The Genus of Generalized Random and Quasirandom Graphs

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Abstract

The *genus* of a graph G is the minimum integer h such that G has an embedding in some surface (closed compact 2-manifold) $\mathbb{S}_h$ of genus h. In this thesis, we will discuss the genus of generalized random and quasirandom graphs. First, by developing a general notion of random graphs, we determine the genus of generalized random graphs. Next, we approximate the genus of dense generalized quasirandom graphs.

Based on analysis of minimum genus embeddings of quasirandom graphs, we provide an Efficient Polynomial-Time Approximation Scheme (EPTAS) for approximating the genus (and non-orientable genus) of dense graphs. More precisely, we provide an algorithm that for a given (dense) graph G of order n and given $\varepsilon > 0$, returns an integer g such that G has an embedding into a surface of genus g, and this is ε-close to a minimum genus embedding in the sense that the minimum genus $\mathsf{g}(G)$ of G satisfies: $\mathsf{g}(G) \le g \le (1+\varepsilon)\mathsf{g}(G)$. The running time of the algorithm is $O(f(\varepsilon)\, n^2)$, where $f(\cdot)$ is an explicit function. Next, we extend this algorithm to also output an embedding (rotation system) of genus g. This second algorithm is an Efficient Polynomial-time Randomized Approximation Scheme (EPRAS) and runs in time $O(f_1(\varepsilon)\, n^2)$.

The last part of the thesis studies the genus of complete 3-uniform hypergraphs, which is a special case of genus of random bipartite graphs, and also a natural generalization of Ringel–Youngs Theorem. Embeddings of a hypergraph H are defined as the embeddings of its associated Levi graph L_H with vertex set $V(H) \sqcup E(H)$, in which $v \in V(H)$ and $e \in E(H)$ are adjacent if and only if v and e are incident in H. The construction in the proof may be of independent interest as a design-type problem.

Keywords: graph embeddings (05C10); genus (57M15); Szemerédi regularity lemma (05C85); random graphs (05C80); hypergraphs (05C65)

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Chapter 1

Introduction

1.1 Motivation

The study of topological graph theory dates back to the middle of the eighteenth century, start with the study of Euler characteristic for polyhedra. It took heightened interest in the latter part of the nineteenth century, with several map-coloring problems. The object is to find the minimum number of colors needed to color all possible maps, where countries that share a border must be colored differently. Probably one of the most famous conjectures in mathematics is *Four Color Conjecture*, that is, four colors are enough to color all possible maps drawn on the plane. The conjecture was made by Francis Guthrie [5, 47] in 1852. The first proof was published by Kempe in 1878, but this contained an error which was found in 1890 [26] by Heawood, who proved five colors are sufficient to color any map drawn on the plane. The conjecture was finally confirmed in 1976 by Appel and Haken [3], after a century of false proofs and refinements of techniques.

The problem of coloring maps on the plane is equivalent to the problem on the sphere. *Heawood Map-coloring Conjecture* [26] generalizes the four color conjecture to every closed compact 2-manifold. We state the orientable case:

$$\chi(\mathbb{S}_h) = \left\lfloor \frac{7+\sqrt{1+48h}}{2} \right\rfloor, \text{ for } h > 0,$$

where h is the genus of the closed compact 2-manifold $\mathbb{S}_h$. The problem was open for almost eight decades, and in 1965 it was given the place of honor for Tietze's *Famous Problems of Mathematics* [68]. The problem was eventually reduced to the *genus computation* for complete graphs and was studied in a series of papers in the twentieth century. In 1968, Ringel and Youngs [53] determined the genus of all complete graphs, which implies the final solution of the Heawood's Conjecture. The complete proof was finally presented in the monograph [52]. Their proof is split in 12 cases, some of the cases were slightly simplified later, but for the most complicated cases, no short proofs are known.

Theorem 1.1.1 (Ringel–Youngs Theorem)**.** *If* $n \geq 3$ *then*

$$\mathsf{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

If $n \geq 5$ *and* $n \neq 7$*, then*

$$\widetilde{\mathsf{g}}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

Given a graph G, determine the genus of G is one of the central problems in topological graph theory. It is also a very hard problem in both mathematics and computer science, since Thomassen [65] proved that the problem is NP-complete. On the other hand, determine the genus of a graph is also very important, by Robertson and Seymour's *graph minor theory.*

Definition 1.1.2. Given two graphs H and G, we say H is a *minor* of G if H can be obtained from G by deleting edges and vertices and by contracting edges.

The following generalized Kuratowski's Theorem [56] tells us, the genus of a graph gives us a rough structure of the graph. Then many NP-complete algorithms become ploynomial-time if we know the underlying graph has bounded genus.

Theorem 1.1.3 (Robertson and Seymour)**.** *Graphs with bounded genus have finitely many forbidden minors.*

The thesis is motived by the problem on approximating the genus of large dense graphs. After applying Szemerédi regularity lemma on graph G, we obtain a partition $\mathcal{P}$ of $V(G)$, and the induced subgraph between any two parts (except an ε-fraction) is random-like. Therefore, the theories of the genus of random and quasirandom graphs are needed. Background of the graph genus problem is introduced in Chapter 5.1.1.

1.2 Structure of the Thesis

The thesis is organized as follows. In the second chapter, we introduce the basic definitions in topological graph theory, and we also present the tools we use to construct the minimum (or near minimum) genus embeddings of generalized random graphs and quasirandom graphs. In chapter three, we determine the genus of random bipartite graphs, and discuss the genus of $\mathbb{H}$-random graphs. In chapter four, we approximate the genus of dense quasirandom graphs. In chapter five, we provide an Efficient Polynomial-Time Approximation Scheme (EPTAS) for approximating the genus (and non-orientable genus) of dense graphs, as well as an Efficient Polynomial-time Randomized Approximation Scheme (EPRAS) for the near optimal embeddings of dense graphs, based on the results in the previous chapters. In chapter six, we consider the minimum genus embeddings of complete 3-uniform hypergraphs, which is a special case of random bipartite graphs we discussed in chapter 3, and also a natural generalization of Ringel–Youngs Theorem. We determine the genus (and non-orientable genus) of K_n^3 when n is even, and discuss the genus when n is odd. Parts of the results in Chapters 3,4,5 and 6 are included in submitted papers [29, 30, 31]. Some of the rest of the results in those chapters and some in-writing future work will be included in [32, 33].

Chapter 2

Embeddings of Graphs

2.1 Notation and Basic Definitions

Through out the dissertation, We will use standard notation in graph theory, topological graph theory and graph limit theory as given in [16], [44] and [36], respectively. We fix the following notation.

- Denote by $\mathbb{N}$ the set of non-negative integers, and $[n]$ is the set $\{1, \ldots, n\}$.
- G is always a graph (1-complexes).
- Given a graph G, suppose $X, Y \subseteq V(G)$. We use $E(X, Y)$ denote the set of edges between X and Y, and $\mathsf{e}(X, Y) = |E(X, Y)|$. We also use $\mathsf{e}(G)$ to denote $|E(G)|$.
- For two sets X and Y, by $X \sqcup Y$ we denote the disjoint union of X and Y , and we set $X \oplus Y = (X \times Y) \cup (Y \times X)$.
- $A(n) \sim B(n)$ means $\lim_{n\to\infty} A(n)/B(n) = 1$, and $A(n) \ll B(n)$ means that $\lim_{n\to\infty} A(n)/B(n) = 0$. If $\lim_{n\to\infty} A(n)/B(n) = c$ for some constant c, we denote it by $A(n) = \Theta(B(n))$. We say an event $A(n)$ happens asymptotically almost surely (abbreviated a.a.s.) if $\mathbb{P}(A(n)) \to 1$ as $n \to \infty$.

Now we give the definition of the genus of graphs, as a natural generalization of planar graphs.

Definition 2.1.1. Given a graph G, let $\mathsf{g}(G)$ be the *genus* of G, that is, the minimum h such that G embeds into the orientable surface $\mathbb{S}_h$ of genus h, and let $\widetilde{\mathsf{g}}(G)$ be the *non-orientable genus* of G which is the minimum c such that G embeds into the non-orientable surface $\mathbb{N}_c$ with crosscap number c. If orientability of a surface is not a concern, we define the *Euler genus* as $\widehat{\mathsf{g}}(G) = \min\{2\mathsf{g}(G), \tilde{\mathsf{g}}(G)\}$. By a surface we mean a compact two-dimensional manifold without boundary.

For any surface S, we say G is *2-cell embedded* in S if in that embedding, each face of G is homeomorphic to an open disk. Similarly, a *k-gon embedding* of G is when every face is bounded by a cycle of length k. In particular, we say an embedding of G is *triangular* if every face is bounded by a triangle, and an embedding is *quadrangular* if every face is bounded by a cycle of length 4.

Every 2-cell embedding (and thus also any minimum genus embedding) of G can be represented combinatorially by using the corresponding *rotation system* $\pi = \{\pi_v \mid v \in V(G)\}$ where a *local rotation* π_v at the vertex v is a cyclic permutation of the neighbours of v. In addition to this, we also add the *signature*, which is a mapping $\lambda : E(G) \to \{1, -1\}$ and describes if the local rotations around the endvertices of an edge have been chosen consistently or not. The signature is needed only in the case of non-orientable surfaces; in the orientable case, we may always assume the signature is trivial (all edges have positive signature). The pair (π, λ) is called the *embedding scheme* for G. For more background on topological graph theory, we refer to [25, 44].

We say that two embeddings $\phi_1, \phi_2 : G \to S$ of a graph G into the same surface $\mathbb{S}$ are *equivalent* (or *homeomorphic*) if there exists a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $\phi_2 = h\phi_1$. By [44, Corollary 3.3.2], (2-cell) embeddings are determined up to equivalence by their embedding scheme (π, λ), and two such embedding schemes (π, λ) and (π', λ') determine equivalent embeddings if and only if they are *switching equivalent*. This means that there is a vertex-set $U \subseteq V(G)$ such that (π', λ') is obtained from (π, λ) by replacing π_u with π_u^{-1} for each $u \in U$ and by replacing $\lambda(e)$ with $-\lambda(e)$ for each edge e with one end in U and the other end in $V(G) \setminus U$.

2.2 Near Minimum Genus Embeddings

Given a graph G, determining the genus of G is one of the fundamental problems in topological graph theory. Youngs [72] showed that the problem of determining the genus of a connected graph G is the same as determining a 2-cell embedding of G with minimum genus. The same holds for the non-orientable genus [48]. That means, in this thesis we only need to consider 2-cell embeddings of graphs. For 2-cell embeddings we have the famous Euler's Formula.

Theorem 2.2.1 (Euler's Formula)**.** *Let G be a graph which is 2-cell embedded in a surface S. If G has n vertices, e edges and f faces in S, then*

$$\chi(S) = n - e + f. \tag{2.2.1}$$

Here $\chi(S)$ is the *Euler characteristic* of the surface S, where $\chi(S) = 2 - 2h$ when $S = \mathbb{S}_h$ and $\chi(S) = 2 - c$ when $S = \mathbb{N}_c$.

Definition 2.2.2. Let G be a simple graph. The *corresponding digraph* of G is a family of random simple digraphs $\mathcal{D}$ obtained from G by randomly orienting each edge.

Specifically, each digraph $D \in \mathcal{D}$ has $V(D) = V(G)$ and If $uv \in E(G)$ then either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ is an edge of D, each has probability $1/2$ and the two events are exclusive. The corresponding digraph $\mathcal{D}$ of a random graph $\mathcal{G}$ is a family of digraphs defined on the same vertex set of graphs in $\mathcal{G}$, and when two vertices u, v produce an edge with probability p in $\mathcal{G}$, then $\overrightarrow{uv}$ occurs with probability $p/2$ and $\overrightarrow{vu}$ occurs with probability $p/2$ in $\mathcal{D}$, and those two events are exclusive.

Definition 2.2.3. Let$D$ be a digraph, a *blossom* of length l with *center* v and *tips* $\{v_1, v_2, \ldots, v_l\}$ is a set $\mathcal{C}$ of l directed cycles $\{C_1, C_2, \ldots, C_l\}$, where $\overrightarrow{v_iv}, \overrightarrow{vv_{i+1}} \in C_i$, for $i = 1, 2, \ldots, l$, with $v_{l+1} = v_1$. A k-blossom is a blossom, all of whose elements are directed k-cycles. A blossom of length l is *simple* if either $l \geq 3$ or $l = 2$ and $C_1 \neq C_2^{-1}$.

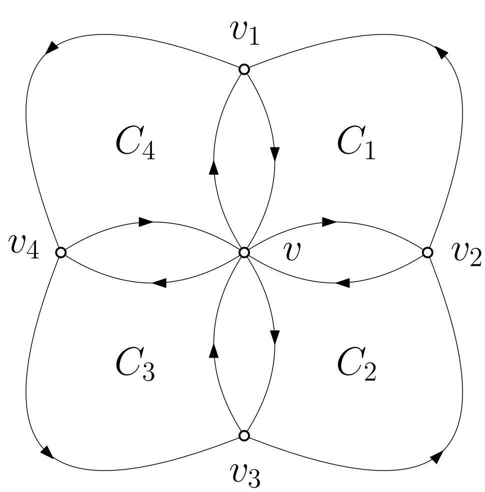

Figure 2.1: A 4-blossom of length 4 with center v and tips v_1, v_2, v_3, v_4.

Let $\mathcal{C}$ be a family of arc-disjoint closed trails in $D \cup D^{-1}$. We say that $\mathcal{C}$ is *blossom-free* if no subset of $\mathcal{C}$ forms a blossom centered at some vertex. The following lemma is a slight strengthening of [58, Lemma 2.1]; the proof is elementary and we omit details.

Lemma 2.2.4. *Let G be a graph and let D be the corresponding digraph. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ is a set of arc-disjoint closed trails in D and D^{-1} (respectively) such that their union $\mathcal{C}_1 \cup \mathcal{C}_2$ is blossom-free in $D \cup D^{-1}$. Then there exist a rotation system Π of G such that every closed trail in $\mathcal{C}_1 \cup \mathcal{C}_2$ is a face of Π.*

For every $\varepsilon > 0$, an *ε-near k-gon embedding* Π is a rotation system of G such that $kf_k(\Pi) \geq 2(1-\varepsilon)|E(G)|$, where $f_k(\Pi)$ is the number of faces of length k of Π.

The following result from [21] (see also [49, 58] where its current formulation appears) will be our main tool for constructing near-optimal embeddings of random graphs and quasirandom graphs.

Theorem 2.2.5. *Let $\varepsilon > 0$ be a real number and $d \geq 2$ be an integer. Then there exist a positive real number δ and an integer N_0 such that for every $N \geq N_0$ the following holds. If Δ is a real number and if $\mathcal{H}$ is a d-uniform hypergraph with $|V(\mathcal{H})| = N$ such that*

1. $|\{x \in V(\mathcal{H}) \mid (1-\delta)\Delta \leq \deg(x) \leq (1+\delta)\Delta\}| \geq (1-\delta)N$,

2. *for every* $x, y \in V(\mathcal{H})$, $|\{e \in E(H) \mid x, y \in e\}| < \delta\Delta$,

3. *at most $\delta N \Delta$ hyperedges of $\mathcal{H}$ contain a vertex $v \in V(\mathcal{H})$ with* $\deg(v) > (1+\delta)\Delta$,

then $\mathcal{H}$ has a matching of size at least $(1-\varepsilon)N/d$. Moreover, for every matching M in $\mathcal{H}$, there exists a matching M' in $\mathcal{H}$ with $M \cap M' = \varnothing$, and with $|M'| \geq (1-\varepsilon)N/d$.

Similarly as for undirected graphs (see [44, Lemma 5.4.2]), we have the following property on digraphs.

Lemma 2.2.6. *Let $D(V, A)$ be a simple digraph and $a, b \in A, a \neq b$. Let $f, g \in \mathbb{Z}^+$. Then there exists a positive integer $K = K(f, g)$, such that if D contains at least K closed trails of length f containing both a and b, then there exist two vertices $u, v \in V(D)$ and g internally disjoint directed paths from u to v, all of the same length l, where $2 \leq l \leq f - 2$.*

Proof. Let $x \in V$ be the head of a and let $y \in V$ be the tail of b. We may assume $x \neq y$. The proof is by induction on $f+g$, with $K(f,g) = \prod_{i=1}^{f-2}((f-i)(f-i-1)g)^{2^{i-1}}$. In the base case when $g = 0$ there is nothing to prove, and when $f = 3$, the claim is easy, so we move to the induction step. Assume now we have $K(f+1, g)$ closed trails of length $f+1$ containing both a and b. Let $\overrightarrow{P_{xy}}$ be the set of paths from x to y on these closed trails. Note that $K(f+1,g) = f(f-1)gK(f,g)^2$. If one of the edges say $\overrightarrow{az}$, is used on $K(f,g)$ of the paths, we can consider the $K(f,g)$ subpaths from z to y and apply induction. Otherwise, there is a subset $\overrightarrow{P'_{xy}}$ of $\overrightarrow{P_{xy}}$ containing $f(f-1)gK(f,g)$ paths, all of which start with different edges. Choose one path in $\overrightarrow{P'_{xy}}$ arbitrarily, call it P.

If at least $fK(f,g)$ of our paths intersect P, there exists $v \in V(P)$ such that at least $K(f,g)$ paths pass though v. Contract all of those directed paths from a to v, we have $K(f,g)$ closed trails of length at most f containing both a and b. For those closed trails of length $f' < f$, we will add closed trails of length $f - f'$ containing x. By induction, we obtain g internally disjoint directed paths.

Finally we suppose that we do not have $fK(f,g)$ paths of $\overrightarrow{P'_{xy}}$ intersecting P. Since P is arbitrary, we may assume the same holds for any P. Then at least $(f-1)g$ of our paths of length at most $f-1$ are internally disjoint. Therefore at least g internally disjoint directed paths having the same length l, where $l \leq f-1$. □

Chapter 3

Genus of Generalized Random Graphs

3.1 Introduction

The random graph (Erdős-Rényi Model [17]) $\mathcal{G}(n,p)$ is a probability space whose objects are all (labelled) graphs defined on a vertex set V of cardinality n, and each possible edge occurs with probability p independently, i.e., a graph $G = (V, E) \in \mathcal{G}(n,p)$ has probability $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$. There are thousands of papers studying properties of random graphs; for more background about this fascinating area, see [2, 7].

Stahl [60] was the first to consider the genus (in fact, the average genus) of random graphs. Almost concurrently, Archdeacon and Grable [4] studied the genus of random graphs in $\mathcal{G}(n,p)$. They obtained the following result when $p = p(n)$ is not too small.

Theorem 3.1.1 ([4])**.** *Let $\varepsilon > 0$ and let $0 < p < 1$ with $p^2(1-p^2) \geq 8(\ln n)^4/n$. Then almost every graph G in $\mathcal{G}(n,p)$ satisfies*

$$(1-\varepsilon)\frac{pn^2}{12} \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{pn^2}{12}$$

and

$$(1-\varepsilon)\frac{pn^2}{6} \leq \widetilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{pn^2}{6}.$$

They also conjectured that almost every graph in $\mathcal{G}_{n,p}$ has an ε-near k-gon embedding (in which all but an ε-fraction of edges lie on the boundary of two k-gonal faces) on some orientable surface and on some non-orientable surface. Rödl and Thomas [58] resolved their conjecture and extended Theorem 3.1.1 to an even broader range of edge-probabilities.

Theorem 3.1.2 ([58]). *Let $\varepsilon > 0$, let $i \geq 1$ be an integer and assume that $n^{-\frac{i}{i+1}} \ll p \ll n^{-\frac{i-1}{i}}$. Then $G \in \mathcal{G}(n,p)$ almost surely satisfies*

$$(1-\varepsilon)\frac{i}{4(i+2)}pn^2 \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{i}{4(i+2)}pn^2$$

and

$$(1-\varepsilon)\frac{i}{2(i+2)}pn^2 \leq \widetilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{i}{2(i+2)}pn^2.$$

In this chapter, we will study the genus of a more general setting of random graphs. In the next section, we determine the genus of a general setting of random bipartite graphs. In the rest of the chapter, we will discuss the genus of $\mathbb{H}$-random graphs.

3.2 Genus of Random Bipartite Graphs

In this section, we define *random bipartite graphs* $\mathcal{G}(n_1, n_2, p)$ as the probability space of all bipartite graphs with (labelled) bipartition $X \sqcup Y$, $|X| = n_1$, $|Y| = n_2$, where each edge xy ($x \in X$, $y \in Y$) appears independently with probability p. In this thesis, we will always assume $n_1 \geq n_2$ for convenience.

3.2.1 Random Bipartite Graphs $\mathcal{G}(n_1, n_2, p)$

Let us first consider the case when n_1 and n_2 have about the same magnitude.

Lemma 3.2.1. *Let $\varepsilon > 0$ and $G \in \mathcal{G}(n_1, n_2, p)$ be a random bipartite graph on vertex set $X \sqcup Y$ with $|X| = n_1 \geq n_2 = |Y|$. If there exist a positive real number c and a positive integer i such that $n_1/n_2 < c$, and $p \gg (n_1n_2)^{-\frac{i}{2i+1}}$, then a.a.s. G has an ε-near $(2i+2)$-gon embedding.*

Proof. Choose $0 < \varepsilon_1 < \frac{1}{2}$, $\varepsilon_0 = \frac{3i+4}{1-\varepsilon_1}\varepsilon_1$, such that $\varepsilon_0 < 1/2$ and $\varepsilon \geq \frac{4\varepsilon_0}{1+\varepsilon_0}$. Let $n = \sqrt{n_1n_2}$. Then $p \gg n^{-\frac{2i}{2i+1}}$. Let us first assume that $p \ll n^{-\frac{2i-\varepsilon_1}{2i+1-\varepsilon_1}}$. Let $D \in \mathcal{D}$ be the corresponding digraph of $\mathcal{G}(n_1, n_2, p)$. Consider the following hypergraph $\mathcal{H}$, where $V(\mathcal{H})$ is the edge set of D and $E(\mathcal{H})$ is the set of closed trails of D of length $2i+2$. Let $d = 2i+2, \delta = \frac{\varepsilon_1}{1-\varepsilon_1}$ and $\Delta = n_1^i n_2^i (\frac{p}{2})^{2i+1}$. We claim that our hypergraph $\mathcal{H}$ satisfies all three conditions in Theorem 2.2.5, a.a.s.

To prove that condition (1) holds, let $N = |V(\mathcal{H})|$. We have

$$\begin{aligned} \mathbb{E}(N) &= n_1n_2p, \\ \mathbb{E}(N^2) &= n_1n_2p(n_1-1)(n_2-1)p + O(n_1^2n_2p^2 + n_2^2n_1p^2). \end{aligned} \tag{3.2.1}$$

By Chebyshev's inequality,

$$\mathbb{P}(|N-\mathbb{E}(N)| \geq \varepsilon_1 n_1 n_2 p) \leq \frac{\mathbb{E}(N^2)-\mathbb{E}^2(N)}{\varepsilon_1^2 \mathbb{E}^2(N)} = O\left(\frac{n_1+n_2}{n_1 n_2}\right) = o(1). \tag{3.2.2}$$

Therefore, we have a.a.s.

$$(1-\varepsilon_1)n_1 n_2 p < N < (1+\varepsilon_1)n_1 n_2 p. \tag{3.2.3}$$

For each pair of vertices $(a,b) \in X \oplus Y$, let $\rho(b,a)$ be the number of directed paths in D from b to a of length $2i+1$, and let U be the number of edges $\overrightarrow{uv}$ of D such that the number of directed paths from v to u of length $2i+1$ is at most $(1-\delta)\Delta$ or at least $(1+\delta)\Delta$. Similarly as above we have

$$\begin{aligned}
\mathbb{E}(\rho(b,a)) &= \binom{n_1-1}{i}\binom{n_2-1}{i}(i!)^2\left(\frac{p}{2}\right)^{2i+1}, \\
\mathbb{E}(\rho^2(b,a)) &= \binom{n_1-1}{i}\binom{n_2-1}{i}\binom{n_1-1-i}{i}\binom{n_2-1-i}{i}(i!)^4\left(\frac{p}{2}\right)^{4i+2} \\
&\quad + O(n_1^{2i}n_2^{2i-1}p^{4i+1} + n_1^{2i-1}n_2^{2i}p^{4i+1}).
\end{aligned} \tag{3.2.4}$$

Using Chebyshev's inequality, since $|\Delta - \mathbb{E}(\rho(b,a))| = o(\mathbb{E}(\rho(b,a)))$, for sufficiently large n,

$$\begin{aligned}
\mathbb{P}\Big(|\rho(b,a)-\Delta| \geq \delta\Delta\Big) &\leq \mathbb{P}\Big(|\rho(b,a)-\mathbb{E}(\rho(b,a))| \geq \frac{\varepsilon_1}{2}\mathbb{E}(\rho(b,a))\Big) \\
&\leq \frac{\mathbb{E}(\rho^2(b,a))-\mathbb{E}^2(\rho(b,a))}{(\frac{\varepsilon_1}{2})^2\mathbb{E}^2(\rho(b,a))} = O\left(\frac{n_1+n_2}{n_1 n_2 p}\right) = o(1).
\end{aligned} \tag{3.2.5}$$

Also for U we have

$$\mathbb{E}(U) = pn_1n_2\mathbb{P}(|\rho(b,a)-\Delta| \geq \delta\Delta) \leq O(n_1+n_2). \tag{3.2.6}$$

Hence by Markov's inequality,

$$\mathbb{P}\left(U \geq \varepsilon_1 \frac{p}{2} n_1 n_2\right) \leq \frac{\mathbb{E}(U)}{\varepsilon_1 \frac{p}{2} n_1 n_2} = O\left(\frac{n_1+n_2}{n_1 n_2 p}\right) = o(1). \tag{3.2.7}$$

This means, together with (3.2.3) and (3.2.5), a.a.s. at least $N - \varepsilon_1 \frac{p}{2} n_1 n_2 > (1-\delta)N$ vertices of $V(\mathcal{H})$ satisfy $(1-\delta)\Delta \leq \deg(x) \leq (1+\delta)\Delta$, so condition (1) holds for $\mathcal{H}$.

To verify (2), let e, f be two edges of D that together belong to at least $\delta\Delta$ hyperedges in $\mathcal{H}$. This means that they are together in many closed trails of length $2i+2$. By Lemma 2.2.6 there exists an integer K only depending on i, such that if we have more than K closed trails of length $2i+2$ containing both e and f, there exist two vertices u and v, and at least $8i+2$ internally disjoint directed paths from u to v of length l, where $2 \le l \le 2i$.

Let B be the number of vertex pairs $(u, v) \in V(D)^2$ such that there exist $8i+2$ internally disjoint directed paths from u to v of length l. Note that $p \ll n^{-\frac{2i-\varepsilon_1}{2i+1-\varepsilon_1}} < n^{-\frac{4i-1}{4i+1}}$ since $\varepsilon_1 < 1/2$. We have

$$\mathbb{E}(B) = O\Big(n_1^{(8i+2)\frac{l-1}{2}+1} n_2^{(8i+2)\frac{l-1}{2}+1} p^{(8i+2)l}\Big) \le o(n^{4l-8i}) = o(1), \text{ when } l \equiv 1 \pmod 2;$$
$$\mathbb{E}(B) = O\Big(n_1^{(8i+2)\frac{l}{2}} n_2^{(8i+2)\frac{l-2}{2}+2} p^{(8i+2)l}\Big) \le o(n_1^{2l} n_2^{2l-8i}) = o(1), \text{ when } l \equiv 0 \pmod 2. \tag{3.2.8}$$

By Markov's inequality, $\mathbb{P}(B \ge 1) \le o(1)$, that implies that no more than K closed trails of D contain both e and f, for every $e, f \in A(D)$, a.a.s. Therefore in our hypergraph $\mathcal{H}$, condition (2) holds for $\mathcal{H}$ when n is large enough.

Finally, let us consider condition (3) of Theorem 2.2.5. Let F be the number of closed trails of length $2i+2$ in D which contain at least one directed edge $\overrightarrow{uv} \in P^\delta$, where P^δ is the set of pairs of vertices $(u, v) \in X \oplus Y$ such that the number of directed trails from v to u of length $2i+1$ is at least $(1+\delta)\Delta$. Each trail $R = x_1 y_1 x_2 y_2 \cdots y_{i+1} x_1$ contributing to F is determined by two sequences of vertices $x_1, x_2, \ldots, x_{i+1} \in X$ and $y_1, y_2, \ldots, y_{i+1} \in Y$. Each such closed trail R has the same probability that it forms a trail contributing to F. There are $2i+2$ candidates for an edge of R being in P^δ. This implies that

$$\mathbb{E}(F) \le n_1^{i+1} n_2^{i+1} (2i+2) \mathbb{P}(R \subseteq A(D))\, \mathbb{P}(\overrightarrow{x_1 y_1} \in P^\delta \mid R \subseteq A(D)). \tag{3.2.9}$$

For $j = 1, \ldots, 2i-1$, let α_j be the number of trails of length $2i+1$ from y_1 to x_1 that contain precisely j edges in R. Then $\alpha = \sum_{j=1}^{2i-1} \alpha_j$ is the number of trails of length $2i+1$ from y_1 to x_1 different from R which contain at least one edge in R.

