Odd disjoint trails and totally odd graph immersions

by

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M.Sc., University of Victoria, 2012
B.Sc., University of Victoria, 2010

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Spring 2017

Approval

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Degree:	**Doctor of Philosophy (Mathematics)**
Title:	**Odd disjoint trails and totally odd graph immersions**
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Date Defended:	**January 10, 2017**

Abstract

The odd edge-connectivity between two vertices in a graph is the maximum number $\lambda_o(u,v)$ of edge-disjoint (u,v)-trails of odd length. In this thesis, we define the perimeter of a vertex-set, a natural upper bound for the odd edge-connectivity between some of its constituent pairs. Our central result is an approximate characterization of odd edge-connectivity: $\lambda_o(u,v)$ is bounded above and below by constant factors of the usual edge-connectivity $\lambda(u,v)$ and/or the minimum perimeter among vertex-sets containing u and v.

The relationship between odd edge-connectivity and perimeter has many implications, most notably a loose packing–covering duality for odd trails. (In contrast, odd paths do not obey any such duality.) For Eulerian graphs, we obtain a second, independent proof of the packing–covering duality with a significantly better constant factor. Both proofs can be implemented as polynomial-time approximation algorithms for $\lambda_o(u,v)$. After observing that perimeter satisfies a submodular inequality, we are able to prove an analogue of the Gomory–Hu Theorem for sets of minimum perimeter and, consequently, to construct an efficient data structure for storing approximate odd edge-connectivities for all vertex pairs in a graph.

The last part of the thesis studies more complicated systems of odd trails. A totally odd immersion of a graph H in another graph G is a representation in which vertices in H correspond to vertices in G and edges in H correspond to edge-disjoint odd trails in G. Using our perimeter version of the Gomory–Hu Theorem, we describe the rough structure of graphs with no totally odd immersion of the complete graph K_t. Finally, we suggest a totally odd immersion variant of Hadwiger's Conjecture and show that it is true for almost all graphs.

Keywords: covering and packing (05C70); paths and cycles (05C38); graph algorithms (05C85); connectivity (05C40); signed graphs (05C22); Eulerian graphs (05C45)

Dedication

To my wife, my cat,
and the rest of my support network.

If I could begin to be
Half of what you think of me,
I could do about anything,
I could even learn how to love.

Rebecca Sugar

Acknowledgements

I would like to thank my supervisor, Bojan Mohar, for introducing me to so many interesting topics in structural graph theory. I am grateful for the guidance, support, and opportunities he has given me as one of his doctoral students. I would also like to acknowledge the financial support from Simon Fraser University and from NSERC through the Alexander Graham Bell Canadian Graduate Scholarship.

Many people helped make my time at SFU productive, rewarding, and memorable. Discussions with Matt DeVos, Hehui Wu, and the rest of the immersions research group inspired several ideas in this thesis. Faculty and staff provided logistical support for all sorts of projects from seminars to social teas. Most importantly, my fellow graduate students offered their friendship and contributed to a positive environment for my studies. I am indebted to more teachers, colleagues, officemates, and friends than I can list here in a single page; thanks to you all.

This thesis is the culmination not just of a four-year doctorate but of a 23-year academic career. At the risk of having to change the answers to some online security questions, I would like to recognize a few teachers who helped shape who I am today: Andy Andrews, Steve Bloom, Barb Boonstra, Ryan Bonderud, Rick Brewster, Cara Cadre, Lenn Carrell, Darryl Chow, Paul Cordonier, Matt DeVos, Peter Dukes, Robb Fry, Judi Gelowitz, Gordon Gore, Marilyn Hannis, Phil Heal, Pavol Hell, Maureen Hove, Jing Huang, Lynda Milham, Gary MacGillivray, Marni Mishna, Kieka Mynhardt, Teresa O'Brien, Tonya Perry, Len Potter, Ross Presta, Linda Rinaldi, Brandi Rusk, Frederic Schuller, Eric Schweizer, Ahmed Sourour, Richard Taylor, Jessie Toews, John Webb, and—most of all—my parents.

Much of this thesis, including these acknowledgements, was written late at night with a cat by my side. My thanks to Sir Digby Chicken-Caesar for his constant company, unconditional love, and occasional editing sugges7yyy77yuuuuuuui.

The last four years have been an important emotional learning experience for me, and I have relied heavily on my support network to get through it. My family and friends deserve special thanks, especially Wanda Boyer, Sebastián González Hermosillo de la Maza, Krystal Guo, Laura Gutierrez Funderburk, Seyyed Aliasghar Hosseini, and Jessica Winters.

Finally, this all would not have been possible without the encouragement and support of my wife and partner Leah Churchley. Thank you from the bottom of my heart.

Table of Contents

List of Figures

List of Symbols

$A \cap B$	intersection of sets
$A \cup B$	union of sets
$A \setminus B$	difference of sets
$A \triangle B$	symmetric difference of sets
$\chi(G)$	chromatic number of G
$\delta(G)$	minimum degree of G
$\delta(X)$	set of edges with one end in a vertex-set X and one end outside X
$\deg(v)$	degree of a vertex v
$E(G)$	edge-set of G
$E(X)$	set of edges of G with both ends in a vertex-set X
$E(X, Y)$	set of edges of G with one end in X and one end in Y
$\mathbf{E}(X)$	expectation of a random variable X
$\mathcal{G}_{n,p}$	random graph on n vertices with edge-probability p
$G[X]$	subgraph of G induced by a vertex-set X
$G + uv$	addition of a new edge uv to a graph G
$G - F$	deletion of an edge-set F from G
$G - X$	deletion of a vertex-set X from G
K_t	complete graph on t vertices
$\lambda(u, v)$	edge-connectivity between vertices (or vertex-sets)

$\lambda_o(u, v)$	odd edge-connectivity between vertices
$N(v)$	set of neighbours of a vertex v
$O(f)$	set of functions g for which $\limsup_{n\to\infty} g(n)/f(n) < \infty$
$o(f)$	set of functions g for which $\lim_{n\to\infty} g(n)/f(n) = 0$
$\Omega(f)$	set of functions g for which $f \in O(g)$
$\omega(f)$	set of functions g for which $f \in o(g)$
$\Theta(f)$	set of functions g for which $f \in O(g)$ and $g \in O(f)$
$p(X, H)$	perimeter of a vertex-set X with respect to a subgraph H
$\mathbf{P}(X)$	probability of an event X
$\tau_o(u, v)$	odd trail covering number
$V(G)$	vertex-set of G

Glossary

almost all graphs

Any family of graphs $\mathcal{F}$ for which $\lim_{n\to\infty} \mathbf{P}(\mathcal{F}) = 1$ under a specified sequence of probability measures, e.g. $\mathcal{G}_{n,p}$.

bipartite graph

A graph whose vertices can be divided into disjoint sets A, B such that every edge of G has one end in A and one end in B. The pair (A, B) is called a **bipartition**.

A graph is bipartite if and only if it contains no odd cycles. Consequently, for any pair of vertices u and v in a bipartite graph, every (u, v)-trail has the same parity.

edge-connectivity

Written $\lambda(u, v)$, the maximum number of edge-disjoint trails between u and v.

According to Menger's Theorem, the edge-connectivity between u and v is equal to the minimum number of edges in a cut separating them.

Erdős–Pósa property

The existence of a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that every graph contains either k vertex-disjoint (or edge-disjoint) subgraphs in a family $\mathcal{G}$ or a set of at most $f(k)$ vertices (respectively, edges) intersecting all such subgraphs.

Erdős and Pósa originally proved this property for vertex-disjoint cycles [31].

Eulerian graph

A graph whose vertices all have even degree. A connected graph is Eulerian if and only if it has an **Euler tour**, i.e., a closed trail using all of its edges [33].

flow

A function $f : E(D) \to \mathbb{R}$ defined for a directed graph D, source and sink vertices s and t, and capacity function $c : E(D) \to \mathbb{R}$, satisfying the following two conditions.

Capacity for all $e \in E(D)$, $0 \leq f(e) \leq c(e)$.

Conservation for all $x \in V(D)$ other than s and t, the total flow $\sum f(e)$ on arcs e entering x is equal to the total flow leaving x.

The **value** of f is the total flow on arcs leaving s minus the total flow entering it.

The Max-Flow Min-Cut Theorem says that the value of the maximum (s,t)-flow is equal to the minimum net capacity of an (s,t)-cut. Finding a maximum flow is a fundamental problem of combinatorial optimization, first solved by Ford and Fulkerson [38] in the 1950s and now known to be computable in $O(nm)$-time [91].

immersion of H in G

A set of vertices $R \subseteq V(G)$ in one-to-one correspondence with $V(H)$, together with a collection $\mathcal{T}$ of edge-disjoint trails in G in correspondence with $E(H)$, where each trail of $\mathcal{T}$ connects the vertices of R that represent the ends of its corresponding edge.

G admits an immersion of H if and only if a graph isomorphic to H can be obtained from G by splitting off pairs of edges and deleting isolated vertices.

internally k-edge-connected graph

A $(k-1)$-edge-connected graph G such that, if $X \subseteq V(G)$ with $|\delta(X)| = k-1$, either $|X| = 1$ or $|V(G) \setminus X| = 1$.

laminar family of sets

A family of sets $\mathcal{F}$ in which every pair of sets $X, Y \in \mathcal{F}$ satisfies either $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$.

maximum bipartite subgraph

A bipartite subgraph of a graph G with at least as many edges as any other bipartite subgraph. A bipartite subgraph of G is maximum if and only if it contains at least half the edges of $\delta(X)$ for every $X \subseteq V(G)$.

Finding a maximum bipartite subgraph is NP-hard [60], although it is easy to find a bipartite subgraph of G with at least $|E(G)|/2$ edges.

odd edge-connectivity

The maximum number $\lambda_o(u,v)$ of edge-disjoint odd trails between vertices u and v.

odd trail covering number

The minimum size $\tau_o(u,v)$ of an edge-set intersecting every odd (u,v)-trail in a graph.

perimeter

For a vertex-set X and subgraph H of G, the quantity

$$p(X,H) = |E(X) \setminus E(H)| + \tfrac{1}{2}|\delta(X)|.$$

If H is bipartite and $u, v \in X$ are on the same side of the bipartition of H, then $p(X,H)$ is an upper bound for the number of edge-disjoint odd (u,v)-trails in G.

quasi-random graph sequence

A sequence of graphs $(G_n) = (G_1, G_2, \dots)$ such that $|V(G_n)| \in \Theta(n)$ and, for some constant $p \in (0,1)$, every vertex-set $U \subseteq V(G_n)$ has $|E(U)| = \frac{p}{2}|U|^2 + o(n^2)$.

There are many other equivalent definitions of quasi-random graphs; see [18, 74, 108]

random graph

A graph sampled from the probability space $\mathcal{G}_{n,p}$ of simple graphs on n vertices, in which every labelled graph with m edges has probability $p^m(1-p)^{\binom{n}{2}-m}$. The parameter $p \in (0,1)$ can be constant or a function of n which tends to zero as $n \to \infty$.

Graphs can be sampled from $\mathcal{G}_{n,p}$ by adding edges at random; each possible edge is included with probability p independently from the others. This model was introduced independently by Erdős and Rényi [32] and by Gilbert [46].

root vertices

The vertices in G corresponding to those of H in an immersion of H in G. Some authors instead use the terms “branch vertices”, “basic vertices”, or “corners”.

splitting off edges

Deleting two adjacent edges xy and yz and replace them with a new edge xz. This is also called “lifting” in the literature.

Splitting off edges may create or remove loops and parallel edges.

strong immersion

An immersion whose trails do not use the root vertices internally.

submodular function

A function of sets $f : 2^X \to \mathbb{R}$ satisfying the inequality

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

totally odd immersion

An immersion whose constituent trails all have odd length.

trail

A walk with no repeated edge, or the subgraph induced by the edges of such a walk. The **length** of a trail is the number of edges. An **odd trail** is one with odd length.

A (u, v)-trail is one between the vertices u and v. A $\{u, v\}$-trail is one whose ends lie in the set $\{u, v\}$; i.e., a (u, v)-trail, a closed (u, u)-trail, or a closed (v, v)-trail.

(u, v)**-cut**

An edge-set $\delta(X)$ consisting of all edges with one end in X and the other not in X, where $u \in X$ and $v \notin X$.

wall

A subcubic planar graph obtained by subdividing some edges of a finite hexagonal tiling. The cycles corresponding to the hexagonal tiles are called **bricks**.

To be precise, a $k \times k$ wall is any graph obtained by the following procedure: starting with the vertices (i, j) for $i \in \{0, \ldots, k\}$, $j \in \{0, 2k + 1\}$, add edges between (i, j) and (i', j') whenever $i = i'$ and whenever $j = j'$ and $i = i' + (-1)^{i+j}$; then delete all vertices of degree one and replace some of the edges with internally vertex-disjoint paths.

Chapter 1

Introduction

The study of path systems has long been a central area of graph theory. For the last century, researchers have studied flows, linkages, subdivisions, immersions, and other structures involving disjoint paths and connectivity. It is typically much harder to characterize the existence of these structures when constraints are added to the paths: for example, Menger's Theorem describes the optimal packing of disjoint paths between two vertices, but there is no similar duality theorem when we restrict the paths to have odd length.

This thesis studies edge-disjoint systems of odd trails. (Recall that a **trail** is a walk with no repeated edge; a **path** is a trail with no repeated vertex. An **odd** trail is one with an odd number of edges.) Although researchers traditionally state their results in terms of paths whenever possible, trails are often more natural for understanding their proofs. Trails are produced, for instance, when edge-disjoint paths are concatenated, when split off edges are restored, and when flows of unit value are discretized. The distinction between paths and trails becomes important under parity restrictions, as Figure 1.1 shows.

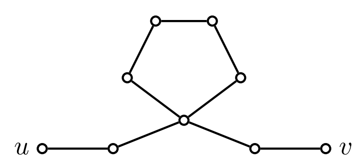

Figure 1.1: There is an odd trail between u and v, but no odd path.

The first half of this thesis shows that packings and coverings of odd trails are closely related; in this regard, odd trails are better behaved than odd paths. We establish an approximate packing–covering duality theorem for general graphs and an improved version for Eulerian graphs. The second half of the thesis studies totally odd graph immersions—collections of odd trails mimicking the structure of a given "template" graph—and finds sufficient conditions for their existence. Our work raises several interesting yet realistic open questions that hint at fruitful new areas to be explored.

This thesis uses the standard terminology of graph theory, for which we refer the reader to [9, 25, 116]. Unless otherwise specified, graphs are allowed to have loops and parallel edges.

1.1 Paths, parity, and packing–covering dualities

Path systems are ubiquitous in graph theory as tools and as important objects of study. Notable examples include disjoint paths [83, 86], network flows [2, 102], graph linkages [64, 107, 109], and others [3, 102]. It is natural to study the existence of these structures under parity constraints: characterizing, for instance, when a graph has vertex- or edge-disjoint odd cycles [62, 65], parity linkages [59, 68, 110], odd minors [44, 69, 72], or odd subdivisions [61]. Several far-reaching theorems have also been discovered for packing vertex-disjoint paths or cycles of non-unit weight in group-labelled graphs [15, 16, 55, 71].

In the late 1920s, Menger proved his now-famous duality theorem: the maximum number of internally disjoint paths between two vertices is equal to the minimum number of vertices separating them [86, 103]. Since then, "min–max" theorems have become a common staple in graph theory, from Kőnig's Theorem about matchings in bipartite graphs (see [9]) to Mader's S-Paths Theorem on paths with ends in a specified vertex-set [83]. The concept of duality applies more generally in the study of network flows—for instance, the Max-Flow Min-Cut Theorem [29, 38], Hu's 2-Commodity Flow Theorem [54, 99], and the Okamura–Seymour Theorem [89]—as well as in linear programming; see [4, 21, 102].

In the 1960s, Erdős and Pósa published a seminal paper relating the maximum number of vertex-disjoint cycles in a graph to the minimum size of a vertex-set intersecting all cycles [31]. Although not a perfect duality in the above sense, their approximate duality opened the door for many more deep and fascinating results. The so-called Erdős–Pósa property has been proved for various families of graphs [94], including long cycles [5, 35], odd cycles in well-connected graphs [93, 112], and families that consist of all graphs having a fixed planar graph as a minor [97]. The generalization of the S-Paths Theorem to group-labelled graphs by Chudnovsky et al. [16] implies the Erdős–Pósa property for many other objects, and a number of similarly flexible theorems have built on their work (e.g. [55, 71]).

Despite their long history, we are only beginning to understand how edge-disjoint packings differ from their vertex-disjoint counterparts. Although Menger's Theorem and the Erdős–Pósa Theorem both have edge versions (see [3, 25]), other packing-covering dualities are not so easily translated to the edge-disjoint setting. Some progress has been made on the edge-disjoint packing of long cycles [11], of odd cycles [62], and of graphs containing a fixed graph as an immersion [78]. It is common to find a closer relationship between edge-disjoint packings and coverings among Eulerian graphs, e.g. [37, 43, 63, 99].

1.2 Odd paths and their challenges

Collections of (u, v)-paths of odd length are natural and important examples of parity-constrained path systems. As with any type of graph structure, a key question to ask is whether the Erdős–Pósa property holds: is there a function f such that every graph has either k vertex-disjoint (or edge-disjoint) instances of the structure or a set of $f(k)$ vertices (or edges) intersecting all such instances? While odd paths satisfy the vertex-disjoint version of the Erdős–Pósa property [16], odd (u, v)-paths to not satisfy the edge-disjoint version. As is often the case (e.g. [95]), topological obstructions provide a reason why the Erdős–Pósa property fails.

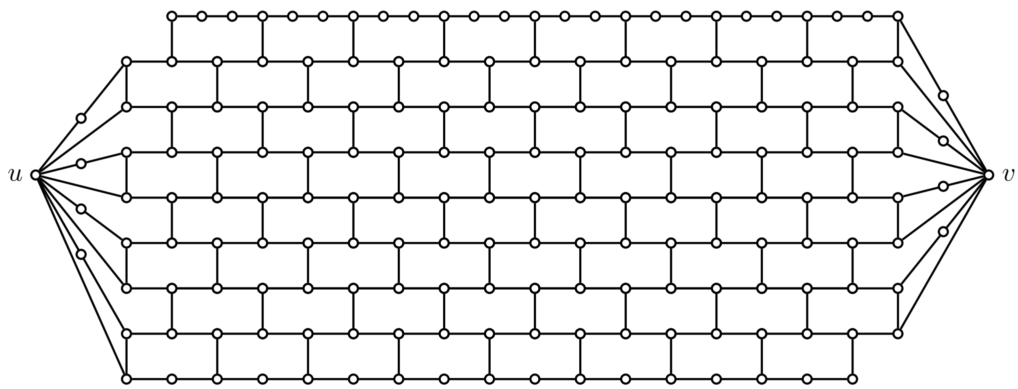

Figure 1.2: A graph W_8 constructed by connecting u and v to opposite sides of a large wall whose "odd bricks" are all in the top layer. This graph does not have two edge-disjoint odd (u, v)-paths, but covering all odd (u, v)-paths requires an edge-set whose size is proportional to the width of the wall.

Fact 1.1. *A graph without two edge-disjoint odd (u, v)-trail may still need arbitrarily many edges to cover all such trails.*

Proof. We describe a family of graphs W_k, of which a typical example is shown in Figure 1.2. Begin with a **wall** consisting of k rows of k cycles—called **bricks**—where only the bricks in the top row have odd length. Then, add two new vertices u and v joined by vertex-disjoint paths to the left and right sides of the wall, respectively. This can be done in such a way that the resulting graph W_k is planar, has no vertices of degree ≥ 4 other than u and v, and every odd (u, v)-path in W_k uses an edge from the top row of bricks.

Omitting some topological details of the proof, any odd (u, v)-path P defines a curve that divides $G - E(P)$ into two subgraphs: a bipartite subgraph below P and a disconnected subgraph above it. Neither subgraph contains an odd (u, v)-path; furthermore, no (u, v)-path of $G - E(P)$ passes through both, as this would require an internal vertex of P to have degree ≥ 4 in G. Consequently, W_k does not have two edge-disjoint odd (u, v)-paths regardless of the value of k.

On the other hand, the size of any edge-set covering all odd (u, v)-paths in W_k is proportional to k. For if F is a set of fewer than $k/4$ edges in W_k, it is easy to see that $W_k - F$ contains two consecutive rows of bricks in W_k as well as a "stack" of bricks extending from the bottom to the top of the wall. An odd (u, v)-path in $W_k - F$ can be routed from u through the first row of bricks, up the stack and around the odd brick at the top of the wall, down the other side of the stack, and along the second row to v. Hence F fails to cover all odd (u, v)-paths in W_k. □

Although edge-disjoint paths can share vertices, a path is not allowed to have any vertex intersections with itself. This inconsistency provides some explanation to why edge-disjoint odd paths are so difficult: because they combine two different concepts of disjointness, they cannot be easily concatenated or otherwise combined. A similar challenge was identified by Bruhn, Heinlein, and Joos [11] in their work on packing long cycles.

The fractional version of odd path packing has been studied with some success [104], but little else is known about this problem. In this thesis, we sidestep the difficulties inherent to odd paths by considering the relaxed problem of packing odd trails. Our main result in Chapter 2 establishes the Erdős–Pósa property for odd (u, v)-trails, suggesting that odd trails are better behaved than odd paths.

1.3 Graph immersions

Our original motivation for studying edge-disjoint (odd) trails comes from the world of graph immersions. An **immersion** of a graph H in another graph G consists of

- a set $R \subseteq V(G)$;
- a collection $\mathcal{T}$ of edge-disjoint trails in G; and
- bijections $\phi : V(H) \to R$ and $\phi' : E(H) \to \mathcal{T}$ where the trail $\phi'(e) \in \mathcal{T}$ corresponding to an edge $e = uv \in E(H)$ connects $\phi(u)$ with $\phi(v)$.

We refer to R as the **root vertices** of the immersion. An immersion is **strong** if its trails do not use the root vertices internally.

