminary Bounds for $f_n$

In this section we are assuming that the disk trajectory is as given by Theorem 1.1.a (*Existence, p.7*), and establish bounds on $f_n$'s and some of their derivatives. Let

$$
\alpha_n(s; E) = \frac{1}{\gamma_2} \frac{(Es)^n}{n!} \quad \text{and} \quad \alpha_{-1}(s; E) = 0.
$$

Note that for $n \geq 1$

$$
E \int_0^s \alpha_{n-1}(\tau; E) \, d\tau = \alpha_n(s; E)
$$

Note also that $\alpha_n$ is increasing in both arguments (we will be routinely replacing $E$'s with larger ones). In the rest of the chapter we will be proving estimates of the form

$$
(...)_n \leq C_k \alpha_n(s; E_k),
$$

where $k$ will essentially be the number of the estimate in the chapter (see Lemma 4.1 below for an example). Every time we choose a value for $C_k$ or $E_k$ with $k \geq 2$, we are implicitly taking it to be at least as large as the previous one:

$$
C_k := Y \quad \text{means} \quad C_k := \max\{Y, C_{k-1}\}.
$$
This will ensure that sequences $C_k$ and $E_k$ are increasing:

$$C_1 \leq C_2 \leq ... \quad \text{and} \quad E_1 \leq E_2 \leq ...$$

### 4.1.1 Preliminary Bounds for $f_n$

We begin by establishing uniform bounds on $f_n$ and $\hat{f}_n$. Recall formula (2.36, p.22):

$$f_{n+1}(x_2, v_2, s, t) = j_n(x_2 - (t-s)v_2, s)\phi(v(s, t) - p(s), v_2); \quad (4.2)$$

the shape function $\phi$ is easily bounded, but the density fluxes require some calculations:

**Lemma 4.1 (Bound for $j_n$).** Let $M$ be defined by Theorem 3.5 (Uniform Bound for $\dot{p}$, p.29). Then for all $z \in [-R, R]$, $s \geq 0$ and $n \geq 0$ we have

$$j_n(z, s) \leq C_1 \alpha_n(s; E_1),$$

where $E_1 = \frac{M^2}{\gamma_2}$ and $C_1 = \|\phi_0\|(1 + M)$.

**Proof.** Using formula (2.35, p.22) we can bound $j_{0\perp}$ and $j_{00}$ as follows:

$$\gamma_2 j_{0\perp}(z, s) \leq \int_0^s \int_R \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) dv_2 d\tau$$

$$= \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s))\hat{\phi}_0(v(\tau, s)) d\tau = \int_{\xi(s)}^p (p(s) - v)\hat{\phi}_0(v) dv$$

$$\gamma_2 j_{00}(s) = \int_{-\infty}^p (p(s) - v)\phi_0(v_1, v_2) dv_2 dv_1 = \int_{-\infty}^p (p(s) - v)\hat{\phi}_0(v) dv$$

Combining the two gives

$$\gamma_2 j_0(z, s) \leq \int_{-\infty}^p (p(s) - v)\hat{\phi}_0(v) dv \leq |p(s)|\|\hat{\phi}_0\|_1 + \|\hat{\phi}_0\|_{L^1(v dv)} \leq \|\phi_0\|(1 + M).$$
We now use formula (2.34, p. 22) and induction to get the bound for \( n \geq 1 \):

\[
j_n(z, s) \leq \frac{1}{\gamma_2} \int_0^s \int_{-R}^R \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau
\]

\[
\leq \frac{M^2}{\gamma_2} \int_0^s \int_{-R}^R j_{n-1}(z - (s - \tau)v_2, \tau) \phi(v(\tau, s) - p(\tau), v_2) \, dv_2 \, d\tau
\]

\[
\leq C_1 E_1 \int_0^s \alpha_{n-1}(\tau; E_1) \hat{\phi}(v(\tau, s) - p(\tau)) \, d\tau
\]

\[
\leq C_1 \alpha_n(s; E_1) \| \hat{\phi} \|_{\infty} \leq C_1 \alpha_n(s; E_1).
\]

Bounds for \( f_n \) and \( \hat{f}_n \) easily follow:

**Theorem 4.1 (Bound for \( f_n \)).** For all \( n \geq 0, x_2 \in [-R, R], v_2 \in \mathbb{R} \) and \( 0 \leq s \leq t \) we have

\[
f_{n+1}(x_2, v_2, s, t) \leq C_1 \alpha_n(s; E_1) \quad \text{and} \quad \hat{f}_{n+1}(x_2, s, t) \leq C_1 \alpha_n(s; E_1)
\]

**Proof.** Applying Lemma 4.1 (Bound for \( j_n \), p. 36) to (4.2) and taking the supremum (integrating over \( v_2 \)) gives the desired result.

### 4.1.2 Bounds for Derivatives of \( f_n \)

This subsection contains the most technical estimates of the thesis. The motivation for them is the need to bound (4.15, p. 53); see the discussion in Remark 4.1 (Difference in Derivatives, p. 53) for an explanation.

For the remainder of this section we define \( a_{\pm} \) as follows:

\[
a_{\pm} = \frac{z \pm R}{s - \tau}.
\]

**Lemma 4.2 (Bound on \( \partial_z j_n(z, s) \)).** There exist constants \( C_2 \) and \( E_2 \) such that for all \( z \in [-R, R], s \geq 0 \) and \( n \geq 0 \) we have

\[
\left| \frac{\partial j_n(z, s)}{\partial z} \right| \leq C_2 \alpha_n(s; E_2).
\]

**Proof.** Throughout the proof we will make use of identity (2.1.b, p. 11) and the bound (2.1.d):

\[
\frac{p(s) - v(\tau, s)}{s - \tau} = \frac{\partial v(\tau, s)}{\partial s} \leq \frac{1}{2} M.
\]

\[
\leq C_1 \alpha_n(s; E_1) \| \hat{\phi} \|_{\infty} \leq C_1 \alpha_n(s; E_1).
\]
Recall that \( j_0(z, s) = j_{0\perp}(z, s) + j_{00}(s) \), so for \( n = 0 \) we only need to work with \( j_{0\perp} \) (2.35, p. 22):

\[
\left| \frac{\partial}{\partial z} j_{0\perp}(z, s) \right| \leq \frac{1}{\gamma_2} \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \left| \frac{p(s) - v(\tau, s)}{s - \tau} \right| \phi_0(v(\tau, s), a_-) \, d\tau
\]

\[
+ \frac{1}{\gamma_2} \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \left| \frac{p(s) - v(\tau, s)}{s - \tau} \right| \phi_0(v(\tau, s), a_+) \, d\tau
\]

\[
\leq \frac{2\|\phi_0\|_\infty}{\gamma_2} \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \left| \frac{\partial v(\tau, s)}{\partial s} \right| \, d\tau
\]

\[
\leq \frac{M\|\phi_0\|}{\gamma_2} \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \, d\tau \leq \frac{M^2\|\phi_0\|}{\gamma_2} = M^2\|\phi_0\|\alpha_0,
\]

so we set \( C_2 = M^2\|\phi_0\| \).

For \( n \geq 1 \) we proceed by induction using formula (2.34, p. 22):

\[
\gamma_2 \frac{\partial j_n}{\partial z}(z, s) = \frac{\partial}{\partial z} \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau
\]

\[
= \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \left( \frac{p(s) - v(\tau, s)}{s - \tau} \right) \left[ f_n(z, a_+, \tau, s) - f_n(z, a_-, \tau, s) \right] \, d\tau
\]

\[= I_1 \]

\[
+ \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \frac{\partial f_n}{\partial z}(z, v_2, \tau, s) \, dv_2 \, d\tau.
\]

\[= I_2 \]

We bound \( I_1 \) using (2.1.d, p. 11) and Theorem 4.1 (Bound for \( f_n \), p. 37):

\[
I_1 = \int_0^s \frac{\partial v(\tau, s)}{\partial \tau} \frac{\partial v(\tau, s)}{\partial s} \left[ f_n(z, a_+, \tau, s) - f_n(z, a_-, \tau, s) \right] \, d\tau \leq 2M^2 C_1 \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau
\]

To bound \( I_2 \) we additionally use the induction hypothesis:

\[
I_2 = \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \frac{\partial f_n}{\partial z}(z, v_2, \tau, s) \, dv_2 \, d\tau
\]

\[
= \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \frac{\partial j_{n-1}(z - (s - \tau)v_2, \tau)}{\partial z} \phi(v(\tau, s) - p(\tau), v_2) \, dv_2 \, d\tau
\]

\[\leq C_2\|\phi\|_\infty M^2 \int_0^s \alpha_{n-1}(\tau; E_2) \, d\tau.
\]
Combining the estimates on $I_1$ and $I_2$ gives

\[
\frac{|\partial j_n(z,s)|}{\partial z} \leq 2M^2C_1 \frac{\int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau}{\gamma_2} + \frac{C_2\|\hat{\phi}\|_{\infty}M^2}{\gamma_2} \int_0^s \alpha_{n-1}(\tau; E_2) \, d\tau \\
\leq C_2 \left( \frac{2M^2}{\gamma_2} + \|\phi\|_{\infty}E_1 \right) \int_0^s \alpha_{n-1}(\tau; E_2) \, d\tau = C_2\alpha_n(s; E_2).
\]

Using the lemma above we can estimate one of the derivatives of $f_n$:

**Theorem 4.2 (Bound on $\partial_t f_{n+1}$).** There exist constants $C_3$ and $E_3$ such that for all $x_2 \in [-R, R]$, $s \geq 0$ and $n \geq 0$ we have

\[
\left| \frac{\partial f_{n+1}}{\partial t}(x_2, v_2, s, t) \right| \leq C_3\alpha_n(s, E_3) \quad \text{and} \quad \left| \frac{\partial \hat{f}_{n+1}}{\partial t}(x_2, s, t) \right| \leq C_3\alpha_n(s, E_3).
\]