Since $n_2 = \Theta(n_1)$, we have

$$\begin{aligned}\mathbb{E}(\alpha \mid R \subseteq A(D)) &= \sum_{j=1}^{2i-1} \mathbb{E}(\alpha_j \mid R \subseteq A(D)) \\ &\leq \sum_{j=1}^{2i-1} \binom{2i+1}{j} j!\, n_1^{2i-j} \Big(\frac{p}{2}\Big)^{2i+1-j} \\ &\leq O(n^{2i-1} p^{2i+1}).\end{aligned} \tag{3.2.10}$$

Now, by Markov's inequality,

$$\mathbb{P}(\alpha \geq 1 \mid R \subseteq A(D)) \leq O(n^{2i-1} p^{2i+1}) \ll O(n^{-\frac{4i}{4i+1}}) = o(1). \tag{3.2.11}$$

In the next argument we will use the following events: Q^δ is the event that the number of trails of length $2i+1$ from y_1 to x_1 that are different from R is at least $(1+\delta)\Delta - 1$; R_E is the event that all edges in R appear in D, possibly with different orientations. There are 2^{2i+2} different orientations $\omega_1, \ldots, \omega_{2^{2i+2}}$ of these edges. We denote by R_E^j the event that these edges are present and have orientation ω_j. Clearly, different events R_E^j are mutually exclusive and R_E is the union of all these events. Note that the following holds:

$$\begin{aligned}\mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta, \alpha = 0 \mid R \subseteq A(D)) &= \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R \subseteq A(D)) \\ &\leq \sum_{j=1}^{2^{2i+2}} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R_E^j) \\ &= 2^{2i+1}\, \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0 \mid R_E) \\ &\leq 2^{2i+1}\, \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta, \alpha = 0) \\ &\leq 2^{2i+1}\, \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta).\end{aligned}$$

We used the fact that $\alpha = 0$ is less likely to happen under the condition that R_E holds and that $\overrightarrow{x_1y_1} \in Q^\delta$ is independent of R_E when $\alpha = 0$.

Combining the above inequalities with (3.2.5), we get

$$\begin{aligned}&\mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta \mid R \subseteq A(D)) \\ &\quad = \mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta, \alpha \geq 1 \mid R \subseteq A(D)) + \mathbb{P}(\overrightarrow{x_1y_1} \in P^\delta, \alpha = 0 \mid R \subseteq A(D)) \\ &\quad \leq o(1) + 2^{2i+1} \mathbb{P}(\overrightarrow{x_1y_1} \in Q^\delta) = o(1).\end{aligned} \tag{3.2.12}$$

Now, together with (3.2.9), $\mathbb{E}(F) \leq o(n_1^{i+1} n_2^{i+1} p^{2i+2})$, and by Markov's inequality,

$$\mathbb{P}(F \geq \delta N \Delta) \leq \frac{2^{2i+1} o(n^{2i+2} p^{2i+2})}{\delta(1-\varepsilon_1) n^{2i+2} p^{2i+2}} = o(1). \tag{3.2.13}$$

This means condition (3) holds for $\mathcal{H}$ a.a.s.

We are now ready to apply Theorem 2.2.5. The theorem tells us that for sufficiently large n, there exists a matching M of $\mathcal{H}$ of size at least $(1-\varepsilon_1)\frac{N}{2i+2}$. Therefore $M^{-1} = \{H^{-1} \mid H \in M\}$ is a matching on $\mathcal{H}^{-1}$ defined on D^{-1}. Again, by Theorem 2.2.5, we have another matching M' in $\mathcal{H}^{-1}$ of size at least $(1-\varepsilon_1)\frac{N}{2i+2}$ such that $M' \cap M^{-1} = \varnothing$. This implies that $M \cup M'$ does not have non-simple blossoms of length 2.

Next we will argue that there is only a small number of simple blossoms. Consider the digraph $D \cup D^{-1}$. Let $2 \leq j \leq \frac{1}{\varepsilon_1}$ be an integer, and let $T(j)$ be the number of simple $(2i+2)$-blossoms of length j in $D \cup D^{-1}$. We have

$$\begin{aligned}
\mathbb{E}(T(j)) &\leq n_1(n_1^{ij} n_2^{ij} p^{j+ij}) + n_2(n_1^{ij} n_2^{ij} p^{j+ij}) \\
&\leq 2n_1(n_1^{ij} n_2^{ij} p^{j+ij}) \leq 2\sqrt{c} n^{1+2ij} p^{j+2ij} \\
&< 2\sqrt{c}\, n^2 p n^{\frac{1}{\varepsilon_1}(2i-\varepsilon_1)} p^{\frac{1}{\varepsilon_1}(2i+1-\varepsilon_1)} \\
&< 2\sqrt{c}\, n^2 p n^{\frac{1}{\varepsilon_1}(2i-\varepsilon_1)} n^{-\frac{1}{\varepsilon_1}(2i-\varepsilon_1)} = O(n^2 p).
\end{aligned} \tag{3.2.14}$$

Hence by Markov's inequality,

$$\mathbb{P}\Bigg(\sum_{j=2}^{1/\varepsilon_1} T(j) \geq \varepsilon_1 p n^2\Bigg) \leq \mathbb{P}\Big(T(j) \geq \varepsilon_1^2 p n^2\Big) \leq o(1). \tag{3.2.15}$$

Therefore, a.a.s. the number of simple $(2i+2)$-blossoms of length at most $1/\varepsilon_1$ in $D \cup D^{-1}$ is at most $\varepsilon_1 p n_1 n_2 = \varepsilon_1 p n^2$. Since $M \cup M'$ has size at least $2(1-\varepsilon_1)\frac{N}{2i+2}$, it has a subset M_1 without simple $(2i+2)$-blossom of length at most $1/\varepsilon_1$ after removing at most $\varepsilon_1 p n^2$ closed trails. By using (3) we have:

$$\begin{aligned}
|M_1| &\geq 2(1-\varepsilon_1)\frac{N}{2i+2} - \varepsilon_1 p n^2 \\
&\geq (1-\varepsilon_1)\frac{N}{i+1} - \frac{\varepsilon_1}{1-\varepsilon_1} N \\
&\geq (1 - \frac{i+2}{1-\varepsilon_1}\varepsilon_1)\frac{N}{i+1}, \qquad \text{a.a.s.}
\end{aligned} \tag{3.2.16}$$

Now we consider the $(2i+2)$-blossoms of length at least $1/\varepsilon_1$ in M_1. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two blossoms of M_1 with center v, by the way we constructed M_1 we could see that the tips of $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot intersect. Therefore, if v has m neighbours in D, at most $\varepsilon_1 m$ different $(2i+2)$-blossoms of length at least $1/\varepsilon_1$ have center v. Thus, the total number of such blossoms is at most $\sum_{v\in V(D)} \deg_G(v)/(1/\varepsilon_1) = 2\varepsilon_1 N$. By removing one of the trails from each such blossom we get a blossom-free subset $M_0 \subseteq M_1$ which satisfies

$$\begin{aligned} |M_0| &\geq |M_1| - 2\varepsilon_1 N \\ &\geq (1 - \frac{3i+4}{1-\varepsilon_1}\varepsilon_1)\frac{N}{i+1} = (1-\varepsilon_0)\frac{N}{i+1}. \end{aligned} \tag{3.2.17}$$

Finally, using M_0 we can obtain an ε_0-near $(2i+2)$-gon embedding of G by using Lemma 2.2.4. This completes the proof when $p \ll n^{-\frac{2i-\varepsilon_1}{2i+1-\varepsilon_1}}$.

For the case $p \geq \Theta\Big(n^{-\frac{2i-\varepsilon_1}{2i+1-\varepsilon_1}}\Big)$, we use a similar argument as used in [58, Lemma 4.8]. Choose an integer $t = t(n)$, such that $n^{-\frac{2i}{2i+1}} \ll p/t \ll n^{-\frac{2i-\varepsilon_1}{2i+1-\varepsilon_1}}$. Let $p_1 = p/t$. Now take a corresponding digraph D of $\mathcal{G}(n_1, n_2, p)$ and partition its edges into t parts, putting each edge in one of the parts uniformly at random. Then each of the resulting digraphs $D_1, D_2, \ldots, D_t$ is a corresponding digraph of $\mathcal{G}(n_1, n_2, p_1)$. By the above, for every $1 \leq j \leq t$, $D_j \cup D_j^{-1}$ has a collection of blossom-free directed $(2i+2)$-trails of size at least $(1-\varepsilon_0)\frac{|A(D_j)|}{i+1}$ a.a.s. That means, if we let q be the probability that $D_j \cup D_j^{-1}$ does not have such set of trails, then $q \to 0$ as $n \to \infty$.

Let $I \subseteq \{1, 2, \ldots, t\}$ be the index set, containing all j, $1 \leq j \leq t$, for which $D_j \cup D_j^{-1}$ does not have a collection of directed blossom-free $(2i+2)$-trails of size at least $(1-\varepsilon_0)\frac{|A(D_j)|}{i+1}$. Then by Markov's inequality, $\mathbb{P}(|I| \geq \sqrt{q}t) \leq \sqrt{q}$. Hence for sufficiently large n, $|I| \leq \varepsilon_0 t$ a.a.s.

Similarly as in the proof of (3.2.3), we see that for each $0 \leq j \leq t$ a.a.s.

$$(1-\varepsilon_0)\frac{1}{2}n^2 p_1 \leq |A(D_j)| \leq (1+\varepsilon_0)\frac{1}{2}n^2 p_1. \tag{3.2.18}$$

Now let Γ be the union of collections of directed blossom-free $(2i+2)$-trails of size at least $(1-\varepsilon_0)\frac{|A(D_j)|}{i+1}$ for $j \notin I$. We have:

$$\begin{aligned} |\Gamma| &\geq (1-\varepsilon_0)\sum_{j\notin I}\frac{|A(D_j)|}{i+1} \geq (1-\varepsilon_0)^2 t(1-\varepsilon_0)\frac{p_1 n^2}{2i+2} \\ &\geq (1-\varepsilon_0)^3\frac{pn^2}{2i+2} \\ &\geq (1-\varepsilon)(1+\varepsilon_0)\frac{pn^2}{2} \geq (1-\varepsilon)\frac{|A(D)|}{i+1}. \end{aligned} \tag{3.2.19}$$

Since the directed closed trails of Γ that belong to any D_j $(j \notin I)$ are blossom-free and any D_k and D_j are edge disjoint for $k \neq j$, Γ is blossom-free. By Lemma 2.2.4, we get a rotation system Π in which every closed trail in Γ is a face of Π. Let f_{2i+2} be the number of faces of length $2i+2$ of Π. We have $(2i+2)f_{2i+2} \geq 2(1-\varepsilon)|E(G)|$, thus Π is an ε-near $(2i+2)$-gon embedding. □

The result of Lemma 3.2.1 has been proved under the assumption that $n_2 = \Theta(n_1)$. However, that assumption can be omitted as long as $n_2 \gg 1$.

Lemma 3.2.2. *Let $\varepsilon > 0$ and $G \in \mathcal{G}(n_1, n_2, p)$ be a random bipartite graph on vertex set $X \sqcup Y$ with $|X| = n_1 \geq n_2 = |Y|$. If $p \gg n_2^{-\frac{2i}{2i+1}}$ where i is a fixed positive integer and $n_2 \gg 1$, then a.a.s. (as $n_2 \to \infty$) G has an ε-near $(2i+2)$-gon embedding.*

Proof. It is sufficient to consider the case $n_1/n_2 \gg 1$. Let $t = \lfloor \frac{n_1}{n_2} \rfloor$, and let $\mathcal{P} = \{X_j\}_{j\in J}$ be the equitable partition of X into t parts, where $J = [t]$. Note that $|X_j| = N_j$ is between n_2 and $2n_2$, for every $j \in J$. Let G_j be the bipartite graph $G[X_j \sqcup Y]$ and let D_j be its corresponding digraph. Choose $\varepsilon_0 > 0$ such that $\varepsilon > \frac{4\varepsilon_0}{1+\varepsilon_0}$. By Lemma 3.2.1 there exists a set M_j of closed trails of length $2i+2$ in $D_j \cup D_j^{-1}$, such that $|M_j| \geq (1-\varepsilon_0)\frac{|A(D_j)|}{i+1}$ and M_j is blossom-free, for each $j \in J$ a.a.s. That means, if we let q_j be the probability that $D_j \cup D_j^{-1}$ does not have such set of closed trails, we have $q_j \to 0$ when $n_2 \to \infty$. The probabilities q_j are almost the same since $|X_j|$ only take at most two different values. We let $q = \max\{q_j \mid j \in J\}$. Define the index set $I \subseteq J$ containing those $j \in J$, for which $D_j \cup D_j^{-1}$ does not have a set of closed trails satisfying the conditions stated above. By Markov's inequality, we have $\mathbb{P}(|I| \geq \sqrt{q}t) \leq \sqrt{q}$. Then, when n is large enough, $|I| \leq \varepsilon_0 t$. Similarly as in the proof of (3.2.3) we have a.a.s.

$$\begin{aligned} (1-\varepsilon_0)N_j n_2 p \leq |A(D_j)| \leq (1+\varepsilon_0)N_j n_2 p, \quad \forall j \in J, \\ (1-\varepsilon_0)n_1 n_2 p \leq |E(G)| \leq (1+\varepsilon_0)n_1 n_2 p. \end{aligned} \tag{3.2.20}$$

Let $M = \bigcup_{j\in J\setminus I} M_j$. Since each M_j ($j \in J \setminus I$) is blossom-free and the edge-sets of different D_j are disjoint, M is also blossom-free. We also have:

$$\begin{aligned} |M| = \sum_{j\in J\setminus I} |M_j| &\geq t(1-\varepsilon_0)(1-\varepsilon_0)\frac{|A(D_j)|}{i+1} \\ &\geq t(1-\varepsilon_0)^3\frac{N_j n_2 p}{i+1} \geq (1-\varepsilon_0)^3\frac{n_1 n_2 p}{i+1} \\ &\geq (1+\varepsilon_0)(1-\varepsilon)\frac{n_1 n_2 p}{i+1} \geq (1-\varepsilon)\frac{|E(G)|}{i+1}. \end{aligned} \tag{3.2.21}$$

Therefore, by Lemma 2.2.4 we get the desired ε-near $(2i+2)$-gon embedding Π a.a.s.

□

Now we are ready to compute the genus of random bipartite graphs.

Theorem 3.2.3. *Let $\varepsilon > 0$ and $G \in \mathcal{G}(n_1, n_2, p)$ be a random bipartite graph and suppose that $i \geq 2$ is an integer. If p satisfies $(n_1n_2)^{-\frac{i}{2i+1}} \ll p \ll (n_1n_2)^{-\frac{i-1}{2i-1}}$, $n_1/n_2 < c$ and $n_2/n_1 < c$ where c is a positive real number, then we have a.a.s.*

$$(1-\varepsilon)\frac{i}{2i+2}pn_1n_2 \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{i}{2i+2}pn_1n_2$$

and

$$(1-\varepsilon)\frac{i}{i+1}pn_1n_2 \leq \widetilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{i}{i+1}pn_1n_2.$$

Proof. To prove the lower bound, we count the number of closed trails of G of length at most $2i$. Let C be the number of such closed trails. We have

$$\mathbb{E}(C) \leq \sum_{j=2}^{i} n_1^j n_2^j p^{2j} = o(n_1n_2p). \tag{3.2.22}$$

Then by Markov's inequality, a.a.s. at most $\frac{1}{4(i-1)}\varepsilon pn_1n_2$ closed trails of G have length at most $2i$. Similarly as in the proof of (3.2.3) we get $|E(G)| \geq (1-\frac{1}{2i}\varepsilon)pn_1n_2$, a.a.s. Let Π be a rotation system of G, and let $f(\Pi)$ be the number of faces, and f' be the number of faces of Π with length at most $2i$. Then $2|E(G)| \geq (2i+2)(f(\Pi)-f')+4f' \geq$

$(2i+2)f(\Pi) - (2i-2)f'$. By the above, $f' \le 2C \le \frac{1}{2(2i-2)}\varepsilon p n_1 n_2$. Now we have a.a.s.

$$\begin{aligned} \mathsf{g}(G,\Pi) &= \frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) + 1 \sim \frac{1}{2}(|E(G)| - f(\Pi)) \\ &\ge \frac{i}{2i+2}|E(G)| - \frac{i-1}{2i+2}f' \\ &\ge \left(1 - \frac{1}{2i}\varepsilon\right)\frac{i}{2i+2}pn_1n_2 - \frac{i-1}{2i+2}\frac{1}{2(i-1)}\varepsilon p n_1 n_2 \\ &= (1-\varepsilon)\frac{i}{2i+2}pn_1n_2. \end{aligned} \tag{3.2.23}$$

For the upper bound, by Lemma 3.2.1 we have an ε'-near $(2i+2)$-gon embedding Π, with $\varepsilon' = \frac{i\varepsilon}{2+\varepsilon}$, and let $f(\Pi)$ be the number of faces. Also, we have $|E(G)| \le (1 + \frac{1}{2}\varepsilon)pn_1n_2$. Therefore,

$$\begin{aligned} \mathsf{g}(G,\Pi) &= \frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) + 1 \sim \frac{1}{2}(|E(G)| - f(\Pi)) \\ &\le \frac{1}{2}(|E(G)| - \frac{2(1-\varepsilon')}{2i+2}|E(G)|) \\ &\le \left(1 + \frac{1}{2}\varepsilon\right)\frac{i+\varepsilon'}{2i+2}pn_1n_2 = (1+\varepsilon)\frac{i}{2i+2}pn_1n_2. \end{aligned} \tag{3.2.24}$$

This completes the proof for the orientable genus. The proof for $\widetilde{\mathsf{g}}(G)$ is essentially the same, where the lower bound uses Euler's Formula as in (3.2.23), while for the upper bound we just observe that $\widetilde{\mathsf{g}}(G) \le 2g(G) + 1$, see [44]. □

Theorem 3.2.4. *Let $\varepsilon > 0$ and $G \in \mathcal{G}(n_1, n_2, p)$ be a random bipartite graph. If $n_1 \ge n_2 \gg 1$ and $p \gg n_2^{-\frac{2}{3}}$, then we have a.a.s.*

$$(1-\varepsilon)\frac{pn_1n_2}{4} \le \mathsf{g}(G) \le (1+\varepsilon)\frac{pn_1n_2}{4}$$

and

$$(1-\varepsilon)\frac{pn_1n_2}{2} \le \widetilde{\mathsf{g}}(G) \le (1+\varepsilon)\frac{pn_1n_2}{2}.$$

Proof. The lower bound follows from [44, Proposition 4.4.4]. For the upper bound, we have the same proof as for (3.2.24), except that we use Lemma 3.2.2 (with $i = 1$) instead of Lemma 3.2.1. □

3.2.2 Bipartite Graphs with a Small Part

Now we consider the case when $G \in \mathcal{G}(n_1, n_2, p)$ where $n_1 \gg 1$ and n_2 is a constant. We say S is a *standard graph* of $\mathcal{G}(n_1, n_2, p)$ if S is a bipartite graph on the vertex set $V(S) = X \sqcup Y$ with $|X| \sim n_1$ and $|Y| = n_2$, and we have expected degree distributions for $\mathcal{G}(n_1, n_2, p)$. This means, for every $Y' \subseteq Y$ with $|Y'| = m$, $|\{x \in X \mid N(x) = Y'\}| = \lfloor p^m(1-p)^{n_2-m} n_1 \rfloor$, where $N(x)$ is the set of neighbours of x. Suppose that c is some constant. Then we say that an embedding Π of G is a *near k-gon embedding* (*with respect to c*) if $2|E(G)| - kf_k(\Pi) \leq c$.

Lemma 3.2.5. *Let S be the standard graph of $\mathcal{G}(n_1, n_2, p)$ where $n_1 \gg 1$ and n_2 is a constant. Suppose that $p \gg n_1^{-\frac{1}{2}}$ and let S' be the bipartite graph obtained by removing all vertices of degree at most one in S. Then S' has a near 4-gon embedding with respect to the constant $c = (4n_2 + 14)2^{n_2}$.*

Proof. Let $V(S) = X(S) \sqcup Y(S)$. Note that $n_1 - 2^{n_2} \leq |X(S)| \leq n_1$ and $|Y(S)| = n_2$. For every $Y' \subseteq Y(S)$, let $F_S(Y') = \{x \in X(S) \mid N(x) = Y'\}$. Now consider all of the 2^{n_2} subsets of $Y(S)$, they give us a partition of $X(S) = \bigsqcup_{Y' \subseteq Y(S)} F_S(Y')$. Note that $S[Y' \sqcup F_S(Y')]$ is a complete bipartite graph for every $Y' \subseteq Y(S)$. If $|Y'| \geq 2$, by [51], we have a near 4-gon embedding of $S[Y' \sqcup F_S(Y')]$. Moreover, there is always a near 4-gon embedding with respect to the constant 14 since in the worst case, we may have one 6-gon and one 8-gon apart from the 4-gons. Let $\mathcal{C}(Y')$ be the set of all facial walks of length 4 in the optimal embedding of $S[Y' \sqcup F_S(Y')]$. We can remove from $\mathcal{C}(Y')$ a collection of at most $|Y'|$ closed trails to make $\mathcal{C}(Y')$ free of blossoms with center in Y'. Therefore, we can remove at most $2^{n_2} n_2$ closed trails of length 4 to make $\bigcup_{Y' \in Y(S), |Y'| \geq 2} \mathcal{C}(Y')$ free of blossoms centered in Y'. An obvious extension of Lemma 2.2.4 shows that the union of these sets for all Y' with $|Y'| \geq 2$ gives rise to a near 4-gon embedding of S' with respect to the constant $c = (4n_2 + 14)2^{n_2}$. □

Lemma 3.2.6. *Let S be the standard graph of $\mathcal{G}(n_1, n_2, p)$ where $n_1 \gg 1$ and n_2 is a constant. Suppose that $p \gg n_1^{-\frac{1}{3}}$, then*

$$\mathsf{g}(S) \sim \frac{n_1 n_2 p}{4} \sum_{i=2}^{n_2-1} \frac{i-1}{i+1} \binom{n_2-1}{i} (-p)^i.$$

In particular, when $n_1^{-\frac{1}{3}} \ll p \ll 1$, $\mathsf{g}(S) = (1 + o(1)) \frac{n_1 p^3}{4} \binom{n_2}{3}$.

Proof. Let Π be the rotation system of S' given by Lemma 3.2.5. Since this gives a near 4-gon embedding, we have

$$\begin{aligned}
\mathsf{g}(S') &\sim \frac{1}{2}(2+|E(S')|-f(\Pi)-|V(S')|) \\
&\sim \frac{1}{2}(|E(S')|-f(\Pi)-n_1+(1-p)^{n_2}n_1+(1-p)^{n_2-1}n_1n_2p) \\
&\sim \frac{1}{2}\left(\frac{|E(S')|}{2}-n_1+(1-p)^{n_2}n_1+(1-p)^{n_2-1}n_1n_2p\right) \\
&\sim \frac{1}{2}\left(\frac{n_1n_2p-(1-p)^{n_2-1}n_1n_2p}{2}-n_1+(1-p)^{n_2}n_1+(1-p)^{n_2-1}n_1n_2p\right) \\
&= \frac{1}{2}\left(\frac{1}{2}n_1n_2p\sum_{i=1}^{n_2-1}\binom{n_2-1}{i}(-p)^i+n_1\sum_{i=2}^{n_2}\binom{n_2}{i}(-p)^i\right) \\
&= \frac{n_1n_2p}{4}\sum_{i=2}^{n_2-1}\binom{n_2-1}{i}(-p)^i\frac{i-1}{i+1}.
\end{aligned} \tag{3.2.25}$$

Since $g(S)=g(S')$, this completes the proof. □

Lemma 3.2.7. *Let $\varepsilon>0$ and $G\in\mathcal{G}(n_1,n_2,p)$ where $n_1\gg 1$ and n_2 is a constant. If $p\gg n_1^{-\frac{1}{3}}$ and S is the standard graph of $\mathcal{G}(n_1,n_2,p)$, we have a.a.s. (as $n_1\to\infty$)*

$$(1-\varepsilon)\mathsf{g}(S)\le\mathsf{g}(G)\le(1+\varepsilon)\mathsf{g}(S).$$

Proof. Let $V(G)=X(G)\sqcup Y(G)$ with $|X(G)|=n_1$ and $|Y(G)|=n_2$. For every $Y'\subseteq Y(G)$, where $|Y'|=m\ge 1$, let $F_G(Y')=\{x\in X(G)\mid N(x)=Y'\}$. Then

$$\begin{aligned}
\mathbb{E}(|F_G(Y')|) &= p^m(1-p)^{n_2-m}n_1, \\
\mathbb{E}(|F_G(Y')|^2) &= p^{2m}(1-p)^{2n_2-2m}n_1(n_1-1)+p^m(1-p)^{n_2-m}n_1.
\end{aligned} \tag{3.2.26}$$

For every $t>0$, by Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}\Big(\Big||F_G(Y')|-\mathbb{E}(|F_G(Y')|)\Big|\ge t\mathbb{E}(|F_G(Y')|)\Big) &\le \frac{\mathbb{E}(|F_G(Y')|^2)-\mathbb{E}^2(|F_G(Y')|)}{t^2\mathbb{E}^2(|F_G(Y')|)} \\
&\sim \frac{p^m(1-p)^{n_2-m}n_1}{t^2p^{2m}(1-p)^{2n_2-2m}n_1^2}=\frac{1}{t^2p^m(1-p)^{n_2-m}n_1}.
\end{aligned} \tag{3.2.27}$$

Suppose now that $p \gg n_1^{-\frac{1}{3}}$ and $m \geq 3$. By taking $t = \frac{\varepsilon}{10 n_2 2^{n_2}} p^{3-m}(1-p)^{m-n_2/2}$ in (3.2.27) we obtain that

$$\mathbb{P}\Big(\Big||F_G(Y')| - \mathbb{E}(|F_G(Y')|)\Big| \geq \tfrac{\varepsilon}{10 n_2 2^{n_2}} p^3 (1-p)^{n_2/2} n_1\Big) \leq \frac{100 n_2^2 4^{n_2}}{\varepsilon^2 p^{6-m} n_1} \leq o(1). \quad (3.2.28)$$

Let S be the standard graph of $\mathcal{G}(n_1, n_2, p)$ with $V(S) = X(S) \sqcup Y(S)$. We may assume that $Y(S) = Y(G) = [n_2]$. Let G' be the subgraph obtained from G by deleting all vertices of degree at most 2 in $X(G)$. Observe that deleting vertices of degree at most 1 does not change the genus and that vertices of degree 2 form (at most) $\binom{n_2}{2}$ "parallel" classes, thus

$$\mathsf{g}(G') \leq \mathsf{g}(G) \leq \mathsf{g}(G') + \binom{n_2}{2}. \quad (3.2.29)$$

For every $Y' \subseteq Y$ (and $|Y'| \geq 3$), we consider $F_{G'}(Y') = F_G(Y')$ and $F_S(Y')$. By (3.2.28), these two sets have almost the same cardinality (a.a.s.). More precisely, we have a.a.s.

$$\begin{aligned}\sum_{|Y'|\geq 3} \Big||F_G(Y')| - |F_S(Y')|\Big| &\leq \sum_{|Y'|\geq 3} \Big(\Big||F_G(Y')| - \mathbb{E}(|F_G(Y')|)\Big| + 1\Big) \\ &\leq 2^{n_2}\big(1 + \frac{2^{n_2}\varepsilon}{10 n_2} p^3 (1-p)^{n_2/2} n_1\big) \leq 2^{n_2} + \frac{\varepsilon}{10 n_2} p^3 n_1.\end{aligned} \quad (3.2.30)$$

If $p \ll 1$, then (3.2.30) implies, in particular, that S can be obtained from G' by adding and deleting at most $n_2 2^{n_2} + \frac{\varepsilon}{10} p^3 n_1$ edges a.a.s. Since adding an edge changes the genus by at most 1, and by Lemma 3.2.6, $\mathsf{g}(S) > \frac{1}{5} p^3 n_1 \gg 1$ (if n_1 is large), we obtain that $(1 - \frac{1}{2}\varepsilon)\mathsf{g}(S) \leq \mathsf{g}(G') \leq (1 + \frac{1}{2}\varepsilon)\mathsf{g}(S)$ a.a.s. Together with (3.2.29) this implies the lemma.

Finally, suppose that $p \gg n_1^{-1/n_2}$. In this case we take $t = \frac{\varepsilon p \Psi(p, n_2)}{15 n_2 2^{n_2}}$ in (3.2.27), where $\Psi(p, n_2)$ is defined in Theorem 3.2.8. Therefrom we conclude that with high probability

$$\Big||F_G(Y')| - |F_S(Y')|\Big| \leq 2 + \frac{\varepsilon p \Psi(p, n_2)}{15 n_2 2^{n_2}} |F_S(Y')|.$$

Now we derive similarly as above that S can be obtained from G' by adding and removing less than $2 n_2 2^{n_2} + \frac{\varepsilon}{10} p \Psi(p, n_2) n_1$ edges a.a.s., which is less than $\frac{\varepsilon}{5} p \Psi(p, n_2) n_1$ when n_1 is sufficiently large. The same conclusion as above follows. □

We have all tools to prove the last main statement.