In the literature, the equivalent definition with "paths" in the place of "trails" is more common. However, the trail version is arguably more natural due to the following characterization of the existence of immersions. To **split off** a pair of adjacent edges $e_1 = xy$ and $e_2 = yz$ in a graph means to delete e_1 and e_2 and add a new edge $e_3 = xz$. Splitting off edges may introduce or remove parallel edges and loops. When the operation is reversed, paths in the split off graph correspond to trails in the original graph.

Fact 1.2. *A graph G admits an immersion of H if and only if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly splitting off pairs of edges and deleting isolated vertices.* □

Immersions have been popular objects of study since Nash-Williams [88] conjectured that in any infinite family of graphs, one admits an immersion of another. This property, referred to as the **well-quasi-ordering** of the immersion order, was eventually proved by Robertson and Seymour [98] using parts of the graph minors project [96]. As a result, every immersion-closed graph class has a recognition algorithm that runs in polynomial time [34], although the enormous constant factors involved may make such algorithms infeasible. A practical algorithm for finding immersions of the complete graph K_4 is implemented in [10].

On the structural side, many questions about immersions arise from analogies to graph subdivisions and minors. (A **subdivision** is an immersion whose edge-disjoint trails are internally vertex-disjoint paths; a graph is a **minor** of another if it can be obtained by edge deletions and contractions.) In the 1940s, Hajós [unpublished] and Hadwiger [51] respectively conjectured that every graph with chromatic number t has a subdivision or a minor of the complete graph on t vertices. Hajós' conjecture turns out to be false [12, 30, 111], but Hadwiger's conjecture lives on as one of the most notorious open questions of graph theory.

Immersions are less intimately related to surface embeddings than subdivisions and minors are: any complete graph can be immersed in a planar graph, for instance, and one can compare Kuratowski's Theorem (see [9]) with the characterization of $\{K_5, K_{3,3}\}$-immersion-free graphs given in [45]. Nevertheless, other similarities between immersions, minors, and subdivisions makes the following conjecture irresistible.

Conjecture 1.3 (Lescure and Meyniel [77], Abu-Khzam and Langston [1])**.** *Every graph with chromatic number t admits a (strong)*[1] *immersion of K_t.*

Progress on this conjecture has been surprisingly rapid; in fact, it seems that the existence of clique immersions may have more to do with the minimum degree than with the chromatic number. Lescure and Meyniel [77] showed that every simple graph with minimum degree 4 or 5 admits a strong immersion of K_5 or K_6, respectively, and DeVos et al. [23] proved that minimum degree 6 guarantees a (not necessarily strong) immersion of K_7. Although this pattern does not continue—when $t \geq 8$, there are examples with minimum degree $t-1$ and no K_t-immersion [20, 22]—every simple graph admits a clique immersion whose order is within a constant factor of the minimum degree of the graph [22]. Building on the proof in [22], Dvořák and Yepremyan [27] showed that a minimum degree of $11t + 7$ is enough to force an immersion of K_t, while Le and Wollan [76] recently announced an improved bound of $7t + 7$. It is an open question to see how much further the minimum degree bound can be decreased.

Question 1.4. *Does every graph with minimum degree $\geq t$ admit an immersion of K_t?*

[1] Lescure and Meyniel [77] originally published the conjecture in 1988, using the term "immersion" to mean strong immersion. Abu-Khzam and Langston [1] amended it in 2003—dropping the requirement that the immersion be strong—in light of results on the "weak" immersion order.

It is worth noting that an affirmative answer to Question 1.4 would imply Conjecture 1.3 [1].

Motivated by recent directions in the graph minors literature, Vergara [114, 115] studied clique immersions in certain classes of dense graphs and showed that every n-vertex graph with independence number two has an immersion of $K_{n/3}$. Gauthier and Wollan [42] report that every such graph actually admits an immersion of $K_{2n/5}$; this result is notable in light of the continuing work on the Duchet–Meyniel Theorem for graph minors [17, 26, 66], where it appears difficult to substantially improve on a $K_{n/3}$-minor without the help of additional hypotheses.

An obvious necessary condition for the existence of a K_t-immersion is the presence of a set of t vertices R such that $\lambda(S, R \setminus S) \geq s(t-s)$ for every s-vertex subset $S \subseteq R$. It is natural to wonder whether a condition of this form might also be sufficient. The rough structure theorem for graphs without K_t-immersions, discovered independently in [24] and [117], points in this direction.

Our contributions can be viewed in the context of **totally odd immersions**—that is, immersions whose trails each have odd length. The first half of this thesis studies collections of edge-disjoint odd (u, v)-trails, which are equivalent to a totally odd immersions of a two-vertex graph consisting of multiple parallel edges. In the second half, we investigate totally odd immersions of complete graphs. We first extend the characterizations of [24] and [117] and describe the rough structure of graphs with no totally odd immersion of a large complete graph (see Section 4.3.2). Then, in Chapter 5, we suggest a totally odd immersion version of Hadwiger's conjecture and show it is true for almost all graphs.

Chapter 2

Packing edge-disjoint odd (u, v)-trails

The main topic of this thesis is the edge-disjoint packing of odd trails in a graph. We define the **odd edge-connectivity** between two vertices u and v, written $\lambda_o(u, v)$, to be the maximum number of edge-disjoint (u, v)-trails of odd length. Clearly, $\lambda_o(u, v)$ is bounded by the **odd trail covering number** $\tau_o(u, v)$, which is the minimum size of an edge-set intersecting every odd (u, v)-trail in the graph. In this first chapter, we prove an approximate duality between $\lambda_o(u, v)$ and $\tau_o(u, v)$—a result which is not possible if "trails" are replaced with "paths".

Theorem 2.1. *If G is a graph, u and v are vertices, and $k \in \mathbb{N}$, then G has either k edge-disjoint odd (u, v)-trails or a set of at most $6k - 2$ edges intersecting all such trails. In other words,*

$$\lambda_o(u, v) \leq \tau_o(u, v) \leq 6\lambda_o(u, v) + 4.$$

In fact, this statement is more coarse than necessary. To explain what we mean by this, we need a new definition.

Definition. *Let G be a graph, X a vertex-set, and H a subgraph of G. The* ***perimeter*** *of X with respect to H is*

$$p(X, H) = |E(X) \setminus E(H)| + \tfrac{1}{2}|\delta(X)|,$$

where $E(X)$ denotes the set of edges with both ends in X, and $\delta(X)$ is the set of edges of G with one end in X and the other end not in X.

Let G, H, and X be as above. If $u, v \in X$, then every (u, v)-trail that leaves H must "cross the perimeter" using either two edges of $\delta(X)$ or one edge of $E(X) \setminus E(H)$. Therefore, $p(X, Y)$ is an upper bound on the maximum number of such edge-disjoint trails. Our main theorem gives a lower bound for the number of edge-disjoint (u, v)-trails each using exactly one edge outside H, expressed in terms of the usual edge-connectivity $\lambda(u,v)$ and a minimum perimeter.

Theorem 2.2. *Let G be a graph, $u,v \in V(G)$, and H a spanning subgraph of G such that $H+uv$ is 2-edge-connected. Then for all $k \in \mathbb{N}$, at least one of the following holds:*

(i) *G has k edge-disjoint (u,v)-trails each using exactly one edge of $G - E(H)$;*

(ii) *H has a (u,v)-cut with at most $3k-1$ edges; or*

(iii) *G has a vertex-set R containing u and v with perimeter*

$$p(R,H) \leq \left(1 + \max_{T \subseteq V(G)\setminus R} \frac{|\delta_G(T)|}{|\delta_H(T)|}\right)(k-1). \tag{2.1}$$

In inequality (2.1), T could be the empty set; if so, we define $\frac{|\delta_G(T)|}{|\delta_H(T)|} = 1$. By $H+uv$ we mean the graph obtained from H by adding a new edge uv. We require H to be a spanning subgraph to avoid the outcome when $\delta_H(T) = \emptyset$, in which case G would be disconnected or inequality (2.1) would be trivial.

When H is bipartite, Theorem 2.2 has significant implications for the odd edge-connectivity between u and v. Specifically, $p(X,H)$ is an upper bound for $\lambda_o(u,v)$ whenever $u,v \in X$ are on the same side of the bipartition of H, as any odd (u,v)-trail must leave H. If H is a maximal[1] bipartite subgraph, any (u,v)-trail that uses exactly one edge of $E(G) \setminus E(H)$ has the same parity. We can therefore use Theorem 2.2 to approximate the odd edge-connectivity $\lambda_o(u,v)$.

Theorem 2.3. *Let G be a graph, $u,v \in V(G)$, and H a maximal bipartite subgraph of G such that $H+uv$ is 2-edge-connected. Then for all $k \in \mathbb{N}$, at least one of the following holds:*

(i) *G has k edge-disjoint odd (u,v)-trails;*

(ii) *H has a (u,v)-cut with at most $3k-1$ edges; or*

(iii) *u and v are on the same side of the bipartition of H, and G has a vertex-set R containing u and v with perimeter*

$$p(R,H) \leq \left(1 + \max_{T \subseteq V(G)\setminus R} \frac{|\delta_G(T)|}{|\delta_H(T)|}\right)(k-1).$$

If we demand that H is **maximum**—meaning that it has the most edges of any bipartite subgraph of G—we can replace the edge-connectivity condition on H and the bounds in (ii) and (iii) with more natural conditions.

[1] A bipartite subgraph H of G is **maximal** if adding any edge of $E(G) \setminus E(H)$ gives rise to an odd cycle in H. Provided G has no isolated vertices, a maximal bipartite subgraph of G is necessarily spanning.

Theorem 2.4. *Let G be a graph, $u, v \in V(G)$, and H a maximum bipartite subgraph of G. If $G + uv$ is 2-edge-connected, then for all $k \in \mathbb{N}$, at least one of the following holds:*

(i) *G has k edge-disjoint odd (u, v)-trails;*

(ii) *G has a (u, v)-cut with at most $6k - 2$ edges; or*

(iii) *u and v are on the same side of the bipartition of H, and G has a vertex-set R containing u and v with perimeter $p(R, H) \leq 3(k - 1)$.*

Our main result, Theorem 2.2, is proved in Section 2.1 and applied in Section 2.2, yielding Theorems 2.1, 2.3, and 2.4. All of our arguments can be implemented as polynomial-time algorithms (notwithstanding our statement of Theorem 2.4 in terms of a maximum bipartite subgraph, which is NP-hard to find). We discuss these algorithms in Section 2.3. Section 2.4 concludes the chapter with a family of examples that show Theorem 2.1 cannot be improved to a perfect packing–covering duality for odd (u, v)-trails.

2.1 Proof of Theorem 2.2

The goal of this section is to establish Theorem 2.2, which relates the maximum number of edge-disjoint (u, v)-trails, each using exactly one edge from outside a given subgraph H, to the edge-connectivity $\lambda_H(u, v)$ and the minimum perimeter $p(X, H)$ among vertex-sets X containing u and v.

The proof comes in four parts. First, in Section 2.1.1, we prove a secondary lemma describing the structure of small edge-cuts in a graph. Then, in Section 2.1.2, we prove Theorem 2.2 for the special case where $u = v$. Section 2.1.3 presents another lemma which allows us to set aside many (u, v)-paths in H without "using up" too much edge-connectivity. Finally, we combine the lemma of Section 2.1.3 with the special case of Section 2.1.2 to prove the general case of Theorem 2.2.

2.1.1 On the structure of small edge-cuts in a graph

Call a vertex-set X (t, r)**-separating** if $t \in X$ and $r \notin X$. It is **tight** if $|\delta(X)|$ is equal to the edge-connectivity $\lambda(t, r)$, and **nearly tight** if $|\delta(X)| \leq \lambda(t, r) + 1$. When we say a vertex-set is **maximal**, we refer to the inclusion order on subsets of $V(G)$. Figure 2.1 illustrates the structure of nearly-tight sets, which we formally describe in Lemma 2.5.

Lemma 2.5. *Let r, t be distinct vertices of a graph G.*

(a) *There is a unique maximal tight (t, r)-separating set T.*

(b) *If S is a maximal nearly-tight (t, r)-separating set, then $T \subseteq S$.*

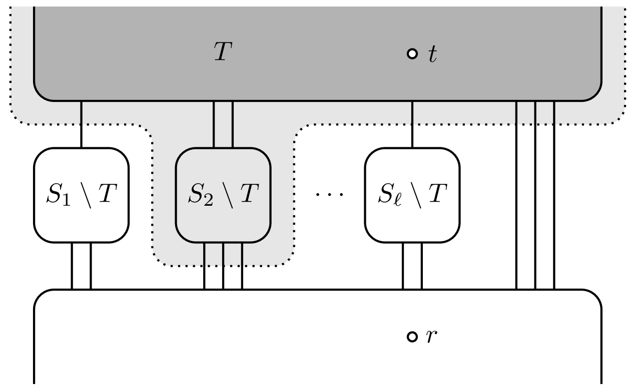

Figure 2.1: Lemma 2.5 describes the structure of (t,r)-cuts of size $\leq \lambda(t,r)+1$. There is a unique maximal tight (t,r)-separating set T, which is the intersection of any two maximal nearly-tight sets S_i, S_j.

(c) *If S_1, S_2 are distinct maximal nearly-tight (t,r)-separating sets, then $S_1 \setminus T$ and $S_2 \setminus T$ are vertex-disjoint.*

(d) *If $S_1, \ldots, S_\ell$ are maximal nearly-tight (t,r)-separating sets, then $|\delta(\bigcup S_i)| \leq \lambda(t,r)+\ell$.*

(e) *If G is 2-edge-connected, then there are at most $\lambda(t,r)$ distinct maximal nearly-tight (t,r)-separating sets.*

Proof. All five statements are consequences of the well-known submodular inequality for edge-cuts (see [9, 39, 79]):

$$|\delta(A \cup B)| + |\delta(A \cap B)| \leq |\delta(A)| + |\delta(B)|. \tag{2.2}$$

If A and B are (t,r)-separating sets, then all four terms in the inequality are greater than or equal to $\lambda(t,r)$.

(a) If A and B are tight (t,r)-separating sets, inequality (2.2) yields $|\delta(A \cup B)| = \lambda(t,r)$, implying $A \cup B$ is also tight. At least one tight set exists by Menger's Theorem; the unique maximal choice for T is the union of all such sets.

(b) Inequality (2.2) for S and T gives $|\delta(S \cup T)| \leq \lambda(t,r)+1$. The maximality of S means it cannot be a proper subset of the nearly-tight set $S \cup T$; hence $T \subseteq S$.

(c) When applied to S_1 and S_2, inequality (2.2) gives us $|\delta(S_1 \cup S_2)| \leq \lambda(t,r)+2$, and the maximality of S_1 rules out the possibility that $|\delta(S_1 \cup S_2)| \leq \lambda(t,r)+1$. So it must be the case that $|\delta(S_1 \cup S_2)| = \lambda(t,r)+2$ and $|\delta(S_1 \cap S_2)| = \lambda(t,r)$. By part (a), the latter implies that $S_1 \cap S_2 \subseteq T$, but we know from part (b) that $T \subseteq S_1, S_2$. So in fact $S_1 \cap S_2 = T$.

(d) Let S be a (t,r)-separating set. Using inequality (2.2), we find $\delta(S \cup S_i) \le \delta(S) + 1$ for every $i \in \{1,\dots,\ell\}$. By induction, it follows that $|\delta(S_1 \cup \cdots \cup S_\ell)| \le \lambda(t,r) + \ell$.

(e) Let $S_1,\dots,S_\ell$ be distinct maximal nearly-tight (t,r)-separating sets. By counting edges, we find, for every $i \in \{1,\dots,\ell\}$, that

$$|\delta(S_i \setminus T)| + |\delta(S_i \cap T)| - |\delta(S_i)| = 2|\delta(S_i \setminus T) \cap \delta(T)|.$$

Assuming G is 2-edge-connected, $|\delta(S_i \setminus T)| \ge 2$. Since $|\delta(S_i)| \le \lambda(t,r) + 1$ and $|\delta(S_i \cap T)| = |\delta(T)| = \lambda(t,r)$, the left-hand side is at least one. In particular, the right-hand side is positive, so at least one of the $\lambda(t,r)$ edges in $\delta(T)$ is used by $\delta(S_i \setminus T)$. By part (c), the vertex-sets $S_i \setminus T$ and $S_j \setminus T$ are disjoint when $i \ne j$, so the edges of $\delta(T)$ used by $\delta(S_i \setminus T)$ are different from the edges used by $\delta(S_j \setminus T)$. It follows that there are at most $|\delta(T)| = \lambda(t,r)$ distinct S_i. □

2.1.2 A special case of Theorem 2.2

In this section, we prove Theorem 2.2 in the special case where u and v are equal. Let G be a graph, H a subgraph of G, and $k \in \mathbb{N}$, and let $r \in V(G)$ be the specified vertex $r = u = v$. In this restricted case, the condition that $H + uv$ is 2-edge-connected just means that H is 2-edge-connected.

The following construction lets us answer questions about the desired trails in G by computing edge-connectivities in an auxiliary graph. The **complement** of H in G is denoted $\overline{H} = G - E(H)$.

Definition. *For any $F \subseteq E(\overline{H})$, let $H \oplus F$ be the graph obtained from H by adding a new vertex t^+ and, for each $e = xy \in F$, adding the new edges xt^+ and yt^+. (This may produce parallel edges incident with t^+.)*

The crucial property of $H \oplus F$ is that $\lambda_{H\oplus F}(t^+, r) = \deg_{H\oplus F}(t^+) = 2|F|$ if and only if G has a collection of edge-disjoint closed trails through r, each using exactly one edge of F and no other edge of $\overline{H}$. Let us call an edge-set $F \subseteq E(\overline{H})$ **inhibited** if $\lambda_{H\oplus F}(t^+, r) < 2|F|$ and **uninhibited** if $\lambda_{H\oplus F}(t^+, r) = 2|F|$.

Claim 2.1.1. *Let $F \subseteq E(\overline{H})$ be an uninhibited set. Let T^+ be the unique maximal tight (t^+, r)-separating set in $H \oplus F$ and let $S_1^+, \dots, S_\ell^+$ be the maximal nearly-tight (t^+, r)-separating sets. If $e \in E(\overline{H}) \setminus F$ and $F_e = F \cup \{e\}$ is inhibited, then e either has at least one end in T^+ or has both ends in some $S_i^+ \setminus T^+$.*

Proof of the claim. By definition, F_e is inhibited by a (t^+, r)-separating set S^+ in $H \oplus F_e$ with $|\delta_{H\oplus F_e}(S^+)| \le 2k - 1$. Consider the quantity

$$s = \left|\delta_{H\oplus F_e}\left(S^+\right)\right| - \left|\delta_{H\oplus F}\left(S^+\right)\right|.$$

The number s counts how many ends of e lie outside S^+; to see this, observe that the only differences between $H \oplus F_e$ and $H \oplus F$ are the two edges between t^+ and the ends of e. The fact that F is uninhibited means $|\delta_{H\oplus F}(S^+)| \geq 2(k-1)$ and hence $s \leq 1$. Therefore, S^+ contains one or both ends of e.

Since $|\delta_{H\oplus F}(S^+)| \leq |\delta_{H\oplus F_e}(S^+)| \leq 2k-1$, we know S^+ is a nearly-tight (t^+, r)-separating set in $H \oplus F$ and hence is contained in some S_i^+. The claim follows immediately if S^+ contains both ends of e. On the other hand, if e has one end in S^+ and the other outside S^+, then $|\delta_{H\oplus F}(S^+)| = |\delta_{H\oplus F_e}(S^+)| - 1 = 2(k-1)$; that is, S^+ is tight in $H \oplus F$. By Lemma 2.5 (a), $S^+ \subseteq T^+$, so one end of e is in T^+. □

Suppose now that $F \subseteq E(\overline{H})$ is a maximal uninhibited edge-set. Then Claim 2.1.1 applies to every $e \in E(\overline{H}) \setminus F$. Let T^+, $S_1^+, \ldots, S_\ell^+$ be the vertex-sets in the above claim, and let $T = T^+ \setminus \{t^+\}$ and $S_i = S^+ \setminus \{t^+\}$ be their restriction to the vertices of G ($i = 1, \ldots, \ell$). We set out to show that the vertex-set $R = V(G) \setminus \bigcup S_i$ satisfies the third outcome of Theorem 2.2. Claim 2.1.1 tells us that no edge of $E(\overline{H}) \setminus F$ has both ends in R, and that the only edges of $E(\overline{H}) \setminus F$ in $\delta_G(R)$ are also in $\delta_G(T)$—in other words, that every edge of $\delta_G(R)$ is in exactly one of F, H, or $\delta_G(T) \cap (E(\overline{H}) \setminus F)$. Hence

$$\begin{aligned} p(R, H) &= |E(R) \cap F| + \tfrac{1}{2}|\delta_G(R)| \\ &= |E(R) \cap F| + \tfrac{1}{2}|\delta_G(R) \cap F| + \tfrac{1}{2}|\delta_G(R) \cap E(H)| \\ &\qquad + \tfrac{1}{2}|\delta_G(R) \cap \delta_G(T) \cap (E(\overline{H}) \setminus F)| \end{aligned} \tag{2.3}$$

By the construction of $H \oplus F$, the first three terms in the right-hand side of Equation (2.3) sum to $\frac{1}{2}|\delta_{H\oplus F}(R)| = \frac{1}{2}|\delta_{H\oplus F}(\bigcup S_i)|$. Observe that

$$\left|\delta_{H\oplus F}\left(\bigcup_{i=1}^{\ell} S_i\right)\right| \leq \lambda_{H\oplus F}\left(t^+, r\right) + \ell \leq 4(k-1),$$

where the first inequality holds by Lemma 2.5 (d) and the second by Lemma 2.5 (e) using our assumption that H, and therefore also $H \oplus F$, is 2-edge-connected. Moreover, since T is tight, the remaining term in Equation (2.3) satisfies

$$\begin{aligned} |\delta_G(R) \cap \delta_G(T) \cap (E(\overline{H}) \setminus F)| &\leq |\delta_G(T) \cap E(\overline{H})| \\ &= |\delta_G(T)| - |\delta_H(T)| \\ &= (|\delta_G(T)| - |\delta_H(T)|) \cdot \frac{2(k-1)}{|\delta_{H\oplus F}(T)|} \\ &\leq (|\delta_G(T)| - |\delta_H(T)|) \cdot \frac{2(k-1)}{|\delta_H(T)|} \\ &= \left(\frac{|\delta_G(T)|}{|\delta_H(T)|} - 1\right) \cdot 2(k-1). \end{aligned}$$

Note that the above edge-set is empty when $T = \emptyset$ (i.e. when $T^+ = \{t^+\}$), which is why we can define $\frac{|\delta_G(\emptyset)|}{|\delta_H(\emptyset)|} = 1$ in the statement of our theorem. Combining the above calculations, we have

$$\begin{aligned} p(R,H) &\leq |E(R) \cap F| + \tfrac{1}{2}|\delta_G(R) \cap F + \tfrac{1}{2}|\delta_G(R) \cap E(H) + \tfrac{1}{2}|\delta_G(R) \cap \delta_G(T) \cap (E(\overline{H}) \setminus F) \\ &= \tfrac{1}{2}|\delta_{H \oplus F}(R) + \tfrac{1}{2}|\delta_G(R) \cap \delta_G(T) \cap (E(\overline{H}) \setminus F) \\ &\leq 2(k-1) + \left(\frac{|\delta_G(T)|}{|\delta_H(T)|} - 1 \right)(k-1) \\ &= \left(1 + \frac{|\delta_G(T)|}{|\delta_H(T)|} \right)(k-1). \end{aligned}$$

This completes the proof of Theorem 2.2 in the special case where $u = v = r$. □

2.1.3 Paths whose deletion reduces $\lambda(u,v)$ by at most one

The following lemma is a cousin to a result of Mader [82], who proved that every connected graph has, between any given pair of vertices u, v, a path whose deletion reduces $\lambda(u,v)$ by one and the local edge-connectivity for every other pair by at most two.