**Proof.** Using equation (4.2, p.36) and Lemma 2.1 (Properties of $v(s,t)$, p.11) we write

\[
\frac{\partial f_{n+1}}{\partial t}(x_2, v_2, s, t) = \frac{\partial}{\partial t} \left( j_n(x_2 - (t-s)v_2, s)\phi(v(s, t) - p(s), v_2) \right) \\
= -v_2 \frac{\partial j_n}{\partial z}(x_2 - (t-s)v_2, s)\phi(v(s, t) - p(s), v_2) \\
+ \frac{\partial v(s, t)}{\partial t} j_n(x_2 - (t-s)v_2, s) \frac{\partial \phi}{\partial v_1}(v(s, t) - p(s), v_2) \\
\leq \left| \frac{\partial j_n}{\partial z} \right| |v_2\phi(v(s, t) - p(s), v_2)| + M|j_n(z, s)| \left| \frac{\partial \phi}{\partial v_1} \right| \\
\leq \left( C_2|v_2\phi(v(s, t) - p(s), v_2)| + MC_1 \left| \frac{\partial \phi}{\partial v_1} \right| \right) \alpha_n(s, E_2). \tag{4.5}
\]

Recall that $\|\phi\| = 1$ by (A4), p.4), so taking absolute values and the supremum in $v_2$ on both sides gives

\[
\left| \frac{\partial f_{n+1}}{\partial t}(x_2, v_2, s, t) \right| \leq (C_2 + MC_1) \|\phi\|\alpha_n(s, E_2) = (C_2 + MC_1) \alpha_n(s, E_2).
\]

Integrating both sides of (4.5) with respect to $v_2$ and taking absolute values gives

\[
\left| \frac{\partial \hat{f}_{n+1}}{\partial t}(x_2, s, t) \right| \leq (C_2 + MC_1) \|\phi\|\alpha_n(s, E_2) = (C_2 + MC_1) \alpha_n(s, E_2).
\]

Thus setting $E_3 = E_2$ and $C_3 = C_2 + MC_1$ finishes the proof. \qed
The next step is estimating $\partial_s f_{n+1}$, which in turn requires a bound on $\partial_s j_n$. Recall definition (2.34, p.22):

$$j_n(z, s) = \frac{1}{\gamma_2} \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau.$$

Before trying to differentiate $j_n$ with respect to $s$ we observe that all but the red $s$ are “good”, that is, we can differentiate using calculus. The red $s$ requires a more careful Lipschitz estimate. To separate the two cases, we will temporarily treat red $s$ as an independent variable:

$$j_n(z, s, \sigma) = \frac{1}{\gamma_2} \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau,$$

and similarly for $j_0^-$. We keep $j_0^0$ as it was defined in (2.35a, p.22). The original derivative is recovered by setting $\sigma = s$:

$$\frac{\partial j_n(z, s)}{\partial s} = \left[ \frac{\partial j_n(z, s, \sigma)}{\partial s} + \frac{\partial j_n(z, s, \sigma)}{\partial \sigma} \right]_{\sigma = s}.$$ (4.7)

The derivative with respect to $s$ can be computed and estimated directly:

**Theorem 4.3 (Bound on $\partial_s j_n(z, s, \sigma)$).** There exist constants $C_4$ and $E_4$ such that for all $z \in [-R, R]$, $s, \sigma \geq 0$ and $n \geq 0$ we have

$$\frac{\partial j_n(z, s, \sigma)}{\partial s} \leq C_4 \alpha_n(s, E_4).$$

**Proof.** We begin with $n = 0$ case: recall that $j_0(z, s, \sigma) = j_0^-(z, s, \sigma) + j_0^+(z, s, \sigma) + j_0^0(s)$. We have

$$\gamma_2 \frac{\partial}{\partial s} j_0^-(z, s, \sigma) = \frac{\partial}{\partial s} \int_0^s \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \, dv_2 \, d\tau$$

$$= \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} \bigg|_{\tau = s} (p(s) - v(s, s)) \phi_0(v(s, s), v_2) \, dv_2$$

$$+ \int_0^s \frac{\partial}{\partial s} \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \, dv_2 \, d\tau$$

$$= j_0^-(z, s, \sigma).$$ (4.8)
The first term (4.8) vanishes because $v(s, s) = p(s)$. The second term $I$ can be written as follows:

$$I = \int_0^s \frac{\partial}{\partial s} \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \ dv_2 \ d\tau$$

$$= - \int_0^s a_- \frac{\partial v(\tau, \sigma)}{\partial \tau} \frac{p(s) - v(\tau, s)}{s - \tau} \phi_0(v(\tau, s), a_-) \ d\tau$$

$$+ \int_0^s \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} \left( \frac{p(s) - \partial v(\tau, s)}{\partial s} \right) \phi_0(v(\tau, s), v_2) \ dv_2 \ d\tau$$

$$+ \int_0^s \int_{-\infty}^{a_-} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) \frac{\partial v(\tau, s)}{\partial s} \frac{\partial \phi_0(v(\tau, s), v_2)}{\partial v_1} \ dv_2 \ d\tau$$

$$= - I_1 + I_2 + I_3.$$
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The estimates for $\partial_s j_{0+}(z,s,\sigma)$ are similar. The last derivative of $j_0$ left to estimate is

\[
\gamma_2 \frac{\partial}{\partial s} j_0(s) = \frac{\partial}{\partial s} \int_{-\infty}^{g(s)} \int_{\mathbb{R}} (p(s) - v_1) \phi_0(v_1, v_2) \, dv_2 \, dv_1 \\
= \frac{\partial \overline{\kappa}(s)}{\partial s} \int_{\mathbb{R}} (p(s) - \kappa(s)) \phi_0(\kappa(s), v_2) \, dv_2 + \int_{-\infty}^{g(s)} \int_{\mathbb{R}} \dot{p}(s) \phi_0(v_1, v_2) \, dv_2 \, dv_1 \\
\leq M^2 \|\hat{\phi}_0\|_{\infty} + M \|\phi_0\|_1 \leq 2M^2 \|\phi_0\|.
\]

Thus in total we get

\[
\left| \frac{\partial j_0}{\partial s}(z, s, \sigma) \right| \leq 2 \left( \frac{M^2 + 2M^2 + M^3}{\gamma_2} \right) \|\phi_0\| \leq 10M^3 \|\phi_0\| \Rightarrow \alpha_0 = C_4 \alpha_0.
\]

The calculations for $n \geq 1$ are similar:

\[
\gamma_2 \frac{\partial}{\partial s^0} j_n(z, s, \sigma) = \int_{-\infty}^{\infty} \frac{\partial v(s, \sigma)}{\partial \tau} \left( p(s) - v(s, s) \right) f_n(z, v_2, s, s) \, dv_2 \\
+ \int_0^s \frac{\partial}{\partial s} \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau \\
=: I_1 + I_2.
\]

We immediately have $I_1 = 0$. For $I_2$ we have

\[
I_2 = \int_0^s \frac{\partial}{\partial s} \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \, dv_2 \, d\tau \\
= \int_0^s \frac{\partial v(\tau, \sigma)}{\partial \tau} \frac{a_- (p(s) - v(\tau, s))}{s - \tau} f_n(z, a_-, \tau, s) \, d\tau \\
- \int_0^s \frac{\partial v(\tau, \sigma)}{\partial \tau} \frac{a_+ (p(s) - v(\tau, s))}{s - \tau} f_n(z, a_+, \tau, s) \, d\tau \\
+ \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} \left( \frac{\partial}{\partial s} [p(s) - v(\tau, s)] f_n(z, v_2, \tau, s) \right) \, dv_2 \, d\tau \\
=: I_{21} - I_{22} + I_{23}.
\]
We estimate $I_{21}$ using (4.4, p.37) and Lemma 4.1 (Bound for $j_n$, p.36):

$$I_{21} \leq \int_0^s \frac{\partial v(\tau, \sigma)}{\partial \tau} \left| \frac{p(s) - v(\tau, s)}{s - \tau} \right| a_{-} f_n(z, a_{-}, \tau, s) \, d\tau$$

$$= \int_0^s \left( \frac{\partial v(\tau, \sigma)}{\partial \tau} \right)^2 j_{n-1}(z - (s - \tau)a_{-}, \tau)a_{-} \phi(v(\tau, s) - p(\tau), a_{-}) \, d\tau$$

$$\leq C_1 M^2 \|\phi\|_{L^\infty(\nu \mathrm{dv})} \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau \leq C_1 M^2 \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau.$$

The estimate for $I_{22}$ is identical. To estimate $I_{23}$ we additionally use Theorem 4.2 (Bound on $\partial_\sigma f_{n+1}$, p.39):

$$I_{23} = \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} \frac{\partial}{\partial s} \left[ (p(s) - v(\tau, s)) f_n(z, v_2, \tau, s) \right] \, dv_2 \, d\tau$$

$$= \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} \left[ \dot{p}(s) - \frac{\partial v(\tau, s)}{\partial s} \right] f_n(z, v_2, \tau, s) \, dv_2 \, d\tau$$

$$+ \int_0^s \int_{a_-}^{a_+} \frac{\partial v(\tau, \sigma)}{\partial \tau} (p(s) - v(\tau, s)) \frac{\partial f_n(z, v_2, \tau, s)}{\partial s} \, dv_2 \, d\tau$$

$$\leq \int_0^s \frac{\partial v(\tau, \sigma)}{\partial \tau} \left| \dot{p}(s) - \frac{\partial v(\tau, s)}{\partial s} \right| \dot{f}_n(z, \tau, s) \, d\tau$$

$$+ \int_0^s \frac{\partial v(\tau, \sigma)}{\partial \tau} |p(s) - v(\tau, s)| \frac{\partial f_n(z, \tau, s)}{\partial s} \, d\tau$$

$$\leq M^2 \int_0^s C_1 \alpha_{n-1}(\tau; E_1) + C_3 \alpha_{n-1}(\tau; E_3) \, d\tau \leq 2C_3 M^2 \int_0^s \alpha_{n-1}(\tau; E_3) \, d\tau.$$