Theorem 3.2.8. *Let $\varepsilon > 0$ and $G \in \mathcal{G}(n_1, n_2, p)$ where $n_1 \gg 1$ and n_2 is a constant.*

(a) *If $p \gg n_1^{-\frac{1}{3}}$ we have a.a.s. (as $n_1 \to \infty$)*

$$(1-\varepsilon)\frac{n_1 n_2 p}{4}\,\Psi(p, n_2) \le \mathsf{g}(G) \le (1+\varepsilon)\frac{n_1 n_2 p}{4}\,\Psi(p, n_2)$$

and

$$(1-\varepsilon)\frac{n_1 n_2 p}{2}\,\Psi(p, n_2) \le \widetilde{\mathsf{g}}(G) \le (1+\varepsilon)\frac{n_1 n_2 p}{2}\,\Psi(p, n_2),$$

where $\Psi(p, n_2) = \sum_{i=2}^{n_2-1} \frac{i-1}{i+1}\binom{n_2-1}{i}(-p)^i$.

(b) *If $n_1^{-\frac{1}{2}} \ll p \ll n_1^{-\frac{1}{3}}$, then a.a.s.*

$$\mathsf{g}(G) = \left\lceil \frac{(n_2-3)(n_2-4)}{12} \right\rceil \quad \textit{and} \quad \widetilde{\mathsf{g}}(G) = \left\lceil \frac{(n_2-3)(n_2-4)}{6} \right\rceil$$

with a single exception that $\widetilde{\mathsf{g}}(G) = 3$ when $n_2 = 7$.

(c) *If $p \ll n_1^{-\frac{1}{2}}$, then a.a.s. $\mathsf{g}(G) = 0$.*

Proof. To prove part (a), we just combine Lemmas 3.2.5, 3.2.6 and 3.2.7.

For case (b), when $Y' \subseteq Y(G)$ with $|Y'| = m \ge 3$, we have

$$\mathbb{E}(|F_G(Y')|) = p^m (1-p)^{n_2-m} n_1 = o(1). \tag{3.2.31}$$

Then by Markov's inequality, $\mathbb{P}(|F_G(Y')| \ge 1) = o(1)$. For the sets $Y_2 \subseteq Y(G)$ with $|Y_2| = 2$, by (3.2.27) we can see that for every $t > 0$, $(1-t)p^2 n_1 \le |F_G(Y_2)| \le (1+t)p^2 n_1$ a.a.s. That means if we remove all vertices with degree 1 in G, we will obtain the complete graph K_{n_2}, in which each edge is replaced by roughly $p^2 n_1$ internally disjoint paths of length 2. By [53] we have $\mathsf{g}(G) = \mathsf{g}(K_{n_2}) = \left\lceil \frac{(n_2-3)(n_2-4)}{12} \right\rceil$ a.a.s. (and similarly for $\tilde{g}(G)$, where the exception occurs when $n_2 = 7$). This proves part (b).

To prove (c), note that when $p \ll n_1^{-1/2}$, none of the subdivided edges of K_{n_2} from case (b) will occur (a.a.s.), and with high probability, every vertex in $X(G)$ will be of degree at most 1. Thus, $\mathsf{g}(G) = 0$ a.a.s. □

Note that in Theorem 3.2.8, when $p = \Theta(n^{-\frac{1}{3}})$, a.a.s. the graph G will be the Levi graph of $mK_{n_1}^3$, where $mK_{n_1}^3$ is the complete 3-uniform multi-hypergraph of order n_1, and each triple has m edges. This problem is hard and of independent interest as a generalization of Ringel-Youngs Theorem. We will discuss it in Chapter 6.

3.3 Genus of Generalized Random Graphs

Now, we define a general notion of random graphs. Let H_m be a weighted complete graph with loops of order m, we may assume it has vertex set $[m]$ and edge set $[m]^2$. The vertex weights of H_m are $c_1, \ldots, c_m$ such that $\sum_{i=1}^m c_i = 1$, and the edge weights are p_{ij}, where $1 \le i \le j \le m$. Let $\mathbb{H}_m(n)$ be a family of graphs which are the blow-up of H_m, that is, for every $G \in \mathbb{H}_m$, $|V(G)| = n$, and there is a partition $V_1, \ldots, V_m$ of $V(G)$ such that $|V_i| = c_i n$, and edge with end vertices in V_i and V_j occurs with probability p_{ij}, for every $1 \le i \le j \le m$.

We consider the genus of $\mathbb{H}_m$-random graphs. Suppose m and c_i are constants for every $1 \le i \le m$, and n is large enough. Since a significantly smaller p_{ij} does not contribute a lot to the genus of $G \in \mathbb{H}_m(n)$ (actually it hides in the error term when n is large enough), we may assume all non-zero p_{ij} have the same magnitude. That is, there is a function $p(n)$ of n, and constants q_{ij} for every $1 \le i \le j \le m$, such that $p_{ij} = q_{ij}p(n)$, i.e. the edges between V_i and V_j in G occur with probability $q_{ij}p(n)$.

3.3.1 $\mathbb{H}_2$-Random Graphs

Random bipartite graphs are actually a special family of $\mathbb{H}_2$-random graphs, where the underlying complete graph H_2 does not have loops. In this section, we deal with a more general random graphs. We assume $c_1 = c_2 = 1/2$ for the convenience.

Lemma 3.3.1. *Let $\varepsilon > 0$, and $G \in \mathbb{H}_2$, where $\frac{i+1}{2}(q_{11} + q_{22}) \ge q_{12}$ and i is odd, $n^{-\frac{i}{i+1}} \ll p(n) \ll n^{-\frac{i-1}{i}}$. Then a.a.s. G has an ε-near $(i+2)$-gon embedding.*

Proof. Note that we only need to consider the case when i is odd, or we can split the graph into a random bipartite graph $\mathcal{G}(n, n, q_{12}p(n))$, and two random graphs $\mathcal{G}(n, q_{11}p(n))$ and $\mathcal{G}(n, q_{22}p(n))$. We first consider the case $\frac{i+1}{2}(q_{11}+q_{22}) \ge q_{12}$. Suppose $0 < \varepsilon_1 < 1/2$ and $\varepsilon_0 = \frac{4i\varepsilon_1+1}{1-\varepsilon_1}$. Assume first $p(n) \ll n^{-\frac{i-1-\varepsilon_1}{i-\varepsilon_1}}$. Let $D \in \mathcal{D}(\mathbb{H}_2)$ be the corresponding digraph of $\mathbb{H}_2$. Now we partition the digraph D into two edge-disjoint digraphs D_1 and D_2. The second digraph D_2 contains some edges in $V_1 \times V_1 \cup V_2 \times V_2$, each of which picked uniformly at random with probability $p = \frac{q_{11}+q_{22}-\frac{2}{i+1}q_{12}}{q_{11}+q_{22}}$. Let $\mathsf{e}_{D_1}(V_1, V_2) = N_1$, $\mathsf{e}_{D_1}(V_1, V_1) + \mathsf{e}_{D_1}(V_2, V_2) = N_2$, $\mathsf{e}_{D_2}(V_1, V_1) = N_3$ and $\mathsf{e}_{D_2}(V_2, V_2) =$

N_4. Similarly as what we proved in Inequality (3.2.3), we have a.a.s.

$$\begin{aligned} (1-\varepsilon_1)q_{12}n^2p(n) \le N_1 \le (1+\varepsilon_1)q_{12}n^2p(n), \\ (1-\varepsilon_1)\frac{1}{i+1}q_{12}n^2p(n) \le N_2 \le (1+\varepsilon_1)\frac{1}{i+1}q_{12}n^2p(n), \\ (1-\varepsilon_1)\frac{1}{2}q_{11}pn^2p(n) \le N_3 \le (1+\varepsilon_1)\frac{1}{2}q_{11}pn^2p(n), \\ (1-\varepsilon_1)\frac{1}{2}q_{22}pn^2p(n) \le N_4 \le (1+\varepsilon_1)\frac{1}{2}q_{22}pn^2p(n). \end{aligned} \tag{3.3.1}$$

Let $d = i+2$ and $\mathcal{H}_1$ be a d-uniform hypergraph, where $V(\mathcal{H}_1)$ is the edge set of D_1 and $E(\mathcal{H}_1)$ is the set of directed closed trails of length d in D_1, which satisfy every edge in $E(\mathcal{H}_1)$ contains one edge in $\mathsf{e}(V_1, V_1)$ or $\mathsf{e}(V_2, V_2)$.

For every pairs $x, y \in V_1 \times V_1 \cup V_2 \times V_2$, let P be the number of directed trails of length $i+1$ from y to x contains all edges in the set $\mathsf{e}_{D_1}(V_1, V_2)$. We have

$$\mathbb{E}(P) = \frac{1}{2^{i+1}}n^i q_{12}^{i+1}p(n)^{i+1},$$

Let $\Delta_1 = \frac{1}{2^{i+1}}n^i q_{12}^{i+1}p(n)^{i+1}$ and $\delta = \frac{2\varepsilon_1}{1-\varepsilon_1}$, then by Chebyshev's inequality, for sufficiently large n we have,

$$\mathbb{P}(|P-\Delta_1| \ge \delta\Delta_1) \le \frac{\mathbb{E}(P^2)-\mathbb{E}^2(P)}{\delta^2\mathbb{E}^2(P)} = O\Big(\frac{1}{np(n)^2}\Big) = o(1). \tag{3.3.2}$$

Let U be the number of edges $\overrightarrow{ab}$ in $\mathsf{e}_{D_1}(V_1, V_1) \cup \mathsf{e}_{D_1}(V_2, V_2)$ such that the number of directed paths from b to a of length $i+1$ is at least $(1+\delta)\Delta_1$ or at most $(1-\delta)\Delta_1$. Therefore,

$$\mathbb{E}(U) = \frac{n^2}{2}q_{12}p(n)\mathbb{P}(|P-\Delta_1| \ge \delta\Delta_1) = O\Big(\frac{n}{p(n)}\Big).$$

Hence by Markov's inequality,

$$\mathbb{P}\Big(U > \frac{\varepsilon_1}{2}q_{12}p(n)n^2\Big) \le O\Big(\frac{1}{np(n)^2}\Big) = o(1).$$

For every pair $(x, y) \in V_1 \oplus V_2$, let P' be the number of directed paths from y to x of length $i+1$ in D_1, and let U' be the number of directed edges $\overrightarrow{ab}$ in $\mathsf{e}_{D_1}(V_1, V_2)$ such that the number of directed paths from b to a of length $i+1$ is at least $(1+\delta)\Delta_1$

or at most $(1-\delta)\Delta_1$. Then similarly we get

$$\mathbb{P}(|P'-\Delta_1| \geq \delta\Delta_1) \leq \frac{\mathbb{E}(P'^2)-\mathbb{E}^2(P')}{\delta^2\mathbb{E}^2(P')} = O\Big(\frac{1}{np(n)}\Big) = o(1),$$
$$\mathbb{P}\Big(U' > \frac{\varepsilon_1}{2}q_{12}n^2p(n)\Big) \leq O\Big(\frac{1}{np(n)^2}\Big) = o(1).$$

Therefore, there exist $V'(\mathcal{H}_1) \subseteq V(\mathcal{H}_1)$ with $|V'(\mathcal{H}_1)| \geq (1-\frac{\varepsilon_1}{1-\varepsilon_1})N_1 + (1-\frac{2\varepsilon_1}{1-\varepsilon_1})N_2 \geq (1-\delta)V(\mathcal{H}_1)$ such that for every $v \in V'(\mathcal{H}_1)$, we have $(1-\delta)\Delta \leq \deg_{\mathcal{H}_1}(x) \leq (1+\delta)\Delta_1$.

Condition (2) in Theorem 2.2.5 holds for $\mathcal{H}_1$ trivially. By a similar argument in the proof of Lemma 4.2.3, condition (3) also holds for $\mathcal{H}_1$ a.s.s. By applying Theorem 2.2.5, we obtain a matching M_1 of $\mathcal{H}_1$ of size at least $(1-\varepsilon_1)\frac{N_1+N_2}{i+2}$. For $\mathcal{H}_1^{-1}$ defined on D_1^{-1}, we have another large matching M_1' disjoint from M_1^{-1} has size at least $(1-\varepsilon_1)\frac{N_1+N_2}{i+2}$. Therefore, $M_1 \cup M_1'$ is a collection of directed triangles in $D_1 \cup D_1^{-1}$ of size at least $2(1-\varepsilon_1)\frac{N_1+N_2}{i+2}$. Since $M_1 \cap M_1' = \varnothing$, it does not contain any non-simple blossoms of size two.

Let $2 \leq j \leq \frac{1}{\varepsilon_1}$ be an integer, and let $B(j)$ be the number of simple $(i+2)$-blossoms of length j in $D_1 \cup D_1^{-1}$, same as what we do in Lemma 3.2.1, for some integer $j/2 \leq a \leq j$ we have

$$\mathbb{E}(B(j)) \leq O\Big(n^{1+j}q_{12}^{2j}p(n)^{2j}\Big(\frac{\max(p_1,p_2)}{p_1+p_2}\Big)^a\Big) \leq o(n^2p(n)),$$

It follows by Markov's inequality that the number of simple $(i+2)$-blossoms of length at most $1/\varepsilon_1$ where each triangle contains vertices in both V_1 and V_2 in $D_1 \cup D_1^{-1}$ is at most $\varepsilon_1 p(n)n^2$, a.a.s.

Obviously at most $2\varepsilon_1(N_1+N_2)$ blossoms have length at least $1/\varepsilon_1$, by removing one triangle in each blossoms, we obtain a blossom-free set $M \subseteq M_1 \cup M_1'$, such that

$$\begin{aligned}|M| &\geq 2(1-\varepsilon_1)\frac{N_1+N_2}{i+2} - \varepsilon_1 q_{12}n^2 - 2\varepsilon_1(N_1+N_2)\\ &= 2(1-\frac{4i\varepsilon_1+1}{1-\varepsilon_1})\frac{N_1+N_2}{i+2} \geq 2(1-\varepsilon_0)\frac{N_1+N_2}{i+2}.\end{aligned}$$

Now consider the digraph D_2, we view D_2 as two disjoint corresponding digraphs $D_2[V_1]$ and $D_2[V_2]$ of $\mathcal{G}(n, p_1'p(n))$ and $\mathcal{G}(n, p_2'p(n))$, respectively, where $p_1' = q_{11}p$ and $p_2' = q_{22}p$. Therefore, by [58, Lemma 1.3], there exists a blossom-free set M' of closed trails of size at least $2(1-\varepsilon_0)\frac{N_3+N_4}{i+2}$.

Since M and M' are edge disjoint, and both M and M' are blossom-free, $M \cup M'$ is also blossom-free and it has size at least $2(1-\varepsilon_0)\frac{N_1+N_2+N_3+N_4}{i+2} = 2(1-\varepsilon_0)\frac{|E(G)|}{i+2}$. Note that when $q_{11}+q_{22}=q_{12}$, we will only get digraph D_1, then we will also get the large blossom-free set of triangles in $D_1 \cup D_1^{-1}$.

Next we choose a real number $t = t(n)$ such that $n^{-\frac{i}{i+1}} \ll p(n)/t \ll n^{-\frac{i-\varepsilon_1}{i+1-\varepsilon_1}}$. By the same argument in Lemma 3.2.1 and by Lemma 2.2.4, we get the desired rotation system Π. □

Lemma 3.3.2. *Let $\varepsilon > 0$, and $G \in \mathbb{H}_2$, where $\frac{i+1}{2}(q_{11}+q_{22}) \le q_{12}$ and i is odd, $n^{-\frac{i}{i+1}} \ll p(n) \ll n^{-\frac{i-1}{i}}$. Then a.a.s. G has an embedding Π, such that*

$$\begin{aligned} &f_{i+2}(\Pi) \ge 2(1-\varepsilon)(\mathsf{e}(V_1,V_1)+\mathsf{e}(V_2,V_2)),\\ &(i+3)f_{i+3}(\Pi) \ge 2(1-\varepsilon)(\mathsf{e}(V_1,V_2)-(i+1)(\mathsf{e}(V_1,V_1)+\mathsf{e}(V_2,V_2))). \end{aligned} \tag{3.3.3}$$

Proof. For the same reason, we also only consider the case when i is odd. Let $D \in \mathcal{D}(\mathbb{H}_2)$ be the corresponding digraph. Now we partition D into two edge disjoint graphs D_1 and D_2. D_1 is obtained by removing edges between V_1 and V_2 uniformly at random with probability $p = \frac{q_{12}-\frac{i+1}{2}(q_{11}+q_{22})}{q_{12}}$. Let N_1 and N_2 be the number of edges in D_1 and D_2, respectively.

Let $0 < \varepsilon_0 < 2$ such that $\varepsilon = 2\varepsilon_0(2-\varepsilon_0)$. By Lemma 3.3.1 and Theorem 3.2.3, there exist two sets M_1 and M_2 with high probability, such that M_1 is a set of directed blossom-free closed trails of length $i+2$ in $D_1 \cup D_1^{-1}$, and M_2 is a set of blossom-free directed closed trails of length $i+3$ in $D_2 \cup D_2^{-1}$. They also satisfy $|M_1| \ge 2(1-\varepsilon_0)\frac{N_1}{i+2}$ and $|M_2| \ge 2(1-\varepsilon_0)\frac{N_2}{i+3}$. Since D_1 and D_2 are edge disjoint, $M = M_1 \cup M_2$ is blossom-free. By Lemma 2.2.4, $M = M_1 \cup M_2$ gives us a rotation system Π.

By Chebyshev's inequality, similarly as the proof of Inequality 3.2.3 we have

$$\begin{gathered} (1-\varepsilon_0)\frac{i+2}{2}n^2(q_{11}+q_{22})p(n) \le N_1 \le (1+\varepsilon_0)\frac{i+2}{2}n^2(q_{11}+q_{22})p(n),\\ (1-\varepsilon_0)n^2 q_{12}pp(n) \le N_2 \le (1+\varepsilon_0)n^2 q_{12}pp(n). \end{gathered}$$

Let $\varepsilon' = \frac{\varepsilon}{2(1-\varepsilon)} > 0$, same as Inequalities 3.3.1 we have

$$\begin{aligned} (1-\varepsilon')\frac{1}{2}q_{11}n^2p(n) \le \mathsf{e}(V_1,V_1) \le (1+\varepsilon')\frac{1}{2}q_{11}n^2p(n),\\ (1-\varepsilon')q_{12}n^2p(n) \le \mathsf{e}(V_1,V_2) \le (1+\varepsilon')q_{12}n^2p(n),\\ (1-\varepsilon')\frac{1}{2}q_{22}n^2p(n) \le \mathsf{e}(V_2,V_2) \le (1+\varepsilon')\frac{1}{2}q_{22}n^2p(n). \end{aligned} \tag{3.3.4}$$

Therefore,

$$(i+2)f_{i+2}(\Pi) = (i+2)|M_1| \geq 2(1-\varepsilon_0)N_1 \geq 2(1-\varepsilon)(1+\varepsilon')\frac{i+2}{2}n^2(q_{11}+q_{22})p(n)$$
$$\geq 2(i+2)(1-\varepsilon)(\mathsf{e}(V_1,V_1)+\mathsf{e}(V_2,V_2)).$$

For faces of length $i+3$, we have

$$\begin{aligned}(i+3)f_{i+3}(\Pi) = (i+3)|M_2| &\geq 2(1-\varepsilon_0)N_2 \\ &\geq 2(1-\varepsilon)(1+\varepsilon')n^2(q_{12}-\frac{i+1}{2}(q_{11}+q_{22}))p(n) \\ &\geq 2(1-\varepsilon)(\mathsf{e}(V_1,V_2)-(i+1)(\mathsf{e}(V_1,V_1)+\mathsf{e}(V_2,V_2))).\end{aligned}$$

This completes the proof. □

Next theorem will give us the minimum genus of $\mathbb{H}_2$.

Theorem 3.3.3. *Let $\varepsilon > 0$, and $G \in \mathbb{H}_2$, where $n^{-\frac{i}{i+1}} \ll p(n) \ll n^{-\frac{i-1}{i}}$. Then a.a.s.*

1. *if i is even or i is odd and $\frac{i+1}{2}(q_{11}+q_{22}) \geq q_{12}$, then*

$$(1-\varepsilon)\frac{ip(n)}{4(i+2)}n^2(q_{11}+q_{22}+2q_{12}) \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{ip(n)}{4(i+2)}n^2(q_{11}+q_{22}+2q_{12}),$$

2. *if i is odd and $\frac{i+1}{2}(q_{11}+q_{22}) < q_{12}$, then*

$$(1-\varepsilon)\mathfrak{F}n^2p(n) \leq \mathsf{g}(G) \leq (1+\varepsilon)\mathfrak{F}n^2p(n),$$

where $\mathfrak{F} = \frac{i+1}{2(i+3)}q_{12} + \frac{i-1}{4(i+3)}(q_{11}+q_{22})$.

Proof. Note that we only need to consider the case when i is odd, or we can split the graph into a random bipartite graph $\mathcal{G}(n,n,q_{12}p(n))$, and two random graphs $\mathcal{G}(n,q_{11}p(n))$ and $\mathcal{G}(n,q_{22}p(n))$. Suppose $\frac{i+1}{2}(q_{11}+q_{22}) \geq q_{12}$ first. We can obtain the lower bound directly by [58]. To verify the upper bound, by Lemma 3.3.1, with high probability, there exists an ε_0-near $(i+2)$-gon embedding Π of G, where we pick $\varepsilon_0 < \frac{i\varepsilon}{2(2+\varepsilon)}$. By Inequalities 3.3.1, $|E(G)| \leq (1+\frac{\varepsilon}{2})\frac{1}{2}n^2(q_{11}+q_{22}+2q_{12})$. By Euler's

Formula,

$$\begin{aligned}
\mathsf{g}(G) &\sim \frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) \\
&\leq \frac{1}{2}(|E(G)| - \frac{2(1-\varepsilon_0)}{i+2}|E(G)|) \\
&\leq \frac{1}{2(i+2)}(i+2\varepsilon_0)(1+\frac{\varepsilon}{2})\frac{1}{2}n^2(q_{11}+q_{22}+2q_{12}) \\
&= (1+\varepsilon)\frac{i}{4(i+2)}n^2(q_{11}+q_{22}+2q_{12}).
\end{aligned}$$

For the case $\frac{i+1}{2}(q_{11}+q_{22}) \geq q_{12}$, pick $\varepsilon_0 > 0$ small enough, apply Lemma 3.3.2 we obtain

$$\begin{aligned}
\mathsf{g}(G) \sim &\frac{1}{2}(|E(G)| - f(\Pi) - |V(G)|) \\
\leq &\frac{1}{2}(\mathsf{e}(V_1,V_1) + \mathsf{e}(V_2,V_2) + \mathsf{e}(V_1,V_2) - 2(1-\varepsilon_0)(\mathsf{e}(V_1,V_1) + \mathsf{e}(V_2,V_2)) \\
&- \frac{2}{i+3}(1-\varepsilon_0)(\mathsf{e}(V_1,V_2) - (i+1)(\mathsf{e}(V_1,V_1) + \mathsf{e}(V_2,V_2)))) \\
= &(\frac{i+1}{2(i+3)} + \frac{\varepsilon_0}{i+3})\mathsf{e}(V_1,V_2) + (\frac{i-1}{2(i+3)} + \frac{4\varepsilon_0}{i+3})(\mathsf{e}(V_1,V_1) + \mathsf{e}(V_2,V_2)) \\
\leq &(\frac{i+1}{2(i+3)} + \frac{\varepsilon_0}{i+3})(1+\varepsilon_0)n^2 q_{12}p(n) \\
&+ (\frac{i-1}{4(i+3)} + \frac{2\varepsilon_0}{i+3})(1-\varepsilon_0)n^2(q_{11}+q_{22})p(n) \\
\leq &(1+\varepsilon)(\frac{i+1}{2(i+3)}q_{12} + \frac{i-1}{4(i+3)}(q_{11}+q_{22})n^2p(n)) \leq (1+\varepsilon)\mathfrak{F}n^2p(n).
\end{aligned}$$

In order to verify the lower bound, let Π be an arbitrary embedding, f' is the number of faces of length $i+2$ of Π. Note that for a face of length $i+2$ of Π contains at least one edge in $E(V_1,V_2)$, it has to contain at least one edge in $E(V_1,V_1) \cup E(V_2,V_2)$ since i is odd. Therefore,

$$f' \leq 2(\mathsf{e}(V_1,V_1) + \mathsf{e}(V_2,V_2)).$$

Also, let f be the number of faces, we have

$$2|E(G)| \geq (i+2)f' + (i+3)(f-f'),$$

which implies $f \leq \frac{2}{i+3}|E(G)| + \frac{1}{i+3}f'$.

Then by Euler's Formula,

$$\begin{aligned}\mathsf{g}(G) &\sim \frac{1}{2}(|E(G)| - f - |V(G)|) \\ &\geq \frac{1}{2}(\frac{i+1}{i+3}|E(G)| - \frac{1}{i+3}f') \\ &\geq \frac{i+1}{2(i+3)}\mathsf{e}(V_1, V_2) + \frac{i-1}{2(i+3)}(\mathsf{e}(V_1, V_1) + \mathsf{e}(V_2, V_2)) \\ &\geq (1-\varepsilon)(\frac{i+1}{2(i+3)}q_{12} + \frac{i-1}{4(i+3)}(q_{11} + q_{22})n^2 p(n)) \geq (1-\varepsilon)\mathfrak{F}n^2 p(n),\end{aligned}$$

This completes the proof. □

We have the following result for the non-orientable embeddings.

Theorem 3.3.4. *Let $\varepsilon > 0$, and $G \in \mathbb{H}_2$, where $n^{-\frac{i}{i+1}} \ll p(n) \ll n^{-\frac{i-1}{i}}$. Then a.a.s.*

1. *if i is even or i is odd and $\frac{i+1}{2}(q_{11} + q_{22}) \geq q_{12}$, then*

$$(1-\varepsilon)\frac{ip(n)}{2(i+2)}n^2(q_{11} + q_{22} + 2q_{12}) \leq \tilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{ip(n)}{2(i+2)}n^2(q_{11} + q_{22} + 2q_{12}),$$

2. *if i is odd and $\frac{i+1}{2}(q_{11} + q_{22}) < q_{12}$, then*

$$2(1-\varepsilon)\mathfrak{F}n^2 p(n) \leq \tilde{\mathsf{g}}(G) \leq 2(1+\varepsilon)\mathfrak{F}n^2 p(n),$$

where $\mathfrak{F} = \frac{i+1}{2(i+3)}q_{12} + \frac{i-1}{4(i+3)}(q_{11} + q_{22})$.

Chapter 4

Genus of Dense Quasirandom Graphs

4.1 Cut Metric and Quasirandomness

Szemerédi Regularity Lemma [61] is one of the most powerful tools in understanding large dense graphs. Szemerédi first used the lemma in his celebrated theorem on long arithmetic progressions in dense subset of integers [62]. Nowadays, the regularity lemma has many connections to other areas of mathematics, for example, analysis, number theory and theoretical computer science. For an overview of applications, we refer to [38, 63]. Regularity Lemma gives us a structural characterization of graphs. Roughly speaking, it says that every large graph can be partitioned into a bounded number of parts such that the graphs between almost every pair of parts is random-like. To make this precise we need some definitions.

Let G be a graph and $X, Y \subseteq V(G)$. We define the *edge density* $d(X,Y) = \mathsf{e}(X,Y)/(|X||Y|)$. We say the pair (X,Y) is *ε-regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have $|d(X',Y') - d(X,Y)| < \varepsilon$. A vertex partition $\mathcal{P} = \{V_i\}_{i=1}^k$ is *equitable* if $V_i \cap V_j = \varnothing$ for every $1 \leq i < j \leq k$, and we have $\big||V_i| - |V_j|\big| \leq 1$. An equitable vertex partition $V_1, \ldots, V_K$ with K parts is *ε-regular* if all but εK^2 pairs of parts (V_i, V_j) $(1 \leq i < j \leq K)$ are ε-regular.

Theorem 4.1.1 (Szemerédi Regularity Lemma)**.** *For every $\varepsilon > 0$ and every integer m, there exists an integer $M = M(m, \varepsilon)$ such that every graph G has an ε-regular partition into K parts, where $m \leq K \leq M$.*

For $\varepsilon > 0$, a partition obtained from the regularity lemma is also called an *ε-Szemerédi partition*. In the original proof of the regularity lemma, the bound M on the number of parts is a tower of twos (We define the tower function $T(n)$ of height n

as follows: $T(1) = 2$, and $T(n) = 2^{T(n-1)}$ for $n \geq 2$.) of height $O(1/\varepsilon^2)$. Unfortunately, this is not far away from the truth. Gowers [24] showed that such an enormous bound is indeed necessary. In this paper, we will always assume that $m \gg 1/\varepsilon$, so $K \gg 1/\varepsilon$ as well.