Lemma 2.6. *If t, u, v are vertices in a connected graph, there is a path between t and one of u and v whose deletion reduces $\lambda(u,v)$ and $\lambda(t,\{u,v\})$ each by at most one.*

Proof. The proof is by induction on the number of edges of G. Suppose G has an edge e that is not incident with t, u, or v. There are three possibilities. First, if e is part of neither a tight (u,v)-cut nor a tight $(t,\{u,v\})$-cut, we can simply delete e and apply induction.

Next suppose e is in a tight (u,v)-cut $\delta(S)$ where (say) $u \in S$ and $v, t \notin S$. By Menger's Theorem, there are $\lambda(u,v)$ edge-disjoint paths each passing through u and an edge of $\delta(S)$. Moreover, the length of the path from u to v through e is at least two, since e is not incident with u. If xw and $wy = e$ are the last two consecutive edges on this path, deleting xw and wy and adding a new edge xy preserves the edge-connectivities $\lambda(u,v)$ and $\lambda(t,\{u,v\})$. To see the latter fact, observe that any maximal collection of $(t,\{u,v\})$-paths can be replaced with one which uses the segments of the (u,v)-paths once they cross the cut $\delta(S)$. As the resulting graph G' has fewer edges than G, by induction it contains a path P' from t to $\{u,v\}$ satisfying $\lambda_{G'-E(P')}(u,v) \geq \lambda_{G'}(u,v) - 1 = \lambda_G(u,v) - 1$ and similarly for $\lambda(t',\{u,v\})$. The conclusion of the lemma is therefore fulfilled by the path in G obtained from P' by replacing xy with xwy (if necessary) and removing any resulting cycles.

The third case, where e is in a tight $(t,\{u,v\})$-cut, is similar to the second. We may henceforth assume that every edge is incident with t, u, or v.

Consider a largest collection of edge-disjoint (u,v)-paths in G. If there is a (u,v)-path in the collection through t, let P be a (u,t)- or (t,v)-subpath. If not, let P be any path of the form ut, tv, uxt, or tyv. In either case, the edges of P intersect at most one path in the collection, so deleting P decreases $\lambda(u,v)$ by at most one. It is also easy to see that deleting P decreases $\lambda(t,\{u,v\})$ by exactly one. This completes the proof of Lemma 2.6. □

2.1.4 The general case of Theorem 2.2

In this section, we combine the results of Sections 2.1.2 and 2.1.3 to complete the proof of Theorem 2.2. Let G be a graph, u, v vertices, H a maximal bipartite subgraph, and $k \in \mathbb{N}$. Suppose $H + uv$ is 2-edge-connected.

Let G' and H' be obtained from G and H respectively by identifying u and v to a single vertex, which we denote by r. (By doing this, each edge joining u and v becomes a loop at r.) The new graph H' is 2-edge-connected because $H + uv$ is. Apply the special case of Theorem 2.2, proved in Section 2.1.2, to G' and H' with the specified vertex $u = v = r$. If we fail to find k edge-disjoint closed (r,r)-trails in G' with the desired properties, we get a vertex-set R' such that $r \in R'$ and

$$p_{G'}(R', H') \leq \left(1 + \max_{T \subseteq V(G') \setminus R'} \frac{|\delta_{G'}(T)|}{|\delta_{H'}(T)|}\right)(k-1).$$

Let $R = (R' \setminus \{r\}) \cup \{u,v\}$. For all vertex-sets $T \subseteq V(G') \setminus R' = V(G) \setminus R$, we have $|\delta_{G'}(T)| = |\delta_G(T)|$ and $|\delta_{H'}(T)| = |\delta_H(T)|$. Furthermore, $p_G(R,H) = p_{G'}(R',H')$. Therefore, the third outcome of Theorem 2.2 holds for G, H, u, v.

On the other hand, suppose we succeed in finding k edge-disjoint closed trails in G' through r, each using exactly one edge outside of H'. Construct another auxiliary graph H^+ from H by adding a new vertex t^+ and edges xt^+, yt^+ for every edge $xy \in E(G) \setminus E(H)$ used by a trail in such a collection. The disjoint (r,r)-trails in G' correspond in H^+ to trails passing through t^+ each having both ends in the set $\{u,v\}$, so $\lambda_{H^+}(t^+, \{u,v\}) = 2k$. We may also assume that $\lambda_{H^+}(u,v) \geq \lambda_H(u,v) \geq 3k$; otherwise, if H has a (u,v)-cut with fewer than $3k$ edges, the second outcome of Theorem 2.2 is satisfied.

Lemma 2.6, repeatedly applied to H^+, t^+, u, v, gives us $2k$ edge-disjoint paths between t^+ and the set $\{u,v\}$, after whose deletion u and v remain k-edge-connected. For each edge x_iy_i outside H' used in obtaining H^+ as above ($i = 1,\dots,k$), consider the trails X_i, Y_i using the edges x_it^+, y_it^+ in H^+, respectively. Let $Z_1,\dots,Z_k$ be edge-disjoint (u,v)-paths in the remaining graph. Construct k edge-disjoint (u,v)-trails as follows: for each i, follow X_i to x_i, take the edge x_iy_i, return to $\{u,v\}$ along the route of Y_i, and append Z_i if necessary to connect u and v. These trails each contain exactly one edge of $E(G) \setminus E(H)$, establishing the first outcome of the theorem. □

2.2 Packing and covering odd trails

When H is bipartite, Theorem 2.2 can be used to obtain lower bounds on the odd edge-connectivity between u and v, where the precision of the bound depends on how well edge-cuts in H approximate edge-cuts in G. In this section, we prove two results of this form—Theorem 2.3 and Theorem 2.4—as well as the approximate packing–covering duality for odd (u,v)-trails stated in Theorem 2.1.

2.2.1 Proof of Theorem 2.3

Let G be a graph, $u,v \in V(G)$, and H a maximal bipartite subgraph of G such that $H+uv$ is 2-edge-connected. Let $k \in \mathbb{N}$.

If u and v are on opposite sides of the bipartition of H, then any (u,v)-path in H has odd length. By Menger's Theorem, either G has k edge-disjoint odd (u,v)-paths, or H has a (u,v)-cut with fewer than k edges. Either result satisfies Theorem 2.3. Suppose, then, that u and v are on the same side of the bipartition of H. Because H is maximal, every (u,v)-trail using exactly one edge outside of H has odd length. The statement then follows immediately from Theorem 2.2. □

2.2.2 Proof of Theorem 2.4

Let G be a graph, $u,v \in V(G)$, H a maximum bipartite subgraph of G, and $k \in \mathbb{N}$. Suppose $G+uv$ is 2-edge-connected. The key element of the following proof is that, as a maximum bipartite subgraph, H contains at least half of the edges of any cut in G.

As in the proof of Theorem 2.3, if u and v are on opposite sides of the bipartition of H, one of the first two desired outcomes falls out of Menger's Theorem. That is, if H does not have k edge-disjoint odd (u,v)-paths, then the minimum (u,v)-cut in H corresponds in G to a cut with at most $2(k-1)$ edges. We may assume, then, that u and v are on the same side of the bipartition of H. This time, we cannot appeal directly to Theorem 2.2 as $H+uv$ is not necessarily 2-edge-connected. However, we can still reduce to Theorem 2.2 using the following argument.

Let X be a vertex-set such that $u,v \notin X$, $|\delta_H(X)| = 1$, and the induced subgraph $H[X]$ is connected. Because $G+uv$ is 2-edge-connected, $|\delta_G(X)| \geq 2$; since H is maximum, in fact $|\delta_G(X)| = 2$. Let $\delta_G(X) = \{e, \overline{e}\}$, where $e = xy \in E(H)$; $\overline{e} = \overline{xy} \notin E(H)$; $x, \overline{x} \in X$; and $y, \overline{y} \notin X$. Let G' be obtained from G by deleting the vertices of X and adding a new edge $e' = y\overline{y}$. Let $H' = H - X$.

Suppose G' and H' satisfy one of the conclusions (if not the hypotheses) of Theorem 2.2.

(i) If G' has k edge-disjoint (u,v)-trails, each using exactly one edge outside H', then at most one trail in such a collection uses e'. Since $H[X]$ is connected, this trail can be extended to a trail in G using exactly one edge outside H (namely $\overline{e}$). Therefore, G has k edge-disjoint (u,v)-trails, each using exactly one edge outside H.

(ii) If H' has a (u, v)-cut $\delta(T')$ with at most $3k - 1$ edges, let

$$T = \begin{cases} T' \cup X & y \in T' \\ T' & y \notin T'. \end{cases}$$

Then $\delta(T)$ is a (u, v)-cut in H with at most $|\delta_H(T)| = |\delta_{H'}(T')| \leq 3k - 1$ edges.

(iii) If G' has disjoint vertex-sets R' and T' such that R' contains u and v and has perimeter

$$p_{G'}(R', H') \leq \left(1 + \frac{|\delta_{G'}(T')|}{|\delta_{H'}(T')|}\right)(k - 1),$$

let $R = R'$ and let T be defined as in the previous case. Observe that $|\delta_{G'}(T')| = |\delta_G(T)|$, $|\delta_{H'}(T')| = |\delta_H(T)|$, and $p_{G'}(R', H') = p_G(R, H)$, so R fulfills the third conclusion of Theorem 2.2 for G and H.

According to the above discussion, if G' and H' satisfy the conclusions of Theorem 2.2, then so do G and H. Repeating the same argument, each time eliminating a vertex-set X for which $u, v \notin X$ and $|\delta_H(X)| = 1$, we eventually arrive at graphs G' and H' for which $G' + uv$ and $H' + uv$ are 2-edge-connected. Since Theorem 2.2 applies to G' and H', we can infer that the conclusion of Theorem 2.2 also holds for G and H.

In the first outcome of Theorem 2.2, G has k edge-disjoint odd (u, v)-trails as we saw in the previous section. In the second outcome, H has a (u, v)-cut of fewer than $3k$ edges, which corresponds to a (u, v)-cut of at most $6k - 2$ edges in G. Finally, in the third outcome, G has a vertex-set R containing u and v with perimeter

$$p(R, H) \leq \left(1 + \max_{T \subseteq V(G) \setminus R} \frac{|\delta_G(T)|}{|\delta_H(T)|}\right)(k - 1) \leq 3(k - 1),$$

using the fact that the maximum bipartite subgraph H satisfies $\frac{|\delta_G(T)|}{|\delta_H(T)|} \leq 2$ for all $T \subseteq V(G)$. This completes the proof of Theorem 2.4. □

2.2.3 Proof of Theorem 2.1

Let G be a graph and $u, v \in V(G)$. We may delete any components of G which do not contain u and v, as these components are irrelevant to the problem. We may also delete any cut-edge which does not separate u and v: no (u, v)-trail in G uses such an edge, and neither does a minimum cover of the odd (u, v)-trails in G.

We may assume G is a connected graph in which every cut-edge separates u and v; in other words, $G + uv$ is 2-edge-connected. Let H be a maximum bipartite subgraph of G. Apply Theorem 2.4 with $k = \lambda_o(u, v) + 1$. As the first outcome of the theorem cannot hold, either G has a (u, v)-cut with fewer than $6(\lambda_o(u, v) + 1) - 2$ edges or G has a vertex-set R

containing u and v for which $p(R,H) \leq 3\lambda_o(u,v)$. In the latter case, the perimeter edges $(E(R)\backslash E(H))\cup\delta(R)$ intersect all odd (u,v)-trails and consist of at most $2p(R,H) \leq 6\lambda_o(u,v)$ edges. Either way, we obtain an odd trail cover that certifies $\tau_o(u,v) \leq 6\lambda_o(u,v) + 4$. □

2.3 Algorithms

The arguments used in this chapter can be implemented as constructive algorithms, which, with a few tweaks, can be made to run in polynomial time. In this section, we describe the algorithms related to each of our main results. All runtimes are expressed in terms of $n = |V(G)|$ and $m = |E(G)|$.

As our proofs make liberal use of edge-connectivity assumptions and tight cuts, it comes as no surprise that the corresponding algorithms rely heavily on subroutines for the maximum flow problem. The most efficient solution we are aware of uses an algorithm by Orlin for sparse graphs and an algorithm by King, Rao, and Tarjan for sufficiently dense graphs. (There are simpler, equally efficient algorithms for finding the maximum number of edge-disjoint (u,v)-paths and a minimum unweighted (u,v)-cut—see [102]—but their suitability depends on the encoding of parallel edges in the input graph.)

Theorem 2.7 (Orlin [91])**.** *A maximum collection of edge-disjoint paths between two vertices in a graph can be found in $O(nm)$-time.* □

If a maximum flow has been found, a minimum cut can be easily obtained through a breadth- or depth-first search of the residual graph.

Corollary 2.8. *A (vertex-minimal) tight set $T \subseteq V(G)$ separating two given vertices can be found in $O(nm)$ time.* □

2.3.1 Algorithm for Theorem 2.2

Theorem 2.9. *There is an $O(nm^2)$-time algorithm which takes as input a graph G, vertices $u, v \in V(G)$, and a spanning subgraph H of G such that $H + uv$ is 2-edge-connected, and outputs an integer k together with the following:*

(1) *a collection of k edge-disjoint (u,v)-trails each using exactly one edge of $G - E(H)$, and either*

(2a) *a (u,v)-cut in H with at most $3k + 2$ edges, or*

(2b) *a vertex-set R containing u and v with perimeter*

$$p(R,H) \leq \left(1 + \max_{T\subseteq V(G)\backslash R} \frac{|\delta_G(T)|}{|\delta_H(T)|}\right) k.$$

Proof. The algorithm begins by replacing G and H with the graphs G' and H' obtained by identifying the vertices u and v to a new vertex r. We enqueue the edges $E(\overline{H'}) = E(G') \setminus E(H')$ and initialize an edge-set $F = \emptyset$. This can all be done in linear time or faster, depending on the encoding of the input graphs.

For each edge $e \in E(\overline{H'}) \setminus F$, let $F_e = F \cup \{e\}$. We compute the edge-connectivity between r and t in the auxiliary graphs $H' \oplus F_e$; if it is at least $2(|F|+1)$, we add e to F and restart the process until we exhaust the queue of the edges. Constructing $H' \oplus F \cup \{e\}$ takes linear time, while computing $\lambda(u,v)$ can be done in $O(nm)$-time. Note that once an edge e is refuted, it will not have sufficient connectivity in any of the later steps, so each edge is processed only once. Therefore, this procedure takes $O(nm^2)$-time in total.

When no more edges can be added to F, we provisionally set $k = |F|$. Next, we construct the unique maximal tight (t^+, r)-separating set T (see Corollary 2.8) and nearly-tight (t^+, r)-separating sets $S_1, \ldots, S_\ell$ in $H' \oplus F$ (note that $\ell \leq 2k$). Each nearly-tight set can be found by contracting the maximal tight set, duplicating one of the edges leaving it, and finding a maximal tight set in the resulting better-connected graph. After $O(m)$ applications of the $O(nm)$-time minimum cut algorithm, we obtain a set R' of perimeter

$$p(R', H'') \leq \left(1 + \max_{T \subseteq V(G') \setminus R'} \frac{|\delta_{G'}(T)|}{|\delta_{H'}(T)|}\right) k.$$

The above procedure also gives us a collection $\mathcal{T}$ of k edge-disjoint trails through r in G', each using exactly one edge of F and no other edge of $\overline{H'}$. Given a maximum (t^+, r)-flow in $H' \oplus F$—which is found when computing the edge-connectivity in the above loop—such a collection is obtained in linear time.

Next, we run an ($O(nm)$-time) maximum flow algorithm to determine $\lambda_H(u,v)$ and compute a minimum cut. Suppose first that $\lambda_H(u,v) \geq 3k$. The last paragraph of Section 2.1.4 describes how to extend the trails in $\mathcal{T}$ to a collection of k edge-disjoint (u,v)-trails satisfying outcome (1) of Theorem 2.9. Using a straightforward implementation for the recursive algorithm outlined in the proof of Lemma 2.6, this procedure takes $O(nm^2)$ time. Moreover, the set R' in G' corresponds to a vertex-set R in G containing u and v satisfying outcome (2b) of the theorem.

It remains to discuss what to do if $\lambda_H(u,v) < 3k$. In this case, we discard some edges from F, and the respective trails from $\mathcal{T}$, until the reduced set of edges (which we still denote by F and set $k = |F|$) satisfies $3k \leq \lambda_H(u,v) \leq 3k+2$. Once again, we are in the same situation as the last paragraph of Section 2.1.4 and we can find in $O(nm^2)$ time k edge-disjoint trails yielding outcome (1) of the theorem. The minimum (u,v)-cut in H gives outcome (2a) as well. □

2.3.2 Algorithms for Theorems 2.1, 2.3, and 2.4

Theorem 2.3 boils down to computing $\lambda_H(u,v)$ or applying Theorem 2.2, depending on whether u and v are on the same or opposite sides of the bipartition of H. By Theorem 2.7 and Theorem 2.9, these take $O(nm^2)$ time. Essentially the same strategy can be used to put Theorem 2.4 into practice, but only if a maximum bipartite subgraph H is already known for G. Unfortunately, such an H is NP-hard to find [60]. However, it is still feasible to find a (not necessarily maximum) bipartite subgraph H for which the conclusions of Theorem 2.4 hold.

Theorem 2.10. *There is an $O(nm^3)$-time algorithm which takes as input a graph G and vertices $u, v \in V(G)$ such that $G + uv$ is 2-edge-connected, and outputs an integer k, together with the following:*

(1) *a collection of k edge-disjoint odd (u,v)-trails, and either*

(2a) *a (u,v)-cut in G with at most $6k+4$ edges, or*

(2b) *a bipartite subgraph H with u and v on the same side of its bipartition and a vertex-set R containing u and v with perimeter $p(R,H) \leq 3k$.*

Proof. Let H be an arbitrary maximal bipartite subgraph of G. In linear time, we can decide whether u, v are on the same or opposite sides of the bipartition of H. If they are on opposite sides, compute a minimum (u,v)-cut $\delta_H(X)$ in H. If $|\delta_G(X)| > 2|\delta_H(X)|$, we find a bipartite subgraph with more edges than H by swapping the H and non-H edges of $\delta_G(X)$. Otherwise, we output the $k = \lambda_H(u,v)$ edge-disjoint odd (u,v)-paths in H and the (u,v)-cut $\delta_G(X)$, which has at most $2k$ edges.

Suppose now that u and v are on the same side of the bipartition of H. Compute the 2-edge-connected components of H in linear time [106]. If we find a vertex-set X for which $|\delta_H(X)| = 1$ but $|\delta_G(X)| > 2$, swap the H and non-H edges of $\delta_G(X)$ to get a bipartite subgraph with more edges. If we find a vertex-set X for which $u, v \notin X$, $|\delta_H(X)| = 1$ and $|\delta_G(X)| = 2$, apply the (linear-time) reduction described in Theorem 2.4. Otherwise, $H + uv$ is 2-edge-connected, so we can apply the $O(nm^2)$-time algorithm of Theorem 2.9 to G, H, u, v. We obtain a collection of k edge-disjoint odd (u,v)-trails and either a small (u,v)-cut in H or a vertex-set of small perimeter. If the latter certificates are not precise enough to satisfy (ii) or (iii), respectively, we again get a bipartite subgraph with more edges than H.

The above algorithm takes $O(nm^2)$ time to either satisfy the theorem or find a bipartite subgraph with more edges than H, in which case we can repeat the algorithm from the beginning with the improved subgraph. After $O(m)$ possible restarts, we obtain suitable outputs for the theorem. □

The algorithm corresponding to Theorem 2.1 simply deletes irrelevant components and cut-edges and applies Theorem 2.10.

Theorem 2.11. *There is a $O(nm^3)$-time 6-approximation algorithm for $\lambda_o(u,v)$.* □

2.4 Concluding remarks

In this chapter, we proved the approximate packing–covering duality for odd (u,v)-trails

$$\lambda_o(u,v) \le \tau_o(u,v) \le 6\lambda_o(u,v) + 4$$

The following examples show that a perfect duality is too much to hope for.

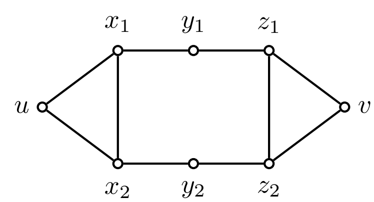

Figure 2.2: A graph with $\lambda_o(u,v) = 1$ and $\tau_o(u,v) = 2$.