Thus in total we have

$$\left| \frac{\partial}{\partial s} j_n(z, s, \sigma) \right| \leq \frac{2M^2 (C_1 + C_3)}{\gamma_2} \int_0^s \alpha_{n-1}(\tau; E_3) \, d\tau \leq \frac{4M^2 C_3}{\gamma_2} \int_0^s \alpha_{n-1}(\tau; E_4) \, d\tau = \alpha_n(s; E).$$

**Theorem 4.4 (Bound on $\partial_\sigma j_n(z, s, \sigma)$).** There exist constants $C_5$ and $E_5$ such that for all $z \in [-R, R]$, $s \geq 0$ and $n \geq 0$ we have

$$\frac{\partial j_n(z, s)}{\partial s} \leq C_5 (\alpha_{n-1}(s, E_5) + \alpha_n(s, E_5) + \alpha_{n+1}(s, E_5)) =: C_5 \beta_n(s, E_5).$$

**Proof.** We will prove the theorem by making use of definition (4.6, p.40) and equality (4.7, p.40). The derivative with respect to $s$ has already been estimated; the derivative with respect to $\sigma$ is more difficult because it is not clear that $\partial_\sigma v(\tau, \sigma)$ is differentiable in $\sigma$. To
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definition of the form $\Delta^\pm \lesssim |\sigma - \sigma'|$. We have

$$\Delta^\pm = \int_0^s \int_{-\infty}^{a_-} \left[ \frac{\partial u(\tau, \sigma)}{\partial \tau} - \frac{\partial u(\tau, \sigma')}{\partial \tau} \right] \left\{ (p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) \right\} dv_2 d\tau.$$ 

Since $\{\cdots\}$ is differentiable and $[\cdots]$ is integrable in $\tau$ for all $\sigma, \sigma'$, we can undo the product rule:

$$\Delta^\pm = \int_0^s \int_{-\infty}^{a_-} \left[ (u(\tau, \sigma) - u(\tau, \sigma'))(p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) \right] dv_2 d\tau$$

$$- \int_0^s \int_{-\infty}^{a_-} \left[ (u(\tau, \sigma) - u(\tau, \sigma'))\frac{\partial}{\partial \tau} \left\{ (p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) \right\} dv_2 d\tau$$

$$=: \Delta_1^\pm - \Delta_2^\pm.$$

For $\Delta_1$ we begin by undoing the Leibniz integral rule:

$$\Delta_1 = \int_0^s \frac{\partial}{\partial \tau} \int_{-\infty}^{a_-} \left[ (u(\tau, \sigma) - u(\tau, \sigma'))(p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) \right] dv_2 d\tau$$

$$- \int_0^s \frac{a_-}{s - \tau} \left[ (u(\tau, \sigma) - u(\tau, \sigma'))(p(s) - v(\tau, s))\phi_0(v(\tau, s), a_-) \right] d\tau$$

$$=: \Delta_1 - \Delta_1.$$

We estimate $\Delta_1^\pm$ using Lemma 2.1.d (Properties of $v(s, t)$, p.11) and Theorem 3.5 (Uniform Bound for $\hat{p}$, p.29):

$$\Delta_1^\pm = \int_0^s \frac{\partial}{\partial \tau} \int_{-\infty}^{a_-} \left[ (u(\tau, \sigma) - u(\tau, \sigma'))(p(s) - v(\tau, s))\phi_0(v(\tau, s), v_2) \right] dv_2 d\tau$$

$$= \int_{-\infty}^{a_-} \left[ (u(s, \sigma) - u(s, \sigma')) \underbrace{(p(s) - v(s, s))}_{=0} \phi_0(v(s, s), v_2) \right] dv_2$$

$$- \int_{-\infty}^{a_-} \left[ (u(0, \sigma) - u(0, \sigma'))(p(s) - v(0, s))\phi_0(v(0, s), v_2) \right] dv_2$$

$$\leq \int_{-\infty}^{\infty} ||\hat{p}||_{\infty} |\sigma - \sigma'| ||p(s)||\phi_0(v(0, s), v_2) dv_2$$

$$\leq M^2 |\sigma - \sigma'| ||\phi_0||_{\infty} \leq M^2 ||\phi_0||||\sigma - \sigma'|.$$
For $\Delta_{12}$ we again use (4.4, p.37):

$$\Delta_{12} = \int_0^s (v(\tau, \sigma) - v(\tau, \sigma')) \left( \frac{p(s) - v(\tau, \sigma)}{s - \tau} \right) a_0(v(\tau, s), a_0) \, d\tau$$

$$\leq \int_0^s \left\| \hat{p}(t) \right\| |\sigma - \sigma'| \left| \frac{\partial v(\tau, s)}{\partial s} \right| a_0(v(\tau, s), a_0) \, d\tau$$

$$\leq M^2 |\sigma - \sigma'| \|\phi_0(\cdot)\|_\infty \int_0^s \, d\tau \leq M^2 \|\phi_0\| |\sigma - \sigma'|.$$ 

For $\Delta_{22}$ we have

$$\Delta_{22} = \int_0^s \int_{-\infty}^{a_0} \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] \frac{\partial}{\partial \tau} \left\{ (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \right\} \, dv_2 \, d\tau$$

$$\leq \left\| \hat{p}(t) \right\| |\sigma - \sigma'| \int_0^s \int_{-\infty}^{a_0} \frac{\partial}{\partial \tau} \left\{ (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \right\} \, dv_2 \, d\tau,$$

where

$$I = \int_{-\infty}^{a_0} \frac{\partial}{\partial \tau} \left\{ (p(s) - v(\tau, s)) \phi_0(v(\tau, s), v_2) \right\} \, dv_2$$

$$= \int_{-\infty}^{a_0} \left[ \frac{\partial v(\tau, s)}{\partial \tau} \right] \phi_0(\cdot) + (p(s) - v(\tau, s)) \frac{\partial (v(\tau, s))}{\partial v_1} \, dv_2 \, d\tau$$

$$\leq 2M^2 \int_{-\infty}^{a_0} \left( \phi_0(v(\tau, s), v_2) + \frac{\partial \phi_0}{\partial v_1}(v(\tau, s), v_2) \right) \, dv_2 \, d\tau \leq 2M^2 \|\phi_0\|.$$ 

Hence we estimate $\Delta_{22} \leq 2M^3 |\phi_0||\sigma - \sigma'|s.$ 

We now have Lipschitz estimates on all parts of $j_{0-}$, so we may conclude that it is Lipschitz continuous. The estimates for $\partial_{\sigma}j_{0+}(z, s, \sigma)$ are similar. Since $j_{00}$ does not depend on $\sigma$ we may conclude that $j_0(z, s, \sigma)$ is almost everywhere differentiable with respect to $\sigma$ and that

$$\left| \frac{\partial j_0}{\partial \sigma}(z, s, \sigma) \right| \leq \lim_{\sigma \to \sigma'} \frac{2|\Delta^-|}{|\sigma - \sigma'|} \leq \lim_{\sigma \to \sigma'} \frac{2|\Delta_{11} - \Delta_{12} - \Delta_{22}|}{|\sigma - \sigma'|}$$

$$\leq \frac{2M^2 + 2M^2s + 4M^3s}{\gamma_2} \|\phi_0\| \leq \frac{2M^2 \|\phi_0\|}{\gamma_2} + \frac{6M^3 \|\phi_0\|}{\gamma_2} s.$$
Combining the estimate above with the base case for Theorem 4.3 (Bound on $\partial_s j_n(z,s,\sigma)$, p.40) gives

$$\left| \frac{\partial j_0}{\partial s}(z,s) \right| \leq \left( 2M^2\|\phi_0\| + C_4 \right) a_0 + \frac{6M^3\|\phi_0\|}{\gamma_2} s.$$  

We now proceed by induction. For $n \geq 1$ let

$$\Delta^n = \gamma_2 \left( j_n(z,s,\sigma) - j_n(z,s,\sigma') \right).$$

Undoing the product rule gives

$$\Delta^n = \int_0^s \int_{a_-}^{a_+} \left[ \frac{\partial v(\tau,\sigma)}{\partial \tau} - \frac{\partial v(\tau,\sigma')}{\partial \tau} \right] \left\{ (p(s) - v(\tau,s)) f_n(z,v_2,\tau,s) \right\} dv_2 d\tau$$

$$= \int_0^s \int_{a_-}^{a_+} \frac{\partial}{\partial \tau} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \left\{ (p(s) - v(\tau,s)) f_n(z,v_2,\tau,s) \right\} dv_2 d\tau$$

$$- \int_0^s \int_{a_-}^{a_+} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \frac{\partial}{\partial \tau} \left\{ (p(s) - v(\tau,s)) f_n(z,v_2,\tau,s) \right\} dv_2 d\tau$$

$$= \Delta^n_1 - \Delta^n_2.$$

To estimate $\Delta^n_1$ we first undo the Leibniz integral rule:

$$\Delta^n_1 = \int_0^s \int_{a_-}^{a_+} \frac{\partial}{\partial \tau} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \left\{ (p(s) - v(\tau,s)) f_n(z,v_2,\tau,s) \right\} dv_2 d\tau$$

$$= \int_0^s \frac{\partial}{\partial \tau} \int_{a_-}^{a_+} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \left\{ (p(s) - v(\tau,s)) f_n(z,v_2,\tau,s) \right\} dv_2 d\tau$$

$$- \int_0^s \frac{a_+}{s-\tau} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \left\{ (p(s) - v(\tau,s)) f_n(z,a_+,\tau,s) \right\} d\tau$$

$$+ \int_0^s \frac{a_-}{s-\tau} \left[ v(\tau,\sigma) - v(\tau,\sigma') \right] \left\{ (p(s) - v(\tau,s)) f_n(z,a_-,\tau,s) \right\} d\tau$$

$$= \Delta^n_{11} - \Delta^n_{1+} + \Delta^n_{1-}.$$
To estimate $\Delta_{1+}^n$ we use (4.4, p.37) and Lemma 4.1 (Bound for $f_n$, p.36):

$$\Delta_{1+}^n = \int_0^s \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] \frac{(p(s) - v(\tau, s))}{s - \tau} a_{\pm} f_n(z, a_{\pm}, \tau, s) \, d\tau$$

$$\leq \int_0^s \|\hat{p}\|_{\infty} |\sigma - \sigma'| \left| \frac{\partial v(\tau, s)}{\partial s} \right| a_{\pm} f_n(z, a_{\pm}, \tau, s) \, d\tau$$

$$\leq M^2 |\sigma - \sigma'| \int_0^s \alpha_{n-1}(\tau, E_1) \|\phi\|_{L^\infty(v \, dv)} \, d\tau \leq C_1 \gamma_2 \alpha_n(s, E_1) |\sigma - \sigma'|.$$