Recall that the cut metric $\mathsf{d}_\square$ between two (edge-weighted) graphs G and H on the same vertex set V is defined by

$$\mathsf{d}_\square(G, H) = \max_{U,W \subseteq V} \frac{|\mathsf{e}_G(U, W) - \mathsf{e}_H(U, W)|}{|V|^2}. \tag{4.1.1}$$

Here $\mathsf{e}_G(U, W)$ denotes the total weight of the edges with one end vertex in U and the other end vertex in W. When G and H are bipartite graphs defined on the vertex set $X \sqcup Y$, we can define the cut metric as

$$\mathsf{d}_\square(G, H) = \max_{U \subseteq X, W \subseteq Y} \frac{|\mathsf{e}_G(U, W) - \mathsf{e}_H(U, W)|}{|X||Y|}. \tag{4.1.2}$$

If $|X| \sim |Y|$ is large, definitions (4.1.1) and (4.1.2) differ by a small constant factor.

By using the language of cut metric, a weaker statement of the Szemerédi partition $\mathcal{P} = \{V_1, \ldots, V_K\}$ is that, for all but at most εK^2 pairs of (V_i, V_j), we have $\mathsf{d}_\square(G[V_i \sqcup V_j], K(V_i, V_j; p_{ij})) < \varepsilon$, where $K(V_i, V_j; p_{ij})$ is the complete bipartite graph defined on the vertex set $V_i \sqcup V_j$ with all edges weighted $p_{ij} = d_G(V_i, V_j)$.

The cut distance gives us a way to describe the similarity between two graphs, and it is widely used in graph limit theory [36]. Another widely used way to describe the similarity between two large graphs is comparing the homomorphism densities of small graphs. Let $\mathsf{hom}(F, G)$ denote the number of homomorphisms of F into G. Then we define the *homomorphism density*:

$$t(F, G) = \frac{\mathsf{hom}(F, G)}{|V(G)|^{|V(F)|}}.$$

To compare the cut distance and homomorphism densities, we have the following fundamental relation. For the more general version to graphons, see [36, 37].

Lemma 4.1.2 (Counting Lemma)**.** *Let G and G' be two graphs defined on the same vertex set. Then for any graph F,*

$$|t(F, G) - t(F, G')| \leq |E(F)| \mathsf{d}_\square(G, G').$$

Quasirandom graphs are graphs which share many properties with random graphs. The definition of quasirandomness was first introduced in a seminal paper by Chung, Graham and Wilson [13]. In that paper, they listed many equivalent definitions of quasirandom graphs, but essentially, quasirandom graphs are graphs close to random graphs in the sense of cut distance. We will introduce a general form of quasirandomness. In order to avoid to use probability, we define the corresponding edge weighted complete graphs.

We will focus on a more general setting of quasirandom graphs. Let P be a $m \times m$ symmetric matrix with non-negative entries. Let $K(n^{(m)}, P)$ be the complete edge weighted graph, which is defined on the vertex set $V_1 \sqcup \cdots \sqcup V_m$, and for every $i \in [m]$ we have $|V_i| = n$ and the weight of edges between V_i and V_j is given by the (i, j)-entry of P. Although by using the same method, we can deal with more general graphs, for the convenience we only consider the case that each part of the graph has the same size. We will always let the diagonal of P be 0 (then $K(n^{(m)}, P)$ is actually a m-partite graph), and all the entries in P are between 0 and 1. We will use p_{ij} to denote the (i, j)-entry of P.

Recall [36] that a *graphon* is a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$. Let $\widetilde{K}_m = K_m(P)$ be the *quotient graph* of $K(n^{(m)}, P)$, that is, $\widetilde{K}_m$ has m vertices, and the weight of the edge ij is p_{ij}. Let $W_m = W_m(P)$ be a *step function* of $\widetilde{K}_m$, that means $\langle W_m, [\frac{i-1}{m}, \frac{i}{m}] \times [\frac{j-1}{m}, \frac{j}{m}] \rangle = p_{ij}$. We say a sequence of graphs $\{G_n\}$ is W_m-*quasirandom* if $G_n \to W_m$ as $n \to \infty$ where the convergence is in the cut-distance metric.

Definition 4.1.3. Given a graph G of order mn, we say G is ε-W_m-*quasirandom* if $\mathsf{d}_\square(G, K(n^{(m)}, P)) < \varepsilon$, and we write $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon)$ in such a case. If $m = 1$, we write just $\mathcal{Q}(n, p, \varepsilon)$.

Given a graph G of order n and an partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$, we say $\mathcal{P}$ is an ε-*quasirandom partition* if for every $i \in [k]$ we have $|V_i| = n/k$, and for every $1 \le i < j \le k$, $\mathsf{d}_\square\Big(G[V_i \cup V_j], K((n/k)^{(2)}, d(V_i, V_j))\Big) < \varepsilon$. Clearly, one can obtain an ε-quasirandom partition from an ε-Szemerédi partition $\mathcal{P}' = \{V_1, \ldots, V_k\}$. By removing all the edges between irregular pairs, we obtain an ε-quasirandom partition of the resulting graph.

To obtain a Szemerédi partition, there are many know results (for example, [1]), and recently Tao [64] provided a probabilistic algorithm which produces an ε-Szemerédi partition with high probability in constant time (depending on ε). In this paper, we will use a more recent deterministic PTAS due to Fox et al. [19].

Theorem 4.1.4 ([19])**.** *There exists an $O_{\varepsilon,\alpha,k}(n^2)$ time algorithm, which, given $\varepsilon > 0$, and $0 < \alpha < 1$, an integer k, and a graph G on n vertices that admits an ε-Szemerédi partition with k parts, outputs a $(1+\alpha)\varepsilon$-Szemerédi partition of G into k parts.*

4.2 Genus of Quasirandom Graphs

Throughout this section, we will assume our graph is large enough, and ε is small. Given $\varepsilon > 0$, we consider a graph $G \in \mathcal{Q}(n,p,\varepsilon)$. Recall that this means that $\mathsf{d}_\square(G, K(n,p)) < \varepsilon$. Suppose that $\varepsilon < p^8/10$. By the Counting lemma 4.1.2, we have the following result.

Lemma 4.2.1. *Suppose that we have $\varepsilon > 0$ and $0 \le p \le 1$, and a simple graph $G \in \mathcal{Q}(n,p,\varepsilon)$. Then we have*

$$2 \sum_{uv \in E(G)} |\mathsf{n}(u,v) - p^2 n| \le \sqrt{13\varepsilon}\, n^3.$$

Proof. By the Counting Lemma, we have

$$2 \sum_{uv \in E(G)} \mathsf{n}(u,v) = \mathsf{hom}(K_3, G) \ge (p^3 - 3\varepsilon)n^3.$$

Let K_4^- be the graph obtained by K_4 by deleting an edge. Let A be the adjacency matrix of G, and a_{ij} the (i,j)-entry of A. Hence by Chauchy-Schwarz inequality,

$$\begin{aligned}\Bigg(2 \sum_{uv \in E(G)} |\mathsf{n}(u,v) - p^2 n|\Bigg)^2 &= \Bigg(\sum_{u,v \in V(G)} |\mathsf{n}(u,v) - p^2 n| a_{uv}\Bigg)^2 \\ &\le n^2 \sum_{u,v \in V(G)} \Big|\mathsf{n}(u,v) - p^2 n\Big|^2 a_{uv}^2 \\ &= n^2 \Bigg(2 \sum_{uv \in E(G)} \mathsf{n}^2(u,v) - 4p^2 n \sum_{uv \in E(G)} \mathsf{n}(u,v) + 2p^4 n^2 |E(G)|\Bigg) \\ &\sim n^2 \Big(\mathsf{hom}(K_4^-, G) - 4p^2 n \sum_{uv \in E(G)} \mathsf{n}(u,v) + 2p^4 n^2 |E(G)|\Big) \\ &\le n^6 (5\varepsilon + 6\varepsilon p^2 + 2\varepsilon p^4) \le 13\varepsilon n^6.\end{aligned}$$

This completes the proof. □

Given an ε-quasirandom graph $G \in \mathcal{Q}(n,p,\varepsilon)$, we choose a real number $t = t(n)$ such that $n^{-1/2} \ll p/t \ll n^{-(1-\varepsilon)/(2-\varepsilon)}$ (we will use this upper bound to limit the number of short blossoms in the graph later). Let $D \in \mathcal{D}(G)$ be the corresponding digraph of G, and we partition its edges into t parts, putting each edge in one of the parts uniformly at random. We call the resulting digraphs $D_1, \ldots, D_t$. Now for each D_i, let $\mathcal{H}_i$ be the 3-uniform hypergraph where $V(\mathcal{H}_i)$ is the edge set of D_i, and $E(\mathcal{H}_i)$ is the set of directed triangles in D_i. For convenience, we write p_1 to denote p/t.

Lemma 4.2.2. *Let $\Delta = np_1^2/4$. Then there exists a real number $\delta > 0$ such that a.a.s.*

$$|\{x \in V(\mathcal{H}_i) \mid (1-\delta)\Delta \le \deg(x) \le (1+\delta)\Delta\}| \ge (1-\delta)|V(\mathcal{H}_i)|.$$

Proof. We first go back to the graph G. By Lemma 4.2.1, we have

$$2 \sum_{uv \in E(G)} |\mathsf{n}(u,v) - np^2| \le \sqrt{13\varepsilon}\, n^3. \tag{4.2.1}$$

Let $\lambda^2 = \frac{\sqrt{14\varepsilon}}{p^2(p-2\varepsilon)}$. For every edge $uv \in E(G)$, we say uv is *unbalanced* if there are at least $(1+\lambda)np^2$ paths of length 2 between u and v, or at most $(1-\lambda)np^2$ paths of length 2 between them. Assume there are at least $\lambda|E(G)|$ unbalanced edges. Then

$$\begin{aligned} 2 \sum_{uv \in E(G)} |\mathsf{n}(u,v) - np^2| &\ge 2\lambda^2 |E(G)| np^2 \\ &> \lambda^2(n^2p - 2\varepsilon n^2)np^2 > \sqrt{13\varepsilon} n^3, \end{aligned}$$

which contradicts (4.2.1).

We say an edge is *balanced* if it is not unbalanced in G. Let $\mathfrak{B}$ be the set of edges in G which are balanced. Since we create D_i by selecting edges uniformly at random from D, then for every $\varepsilon > 0$, we have

$$\mathbb{E}(|\mathfrak{B} \cap E(D_i)|) = \frac{|\mathfrak{B}|}{t}.$$
$$\mathbb{E}(|\mathfrak{B} \cap E(D_i)|^2) = \frac{|\mathfrak{B}|^2 - |\mathfrak{B}|}{t^2}.$$

Then by Chebyshev's inequality,

$$\mathbb{P}\Big(\Big||\mathfrak{B} \cap E(D_i)| - \mathbb{E}(|\mathfrak{B} \cap E(D_i)|)\Big| > \varepsilon\frac{|\mathfrak{B}|}{t}\Big) \le \frac{|\mathfrak{B}|t^2}{t^2\varepsilon^2|\mathfrak{B}|^2} = \frac{1}{\varepsilon^2|\mathfrak{B}|} = o(1), \tag{4.2.2}$$

which means a.a.s. that D_i contains at least $(1-\varepsilon)|\mathfrak{B}|/t$ edges which are balanced in G. In the graph D_i, for every $e \in E(D_i)$, let $\mathcal{T}_i(e)$ be the set of directed triangles which contain e. Also in the graph G, let $\mathcal{T}(e)$ be the set of triangles which contain e. Therefore,

$$\mathbb{E}(|\mathcal{T}_i(e)|) = \frac{\mathcal{T}_G(e)}{4t^2}, \qquad \mathbb{E}(|\mathcal{T}_i(e)|^2) = \frac{\mathcal{T}_G^2(e) - \mathcal{T}_G(e)}{16t^4}.$$

Then by Chebyshev's inequality,

$$\mathbb{E}\Big(\Big||\mathcal{T}_i(e)| - \mathbb{E}(|\mathcal{T}_i(e)|)\Big| > \varepsilon\mathbb{E}(|\mathcal{T}_i(e)|)\Big) \leq \frac{\mathbb{E}^2(|\mathcal{T}_i(e)|) - \mathbb{E}(|\mathcal{T}_i(e)^2|)}{\varepsilon^2\mathbb{E}^2(|\mathcal{T}_i(e)|)} = \frac{1}{|\mathcal{T}_G(e)|} = o(1).$$

This means at least $(1-\varepsilon)^2|\mathfrak{B}|/t$ edges in D_i are contained in at least $(1-\varepsilon)(1-\lambda)\Delta$ directed triangles and in at most $(1+\varepsilon)(1+\lambda)\Delta$ triangles.

Since $|\mathfrak{B}|/t \geq (1-\lambda)|E(G)|/t$, and a.a.s. $|V(\mathcal{H}_i)| < (1+\varepsilon)|E(G)|/t$ (also by Chebyshev's inequality), we choose $\delta \geq \delta_0 := \max\{\lambda+\varepsilon+\varepsilon\lambda, \psi(\varepsilon,\lambda)\}$, where $\psi(\varepsilon,\lambda) = \frac{\lambda(1-\varepsilon)^2+\varepsilon(3-\varepsilon)}{1+\varepsilon}$. Therefore,

$$\begin{aligned}
&(1+\varepsilon)(1+\lambda)\Delta = (1+\lambda+\varepsilon+\varepsilon\lambda)\Delta < (1+\delta)\Delta,\\
&(1-\varepsilon)(1-\lambda)\Delta = (1-\lambda-\varepsilon+\varepsilon\lambda)\Delta > (1-\delta)\Delta,\\
&(1-\varepsilon)^2\frac{|\mathfrak{B}|}{t} \geq (1-\varepsilon)^2(1-\lambda)\frac{|V(\mathcal{H}_i)|}{1+\varepsilon} = (1-\psi(\varepsilon,\lambda))|V(\mathcal{H}_i)| \geq (1-\delta)|V(\mathcal{H}_i)|,
\end{aligned}$$

which completes the proof. □

Now we fix the value Δ in Lemma 4.2.2. The lemma shows that a.a.s. $\mathcal{H}_i$ satisfies condition (1) of Theorem 2.2.5. Condition (2) holds trivially since for every $a, b \in V(\mathcal{H}_i)$, at most one triangle of G contains both a and b as its edges. Condition (3) holds by the following lemma.

Lemma 4.2.3. *Let F_i be the number of directed triangles in D_i which contain at least one directed edge $\overrightarrow{uv} \in P^\delta$, where P^δ is the set of pairs of vertices $(u,v) \in V^2(D_i)$ such that the number of directed paths from v to u of length 2 in D_i is at least $(1+\delta)\Delta$. Then a.a.s. $F_i < \delta\Delta|V(\mathcal{H}_i)|$.*

Proof. By using probabilistic method, we can almost pass the counting properties from G to D_i, then it suffices to consider the dense graph G. Let F be the number of triangles which contain at least one unbalanced edge in G. Since D_i is constructed randomly, if $F < (1-\varepsilon)\delta np^2|E(G)|$, then $F_i < \delta\Delta|V(\mathcal{H}_i)|$ a.a.s.

By Lemma 4.2.1, we have

$$2\sum_{uv\in E(G)\setminus\mathfrak{B}} \mathsf{n}(u,v) < \sqrt{13\varepsilon}\, n^3 + \frac{2\sqrt{13\varepsilon}}{\delta} n^3.$$

Therefore, we let $\delta > \max\{\delta_0, \phi(\varepsilon)\}$, where $\phi(\varepsilon) = \frac{\sqrt{13\varepsilon}+\sqrt{13\varepsilon+16p^2\sqrt{13\varepsilon}}}{4p^2(p-2\varepsilon)(1-\varepsilon)}$. Now Lemma 4.2.2 still holds, and we have the following quadratic function always positive for $x \geq \delta$:

$$q(x) = 2p^2(p-2\varepsilon)(1-\varepsilon)x^2 - \sqrt{13\varepsilon}x + 4p^2.$$

This implies

$$F < \sum_{uv\in E(G)\setminus\mathfrak{B}} \mathsf{n}(u,v) < \delta n^3 p^2 (p-2\varepsilon)(1-\varepsilon) < (1-\varepsilon)\delta n p^2 |E(G)|,$$

which completes the proof. □

Now we are going to apply Theorem 2.2.5 to construct a large set of blossom-free triangles. Using them together with Lemma 2.2.4, we will be able to obtain a near triangular embedding. Specifically, we say that an embedding of a graph is ε-near triangular (ε-near quadrangular) if all but at most $2\varepsilon|E(G)|$ edges of G belong to two triangular (quadrangular) faces.

Theorem 4.2.4. *For every $\varepsilon > 0$, every graph $G \in \mathcal{Q}(n,p,\varepsilon)$ admits a 9ε-near triangular embedding of G.*

Proof. Suppose that $G \in \mathcal{Q}(n,p,\varepsilon)$. We will use the notation introduced earlier. For every $i \in [t]$, $\mathcal{H}_i$ satisfies conditions (1)–(3) in Theorem 2.2.5 a.a.s. That means, if q is the probability that $\mathcal{H}_i$ does not satisfy all of the conditions, then $q \to 0$ as $n \to \infty$. Let $I \subseteq [t]$ be the index set, such that for every $j \in I$, $\mathcal{H}_j$ does not satisfy all the conditions listed in Theorem 2.2.5. By Markov's inequality we obtain $|I| \leq \varepsilon t$ a.a.s. Note that "a.a.s." here comes from the way we construct D_i, that means there exists a construction of D_i for every $i \in [t]$ such that $|I| \leq \varepsilon t$. For the convenience, we denote $J = [t] \setminus I$.

For every $i \in J$, apply Theorem 2.2.5. For sufficiently large n, there exists a matching M_i of $\mathcal{H}_i$ of size at least $(1-\varepsilon)\frac{|E(D_i)|}{3}$. Let $\mathcal{H}_i^{-1}$ be the 3-uniform hypergraph defined on the digraph D^{-1}, then M_i^{-1} is a large matching in $\mathcal{H}_i^{-1}$. Apply Theorem 2.2.5 again, we obtain another matching M_i' in $\mathcal{H}_i^{-1}$ which is disjoint with M_i^{-1}, and it has size at least $(1-\varepsilon)\frac{|E(D_i)|}{3}$. Therefore, in the graph $D_i \cup D_i^{-1}$, $M_i \cup M_i'$ has size at least $2(1-\varepsilon)\frac{|E(D_i)|}{3}$, and does not have non-simple blossoms of length 2.

In order to apply Lemma 2.2.4 to obtain a rotation system and a surface embedding, we have to remove one of the cycles from each of the blossoms that appear in $M_i \cup M_i'$. We first consider short blossoms, that means, blossoms of length at most $1/\varepsilon$. We use $\overrightarrow{\mathcal{B}_j}$ to denote 3-blossom-graphs of length j in $D \cup D^{-1}$, and $\mathcal{B}_j$ is the underlying simple graph in G, where j is an integer such that $2 \le j \le 1/\varepsilon$. By Counting Lemma, we have

$$\mathsf{hom}(\mathcal{B}_j, G) \le n^{j+1}p^{2j} + 2j\varepsilon n^{j+1}.$$

By the way we construct D_i, for every $i \in [t]$, given a 3-blossom simple graph $\mathcal{B}_j$, $\mathbb{P}(\overrightarrow{\mathcal{B}_j} \in D_i \cup D_i^{-1}) = \frac{1}{2^{j-1}t^{2j}}$. Let $N_i(j)$ denote the number of $\overrightarrow{\mathcal{B}_j}$ in $D_i \cup D_i^{-1}$. Then by Chebyshev's inequality

$$N_i(j) < (1+\varepsilon)\left(\frac{n^{j+1}p_1^{2j}}{2^{j-1}} + \frac{2j\varepsilon n^{j+1}p_1^{2j}}{p^{2j}2^{j-1}}\right) \ll n^2 p_1, \quad a.a.s.$$

That means $\sum_{j=1}^{1/\varepsilon} N_i(j) < \varepsilon(1-\varepsilon)n^2p_1/2 < \varepsilon|E(D_i)|$, a.a.s. Then we can remove a triangle from each blossom of length at most $1/\varepsilon$, we obtain a subset of $M_i \cup M_i'$ of size at least $2(1-3\varepsilon)\frac{|E(D_i)|}{3}$.

In the next step we will argue that there is only a small number of long blossoms. If $\mathcal{B}$ and $\mathcal{B}'$ are two blossoms with center v in $M_i \cup M_i'$, it is clear that the tips of $\mathcal{B}$ and $\mathcal{B}'$ are disjoint. Therefore if v has l neighbours in D_i, at most εl different 3-blossoms of length at least $1/\varepsilon$ have center v. Therefore, the total number of long blossoms is at most $2\varepsilon|E(D_i)|$. By removing one of the triangles from each long blossom, we finally obtain a blossom-free subset $\mathfrak{M}_i$ of $M_i \cup M_i^{-1}$, such that

$$|\mathfrak{M}_i| \ge 2(1-3\varepsilon)\frac{|E(D_i)|}{3} - 2\varepsilon|E(D_i)| = 2(1-6\varepsilon)\frac{|E(D_i)|}{3}.$$

Note that for every $i \in J$, by Chebyshev's inequality, there exists a set $J' \subseteq J$ such that $|J'| \ge (1-\varepsilon)|J|$ and for every $i \in J'$,

$$(1-\varepsilon)\frac{|E(G)|}{t} \le |E(D_i)| \le (1+\varepsilon)\frac{|E(G)|}{t}.$$

Now we take the union of all $\mathfrak{M}_i$ with $i \in J'$, the union set is blossom-free since all pairs of the graphs D_i and D_j with $i \ne j$ are edge disjoint. Now, applying Lemma

2.2.4, we obtain an embedding Π from $\bigcup_{i\in J'} \mathfrak{M}_i$ such that

$$\begin{aligned}3f_3(\Pi) &\geq 3\sum_{i\in J'} |\mathfrak{M}_i| \geq 2(1-6\varepsilon)(1-\varepsilon)^3|E(G)|\\ &\geq 2(1-9\varepsilon)|E(G)|,\end{aligned}$$

which completes the proof. □

Theorem 4.2.5. *Suppose that $\varepsilon > 0$, and every $G \in \mathcal{Q}(n, p, \varepsilon/18)$. Then we have.*

$$\frac{\mathsf{e}(G)}{6} \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{6}$$

and

$$\frac{\mathsf{e}(G)}{3} \leq \widetilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{3}.$$

Proof. By Theorem 4.2.4, G has an $\varepsilon/2$-near triangular embedding Π. Therefore,

$$\begin{aligned}\mathsf{g}(G) \leq \mathsf{g}(G,\Pi) &\sim \frac{1}{2}(\mathsf{e}(G) - f(\Pi)) \leq \frac{1}{2}(\mathsf{e}(G) - f_3(\Pi))\\ &\leq \frac{1}{2}(\mathsf{e}(G) - 2(1-\frac{\varepsilon}{2})\mathsf{e}(G)) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{6}.\end{aligned}$$

The lower bound for $\mathsf{g}(G)$ follows by [44, Proposition 4.4.4] directly, and the error term comes from Euler's formula. For the non-orientable genus, since $\widetilde{\mathsf{g}}(G) \leq \mathsf{g}(G)+1$, we can just simply multiply by 2 the formula for the orientable genus. □

4.3 Genus of Bipartite and Tripartite Quasirandom Graphs

Given $\varepsilon > 0$, we consider a graph $G \in \mathcal{Q}(n^{(2)}, p, \varepsilon)$. This means G is defined on the vertex set $V_1 \sqcup V_2$, each V_i of size n, and $\mathsf{d}_\square(G, K(n^{(2)}, p)) < \varepsilon$. The following lemma is an analogous result to Lemma 4.2.1.

Lemma 4.3.1. *Let $\varepsilon > 0$ and let $G_2 \in \mathcal{Q}(n^{(2)}, P, \varepsilon)$. For every vertices $u \in V_1$ and $v \in V_2$, we have*

$$2\sum_{uv\in E(G_2)} |\mathsf{p}_3(u,v) - n^2p^3| \leq \sqrt{17\varepsilon}\, n^4.$$

Proof. First by Counting Lemma, we have

$$2\sum_{uv\in E(G_2)} \mathsf{p}_3(u,v) = \mathsf{hom}(C_4, G_2) \geq (p^4 - 4\varepsilon)n^4.$$

Therefore, by Cauchy-Schwartz inequality,

$$\begin{aligned}\left(2\sum_{uv\in E(G_2)}|\mathsf{p}_3(u,v)-n^2p^3|\right)^2 &\le 2n^2\sum_{uv\in E(G_2)}\left|\mathsf{p}_3(u,v)-n^2p^3\right|^2\\ &\le n^2(n^6p^7+8\varepsilon n^6+n^6p^7+\varepsilon n^6p^6-2n^6p^7+8\varepsilon n^6p^3)\\ &< 17\varepsilon n^8,\end{aligned}$$

and this proves the lemma. □

Like what we did for quasirandom graphs in the proof of Theorem 4.2.4, in order to bound the number of short blossoms, we choose an integer $t=t(n)$ and let $p_1=p/t$, such that $n^{-\frac{2}{3}}\ll p_1\ll n^{-\frac{2-\varepsilon}{3-\varepsilon}}$. Let $D\in\mathcal{D}(G_2)$ be the corresponding digraph of G_2. We partition its edges into t parts uniformly at random, the resulting edge disjoint digraphs are $D_1,\ldots,D_t$. For each D_i, let $\mathcal{H}_i$ be the 4-uniform hypergraph such that $V(\mathcal{H}_i)=E(D_i)$ and the edge set of $\mathcal{H}_i$ is the set of directed closed trails of length 4. Now we are going to check condition (1) of Theorem 2.2.5.

Lemma 4.3.2. *Let $\Delta_2=n^2p_1^3/8$, then there exists a real number $\delta>0$ such that a.a.s.*

$$|\{x\in V(\mathcal{H}_i)\mid(1-\delta)\Delta_2\le\deg(x)\le(1+\delta)\Delta_2\}|\ge(1-\delta)|V(\mathcal{H}_i)|.$$

Proof. Recall that $p=tp_1$. Suppose $\lambda^2=\frac{\sqrt{18\varepsilon}}{p^3(p-\varepsilon)}$ and uv is an edge in G_2. We say uv is *balanced* if $\mathsf{p}_3(u,v)$ is at least $(1-\lambda)n^2p^3$ and at most $(1+\lambda)n^2p^3$, otherwise it is *unbalanced*. Assume that at least $\lambda|E(G_2)|$ edges are unbalanced. Then we have

$$2\sum_{uv\in E(G_2)}|\mathsf{p}_3(u,v)-n^2p^3|\ge\lambda^2n^2p^3(n^2p-\varepsilon n^2)\ge\sqrt{18\varepsilon}\,n^4,$$

which contradicts Lemma 4.3.1. Then in the graph G_2, at least $(1-\lambda)|E(G_2)|$ edges are balanced.

For the graph D_i, by Chebyshev's inequality, it satisfies

$$(1-\varepsilon)\frac{|E(G)|}{t}\le|E(D_i)|\le(1+\varepsilon)\frac{|E(G)|}{t},\quad a.a.s.$$

Let U be the number of balanced edges in G_2. Then a.a.s. D_i contains at least $(1-\varepsilon)U/t$ edges which are balanced in G_2, and most of them are still in D_i. To be more precise, at least $(1-\varepsilon)^2U/t$ edges are contained in at least $(1-\varepsilon)(1-\lambda)n^2p_1^3/8$ directed cycles of length 4, and are contained in at most $(1+\varepsilon)(1+\lambda)n^2p_1^3/8$ directed cycles of length 4. Now we let $\delta \geq \max\{\lambda + \varepsilon + \varepsilon\lambda, \psi(\varepsilon, \lambda)\}$, where $\psi(\varepsilon, \lambda)$ is given in Lemma 4.2.2. We have a.a.s.

$$
\begin{aligned}
(1-\varepsilon)^2 \frac{U}{t} &\geq (1-\varepsilon)^2(1-\lambda)\frac{|E(G_2)|}{t} \\
&\geq \frac{(1-\varepsilon)^2(1-\lambda)}{1+\varepsilon}|V(\mathcal{H}_i)| \geq (1-\delta)|V(\mathcal{H}_i)|.
\end{aligned}
$$

Also for the number of 4-cycles, we have $1+\delta > (1+\varepsilon)(1+\lambda)$ and $1-\delta < (1-\varepsilon)(1-\lambda)$. This completes the proof. □

For the Condition (2) in Theorem 2.2.5, a.a.s. D_i contains at most $t^2 n$ cycles of length 4 that contain two edges. Since $t \ll n^{1/3}$, this is definitely less than $\delta\Delta_2 = \Theta(n^2)$. In order to verify Condition (3), we have the following lemma. The proof is same as what we did in Lemma 4.2.3, and we omit the details.