Fact 2.12. *Let G_k be the graph obtained from $2k+1$ internally vertex-disjoint (u,v)-paths $ux_iy_iz_iv$, $i = 1,\dots,2k$, by adding the edges $x_{j-1}x_j$ and $z_{j-1}z_j$, $j = 2,4,\dots,2k$.*

(i) *G_k has exactly $\lambda_o(u,v) = k$ edge-disjoint odd (u,v)-trails; but*

(ii) *G_k has no (u,v)-cut with fewer than $\lambda(u,v) = 2k+1$ edges; and*

(iii) *G_k does not have a bipartite subgraph H with u and v on the same side of the bipartition and a vertex-set X containing u and v with perimeter $p(X,H) < 2k$.*

Proof. Observe that each odd (u,v)-trail uses at least one of the "extra" edges $x_{j-1}x_j$ or $z_{j-1}z_j$. Suppose that an odd (u,v)-trail P contains $x_{j-1}x_j$. If P also contains z_{j-1} or z_j, then every odd (u,v)-trail that passes through any of the vertices $x_{j-1}, x_j, z_{j-1}, z_j$ has at least one edge in common with P. On the other hand, if P passes through neither z_{j-1} nor z_j, then it must contain two other vertices x_i, z_i for some $i \in \{1,\dots,2k+1\}$, and any odd (u,v)-trail passing through any of the vertices x_{j-1}, x_j, x_i, z_i has at least one edge in common with P. In either case, P prevents four out of the $4k+2$ such vertices from being in a disjoint odd (u,v)-trail. Thus G_k has at most k edge-disjoint odd (u,v)-trails. On the other hand, it is easy to construct a collection of exactly k such trails.

It is obvious from the construction that $\lambda(u,v) = 2k+1$. For the third statement, note that $p(X,H)$ is an upper bound for the number of edge-disjoint odd closed trails through u and/or v, and G_k has $2k$ such triangles: $ux_{j-1}x_j$ and $vz_{j-1}z_j$ for $j = 2,4,\dots,2k$. □

It remains to be seen how far our bounds for the ratio between odd edge-connectivity and odd trail covering number can be improved.

Question 2.13. *What is the smallest constant* c *such that* $\lambda_o(u,v) \leq \tau_o(u,v) \leq c\lambda_o(u,v) + O(1)$ *for all graphs* G *and vertices* u, v*?*

In this chapter, we used a "greedy" approach to show that $c \leq 6$; the examples described above demonstrate that $c \geq 2$. The recent master's thesis of Ibrahimpur [56] built on our methods in [19] to show $c \leq 5$.

Chapter 3

Packing edge-disjoint odd (u, v)-trails in Eulerian graphs

In the last chapter, we showed that $\lambda_o(u, v) \leq \tau_o(u, v) \leq 6\lambda_o(u, v)$, where $\lambda_o(u, v)$ is the odd edge-connectivity and $\tau_o(u, v)$ is the odd trail covering number. Vertices of degree three are a limiting factor in our proof (see Lemma 2.6) as well as our most extreme examples (see Section 2.4). It is natural to wonder what happens if degree-three vertices are forbidden, but the answer is unsatisfying: one can "artificially" increase the minimum degree of a graph without changing the odd edge-connectivities (see Figure 3.1). With this in mind, another obvious question arises: is it easier to pack odd (u, v)-trails if we exclude all vertices of odd degree?

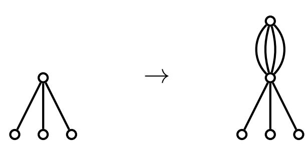

Figure 3.1: Attaching a bipartite graph at each vertex can increase the minimum degree of a graph without affecting the odd edge-connectivities. However, it does not decrease the number of vertices of odd degree.

In this chapter, we investigate odd trail packings in Eulerian graphs and answer this question in the affirmative.

Theorem 3.1. *An Eulerian graph has either k edge-disjoint odd (u, v)-trails or a set of fewer than $\frac{5}{2}k$ edges intersecting all such trails. In other words,*

$$\lambda_o(u, v) \leq \tau_o(u, v) \leq \tfrac{5}{2}\lambda_o(u, v).$$

As was the case in Chapter 2, the approximate packing–covering duality comes from a family of bounds for $\lambda_o(u, v)$ involving the perimeter and edge-connectivity.

Theorem 3.2. *Let $\alpha \in (0, 1]$; G an Eulerian graph; $u, v \in V(G)$; and $k \in \mathbb{N}$. Then either:*

(i) *G has k edge-disjoint odd (u, v)-trails;*

(ii) *G has a (u, v)-cut with fewer than $(3 - 2\alpha)k$ edges; or*

(iii) *G has a bipartite subgraph H with u, and v on the same side of the bipartition and a vertex-set X containing u and v with perimeter $p(X, H) < (1 + \alpha)k$.*

In cases (ii) and (iii), Theorem 3.2 certifies that G does not have too many odd trails. (The duality is not perfect; as we discuss in Section 3.4, a perfect duality of this form is impossible even for Eulerian graphs.) The parameter α gives us flexibility to choose which outcome produces a more precise certificate. For a packing–covering duality, this trade-off is balanced when $\alpha = \frac{1}{4}$, yielding Theorem 3.1.

The proof of Theorem 3.2 begins with perfect duality for edge-disjoint odd closed trails through a given vertex, which follows from a result of Chudnovsky et al. [16].

Lemma 3.3. *The maximum number of edge-disjoint odd (r, r)-trails in an Eulerian graph G is equal to the minimum value of $p(R, H)$ among all bipartite subgraphs H of G and vertex-sets R containing r.*

More generally, we can characterize the maximum number of edge-disjoint odd trails having ends in a specified vertex-set. To complete the proof of Theorem 3.2, we show that a sufficiently large collection of odd trails with ends in $\{u, v\}$ can be modified to produce the desired odd (u, v)-trails.

The two halves of our proof of Theorem 3.2 are given in Section 3.1 and Section 3.2, respectively. Section 3.3 discusses the algorithmic implications of our arguments, while Section 3.4 compares the Eulerian and general cases and explores the limits of Theorem 3.2.

3.1 Packing odd closed trails

In this section, we prove Lemma 3.3, which characterizes the maximum number of edge-disjoint odd trails through a given vertex.

Let H be a graph whose edges are labelled by the elements of a group Γ and let $A \subseteq V(H)$. A **non-zero A-path** is a path whose ends are both in A and for which, roughly speaking, the product of the edge labels along the path is not the identity of the group Γ. Similarly, a **non-zero cycle** is a cycle in H for which the product of the edge labels is not the identity. (We are mostly interested in the case where $\Gamma = \mathbb{Z}_2$, where these rough definitions will suffice; in the more general case, some care must be taken with the order of multiplication and the orientations of the edges along the path.[1])

[1]In the more general setting, H is a symmetric simple digraph with a labelling $\gamma : E(H) \to \Gamma$ in which $\gamma(uv) = \gamma(vu)^{-1}$ for all $uv \in E(H)$. The weight of an (oriented) path $P : v_0e_1v_1e_2v_2 \ldots e_kv_k$ is $\prod_{i=1}^{k} \gamma(e_i)$. A non-zero A-path is a path whose ends are both in A and whose weight is not the identity of Γ.

Chudnovsky et al. [16] proved the following characterization of optimal vertex-disjoint packings of non-zero A-paths in group-labelled graphs.

Theorem 3.4 (Chudnovsky et al. [16]). *Let H be a graph whose edges are labelled by the elements of a group Γ and let $A \subseteq V(H)$. The maximum number of vertex-disjoint non-zero A-paths in H is equal to the minimum value of*

$$\pi(S, D) = |S| + \sum_{\substack{\text{components } K \\ \text{of } H-S-D}} \left\lfloor \frac{|(A \cup V(D)) \cap V(K)|}{2} \right\rfloor$$

where the minimum is taken over all vertex-sets $S \subseteq V(H)$ and all edge-sets D containing no non-zero cycles and no non-zero A-paths. □

The proof of Lemma 3.3 applies Theorem 3.4 to an auxiliary graph $\widetilde{L}(G)$ obtained from G by subdividing each edge once and taking the line graph of the result. To be precise, the vertices of $\widetilde{L}(G)$ are the pairs $(x, e) \in V(G) \times E(G)$ such that x and e are incident in G. (If G has a loop e at x, the graph has two copies of the vertex (x, e).) The edges of $\widetilde{L}(G)$ join (x, e) to (y, f) whenever $x = y$ or $e = f$. For notational convenience, let $\widetilde{L}(x)$ denote the vertex-set $\{(x, e) : e \text{ is incident with } x\}$ of $\widetilde{L}(G)$ whenever $x \in V(G)$.

We are interested in edge-disjoint odd closed trails through a specified vertex r in G. These are equivalent to vertex-disjoint non-zero $\widetilde{L}(r)$-paths in $\widetilde{L}(G)$ under the following $\mathbb{Z}_2$-labelling of the edges: each edge of the form $(x, e)(y, e)$ has label 1, while every other edge, which has the form $(x, e)(x, f)$, has label 0. According to Theorem 3.4 $\widetilde{L}(G)$ has a vertex-set S and an edge-set D with no non-zero cycles and no non-zero A-paths, such that $\pi(S, D)$ is equal to the maximum number of odd closed trails through r in G. Suppose such a pair (S, D) is chosen according to the following criteria in decreasing order of priority:

1. S has as few vertices as possible.
2. $V(D)$ has as few vertices as possible.
3. D has as many edges as possible.

Claim 3.1.1. *If $x \in V(G)$, then $V(D)$ uses all or none of the vertices in $\widetilde{L}(x) \setminus S$.*

Proof of the claim. Consider the effect of removing a vertex $\widetilde{x} \in V(D) \cap (\widetilde{L}(x) \setminus S)$ and all incident edges from the subgraph D. Because $\widetilde{L}(x)$ contains a vertex which is in neither S nor $V(D)$, all the vertices of $\widetilde{L}(x) \setminus S$ are in the same component of $\widetilde{L}(G) - S - D$. Observe that $\widetilde{x}$ has at most one neighbour outside of $\widetilde{L}(x)$. Therefore, if the components of $\widetilde{L}(G) - S - D$ and of $\widetilde{L}(G) - S - (D - \widetilde{x})$ do not induce the same partition on the vertices, the only difference is that one component of the latter contains two components of the former. If the partitions are the same, then $\pi(S, D - \widetilde{x}) \leq \pi(S, D)$ follows immediately from the definition; otherwise, it is a consequence of the inequality of floors $\left\lfloor \frac{k_1+k_2-1}{2} \right\rfloor \leq \left\lfloor \frac{k_1}{2} \right\rfloor + \left\lfloor \frac{k_2}{2} \right\rfloor$. This contradicts the assumption that D has as few vertices as possible. □

Claim 3.1.2. *If $x \in V(G)$, then D either contains or is disjoint from the subgraph of $\widetilde{L}(G)$ induced by $\widetilde{L}(x) \setminus S$.*

Proof of the claim. The proof of the previous claim actually proves a slightly stronger statement: the subgraph of $\widetilde{L}(G)$ induced by $\widetilde{L}(x)$ is disconnected after the edges of D are deleted. In particular, if $(x, e), (x, f)$ are vertices in $V(D) \cap (\widetilde{L}(x) \setminus S)$, there is a zero-weight path in D connecting them. Any zero-weight edge $(x, e)(x, f)$ can therefore be added to D without creating a non-zero cycle or non-zero $\widetilde{L}(r)$-path. The edge-maximality of D means it contains all such edges. □

Claim 3.1.3. $S = \emptyset$.

Proof of the claim. Let $\widetilde{x} = (x, e) \in S$. If D does not intersect $\widetilde{L}(x)$, then removing $\widetilde{x}$ from S merges at most two components of $\widetilde{L}(G) - S - D$. As we saw in the proof of Claim 3.1.1, this implies $\pi(S \setminus \{\widetilde{x}\}, D) \leq \pi(S, D)$ and contradicts the minimality of S. On the other hand, if D intersects $\widetilde{L}(x)$, we could improve our choice by taking $\left(S \setminus \widetilde{L}(x), D \cup E(\widetilde{L}(x))\right)$ instead of (S, D): because the increase of $|N(D)|$ in each component is offset by an equivalent decrease in $|S|$, this would not increase π. In either case, we can safely remove $\widetilde{x}$ from S; since we chose S to be minimal, $S = \emptyset$. □

Let R be the set of all vertices $x \in V(G)$ such that $V(D)$ intersects $\widetilde{L}(x)$. As $\widetilde{L}(G) - S - D = \widetilde{L}(G) - D$ contains no non-zero cycles, it corresponds to a subgraph in G whose restriction to R is bipartite. Let H be the corresponding bipartite subgraph of $G[R]$. The above claims allow us to classify each nontrivial component of $\widetilde{L}(G) - D$ into two types. The first type of component is a single edge of the form $\widetilde{e} = (x, e)(y, e)$ with both ends in $V(D)$; the corresponding edge e has both ends in R and is therefore contained in $E(R) \setminus E(H)$. The second type of component is a subgraph induced by a union of vertex-sets of the form $\widetilde{L}(x)$ together with the edges connecting them to the rest of $\widetilde{L}(G)$. To be precise, the vertices in such a component are formed from the union of $\bigcup_{x \in V(K)} \widetilde{L}(x)$ with $\{(x, e) \in V(D) : e \in \delta(V(K))\}$ for some component K of $G - R$. We now compute

$$\begin{aligned}
\pi(S, D) &= |S| + \sum_{\substack{\text{components } K \\ \text{of } \widetilde{L}(G) - S - D}} \left\lfloor \frac{|(A \cup V(D)) \cap V(K)|}{2} \right\rfloor \\
&= 0 + \sum_{\substack{\widetilde{e} \in E(\widetilde{L}(G) - S - D) \\ \text{with ends in } V(D)}} \left\lfloor \frac{2}{2} \right\rfloor + \sum_{\substack{\text{components } \widetilde{K} \\ \text{of } \widetilde{L}(G) - S - D \\ \text{of the second type}}} \left\lfloor \frac{|V(D) \cap V(\widetilde{K})|}{2} \right\rfloor \\
&= |E(R) \setminus E(H)| + \sum_{\substack{\text{components } K \\ \text{of } G - R}} \left\lfloor \frac{|\delta(V(K))|}{2} \right\rfloor .
\end{aligned}$$

When G is Eulerian, $|\delta(V(K))|$ is always even, so

$$\begin{aligned}\pi(S, D) &= |E(R) \setminus E(H)| + \tfrac{1}{2}|\delta(R)| \\ &= p(R, H).\end{aligned}$$

This completes the proof of Lemma 3.3. It is worth noting that we have actually shown a more general lemma which may be useful for approximating the odd edge-connectivity in graphs which have no small cuts consisting of an odd number of edges.

Lemma 3.5. *The maximum number of edge-disjoint odd (r, r)-trails in a graph G is equal to the minimum value of*

$$|E(R) \setminus E(H)| + \sum_{\substack{\text{components } K \\ \text{of } G-R}} \left\lfloor \frac{|\delta(V(K))|}{2} \right\rfloor .$$

among all vertex-sets R containing r and bipartite subgraphs H of G. □

The above argument can be implemented in polynomial time using an algorithm due to Chudnovsky, Cunningham, and Geelen [15]. We discuss this procedure in Section 3.3.

3.2 Proof of Theorem 3.2

Let G be an Eulerian graph, $u, v \in V(G)$, and $\alpha \in (0, 1]$; after discarding any irrelevant components, we may assume G is connected.

As in the proof of Theorem 2.2, construct an auxiliary graph G' by identifying u and v to a single vertex, which we call r. Lemma 3.3 counts the number of edge-disjoint odd closed trails through r that can be packed in G'. By construction, these are equivalent to edge-disjoint odd $\{u, v\}$**-trails** in G—i.e., trails whose ends lie in the set $\{u, v\}$. If there are fewer than $(1 + \alpha)k$ such trails, then Lemma 3.3 guarantees the existence of a vertex-set R and bipartite subgraph H' in G' such that $r \in R$ and $p(R, H') < (1 + \alpha)k$. In this case, the third outcome of Theorem 3.2 is satisfied by the vertex-set $(R \setminus \{r\}) \cup \{u, v\}$ and the bipartite subgraph H obtained from H' by reversing the identification of u and v.

Suppose, then, that G has at least $\lceil (1 + \alpha)k \rceil$ odd $\{u, v\}$-trails. Let $\mathcal{S}$ be a collection of such trails and write $\mathcal{S}_u$, $\mathcal{S}_{uv}$, and $\mathcal{S}_v$ for the closed (u, u)-trails, (u, v)-trails, and closed (v, v)-trails in $\mathcal{S}$, respectively. We declare victory if $|\mathcal{S}_{uv}| \geq k$, since a collection of k odd (u, v)-trails satisfies Theorem 3.2. Therefore, we may assume that $\mathcal{S}_u \cup \mathcal{S}_v$ is nonempty for the remainder of the proof.

3.2.1 Absorbing components of $G - E(\mathcal{S})$

We now investigate some simple local modifications that enlarge the edge-set of $\mathcal{S}$ without decreasing the number of trails.

Claim 3.2.1. *If $G - E(\mathcal{S})$ is not Eulerian, there is a collection $\mathcal{T}$ of odd $\{u, v\}$-trails with $|\mathcal{T}| \geq |\mathcal{S}|$ and $E(\mathcal{T}) \supsetneq E(\mathcal{S})$.*

Proof of the claim. The only vertices that possibly have odd degree in $G - E(\mathcal{S})$ are u and v, and if they do, they are connected by a (u, v)-trail T in $G - E(\mathcal{S})$. If T has odd length, we immediately obtain a better collection $\mathcal{T} = \mathcal{S} \cup \{T\}$. Otherwise, if T is even and (say) $\mathcal{S}_u$ is nonempty, we can form an odd (u, v)-trail by concatenating T with an odd closed trail $T_u \in \mathcal{S}_u$. A larger collection $\mathcal{T}$ is then obtained from $\mathcal{S}$ by replacing T_u with the new trail. □

For the following claims, a component of $G - E(\mathcal{S})$ is considered **nontrivial** if it has at least one edge.

Claim 3.2.2. *If $G - E(\mathcal{S})$ has a nontrivial component with an even number of edges, there is a collection $\mathcal{T}$ of odd $\{u, v\}$-trails with $|\mathcal{T}| \geq |\mathcal{S}|$ and $E(\mathcal{T}) \supsetneq E(\mathcal{S})$.*

Proof of the claim. Let K be such a component. Since G is connected, K shares a vertex x with some trail $T \in \mathcal{S}$. We may assume K is Eulerian by Claim 3.2.1. Let T' be the trail obtained by following T to x, detouring along an Euler tour of K, then resuming the route of T from x to its other end. (See Figure 3.2.) As T' has odd length, replacing T with T' produces a collection of odd trails $\mathcal{T}$ as desired. □

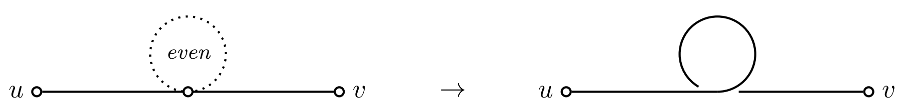

Figure 3.2: In Claim 3.2.2, a component of $G - E(\mathcal{S})$ disappears after its even Euler tour is added to the route of a trail in $\mathcal{S}$.

Claim 3.2.3. *If $G - E(\mathcal{S})$ has distinct nontrivial components touching the same trail of $\mathcal{S}$, there is a collection $\mathcal{T}$ of odd $\{u, v\}$-trails with $|\mathcal{T}| \geq |\mathcal{S}|$ and $E(\mathcal{T}) \supsetneq E(\mathcal{S})$.*

Proof of the claim. Suppose K_1, K_2 are distinct components of $G - E(\mathcal{S})$ touching $T \in \mathcal{S}$; that is, they contain vertices x_1, x_2 on T, respectively. By the above claims, we may assume K_1 and K_2 are Eulerian and each have an odd number of edges. Let T' be the odd trail obtained by following T to x_1, taking an Euler tour of K_1, following T from x_1 to x_2, taking an Euler tour of K_2, and finally following T from k_2 to its other end. (See Figure 3.3.) Replacing T with T' yields the better collection $\mathcal{T}$. □

If K is a nontrivial component of $G - E(\mathcal{S})$, let $K^{(0)} = V(K)$. Then, for each $i \geq 1$, let $K^{(i)}$ be the union of $K^{(i-1)}$ and every vertex appearing between two (not necessarily distinct) vertices $x, y \in K^{(i-1)}$ on trails $T \in \mathcal{S}$. The set $\langle K \rangle = K^{(|V(G)|)}$ obtained at the end of this process has two important properties. First, $\delta(\langle K \rangle)$ contains at most two edges

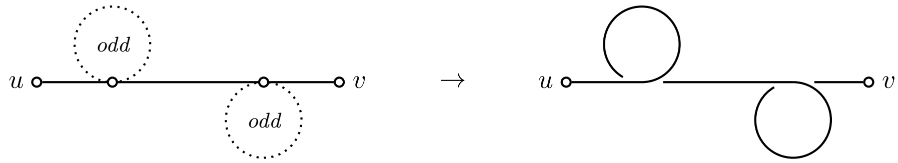

Figure 3.3: In Claim 3.2.3, two (Eulerian) components of $G - E(\mathcal{S})$ are absorbed if each has an odd number of edges.

of any trail of $\mathcal{S}$. Second, as the next claim shows, if any trail passes through two distinct "extended components", the trails $\mathcal{S}$ can be rerouted in order for Claim 3.2.3 to be applied to their respective components.

Claim 3.2.4. *If $G - E(\mathcal{S})$ has distinct nontrivial components K_1 and K_2 for which some trail in $\mathcal{S}$ passes through $\langle K_1 \rangle$ and $\langle K_2 \rangle$, there is a collection $\mathcal{T}$ of odd $\{u, v\}$-trails with $|\mathcal{T}| \geq |\mathcal{S}|$ and $E(\mathcal{T}) \supsetneq E(\mathcal{S})$.*

Proof of the claim. Let j be the smallest integer such that a trail $T \in \mathcal{S}$ touches both ${K_1}^{(j)}$ and $\langle K_2 \rangle$, where ${K_1}^{(j)}$ is defined as in the construction of $\langle K_1 \rangle$. Similarly, let j' be the smallest integer for which T contains a vertex in ${K_2}^{(j')}$. Let $x_1^{(j)}$ and $x_2^{(j')}$ be vertices on T that lie in ${K_1}^{(j)}$ and ${K_2}^{(j')}$, respectively.