For $\Delta_{11}^n$ we have

$$\Delta_{11}^n = \int_0^s \frac{\partial}{\partial \tau} \int_{a_-}^{a_+} \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] \left\{ \left( p(s) - v(\tau, s) \right) f_n(z, v_2, \tau, s) \right\} \, dv_2 \, d\tau$$

$$= \int_{-\infty}^{\infty} \left[ v(s, \sigma) - v(s, \sigma') \right] \left\{ \left( p(s) - v(s, s) \right) f_n(z, v_2, s, s) \right\} \, dv_2$$

$$- \int_{(z-R)/s}^{(z+R)/s} \left[ v(0, \sigma) - v(0, \sigma') \right] \left\{ (p(s) - v(0, s)) f_n(z, v_2, 0, s) \right\} \, dv_2$$

$$\leq \int_\mathbb{R} \|\hat{p}\|_{\infty} |\sigma - \sigma'| (p(s) f_n(z, v_2, 0, s) \, dv_2$$

$$\leq M^2 |\sigma - \sigma'| \hat{f}_n(z, 0, s) \leq C_1 \alpha_{n-1}(0, E_1) |\sigma - \sigma'| \leq C_1 \alpha_{n-1}(s, E_1) |\sigma - \sigma'|.$$

In total we get

$$\Delta_1^n \leq C_1 \alpha_{n-1}(s, E_1) |\sigma - \sigma'| + 2C_1 \gamma_2 \alpha_n(s, E_1) |\sigma - \sigma'|$$

$$\leq C'_1 \gamma_2 \left( \alpha_{n-1}(s, E_1) + \alpha_n(s, E_1) \right) |\sigma - \sigma'|,$$

where $C'_1 = C_1 / \gamma_2 + 2C_1$. For $\Delta_2^n$ we have

$$\Delta_2^n = \int_0^s \int_{a_-}^{a_+} \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] \frac{\partial}{\partial \tau} \left\{ \left( p(s) - v(\tau, s) \right) f_n(z, v_2, \tau, s) \right\} \, dv_2 \, d\tau$$

$$= \int_0^s \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] (p(s) - v(\tau, s)) \int_{a_-}^{a_+} \frac{\partial f_n(z, v_2, \tau, s)}{\partial \tau} \, dv_2 \, d\tau$$

$$- \int_0^s \left[ v(\tau, \sigma) - v(\tau, \sigma') \right] \frac{\partial v(\tau, s)}{\partial \tau} \int_{a_-}^{a_+} f_n(z, v_2, \tau, s) \, dv_2 \, d\tau$$

$$=: \Delta_{21}^n - \Delta_{22}^n.$$
The estimate for $\Delta_{22}^n$ follows readily from Theorem 4.1 (Bound for $f_n$, p. 37):

$$\Delta_{22}^n \leq \|p\|_\infty^2 |\sigma - \sigma'| \int_0^s \hat{f}_n(z, \tau, s) \, d\tau \leq C_1 M^2 |\sigma - \sigma'| \int_0^s \alpha_{n-1}(\tau, E_1) \, d\tau. \quad (4.9)$$

To estimate $\Delta_{21}^n$ we use the induction hypothesis and Lemma 4.2 (Bound on $\partial_z j_n(z, s)$, p. 37) to write

$$\frac{\partial f_n}{\partial \tau}(z, v_2, \tau, s) = \frac{\partial}{\partial \tau} \left( j_{n-1}(z - (s - \tau) v_2, \tau) \phi(v(\tau, s) - p(\tau), v_2) \right) \quad (4.10)$$

$$= \frac{\partial j_{n-1}}{\partial z}(z - (s - \tau) v_2, \tau) v_2 \phi(v(\tau, s) - p(\tau), v_2)$$

$$+ \frac{\partial j_{n-1}}{\partial \tau}(z - (s - \tau) v_2, \tau) \phi(v(\tau, s) - p(\tau), v_2)$$

$$+ j_{n-1}(z - (s - \tau) v_2, \tau) \left( \frac{\partial v(\tau, s)}{\partial \tau} - \hat{p}(\tau) \right) \frac{\partial \phi}{\partial v_1}(v(\tau, s) - p(\tau), v_2)$$

$$\leq C_2 \alpha_n(\tau; E_2) v_2 \phi(\ldots) + C_5 \beta_{n-1}(\tau; E_5) \phi(\ldots) + 2MC_1 \alpha_{n-1}(\tau; E_1) \left| \frac{\partial \phi}{\partial v_1}(\ldots) \right|$$

$$\leq 2C_5 \beta_{n-1}(\tau; E_5) \left[ v_2 \phi(\ldots) + \phi(\ldots) + \left| \frac{\partial \phi}{\partial v_1}(\ldots) \right| \right].$$

Integrating both sides gives

$$\int_{a^-}^{a^+} \frac{\partial f_n}{\partial \tau}(z, v_2, \tau, s) \, dv_2 \leq 2C_5 \beta_{n-1}(\tau, E_5) \int_{a^-}^{a^+} v_2 \phi(\ldots) + \phi(\ldots) + \left| \frac{\partial \phi}{\partial v_1}(\ldots) \right| \, dv_2$$

$$\leq 2C_5 \beta_{n-1}(\tau, E_5) \|\phi\| = 2C_5 \beta_{n-1}(\tau, E_5).$$

The estimate for $\Delta_{21}^n$ readily follows:

$$\Delta_{21}^n \leq 2C_5 \int_0^s \left[ \varphi(\tau, \sigma) - \varphi(\tau, \sigma') \right] \left( p(s) - v(\tau, s) \right) \beta_{n-1}(\tau, E_5) \, d\tau$$

$$\leq 4C_5 M^2 |\sigma - \sigma'| \int_0^s \beta_{n-1}(\tau, E_5) \, d\tau. \quad (4.11)$$

Finally, combining (4.9, p. 48) with the bound above gives

$$\Delta_2^n \leq \left( 4C_5 M^2 \int_0^s \beta_{n-1}(\tau, E_5) \, d\tau + C_1 M^2 \int_0^s \alpha_{n-1}(\tau, E_1) \, d\tau \right) |\sigma - \sigma'|$$

$$\leq 5C_5 M^2 \left( \int_0^s \beta_{n-1}(\tau, E_5) \, d\tau \right) |\sigma - \sigma'| = \frac{C_5 \gamma_2}{4} \beta_n(s, E_5) |\sigma - \sigma'|,$$
where we have chosen $E_5 = 20M^2/\gamma_2$. We can now finish the estimate:

$$\frac{\Delta^n}{|\sigma - \sigma'|} \leq C_1 \gamma_2 \left( \alpha_{n-1}(s, E_1) + \alpha_n(s, E_1) \right) + \frac{C_5 \gamma_2}{4} \beta_n(s, E_5) \leq \frac{C_5 \gamma_2}{2} \beta_n(s, E_5),$$

where $C_5$ was chosen large enough to satisfy inequality above and (4, p. 46). Equivalently,

$$j_n(z, s, \sigma) - j_n(z, s, \sigma') = \frac{\Delta^n}{\gamma_2} \leq \frac{C_5}{2} \beta_n(s, E_5) |\sigma - \sigma'|,$$

so $j_n(z, s, \sigma)$ is indeed Lipschitz in $\sigma$. Therefore, it is almost everywhere differentiable with

$$\left| \frac{\partial j_n(z, s, \sigma)}{\partial \sigma} \right| \leq \frac{C_5}{2} \beta_n(s, E_5).$$

Combining the bound above with Theorem 4.3 (Bound on $\partial_k j_n(z, s, \sigma)$, p. 40) gives the desired result.

\[
\text{Corollary 4.1 (Bound on $\partial_s f_n$). For all } x_2 \in [-R, R], n \geq 1 \text{ and } 0 \leq s \leq t \text{ we have}
\]

$$\frac{\partial j_n}{\partial s}(x_2, s, t) \leq 2C_5 \beta_{n-1}(s; E_5),$$

\[
\text{Proof. The result is obtained by using equality (4.10) and increasing } C_5 \text{ if needed.}
\]

\[
\text{Lemma 4.3 (Smoothness of } G_n) \text{. Suppose } p \in W^{1,\infty}(0,T) \text{ is a solution for (1.1, p. 2). Then}
\]

the drag forces due to recollisions $G_n(t; p)$ are Lipschitz in $t$, and there exist constants $C'_5$ and $E'_5$ such that for all $n \geq 0$

$$\left| \frac{d}{dt} G_n(t; p) \right| \leq C'_5 \beta_n(t; E'_5). \quad (4.12)$$

\[
\text{Proof. Since the outer integral in formula (2.31, p. 21) is over a fixed domain, we might as}
\]

well work with the pointwise drag functions:

$$G_n(x_2, t; p) = \int_0^t \int_{a^-}^{a^+} \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^2 + \frac{\beta_2}{\gamma_2} (p(t) - v(s, t)) \right] f_n(x_2, v_2, s, t) dv_2 ds.$$

The only difference between the expression above and (2.34, p. 22) is the quadratic term $(p(t) - v(s, t))^2$, which is exactly as smooth as $(p(t) - v(s, t))$. This means we could repeat all calculations in this section with $G_n(x_2, t; p)$ instead of $j_n(z, s)$, and obtain a similar bound (4.12).

\[
\text{Now Theorem 1.1.c (Regularity, p. 7) is a consequence of Lemma 4.3 and assumption (A3).}
\]
4.2 Lipschitz Bounds for \( f_n \)

Let
\[
\{ p, \eta^{(p)}_n, \{ f_n^{(p)} \}_{n=0}^\infty \} \quad \text{and} \quad \{ q, \eta^{(q)}_n, \{ f_n^{(q)} \}_{n=0}^\infty \}
\]
be two solutions to the system (1.1). This section is devoted to obtaining a Lipschitz estimate for \( f_n^{(c)} \), which will be used later to obtain a Lipschitz estimate for the drag force.