Lemma 4.3.3. *Let F_i be the number of directed cycles of length 4 in D_i which contain at least one directed edge $\overrightarrow{uv} \in P^\delta$, where P^δ is the set of pairs of vertices $(u, v) \in V_1 \oplus V_2$ such that the number of directed paths from v to u of length 3 is at least $(1+\delta)\Delta_2$. Then with high probability, $F_i < \delta\Delta_2|V(\mathcal{H}_i)|$.*

Then for each $\mathcal{H}_i$, it satisfies condition (1)–(3) a.a.s. Then we can apply Theorem 2.2.5 to obtain a rotation system.

Theorem 4.3.4. *Let $\varepsilon > 0$ and $G_2 \in \mathcal{Q}(n^{(2)}, P, \varepsilon)$, then G_2 has a 10ε-near quadrangular embedding.*

Proof. By the same argument as was used in the proof of Theorem 4.2.4, there exists a construction of $D_1, \dots, D_t$ by partitioning the edges uniformly at random from the corresponding digraph D. By Markov's inequality, we have $|I| \leq \varepsilon t$, where $I \subseteq [t]$ is the index set and every $\mathcal{H}_i$ with $i \in I$ fails to satisfy some of the conditions listed in Theorem 2.2.5. We denote the set $[t] \setminus I$ by J.

Then for every $i \in J$, there exists a matching M_i of $\mathcal{H}_i$ of size at least $(1-\varepsilon)\frac{|E(D_i)|}{4}$, and another matching M_i' in $\mathcal{H}_i^{-1}$ which is disjoint with M_i^{-1}, and it also has size at least $(1-\varepsilon)\frac{|E(D_i)|}{4}$. Then in the graph $D_i \cup D_i^{-1}$, $M_i \cup M_i'$ has size at least $(1-\varepsilon)\frac{|E(D_i)|}{2}$, and does not have non-simple blossoms of length 2.

Now we proceed with removing all the blossoms in $M_i \cup M_i'$. For all of the blossoms of length at most $1/\varepsilon$, let $\overrightarrow{\mathcal{B}_j}$ be the 4-blossom-graphs of length j in $D \cup D^{-1}$, and $\mathcal{B}_j$ is the underlying simple graph in G, where $2 \leq j \leq 1/\varepsilon$. Therefore,

$$
\mathsf{hom}(\mathcal{B}_j, G) \leq n^{2j+1}p^{3j} + 3j\varepsilon n^{2j+1}.
$$

For every $i \in [t]$, giving a 4-blossom simple graph $\mathcal{B}_j$, $\mathbb{P}(\overrightarrow{\mathcal{B}_j} \in D_i \cup D_i^{-1}) = \frac{1}{2^{j-1}t^{3j}}$. Let $N_i(j)$ be the number of $\overrightarrow{\mathcal{B}_j}$ in $D_i \cup D_i^{-1}$. Then we have

$$N_i(j) < (1+\varepsilon)\Big(\frac{n^{2j+1}p_1^{3j}}{2^{2j-1}} + \frac{3j\varepsilon n^{2j+1}p_1^{3j}}{p^{3j}2^{2j-1}}\Big) \ll n^2 p_1, \quad a.a.s.$$

That means $\sum_{j=1}^{1/\varepsilon} N_i(j) < \varepsilon(1-\varepsilon)n^2 p_1 < \varepsilon|E(D_i)|$, a.a.s.

Now we are going to remove one cycle from each of the blossoms. Since the number of 4-blossoms of length at least $1/\varepsilon$ is bounded by $2\varepsilon|E(D_i)|$, we obtain a blossom-free subset of $M_i \cup M_i'$ of size at least $(1-7\varepsilon)\frac{|E(D_i)|}{2}$, a.a.s. By Lemma 2.2.4 there exists a construction of D_i for every $i \in [t]$, and $(1-\varepsilon)^2 t$ of them have almost the correct number of edges, and they give rise to a 7ε-near quadrangular embedding. Then G has an embedding Π such that

$$4f_4(\Pi) \geq 2(1-7\varepsilon)(1-\varepsilon)^3|E(G)| \geq 2(1-10\varepsilon)|E(G)|,$$

which completes the proof. □

Now we are going to compute the genus of $G_2 \in \mathcal{Q}(n^2, p, \varepsilon)$.

Theorem 4.3.5. *Let $\varepsilon > 0$, there exist a real number $\lambda > 0$ such that for every positive number $\tau < \lambda$ and every $G_2 \in \mathcal{Q}(n^2, p, \tau)$, we have.*

$$\frac{\mathsf{e}(G_2)}{4} \leq \mathsf{g}(G_2) \leq (1+\varepsilon)\frac{\mathsf{e}(G_2)}{4},$$

and

$$\frac{\mathsf{e}(G_2)}{2} \leq \widetilde{\mathsf{g}}(G_2) \leq (1+\varepsilon)\frac{\mathsf{e}(G_2)}{2}.$$

Proof. It suffcies to consider the orientable genus. The lower bound follows by [44, Proposition 4.4.4]. For the upper bound, we use Theorem 4.3.4 which gives an $\varepsilon/2$-near quadrangular embedding Π. Therefore,

$$\mathsf{g}(G_2) \leq \mathsf{g}(G_2, \Pi) \leq \frac{1}{2}(\mathsf{e}(G_2) - f_4(\Pi)) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{4},$$

which completes the proof. □

In the next case, we will focus on tripartite quasirandom graphs. We use $\mathcal{Q}(n^{(3)}, p, \varepsilon)$ to denote the family of quasirandom graphs defined on the vertex set $V_1 \sqcup V_2 \sqcup V_3$ with $|V_i| = n$ for every $1 \leq i \leq 3$, and for every $G \in \mathcal{Q}(n^{(3)}, p, \varepsilon)$, we have $\mathsf{d}_\square(G, K(n^{(3)}, p)) < \varepsilon$. We start with an analogous result to Lemma 4.2.1.

Lemma 4.3.6. *Let $\varepsilon > 0$ and $G_3 \in \mathcal{Q}(n^{(3)}, p, \varepsilon)$, where $V(G_3) = V_1 \sqcup V_2 \sqcup V_3$. Then for every $i \neq j$ and $u \in V_i$, $v \in V_j$, we have*

$$2 \sum_{uv \in E(G_3)} |\mathsf{n}(u,v) - np^2| \leq \sqrt{13\varepsilon}\, n^3.$$

Proof. First apply the Counting Lemma,

$$2 \sum_{uv \in E(G_3)} \mathsf{n}(u,v) = \mathsf{hom}(K_3, G_3) \geq (6p^3 - 3\varepsilon)n^3.$$

Therefore,

$$\begin{aligned}\left(2 \sum_{uv \in E(G_3)} |\mathsf{n}(u,v) - np^2|\right)^2 &\leq 2n^2 \sum_{uv \in E(G_2)} \left|\mathsf{n}(u,v) - np^2\right|^2 \\ &\leq n^2\Big((6p^5 + 5\varepsilon)n^4 + n^2p^4(6n^2p + \varepsilon n^2) - 2n^2p(6p^3n^3 - 3\varepsilon n^3)\Big) \\ &\leq n^6(5\varepsilon + 2\varepsilon p^4 + 6\varepsilon) \leq 13\varepsilon n^6,\end{aligned}$$

and this proves the lemma. □

Similarly as what we did before, we choose an integer $t = t(n)$ and let $p_1 = p/t$, such that $n^{-\frac{1}{2}} \ll p_1 \ll n^{-\frac{1-\varepsilon}{2-\varepsilon}}$. Let $D \in \mathcal{D}(G_3)$ be the corresponding digraph of G_3. We partition the edge set of D into t parts uniformly at random, and the resulting edge disjoint digraphs are denoted by $D_1, \ldots, D_t$. For each D_i, we define $\mathcal{H}_i$ be the 3-uniform hypergraph for every $i \in [t]$, such that $V(\mathcal{H}_i) = E(D_i)$ and the edge set of $\mathcal{H}_i$ is the set of directed triangles in D_i. The following lemma implies that Condition (1) in Theorem 2.2.5 holds a.a.s. in $\mathcal{H}_i$.

Lemma 4.3.7. *Let $\Delta_3 = np_1^2/4$, then for every $i \in [t]$ there exists a real number $\delta > 0$ such that a.a.s.*

$$|\{x \in V(\mathcal{H}_i) \mid (1-\delta)\Delta_3 \leq \deg(x) \leq (1+\delta)\Delta_3\}| \geq (1-\delta)|V(\mathcal{H}_i)|.$$

Proof. Suppose $\lambda^2 = \frac{\sqrt{14\varepsilon}}{p^2(3p-\varepsilon)}$. In order to use Counting Lemma, we go back to graph G_3. Suppose uv is an edge in G_3. Similarly as what we did in quasirandom graphs, we say uv is *balanced* if $\mathsf{n}(u,v)$ is at least $(1-\lambda)np^2$ and at most $(1+\lambda)np^2$, otherwise it is *unbalanced*. Assume that at least $\lambda|E(G_2)|$ edges are unbalanced. Then we have

$$2 \sum_{uv \in E(G_2)} |\mathsf{n}(u,v) - np^2| \geq \lambda^2 np^2(3n^2p - \varepsilon n^2) \geq \sqrt{14\varepsilon}\, n^3,$$

which contradicts Lemma 4.3.6. This means that in the graph G_3, at least $(1-\lambda)|E(G_3)|$ edges are balanced.

Note that the graph D_i has correct number of edges a.a.s. This means

$$(1-\varepsilon)\frac{|E(G_3)|}{t} \leq |E(D_i)| \leq (1+\varepsilon)\frac{|E(G_3)|}{t}, \quad a.a.s.$$

By a similar argument as we used before, we let U be the number of balanced edges in G_3. Then D_i contains at least $(1-\varepsilon)U/t$ balanced edges in G_3, and at least $(1-\varepsilon)^2U/t$ edges are contained in at least $(1-\varepsilon)(1-\lambda)np_1^2/4$ directed triangles and at most $(1+\varepsilon)(1+\lambda)np_1^2/4$ triangles in D_i, a.a.s. Now we let $\delta \geq \max\{\lambda+\varepsilon+\varepsilon\lambda, \psi(\varepsilon,\lambda)\}$, where $\psi(\varepsilon,\lambda)$ is defined in Lemma 4.2.2. This completes the proof. □

Since edges in $\mathcal{H}_i$ are triangles, Condition (2) in Theorem 2.2.5 holds trivially. For Condition (3), we have the following lemma. The proof follows almost the same step as were used in the proof of Lemma 4.2.3. We omit the details.

Lemma 4.3.8. *Let F_i be the number of directed triangles in D_i which contain at least one directed edge $\overrightarrow{uv} \in P^\delta$, where P^δ is the set of pairs of vertices $(u,v) \in V_i \times V_j$ such that the number of directed paths from v to u of length 2 is at least $(1+\delta)\Delta_3$, for every $i \neq j$. Then with high probability, $F_i < \delta\Delta_3|V(\mathcal{H}_i)|$.*

Now we are going to construct an embedding of G_3 by using Lemma 2.2.4 and Theorem 2.2.5.

Theorem 4.3.9. *Let $\varepsilon > 0$, and $G_3 \in \mathcal{Q}(n^{(3)}, p, \varepsilon)$. Then G_3 has a 9ε-near triangular embedding.*

Proof. There exists a construction of $D_1, \ldots, D_t$ such that set $I \subseteq [t]$ containing all indices i for which $\mathcal{H}_i$ fails to satisfy conditions (1)–(3) of Theorem 2.2.5 is small, $|I| < \varepsilon t$. Let $J = [t] \setminus I$.

Now we apply Theorem 2.2.5 for every $\mathcal{H}_i$ where $i \in J$. As before, there exists a matching M_i of $\mathcal{H}_i$ of size at least $(1-\varepsilon)\frac{|E(D_i)|}{3}$, and another matching M_i' in $\mathcal{H}_i^{-1}$ which is disjoing with M_i^{-1}, and it also has size at least $(1-\varepsilon)\frac{|E(D_i)|}{3}$. Then in the graph $D_i \cup D_i^{-1}$, $M_i \cup M_i'$ has size at least $2(1-\varepsilon)\frac{|E(D_i)|}{3}$, and does not contain non-simple blossoms of length 2.

As before, we are going to remove a cycle from each of the blossoms. For the short blossoms of length at most $1/\varepsilon$, we let $\overrightarrow{\mathcal{B}_j}$ be the 3-blossom-graphs of length j in $D \cup D^{-1}$, and let $\mathcal{B}_j$ be the underlying simple graph in G. Then,

$$\mathsf{hom}(\mathcal{B}_j, G) \leq 3n^{j+1}p^{2j} + 2j\varepsilon n^{j+1}.$$

For every $i \in [t]$, given a 3-blossom simple graph B_j, $\mathbb{P}(\overrightarrow{\mathcal{B}_j} \in D_i \cup D_i^{-1}) = \frac{1}{2^{j-1}t^{2j}}$. Let $N_i(j)$ be the number of $\overrightarrow{\mathcal{B}_j}$ in $D_i \cup D_i^{-1}$. Similarly as before we have a.a.s. $N_i(j) \ll n^2 p_1$, which implies $\sum_{j=1}^{1/\varepsilon} N_i(j) < \varepsilon(1-\varepsilon)3n^2 p_1 < \varepsilon|E(D_i)|$, a.a.s.

Note that the number of blossoms of length at least $1/\varepsilon$ is bounded by $2\varepsilon|E(D_i)|$. Therefore, we can obtain a blossom-free subset of $M_i \cup M_i'$ of size at least $(1-6\varepsilon)\frac{|E(D_i)|}{2}$ a.a.s. By Lemma 2.2.4 there exists a construction of D_i for every $i \in [t]$, such that $(1-\varepsilon)^2 t$ of them have almost the correct number of edges, and they admit a 6ε-near triangular embedding. Then G has an embedding Π such that

$$3f_3(\Pi) \geq 2(1-6\varepsilon)(1-\varepsilon)^3|E(G)| \geq 2(1-9\varepsilon)|E(G)|,$$

which completes the proof. □

We have the following consequence on the genus of $G_3 \in \mathcal{Q}(n^{(3)}, p, \varepsilon)$.

Theorem 4.3.10. *Let $\varepsilon > 0$, and $G_3 \in \mathcal{Q}(n^{(3)}, p, \varepsilon/18)$. Then we have a.a.s.*

$$(1-\varepsilon)\frac{\mathsf{e}(G)}{6} \leq \mathsf{g}(G_3) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{6}$$

and

$$(1-\varepsilon)\frac{\mathsf{e}(G)}{3} \leq \widetilde{\mathsf{g}}(G_3) \leq (1+\varepsilon)\frac{\mathsf{e}(G)}{3}.$$

Proof. By using the same argument as in Theorem 4.2.5, the upper bound follows from Theorem 4.3.9, and the lower bound follows from [44, Proposition 4.4.4]. □

4.4 Genus of Multipartite Quasirandom Graphs

We first consider partition of quasirandom graphs into several quasirandom subgraphs. We need this process throughout our approximation algorithm when we try to construct an embedding. Given a quasirandom graph G, the following lemma shows that we can partition the graph into a number of edge disjoint graphs with prescribed edge densities, each of which is also quasirandom.

Lemma 4.4.1. *Let $\varepsilon > 0$ and $G \in \mathcal{Q}(n^{(2)}, p, \varepsilon)$. Suppose k is a constant and $c_1, c_2, \ldots, c_k$ are positive real numbers such that $\sum_{i=1}^{k} c_i = 1$. Then there exists an edge partition of G into k edge-disjoint graphs $G_1, G_2, \ldots, G_k$, such that for every $i \in [k]$, $G_i \in \mathcal{Q}(n^{(2)}, c_i p, 3c_i\varepsilon)$.*

Proof. We first consider a random partition, that is, for every edge $e \in E(G)$, we have $\mathbb{P}(e \in E(G_i)) = c_i$. Suppose G is defined on the vertex set $V_1 \sqcup V_2$. Then for every $X \subseteq V_1$ and $Y \subseteq V_2$, by Chebyshev's inequality, we have a.a.s.

$$(1-\varepsilon)c_i \mathsf{e}_G(X,Y) \leq \mathsf{e}_{G_i}(X,Y) \leq (1+\varepsilon)c_i \mathsf{e}_G(X,Y).$$

That is

$$|\mathsf{e}_{G_i}(X,Y) - \mathsf{e}_G(X,Y)c_i| \leq c_i\varepsilon n^2 + c_i\varepsilon^2 n^2 + c_i\varepsilon \mathsf{e}_G(X,Y) < 3c_i\varepsilon n^2.$$

This implies $\mathsf{d}_\square(G_i, K(n^{(2)}, c_i p)) < 3c_i\varepsilon$. Thus, if n is large enough, there exists a random partition such that for every $i \in [k]$, $G_i \in \mathcal{Q}(n^{(2)}, c_i p, 3c_i\varepsilon)$. □

For any non-negative edge weighted simple graph H, let d_e be the weight of the edge e. Let $\mathcal{T}$ be the set of all triangles (of positive edge weight) in the graph H. Now we consider the following linear program with indeterminate $\{t(T) \mid T \in \mathcal{T}\}$:

$$\begin{aligned} &\nu(H) = \max \sum_{T \in \mathcal{T}} t(T), \\ &\sum_{T \ni e, T \in \mathcal{T}} t(T) \leq d_e, \quad \text{for every edge } e \text{ of } H, \\ &t(T) \geq 0, \quad \text{for every } T \in \mathcal{T}. \end{aligned} \tag{4.4.1}$$

Given $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon)$, let p_{ij} be the (i,j)-entry of P. Suppose G is defined on the vertex set $V_1 \sqcup \cdots \sqcup V_m$ and let H be the quotient graph of G. That is, $|V(H)| = m$, and for every $i, j \in V(H)$, the edge weight $w(ij) = p_{ij}$. With all tools in hand, we have the following theorem on the genus of ε-W_m-quasirandom graphs. Suppose $\nu(H)$ is the maximum obtained from the linear program (4.4.1).

Theorem 4.4.2. *Let $\varepsilon > 0$ and $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon/30)$. Suppose H is the quotient graph of G. Then*

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{4} \leq \mathsf{g}(G) < (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{4} + nm^2$$

and

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{2} \leq \mathsf{g}(G) < (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{2} + 2nm^2.$$

Proof. We first consider the upper bound. Suppose G is defined on the vertex set $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_m$. Suppose $\nu(H)$ is defined as above together with a set of triangles

$\{T_{ijk}\}$ for every $1 \le i < j < k \le m$. Let $d_{ijk} = t(T_{ijk})$. For every V_i and V_j, let $b_{ij} = p_{ij} - \sum_{k\neq i,j} d_{ijk}$, where p_{ij} is the edge weight of edge ij in H. Now for every $1 \le i < j \le m$, we randomly partition the edges between V_i and V_j into at most $m-1$ parts $E_1^{ij}, \ldots, E_m^{ij}, E_0^{ij}$, using probabilities $d_{ij1}/p_{ij}, \ldots, d_{ijm}/p_{ij}, b_{ij}/p_{ij}$, respectively.

By Lemma 4.4.1 we partition the graph G into $O(m^2)$ $\varepsilon/10$-quasirandom bipartite graphs and $O(m^3)$ $\varepsilon/10$-quasirandom tripartite graphs. That is, pick V_i, V_j and V_k, there exist a subgraph G_k^{ij} defined on $V_i \cup V_j$, a subgraph G_i^{jk} defined on $V_j \cup V_k$ and a subgraph G_j^{ik} defined on $V_i \cup V_k$ such that $G_k^{ij}, G_i^{jk}, G_j^{ik} \in \mathcal{Q}(n^{(2)}, d_{ijk}, \varepsilon/10)$. Combine them together, we obtain a tripartite graph in $\mathcal{Q}(n^{(3)}, d_{ijk}, \varepsilon/10)$.

Now we embed the graph G by the partition we constructed. For the tripartite parts, we embed them as quasirandom tripartite graphs, and for the bipartite parts, we embed them as quasirandom bipartite graphs. By Theorems 4.3.5 and Theorem 4.3.10 we obtain a rotation system Π for the disjoint union of these graphs whose genus is

$$\begin{aligned}\sum_i \mathsf{g}(G_i) &\le (1+\varepsilon)\Big(\sum_{i<j} \frac{n^2 b_{ij}}{4} + \sum_{i<j<k} \frac{n^2 d_{ijk}}{2}\Big)\\ &= (1+\varepsilon)\Big(\sum_{i<j} \frac{n^2 p_{ij}}{4} - \sum_{i<j<k} \frac{n^2 d_{ijk}}{4}\Big) = (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{4}.\end{aligned}$$

To obtain an embedding of G we identify pairwise copies of the same vertex in different copies of the quasirandom subgraphs. Each identification can increase the genus by at most 1, and the number of identification is at most m for each of the vertices. This justifies the added term nm^2 in the upper bound.

Now we consider the lower bound for $\mathsf{g}(G)$. For any embedding Π of G, let $\mathfrak{T}_1(\Pi)$ and $\mathfrak{T}_2(\Pi)$ be the subsets of $E(G)$ such that for every $e \in \mathfrak{T}_l(\Pi)$, there exist l triangular faces in Π that contain e ($l = 1, 2$). For every $1 \le i < j \le m$, let $\mathfrak{E}_{ij}^l = \mathfrak{T}_l(\Pi) \cap E_G(V_i, V_j)$, and let $\mathfrak{T}_{ijk}^l$ be the triangles in $\mathfrak{T}_l(\Pi)$ whose vertices lie in V_i, V_j and V_k. Let ${}^k\mathfrak{E}_{ij}^l = \mathfrak{E}_{ij}^l \cap \mathfrak{T}_{ijk}^l$, ${}^j\mathfrak{E}_{ik}^l = \mathfrak{E}_{ik}^l \cap \mathfrak{T}_{ijk}^l$ and ${}^i\mathfrak{E}_{jk}^l = \mathfrak{E}_{jk}^l \cap \mathfrak{T}_{ijk}^l$, for every $1 \le i < j < k \le m$ and $l = 1, 2$. Let

$$\mathsf{m}_{ijk} = \min\{|{}^k\mathfrak{E}_{ij}^2| + |{}^k\mathfrak{E}_{ij}^1|/2, |{}^i\mathfrak{E}_{jk}^2| + |{}^i\mathfrak{E}_{jk}^1|/2, |{}^j\mathfrak{E}_{ik}^2| + |{}^j\mathfrak{E}_{ik}^1|/2\}.$$

Then the number of triangles in $\mathfrak{T}_{ijk}^1 \cup \mathfrak{T}_{ijk}^2$ is at most $2\mathsf{m}_{ijk}$.

Let us now consider the quotient graph H. For every triangle T_{ijk} in H, where $1 \le i < j < k \le m$, let $t_1(T_{ijk}) = \mathsf{m}_{ijk})/n^2$. Then $\{t_1(T_{ijk})\}$ is an admissible solution of the linear program (4.4.1), but it may not be optimal. Suppose all the $\{t_1(T_{ijk})\}$ we obtained indeed give us the maximum solution in linear program (4.4.1), we define a new partition $\mathcal{P}'$ by keeping all the number of edges given in $\{t_1(T_{ijk})\}$, but partition it randomly. Then by Theorem 4.3.9, in each subgraph $\mathfrak{T}^1_{ijk} \cup \mathfrak{T}^2_{ijk}$, the number of triangles is at least $2(1-\varepsilon)\mathsf{m}_{ijk}$. This implies the lower bound of $\mathsf{g}(G)$. □

Let us observe that Theorem 4.4.2 also gives a formula for the genus of *dense* $\mathbb{H}_m$-random graphs mentioned in Chapter 3.

Chapter 5

Genus of Large Dense Graphs

5.1 Introduction

5.1.1 The Graph Genus Problem

The genus is a natural measure how far is G from being planar. Determining the genus of a graph is one of fundamental problems in graph theory with wide range of applications in computing and design of algorithms. Algorithmic interest comes from the fact that graphs of bounded genus share many properties with planar graphs and thus admit efficient algorithms for many problems that are difficult for general graphs [14, 18, 15].

The genus of graphs played an important role in the developments of graph theory through its relationship to the *Heawood problem* asking what is the largest chromatic number of graphs embedded in a surface of genus g. This problem was eventually reduced to the genus computation for complete graphs and resolved by Ringel and Youngs in 1968, see [52]. Further importance of the genus became evident in the graph minors theory of Robertson and Seymour with developments of structural graph theory that plays an important role in stratification of complexity classes [15].

The *genus problem* is the computational task of deciding whether the genus of a given graph G is smaller than a given integer k. The question about its computational complexity was listed among the 12 open problems in the monograph by Garey and Johnson [23] in 1979. Half of these problems were resolved by 1981, three of them (including graph isomorphism) are still unresolved, while three of them have been answered with considerable delay. The genus problem was among the latter three problems. It was resolved in 1989 when Thomassen [65] proved that it is NP-complete. Later, Thomassen simplified his proof in [66] by showing that the question whether G triangulates an (orientable) surface is NP-complete. In 1997 he also proved that the genus problem for cubic graphs is NP-complete [67]. Mohar [43] proved that the genus is NP-complete even if we restrict our attention to *apex graphs*, i.e. graphs which become planar by removing a single vertex.

Measuring graphs by their genus is *fixed parameter tractable*. It follows from the Robertson and Seymour theory of graph minors (and their $O(n^3)$ algorithm for testing H-minor inclusion for any fixed graph H [54, 55]) that for every fixed k, there is an $O(n^3)$ algorithm for testing whether a given graph G has genus at most k. The time complexity in their cubic-time algorithm involves a huge constant depending on H, and the algorithm needs the list of the forbidden minors for genus k. Notably, this is "an impossible task" since the number of surface obstructions is huge (see, e.g., [46] for the up to date results about the surface of genus 1). Moreover, the Robertson–Seymour theory has a non-constructive element. The constants involved in their estimates about forbidden minors are not computable through their results. This deficiency was repaired with the results of Mohar [41, 42], who found a linear-time algorithm for embedding graphs in any fixed surface. His result generalizes the seminal linear-time algorithms for planarity testing by Hopcroft and Tarjan [27] and by Booth and Lueker [8]. It also generalizes to any surface the linear time algorithms that actually construct an embedding in the plane [11] or find a Kuratowski subgraph when an embedding does not exist [71]. Mohar's algorithm gives a constructive proof for the finite number of forbidden minors for surface embeddability. The price paid for this is that the algorithms are complicated and hard to implement. A different linear-time FPT algorithm based on structural graph theory (reducing a graph to have bounded tree-width) has been found by Kawarabayashi, Mohar, and Reed [28]. This algorithm includes as a subroutine a linear-time algorithm for computing the genus of graph of bounded tree-width, which turned to be a difficult task by itself.

A large body of research has been done on approximating the genus by means of polynomial-time algorithms. Graphs whose genus is $\Theta(n)$ (n being the number of vertices) admit a constant factor approximation algorithm. This is an easy consequence of Euler's formula, see [10]. This case includes graphs whose (average) degree is at least d for some $d > 6$.

For graphs of bounded degree, $\Delta = \Delta(G) \leq \Delta_0$, other approaches have been found. Chen et al. [10] describe a factor $O(\sqrt{n})$ algorithm. Chekuri and Sidiropoulos [9] found a polynomial-time algorithm that finds and embedding into a surface whose Euler genus is at most $(\Delta\, \widehat{\mathsf{g}}(G) \log n)^{O(1)}$. Here the approximation factor depends on Δ, polylog factor in n and polynomial factor of the Euler genus itself.

Some other results give additional insight into approximating the genus when the average degree is bounded. For example, the aforementioned paper of Mohar [43] yields a polynomial-time constant factor approximation for the genus of apex graphs (whose maximum degree can be arbitrarily large, but their average degree is less than 8). This result was extended to k-apex graphs in [35].

Kawarabayashi and Sidiropoulos [35] removed the dependence on the maximum degree needed in Chekuri and Sidiropoulos approximation. With a very clever approach they were able to design a polynomial-time algorithm that approximates the Euler genus of any graph within a factor of $O(\widehat{\mathsf{g}}^{255} \log^{189} n)$. A corollary of their result is that the genus can be approximated within factor $O(n^{1-\alpha})$ for some constant $\alpha > 0$, see [35]. A predecessor to this result was published by Makarychev, Nayyeri, and Sidiropoulos [39], who proved that for a graph G possessing a Hamiltonian path (which, unfortunately, needs to be given as part of the input), one can efficiently approximate the Euler genus within factor $(\mathsf{g}(G) \log n)^{O(1)}$. Here the quality of approximation depends on the orientable genus together with a *polylog*(n) factor.

5.1.2 Overview of the Algorithm

The main results of this paper provide a Efficient Polynomial-Time Approximation Scheme (EPTAS) for approximating the genus of dense graphs. A graph is *α-dense* if $|E(G)| \geq \alpha n^2$. By saying a graph G is *dense* we mean it to be α-dense for some fixed $\alpha > 0$. While a constant factor approximation is trivial for this class of graphs, approximations with factor arbitrarily close to 1 provided in this paper need a sophisticated algorithm and complicated mathematical justification.