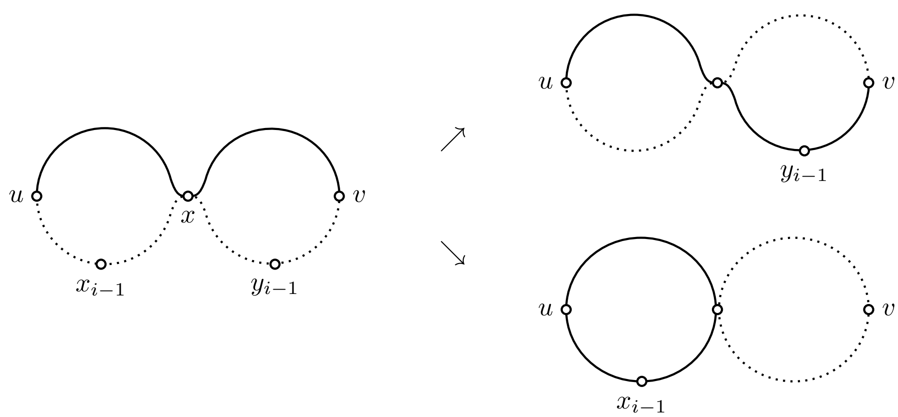

Figure 3.4: A trail through $\langle K_1 \rangle$ can be rerouted to pass through a vertex added earlier to $\langle K \rangle$.

We first reroute the trails of $\mathcal{S}$ so that one of them passes through $x_2^{(j')}$ and a vertex of K_1. Let $x_j = x_1^{(j)}$ and $T_j = T$. Then, for $i = j, \ldots, 1$, execute the following loop. Suppose x_i appears between the vertices $x_{i-1}, y_{i-1} \in K^{(i-1)}$ on a trail $X_i \in \mathcal{S}$. If $X_i = T_i$, let $T_{i-1} = T_i$ and continue the loop. Otherwise, if $X_i \neq T_i$, observe that (any occurrence of) the vertex $x_1^{(i)}$ splits X_i into two subtrails of opposite parity. By swapping the subtrails of the same parity between X_i and T_i we obtain two new odd $\{u, v\}$-trails T_{i-1}, X_{i-1}. (See Figure 3.4.) Since x_{i-1} and y_{i-1} appear on different sides of x_i on X_i, the new trails T_{i-1}, X_{i-1} each contain one of them. Moreover, either T_{i-1} or X_{i-1} passes through $x_2^{(j')}$. Without loss of generality,

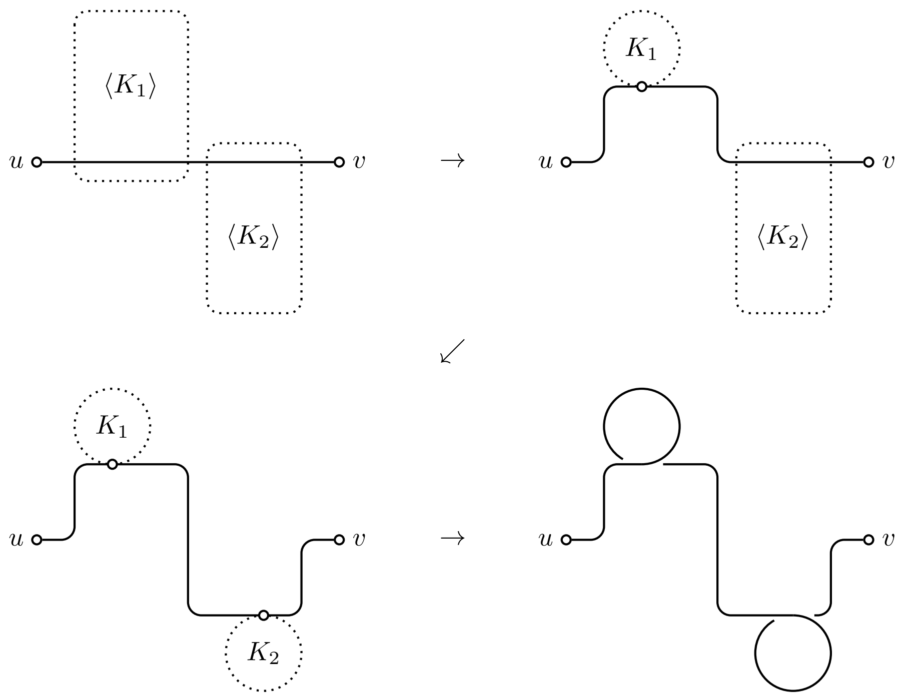

Figure 3.5: The proof of Claim 3.2.4 goes through two rounds of rerouting to sculpt a trail that can absorb the components K_1 and K_2.

say T_{i-1} contains x_{i-1} and $x_2^{(j')}$. We modify $\mathcal{S}$ by replacing $\{T, X\}$ with $\{T_{i-1}, X_{i-1}\}$ and continue the loop. At the end of the loop, the resulting collection $\mathcal{T}'$ has a trail T_0 passing through $x_2^{(j')}$ and a vertex x_0 of ${K_1}^{(0)} = V(K_1)$

The choice of j and j' imply that no trail modified in the above procedure is involved in the construction of ${K_2}^{(j')}$. (The modifications affect T and some trails through ${K_1}^{(j-1)}$, while the construction of ${K_2}^{(j')}$ depends only on trails that share at least two vertices with ${K_2}^{(j'-1)}$. The choice of j' implies T does not touch ${K_2}^{(j'-1)}$, and the choice of j ensures that none of the other affected trails do either.) Therefore, we are justified in saying that $x_2^{(j')} \in {K_2}^{(j')}$ even after we replace $\mathcal{S}$ with the collection $\mathcal{T}'$ of $\{u, v\}$-trails obtained at the end of the above process. Observe that $|\mathcal{T}'| = |\mathcal{S}|$, $E(\mathcal{T}') = E(\mathcal{S})$.

The above argument can be applied a second time to reroute the trails of $\mathcal{T}'$ so that one passes through vertices of K_1 and K_2. Claim 3.2.3 then yields the desired collection $\mathcal{T}$. □

3.2.2 Simplifying nontrivial 2-edge-cuts

In the next section, it is convenient to consider a simplified version of the graph. To be precise, if X is a vertex-set in G with $u, v \notin X$, $|\delta(X)| \leq 2$, and such that $G[X]$ is connected, let us modify G as follows. If $|\delta(X)| \leq 1$, we can delete it as no (u, v)-trail passes through X. Otherwise, suppose $|\delta(X)| = 2$. If $G[X]$ is not bipartite, we contract X to a single vertex with a loop. If $G[X]$ is bipartite, then every path in $G[X]$ between the ends of the two edges in $\delta(X)$ has the same parity. If the parity is even, contract X to a single vertex without a loop; if it is odd, delete X entirely and replace $\delta(X)$ with an edge. (See Figure 3.6.) It is easy to see that these operations preserve the number of trails in $\mathcal{S}$. From now on, we assume that $G + uv$ is internally 3-edge-connected.

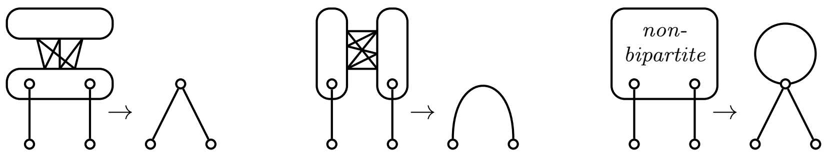

Figure 3.6: If G has a nontrivial 2-edge-cut that does not separate u and v, we can contract it away without affecting the number of odd $\{u, v\}$-trails.

3.2.3 Bounding the remaining components

After repeatedly applying the above claims, we can assume a number of useful properties. For instance, every nontrivial component of $G - E(\mathcal{S})$ can now be assumed to be an Eulerian subgraph with an odd number of edges. Each such component K shares a vertex with at least one trail of $\mathcal{S}$, but no two lie on the same trail. More generally, the vertex-sets $\langle K \rangle$ constructed from these components are pairwise disjoint and no two interact with the same trail in $\mathcal{S}$.

Let K_1, ..., K_ℓ be those nontrivial components of $G - E(\mathcal{S})$ whose extension $\langle K_i \rangle$ $(i = 1, \ldots, \ell)$ is not just a vertex (with loops).

Claim 3.2.5. *Let $\mathcal{S}$ and $K_1, \ldots, K_\ell$ be as above. If $u \in \langle K_i \rangle$ and $v \in \langle K_j \rangle$ where $i, j \in \{1, \ldots, \ell\}$, then $i = j = \ell = 1$.*

Proof of the claim. If $i = j$, the construction adds all of $V(\mathcal{S})$ to $\langle K_i \rangle$ since $u, v \in \langle K_i \rangle$. By Claim 3.2.4 and the assumption that G is connected, $G - E(\mathcal{S})$ has no other nontrivial components, so $\ell = 1$.

On the other hand, suppose $i \neq j$. Because $\langle K_i \rangle$ contains u, it contains all of $V(\mathcal{S}_u)$ by construction. Likewise, $\langle K_j \rangle$ contains $V(\mathcal{S}_v)$. Note that $v \notin \langle K_i \rangle$, so the fact that G is connected means $\langle K_i \rangle$ has at least one edge leaving it. This edge cannot be in any component $K_1, \ldots K_\ell$, so it is either on a (u, v)-trail in $\mathcal{S}$ or a (v, v)-trail in $\mathcal{S}$. Either way, there is a trail in $\mathcal{S}$ containing vertices of both $\langle K_i \rangle$ and $\langle K_j \rangle$. This contradicts Claim 3.2.4. □

Claim 3.2.6. *Let $\mathcal{S}$ and $K_1, \ldots, K_\ell$ be as above. Then at least two trails of $\mathcal{S}$ pass through $\langle K_i \rangle$ for each $i \in \{1, \ldots, \ell\}$.*

Proof of the claim. Each $T \in \mathcal{S}$ contains at most two edges of $\delta(\langle K_i \rangle)$—after all, the construction adds to $\langle K_i \rangle$ any subtrail of T that leaves and re-enters it. If $\langle K_i \rangle$ touches only one trail, this implies $|\delta(\langle K_i \rangle)| \leq 2$. But we assumed in Section 3.2.2 that G is internally 3-edge-connected; this contradicts the assumption that $\langle K_i \rangle$ has more than one vertex. □

Let $\mathcal{S}$ and $K_1, \dots, K_\ell$ be as above. By Claim 3.2.5, we may assume that $\langle K_2 \rangle, \dots, \langle K_\ell \rangle$ are disjoint from $\{u, v\}$. If $|\mathcal{S}| - k + 1 \leq \ell$, define

$$\mathbf{K} = \bigcup_{i=|\mathcal{S}|-k+1}^{\ell} \langle K_i \rangle.$$

Otherwise, let $\mathbf{K} = \emptyset$.

Claim 3.2.7. $|\delta(\mathbf{K})| \leq (2 - 2\alpha)k$.

Proof of the claim. The statement is obvious if $\mathbf{K} = \emptyset$. Suppose, then, that $\ell \geq |\mathcal{S}| - k + 1 \geq 2$. By Claim 3.2.5, no K_i contains both u and v for $i = 1, \dots, \ell$.

According to Claim 3.2.3, each trail $T \in \mathcal{S}$ passes through at most one $\langle K_i \rangle$, and by construction, $|E(T) \cap \delta(\langle K_i \rangle)| \leq 2$. Moreover, Claim 3.2.6 states that $\langle K_i \rangle$ is visited by at least two trails of $\mathcal{S}$. Therefore, $|\delta(\langle K_i \rangle)| \geq 4$ for each extended component $\langle K_i \rangle$ disjoint from $\{u, v\}$. If this applies to every $\langle K_i \rangle$, we may compute

$$\begin{aligned}
|\delta(\mathbf{K})| &\leq \sum_{i=|\mathcal{S}|-k+1}^{\ell} |\delta(\langle K_i \rangle)| \\
&= \sum_{i=1}^{\ell} |\delta(\langle K_i \rangle)| - \sum_{i=1}^{|\mathcal{S}|-k} |\delta(\langle K_i \rangle)| \\
&\leq \sum_{i=1}^{\ell} \sum_{T \in \mathcal{S}} |E(T) \cap \delta(\langle K_i \rangle)| - \sum_{i=1}^{|\mathcal{S}|-k} |\delta(\langle K_i \rangle)| \\
&\leq 2|\mathcal{S}| - 4(|\mathcal{S}| - k) \\
&\leq (2 - 2\alpha)k.
\end{aligned}$$

If not, suppose without loss of generality that $\langle K_1 \rangle$ contains u but not v. By Claim 3.2.5, $\langle K_i \rangle$ does not contain u or v for $i = 2, \dots, \ell$. If we remove the terms involving $\langle K_1 \rangle$ from the above computation, we find

$$\begin{aligned}
|\delta(\mathbf{K})| &\leq \sum_{i=2}^{\ell} \sum_{T \in \mathcal{S}} |E(T) \cap \delta(\langle K_i \rangle)| - \sum_{i=2}^{|\mathcal{S}|-k} |\delta(\langle K_i \rangle)| \\
&\leq 2\,|\{T \in \mathcal{S} \text{ not touching } \langle K_1 \rangle\}| - 4(|\mathcal{S}| - k - 1) \\
&\leq 2(|\mathcal{S}| - 2) - 4(|\mathcal{S}| - k - 1) \\
&\leq (2 - 2\alpha)k.
\end{aligned}$$

In both cases we establish the desired bound. □

Notice that $\mathcal{S}$ comprises at least $\lceil (1+\alpha)k \rceil$ odd trails with ends in $\{u, v\}$, even though the lemma only requires k odd trails (albeit ones with prescribed ends). We now spend the "extra" αk trails in $\mathcal{S}$ to eliminate the nontrivial components $K_1, \ldots, K_{|\mathcal{S}|-k}$ of $G - E(\mathcal{S}) - \mathbf{K}$ whose extensions contain more than one vertex.

Let $i \in \{1, \ldots, \min\{\ell, |\mathcal{S}|-k\}\}$, and let $T_i \in \mathcal{S}$ share a vertex x with K_i. Such a trail exists because G is connected. Moreover, provided $|\mathcal{S}| \geq 2$, there exists another $T_i' \in \mathcal{S}$ such that $E(T_i) \cup E(T_i') \cup E(K_i)$ induces a connected subgraph. Since every vertex other than u and v in this subgraph has even degree, there is a trail T'' for which $E(T'') = E(T_i) \cup E(T_i') \cup E(K_i)$. Note that T'' has an odd number of edges. Replacing T_i, T_i' with T'' thus absorbs the edges of K_i into $E(\mathcal{S})$ at the expense of reducing $|\mathcal{S}|$ by one. Let $\mathcal{T}$ be the collection of odd $\{u, v\}$-trails obtained after repeating the above process $\min\{\ell, |\mathcal{S}| - k\}$ times.

Claim 3.2.8. *$|\mathcal{T}| \geq k$. Moreover, every (non-loop) edge of $G - E(\mathcal{T})$ is contained in $G[\mathbf{K}]$.*

Proof of the claim. The size of $|\mathcal{T}|$ is $|\mathcal{S}| - \min\{\ell, |\mathcal{S}| - k\} \geq k$. By definition, each component of $G - E(\mathcal{T})$ is either a single vertex or equal to some component K_i in $G - E(\mathcal{S})$ where $|\mathcal{S}| - k < i \leq \ell$. □

3.2.4 Turning $\{u, v\}$-trails into (u, v)-trails

In the final portion of the proof, we rearrange the edges of $\mathcal{T}$ to maximize the number of trails having both u and v as ends. As we did with $\mathcal{S}$, let $\mathcal{T}_u$, $\mathcal{T}_{uv}$, and $\mathcal{T}_v$ denote the closed (u, u)-trails, (u, v)-trails, and closed (v, v)-trails in $\mathcal{T}$, respectively.

Claim 3.2.9. *If $\mathcal{T}_u$ and $\mathcal{T}_v$ are not vertex-disjoint, there is a collection of odd $\{u, v\}$-trails $\mathcal{T}'$ with $|\mathcal{T}'| = |\mathcal{T}|$, $E(\mathcal{T}') = E(\mathcal{T})$, and $|\mathcal{T}'_{uv}| > |\mathcal{T}_{uv}|$.*

Proof of the claim. Suppose to the contrary that $T_u \in \mathcal{T}_u$ and $T_v \in \mathcal{T}_v$ share a vertex $x \in V(T_u) \cap V(T_v)$. The edges of T_u can be partitioned into two (u, x)-trails of opposite parity; similarly, T_v splits into two (x, v)-trails of opposite parity. By concatenating the appropriate subtrails, we obtain two odd (u, v)-trails T_1, T_2 using only the edges of T_u and T_v. The desired collection $\mathcal{T}'$ is obtained by replacing T_u and T_v with T_1 and T_2. □

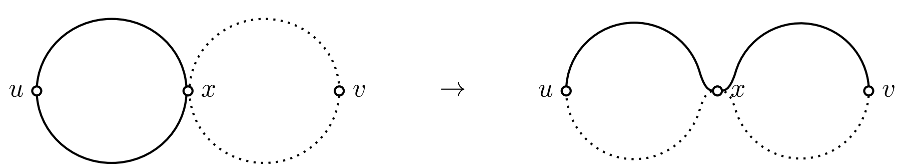

Figure 3.7: In Claim 3.2.9, intersecting odd closed trails can be replaced with odd (u, v)-trails.

Claim 3.2.9 also applies if we can rearrange the trails of $\mathcal{T}$ to satisfy its hypothesis. To this end, we define the closure $\langle \mathcal{T}_u \rangle$ to be the vertex-set obtained from $V(\mathcal{T}_u)$ by repeatedly applying a similar rule to the one we used above to construct the "extended components" $\langle K \rangle$: whenever $T \in \mathcal{T}_{uv}$ passes through a vertex y that has already been added to $\langle \mathcal{T}_u \rangle$, add each (u, y)-subtrail of T to $\langle \mathcal{T}_u \rangle$.

Claim 3.2.10. *If $\langle \mathcal{T}_u \rangle$ and $V(\mathcal{T}_v)$ intersect, there is a collection of odd $\{u, v\}$-trails $\mathcal{T}'$ with $|\mathcal{T}'| = |\mathcal{T}|$, $E(\mathcal{T}') = E(\mathcal{T})$, and $|\mathcal{T}'_{uv}| > |\mathcal{T}_{uv}|$.*

Proof of the claim. Let $x \in \langle \mathcal{T}_u \rangle \cap \mathcal{T}_v$. It suffices to show that we can modify the trails of $\mathcal{T}$ so that one of the closed (u, u)-trails uses x; the desired collection $\mathcal{T}'$ can then be obtained using Claim 3.2.9. Let ${\mathcal{T}_u}^{(i)}$ denote the partially-constructed vertex set after i steps; the proof is by induction on the step in which x is added to $\langle \mathcal{T}_u \rangle$.

If $x \in V(\mathcal{T}_u)$, there is nothing to prove. Let $i \geq 1$ and suppose x appears on a trail $T_i \in \mathcal{T}$ between u and some $t \in {\mathcal{T}_u}^{(i-1)}$. By induction, there is an odd closed (u, u)-trail $C_i \in \mathcal{T}_u$ which uses t. This trail can be divided into edge-disjoint trails of opposite parity between u and t, and one of these subtrails has the same parity as the (u, t)-subtrail of T_i. Construct an odd closed (u, u)-trail C_{i-1} through x by taking T_i from u to t and returning to u via the appropriate subtrail of C_{i-1}. The edges left over form a trail T_{i-1} from u to t to v; because $|E(C_i)| + |E(T_i)|$ is even, this trail is also odd. We modify $\mathcal{T}$ by replacing $\{C_i, T_i\}$ with $\{C_{i-1}, T_{i-1}\}$.

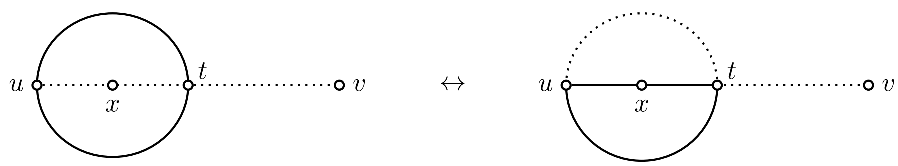

Figure 3.8: Claim 3.2.10 rearranges $\mathcal{T}$ so that $x \in \langle \mathcal{T}_u \rangle$ is on an odd closed trail in $\mathcal{T}_u$.

The above modifications do not change the size, edge-set, and number of (u, v)-trails in $\mathcal{T}$. They also do not affect the trails $\mathcal{T}_v$, so after their application, the vertex x is on a closed (u, u)-trail as well as a closed (v, v)-trail. We may therefore obtain $\mathcal{T}'$ using Claim 3.2.9. $\square$

3.2.5 Completing the proof

Suppose $\mathcal{T}_u$ is nonempty. (If not, apply the following argument with $\mathcal{T}_v$ in place of $\mathcal{T}_u$; if both are empty, then $|\mathcal{T}_{uv}| \geq k$ and we are already done.) By construction of the closure, $\delta(\langle \mathcal{T}_u \rangle)$ contains no edges of $\mathcal{T}_u$ and exactly one edge from each trail in $\mathcal{T}_{uv}$. After locally modifying $\mathcal{T}$ by repeatedly applying Claim 3.2.10, we ensure that $\langle \mathcal{T}_u \rangle$ and $V(\mathcal{T}_v)$ are disjoint, so $\delta(\langle \mathcal{T}_u \rangle)$ also contains no edge of $\mathcal{T}_v$. The only remaining edges that might be in $\delta(\langle \mathcal{T}_u \rangle)$ are in $G - E(\mathcal{T})$; by Claim 3.2.8, $G[\mathbf{K}]$ contains all of these. Therefore, if $\mathbf{U} = \langle T_u \rangle \cup \mathbf{K}$, we have

$$|\delta(\mathbf{U})| \leq |\delta(\langle \mathcal{T}_u \rangle) \cap E(\mathcal{T})| + |\delta(\mathbf{K})| \leq |\mathcal{T}_{uv}| + (2 - 2\alpha)k.$$

Because $v \notin \langle \mathcal{T}_u \rangle$ and $\mathbf{K}$ is disjoint from $\{u, v\}$, the vertex-set $\mathbf{U}$ separates u and v. Therefore, we either have $\mathcal{T}_{uv} \geq k$ or G has a cut of fewer than $(3 - 2\alpha)k$ edges. This completes the proof of Theorem 3.2. $\square$

3.3 Algorithm for packing odd (u, v)-trails in Eulerian graphs

The arguments used in this chapter imply a polynomial-time algorithm for approximating the odd edge-connectivity between two vertices in an Eulerian graph. In this section, we establish the runtime of this algorithm.