Recall definition (2.36, p. 22): \( f_n^{(p)}(x_2, v_2, s, t) = j_n^{(p)}(x_2 - (t - s)v_2, s))\phi(v_p(s, t) - p(s), v_2). \)

**Theorem 4.5** (Lipschitz Bound for \( f_n \)). There exist constants \( C_6 \) and \( E_6 \) such that for all \( x_2 \in [-R, R] \), \( 0 \leq s \leq t \) and \( n \geq 0 \) we have
\[
\left| j_n^{(p)}(x_2, s, t) - j_n^{(q)}(x_2, s, t) \right| \leq \| p - q \| C_6 \beta_n(s; E_6).
\]

**Proof.** Let \( z = x_2 - (t - s)v_2 \), then the difference is:
\[
f_{n+1}^{(p)}(x_2, v_2, s, t) - f_{n+1}^{(q)}(x_2, v_2, s, t) = j_n^{(p)}(z, s) - j_n^{(q)}(z, s) \phi(v_p(s, t) - p(s)) \]
\[
= \Delta_j + j_n^{(q)}(z, s) \left( \phi(v_p(s, t) - p(s)) - \phi(v_q(s, t) - q(s)) \right).
\]

\( \Delta \phi \) is easy to estimate:
\[
\Delta \phi \leq C_1 \alpha_n(s; E_1) \| \nabla \phi \|_\infty \left( \left( v_p(s, t) - p(s) \right) - \left( v_q(s, t) - q(s) \right) \right)
\]
\[
\leq 2C_1 \alpha_n(s; E_1) \| p - q \|.
\]

(4.13)

For the base case \( n = 0 \) we use formulas (2.35, p. 22) and (4.3, p. 37) to write
\[
\Delta_j = j_0^{(p)}(z, s) - j_0^{(q)}(z, s) = \left[ j_0^{(p)}(z, s) - j_0^{(q)}(z, s) \right] + \left[ j_0^{(p)}(z, s) - j_0^{(q)}(z, s) \right].
\]

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The difference in the $\gamma_0$ terms can be estimated as follows:

$$
\gamma_2 \Delta_{\gamma_0} = \int_{-\infty}^{\infty} (p(s) - v_1) \hat{\phi}_0(v_1) \, dv_1 - \int_{-\infty}^{\infty} (q(s) - v_1) \hat{\phi}_0(v_1) \, dv_1 \tag{4.14a}
$$

$$
\leq \int_{-\infty}^{\infty} (p(s) - v_1) \hat{\phi}_0(v_1) \, dv_1 + \int_{-\infty}^{\infty} |q(s) - p(s)| \hat{\phi}_0(v_1) \, dv_1 \tag{4.14b}
$$

$$
\leq |\mathcal{E}_p(s) - \mathcal{E}_q(s)|(1 + M)\|\phi_0\| + |p(s) - q(s)|\|\phi_0\| \tag{4.14c}
$$

$$
\leq (2 + M)\|\phi_0\|\|p - q\|_{\infty}. \tag{4.14d}
$$

To estimate the difference in the $\gamma_{\perp}$ term we consider the $\gamma_{\perp}$ case; by definition (2.35, p.22) we have

$$
\gamma_2 \Delta_{\gamma_{\perp}} = \int_{0}^{s} \int_{a_{+}}^{\infty} \frac{\partial v_p(\tau, s)}{\partial \tau} (p(s) - v_p(\tau, s)) \hat{\phi}_0(v_p(\tau, s), v_2) \, dv_2 \, d\tau
$$

$$
- \int_{0}^{s} \int_{a_{+}}^{\infty} \frac{\partial v_q(\tau, s)}{\partial \tau} (q(s) - v_q(\tau, s)) \hat{\phi}_0(v_q(\tau, s), v_2) \, dv_2 \, d\tau
$$

$$
= \int_{0}^{s} \int_{a_{+}}^{\infty} \frac{\partial}{\partial \tau} \left[ v_p(\tau, s) - v_q(\tau, s) \right] (q(s) - v_q(\tau, s)) \hat{\phi}_0(v_q(\tau, s), v_2) \, dv_2 \, d\tau
$$

$$
+ \int_{0}^{s} \int_{a_{+}}^{\infty} \frac{\partial v_p(\tau, s)}{\partial \tau} \left[ (p(s) - v_p(\tau, s)) - (q(s) - v_q(\tau, s)) \right] \phi_0(v_q(\tau, s), v_2) \, dv_2 \, d\tau
$$

$$
+ \int_{0}^{s} \int_{a_{+}}^{\infty} \frac{\partial v_p(\tau, s)}{\partial \tau} (p(s) - v_p(\tau, s)) \left[ \phi_0(v_q(\tau, s), v_2) - \phi_0(v_p(\tau, s), v_2) \right] \, dv_2 \, d\tau
$$

$$
= \Delta_{0+}^{1} + \Delta_{0+}^{2} + \Delta_{0+}^{3}.
$$

To estimate $\Delta_{0+}^{1}$ we undo the Leibniz integral rule:

$$
\Delta_{0+}^{1} = \int_{0}^{s} \frac{\partial}{\partial \tau} \int_{a_{+}}^{\infty} \frac{\partial}{\partial \tau} \left[ v_p(\tau, s) - v_q(\tau, s) \right] (q(s) - v_q(\tau, s)) \phi_0(v_q(\tau, s), v_2) \, dv_2 \, d\tau
$$

$$
+ \int_{0}^{s} \frac{a_{+}}{s - \tau} \left[ v_p(\tau, s) - v_q(\tau, s) \right] (q(s) - v_q(\tau, s)) \phi_0(v_q(\tau, s), a_{+}) \, d\tau
$$

$$
= \Delta_{0+}^{11} + \Delta_{0+}^{12}.
$$
We can simplify $\Delta_{1+}^{11}$ by using Fundamental Theorem of Calculus:

$$|\Delta_{0+}^{11}| = \left| \int_{a_+|_s=0}^\infty \left[ v_p(s, s) - v_q(s, s) \right] \left( q(s) - v_q(s, s) \right) \phi_0(v_q(s, s), v_2) \, dv_2 \right. \\
- \left. \int_{(z+R)/s}^\infty \left[ v_p(0, s) - v_q(0, s) \right] \left( q(s) - v_q(0, s) \right) \phi_0(v_q(0, s), v_2) \, dv_2 \right|$$

$$\leq 2M\|p - q\|_{\infty} \int R \phi_0(v_q(0, s), v_2) \, dv_2 \leq 2M\|p - q\|_{\infty}\|\phi_0\|.$$

For $\Delta_{0+}^{12}$ we have

$$|\Delta_{0+}^{12}| = \left| \int_0^s \left[ v_p(\tau, s) - v_q(\tau, s) \right] \left( q(s) - v_q(\tau, s) \right) a_+ \phi_0(v_q(\tau, s), a_+) \, d\tau \right|$$

$$\leq \|p - q\|_{\infty} \int_0^s \frac{\partial v_q(\tau, s)}{\partial s} \left| a_+ \phi_0(v_q(\tau, s), a_+) \right| \, d\tau \leq M\|\phi_0\||p - q\|_{\infty},$$

and thus $|\Delta_{0+}^{1+}| \leq 3M\|\phi_0\||p - q\|_{\infty}$. The estimates for $\Delta_{0+}^{2+}$ and $\Delta_{0+}^{3+}$ are easier:

$$|\Delta_{0+}^{2+}| \leq 2\|p - q\|_{\infty} \int_0^s \frac{\partial v_q(\tau, s)}{\partial s} \phi_0(v_q(\tau, s), v_2) \, dv_2 \, d\tau \leq 2M\|p - q\|_{\infty}\|\phi_0\|,$$

$$|\Delta_{0+}^{3+}| \leq 2M \int_0^s \frac{\partial v_p(\tau, s)}{\partial s} \left\| \nabla \phi_0 \right\|_{\infty} \left| v_p(\tau, s) - v_q(\tau, s) \right| \, dv_2 \, d\tau \leq 2M^2\|p - q\|_{\infty}\|\phi_0\|.$$
For $n \geq 1$ we write
\[
\gamma_2 \Delta_j = \gamma_2 \left( j_n^{(p)}(z, s) - j_n^{(q)}(z, s) \right)
\]
\[
= \int_0^s \int_{a_-}^{a_+} \frac{\partial v_p}{\partial \tau}(\tau, s) \left( p(s) - v_p(\tau, s) \right) f_n^{(p)}(z, v_2, \tau, s) \, dv_2 \, d\tau
\]
\[
- \int_0^s \int_{a_-}^{a_+} \frac{\partial v_q}{\partial \tau}(\tau, s) \left( q(s) - v_q(\tau, s) \right) f_n^{(q)}(z, v_2, \tau, s) \, dv_2 \, d\tau
\]
\[
= \int_0^s \int_{a_-}^{a_+} \left[ \frac{\partial v_p}{\partial \tau}(\tau, s) - \frac{\partial v_q}{\partial \tau}(\tau, s) \right] \left\{ \left( q(s) - v_q(\tau, s) \right) f_n^{(q)}(z, v_2, \tau, s) \right\} \, dv_2 \, d\tau
\]
(4.15)
\[
+ \int_0^s \int_{a_-}^{a_+} \frac{\partial v_p}{\partial \tau}(\tau, s) \left[ \left( p(s) - v_p(\tau, s) \right) - \left( q(s) - v_q(\tau, s) \right) \right] f_n^{(q)}(z, v_2, \tau, s) \, dv_2 \, d\tau
\]
\[
+ \int_0^s \int_{a_-}^{a_+} \frac{\partial v_q}{\partial \tau}(\tau, s) \left( p(s) - v_p(\tau, s) \right) \left[ f_n^{(p)}(z, v_2, \tau, s) - f_n^{(q)}(z, v_2, \tau, s) \right] \, dv_2 \, d\tau
\]
\[
= \Delta_j^1 + \Delta_j^2 + \Delta_j^3.
\]