Given a (dense) graph G of order n and the allowed approximation error $\varepsilon > 0$, we want to find an integer g and an embedding of G into a surface of genus g which is close to a minimum genus embedding, i.e. $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$. It is easy to see that (after appropriate rescaling of ε) this problem is equivalent to the following one, where the assumption on density is left out.

> Approximating Genus Dense.
> *Input:* A graph G of order n and a real number $\varepsilon > 0$.
> *Output:* An integer g and either a conclusion that $\mathsf{g}(G) \leq g < \varepsilon n^2$, or that $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$.

In order to obtain EPTAS for the genus of dense graphs, we outline an algorithm whose time complexity is $O(f(\varepsilon) n^{O(1)})$, where $f(\cdot)$ is an arbitrary positive function. In fact, the polynomial dependence on n in our algorithm is quadratic.

Theorem 5.1.1. *The problem* Approximating Genus Dense *can be solved in time* $O(f(\varepsilon)\, n^2)$*, where* $f(\cdot)$ *is an explicit positive function.*

The dependence on ε^{-1} in the time complexity of our algorithms is super-exponential. In fact, the value $f(\varepsilon)$ is a tower of exponents of height $O(\varepsilon^{-1})$. There are two steps where non-polynomial dependence on ε^{-1} occurs. The main one is with "small" graphs, where n is considered small in terms of a huge function of ε^{-1}. This case has linear complexity in terms of n [41, 42], but it involves a large constant factor which is increasing super-exponentially fast with ε^{-1}. This is where we are prevented of designing an FPTAS.

Our second computational result extends the previous one by constructing an embedding whose genus is close to the minimum genus. Formally, we consider the problem:

Approximate Genus Embedding Dense.
Input: A graph G of order n and a real number $\varepsilon > 0$.
Output: Rotation system of a 2-cell embedding of G, whose genus g is close to $\mathsf{g}(G)$: either $\mathsf{g}(G) \le g \le (1+\varepsilon)\mathsf{g}(G)$, or $\mathsf{g}(G) \le g < \varepsilon n^2$.

As a solution we provide an Efficient Polynimial-time Randomized Approximation Scheme (EPRAS) [69].

Theorem 5.1.2. *There is a randomized algorithm for* Approximate Genus Embedding Dense *which returns an embedding of the input graph* G *of genus* g*, such that either* $\mathsf{g}(G) \le g \le (1+\varepsilon)\mathsf{g}(G)$*, or* $\mathsf{g}(G) \le g < \varepsilon n^2$*. The time spent by the algorithm is* $O(f_1(\varepsilon)\, n^2)$*, where* $f_1(\cdot)$ *is an explicit positive function.*

There are two parts in the embedding algorithm that are nondeterministic. One of them uses random partition of the edges of G. This part can be derandomized (yielding a cubic polynomial dependence on n), but for the other one we do not see how to derandomize it. This part finds a large matching in a 3-uniform (or 4-uniform) hypergraph, and existence of a large matching and its construction relies on the Lovász Local Lemma (for which one could use a randomized algorithm by Moser and Tardos [45]). We use another randomized solution using Rödl nibble [57] which yields quadratic dependence on n.

Algorithms in Theorems 5.1.1 and 5.1.2 are based on analysis of minimum genus embeddings of quasirandom graphs. We partition the input graph into a bounded number of quasirandom subgraphs, which are preselected in such a way that they admit embeddings using as many triangles and quadrangles as faces as possible. The starting partition is obtained through an algorithmic version of the Szemerédi Regularity Lemma (due to Frieze and Kannan [22] and to Fox, Lovász, and Zhao [19, 20]).

We use the notion of quasirandomness inspired by the seminal paper of Chung, Graham, and Wilson [13] (see also [12]), and we need it in two special cases of bipartite and tripartite graphs, respectively. In order to define it, we need some notation.

Let G be a graph, and $X, Y \subseteq V(G)$. We define the *edge density* between X and Y as the number $d(X,Y) = \mathsf{e}(X,Y)/(|X||Y|)$, where $\mathsf{e}(X,Y)$ is the number of edges with one end in X and another end in Y. If G is a (large) bipartite graph with balanced bipartition $V(G) = V_1 \cup V_2$ ($||V_1| - |V_2|| \leq 1$ and $n = |V_1| + |V_2|$), we say that G is *ε-quasirandom* if for every $X \subseteq V_1$ and every $Y \subseteq V_2$, we have

$$|\mathsf{e}(X,Y) - |X||Y|d(V_1,V_2)| \leq \varepsilon |V_1||V_2|.$$

This is equivalent to saying that the number of 4-cycles with vertices in $X \cup Y$ is close to what one would expect in a random bipartite graph with the same edge density, i.e. $d^4(V_1,V_2)|V_1|^2|V_2|^2$, with an error of at most $4\varepsilon|V_1|^2|V_2|^2$.

First we prove the following result.

Theorem 5.1.3. *Suppose that G is a bipartite ε-quasirandom graph with edge density $d = d(V_1,V_2)$. If $\varepsilon < d^8/10$ and $n = |V(G)| \geq \Theta(\varepsilon^{-3/2})$, then*

$$\tfrac{1}{4}(1-10\varepsilon)\mathsf{e}(G) \leq \mathsf{g}(G) \leq \tfrac{1}{4}(1+10\varepsilon)\mathsf{e}(G).$$

The above theorem says, roughly speaking, that G admits an embedding in which almost all faces are quadrilaterals. More precisely, almost all edges are contained in two quadrangular faces.

A similar result holds for *tripartite ε-quasirandom graphs*. Here we have three almost equal parts V_1, V_2, V_3, and the graph between any two of them is bipartite ε-quasirandom. Here we only need the corresponding embedding result when the densities between the three parts are the same.

Theorem 5.1.4. *Suppose that G is a tripartite ε-quasirandom graph with edge densities $d = d(V_1,V_2) = d(V_1,V_3) = d(V_2,V_3)$. If $\varepsilon < d^8/10$ and $n = |V(G)| \geq \Theta(\varepsilon^{-3/2})$, then*

$$\tfrac{1}{6}(1-10\varepsilon)\mathsf{e}(G) \leq \mathsf{g}(G) \leq \tfrac{1}{6}(1+10\varepsilon)\mathsf{e}(G).$$

Similarly as for Theorem 5.1.3, the outcome of the above theorem is that G admits an embedding in which almost all edges are contained in two triangular faces.

Theorems 5.1.3 and 5.1.4 have extensions to multipartite case with possibly non-equal edge densities and non-empty quasirandom graphs in the parts of the vertex partition. For this extension see the main part of the paper.

Proofs of Theorems 5.1.3 and 5.1.4 build on the approach introduced by Archdeacon and Grable [4] and Rödl and Thomas [58]. The main ingredient is to find two disjoint almost perfect matchings in a 3-uniform (or in a 4-uniform) hypergraph associated with short cycles in G. One difference is that there may be too many short cycles, in which case the matchings obtained from these hypergraphs may not form a set which could be realized as facial cycles of an embedding of the graph. This has to be dealt with accordingly. The proof uses an old result of Frankl and Rödl [21].

From the above expressions about the genus of quasirandom graphs, there is just one major step left. We show that G can be partitioned into a constant number of bipartite and tripartite ε-quasirandom subgraphs such that almost all triangles of G belong to the tripartite ε-quasirandom subgraphs in the partition. For each of these ε-quasirandom subgraphs, Theorem 5.1.4 or Theorem 5.1.3 can be applied. The main result is the following version of Szemerédi Regularity Lemma.

Theorem 5.1.5. *There exists a computable function $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ such that the following holds. For every $\varepsilon > 0$ and every positive integer m there is an integer K, where $m \le K \le s(m, \varepsilon)$ such that every graph of order $n \ge m$ has an equitable partition of its vertices $V(G) = V_1 \cup \cdots \cup V_K$ into K parts and G admits a partition into $O(K^2)$ bipartite ε-quasirandom subgraphs G_{ij} $(1 \le i < j \le K)$ with $V(G_{ij}) = V_i \cup V_j$, and into $O(K^3)$ tripartite ε-quasirandom subgraphs G_{ijk} $(1 \le i < j < k \le K)$ with $V(G_{ijk}) = V_i \cup V_j \cup V_k$ with equal densities between their parts, and one additional subgraph G_0 with at most εn^2 edges. Moreover, the union of all bipartite constituents G_{ij} is triangle-free.*

To obtain such a partition $V(G) = V_1 \cup \cdots \cup V_K$ we start with an ε-regular partition obtained from the Szemerédi Regularity Lemma. Such a partition can be constructed in quadratic time by using an algorithm of Fox, Lovász, and Zhao [20]. The edges in irregular pairs and all edges in subgraphs $G[V_i]$ $(1 \le i \le K)$ are put into the subgraph G_0. All the remaining edges belong to bipartite subgraphs joining pairs V_i and V_j $(1 \le i < j \le K)$. Let d_{ij} be the edge density for each such bipartite subgraph. We represent the partition by a weighted graph H on vertices $\{1, \ldots, K\}$, where each edge ij has weight d_{ij}.

Let $\mathcal{T}$ be the set of all triangles in the quotient graph H (of positive edge weight). For every triangle $T = abc \in \mathcal{T}$, let $d(T) = \min\{d_{ab}, d_{bc}, d_{ac}\}$. Now we consider the following linear program with indeterminates $\{t(T) \mid T \in \mathcal{T}\}$:

$$\begin{aligned} &\nu(H) = \max \sum_{T \in \mathcal{T}} t(T), \\ &\sum_{T \ni e, T \in \mathcal{T}} t(T) \le d_e, \quad \text{for every edge } e \text{ of } H, \\ &t(T) \ge 0, \quad \text{for every } T \in \mathcal{T}. \end{aligned} \tag{5.1.1}$$

We consider an optimum solution $(t(T) \mid T \in \mathcal{T})$ of this linear program. For each $T = abc \in \mathcal{T}$ we now define G_T as a subgraph of $G[V_a \cup V_b \cup V_c]$ by taking a random set of edges with density $t(T)$ from each of the three bipartite graphs between V_a, V_b, V_c. The edges between the sets V_i and V_j ($1 \le i < j \le K$) that remain after removing all tripartite subgraphs G_T form bipartite quasirandom subgraphs.

From the Partition Theorem 5.1.5 it is not hard to see that

$$\mathsf{g}(G) \le \sum_{i,j} \mathsf{g}(G_{ij}) + \sum_{i,j,k} \mathsf{g}(G_{ijk}) + nK^2 + \varepsilon n^2.$$

By using Theorems 5.1.3 and 5.1.4 we derive the main result:

Corollary 5.1.6. *Let G be a graph that is partitioned as stated in Theorem 5.1.1, let $\nu = \nu(H)$ be the optimum value of the linear program (5.1.1), and let $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ be the function from Theorem 5.1.5. If $n = |V(G)| \ge \Theta(s(4\varepsilon^{-1}, \varepsilon) \cdot \varepsilon^{-3/2})$, then the genus of G satisfies:*

$$(1-\varepsilon)\frac{1}{4}\left(\mathsf{e}(G) - \frac{\nu n^2}{K^2}\right) \le \mathsf{g}(G) \le (1+\varepsilon)\frac{1}{4}\left(\mathsf{e}(G) - \frac{\nu n^2}{K^2}\right) + nK^2 + \varepsilon n^2.$$

Application of Theorems 5.1.3 and 5.1.4 requires that the densities are not too small. If this is not the case, we just add the edges to G_0.

In order to apply the corollary to obtain an ε-approximation to the genus, we use the corollary with the value $\frac{1}{2}\varepsilon$ playing the role of ε. If $n \ge \Theta(s^2(4\varepsilon^{-1}, \varepsilon) \cdot \varepsilon^{-1})$, then the last two terms in the corollary are bounded by $\frac{3}{4}\varepsilon n^2$. Now, if ε is much smaller than the lower bound α on the density of G, we get an ε-approximation of the genus of G.

Although we have not mentioned anything about the nonorientable genus before, the same results hold for the nonorientable genus, where all formulas about the genus need to be multiplied by 2.

On a high level, our EPTAS for Approximating Genus Dense works as follows.

Phase 0. Check whether the graph is dense enough: If $|E(G)| \leq \varepsilon n^2$, we return the information that $\mathsf{g}(G) < \varepsilon n^2$ and stop.

Phase 1. Let $m = 2\varepsilon^{-1}$ and let $M = s(m, \varepsilon)$ where the function s is from Theorem 5.1.5. If $|G| = O(M^2\varepsilon^{-1})$, we compute the genus of G exactly and return the result. Otherwise we proceed with the next step.

Phase 2. We find a Szemerédi partition of G into K parts, where $m \leq K \leq M$, and according to Theorem 5.1.5 partition G into edge-disjoint subgraphs $G_0, G_1, \ldots, G_N$, where $N = O(K^3)$, each of them except G_0 of order $\Theta(n/K)$ such that the following holds:

(i) G_0 has at most $\frac{1}{2}\varepsilon n^2$ edges.

(ii) All other subgraphs are either bipartite or tripartite ε-quasirandom.

(iii) The union of bipartite subgraphs contains no triangles.

Phase 3. Determine the densities d_{ij}, $1 \leq i < j \leq K$, and solve the linear program (5.1.1). Let $\nu = \nu(H)$ be the optimal value computed. Return the value $g = \frac{1}{4}\mathsf{e}(G) - \frac{\nu n^2}{4K^2}$.

The heart of the algorithm lies in Phase 2. However, Phase 3 is the most challenging mathematical part and has complicated justification. For the partition of G into $G_0, G_1, \ldots, G_N$ we could use algorithmic version of the Szemerédi Regularity Lemma due to Frieze and Kannan [22]. But it is more convenient to use a recent strengthening of Frieze-Kannan partitions due to Fox, Lovász, and Zhao [19]. Their result provides an *ε-regular partition* (in the sense of Szemerédi) of $V(G)$ into sets $V_1, \ldots, V_K$ of size n/K (we neglect rounding of non-integral values as they are not important for the exposition) such that the majority of pairs (V_i, V_j) $(1 \leq i < j \leq K)$ are *ε-regular*. Each such pair induces a bipartite ε-quasirandom graph. The edges in pairs that are not ε-regular can be added to G_0 together with all edges in $\bigcup_{i=1}^{K} G[V_i]$. So from now on, we assume that all edges of the graph are in ε-regular pairs (V_i, V_j) $(1 \leq i < j \leq K)$.

The second, most difficult step, is to analyse the quotient graph determined by the partition. In this step we use a linear programming approach to find for each triple $T =$

(i, j, k), $1 \leq i < j < k \leq K$, the number $t(T) \geq 0$ and an ε-quasirandom subgraph $G_T \subseteq G[V_i \cup V_j \cup V_k]$ with $t(T)(n/K)^2$ edges between each pair $(V_i, V_j), (V_i, V_k)$, and (V_j, V_k). The graphs G_T are then used to obtain as many triangular faces as possible (up to the allowed error) for the embedding of G. The edges that remain form quasirandom bipartite parts G_{ij} between pairs (V_i, V_j) $(1 \leq i < j \leq K)$. We use those to obtain as many quadrangular faces as possible (up to the allowed error) for the embedding of G.

Finally, the near-triangular embeddings of all G_T and near-quadrangular embeddings of all G_{ij} are used to produce a near-optimal embedding of G. The description of this part is in the main part of the paper.

Let us now comment on the main issues in the algorithmic part and in the theoretical justification. First, we use known regularity partition results to find a partition $V_1, \ldots, V_K$. The algorithm runs in quadratic time with a decent (but superexponential) dependence on $1/\varepsilon$. The linear programming part to determine triangle densities $t(T)$ is done on a constant size linear program and a rounding error of magnitude $O(\varepsilon)$ is allowed. Having gathered all the information about the required edge densities in the partition, the partition of G into subgraphs G_T and G_{ij} uses a randomized scheme, although derandomization is possible. For the computation of the approximate value for the genus (Corollary 5.1.6), the partition is not needed. We just need to know that it exists, and we need to know the edge densities between the regular pairs of the partition. Thus this part is deterministic.

For the justification that the graphs G_T and G_{ij} admit almost triangular and almost quadrangular embeddings, respectively, we use the quasirandomness condition. The proof is based on a theorem by Frankl and Rödl [21] giving a large matching in a dense 3- or 4-uniform hypergraph (respectively). The hyperedges in the hypergraph correspond to cycles of length 3 and 4 (respectively) in the considered subgraph G_T or G_{ij}. Two such matchings are needed in order to combine them into an embedding of the graph, most of whose faces will be the triangles or quadrangles of the two hypergraph matchings. Quasirandomness is used to show that such matchings exists and that they have additional properties needed for them to give rise to an embedding. To obtain such a matching, we can follow the proof of Frankl and Rödl, but the proof uses the Lovasz Local Lemma. In order to make a construction, we may apply the algorithmic version of the Lovasz Local Lemma that was obtained by Moser and Tardos [45]. Alternatively, we may apply the randomized algorithm of Rödl and Thoma [57] which uses the Rödl nibble method. A similar algorithm was obtained by Spencer [59]. Both of these latter algorithms use greedy selection and run in quadratic time, but they are both randomized. This is the essential part where we are not able to provide corresponding derandomized version.

5.2 Algorithms and Analysis

In this section, we will analyse properties on an large dense graph G. Section 5.4.1 provides a deterministic EPTAS for the problem Approximating Genus Dense. In the Section 5.4.2, we will discuss how to construct a near minimum genus embedding for the problem Approximate Genus Embedding Dense.

5.2.1 EPTAS for the Genus of Dense Graphs

Suppose G has n vertices and suppose we have an equitable partition $\mathcal{P}$ of $V(G)$ into K parts, $V(G) = V_1 \sqcup \cdots \sqcup V_K$. We use $H = G/\mathcal{P}$ denote an edge-weighted complete graph with K vertices, $V(H) = \{v_1, \ldots, v_K\}$. The edge between v_i and v_j in H has weight equal to the edge density $d_{ij} = d(V_i, V_j)$ between V_i and V_j in G.

Theorem 5.2.1 (Szemerédi Regularity Lemma)**.** *There exists a computable function $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ such that the following holds. For every $\varepsilon > 0$, every positive integer m, every graph G of order $n \geq m$ has an ε-Szemerédi partition $\mathcal{P}$ of its vertices $V(G) = V_1 \cup \cdots \cup V_K$ into K parts, where $m \leq K \leq s(m, \varepsilon)$.*

Given a graph G, the following theorem shows the relation between the genus of G and $\nu(H)$, where H is the quotient graph of G with respect to its Szemerédi partition.

Theorem 5.2.2. *For every $\varepsilon > 0$, there exist a positive number $\varepsilon' > 0$ and an ε'-Szemerédi partition $\mathcal{P}$ defined in Theorem 5.2.1. Suppose $H = G/\mathcal{P}$ be the quotient graph, and $\nu(H)$ is an optimal solution of linear program (4.4.1). Then*

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4} \leq \mathsf{g}(G) \leq (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4},$$

and

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{2} \leq \widetilde{\mathsf{g}}(G) \leq (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{2},$$

where $n_0 = n/K$.

Proof. Take $m = 4/\varepsilon$ and apply regularity lemma with ε'. We first remove all the edges in each $G[V_i]$ for every $i \in [K]$. Let G' be the resulting graph. Since genus property is edge-Lipschitz, we have that $0 \leq \mathsf{g}(G) - \mathsf{g}(G') \leq \frac{1}{8}\varepsilon n^2$.

For the quotient graph H, by the linear program (4.4.1), we obtain a real number $\nu(H)$ and a set of triangles $\{T_{ijk}\}$ for every $1 \le i < j < k \le K$. Let $d_{ijk} = t(T_{ijk})$, where the function $t(\cdot)$ is defined as an optimal solution of the linear program. For every $1 \le i < j \le K$, let $b_{ij} = d_{ij} - \sum_{k \ne i,j} d_{ijk}$, where d_{ij} is the edge density between V_i and V_j in G. We assume first that in the partition $\mathcal{P}$, for every $1 \le i < j \le K$, the graph induced on V_i and V_j is ε'-regular. Then we partition the edge set $E(V_i, V_j)$ into at most $K-1$ parts ${}^l\mathcal{E}_{ij}$ for all $l \ne i, j$ and $\mathcal{B}_{ij}$ randomly, each with probability $d_{ij1}/d_{ij}, \dots, d_{ijK}/d_{ij}, b_{ij}/d_{ij}$. For convenience, we denote the graphs with vertex set $V_i \cup V_j$ which has edge set ${}^l\mathcal{E}_{ij}$ by ${}^l\mathcal{G}_{ij}$, and the graph with vertex set $V_i \cup V_j$ which has edge set $\mathcal{B}_{ij}$ by $\mathcal{G}_{ij}$. Then by Lemma 4.4.1, we have $\mathsf{d}_\square({}^l\mathcal{G}_{ij}, K(n_0^{(2)}, d_{ijl})) < 3\varepsilon' d_{ijl}/d_{ij}$ and $\mathsf{d}_\square(\mathcal{G}_{ij}, K(n_0^{(2)}, b_{ij})) < 3\varepsilon' b_{ij}/d_{ij}$. We also let $\mathcal{T}_{ijk}$ be the graph defined on the vertex set $V_i \cup V_j \cup V_k$, with edge set ${}^i\mathcal{E}_{jk} \cup {}^j\mathcal{E}_{ik} \cup {}^k\mathcal{E}_{ij}$, hence $\mathsf{d}_\square(\mathcal{T}_{ijk}, K(n_0^{(3)}, d_{ijk})) < 3\varepsilon' m_{ijk}$, where $m_{ijk} = \max\{d_{ijk}/d_{ij}, d_{ijk}/d_{ik}, d_{ijk}/d_{jk}\}$.

We are now going to compute the genus of G'. We construct an embedding Π such that all the $\mathcal{T}_{ijk}$ $(1 \le i < j < k \le K)$ embeds as tripartite quasirandom graphs and all the $\mathcal{B}_{ij}$ embeds as bipartite quasirandom graphs. Apply Theorem 4.3.5 and Theorem 4.3.10 we have

$$\begin{aligned}
\mathsf{g}(G', \Pi) &\le (1+30\varepsilon')\Big(\sum_{1 \le i < j < k \le K} \frac{\mathsf{e}(\mathcal{T}_{ijk})}{6} + \sum_{1 \le i < j \le K} \frac{\mathsf{e}(\mathcal{B}_{ij})}{4} \Big) \\
&\le (1+30\varepsilon')\Big(\sum_{1 \le i < j < k \le K} \frac{3n_0^2 d_{ijk} + 27\varepsilon' m_{ijk} n_0^2}{6} + \sum_{1 \le i < j \le K} \frac{n_0^2 b_{ij} + 3\varepsilon' \frac{b_{ij}}{d_{ij}} n_0^2}{4} \Big).
\end{aligned}$$

To evaluate the error terms, we let $\mathfrak{p} = \max\{d_{ij}\}$ and $\mathfrak{q} = \min\{d_{ij} : d_{ij} > \sqrt{\varepsilon'}\}$ where the maximum and minimum are taken over all i, j, $1 \le i < j \le K$. We assume first that all positive d_{ij} in G' are greater than $\sqrt{\varepsilon'}$. Then,

$$\begin{aligned}
\mathsf{g}(G', \Pi) &\le (1+30\varepsilon')\Big(\sum_{1 \le i < j < k \le K} \frac{n_0^2 d_{ijk}}{2} + \sum_{1 \le i < j \le K} \frac{n_0^2 b_{ij}}{4} + \frac{21\mathfrak{p}\varepsilon' n^2}{4\mathfrak{q}} \Big) \\
&= (1+30\varepsilon')\Big(\sum_{1 \le i < j \le K} \frac{n_0^2 d_{ij}}{4} - \sum_{1 \le i < j < k \le K} \frac{n_0^2 d_{ijk}}{4} + \frac{21\mathfrak{p}\varepsilon' n^2}{4\mathfrak{q}} \Big) \\
&\le (1+30\varepsilon')\Big(\frac{\mathsf{e}(G') - \nu(H) n_0^2}{4} + \frac{42\mathfrak{p} + \mathfrak{q}}{8\mathfrak{q}} \varepsilon' n^2 \Big).
\end{aligned}$$

However, in the partition $\mathcal{P}$ of G', not all the pairs (V_i, V_j) are ε'-regular, and some of the densities d_{ij} can be very small (compared with $\sqrt{\varepsilon'}$). By removing all the edges between irregular pairs and edges with density less than $\sqrt{\varepsilon'}$, we obtain a new graph G''. We let H_0 be the quotient graph $G''/\mathcal{P}$. Note that $\nu(H_0)$ is also a solution of H of the following system of linear inequalities:

$$\begin{aligned} &\mu(H) = \sum_{T\in\mathcal{T}} t(T), \\ &\sum_{T\ni e, T\in\mathcal{T}} t(T) \le d_e, \quad \text{for every edge } e \text{ of } H_0, \\ &t(T) \ge 0, \quad \text{for every } T \in \mathcal{T}. \end{aligned} \tag{5.2.1}$$

Clearly $\nu(H_0) \le \mu(H) \le \nu(H)$. Therefore,

$$\begin{aligned} \mathsf{g}(G) &\le (1+30\varepsilon')\Big(\frac{\mathsf{e}(G)-\nu(H)n_0^2}{4} + \frac{43}{8}\sqrt{\varepsilon'}n^2\Big) + \varepsilon' n^2 + \frac{\sqrt{\varepsilon'}}{2}n^2 + \frac{\varepsilon}{8}n^2 \\ &\le (1+\varepsilon)\Big(\frac{\mathsf{e}(G)-\nu(H)n_0^2}{4}\Big). \end{aligned} \tag{5.2.2}$$

For the lower bound, note that the lower bound on $\mathsf{g}(G'')$ is also a lower bound on $\mathsf{g}(G)$, thus it suffices to consider the graph G''. We first show that $\frac{\mathsf{e}(G'')-\nu(H_0)}{4}$ is also a lower bound on $\mathsf{g}(G'')$ (up to a constant factor $(1-o(1))$). The proof uses a similar argument as we used in the proof of Theorem 4.4.2.

For any embedding Π of G'', let $\mathfrak{T}_1(\Pi)$ and $\mathfrak{T}_2(\Pi)$ be the subsets of $E(G'')$ such that for every $e \in \mathfrak{T}_l(\Pi)$, there exist l triangular faces in Π that contain e. For every $1 \le i < j \le K$, we define $\mathfrak{E}^l_{ij} = \mathfrak{T}_l(\Pi) \cap E_{G''}(V_i, V_j)$, and let $\mathfrak{T}^l_{ijk}$ be the sets of triangles in $\mathfrak{T}_l(\Pi)$ whose vertices lie in V_i, V_j and V_k. We also define ${}^k\mathfrak{E}^l_{ij} = \mathfrak{E}^l_{ij} \cap \mathfrak{T}^l_{ijk}$, ${}^j\mathfrak{E}^l_{ik} = \mathfrak{E}^l_{ik} \cap \mathfrak{T}^l_{ijk}$ and ${}^i\mathfrak{E}^l_{jk} = \mathfrak{E}^l_{jk} \cap \mathfrak{T}^l_{ijk}$, for every $1 \le i < j < k \le K$ and $l = 1, 2$. Let

$$\mathsf{m}_{ijk} = \min\{|{}^k\mathfrak{E}^2_{ij}| + |{}^k\mathfrak{E}^1_{ij}|/2, |{}^i\mathfrak{E}^2_{jk}| + |{}^i\mathfrak{E}^1_{jk}|/2, |{}^j\mathfrak{E}^2_{ik}| + |{}^j\mathfrak{E}^1_{ik}|/2\},$$

then $\mathfrak{T}_{ijk}$ contains at most $2\mathsf{m}_{ijk}$ triangles.

Now we consider the quotient graph H_0. For every triangle T_{ijk} in H_0, let $t(T_{ijk}) = \mathsf{m}_{ijk}/n_0^2$. This is a solution of the linear inequalities (5.2.1) of H_0. Under this random partition (by the values of $t(T_{ijk})$), the number of trianglar faces in the subembedding of $G''[V_i \cup V_j \cup V_k]$ is at least $2(1-10\varepsilon'\frac{t(T_{ijk})}{d_{ij}})\mathsf{m}_{ijk}$. Therefore, by Theorem 4.3.5 and 4.3.10 we have

$$\mathsf{g}(G'') \ge (1-30\varepsilon')\Big(\frac{\mathsf{e}(G'')-\nu(H_0)n_0^2}{4}\Big).$$

In the next step, we will compare $\nu(H)$ and $\nu(H_0)$. Clearly, H_0 is obtained from H be deleting edges between irregular pairs and edges with small weights. The total weight of edges we delete is at most $\varepsilon' K^2 + \frac{K^2}{2}\sqrt{\varepsilon'}$. Therefore,

$$\nu(H_0) + 3\Big(\varepsilon' K^2 + \tfrac{K^2}{2}\sqrt{\varepsilon'}\Big) \leq \nu(H).$$

Then we have

$$\begin{aligned}
\mathsf{g}(G) &\geq (1-30\varepsilon')\Big(\frac{\mathsf{e}(G) - \frac{1}{2K}n^2 - \varepsilon' n^2 - \frac{\sqrt{\varepsilon'}}{2}n^2 - \nu(H)n_0^2 + 3\Big(\varepsilon' K^2 + \frac{K^2}{2}\sqrt{\varepsilon'}\Big)n_0^2}{4}\Big) \\
&\geq (1-30\varepsilon')\Big(\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4} + \frac{n^2(2\varepsilon' + \sqrt{\varepsilon'} - \frac{\varepsilon}{8})}{4}\Big) \\
&\geq (1-\varepsilon)\Big(\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4}\Big)
\end{aligned} \tag{5.2.3}$$

Take ε' be the solution of inequalities (5.2.2) and (5.2.3), this completes the proof. □

With all tools in hand, we are going to provide the algorithm Approximating Genus Dense. Given $\varepsilon > 0$ and a graph G of order n, we do the following:

Step 1. Let $\tau = 3\varepsilon'/2$, where ε' is defined in Theorem 5.2.2. Pick $\alpha = 1/2$, $m = 4/\varepsilon$ and, apply the algorithm in Theorem 4.1.4 with integer k taking values from m to $s(\tau, m)$. Then the algorithm will output an ε'-Szemerédi partition into K parts, where $m \leq K \leq s(\tau, m)$.