Theorem 3.6. *For Eulerian graphs, there is a* 2.5*-approximation algorithm for* $\lambda_o(u, v)$ *that runs in time* $O(nm^2 + m^{2+\omega})$*, where* $n = |V(G)|$*,* $m = |E(G)|$*, and* ω *is the matrix multiplication exponent.*

The first half of the algorithm implements Lemma 3.3. After constructing an auxiliary $\mathbb{Z}_2$-labelled graph $\widetilde{L}(G)$ in $O(m^2)$ time, the lemma boils down to finding an optimal packing of non-zero A-paths in $\widetilde{L}(G)$. There are several algorithms that can do this [15, 92, 119]. The method of Yamaguchi [119] is (apparently) asymptotically fastest; its runtime when applied to $\widetilde{L}(G)$ is $O\Big(|E(\widetilde{L}(G))| \cdot |V(\widetilde{L}(G))|^{\omega}\Big) = O(m^{2+\omega})$, where $\omega < 2.373$ [75] is the exponent for fast matrix multiplication.

Once an optimal vertex-disjoint packing for non-zero A-paths in $\widetilde{L}(G)$ is known, it takes linear time to transform it into a collection of edge-disjoint odd trails in G with ends in $\{u, v\}$. The second half of the algorithm takes this collection of odd trails $\mathcal{S}$ and follows the operations described in the proof of Theorem 3.2. Here, we discuss the runtime of the procedures used.

Claim 3.2.1 takes $O(n + m)$ time to compute the components of $G - E(\mathcal{S})$ and examine the degrees of u and v. It is applied only once.

Claim 3.2.2 counts the number of edges in each component of $G - E(\mathcal{S})$. All components with an even number of edges are absorbed into $\mathcal{S}$ in time $O(n + m)$, after which the claim does not have to be applied again.

Claim 3.2.4 requires the computation of some extended components $\langle K \rangle$. At each of the n steps in the construction, we check in time $O(m)$ if any trail touches two of the partially-constructed $\langle K \rangle$. If so, it only takes linear time to rearrange the edges of $\mathcal{S}$ and absorb two components according to Claim 3.2.3, provided we store in memory the details of the extended components' construction. Otherwise, we add the appropriate subtrails to their extended components and repeat this process until the construction is finished. Therefore, Claim 3.2.4 can be implemented in $O(nm)$ time per application. The proof of Theorem 3.2 asks us to repeat this claim whenever possible; since doing so reduces the number of edges of $G - E(\mathcal{S})$ by at least one, Claim 3.2.4 is applied $O(m)$ times, for a total runtime of $O(nm^2)$. This estimate of the runtime may be improved with more detailed analysis.

Claim 3.2.7 and Claim 3.2.8 require us to construct the vertex-set $\mathbf{K}$ and collection of odd trails $\mathcal{T}$, respectively. The former can be done in $O(n + m)$ time by listing the components of $G - E(\mathcal{S})$, while the latter involves up to $|\mathcal{S}| - k < m$ trail replacements, each taking $O(n+m)$ time. This step is finished within $O(nm + m^2)$ time.

Claim 3.2.10 makes us construct $\langle \mathcal{T}_u \rangle$ (or $\langle T_v \rangle$, if $\mathcal{T}_u$ is empty). Like Claim 3.2.4, this can be implemented in $O(nm)$ time. If $\langle \mathcal{T}_u \rangle$ and $\mathcal{T}_v$ are not vertex-disjoint, the algorithm rearranges the edges of the trails in linear time and we have to recompute $\langle \mathcal{T}_u \rangle$. However, this happens at most $k < m$ times before $\langle \mathcal{T}_u \rangle$ and $\mathcal{T}_v$ are disjoint. Therefore, this phase of the algorithm takes $O(nm^2)$ time.

At the end of this process, which takes $O(nm^2)$ time in total, we obtain a collection of edge-disjoint odd (u, v)-trails whose size is bounded below by $\frac{\lambda(u,v)}{3-2\alpha}$ and $\frac{p(X,H)}{1+\alpha}$, where X and H are obtained from $\widetilde{L}(G)$ as in the proof of Lemma 3.3. If we set $\alpha = \frac{1}{4}$, we have $\frac{2}{5}\tau_o(u, v) \le \lambda_o(u, v) \le \tau_o(u, v)$, meaning we have approximated $\lambda_o(u, v)$ to within a factor of $\frac{5}{2}$. This completes the proof of Theorem 3.6. □

3.4 Concluding remarks

We have shown that an Eulerian graph with no k edge-disjoint odd (u, v)-trails has either a (u, v)-cut of size $(3 - 2\alpha)k$ or a certificate that there are no more than $(1 + \alpha)k$ edge-disjoint odd trails with ends in $\{u, v\}$. In contrast, the (non-Eulerian) extreme examples in Section 2.4 satisfy $\lambda(u, v) = 2k + 1$ and have no fewer than $2k$ edge-disjoint odd $\{u, v\}$-trails, but only admit k edge-disjoint odd (u, v)-trails. Consequently, those graphs do not satisfy Theorem 3.2 for any choice of α, demonstrating a fundamental difference between the Eulerian and non-Eulerian cases.

Theorem 3.2 is not necessary tight; we have not found an Eulerian example that achieves both the edge-connectivity and perimeter bounds for any particular α. However, the following family puts nontrivial lower bounds on what we might hope to achieve. The "double-bowtie" graph consists of three triangles uxw, yvw', and xyz. Let G_ℓ be the graph obtained by "gluing together" ℓ disjoint copies of the double-bowtie as illustrated in Figure 3.9 for $\ell = 3$.

Fact 3.7. *Let $k = \lfloor 4\ell/3 \rfloor$. Then*

(i) *G_ℓ has $\lambda_o(u, v) = k$ edge-disjoint odd (u, v)-trails; but*

(ii) *G has no (u, v)-cut with fewer than $\lambda(u, v) = 2\ell \ge \frac{3}{2}\lfloor 4\ell/3 \rfloor = \frac{3}{2}k$ edges; and*

(iii) *G does not have a bipartite subgraph and vertex-set X such that $u, v \in X$ are on the same side of the bipartition of H and $p(X, H) < \frac{3}{2}k$.*

Proof. A simple induction argument proves the first statement, and the second statement is clear. For the third statement, observe that G_ℓ has $2\ell \ge \frac{3}{2}k$ edge-disjoint odd $\{u, v\}$-trails: namely, the triangles through u and v. □

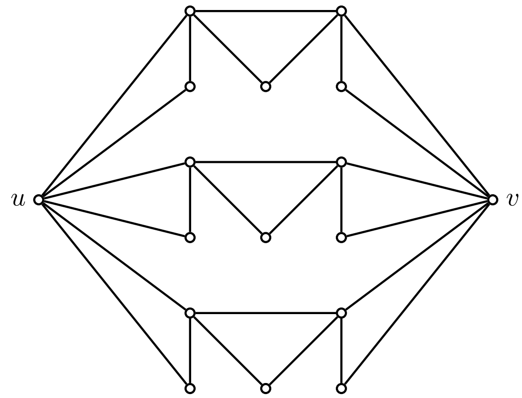

Figure 3.9: The graph H_3 obtained by gluing together three double-bowties.

It follows that Theorem 3.2 cannot be refined to a perfect duality theorem. This matter is discussed further in Chapter 6.

Chapter 4

Perimeter and submodularity

In Chapter 2, we showed that the odd edge-connectivity between two vertices is related to a measure called the perimeter of a vertex-set:

$$p(X, H) = |E(X) \setminus E(H)| + \tfrac{1}{2}|\delta(X)|.$$

The results in this chapter suggest that perimeter plays an even more fundamental role. To be precise, we prove that for every subgraph H of G, the perimeter with respect to H is a submodular function.

Theorem 4.1. *Let X, Y be vertex-sets and H a subgraph of G. Then*

$$p(X, H) + p(Y, H) \geq p(X \cup Y, H) + p(X \cap Y, H).$$

Submodular functions are a central concept in the field of combinatorial optimization; see [41, 85, 102] and the references therein. Polynomial-time algorithms are known for minimizing submodular functions in general [57, 90, 101], implying polynomial-time algorithms for finding sets of minimum perimeter. In fact, the simple definition of perimeter allows us to use considerably more efficient cut-based methods [73] for this task (see Section 4.1).

Theorem 4.2. *There is an $O(nm)$-time algorithm which, given a graph G, subgraph H, and vertex x, outputs a vertex-set X containing x that minimizes the perimeter $p(X, H)$ over all such sets.*

Submodular functions also find widespread use in structural graph theory [39]. For instance, the submodular inequality for edge-cuts is at the heart of many important edge-connectivity results including Mader's Splitting Theorem [81] and the Gomory–Hu Theorem [48], which constructs a tree representation for a special collection of minimum cuts in a graph. The Gomory–Hu tree can be used as an efficient data structure to store the local edge-connectivity between all pairs of vertices, to describe the global connectivity structure of a graph [24], or as a practical strategy for graph clustering [36, 100, 118].

A family of sets $\mathcal{F}$ is **laminar** if, for all $X, Y \in \mathcal{F}$, either $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$.

Theorem 4.3 (Gomory and Hu [48]). *Let G be a graph.*

1. *There exists a laminar family $\mathcal{C}$ of $|V(G)| - 1$ distinct vertex-sets such that, for any two vertices $x \neq y$, some $U \in \mathcal{C}$ induces a minimum (x, y)-cut $\delta(U)$ in G.*

2. *There exists a tree T on the same vertices as G (though not necessarily a subgraph) in which the sets of $\mathcal{C}$ are represented as fundamental cuts: for each edge $e = xy \in E(T)$, $\mathcal{C}$ contains the set of vertices of one of the two components of $T - e$.*

3. *There exists a function $c : E(T) \to \mathbb{R}$ such that for every $u, v \in V(G)$, the edge-connectivity $\lambda_G(u, v)$ is the minimum value of $c(e)$ among edges e on the unique (u, v)-path in T.* □

One important consequence of the submodularity of perimeter is a new version of the Gomory–Hu Theorem: there is a laminar family containing, for each vertex x, a set of minimum perimeter (with respect to a given subgraph H) among all sets containing x.

Theorem 4.4. *Let G be a graph and H a subgraph of G.*

1. *There exists a laminar family of vertex-sets $\mathcal{L}_H$ such that, for all vertices $x \in V(G)$, some $V_x \in \mathcal{L}_H$ minimizes the perimeter $p(X, H)$ over all vertex-sets X containing x.*

2. *There exists a vertex-disjoint collection of rooted trees F_H on the same vertices as G (though not necessarily a subgraph) in which the sets of $\mathcal{L}_H$ are represented as follows: for every $V_x \in \mathcal{L}_H$, there is an ancestor y of x in F_H whose descendants (including y) make up V_x.*

3. *There exists a function $f_H : V(F_H) \to \mathbb{N}$ such that every $x \in V(G)$ has an ancestor y in F_H (possibly $y = x$) such that $f_H(y) = \min_{x \in X} p(X, H)$.*

We present an $O(n^2 m)$-time algorithm to construct the laminar family $\mathcal{L}_H$ and the corresponding data structure.

The remainder of this chapter presents two important applications. In Section 4.3.1, we combine Theorem 2.2 with Theorem 4.4 to construct, in polynomial time, a data structure caching approximate odd edge-connectivities for all vertex pairs in a graph. Finally, in Section 4.3.2, we extend the result of [24] to describe the rough structure of graphs with no totally odd immersion of K_t.

Theorem 4.5. *If a graph G has no totally odd immersion of K_t, then its vertices can be partitioned into $V(G) = V_1 \ldots, V_\ell$ such that, for all $i = 1, \ldots, \ell$,*

- $\lambda(u, v) < 6t(t-1)$ *for all $u \in V_i$ and $v \notin V_i$; and*

- *either* $|V_i| < t$ *or* $p(V_i, H) < \frac{3}{2}t(t-1)$.

This approximately characterizes graphs with no totally odd immersion of a large complete graph: if G has no totally odd immersion of K_t, the vertex-partition in Theorem 4.5 certifies that G has no totally odd immersion of K_{6t^2}.

4.1 Proofs of Theorem 4.1 and 4.2

In this section, we furnish two different proofs that imply Theorem 4.1. The first proof directly establishes the submodular inequality directly for perimeter. The second proof follows a general construction from [73] to show how perimeter can be represented as the capacity of edge-cuts in a weighted digraph. This property is stronger than submodularity and can be used algorithmically to quickly find sets of minimum perimeter, yielding Theorem 4.2.

4.1.1 Direct proof of the submodular inequality

The submodular inequality for edge-cuts follows from the edge-counting $|\delta(X)| + |\delta(Y)| = |\delta(X \cup Y)| + |\delta(X \cap Y)| + 2|E(X \setminus Y, Y \setminus X)|$; see [9, 39, 79]. Using the same identity, we compute

$$\begin{aligned} p(X,H) + p(Y,H) &= |E(X) \setminus E(H)| + |E(Y) \setminus E(H)| + \tfrac{1}{2}|\delta(X)| + \tfrac{1}{2}|\delta(Y)| \\ &= |E(X) \setminus E(H)| + |E(Y) \setminus E(H)| + |E(X \setminus Y, Y \setminus X)| \\ &\qquad + \tfrac{1}{2}|\delta(X \cup Y)| + \tfrac{1}{2}|\delta(X \cap Y)| \\ &\geq |E(X) \setminus E(H)| + |E(Y) \setminus E(H)| + |E(X \setminus Y, Y \setminus X) \setminus E(H)| \\ &\qquad + \tfrac{1}{2}|\delta(X \cup Y)| + \tfrac{1}{2}|\delta(X \cap Y)| \end{aligned}$$

Because $E(X) \cup E(Y) \cup E(X \setminus Y, Y \setminus X) = E(X \cup Y)$, we obtain from the principle of inclusion and exclusion that

$$\begin{aligned} p(X,H) + p(Y,H) &\geq |E(X \cup Y) \setminus E(H)| + |E(X \cap Y) \setminus E(H)| + \tfrac{1}{2}|\delta(X \cup Y)| + \tfrac{1}{2}|\delta(X \cap Y)| \\ &= p(X \cup Y, H) + p(X \cap Y, H). \end{aligned}$$

Therefore, the perimeter measure is submodular.

4.1.2 A representation of perimeter by directed edge-cuts

Let G be a graph and H a subgraph. We construct a digraph D on the vertices $V(G) \cup \{s,t\}$ and define edge capacities $c : E(D) \to \mathbb{R}$ such that, for every $X \subseteq V(G)$,

$$c(\delta_D(X \cup \{s\})) - \frac{|E(G)| - |E(H)|}{2} = p(X,H).$$

In other words, the perimeter of a vertex-set X is, up to an additive constant, equal to the capacity of a corresponding edge-cut $\delta(X \cup \{s\})$ in D.

The construction is as follows. Let $V(D) = V(G) \cup \{s, t\}$. For every edge $e = xy \in E(H)$, add to D two new arcs sx and sy each with capacity $\frac{1}{2}$. Then, for every edge $e = xy \in E(G) \setminus E(H)$, add to D an arc sx with capacity $\frac{1}{2}$, an arc xy with capacity 1, and an arc yt with capacity $\frac{1}{2}$. These cases are illustrated in Figure 4.1.

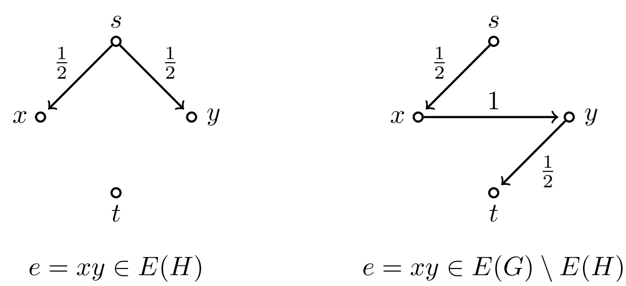

Figure 4.1: For every edge $e = xy$ in a graph G, we add arcs to the digraph D depending on whether $e \in E(H)$ or $e \notin E(H)$.

Let $X \subseteq V(G)$. By considering the contribution of each type of edge to the capacity of $\delta(X \cup \{s\})$, we calculate

$$\begin{aligned} c(\delta(X \cup \{s\})) &= \left(\tfrac{1}{2} + \tfrac{1}{2}\right) |E(X) \cap E(H)| + \tfrac{1}{2}|\delta_G(X) \cap E(H)| \\ &\qquad + \tfrac{1}{2}|E(X) \setminus E(H)| + \tfrac{1}{2}|E(V(G) \setminus X) \setminus E(H)| + |\delta_G(X) \setminus E(H)| \\ &= |E(X) \cap E(H)| + \tfrac{1}{2}|\delta(X) \cap E(H)| + \tfrac{1}{2}|\delta(X) \setminus E(H)| + \tfrac{1}{2}|E(G) \setminus E(H)| \\ &= |E(X) \cap E(H)| + \tfrac{1}{2}|\delta(X)| + \tfrac{1}{2}|E(G) \setminus E(H)| \\ &= p(X, H) + \tfrac{1}{2}|E(G) \setminus E(H)| \end{aligned}$$

Because the capacity of edge-cuts in a weighted digraph is a submodular function and $\frac{1}{2}|E(G) \setminus E(H)|$ is a constant, it follows that perimeter is submodular.

Given a graph G and subgraph H, the weighted digraph D used in the above proof can be constructed in linear time. Let $x \in V(G)$ and consider adding an arc of infinite capacity from s to x in D. In the resulting digraph D_x, a vertex-set $X \subseteq V(G)$ gives rise to a minimum capacity edge-cut $\delta(X \cup \{s\})$ if and only if it minimizes the perimeter $p(X, H)$ over all sets containing x in G. Therefore, we can find sets of minimum perimeter in G by constructing D_x and using an $O(nm)$-time algorithm for finding a minimum-capacity cut [91, 102]. This gives Theorem 4.2. □

4.2 Proof of Theorem 4.4

In this section, we prove Theorem 4.4 and discuss its corresponding algorithm. The proof comes in three parts.

4.2.1 The laminar family $\mathcal{L}_H$

Let G be a graph and H a fixed subgraph. For each $x \in V(G)$, choose a minimal vertex-set V_x containing x such that

$$p(V_x, H) = \min_{x \in X \subseteq V(G)} p(X, H).$$

Define $\mathcal{L}_H = \{V_x : x \in V(G)\}$. We claim that $\mathcal{L}_H$ is laminar; that is, for $V_x, V_y \in \mathcal{L}_H$, either $V_x \subseteq V_y$, $V_y \subseteq V_x$, or $V_x \cap V_y = \emptyset$.

Suppose $x \in V_y$. As V_x has minimum perimeter among sets containing x, we have $p(V_x, H) \leq p(V_y, H)$. Likewise, $p(V_x, H) \leq p(V_x \cup V_y, H)$. Substituting these into the submodular inequality of Theorem 4.1, we obtain $p(V_x \cap V_y, H) \leq p(V_x, H)$. The minimality of V_x means that every proper subset has greater perimeter, so $V_x \subseteq V_y$. Similarly, if $y \in V_x$ we obtain $V_y \subseteq V_x$.

On the other hand, suppose $y \notin V_x$ and $x \notin V_y$. An edge-counting argument yields

$$\begin{aligned}
p(V_x \setminus V_y, H) &= |E(V_x \setminus V_y) \setminus E(H)| + \tfrac{1}{2}|\delta(V_x \setminus V_y)| \\
&\leq |E(V_x) \setminus E(H)| + \tfrac{1}{2}|\delta(V_x \setminus V_y)| \\
&\leq |E(V_x) \setminus E(H)| + \tfrac{1}{2}|\delta(V_x)| + \tfrac{1}{2}|E(V_x \setminus V_y, V_x \cap V_y)| - \tfrac{1}{2}|E(V_y \setminus V_x, V_x \cap V_y)| \\
&= p(V_x, H) + \tfrac{1}{2}|E(V_x \setminus V_y, V_x \cap V_y)| - \tfrac{1}{2}|E(V_y \setminus V_x, V_x \cap V_y)|.
\end{aligned}$$

The same inequality is true with the roles of V_x and V_y reversed. Without loss of generality, assume $|E(V_x \setminus V_y, V_x \cap V_y) \leq |E(V_y \setminus V_x, V_x \cap V_y)|$; then the above calculation reduces to $p(V_x \setminus V_y, H) \leq p(V_x, H)$. But V_x was assumed to be a vertex-minimal set of minimum perimeter, so $V_x \setminus V_y$ cannot be a proper subset of V_x. In other words, $V_x \cap V_y = \emptyset$. □

4.2.2 The rooted forest F_H

It is well-known that laminar families can be represented by disjoint rooted trees [28, 102], but we provide a proof for the sake of completeness. Recursively construct F_H as follows. Begin with $V(F_0) = \emptyset$. Then, for each $i = 1, \ldots, |V(G)|$, choose a minimal vertex-set $V_x \in \mathcal{L}_H$ for which $x \notin V(F_{i-1})$. Let F_i be obtained from F_{i-1} by adding x as a parent to each rooted tree in F_{i-1} that intersects V_x. If V_x contains the entirety of each such tree, then the set of descendants of x in F_i is equal to $V_x \cap F_i$. This is clearly true when $i = 1$. The addition of x maintains this property: because V_x is minimal and $\mathcal{L}_H$ is laminar, every $V_y \in \mathcal{L}_H$ not already contained in $V(F_{i-1})$ is either disjoint from or a superset of V_x. Consequently, such a V_y contains all or none of the vertices of each tree F_i. Let $F_H = F_n$. By induction on i, the set of descendants in F_H of a vertex x is of the form $V_x \cap F_i$ where $V_x \in \mathcal{L}_H$ was chosen in step i of the above construction. It remains to show that each set of $\mathcal{L}_H$ can be represented this way.