**Remark 4.1 (Difference in Derivatives).** The term $\Delta_j^1$ above is the most difficult one to estimate: the difference of derivatives
\[
\left[ \frac{\partial v_p}{\partial \tau} - \frac{\partial v_q}{\partial \tau} \right]
\]
contained in it does not obey the needed Lipschitz estimate by itself. Luckily, it is sitting inside an integral in (4.15, p. 53), so we would expect integration and differentiation to “cancel out”. We cannot apply the FTC directly because of the $\{ \cdots \}$ term, but we can try to integrate by parts to shift the derivative from $(\nu_p - \nu_q)$ to $\{ \cdots \}$. The need to differentiate $\{ \cdots \}$ is the reason we have spent so much time bounding derivatives in subsection 4.1.2.
The actual calculation calls for intertwining integration by parts with reversed Leibniz integral rule. First we use the product rule to write

\[
\Delta^1_1 = \int_0^s \int_{a_-}^{a_+} \frac{\partial}{\partial \tau} \left[ \left( v_p(\tau, s) - v_p(\tau, s) \right) (q(s) - v_q(\tau, s)) f_n(q)(z, v_2, \tau, s) \right] dv_2 d\tau
\]

\[
= \int_0^s \int_{a_-}^{a_+} \frac{\partial}{\partial \tau} \left[ \left( v_p(\tau, s) - v_p(\tau, s) \right) (q(s) - v_q(\tau, s)) f_n(q)(z, v_2, \tau, s) \right] dv_2 d\tau
\]

\[
- \int_0^s \int_{a_-}^{a_+} \frac{\partial}{\partial \tau} \left( (q(s) - v_q(\tau, s)) f_n(q)(z, v_2, \tau, s) \right) dv_2 d\tau
\]

\[
=: I_1 - I_2.
\]

The hard-earned Corollary 4.1 (*Bound on \( \partial_s f_n, p.49 \)) is used to estimate \( I_2 \):

\[
I_2 = \int_0^s \int_{a_-}^{a_+} \left[ v_p(\tau, s) - v_p(\tau, s) \right] (q(s) - v_q(\tau, s)) \frac{\partial f_n(q)}{\partial \tau}(z, v_2, \tau, s) dv_2 d\tau
\]

\[
- \int_0^s \int_{a_-}^{a_+} \left[ v_p(\tau, s) - v_p(\tau, s) \right] \frac{\partial v_q(\tau, s)}{\partial \tau} f_n(q)(z, v_2, \tau, s) dv_2 d\tau
\]

\[
\leq 2M \int_0^s \left[ v_p(\tau, s) - v_p(\tau, s) \right] \frac{\partial f_n(q)}{\partial \tau}(z, v_2, \tau, s) d\tau
\]

\[
+ M \int_0^s \left[ v_p(\tau, s) - v_p(\tau, s) \right] \hat{f}_n(q)(z, v_2, \tau, s) d\tau
\]

\[
\leq 4C_5 M \| q - q \| \int_0^s \beta_{n-1}(\tau; E_5) d\tau + C_1 M \| q - q \| \int_0^s \alpha_{n-1}(\tau; E_1) d\tau
\]

\[
\leq 5C_5 M \| q - q \| \int_0^s \beta_{n-1}(\tau; E_5) d\tau.
\]
To estimate $I_1$ we first undo the Leibniz integral rule:

$$I_1 = \int_0^s \frac{\partial}{\partial \tau} \int_{a_+}^{a_-} (u_p(\tau, s) - u_p(\tau, s)) (q(s) - v_q(\tau, s)) f_n^{(q)}(z, v_2, \tau, s) \, dv_2 \, d\tau$$

$$- \int_0^s \frac{a_+}{s - \tau} (u_p(\tau, s) - u_p(\tau, s)) (q(s) - v_q(\tau, s)) f_n^{(q)}(z, a_+, \tau, s) \, d\tau$$

$$+ \int_0^s \frac{a_-}{s - \tau} (u_p(\tau, s) - u_p(\tau, s)) (q(s) - v_q(\tau, s)) f_n^{(q)}(z, a_-, \tau, s) \, d\tau$$

$$= I_{11} - I_{1+} + I_{1-}.$$

Now $I_{11}$ falls victim to the Fundamental Theorem of Calculus:

$$|I_{11}| = \left| \int_{-\infty}^{\infty} (u_p(s, s) - u_p(s, s)) (q(s) - v_q(s, s)) f_n^{(q)}(z, v_2, s, s) \, dv_2 \right|_{=0}$$

$$- \int_{(z - R)/s}^{(z + R)/s} (u_p(0, s) - u_p(0, s)) (q(s) - v_q(0, s)) f_n^{(q)}(z, v_2, 0, s) \, dv_2$$

$$\leq \int_R |u_p(0, s) - u_p(0, s)| |q(s) - v_q(0, s)| f_n^{(q)}(z, v_2, 0, s) \, dv_2$$

$$\leq 2M \|p - q\| f_n^{(q)}(z, 0, s) \leq 2MC_1 \|p - q\| \alpha_{n-1}(s; E_1).$$

To estimate $I_\pm$ we use (4.4, p.37):

$$I_\pm = \int_0^s (u_p(\tau, s) - u_p(\tau, s)) \left( \frac{q(s) - v_q(\tau, s)}{s - \tau} \right) a_\pm f_n^{(q)}(z, a_\pm, \tau, s) \, d\tau$$

$$\leq \int_0^s (u_p(\tau, s) - u_p(\tau, s)) \left| \frac{\partial v_q}{\partial s} \right| a_\pm f_n^{(q)}(z, a_\pm, \tau, s) \, d\tau$$

$$\leq MC_1 \|p - q\| \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau.$$

Combining estimates (4.16), (4.17) and (4.18) gives

$$\Delta_j^1 \leq \left( 2MC_1 \alpha_{n-1}(s; E_1) + 2MC_1 \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau + 5C_5M \int_0^s \beta_{n-1}(\tau; E_5) \, d\tau \right) \|p - q\|$$

$$\leq 6C_5M \|p - q\| \int_0^s \beta_{n-1}(\tau; E_5) \, d\tau.$$
The term $\Delta_2^j$ is the easiest to estimate:

$$
\Delta_2^j = \int_0^s \frac{\partial v_p}{\partial \tau}(\tau, s) \left[ (p(s) - v_p(\tau, s)) - (q(s) - v_q(\tau, s)) \right] \int_{a_{-}}^{a_{+}} f_n(q)(z, v_2, \tau, s) \, dv_2 \, d\tau
$$

$$
\leq 2 \|p - q\| \int_0^s \frac{\partial v_p}{\partial \tau}(\tau, s) \int_{a_{-}}^{a_{+}} f_n(q)(z, \tau, s) \, dv_2 \, d\tau
\leq 2 \|p - q\| MC_1 \int_0^s \alpha_{n-1}(\tau; E_1) \, d\tau. \quad (4.19)
$$

The estimate for $\Delta_3^j$ is actually recursive:

$$
\Delta_3^j = \int_0^s \frac{\partial v_p}{\partial \tau}(\tau, s) \left[ (p(s) - v_p(\tau, s)) - (q(s) - v_q(\tau, s)) \right] \int_{a_{-}}^{a_{+}} \left[ f_n^{(p)}(z, v_2, \tau, s) - f_n^{(q)}(z, v_2, \tau, s) \right] \, dv_2 \, d\tau
$$

$$
\leq M^2 \int_0^s \left| \hat{f}_n^{(p)}(z, \tau, s) - \hat{f}_n^{(q)}(z, \tau, s) \right| \, d\tau \leq \|p - q\| M^2 C_6 \int_0^s \beta_{n-1}(\tau; E_6) \, d\tau \quad (4.20)
$$

Adding estimates (4.19, p.56) and (4.20, p.56) to the bound above gives

$$
\Delta_j \leq \frac{\|p - q\|}{\gamma_2} \int_0^s \left( 6C_5 M \beta_{n-1}(\tau; E_5) + 2MC_1 \alpha_{n-1}(\tau; E_1) + M^2 C_6 \beta_{n-1}(\tau; E_6) \right) \, d\tau
$$

$$
\leq \|p - q\| \frac{2M^2 C_6}{\gamma_2} \int_0^s \beta_{n-1}(\tau; E_6) \, d\tau = \|p - q\| C_6 \beta_n(s; E_6),
$$

where we have chosen $E_6 = 4M^2 / \gamma_2$. We can now finish the estimate:

$$
\left| \hat{f}_{n+1}^{(p)}(x_2, s, t) - \hat{f}_{n+1}^{(q)}(x_2, s, t) \right| \leq \|p - q\| \frac{C_6}{2} \beta_n(s; E_6) + 2C_1 \alpha_n(s; E_1) \|p - q\|
$$

$$
\leq \|p - q\| C_6 \beta_n(s; E_6)
$$

for large enough $C_6$. \qed
4.3 Lipschitz Bounds for the Drag Force

Theorem 4.6. There exists a constant $L_0$ such that for all $t \in [0, T]$,

$$|G_0(t;p) - G_0(t;q)| \leq L_0 \|p - q\|_\infty.$$  

The proof is virtually identical to that for the base case of Theorem 4.5 (Lipschitz Bound for $f_n$, p.50).