Step 2. Consider the quotient graph $H = G/\mathcal{P}$. Solve the linear program (4.4.1) on H to obtain $\nu(H)$.

Step 3. Output $g = (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4}$, where $n_0 = n/K$.

5.2.2 EPRAS for Embeddings of Dense Graphs

Now, we turn to our algorithm for Approximate Genus Embedding Dense where the added feature is to construct an embedding. Given $\varepsilon > 0$ and a graph G of order n, we can apply Approximating Genus Dense to get g such that $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$. We are going to construct a rotation system Π of G, whose genus satisfies the same bound. Our algorithm proceeds as follows:

Step 1. Apply Approximating Genus Dense, we obtain an $r(\varepsilon)$-Szemerédi partition $\mathcal{P}$ into K parts, where $r(\varepsilon)$ is the value of ε' in Theorem 5.2.2. Determine the quotient graph H and compute the value $\nu(H)$ as well as the family of triangles $\mathfrak{T}$ in H and their balanced edge densities $t(T)$, $T \in \mathfrak{T}$. We randomly partition the graph into $\mathsf{b} = O(K^2)$ bipartite graphs $\mathcal{B}_{ij}$ and $\mathsf{t} = O(K^3)$ tripartite graphs $\mathcal{T}_{ijk}$ as we defined in Theorem 5.2.2. Then with high probability, for any $\tau > 0$, at least

$(1-r(\varepsilon)-\tau)\mathsf{b}$ bipartite graphs are quasirandom and at least $(1-r(\varepsilon)-\tau)\mathsf{t}$ tripartite graphs are quasirandom. Note that $r(\varepsilon)$ appears here because of the irregular pairs.

STEP 2. Let $t_1 = n^{\frac{2-\varepsilon}{4-\varepsilon}}$. For every $1 \le i < j < k \le K$, we partition the edge set of the graph $\mathcal{T}_{ijk}$ into t_1 parts uniformly at random. These t_1 sets of edges give us t_1 graphs ${}^1\mathcal{T}_{ijk}, \dots, {}^{t_1}\mathcal{T}_{ijk}$. Then for any $\tau > 0$, with high probability at least $(1-\tau)t_1$ graphs are still quasirandom. For every $x \in [t_1]$, let ${}^xD_{ijk}$ be the corresponding digraph of ${}^x\mathcal{T}_{ijk}$, and let ${}^x\mathcal{H}_{ijk}$ be the hypergraph as defined in Section 3.

STEP 3. Let $t_2 = n^{\frac{4-\varepsilon}{6-\varepsilon}}$. For every $1 \le i < j \le K$, we partition the edge set of the graph $\mathcal{B}_{ij}$ into t_2 parts uniformly at random. These t_2 sets of edges give us t_2 graphs ${}^1\mathcal{B}_{ij}, \dots, {}^{t_2}\mathcal{B}_{ij}$. Then for any $\tau > 0$, with high probability at least $(1-\tau)t_2$ graphs are still quasirandom. For every $y \in [t_2]$, let ${}^yD_{ij}$ be the corresponding digraph of ${}^y\mathcal{B}_{ij}$, and let ${}^y\mathcal{H}_{ij}$ be the hypergraph as defined in Section 3. Let h be the total number of hypergraphs.

STEP 4. Apply random greedy algorithm [57] on 3-uniform hypergraphs ${}^x\mathcal{H}_{ijk}$ for every $x \in [t_1]$ and every $1 \le i < j < k \le K$, and on 4-uniform hypergraphs ${}^y\mathcal{H}_{ij}$ for every $y \in [t_2]$ and every $1 \le i < j \le K$. Note that the expected running time here is still quadratic even though we need to run the algorithm $t_1 + t_2$ times. This is because the running time is linear on the number of vertices of the hypergraph, and with high probability, we have $|V({}^x\mathcal{H}_{ijk})| = \Theta(\frac{n^2}{t_1})$ and $|V({}^y\mathcal{H}_{ij})| = \Theta(\frac{n^2}{t_2})$. Then for every $\tau > 0$, with high probability the algorithm will output a τ-near perfect matching in at least $(1-r(\varepsilon)-\tau)\mathsf{h}$ hypergraphs. Let $\mathfrak{M}$ be the set of hyperedges (triangles and 4-cycles) such that for every $e \in \mathfrak{M}$, e is output by the algorithm as an element of a τ-near perfect matching in a hypergraph $\mathcal{H}$.

STEP 5. For each hypergraph $\mathcal{H}$ we defined in STEP 4, consider $\mathcal{H}^{-1}$. Delete all the edges contained in $\mathfrak{M}$ (with inverse direction) from $\mathcal{H}^{-1}$. By [58, Theorem 3.3], the resulting hypergraph still satisfies Conditions (1)–(3) in Theorem 2.2.5. Apply random greedy algorithm again. For every $\tau > 0$, with high probability the algorithm will output a τ-near perfect matching in at least $(1-r(\varepsilon)-\tau)\mathsf{h}$ hypergraphs. We also put edges in these near perfect matchings into $\mathfrak{M}$.

STEP 6. Output the rotation $\Pi = \{\pi_v \mid v \in V(G)\}$ which is constructed as follows. For every vertex $v \in V(G)$, if there exists a hyperedge e such that $v \in e$ and $e \in \mathfrak{M}$, suppose v_1 is one of the neighbours (in the graph G) of v in e, put v_1 in the rotation system π_v. If there exists another edge e_1 in $\mathfrak{M}$ such that e_1 contains both v and v_1, put the other neighbour of v in e_1 into π_v clockwise following v_1, and do it recursively. If at some point, we cannot find any other edges containing v in $\mathfrak{M}$, we move to the other vertices. If v_l is a neighbour of v, and when we try to add v_l in π_v, the hyperedges in $\mathfrak{M}$ we processed form a blossom, then remove this hypergraph from $\mathfrak{M}$. By the proofs in Section 3, with high probability, there is only a small number of blossoms. If for a vertex $u \in V(G)$, there are some edges that have not been put in π_u during this process, we put them in π_u arbitrarily in order to obtain the rotation system π_u. By Theorem 5.2.2, we have $(1+\varepsilon)\mathsf{g}(G) \geq \mathsf{g}(G, \Pi)$.

Chapter 6

Genus of Complete 3-Uniform Hypergraphs

6.1 Embeddings of Hypergraphs

In this chapter, we will discuss the embeddings of complete 3-uniform hypergraphs, which is a natural generalization of Ringel-Youngs Theorem. The problem was first discussed by Jungerman, Stahl and White [34]. The genus problems of hypergraphs are tightly related with the genus of bipartite graphs, 2-complexes, block designs and finite geometry. We refer to [6, 50, 70] for more background.

Let H be a hypergraph. In order to study the embeddings of H, let us first consider its associated Levi graph.

Definition 6.1.1. The associated *Levi graph* of a hypergraph H is the bipartite graph L_H defined on the vertex set $V(H) \cup E(H)$, in which $v \in V(H)$ and $e \in E(H)$ are adjacent if and only if v and e are incident in H.

In this paper, we use K_n^3 to denote the complete 3-uniform hypergraph of order n, and we denote its Levi graph by L_n. The vertices of L_n corresponding to $V(K_n^3) = [n]$ will be denoted by X_n and the $\binom{n}{3}$ vertices corresponding to the edges of K_n^3 will be denoted by Y_n. Following [70, Chapter 13], we define embeddings of a hypergraph H in surfaces as the 2-cell embeddings of its Levi graph L_H. That means we have the following definition (See Figure 6.1 as an example).

Definition 6.1.2. Suppose H is a hypergraph, we define the *genus* $\mathtt{g}(H)$ (the *non-orientable genus* $\widetilde{\mathtt{g}}(H)$, and the *Euler genus* $\widehat{\mathtt{g}}(H)$) as the genus (non-orientable genus, and Euler genus, respectively) of L_H.

Since L_H is bipartite, we have the following simple corollary of Euler's Formula (see [44, Proposition 4.4.4]):

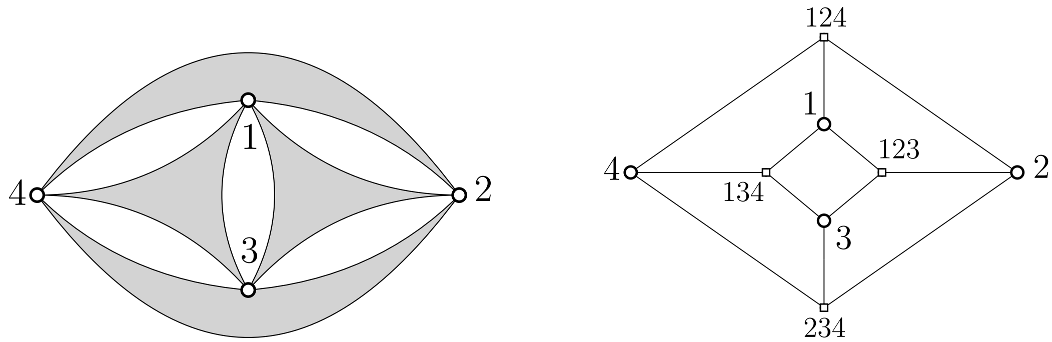

Figure 6.1: Planar embeddings of K_4^3 and its Levi graph L_4.

Lemma 6.1.3. *Let H be a 3-uniform hypergraph with n vertices and e edges. Then*

$$\widehat{\mathsf{g}}(H) \geq \frac{1}{2}e - n + 2. \tag{6.1.1}$$

Moreover, equality holds in (6.1.1) if and only if the Levi graph L_H admits a quadrilateral embedding in some surface.

In the case of the complete 3-uniform hypergraphs we obtain:

Proposition 6.1.4. *For every $n \geq 4$ we have $\widehat{\mathsf{g}}(K_n^3) \geq \left\lceil \frac{(n-2)(n+3)(n-4)}{12} \right\rceil$.*

6.2 Hypergraphs of Even Order

In this section, we assume $n \geq 4$ is an even integer.

6.2.1 Minimum Genus Embeddings of Hypergraphs

For each i $(1 \leq i \leq n)$, let $K_n - i$ be the labelled complete graph defined on the vertex set $[n] \setminus \{i\}$. Suppose T_i and T_i' are Eulerian circuits in $K_n - i$. If T_i' is the reverse of T_i, we denote it by T_i^{-1} and view them to be equivalent. Two families $\mathcal{F}, \mathcal{F}'$ of circuits are *equivalent* if there is a bijection $f : \mathcal{F} \to \mathcal{F}'$ such that for each $C \in \mathcal{F}$ either $f(C) = C$ or $f(C) = C^{-1}$.

Suppose T_i is an Eulerian circuit in $K_n - i$ and T_j in $K_n - j$, where $j \neq i$. Define a *transition* through j in T_i as a subtrail of T_i consisting of two consecutive edges aj and jb, and we denote it simply by ajb (which may sometimes be written as a, j, b). We say that T_i and T_j are *compatible* if for every transition ajb in T_i, there is a transition aib or bia in T_j, and T_i and T_j are *strongly compatible* if for every transition ajb in T_i, there is the transition bia in T_j. Note that this gives a bijective correspondence between $\frac{n-2}{2}$ transitions through j in T_i and $\frac{n-2}{2}$ transitions through i in T_j. We call a set of trails $\{T_1, \ldots, T_n\}$ an *embedding set* if T_i is an Eulerian circuit in $K_n - i$ for each $i = 1, \ldots, n$ and any two of them are compatible. An embedding set is *strong* if for every $i \neq j$, T_i and T_j are strongly compatible. In our construction of embeddings, we will use different rules when specifying Eulerian circuits for odd and even values of i, and we will say that $i \in [n]$ is an *odd vertex* (or *even vertex*) when i is odd (or even) viewed as an integer.

The following result is our main tool.

Theorem 6.2.1. *Let $n \geq 4$ be an even integer. There exists a bijection between equivalence classes of the (labelled) quadrilateral embeddings of the Levi graph L_n of K_n^3 and the equivalence classes of embedding sets of size n. Under this correspondence, strong embedding sets correspond to orientable quadrilateral embeddings.*

Proof. Suppose $\Pi = \{\pi_v \mid v \in V(L_n)\}$ is a quadrilateral embedding of L_n. Recall that $X_n = [n]$ and $Y_n = \binom{[n]}{3}$ is the bipartition of L_n. For every vertex $i \in X_n$, consider the local rotation π_i around i. Note that the neighbors of i are all $\binom{n-1}{2} =: N$ triples of elements of $[n]$ which contain i, and all of them have degree 3 in L_n.

Each pair of consecutive vertices (triples) in π_i determines a 4-face with two vertices in X_n, say i and j. Then both triples are adjacent to i and to j in L_n, so they both contain i and j. Let us now consider the two 4-faces containing the edge joining i and a triple ijk. Since this triple is adjacent to vertices j and k in L_n, one of the neighbors of i preceding or succeeding ijk in the local rotation π_i contains j and the other one contains k. Therefore there is a sequence $a_1, a_2, \ldots, a_N$ such that a_j is the common element between the jth and $(j+1)$st neighbor of i in π_i. Moreover, the jth neighbor of i is the triple $ia_{j-1}a_j$ (where $a_0 = a_N$). Clearly, the cyclic sequence $T_i = (a_1 a_2 \ldots a_N)$ is an Eulerian circuit in $K_n - i$ since the consecutive pairs $a_{j-1}a_j$ $(1 \leq j \leq N)$ run over all pairs in $[n] \setminus \{i\}$.

Suppose iaj and ijk are consecutive neighbors of the vertex i in π_i and assume $iaj \to ijk$ is clockwise. See Figure 6.2 for clarification. That means, ajk is a transition in T_i. Now consider the local rotation π_j. Clearly, iaj and ijk are consecutive vertices in π_j. Moreover, assuming the local rotations around i and j are chosen consistently with the clockwise orientation in the face containing i, iaj, j, ijk, we have $ijk \to iaj$ is clockwise. That means that T_i and T_j are compatible (strongly in the orientable case). Therefore, $\{T_1, \ldots, T_n\}$ form an embedding set (or strong embedding set).

This gives a correspondence $(\pi, \lambda) \mapsto \{T_1, \ldots, T_n\}$. Let us first observe that equivalent embedding schemes (obtained by switching over a vertex-set $U \subseteq V(L_n)$)

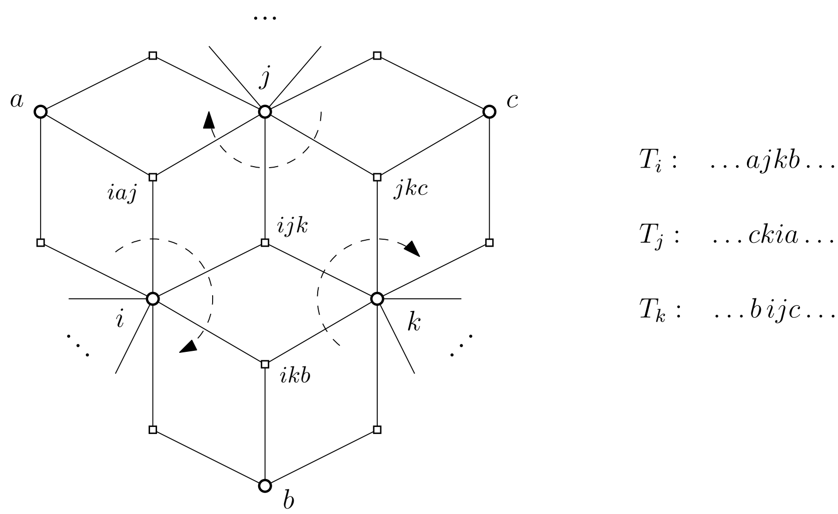

Figure 6.2: A quadrilateral embedding around a vertex $ijk \in Y_n$. The chosen clockwise rotation around vertices i, j, k is indicated by the dashed circular arcs.

correspond to changing the Eulerian circuits T_u with their inverse circuits T_u^{-1} for $u \in U \cap X_n$. Thus the correspondence preserves equivalence.

To see that the described correspondence is injective, consider two quadrilateral embeddings with schemes $\Pi^1 = (\pi^1, \lambda_1)$ and $\Pi^2 = (\pi^2, \lambda_2)$, whose embedding set $\{T_1, \ldots, T_n\}$ is the same. This in particular means that π^1 and π^2 agree on X_n. Clearly, this implies that the set of quadrangular faces is the same for both embeddings (for every π_i^1-consecutive neighbors $iaj \to ijk$ the corresponding 4-face has vertices i, iaj, j, ijk). By [44, Corollary 3.3.2], this implies that Π^1 and Π^2 are equivalent.

In order to show the map is surjective, suppose $\mathcal{E} = \{T_1, \ldots, T_n\}$ is an embedding set. We have to show that there is a quadrilateral embedding of K_n^3 such that this embedding returns an equivalent embedding set under the correspondence described in the first part of the proof. The quadrilateral embedding will be given by an embedding scheme $\Pi = (\pi, \lambda)$ which is determined as follows.

For $i \in X_n$, let T_i be the circuit $a_0a_1a_2 \ldots a_N$, where $a_0 = a_N$. Then we define the rotation π_i around the vertex i as the cyclic permutation:

$$\pi_i = (ia_0a_1, ia_1a_2, ia_2a_3, \ldots, ia_{N-1}a_N).$$

For each triple $ijk \in Y_n$ (where $i < j < k$), set $\pi_{ijk} = (i, j, k)$. Finally, define the signature as follows. Given $i < j < k$, let e_1, e_2, and e_3 be the edges joining ijk with the vertex i, j, and k, respectively. We set $\lambda(e_1) = 1$ if the edge jk appears in the direction from j to k in T_i. Otherwise, set $\lambda(e_1) = -1$. Similarly, set $\lambda(e_2) = 1$

($\lambda(e_3) = 1$) if and only if the edge ki (ij) appears in T_j (T_k) in the direction from k to i (from i to j). By these rules it is clear that equivalent embedding sets give equivalent embedding schemes, and that Π will give back the same embedding set. It remains to see that the embedding Π is quadrilateral. To see this, consider a triple ijk ($i < j < k$) and the faces around it. Figure 6.2 should help us to visualize the situation. By changing the embedding set $\mathcal{E}$ to an equivalent embedding set (by possibly changing T_i, T_j, T_k to their inverses), we may assume that T_i traverses jk in the direction from j to k, T_j traverses ki from k to i, and T_k traverses ij from i towards j. Then $\lambda(e_1) = \lambda(e_2) = \lambda(e_3) = 1$. Let $T_i : \ldots ajkb \ldots$ and $T_j : \ldots ckia \ldots$. Here we used compatibility condition to conclude that kia is a transition in T_j. Compatibility condition implies that $T_k : \ldots bijc \ldots$. This implies that the faces around ijk are precisely as shown in the figure. Since ijk was arbitrary, we conclude that all faces are quadrilaterals, which we were to prove. □

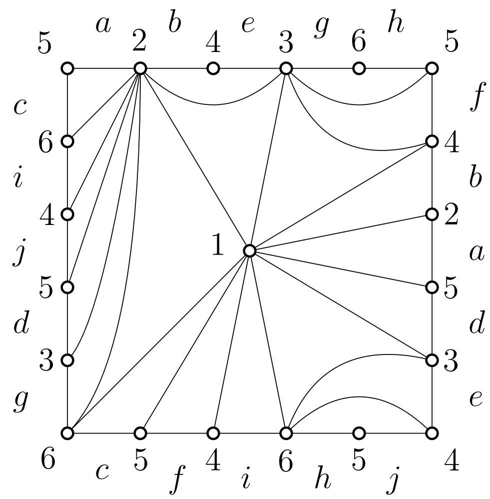

Figure 6.3: A minimum genus orientable embedding of K_6^3.

Example 6.2.2. The genus of K_6^3 is 3.

Here we give a construction of an optimal orientable embedding of K_6^3, see Figure 6.3. Identity the same number as end vertices in edges with same letter , we will get an embedding on the triple torus. Its strong embedding set $\mathcal{E}_6$ is the following:

$$T_1 = 3,4,2,5,3,6,4,5,6,2;$$
$$T_2 = 4,3,1,6,3,5,4,6,5,1;$$
$$T_3 = 1,2,4,6,1,5,2,6,5,4;$$
$$T_4 = 2,1,3,5,1,6,2,5,6,3;$$
$$T_5 = 6,1,4,3,6,4,2,3,1,2;$$
$$T_6 = 5,2,4,1,3,4,5,3,2,1.$$

6.2.2 Genus of Hypergraphs

Next, we will construct minimum genus embeddings of K_n^3 when n is even.

Theorem 6.2.3. *If $n \geq 4$ is even, then*

$$\mathsf{g}(K_n^3) = \frac{(n-2)(n+3)(n-4)}{24}.$$

Proof. By Proposition 6.1.4, it suffies to show that $\mathsf{g}(K_n^3) \leq \frac{(n-2)(n+3)(n-4)}{24}$. We will prove it by induction on n. The base case when $n = 4$ is clear from Figure 6.1, so we proceed with the induction step.

Assume L_n quadrangulates some orientable surface, and $\mathcal{E}_n = \{T_1, \ldots, T_n\}$ is the corresponding strong embedding set, where T_i is an Eulerian trail in $K_n - i$ ($i \in [n]$). Now we consider L_{n+2} with two new vertices in $X_{n+2} = [n+2]$. For brevity we will write $x = n+1$ and $y = n+2$. For every odd vertex $1 \leq i \leq n-1$, T_i contains $\frac{n-2}{2}$ transitions of the form $a, i+1, b$. We arbitrarily pick one of those transitions, and denote it by $a_i, i+1, b_i$. In the next step, we are going to insert a trail E_i between $i+1$ and b_i in T_i to get a closed Eulerian trail in $K_{n+2} - i$. The new, longer trail will be denoted by T_i'. For the trail T_{i+1} in $\mathcal{E}_n$, since $a_i, i+1, b_i$ is a transition in T_i, the transition b_i, i, a_i is contained in T_{i+1} by the strong compatibility condition. Similarly as what we do for T_i, we will insert a trail E_{i+1} between i and a_i in T_{i+1}, and the new longer trail we get is denoted by T_{i+1}'.

For every odd vertex $1 \leq i \leq n-1$, let σ_i be the permutation of the set $[n] \setminus \{i, i+1\}$ that is obtained from the sequence $1, 2, \ldots, n$ by removing i and $i+1$ and by switching the pairs $2j-1, 2j$ for $j = 1, \dots, \frac{i-1}{2}$. Specifically:

$$\begin{aligned}
\sigma_1 &= 3, 4, 5, 6, \dots, n-1, n; \\
\sigma_3 &= 2, 1, 5, 6, \dots, n-1, n; \\
&\dots \\
\sigma_i &= 2, 1, 4, 3, \dots, i-1, i-2, i+2, i+3, i+4, \dots, n-1, n; \\
&\dots \\
\sigma_{n-1} &= 2, 1, 4, 3, \dots, n-2, n-3.
\end{aligned}$$

We construct E_i as follows. We start with x, and then insert y and x consecutively in the interspace of numbers in σ_i, and add $x, y, i+1$ at the end, for every odd vertex $1 \le i \le n-1$. For the case E_{i+1}, we start with y, insert x and y (alternating) in the interspace of numbers in σ_i, and add y, x, i at the end. To be more precise, we get the following:

$$\begin{aligned}
E_1 &= x, 3, y, 4, x, 5, y, 6, \dots, x, n-1, y, n, x, y, 2; \\
E_2 &= y, 3, x, 4, y, 5, x, 6, \dots, y, n-1, x, n, y, x, 1; \\
&\dots \\
E_i &= x, 2, y, 1, \dots, x, i-1, y, i-2, x, i+2, y, \dots, x, n-1, y, n, x, y, i+1; \\
E_{i+1} &= y, 2, x, 1, \dots, y, i-1, x, i-2, y, i+2, x, \dots, y, n-1, x, n, y, x, i; \\
&\dots \\
E_{n-1} &= x, 2, y, 1, x, 4, y, 3, \dots, x, n-2, y, n-3, x, y, n; \\
E_n &= y, 2, x, 1, y, 4, x, 3, \dots, y, n-2, x, n-3, y, x, n-1.
\end{aligned}$$

It is easy to see that T'_i and T'_{i+1} are Eulerian trails in $K_{n+2} - i$ and $K_{n+2} - (i+1)$. To verify the strong compatibility of these Eulerian trails, note that our construction preserves almost all transitions in $\mathcal{E}_n$, except for every odd i we break the transition $a_i, i+1, b_i$ in T_i, and the transition b_i, i, a_i in T_{i+1}. That means we only need to check the strong compatibility of transitions in E_i. If j and i are both odd and $j < i$, this is true since x, i, y is a transition in E_j and y, j, x is a transition in E_i. Similar observations hold in the other three cases depending on the parities of j and i. This shows that T'_a and T'_b are strongly compatible for every $1 \le a < b \le n$.

In the final step, we will construct Eulerian trails T'_x and T'_y, such that $\mathcal{E}_{n+2} = \{T'_1, \dots, T'_n, T'_x, T'_y\}$ is a strong embedding set. We have to fix some transitions in T'_x

and T'_y in order to get the strong compatibility with trails T'_j $(1 \le j \le n)$. We list these transitions in the following tables, where we assume $3 \le i \le n-3$ is an odd vertex.

Transitions in T'_x through odd vertices $(3 \le i \le n-3)$				
$3\ 1\ 2$		$2\ i\ i+1$		$2\ n-1\ n$
$5\ 1\ 4$		$4\ i\ 1$		$4\ n-1\ 1$
$7\ 1\ 6$		$6\ i\ 3$		$6\ n-1\ 3$
$9\ 1\ 8$	$\dots$	$\vdots$	$\dots$	
		$i-1\ i\ i-4$		
$\vdots$		$i+2\ i\ i-2$		$\vdots$
		$i+4\ i\ i+3$		
		$\vdots$		$n-4\ n-1\ n-7$
$n-1\ 1\ n-2$		$n-1\ i\ n-2$		$n-2\ n-1\ n-5$
$y\ 1\ n$		$y\ i\ n$		$y\ n-1\ n-3$

Transitions in T'_x through even vertices $(4 \le i+1 \le n-2)$				
$4\ 2\ 3$		$1\ i+1\ 2$		$1\ n\ 2$
$6\ 2\ 5$		$3\ i+1\ 4$		$3\ n\ 4$
$8\ 2\ 7$		$\vdots$		
		$i-2\ i+1\ i-1$		
$\vdots$	$\dots$	$i+3\ i+1\ i+2$	$\dots$	$\vdots$
		$\vdots$		
$n\ 2\ n-1$		$n\ i+1\ n-1$		$n-3\ n\ n-2$
$1\ 2\ y$		$i\ i+1\ y$		$n-1\ n\ y$

If T'_x has all the transitions listed in the tables above, then it is strongly compatible with T'_j for every $1 \le j \le n$. Since each pair of two different numbers will consecutively appear in T'_x exactly once, the above tables give us the following $n/2$ subtrails $\{A_1, \dots, A_{\frac{n}{2}}\}$ in T'_x where we also let $3 \le i \le n-3$ be odd.

$$A_1 = y, 1, n, 2, n-1, n, y;$$
$$\dots$$
$$A_{\frac{i+1}{2}} = y, i, n, i+1, F_i(1), F_i(2), \dots, F_i(\tfrac{i-1}{2}), n-i, n+1-i, y;$$
$$\dots$$
$$A_{\frac{n}{2}} = y, F_{n-1}(1), F_{n-1}(2), \dots, F_{n-1}(\tfrac{n}{2}), 2, y.$$

where $F_i(j)$ $(3 \leq i \leq n-3)$ is a subtrail of length 4 such that $F_i(j) = n+1-2j, i-2j, n-2j, i+1-2j$ and $F_{n-1}(j) = n+1-2j$.

Similarly, the following tables state the transitions in T'_y forced by strong compatibility with T'_j for every $1 \leq j \leq n$.