Let $V_y \in \mathcal{L}_H$. Observe that $y \notin V_x$ for any $V_x \in \mathcal{L}_H$ that is a proper subset of V_y; otherwise, we would have a contradiction either to the vertex-minimality of V_y or the perimeter minimality of V_x. Consider the last step i in which a vertex $x \in V_y$ is added to F_i. Because $x \in V_x \cap V_y$ and $\mathcal{L}_H$ is laminar, we must have $V_x \subseteq V_y$; on the other hand, the choice of x and the laminarity of $\mathcal{L}_H$ means $y \in V_x$. We conclude that $V_x = V_y$, so V_y is represented in F_H as the set of descendants of x (which is an ancestor of y). □

4.2.3 The data structure f_H

For each $x \in V(F_H)$, define $f_H = p(X, H)$, where X is the set of x and all its descendants in F_H. The third statement of Theorem 4.4 then follows from parts (1) and (2), above. □

4.2.4 Algorithm

To construct $\mathcal{L}_H$, the above proofs tell us that we need only find, for each $x \in V(G)$, the (unique) vertex-minimal set containing x of minimum perimeter. In Section 4.1, we showed that sets of minimum perimeter correspond to minimum cuts in an auxiliary digraph. After finding a maximum flow in the digraph, a breadth- or depth-first search in the residual graph allows us to find not just a minimum cut but one whose associated vertex-set is minimal. Therefore, the laminar family $\mathcal{L}_H$ can be constructed in $O(n^2m)$-time by applying the algorithm of Theorem 4.2 to each of the n vertices in G.

Once $\mathcal{L}_H$ is found, F_H can be constructed in $O(n \log n)$ time by sorting the sets V_x in non-decreasing order and applying the recursive construction described in the second part of the above proof. Finally, f_H is generated by recording the perimeter of each set V_x, which can be done without penalty during the construction of $\mathcal{L}_H$. In total, the implementation of Theorem 4.4 runs in $O(n^2m)$ time.

4.3 Applications

In this section, we give two applications of our perimeter version of the Gomory–Hu Theorem. First, we describe a space-efficient data structure that holds approximate odd edge-connectivities for every pair of vertices in a graph. Then, we turn our attention to totally odd immersions and prove Theorem 4.5, a rough structure theorem for graphs with no totally odd immersion of a large complete graph.

4.3.1 A data structure approximating odd edge-connectivity for all pairs

Theorem 2.2 relates the odd edge-connectivity between two vertices u and v to the usual edge-connectivity $\lambda(u, v)$ and the perimeter of vertex-sets containing both vertices. Using Theorem 4.4, we can efficiently store all the perimeter values necessary to approximate the odd edge-connectivity between any two vertices in a graph.

Theorem 4.6. *Let G be a 2-edge-connected graph, H a maximum bipartite subgraph, and $\mathcal{L}_H = \{V_x : x \in V(G)\}$ the laminar family defined by Theorem 4.4. For all $u, v \in V(G)$ on opposite sides of H, we have*

$$\tfrac{1}{2}\lambda(u,v) \leq \lambda_o(u,v) \leq \lambda(u,v).$$

For all $u, v \in V(G)$ on the same side of H such that $v \in V_u$,

$$\min\left\{\frac{\lambda(u,v)-4}{6}, \frac{p(V_u,H)}{3}\right\} \leq \lambda_o(u,v) \leq \min\{\lambda(u,v), p(V_u,H)\}.$$

For all $u, v \in V(G)$ on the same side of H such that $v \notin V_u$ and $u \notin V_v$,

$$\frac{\lambda(u,v)-4}{6} \leq \lambda_o(u,v) \leq \lambda(u,v).$$

Proof. Clearly, $\lambda(u,v)$ is always an upper bound for $\lambda_o(u,v)$. If u and v are on opposite sides of the bipartition of H, then $\lambda_o(u,v) \geq \lambda_H(u,v) \geq \frac{\lambda_G(u,v)}{2}$ because the maximum bipartite subgraph H contains a majority of the edges of every minimum (u,v)-cut. If u and v are on the same side of the bipartition, we apply Theorem 2.4 with $k = \lambda_o(u,v) + 1$. If we obtain a (u,v)-cut with fewer than $6(\lambda_o(u,v)+1) - 2$ edges, it immediately follows that $\lambda_o(u,v) \geq \frac{\lambda(u,v)-4}{6}$. Otherwise, Theorem 2.4 guarantees the existence of a vertex-set X containing u and v with perimeter $p(X,H) \leq 3\lambda_o(u,v)$. If $v \notin V_u$ then

$$\lambda_o(u,v) \geq \frac{p(X,H)}{3} \geq \frac{p(V_u,H)}{3} \geq \frac{|\delta(V_u)|}{6} > \frac{\lambda(u,v)-4}{6}.$$

On the other hand, if $u, v \in V_u$, then $p(V_u,H)$ is an upper bound for $\lambda_o(u,v)$, while $\lambda_o(u,v) \geq \frac{p(X,H)}{3} \geq \frac{p(V_u,H)}{3}$ as before. This completes the proof. □

In Section 4.2 we described how to obtain the laminar family $\mathcal{L}_H$ used in Theorem 4.6 assuming H is already known. Unfortunately, it is NP-hard to find a maximum bipartite subgraph [60]. However, a version of Theorem 4.6 still holds (with different approximation factors) when H is replaced with an arbitrary maximal bipartite subgraph and Theorem 2.4 is replaced with Theorem 2.2 in the proof. In particular, the conclusion of Theorem 4.6 is true when H is a bipartite graph satisfying the following conditions:

(i) For every $u, v \in V(G)$ and every minimum (u,v)-cut $\delta(X)$ in G, H contains at least half of the edges in $\delta(X)$.

(ii) For every $u, v \in V(G)$ that are in the same part of the bipartition of H and have $\lambda(u,v) > 6\lambda_o(u,v) + 4$, there is a set X containing u, v with perimeter $p(X,H) \leq 3\lambda_o(u,v)$.

Only a maximum bipartite subgraph satisfies condition (i) for all vertex-sets in G. But since (i) is only required to hold for minimum (u, v)-cuts, it is possible to find a satisfactory H in polynomial time. As for (ii), the strategy used in Section 2.3 shows how to efficiently achieve this property: when finding a collection of edge-disjoint odd (u, v)-trails, we either find a set X satisfying (ii) for u and v, or we find a bipartite subgraph H with more edges—in which case we can repeat the algorithm from the beginning with the improved subgraph.

Theorem 4.7. *There is a polynomial-time algorithm which, given a 2-edge-connected graph G, constructs a bipartite subgraph H, a rooted forest F_H and a function f_H satisfying the conclusion of Theorem 4.6. Given F_H, f_H, the usual Gomory–Hu tree T, and associated cut-weight function c—which collectively take up $O(n)$-space—there is an $O(n)$-time 6-approximation algorithm to compute $\lambda_o(u, v)$ for any $u, v \in V(G)$.*

The construction of H discussed above runs the $O(nm^3)$-time algorithm described in Theorem 2.10 for each of the $O(n^2)$ pairs of vertices in G (persisting with the same bipartite subgraph H for each pair). In other words, constructing the space-efficient data structure (F_H, f_H)has the same $O(n^3m^3)$ time complexity as computing approximate odd edge-connectivities for all pairs of vertices in the graph. If, however, a bipartite subgraph satisfying the properties listed above could be quickly precomputed, Theorem 4.6 has the potential to provide significant efficiency gains: given the correct bipartite subgraph H, the theorem can be implemented in just $O(n^2m)$ time (see Section 4.2).

By carefully keeping track of the minimum cuts and minimum-perimeter sets considered during the execution of our H-improving strategy of Section 2.3, we may achieve some speedup in estimating the time complexity of the above methods. There may be other strategies for precomputing a suitable bipartite subgraph H; this question is discussed further in Chapter 6.

4.3.2 The rough structure of graphs with no totally odd K_t-immersion

In this section, we extend the argument used by DeVos et al. [24] to describe the rough structure theorem of graphs without an immersion of the complete graph K_t. Our extension describes the structure when we exclude totally odd K_t-immersions.

If $\mathcal{C}$ is a laminar family of subsets of a set V, we say that $\mathcal{C}$ **induces a partition of** V such that $x, y \in V$ are in the same part if and only if for every $C \in \mathcal{C}$, either $\{x, y\} \subseteq C$ or $\{x, y\} \cap C = \emptyset$.

Theorem 4.8. *Let G be a graph and let H be a maximum bipartite subgraph of G. There exists a laminar family of vertex-sets $\mathcal{C}'$ and a disjoint family of vertex-sets $\mathcal{L}'_H$ such that*

- $|\delta(X)| < 6t(t-1)$ *for all* $X \in \mathcal{C}'$*;*
- $p(Y, H) < \frac{3}{2}t(t-1)$ *for all* $Y \in \mathcal{L}'_H$*; and*

- *if a set Z in the partition of $V(G)$ induced by $\mathcal{C}'$ has at least t vertices and is not a subset of any $Y \in \mathcal{L}'_H$, then G contains a totally odd immersion of K_t with all of its root vertices in Z.*

It is not hard to see that this statement implies Theorem 4.5. To prove Theorem 4.8, we first observe the following sufficient conditions for a graph to have a totally odd immersion of K_t.

Claim 4.3.1. *Suppose G is the graph obtained from a star $K_{1,t-1}$ by replacing each edge with $t-1$ paths of any parity (where all $(t-1)^2$ of these paths are edge-disjoint) and adding $\binom{t}{2}$ loops on the center vertex of the star. Then G admits a totally odd immersion of K_t.*

Proof of the claim. Let v_0 be the center vertex and $v_1, \ldots, v_{t-1}$ the remaining vertices of the original star. Denote by $P_{i,1}, \ldots, P_{i,t-1}$ the $t-1$ edge-disjoint paths between v_0 and v_i for $i = 1, \ldots, t-1$, and let the $\binom{t}{2}$ loops on v_0 be labelled $\ell_{i,j}$ for $0 \leq i < j < t$. A totally odd immersion of K_t is constructed as follows. The root vertices of the immersion are $v_0, \ldots, v_{t-1}$. When $1 \leq i < j < t$, the (v_i, v_j)-trail of the immersion is obtained by concatenating $P_{i,j}$ and $P_{j,i}$, including the loop $\ell_{i,j}$ if necessary to fix the parity. Similarly, the (v_0, v_i)-trail of the immersion uses the path $P_{i,i}$ together with the loop $\ell_{0,i}$ if necessary to fix the parity. □

Claim 4.3.2. *Suppose G is a graph with a vertex-set $Z = \{v_0, \ldots, v_{t-1}\}$ such that v_0 has $\binom{t}{2}$ loops and $\lambda(v_i, v_j) \geq (t-1)^2$ for all $i, j \in \{0, \ldots, t-1\}$. Then G admits a totally odd immersion of K_t.*

Proof of the claim. G contains a subgraph of the type described in Claim 4.3.1. To see this, add a vertex x to G having $t-1$ parallel edges between x and each v_i for $i = 1, \ldots, t-1$. Applying Menger's Theorem to the resulting graph, we obtain $(t-1)^2$ edge-disjoint paths from v_0 to x, and therefore $t-1$ paths from v_0 to each other v_i. Otherwise, some cut of size $< (t-1)^2$ would separate some vertices of Z, in contradiction to their pairwise edge-connectivity. □

Claim 4.3.3. *Suppose G is a graph, H is a maximum bipartite subgraph of G, and $Z = \{v_0, \ldots, v_{t-1}\}$ is a vertex-set such that $\lambda(v_i, v_j) \geq 6t(t-1)$ for all $i, j \in \{0, \ldots, t-1\}$ and $p(X, H) \geq \frac{3}{2}t(t-1)$ for all vertex-sets X containing v_0. Then G admits a totally odd immersion of K_t.*

Proof of the claim. Apply Theorem 2.4 with $u = v = v_0$ to find $\frac{t(t-1)}{2} = \binom{t}{2}$ edge-disjoint closed trails through v_0, each using exactly one edge of $E(G) \setminus E(H)$. Let $F \subseteq E(G) \setminus E(H)$ be the edges used by such a collection of trails, and let H^+ be the graph obtained from H by first adding a new vertex z^+ and then adding edges xz^+, yz^+ for each $xy \in F$. A theorem of Mader [82, Corollary 1] shows us how to find a (v_0, z^+)-path in H^+ whose deletion decreases

$\lambda(v_0, z^+)$ by at most one and $\lambda(u, v)$ by at most two for all other $u, v \in V(H^+)$. Repeating this process, we obtain $t(t-1)$ edge-disjoint (v_0, z^+)-paths $\mathcal{P}$ in H^+ such that, for all i, j,

$$\begin{aligned}\lambda_{H-E(\mathcal{P})}(v_i, v_j) &= \lambda_{H^+-E(\mathcal{P})}(v_i, v_j) \\ &\geq \lambda_{H^+}(v_i, v_j) - 2t(t-1) \\ &\geq \lambda_H(v_i, v_j) - 2t(t-1) \\ &\geq \frac{\lambda_G(v_i, v_j)}{2} - 2t(t-1) \\ &\geq (t-1)^2\end{aligned}$$

In G, the paths $\mathcal{P}$ can be combined with the edges of F to create $\binom{t}{2}$ edge-disjoint odd closed (v_0, v_0)-trails. If we replace the edges of these trails with loops on v_0, the resulting graph satisfies the hypothesis of Claim 4.3.2 and therefore admits a totally odd immersion of K_t. A totally odd immersion in G is obtained by restoring the loops to the corresponding odd closed trails. □

We now prove Theorem 4.8. First, let $\mathcal{C}$ be the laminar family of vertex-sets described in the Gomory–Hu Theorem (Theorem 4.3). Take $\mathcal{C}' = \{X \in \mathcal{C} : |\delta(X)| < 6t(t-1)\}$. Next, let $\mathcal{L}_H$ be the laminar family of vertex-sets described in Theorem 4.4, and take $\mathcal{L}'_H$ to be the (vertex-disjoint) collection of maximal sets Y of $\mathcal{L}_H$ such that $p(Y, H) < \frac{3}{2}t(t-1)$. Suppose Z is some vertex-set in the partition associated with $\mathcal{C}'$ with at least t vertices. Any cut separating vertices of Z has at least $6t(t-1)$ edges. Thus $\lambda(v_i, v_j) \geq 6t(t-1)$ for all $v_i, v_j \in Z$. In particular, no vertex-set $Y \in \mathcal{L}'_H$ separates vertices in Z, because $|\delta(Y)| \leq 2p(Y, H) < 3t(t-1)$ is too small. Therefore, if $|Z| \geq t$ and no vertex-set $Y \in \mathcal{L}'_H$ contains Z, then $p(X, H) \geq \frac{3}{2}t(t-1)$ for all vertex-sets X intersecting Z. Claim 4.3.3 applies to find a totally odd immersion of K_t. This completes the proof of Theorem 4.8.

Chapter 5

Sufficient conditions for totally odd immersions of large cliques

If a graph has chromatic number t, must it contain (in some sense) the complete graph of order t? This central question of graph colouring theory has been asked in several ways, with varying degrees of success. Hajós conjectured that every t-chromatic graph has K_t as a subdivision, but was proven wrong in [12]. Hadwiger [51] suggested instead that every t-chromatic graph has K_t as a minor; this has not yet been decided one way or the other, and is now widely considered one of the deepest open problems of graph theory. Considerable progress has been made towards Hadwiger's Conjecture (see [105]) and its "odd" variant (see [44, 67, 70]), but the problem is still far from solved.

An illuminating first approach to Hajós's and Hadwiger's Conjectures is to test them on random graphs. Almost every graph in $\mathcal{G}_{n,p}$ has chromatic number $O\left(\frac{np}{\log np}\right)$ [6, 80]; in comparison, almost every graph in $\mathcal{G}_{n,1/2}$ has a has a minor of a complete graph on $\Omega(n/\sqrt{\log n})$ vertices [8], but only admits a subdivision of the complete graph on $\Theta(\sqrt{n})$ vertices [7, 30]. This means that Hajós's Conjecture fails for almost all graphs, while Hadwiger's Conjecture is true on average.

Because of the similarities between subdivisions, minors, and immersions, it is natural to consider the immersion version of the above conjectures.

Conjecture 5.1 (Lescure and Meyniel [77], Abu-Khzam and Langston [1])**.** *Every graph with chromatic number t admits an immersion of the complete graph K_t.*

As a random graph is expected to have a clique immersion of linear size, Conjecture 5.1 is also true on average. Indeed, almost all graphs in $\mathcal{G}_{n,p}$ have minimum degree $\approx pn$, so the minimum degree conditions in [22, 27, 76] imply an immersion of a complete graph with $\Theta(pn)$ root vertices. Here, we propose a totally odd variant of Conjecture 5.1.

Conjecture 5.2. *Every graph with chromatic number t admits a totally odd immersion of the complete graph K_t.*

In this chapter, we attack Conjecture 5.2 from two different angles. First, we test it on random graphs and show that it is true for almost all graphs.

Theorem 5.3. *Let $\epsilon \in (0,1)$ be constant and $p = p_n \in (0,1)$. Provided $p_n \in \omega(\sqrt{\log n}/n)$, almost every graph in $\mathcal{G}_{n,p}$ admits a totally odd strong immersion of the complete graph on $\lfloor \gamma pn \rfloor$ vertices, where*

$$\gamma = \frac{(1-\epsilon)\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}.$$

This gives a totally odd strong immersion of a complete graph on roughly $p^{3/2}n$ vertices, which is asymptotically larger than the chromatic number so long as p is not too small.

Corollary 5.4. *Let $p = p_n \in (0,1)$ satisfy $\limsup_{n\to\infty} p_n < 1$. If $p_n \in \omega\Big((\log n)^{-2/3}\Big)$, then almost every graph G in $\mathcal{G}_{n,p}$ admits a totally odd strong immersion of K_t, where t is the chromatic number of G.* □

Our proof of Theorem 5.3 is given in Section 5.1.2. Our arguments also apply to show that quasi-random graphs admit totally odd clique immersions of linear size; see Section 5.3.

In Section 5.2, we take a different approach to studying the relationship between totally odd immersions and graph colourings. It has already been observed that mere minimum degree conditions are enough to guarantee large clique immersions[1]. This is not the case for totally odd immersions: the complete bipartite graph $K_{n,n}$ admits a immersion of K_n despite its low chromatic number, but has no totally odd immersion of K_3.

With this in mind, it is interesting to see how the colouring structure of a graph can be used to find totally odd clique immersion. We present in Section 5.2 a simple argument to construct immersions whose root vertices have different colours under a particular kind of proper colouring, provided vertex pairs in the same colour class have many neighbours in common.

5.1 Clique immersions in random graphs

5.1.1 Concentration bounds

Before embarking on our proof of Theorem 5.3 about totally odd clique immersions in random graphs, it is helpful to make note of a few consequences of the Chernoff[2] bound. (See [87] for a more complete treatment of this technique.)

[1] This is not to say that the colouring structure is completely irrelevant for finding immersions: Kempe chain arguments can augment the known bounds to find immersions of somewhat larger cliques in t-chromatic graphs than would be expected from their minimum degree alone [76].

[2] Chernoff first published his eponymous bound in [14] but later credited Rubin with its discovery [13].