Theorem 4.7 (Lipschitz Bound for $G_{\text{rec}}$). There exist constants $C_7$ and $E_7$ such that for all $t \in [0, T]$,

$$|G_{\text{rec}}(t;p) - G_{\text{rec}}(t;q)| \leq C_7 e^{E_7 t} \|p - q\|_\infty.$$  

Proof. Most of the proof is very similar to that of Theorem 4.5 (Lipschitz Bound for $f_n$, p.50), except that we do not even need to use induction anymore. By repeating what we have done for Theorem 4.5 (Lipschitz Bound for $f_n$, p.50) we can show that

$$|G_n(t;p) - G_n(t;q)| \leq C_7 \gamma_2 \frac{\beta_n(t; E_7)}{3} \|p - q\|_\infty$$

for large enough $C_7$ and $E_7 = E_6$. To obtain the desired result we need to bound the series:

$$|G_{\text{rec}}(t;p) - G_{\text{rec}}(t;q)| \leq \sum_{n=1}^{\infty} |G_n(t;p) - G_n(t;q)|$$

$$\leq C_7 \frac{\gamma_2}{3} \|p - q\|_\infty \sum_{n=1}^{\infty} \beta_n(t; E_7)$$

$$\leq C_7 \frac{\gamma_2}{3} \|p - q\|_\infty \sum_{n=1}^{\infty} \left( \alpha_{n-1}(t; E_7) + \alpha_n(t; E_7) + \alpha_{n+1}(t; E_7) \right)$$

$$\leq C_7 \gamma_2 \|p - q\|_\infty \sum_{n=0}^{\infty} \alpha_n(t; E_7)$$

$$= C_7 \|p - q\|_\infty \sum_{n=0}^{\infty} \frac{(E_7 t)^n}{n!} = C_7 e^{E_7 t} \|p - q\|_\infty.$$
4.3.1 Uniqueness

We now have all the ingredients required to prove Theorem 1.1.b (Uniqueness, p. 7):

Proof. Suppose \( p \) and \( q \) are both solutions to (1.1). Then using (3.6, p. 24) we write

\[
\frac{d}{d\tau} |p(\tau) - q(\tau)| = |\dot{p}(\tau) - \dot{q}(\tau)| \text{sgn}(p(\tau) - q(\tau))
\]

\[
\leq |F(\eta_p(\tau), \tau) - F(\eta_q(\tau), \tau)| + |G_0(\tau; p) - G_0(\tau; q)| + |G_{rec}(\tau; p) - G_{rec}(\tau; q)|
\]

\[
\leq \left( T \text{Lip}_F + L_0 + C_7 e^{F_7 T} \right) \|p - q\|_{L^\infty(0,\tau)}
\]

Integrating both sides from 0 to \( s \) gives

\[
|p(s) - q(s)| \leq L \int_0^s \|p - q\|_{L^\infty(0,\tau)} \ d\tau;
\]

taking the supremum over \( s \in [0,t] \) gives

\[
\|p - q\|_{L^\infty(0,t)} \leq L \int_0^t \|p - q\|_{L^\infty(0,\tau)} \ d\tau,
\]

and now we have \( p = q \) on \([0,T]\) by Gronwall’s inequality. Since the disk trajectory uniquely determines the gas distribution by Section 3.4 (Construction of the Density Function, p. 32), the entire solution is unique. \( \square \)
Chapter 5

Numerical Experiments

5.1 Introduction

5.1.1 Existing Results

Systems related to (1.1, p. 2), with various boundary conditions and different assumptions for the gas, have been investigated numerically in a series of papers by T. Tsuji and K. Aoki [1, 15, 16]. The computation described in [1] was carried out with quadruple precision, but we will recover the same results, at least in one- and two-dimensional cases, using the standard double precision.

The setup considered in [1] is as follows: the disk is attached to the origin with a spring, and the restoring force $F$ is given by Hooke’s Law:

$$F(x, t) = -kx.$$ 

If the gas is taken to be collisionless, the displacement of the disk $\eta(t)$ decays to its equilibrium state $\eta = 0$ algebraically in time; the exact rate dependent on the dimension $d$: 

$$|\eta(t)| = \mathcal{O}(t^{-d-1}) \text{ as } t \to \infty.$$ 

(5.1)

Similar behaviour in the case of constant forcing was analytically shown in [2]: now the velocity of the disk is decaying to equilibrium velocity $p_\infty$:

$$|p_\infty - p(t)| = \mathcal{O}(t^{-d-1}) \text{ as } t \to \infty.$$ 

(5.2)

The two cases above will be used as benchmarks for the calculations in this chapter.
5.1.2 Overview

An important question for any model is its feasibility for numerical computations. In this chapter we present a numerical method for solving (1.1, p.2) and discuss properties of the obtained solutions. Let

$$a_{\pm} = \frac{x_{2} \pm R}{t - s} \quad \text{and} \quad b_{\pm} = \frac{z \pm R}{s - \tau}.$$ 

The equation that we ultimately need to discretize is

$$\dot{p}(t) = F(\eta(t), t) + G_{0}^{d}(t; p) + \sum_{n=1}^{\infty} G_{n}^{d}(t; p) - \sum_{n=1}^{\infty} G_{0}^{r}(t; p),$$

(5.3)

where $G_{0}^{r}$ is as given by (2.32, p.21) and the precollisional drag forces are given by

$$G_{n}^{r}(t; p) = \int_{0}^{t} \frac{\partial v(s, t)}{\partial s} \left[ (p(t) - v(s, t))^{2} + \frac{\beta_{2}}{\gamma_{2}} (p(t) - v(s, t)) \right] \int_{a_{-}}^{a_{+}} f_{n}^{r}(x_{2}, v_{2}, s, t) \, dv_{2} \, dx_{2} \, ds,$$

where $f_{1}^{r}$ is as given by (2.35, p.22) and (2.36, p.22), and $f_{n}^{r}$ for $n \geq 2$ are given recursively by (2.34, p.22):

$$f_{n+1}^{r}(x_{2}, v_{2}, s, t) = j_{n}^{r}(x_{2} - (t - s)v_{2}, s) \phi(v(s, t) - p(s), v_{2}),$$

$$j_{n}^{r}(z, s) = \frac{1}{\gamma_{2}} \int_{0}^{s} \frac{\partial v(\tau, s)}{\partial \tau} (p(s) - v(\tau, s)) \int_{b_{-}}^{b_{+}} f_{n}^{r}(z, v_{2}, \tau, s) \, dv_{2} \, d\tau.$$

(5.4)

We reduce the original system (1.1) to one equation because the position of the disk $\eta$ is obtained by simply integrating its velocity, and the gas equation (1.1c) was accounted for in (2.33, p.22). We won’t be solving for the density of the gas away from the disk.

The main question that we need to answer concerns truncation of the sequence of precollisions: the full drag force in (5.3) is given in terms of a series, which will have to be truncated if we want to compute solutions numerically. Equivalently, we can truncate the density function $f$: let

$$f_{\text{rec}}^{(N_{f})} = \sum_{n=1}^{N_{f}} f_{n}(x, v, t), \quad \text{and} \quad G_{\text{rec}}^{(N_{f})}(t; p) = \sum_{n=1}^{N_{f}} G_{n}(t; p)$$

be that truncation. The estimates in Theorem 4.1 (Bound for $f_{n}$, p.37) imply that the series indeed converges, and the exact form of $\alpha_{n}$ (4.1, p.35) suggests a rapid decay of the terms corresponding to large $n$. However, for smaller $n$ at large time $s$, the $\alpha_{n}$ look rather large. My conjecture for this chapter is that bounds like Theorem 4.1 (Bound for $f_{n}$, p.37) are quite far from being sharp and that the sequence $f_{\text{rec}}^{(N_{f})}$ converges extremely rapidly. So
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much so that truncation to a single precollision—

\[ f_{\text{rec}} \approx f_1(x, v, t) \]  

(5.5)

—produces qualitatively correct results with any forcing \( F \), while inclusion of only one additional term—

\[ f_{\text{rec}} \approx f_1(x, v, t) + f_2(x, v, t) \]  

(5.6)

—gives all the accuracy one could hope to recover numerically.

5.1.3 Change of Variables

Before discretizing (5.3) we will alter it by another change of variables.

Note that the integration domain \([b_-, b_+]\) in (5.4) is rather infinite-looking when \( s - \tau \) is small. Its dependence on the other variable of integration \( \tau \) might pose a problem too. To avoid integrating over \([b_-, b_+]\) we will identify the particles by the location on the disk where they have collided previously, rather than their transverse velocity: a particle at \((z, v_2, \tau, s)\) must have precollided at position

\[ y = z - v_2(s - \tau), \]

so we can use the change of variables

\[ v_2 = \frac{z - y}{s - \tau}, \quad dv_2 = -\frac{dy}{s - \tau}, \quad [b_-, b_+] \rightarrow [R, -R]. \]

Then (5.4) becomes

\[ j_n(z, s) = \frac{1}{\gamma_2} \int_0^s \frac{\partial \nu(\tau, s)}{\partial \tau} \left( \frac{p(s) - v(\tau, s)}{s - \tau} \right) \int_{-R}^{R} f_n(z, \frac{z - y}{s - \tau}, \tau, s) dy d\tau. \]

Next we use Remark 2.2 (Another way to write \( \partial_s \nu \), p. 19) and identity (2.1.b, p. 11) to write

\[ j_n(z, s) = \frac{1}{\gamma_2} \int_0^s \chi(\tau; \Phi_s) \left( \frac{v(\tau, s) - p(\tau)}{s - \tau} \right) \left( \frac{p(s) - v(\tau, s)}{s - \tau} \right) \int_{-R}^{R} f_n(z, \frac{z - y}{s - \tau}, \tau, s) dy d\tau; \]

(5.7)
this is the expression for $j_n$ we will ultimately discretize. The drag force $G_n$ is rewritten similarly:

$$
G_n(t) = \int_0^t \chi(s; \Phi_t) \left( \frac{v(s,t) - p(s)}{t-s} \right) \left( \frac{p(t) - v(s,t)}{t-s} \right) \left( p(t) - v(s,t) + \frac{\beta_2}{\gamma_2} \right) \times 
\times \int_{-R}^R \int_{-R}^R f_n(z, \frac{z-y}{t-s}, s, t) \, dy \, dz \, ds.
\tag{5.8}
$$

### 5.1.4 Discretization

Let $\tilde{p}_j$ and $\tilde{\eta}_j$ be the numerically computed values of the velocity and position of the disk at the $j$th time step:

$$
p(j \Delta t) \approx \tilde{p}_j \quad \text{and} \quad \eta(j \Delta t) \approx \tilde{\eta}_j.
$$

We will be solving (5.3, p.60) in a time-stepping manner: once the right hand side of (5.3) is computed, we advance $p$ and $\eta$ with the 2-step Adams-Bashforth and Adams-Moulton methods respectively [17]:

$$
\tilde{p}_{j+1} = \tilde{p}_j + \frac{\Delta t}{2} \left( 3\dot{p}_j - \dot{p}_{j-1} \right)
$$

$$
\tilde{\eta}_{j+1} = \tilde{\eta}_j + \frac{\Delta t}{12} \left( 5\ddot{p}_{j+1} + 8\dot{p}_j - \dot{p}_{j-1} \right).
$$

We denote the total number of time steps by $N_t$. Time stepping provides an equispaced grid for integration over the history, which we use to approximate the outer integrals in (5.7) and (5.8) with the trapezoidal rule.