Transitions in T'_y through odd vertices $(3 \leq i \leq n-3)$				
$4\ 1\ 3$		$1\ i\ 2$		$1\ n-1\ 2$
$6\ 1\ 5$		$3\ i\ 4$		$3\ n-1\ 4$
$8\ 1\ 7$		$\vdots$		
		$i-2\ i\ i-1$		
$\vdots$	$\dots$	$i+3\ i\ i+2$	$\dots$	
		$\vdots$		$\vdots$
$n\ 1\ n-1$		$n\ i\ n-1$		$n-3\ n-1\ n-2$
$2\ 1\ x$		$i+1\ i\ x$		$n\ n-1\ x$

Transitions in T'_y through even vertices $(4 \leq i+1 \leq n-2)$				
$3\ 2\ 1$		$2\ i+1\ i$		$2\ n\ n-1$
$5\ 2\ 4$		$4\ i+1\ 1$		$4\ n\ 1$
$7\ 2\ 6$		$6\ i+1\ 3$		$6\ n\ 3$
$9\ 2\ 8$	$\dots$	$\vdots$	$\dots$	
		$i-1\ i+1\ i-4$		
$\vdots$		$i+2\ i+1\ i-2$		$\vdots$
		$i+4\ i+1\ i+3$		
		$\vdots$		$n-4\ n-2\ n-7$
$n-1\ 2\ n-2$		$n-1\ i+1\ n-2$		$n-2\ n\ n-5$
$x\ 2\ n$		$x\ i+1\ n$		$x\ n\ n-3$

The above tables also give us the following $n/2$ subtrails $\{B_1, \dots, B_{\frac{n}{2}}\}$ in T'_y.

$B_1 = x, 2, n, n-1, x;$

$\dots$

$B_{\frac{i+1}{2}} = x, i+1, n, G_i(1), G_i(2), \dots, G_i(\frac{i-1}{2}), n-i, x \qquad (i \text{ is odd, } 3 \leq i \leq n-3);$

$\dots$

$B_{\frac{n}{2}} = x, n, n-3, G_{n-1}(1), G_{n-1}(2), \dots, G_{n-1}(\frac{n-4}{2}), 3, 2, 1, x.$

where $G_i(j)$ $(3 \le i \le n-3)$ is a subtrail of length 4 such that $G_i(j) = i-2j, n+1-2j, i+1-2j, n-2j$, and $G_{n-1}(j)$ is a subtrail of length 3 where $G_{n-1}(j) = n+1-2j, n-2j, n-2j-3$.

Finally we are going to combine those subtrails of T'_x and T'_y. Note that any combination will give us a closed Eulerian trail on $K_{n+2}-x$ or $K_{n+2}-y$ (respectively), since if ab (or ba) appears twice in the subtrails of T'_x, then either T'_a or T'_b is not an Eulerian trail. We let $T'_x = A_1, A_2, \dots, A_{\frac{n}{2}}$ and $T'_y = B_1, B_2, \dots, B_{\frac{n}{2}}$, that means the construction is the following:

$$T'_x = y, 1, \dots, n, y, 3, \dots, n-2, y, 5, \dots, 4, y, n-1, \dots, 2;$$
$$T'_y = x, 2, \dots, n-1, x, 4, \dots, 5, x, n-2, \dots, 3, x, n, \dots, 1.$$

It remains to show that T'_x and T'_y are strongly compatible. This is true because for every odd vertex $3 \le i \le n-1$, we can see that $n+3-i, y, i$ is a transition in T'_x and $i, x, n+3-i$ is a transition in T'_y, as well as $2, y, 1$ is a transition in T'_x and $1, x, 2$ is a transition in T'_y. This completes the proof. □

Lemma 6.2.4. *The non-orientable genus of K_6^3 is* 6.

Proof. By Lemma 6.1.3, we have $\tilde{\mathsf{g}}(K_6^3) \ge 6$. Then it suffices to provide a construction of an embedding of K_6^3 in some non-orientable surfaces of genus 6. By Theorem 6.2.1, we only need to construct an embedding set, but not a strong embedding set. Here is the construction of the embedding set $\mathcal{E}_6$.

$$T_1 = 4, 2, 5, 3, 6, 4, 5, 6, 2, 3;$$
$$T_2 = 4, 6, 5, 1, 4, 3, 1, 6, 3, 5;$$
$$T_3 = 1, 2, 4, 6, 1, 5, 2, 6, 5, 4;$$
$$T_4 = 5, 1, 6, 2, 5, 6, 3, 2, 1, 3;$$
$$T_5 = 6, 3, 4, 2, 3, 1, 2, 6, 4, 1;$$
$$T_6 = 2, 1, 5, 3, 2, 5, 4, 3, 1, 4.$$

Note that $6, 3, 4$ is a transition in T_5 and $6, 5, 4$ is a transition in T_3, and also $2, 3, 1$ is a transition in T_5 and $1, 5, 2$ is a transition in T_3. That means, neither T_3 or T_3^{-1} is strongly compatible with T_5. It is not hard to see that the set $\mathcal{E}_6$ is compatible. □

Theorem 6.2.5. *If $n \geq 6$ is even, then*

$$\widetilde{\mathrm{g}}(K_n^3) = \frac{(n-2)(n+3)(n-4)}{12}.$$

Proof. The proof follows the same inductive construction we used in the proof of Theorem 6.2.3. Instead of using K_4^3 as the base step, we use Lemma 6.2.4 as the base step. Therefore, by the way we constructed the embedding set of K_n^3, Eulerian trails T_3 and T_5 will always be compatible, but they will never be strongly compatible. □

6.2.3 Number of Non-Isomorphic Embeddings

We say that two embeddings $\phi_1, \phi_2 : G \to S$ are *isomorphic* if there is an automorphism α of G such that the embeddings ϕ_1 and $\phi_2\alpha$ are equivalent. In this section we will show how to obtain many non-isomorphic optimal embeddings of K_n^3 when n is even.

It is easy to see that the number of non-equivalent (2-cell) embeddings of K_n^3 in some surface is equal to

$$2^{\binom{n}{3}} 2^{3\binom{n}{3}} \left(\left(\binom{n}{3} - 1 \right)! \right)^n 2^{-n} = 2^{\frac{1}{2}n^4 \log n(1-o(1))}.$$

The genera of all these embeddings take only $O(n^3)$ different values, but the majority of them will have their genus much larger than the minimum possible genus. The number of minimum genus embeddings is indeed much smaller as made explicit in the following.

Lemma 6.2.6. *The number of non-equivalent embeddings of K_n^3 into a surface of Euler genus $\frac{1}{6}(n-2)(n+3)(n-4)$ is at most $2^{\frac{1}{4}n^3 \log n(1+o(1))}$, where the logarithm is taken base 2.*

Proof. We may assume that n is even since otherwise there are no such embeddings. By Theorem 6.2.1, optimal embeddings of K_n^3 into surfaces of Euler genus $\frac{1}{6}(n-2)(n+3)(n-4)$ are quadrilateral and are in a bijective correspondence with embedding sets. These are sets of Eulerian circuits satisfying compatibility conditions. Their number can be estimated as follows.

Suppose that compatible Eulerian circuits $T_1, \ldots, T_{k-1}$ are already chosen ($1 \leq k \leq n$). To construct the next circuit T_k, we start by an arbitrary edge in $K_n - k$. If we come to a vertex $i < k$ when following the last chosen edge, the transition is determined by compatibility with T_i. On the other hand if we come to a vertex $i > k$

for the rth time, there are (at most) $n-1-2r$ edges which can be chosen as the next edge on the trail. All together, when passing through such a vertex i, we have at most $(n-3)(n-5)(n-7)\cdots 3\cdot 1=(n-3)!!$ choices. Therefore the number of ways to choose T_k is at most $((n-3)!!)^{n-k}$. Thus the number of embedding sets is at most:

$$((n-3)!!)^{(n-1)+(n-2)+\cdots+1+0} = 2^{\frac{1}{4}n^3 \log n(1+o(1))}$$

and this completes the proof. □

Note that in the proof we are actually giving a bound on compatible closed trail decompositions. Nevertheless, this estimate may be rather tight, since the number of Eulerian circuits in K_{n-1} is $2^{\frac{1}{2}n^2 \log n(1+o(1))}$, see [40, Theorem 4].

Now we will turn to a lower bound on the number of non-isomorphic minimum genus embeddings that can be obtained by a simple generalization of the construction in our proofs of Theorems 6.2.3 and 6.2.5.

Theorem 6.2.7. *If n is even, there exist at least $2^{\frac{1}{4}n^2 \log n(1-o(1))}$ non-isomorphic optimal embeddings of K_n^3 in each, the orientable and the non-orientable surface of Euler genus $\frac{1}{6}(n-2)(n+3)(n-4)$.*

Proof. Let I_n be the number of non-isomorphic optimal embeddings of K_n^3. Here we will only deal with the orientable case; for the non-orientable embeddings, arguments are the same.

Recall that in the construction of the embedding set $\mathbb{E}_n = \{T_1', \dots, T_n'\}$ of K_n^3, for every odd vertex $i \in [n-2]$ we arbitrarily pick a transition $a_i, i+1, b_i$ in $T_i \in \mathbb{E}_{n-2}$, and insert a subtrail E_i. Since $i+1$ appears exactly $\frac{n-4}{2}$ times in T_i, different choice of transitions through $i+1$ will give us different trails T_i' and T_{i+1}'. Also, the choice of consecutive odd-even pairs $i, i+1$ gives us a perfect matching of K_{n-2}. It is easy to see that any perfect matching of K_{n-2} can be used as such a pairing and this will give us different embedding sets $\mathbb{E}_n$. Moreover, fixing a perfect matching, for example, $i, i+1$ for every odd i, we can exchange i and $i+1$ to get a new embedding set. Note that x and y are symmetric in our construction, and can be exchanged.

Let I_n denote the resulting number of inequivalent embedding sets. Then we have:

$$I_n \geq \frac{1}{2}\left(\frac{n-4}{2}\right)^{\frac{n-2}{2}} (n-3)!!\; 2^{\frac{n-2}{2}} I_{n-2}.$$

Therefore,

$$\begin{aligned}
\log I_n &\geq \log \prod_{k=2}^{\frac{n-2}{2}} (k-1)^k (2k-1)!!\ 2^{k-1} \\
&\geq \log \frac{(\frac{n-4}{2})!^{\frac{n-2}{2}} 2^{\frac{n}{2}(\frac{n}{2}-2)} \prod_{k=2}^{\frac{n-4}{2}} k!}{\prod_{k=1}^{\frac{n-6}{2}} k!} \\
&= \frac{n(n-4)}{4} \log \frac{n-4}{2e} + O(n^2) \\
&= \tfrac{1}{4} n^2 \log n\, (1 - o(1)).
\end{aligned}$$

That means that there are at least $2^{\frac{1}{4}n^2 \log n\,(1+o(1))}$ inequivalent minimum genus embeddings.

If ϕ_1 and ϕ_2 are non-equivalent but isomorphic embeddings, then there is an automorphism α such that ϕ_1 is equivalent with $\phi_2\alpha$. Each such automorphism is determined by the values $\alpha(1)$ and $\alpha(123)$, and by noting whether α preserves or reverses the local rotation around the vertex 1. This means that the number of isomorphic embeddings is polynomial in n, and thus the number of isomorphism classes of embeddings decreases by a factor that can be hidden in the $o(1)$ term in the $(1 - o(1))$ factor. This completes the proof. □

6.2.4 Hypergraphs with Multiple Edges

Now we are going to investigate the genus of complete 3-uniform graphs with multiple edges. These results will partially answer the question the authors asked in Theorem 3.2.8. In that work, the genus of random bipartite graphs $\mathcal{G}(n_1, n_2, p)$ is considered, where $n_1 \gg 1$ and n_2 is a constant, and the edge probabilities are $p = \Theta(n_1^{-1/3})$. In that regime, the following hypergraph occurs. Let mK_n^3 be the complete 3-uniform hypergraph where each triple occurs m times, i.e., each edge of K_n^3 has multiplicity m. In this situation, each trail T_i^m in the embedding set $\mathbb{E}_n^m$ is an Eulerian circuit in $m(K_n - i)$. Similarly, we say two trails T_i and T_j are *strongly compatible* (*compatible*) if transitions ajb appear in T_i exactly t times, then transitions bia (aib or bia) appear in T_j exactly t times. It is easy to see that Theorem 6.2.1 is still true in this case, the proof is similar and we omit the details. Therefore, we have the following result.

Theorem 6.2.8. *If $n \geq 4$ is even and $m \geq 2$, then $\mathsf{g}(mK_n^3) = \frac{(n-2)(mn(n-1)-12)}{24}$ and $\widetilde{\mathsf{g}}(mK_n^3) = \frac{(n-2)(mn(n-1)-12)}{12}$.*

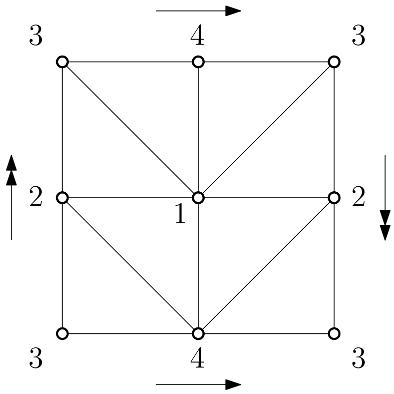

Figure 6.4: A non-orientable embedding of $2K_4^3$ on the Klein bottle.

Proof. The lower bound follows by Lemma 6.1.3. To see the upper bound, we will give an inductive construction on m.

Suppose $\mathbb{E}_n^m = \{T_1^m, \ldots, T_n^m\}$ is a (strong) embedding set of mK_n^3, and suppose that $\mathbb{E}_n = \{T_1, \ldots, T_n\}$ is a (strong) embedding set of K_n^3. For every odd $i \in [n]$, suppose the transition $a_i, i+1, b_i$ is in both T_i^m and T_i. Note that by our construction, such transition exists, and actually we have at least $\frac{n-2}{2}$ such transitions for every i. We arbitrarily pick one such transition $a_i, i+1, b_i$. Since T_i also contains the transition $a_i, i+1, b_i$, we break T_i between $i+1$ and b_i, and we write T_i by starting with b_i and end with $a_i, i+1$. We also break transition $a_i, i+1, b_i$ in T_i^m, and insert T_i between $i+1$ and b_i. For the case $i+1$, we do the same things on transition b_i, i, a_i. Therefore, we will get a (strong) embedding set $\mathbb{E}_n^{m+1}$ of $(m+1)K_n^3$. It is easy to verify the (strong) compatibility among trails in $\mathbb{E}_n^{m+1}$.

The described construction works in all cases except when $n = 4$, and we look for the non-orientable embeddings of mK_4^3. Since $\mathsf{g}(K_4^3) = 0$, the base case of induction for the non-orientable genus of mK_4^3 is when $m = 2$. In this case, we construct the following non-orientable embedding set $\mathbb{E}_4^2$ on the Klein bottle (see Figure 6.4 for the corresponding embedding):

$$T_1^2 : 3, 2, 4, 2, 3, 4;$$
$$T_2^2 : 4, 1, 3, 4, 1, 3;$$
$$T_3^2 : 1, 4, 2, 1, 4, 2;$$
$$T_4^2 : 2, 3, 1, 3, 2, 1.$$

The induction step follows the same argument as when $n \geq 6$. $\square$

Let us observe that the embedding of mK_n^3 described in the proof of Theorem 6.2.8 (with the exception of the non-orientable case when $n = 4$) is just a branched covering from a quadrilateral embedding of K_n^3 where each vertex $i \in X_n$ is a branch point with branching degree m.

6.3 Hypergraphs of Odd Order

Since a complete graph having even number of vertices does not have any Eulerian trails, we have the following observation.

Lemma 6.3.1. *If n is even, the shortest closed walk in K_n that cover all of its edges has at least $\frac{n}{2}$ repeated edges, and any two of those repeated edges are not adjacent in the walk with minimal length.*

Proof. For every $v \in K_n$, $\deg(v) = n - 1$ is odd. In order to visit all the edges, for every vertex v, at least one repeated edge is incident by v. Thus the walk has at least $\frac{n}{2}$ repeated edges. If ab and bc are consecutive edges in the walk with minimal length and both of them are repeated edges, we can replace them by ac, this will give us a shorter walk. □

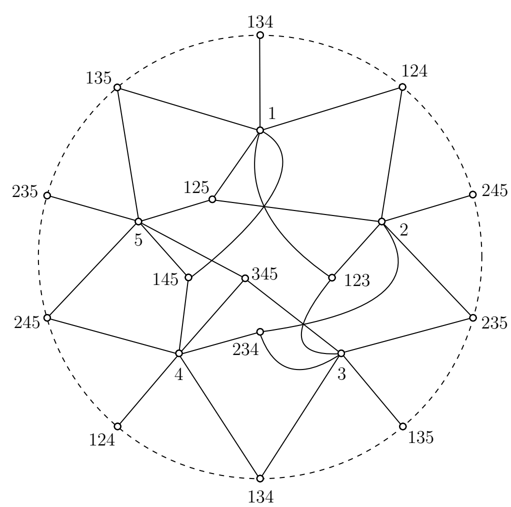

Figure 6.5: An optimal non-orientable embedding of K_5^3.

Theorem 6.3.2. $\widehat{\mathsf{g}}(K_5^3) = \widetilde{\mathsf{g}}(K_5^3) = 4$.

Proof. To prove that 4 is the lower bound, by Theorem 6.2.1 and Lemma 6.3.1, for every vertex i, where $1 \le i \le 5$, at most 4 faces of length 4 contain i. Thus, in any embedding of K_5^3, it has at most 10 faces of length 4. Since $n(n-1)$ is not divisible by 3 when $n=5$, the remaining faces cannot by all 6-gons. By Euler's formula, we have $\widehat{\mathsf{g}}(K_5^3) \leq 4$.

The proof of the upper bound is by construction, see Figure 6.5. The embedding is on a projective plane together with 3 crosscaps, it has 10 faces of length 4, 2 faces of length 6 and one face of length 8. □

Theorem 6.3.3. *If $n \geq 5$ is odd, then*

$$\widehat{\mathsf{g}}(K_n^3) \geq \left\lceil \frac{(n-3)(n^2+n-8)}{12} \right\rceil.$$

Proof. Recall that $V(L_n) = X_n \sqcup Y_n$ where $X_n = [n]$ and $Y_n = \binom{[n]}{3}$. For every $v \in X_n$, it has $\binom{n-1}{2}$ neighbors. By Lemma 6.3.1, at least $\frac{n-1}{2}$ pairs of them are contained in the faces other than cycles of length 4. Therefore, suppose f be the number of faces in the minimum genus embeddings of K_n^3, we have

$$\begin{aligned} f &\leq \frac{n\left(\binom{n-1}{2} - \frac{n-1}{2}\right)}{2} + \frac{n(n-1)}{6} \\ &= \frac{3}{2}\binom{n}{3} - \frac{1}{6}\binom{n}{2}. \end{aligned}$$

Thus, by using Euler's Formula, we obtain

$$\begin{aligned} \widehat{\mathsf{g}}(K_n^3) &\geq \left\lceil 2 - n - \binom{n}{3} - \frac{3}{2}\binom{n}{3} + \frac{1}{6}\binom{n}{2} + 3\binom{n}{3} \right\rceil \\ &= \left\lceil \frac{(n-3)(n^2+n-8)}{12} \right\rceil, \end{aligned}$$

which completes the proof. □

Bibliography

[1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma. *J. Algorithms*, 16(1):80–109, 1994.

[2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, fourth edition, 2016.

[3] K. Appel and W. Haken. Every planar map is four colorable. *Bull. Amer. Math. Soc.*, 82(5):711–712, 1976.

[4] D. Archdeacon and D. A. Grable. The genus of a random graph. *Discrete Math.*, 142(1-3):21–37, 1995.

[5] M. Behzad, G. Chartrand, and L. Lesniak-Foster. *Graphs & digraphs*. PWS Publishers, Boston, Mass., 1979.

[6] L. W. Beineke and R. J. Wilson, editors. *Topics in topological graph theory*, volume 128 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009.

[7] B. Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[8] K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *J. Comput. System Sci.*, 13(3):335–379, 1976. Working Papers presented at the ACM-SIGACT Symposium on the Theory of Computing (Albuquerque, N. M., 1975).

[9] C. Chekuri and A. Sidiropoulos. Approximation algorithms for Euler genus and related problems. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science—FOCS 2013*, pages 167–176. IEEE Computer Soc., Los Alamitos, CA, 2013.

[10] J. Chen, S. P. Kanchi, and A. Kanevsky. A note on approximating graph genus. *Inform. Process. Lett.*, 61(6):317–322, 1997.

[11] N. Chiba, T. Nishizeki, S. Abe, and T. Ozawa. A linear algorithm for embedding planar graphs using PQ-trees. *J. Comput. System Sci.*, 30(1):54–76, 1985.

[12] F. Chung. From quasirandom graphs to graph limits and graphlets. *Adv. in Appl. Math.*, 56:135–174, 2014.

[13] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[14] É. Colin de Verdière. Computational topology of graphs on surfaces. *CoRR*, abs/1702.05358, 2017.

[15] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized algorithms*. Springer, 2015.

[16] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[17] P. Erdős and A. Rényi. On random graphs. I. *Publ. Math. Debrecen*, 6:290–297, 1959.

[18] J. Erickson, P. N. Klein, D. Marx, and C. Mathieu. Algorithms for Optimization Problems in Planar Graphs (Dagstuhl Seminar 16221). *Dagstuhl Reports*, 6(5):94–116, 2016.

[19] J. Fox, L. M. Lovász, and Y. Zhao. On regularity lemmas and their algorithmic applications. *Combin. Probab. Comput.*, 26(4):481–505, 2017.

[20] J. Fox, L. M. Lovász, and Y. Zhao. A fast new algorithm for weak graph regularity. *arXiv:1807.05031*, 2018.

[21] P. Frankl and V. Rödl. Near perfect coverings in graphs and hypergraphs. *European J. Combin.*, 6(4):317–326, 1985.

[22] A. Frieze and R. Kannan. Quick approximation to matrices and applications. *Combinatorica*, 19(2):175–220, 1999.

[23] M. R. Garey and D. S. Johnson. *Computers and intractability*. W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness.

[24] W. T. Gowers. Lower bounds of tower type for Szemerédi's uniformity lemma. *Geom. Funct. Anal.*, 7(2):322–337, 1997.

[25] J. L. Gross and T. W. Tucker. *Topological graph theory*. Dover Publications, Inc., Mineola, NY, 2001. Reprint of the 1987 original [Wiley, New York; MR0898434 (88h:05034)] with a new preface and supplementary bibliography.

[26] P. J. Heawood. Map-colour theorem. *Quart. J. Pure Appl. Math.*, 24(4):332–338, 1890.

[27] J. Hopcroft and R. Tarjan. Efficient planarity testing. *J. Assoc. Comput. Mach.*, 21:549–568, 1974.

[28] K. i. Kawarabayashi, B. Mohar, and B. Reed. A simpler linear time algorithm for embedding graphs into an arbitrary surface and the genus of graphs of bounded tree-width. In *49th Annual IEEE Symposium on Foundations of Computer Science (FOCS'08)*, pages 771–780, 2008.

[29] Y. Jing and B. Mohar. The genus of a random bipartite graph. *arXiv:1712.09989, submitted*, 2017.

[30] Y. Jing and B. Mohar. Efficient polynomial-time approximation scheme for the genus of dense graphs. To appear in *2018 IEEE 59th Annual Symposium on Foundations of Computer Science—FOCS 2018,* IEEE Computer Soc., Paris, France, 2018.

[31] Y. Jing and B. Mohar. The genus of complete 3-uniform hypergraphs. *arXiv:1805.01557, submitted*, 2018.

[32] Y. Jing and B. Mohar. The genus of complete 3-uniform hypergraphs. The odd case. *In preparation*, 2018.

[33] Y. Jing and B. Mohar. On the genus of generalized random graphs. *In preparation*, 2018.

[34] M. Jungerman, S. Stahl, and A. T. White. Imbeddings of hypergraphs. *Congr. Numer.*, 29:545–557, 1980.

[35] K.-i. Kawarabayashi and A. Sidiropoulos. Beyond the Euler characteristic: approximating the genus of general graphs [extended abstract]. In *STOC'15—Proceedings of the 2015 ACM Symposium on Theory of Computing*, pages 675–682. ACM, New York, 2015.

[36] L. Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.

[37] L. Lovász and B. Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.

[38] L. Lovász and B. Szegedy. Szemerédi's lemma for the analyst. *Geom. Funct. Anal.*, 17(1):252–270, 2007.

[39] Y. Makarychev, A. Nayyeri, and A. Sidiropoulos. A pseudo-approximation for the genus of Hamiltonian graphs. In *Approximation, randomization, and combinatorial optimization*, volume 8096 of *Lecture Notes in Comput. Sci.*, pages 244–259. Springer, Heidelberg, 2013.

[40] B. D. McKay and R. W. Robinson. Asymptotic enumeration of Eulerian circuits in the complete graph. *Combin. Probab. Comput.*, 7(4):437–449, 1998.

[41] B. Mohar. Embedding graphs in an arbitrary surface in linear time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing*, STOC '96, pages 392–397, New York, NY, USA, 1996. ACM.

[42] B. Mohar. A linear time algorithm for embedding graphs in an arbitrary surface. *SIAM J. Discrete Math.*, 12(1):6–26, 1999.

[43] B. Mohar. Face covers and the genus problem for apex graphs. *J. Combin. Theory Ser. B*, 82(1):102–117, 2001.

[44] B. Mohar and C. Thomassen. *Graphs on surfaces.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[45] R. A. Moser and G. Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010.

[46] W. Myrvold and J. Woodcock. A large set of torus obstructions and how they were discovered. *Electron. J. Combin.*, 25:Paper 1.16, 2018.

[47] O. Ore. *The four-color problem.* Pure and Applied Mathematics, Vol. 27. Academic Press, New York-London, 1967.

[48] T. D. Parsons, G. Pica, T. Pisanski, and A. G. S. Ventre. Orientably simple graphs. *Math. Slovaca*, 37(4):391–394, 1987.

[49] N. Pippenger and J. Spencer. Asymptotic behavior of the chromatic index for hypergraphs. *J. Combin. Theory Ser. A*, 51(1):24–42, 1989.

[50] J. M. Rahn. The genus of a block design. In *Proceedings of the Sundance conference on combinatorics and related topics (Sundance, Utah, 1985)*, volume 50, pages 255–268, 1985.

[51] G. Ringel. Das Geschlecht des vollständigen paaren Graphen. *Abh. Math. Sem. Univ. Hamburg*, 28:139–150, 1965.

[52] G. Ringel. *Map color theorem.* Springer-Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 209.

[53] G. Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.*, 60:438–445, 1968.

[54] N. Robertson and P. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63(1):65–110, 1995.

[55] N. Robertson and P. Seymour. Graph minors. XXII. Irrelevant vertices in linkage problems. *J. Combin. Theory Ser. B*, 102(2):530–563, 2012.

[56] N. Robertson and P. D. Seymour. Graph minors. VIII. A Kuratowski theorem for general surfaces. *J. Combin. Theory Ser. B*, 48(2):255–288, 1990.

[57] V. Rödl and L. Thoma. Asymptotic packing and the random greedy algorithm. *Random Structures Algorithms*, 8(3):161–177, 1996.

[58] V. Rödl and R. Thomas. On the genus of a random graph. *Random Structures Algorithms*, 6(1):1–12, 1995.

[59] J. Spencer. Asymptotic packing via a branching process. *Random Structures Algorithms*, 7(2):167–172, 1995.

[60] S. Stahl. On the average genus of the random graph. *J. Graph Theory*, 20(1):1–18, 1995.

[61] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975. Collection of articles in memory of JuriĭVladimirovič Linnik.

[62] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[63] T. Tao. Szemerédi's regularity lemma revisited. *Contrib. Discrete Math.*, 1(1):8–28, 2006.

[64] T. Tao. *An epsilon of room, II.* American Mathematical Society, Providence, RI, 2010. Pages from year three of a mathematical blog.

[65] C. Thomassen. The graph genus problem is NP-complete. *J. Algorithms*, 10(4):568–576, 1989.

[66] C. Thomassen. Triangulating a surface with a prescribed graph. *J. Combin. Theory Ser. B*, 57(2):196–206, 1993.

[67] C. Thomassen. The genus problem for cubic graphs. *J. Combin. Theory Ser. B*, 69(1):52–58, 1997.

[68] H. Tietze. *Famous problems of mathematics. Solved and unsolved mathematical problems from antiquity to modern times.* Authorized translation from the second (1959) revised German edition. Edited by B. K. Hofstadter and H. Komm. Graylock Press, New York, 1965.

[69] V. V. Vazirani. *Approximation algorithms.* Springer-Verlag, Berlin, 2001.

[70] A. T. White. *Graphs of groups on surfaces*, volume 188 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 2001. Interactions and models.

[71] S. G. Williamson. Depth-first search and Kuratowski subgraphs. *J. Assoc. Comput. Mach.*, 31(4):681–693, 1984.

[72] J. W. T. Youngs. Minimal imbeddings and the genus of a graph. *J. Math. Mech.*, 12:303–315, 1963.