Integration over the disks in (5.7) and (5.8) is performed using Clenshaw–Curtis quadrature [18], with the nodes and weights obtained from the Chebfun `chebpts` command. We denote the number of nodes on the disk by $N_{d}$. Note that positions of the disk $\tilde{\eta}_j$ and the Chebyshev points on the disk implicitly determine the effective velocity grid. The dashed lines in the diagram above represent (some of) the trajectories of the particles whose velocities are represented in this way.

For $G_0$ (2.32, p.21) and $j_0$ (2.35, p.22) integration in transverse velocity $v_2$ will be performed exactly.
5.1.5 Convergence

We would like to assess the convergence rate of the method we proposed. Unfortunately, we do not have an exact solution to compare the numerical solution to, even in simpler cases. We will get around this issue by using the approach described in Appendix A.6.3 of [17]. Let $\tilde{p}_{\Delta t}$ be the solution computed with temporal resolution $\Delta t$. If we choose a sufficiently large $N_d$, the time-stepping error should be dominant. We expect second order convergence for our method, that is

$$E(\Delta t) = \|\tilde{p}_{\Delta t}(t) - p(t)\|_\infty = C(\Delta t)^2,$$

where $C = O(1)$. We estimate $E(\Delta t)$ as follows:

$$E(\Delta t) \approx \tilde{E}(\Delta t) = \|\tilde{p}_{\Delta t} - \tilde{p}_{\Delta t/2}\|_\infty.$$

If the convergence of our method is indeed of second order, we should see

$$\tilde{E}(\Delta t) = \frac{3C}{4}(\Delta t)^2.$$

We will now assess the convergence rate of the method for the two-dimensional case using the following test problem:

$$\dot{p}(t) = 10t(1 - t) + G[f], \quad p(0) = 0.5,$$

$$\dot{\eta}(t) = p(t), \quad \eta(0) = 0,$$

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = 0, \quad f(x, v, 0) = \frac{1}{4}e^{-v_1^2 - v_2^2},$$

$$\phi(v_1, v_2) = e^{-v_1^2 - v_2^2}.$$

The external forcing and the initial gas density were chosen to induce an oscillation in the solution. We set the final time $T = 0.75$, the number of allowed precollisions $N_f := 2$ and discretize the disk with $N_d = 32$ points.

The results of the convergence study are presented in Figure 5.1a: we see that the method is second order accurate, as expected.

Another important property of an algorithm is its time-complexity. Since we have to integrate over all previous positions of the disk at each time step, we would expect the total time of the computation to be quadratic in the number of time steps. Figure 5.1b confirms that this is indeed the case.
5.2 Results

5.2.1 Hooke’s Law Forcing

We will now consider the case investigated in [1]: the decay rate of $|\eta(t)|$ to 0 with the external force and the relevant gas distributions given by

$$F(x,t) = -kx,$$

$$\phi_0(v_1, v_2) = me^{-v_1^2 - v_2^2},$$

$$\phi(v_1, v_2) = e^{-v_1^2 - v_2^2}.$$ 

The stiffness coefficient $k$ and scaling parameter $m$ were selected to induce one transient oscillation. Different choices would lead to a different—but always finite—number of oscillations, or no oscillations at all. We have verified that in all cases the long-time decay rate remains the same. We have used

$$\Delta t = 0.01, \quad N_d = 16, \quad N_f = 4,$$  \hspace{1cm} (5.10)

and also verified that all stated results hold when the resolution is refined. We compute the decay rate as follows:

$$\text{decay rate} = \frac{d \log_2 |\eta(t)|}{d \log_2 t}.$$ 

The results are presented in the figures below. We do indeed observe the expected decay rate (5.1, p.59).
Figure 5.2: Hooke’s law forcing. Our results agree with those presented in Figure 3 of [1].

5.2.2 Constant Forcing

A similar case is the one investigated analytically in [2]: the external force is kept constant and the decay rate of $|p_\infty - p(t)|$ is considered. The equilibrium velocity is implicitly given by the following condition:

$$G(p_\infty, \phi_0) + F = 0.$$ 

We have chosen $F$ such that $p_\infty = 2$ and set $p_0 = 1$. The relevant gas densities are

$$\phi_0(v_1, v_2) = \phi(v_1, v_2) = e^{-v_1^2-v_2^2}.$$ 

We again compute solutions numerically with parameters (5.10) and present the results in Figures 5.3 and 5.4. As we can see in Figure 5.3b, the expected decay rate (5.2, p.59) is indeed obtained.

Moreover, we can compute the disk velocities with only one recollisions allowed and with four recollision allowed, and compare them. We will use $p_{N_f}$ to denote solutions to the problem computed with different numbers of precollisions allowed; the case with no precollisions will not be considered, so $p_0$ is still reserved for the initial velocity of the disk.

In Figure 5.4a we present the drag forces generated by first, second, etc. collisions. Note that $G_n$ for $n \geq 2$ are negligibly small compared to $G_1$. For this reason the calculation performed with $N_f = 1$ gives a virtually identical result: the maximum relative difference between disk velocities obtained in this way is only about 0.05%. 

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5.2.3 Computed Examples: Known Disk Trajectory

In this subsection we will investigate the drag forces experienced by the disk moving along two special types of trajectories: monotone acceleration and oscillatory acceleration. Specifically, we consider

\[ p^{(ma)}(t) = 1 + t + t^2 \quad \text{and} \quad p^{(oa)}(t) = 1 + 2t + \sin(6t). \]

We take \( p_1 = p^{(\cdot)} \) as given and proceed with the following steps:
(1) Compute the external force $F = F(t)$ that is required to achieve the given trajectory:

$$F(t) = \dot{p}_1(t) - G^l(t; p_1) - G^r(t; p_1) + G^l_0(t; p_1) + G^r_0(t; p_1).$$

(2) Use this $F$ to compute $p_{N_f}$ for $N_f = 2..5$:

$$\dot{p}_{N_f}(t) = F(t) + G^l_0(t; p_{N_f}) + \sum_{n=1}^{N_f} G^l_n(t; p_{N_f}) - G^r_0(t; p_{N_f}) - \sum_{n=1}^{N_f} G^r_n(t; p_{N_f}).$$

(3) Compare solutions obtained with different $N_f$ and determine how many precollisions we need to consider.

Results for the monotone and oscillatory cases are presented in Figures 5.5 and 5.6 respectively; the computations were performed with $\Delta t = 10^{-3}$ and $N_d = 16$, and all stated results hold when the resolution is refined.

We can see from Figures 5.5a and 5.6c that solutions produced with $N_f = 1$ are qualitatively the same as the ones with $N_f = 5$: they agree to 4 decimal places.

Solutions computed with $N_f = 2$ differ from the ones with $N_f = 5$ by only about $10^{-7}$ in the monotone case and about $10^{-6}$ in the oscillatory case. This is on a par with the discretization error, so including more precollisions would not actually produce a more accurate result.

Figure 5.5: Drag forces and their relative importance with the disk trajectory $p(t) = 1+t+t^2$. 

(a) Differences between disk velocities computed with different $N_f$. 

(b) Drag forces from the right. There is no recollisional drag force from the left since the disk is monotonically accelerating.
An explanation for this phenomenon is presented in Figures 5.5b, 5.6b and 5.6d: the drag forces $G_n$ with $n \geq 2$ are minuscule compared to $G_1$ and $G_0$, and thus do not make much difference.

Figure 5.6: Drag forces and their relative importance with the disk trajectory $p(t) = 1 + 2t + \sin(6t)$. 

(a) Disk velocity and a few averaged velocities.

(b) Drag forces from the right.

(c) Differences between disk velocities computed with different $N_f$.

(d) Drag forces from the left.
5.3 Conclusion

We have presented a numerical method to solve the system (1.1, p. 2) in one and two dimensions. The results we obtained using this method agree well with the literature, so we may conclude that the method is indeed reliable.

Moreover, we have provided strong numerical support for claims (5.5, p. 61) and (5.6, p. 61): the dynamics of the disk can indeed be computed quite accurately by considering only one or two precollisions.

Chapters 1-4 did not mention the three-dimensional case, so we did not work with it numerically either. It is possible to alter the existing scripts to allow for the radially symmetric three-dimensional case. However, without such a symmetry assumption, the computation on a laptop would not be feasible.
Chapter 6

Discussion and Future Directions

The cornerstone of this thesis was the identification of the key aspects of the physical system and transformation of the model to reflect them. We have demonstrated how this approach can lead to new analytical results and a deeper understanding of the system.

The calculations presented here, tedious as they were, did not rely on any assumptions other than the “physicality” of the data. In particular, unlike all existing analytical literature, we did not require any smallness assumptions; I believe this was made possible by capturing the correct structure of the physical system with the system of equations.

We have also presented a numerical method for solving the system of equations that respects and relies on the recursive structure established earlier. It allowed us to simulate the system on a laptop, which is somewhat unusual for kinetic theory, and provide additional intuition about the solutions to the system.

The most important shortcoming of this work, at least from the modelling point of view, is the neglect of the interactions between the gas particles; it will be the main focus of future work related to this problem. Another possible direction is further development of the numerical method, such as introduction of the three-dimensional case and, again, interaction between gas particles. Finally, one could attempt to come up with a history-independent approximation for the system (1.1), which would be easier to work with. We believe a useful approximation of this sort can be obtained at least for monotone solutions of the system.
Bibliography