Theorem 5.5 (Chernoff [14])**.** *Let $\epsilon \in (0,1)$. If B is a random variable following a binomial distribution, then*

$$\mathbf{P}(B \leq (1-\epsilon)\mathbf{E}(B)) < e^{-c\mathbf{E}(B)},$$

where $c > 0$ depends only on ϵ. □

Corollary 5.6. *Let $\epsilon \in (0,1)$ be constant and $p = p_n \in (0,1)$. Almost every graph in $\mathcal{G}_{n,p}$ has minimum degree at least $(1-\epsilon)pn$.*

Proof. The expected degree of any vertex is pn, so by Theorem 5.5,

$$\begin{aligned}
\mathbf{P}(\delta(G) \leq (1-\epsilon)pn) &\leq \sum_{v \in V(G)} \mathbf{P}(\deg(v) \leq (1-\epsilon)pn) \\
&= \sum_{v \in V(G)} \mathbf{P}(\deg(v) \leq (1-\epsilon)\mathbf{E}(\deg(v))) \\
&< \sum_{v \in V(G)} e^{-c\mathbf{E}(\deg(v))} \\
&= \sum_{v \in V(G)} e^{-cpn} \\
&= ne^{-cpn} \to 0.
\end{aligned}$$

Thus almost every graph in $\mathcal{G}_{n,p}$ has minimum degree at least $(1-\epsilon)pn$. □

Corollary 5.7. *Let $\epsilon \in (0,1)$ be constant, $p = p_n \in (0,1)$, and $m = m_n \in \mathbb{Z}^+$. Provided $p_n m_n \in \omega(\log n)$, almost every graph in $\mathcal{G}_{n,p}$ is such that, for every pair of (not necessarily disjoint) vertex-sets X, Y with $|X|, |Y| = m$, there are at least $\frac{(1-\epsilon)pm^2}{2}$ edges with one end in X and the other in Y.*

Proof. For vertex-sets X, Y, let B_{XY} be the random variable counting the number of edges with one end in X and the other in Y. That is, let

$$B_{XY} = |E(X \setminus Y, Y \setminus X) \cup E(X \cap Y, X \triangle Y) \cup E(X \cap Y)|.$$

Observe that B_{XY} follows a binomial distribution with expectation

$$\begin{aligned}
\mathbf{E}(B_{XY}) &= p\left(|X \setminus Y||Y \setminus X| + |X \cap Y||X \setminus Y| + |X \cap Y||Y \setminus X| + \binom{|X \cap Y|}{2}\right) \\
&\geq \tfrac{p}{2}\Big(|X \setminus Y||Y \setminus X| + |X \cap Y||X \setminus Y| + |X \cap Y||Y \setminus X| + |X \cap Y|^2 - |X \cap Y|\Big) \\
&= \tfrac{p}{2}\Big(|X \cap Y| + |X \setminus Y|\Big)\Big(|X \cap Y| + |Y \setminus X|\Big) - \tfrac{p}{2}|X \cap Y| \\
&= \tfrac{p}{2}|X||Y| - \tfrac{p}{2}|X \cap Y| \\
&\geq \tfrac{p}{2}(m^2 - m) \\
&\geq \frac{(1-\epsilon')pm^2}{2}
\end{aligned}$$

where the last inequality holds for any constant $\epsilon' \in (0,1)$ provided n is large enough. In particular, we can choose ϵ' such that $(1-\epsilon')^2 = (1-\epsilon)$. Then the Chernoff bound in Theorem 5.5 controls the probability that the statement fails for some X, Y:

$$\begin{aligned}
\sum_{\substack{X,Y\subseteq V(G)\\ |X|,|Y|=m}} \mathbf{P}\left(B_{XY} \leq \frac{(1-\epsilon)pm^2}{2}\right) &= \sum_{\substack{X,Y\subseteq V(G)\\ |X|,|Y|=m}} \mathbf{P}\left(B_{XY} \leq \frac{(1-\epsilon')^2pm^2}{2}\right) \\
&\leq \sum_{\substack{X,Y\subseteq V(G)\\ |X|,|Y|=m}} \mathbf{P}\big(B_{XY} \leq (1-\epsilon')\mathbf{E}(B_{XY})\big) \\
&< \sum_{\substack{X,Y\subseteq V(G)\\ |X|,|Y|=m}} e^{-c\mathbf{E}(B_{uv})} \\
&\leq \sum_{\substack{X,Y\subseteq V(G)\\ |X|,|Y|=m}} e^{-c(1-\epsilon')pm^2/2} \\
&\leq \binom{n}{m}^2 e^{-c(1-\epsilon')pm^2/2} \\
&< e^{2m\log n - c(1-\epsilon')pm^2/2}
\end{aligned}$$

The exponent $2m\log n - c(1-\epsilon')pm^2/2$ tends to $-\infty$ as n increases because $p_n m_n \in \omega(\log n)$ and c, ϵ' are constant with respect to n. Therefore, the probability that the statement fails asymptotically approaches zero, and the claim holds for almost all graphs in $\mathcal{G}_{n,p}$. □

5.1.2 Proof of Theorem 5.3

We are now prepared to prove our main theorem about clique immersions in random graphs. Recall the statement of Theorem 5.3: we are asked to show that almost every graph in $\mathcal{G}_{n,p}$ admits a totally odd strong immersion of a complete graph on $\lfloor \gamma pn \rfloor$ vertices, where

$$\gamma = \frac{(1-\epsilon)\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}.$$

To this end, let R be a set of $\lfloor \gamma pn \rfloor$ vertices each with degree at least $(1-\epsilon)pn$; such a set exists by Corollary 5.6. We will construct an immersion with root vertices R. First, for every pair of adjacent $u, v \in R$, let the edge uv be the (u,v)-trail of the immersion. Then, for each nonadjacent pair $u, v \in R$, we obtain a (u,v)-trail of length three as follows.

Let X be a set of $m = \deg(u) - |R| \geq (1-\epsilon-\gamma)pn$ vertices x in $N(u) \setminus R$ for which the edge ux has not yet been used in the immersion. Likewise, let Y be a set of m vertices y in $N(v) \setminus R$ for which vy has not yet been used. Because $p < 1$, $\limsup_{n\to\infty} \gamma < 1$, and in turn $p_n m_n \in \Omega\big(p_n^2 n - \gamma p_n^2 n\big) \subseteq \Omega\big(p_n^2 n\big) \subseteq \omega(\log n)$. We may therefore apply Corollary 5.7; setting

$$B_{XY} = |E(X \setminus Y, Y \setminus X) \cup E(X \cap Y, X \bigtriangleup Y) \cup E(X \cap Y)|,$$

we find

$$
\begin{aligned}
B_{XY} &\geq \frac{(1-\epsilon)pm^2}{2} \\
&\geq \frac{(1-\epsilon)p(1-\epsilon-\gamma)^2p^2n^2}{2} \\
&= \frac{p^2n^2}{2}\left((1-\epsilon-\gamma)\sqrt{p(1-\epsilon)}\right)^2 \\
&= \frac{p^2n^2}{2}\left(\left(1-\epsilon-\frac{(1-\epsilon)\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}\right)\sqrt{p(1-\epsilon)}\right)^2 \\
&= \frac{p^2n^2}{2}\left(\left(1-\frac{\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}\right)(1-\epsilon)\sqrt{p(1-\epsilon)}\right)^2 \\
&= \frac{p^2n^2}{2}\left(\left(\frac{1+\sqrt{p(1-\epsilon)}-\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}\right)(1-\epsilon)\sqrt{p(1-\epsilon)}\right)^2 \\
&= \frac{p^2n^2}{2}\left(\frac{(1-\epsilon)\sqrt{p(1-\epsilon)}}{1+\sqrt{p(1-\epsilon)}}\right)^2 \\
&= \frac{(\gamma pn)^2}{2}
\end{aligned}
$$

In other words, there are at least $\binom{\lfloor \gamma pn \rfloor}{2}$ edges with one end in X and the other in Y. Consequently, at least one such edge is unused by $E(\mathcal{T})$, since each trail in $\mathcal{T}$ uses at most one edge in $E(V(G) \setminus R)$. So there is a path $uxyv$ in $G - E(\mathcal{T})$, where $x \in X$ and $y \in Y$. We add this path to $\mathcal{T}$ as the (u, v)-trail in the immersion. After repeating this procedure for every nonadjacent pair u, v, we have constructed the trails $\mathcal{T}$ of a totally odd immersion of $K_{\gamma pn}$ with root vertices R. This completes the proof of Theorem 5.3. □

5.2 Forcing clique immersions with colourings and common neighbourhoods

The following results illustrate how the colouring structure of a graph can be used to find totally odd clique immersions. In fact, our simple methods find a very special type of immersion: a strong immersion in which all of the trails are paths of length 1 or 3.

Theorem 5.8. *Let G be a graph, let $r_1, \dots, r_\ell$ and $x_1, \dots, x_\ell$ be distinct vertices in G. Suppose there are $\binom{\ell}{2}$ edges e_{ij}, $1 \leq i < j \leq \ell$, such that:*

(1) *$e_{ij} \in \{r_ir_j, r_ix_j, x_ir_j\}$ for all $1 \leq i < j \leq \ell$;*

(2) *$e_{ij} = r_ir_j$ whenever r_i and r_j are adjacent; and*

(3) *for every* $I \subseteq \{1, \ldots, \ell\}$,

$$\sum_{i \in I} (|N(r_i) \cap N(x_i)| - \deg_H(r_i)) \geq 2\binom{|I|}{2} - 2|E(\{r_i : i \in I\})|$$

where H *denotes the subgraph of* G *with* $E(H) = \{e_{ij} : 1 \leq i < j \leq \ell\}$

Then G *admits a totally odd strong immersion of* K_ℓ *with root vertices* R.

Proof. Write $R = \{r_1, \ldots, r_\ell\}$. If $G[R]$ is a clique, we are done. Otherwise, if $r_i, r_j \in R$ and (say) $e_{ij} = r_i x_j$, we would like to find a vertex $y_{ij} \in N(r_j) \cap N(x_j)$ such that $r_j y_{ij} \notin E(H)$; if so, we might use the odd path $r_i x_j y_{ij} r_j$ as the trail connecting r_i and r_j in the immersion. In fact, for a fixed i, j, there are at least $|N(r_j) \cap N(x_j)| - \deg_H(r_j)$ choices for such a y_{ij}.

To construct our immersion, we have to simultaneously choose y_{ij} as above for every nonadjacent pair $r_i, r_j \in R$, ensuring that no edge $x_j y_{ij}$ is enlisted twice. (If so, it is clear that the edges $r_j y_{ij}$ will also be distinct.) To this end, construct an auxiliary bipartite graph, with bipartition (X, Y), as follows. The vertex-set X consists of ordered pairs (r_i, r_j) of nonadjacent vertices in R, where the order of each pair is chosen according to which vertex is an end of e_{ij} (or e_{ji}, as appropriate). To be precise, $(r_i, r_j) \in X$ whenever $r_i x_j \in E(H)$. On the other side of the bipartition, the vertex-set Y consists of unordered pairs of vertices $\{x_j, y\}$ for which $y \in V(G) \setminus R$ is a common neighbour of r_j and x_j in $G - E(H)$. Finally, the edges of the auxiliary graph connect the vertices $(r_i, r_j) \in X$ to $\{x_j, y\} \in Y$ if and only if $r_i x_j y r_j$ is a path in G. As it turns out, a matching in this auxiliary graph corresponds to an edge-disjoint collection of paths in G.

To construct our immersion, we apply Hall's Theorem [52] (see also [9]) to find a matching saturating X. For any $S \subseteq X$, let $I_S = \{i, j : (r_i, r_j) \in S\}$. The size of the neighbourhood of S in the auxiliary graph is dependent only on I_S, and is greater than or equal to $\frac{1}{2} \sum_{i \in I_S} (|N(r_i) \cap N(x_i)| - \deg_H(r_i))$. By hypothesis, this number is at least $\binom{|I|}{2} - |E(\{r_i : i \in I\})| \geq |S|$, so Hall's Theorem gives us a matching saturating X. In G, this corresponds to an edge-disjoint collection of (odd) paths connecting the nonadjacent pairs r_i, r_j of R. Together with the edges of $G[R]$, these paths form the trails of an immersion of K_ℓ with root vertices R. □

Although the edges e_{ij} in the above theorem seem a little mysterious, they are very handy for finding totally odd clique immersions in graphs where special colourings are known. A **Grundy colouring** of a graph G is a proper colouring with colours $1, 2, \ldots, \ell$ such that for every vertex x the colour of x is the smallest colour not used by any neighbour of x; see [49, 58]. In other words, a Grundy colouring partitions the vertices of the graph into independent sets $X_1, \ldots, X_\ell$ such that every vertex in X_i has neighbours in $X_1, \ldots, X_{i-1}$. The **Grundy number** $\Gamma(G)$ is the largest number ℓ for which G has a Grundy ℓ-colouring; it is not hard to see that $\Gamma(G) \geq \chi(G)$.

Corollary 5.9. *If G is a graph with Grundy number $\Gamma(G)$ and no independent set of size 3, such that every pair of non-adjacent vertices has at least $\ell \leq \Gamma(G)$ common neighbours, then G admits a totally odd immersion of the complete graph K_ℓ.*

Proof. Let $X_1, \ldots, X_\ell$ be the colour classes in a Grundy colouring of G and let $R = \{r_1, \ldots, r_\ell\}$ be any set of differently-coloured vertices (setting $r_i \in X_i$ for $i = 1, \ldots, \ell$). Since the independence number of G is at most two, each $|X_i| \leq 2$; for every $i = 1, \ldots, \ell$ having $|X_i| = 2$, let $X_i = \{r_i, x_i\}$.

By the definition of Grundy colourings, each $r_j \in R$ is incident with an edge $e_{ij} \in \{r_i r_j, x_i r_j\}$. We may assume that e_{ij} is chosen from $G[R]$ whenever possible, in order to satisfy the hypotheses of Theorem 5.8. Finally, for all $I \subseteq \{1, \ldots, \ell\}$,

$$\begin{aligned}
\sum_{i \in I} (|N(r_i) \cap N(x_i)| - \deg_H(r_i)) &\geq \sum_{i \in I} (\ell - i + 1) \\
&\geq \sum_{j=1}^{|I|} j \\
&\geq \frac{|I|(|I|+1)}{2} \\
&\geq 2\binom{|I|}{2} - 2|E(\{r_i : i \in I\})|
\end{aligned}$$

where the last inequality follows from Turán's theorem [84, 113]; see also [9]. The result then follows from Theorem 5.8. □

5.3 Concluding remarks

In Corollary 5.9, we gave a sufficient condition for a graph with independence number 2 to have a totally odd immersion of $K_{n/2}$ based on the number of common neighbours of any pair of nonadjacent vertices. As it turns out, this common-neighbours condition is tight.

Fact 5.10. *There exists:*

- *a graph on $n = 4k$ vertices, no independent set of size 3, and such that every pair of non-adjacent vertices has at least $2k - 2 = \frac{n-1}{2}$ common neighbours,*
- *a Grundy colouring of the graph of order $\frac{n}{2}$, and*
- *and a set R of $\frac{n}{2}$ vertices of different colours,*

such that there is no totally odd immersion of $K_{n/2}$ with root vertices R.

Proof. Let G_k be a graph with vertices $\{u_1, \ldots, u_k\} \cup \{v_1, \ldots, v_k\} \cup \{u'_1, \ldots, u'_k\} \cup \{v'_1, \ldots v'_k\}$ such that $u_i u'_i$, $v_i v'_i$, $u_i v_j$, and $u'_i v'_j$ are *not* edges for any $i, j = 1, \ldots, k$, but all other

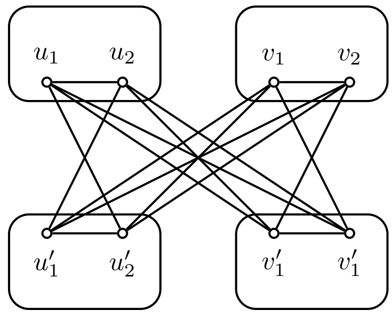

Figure 5.1: A graph with $\chi(G) = |V(G)|/2$ and a set $\{u_1, u_2, \dots, v_1, v_2, \dots\}$ of differently-coloured vertices which are not the root vertices of any totally odd immersion of $K_{|V(G)|/2}$.

possible edges in the graph are present. (The graph G_2 is illustrated in Figure 5.1.) Any nonadjacent vertices in G_k have at least $2k-2$ common neighbours, so G_k only narrowly misses the common-neighbours condition in Corollary 5.9 for $\ell = 2k = \chi(G_k) \leq \Gamma(G_k)$. However, there is no totally odd immersion of the complete graph K_{2k} with root vertices $R = \{u_1, \dots, u_k, v_1, \dots, v_k\}$: each trail of such an immersion would contain an edge from either $E(R)$ or $E(V(G) \setminus R)$, but there are only $4\binom{k}{2} < \binom{2k}{2}$ such edges. □

It is worth noting that a more general version of Corollary 5.9 holds; we can extend the two-vertex sets $\{r_i, x_i\}$ to colour classes of arbitrary size—replacing Hall's Theorem in the proof of Theorem 5.8 with Haxell's theorem on independent transversals [53]— as long as we demand more common neighbours between vertices in the same colour class. This can also be used to prove that almost every graph in $\mathcal{G}_{n,p}$ satisfies Conjecture 5.2: almost every graph is such that every pair of vertices has a linear number of common neighbours, which is asymptotically larger than the chromatic number. However, the methods of Section 5.1.2 generally produce clique immersions of larger complete graphs.

Theorem 5.3 can also be applied in more broad circumstances than its statement. As our proof only depends on Corollary 5.6 and Corollary 5.7, we can use the same argument to find a large totally odd clique immersion in any graph having sufficiently many vertices close to the average degree and sufficiently many edges between any pair of vertex-sets of the appropriate size. These properties are fulfilled by any quasi-random family of dense graphs.

Definition (see [18, 74, 108])**.** *Let $(G_n) = (G_1, G_2, \dots)$ be a sequence of graphs where each G_n has n vertices. Let $p \in (0,1)$ be fixed. We call (G_n) a **quasi-random** sequence if, each subset $U \subseteq V(G_n)$, $|E(U)| = \frac{p}{2}|U|^2 + o(n^2)$.*

A quasi-random sequence of graphs satisfies analogues of Corollary 5.6 and Corollary 5.7.

Fact 5.11. *Let $\epsilon \in (0,1)$ and $p \in (0,1)$ be constant. Then in a quasi-random sequence (G_n), each graph G_n has a set of $\Omega(pn)$ vertices of degree at least $(1-\epsilon)pn$.* □

Fact 5.12. *Let $p \in (0,1)$ be constant. Then in a quasi-random sequence (G_n), for each graph G_n and $X, Y \subseteq V(G_n)$ of size $|X| = |Y| \in \Omega(n)$, there are $\Omega(pn^2)$ edges with an end in X and the other in Y.*

Proof. The number of such edges is bounded below by $|E(X \cap Y)| = \frac{p}{2}|X \cap Y|^2 + o(n^2)$ and by $|E(X \setminus Y, Y \setminus X)| = |E(X \cup Y)| - |E(X \setminus Y)| - |E(Y \setminus X)| = \frac{p}{2}|X \setminus Y||Y \setminus X| + o(n^2)$. Either $|X \cap Y|$ has linear size or $|X \setminus Y|, |Y \setminus X|$ both do. Therefore, at least one of these quantities is $\Omega(pn^2)$. □

A quasi-random version of Theorem 5.3 can be obtained by inserting these facts into the proof in place of Corollary 5.6 and Corollary 5.7.

Theorem 5.13. *Let $p \in (0,1)$ be constant and let (G_n) be a quasi-random sequence of graphs with edge-density p. Then each graph G_n admits a totally odd strong immersion of a complete graph with $\Omega(pn)$ vertices.* □

Chapter 6

Open problems

In this thesis, we proved two packing–covering dualities for odd trails, introduced the perimeter measure, and found sufficient conditions for a graph to have a totally odd immersion of a large clique. These accomplishments bring us closer to understanding odd trails and totally odd immersions, but we have only scratched the surface. In this chapter, we review the main results of our work and recommend a number of open problems in this area.

6.1 Packing edge-disjoint odd (u, v)-trails

In Chapter 2 and Chapter 3, we showed how the odd edge-connectivity between two vertices can be approximated in terms of edge-connectivity and the minimum perimeter among sets containing u and v. (See Figure 6.1.)

However, our results are not known to be tight. For general graphs, the most extreme examples we know of (see Section 2.4) have

$$\lambda(u, v) = \min_{u,v \in X} p(X, H) = 2\lambda_o(u, v).$$

Question 6.1. *What is the minimum constant c such that $\lambda_o(u, v) \leq \tau_o(u, v) \leq c\lambda_o(u, v)$ for all graphs G and vertices u, v?*

We know, by our results in Chapter 2, that $2 \leq c \leq 6$; Ibrahimpur [56] has recently improved on the upper bound, proving that $\tau_o(u, v) \leq 5\lambda_o(u, v) + 2$. Similarly, the worst known Eulerian family (see Section 3.4) leaves a gap with Theorem 3.2:

$$\lambda(u, v) = \min_{u,v \in X} p(X, H) = 1.5\lambda_o(u, v).$$

Question 6.2. *What is the minimum constant c such that $\lambda_o(u, v) \leq \tau_o(u, v) \leq c\lambda_o(u, v)$ for all Eulerian graphs G and vertices u, v?*

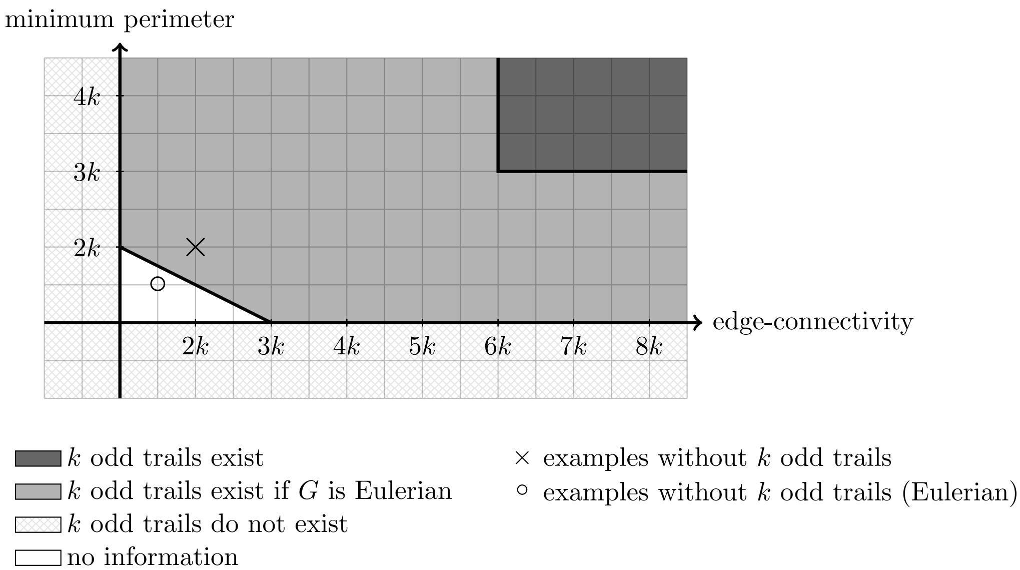

Figure 6.1: The existence of disjoint odd (u, v)-trails depends on $\lambda(u, v)$ and $\min p(X, H)$. Theorem 2.4 applies in the darkest region and Theorem 3.2 in the light grey region. The extreme examples are described in Section 2.4 and Section 3.4.

Our results in Chapter 3 show that $1.5 \leq c \leq 2.5$.

This thesis presents several polynomial-time algorithms (see Theorems 2.3, 3.3, and 4.3.1) for approximating the odd edge-connectivity between two vertices. It would be interesting to see if any of them can be refined to an exact algorithm.

Question 6.3. *Is there a polynomial-time algorithm to compute $\lambda_o(u, v)$ exactly?*

As we remarked in Chapter 4, the maximum bipartite subgraph used in Theorem 2.4 and Theorem 4.6 is NP-hard to find [60]. Our algorithms work around this by using an arbitrary bipartite subgraph H and restarting whenever we finds a way to increase the number of edges in H, which adds an extra factor of m to the runtime.

The reason Theorem 2.4 needs a maximum bipartite subgraph is to guarantee $\frac{|\delta_G(X)|}{|\delta_H(X)|} \leq 2$ for all $X \subseteq V(G)$, a condition which is equivalent to being maximum.

Fact 6.4. *A bipartite subgraph H of a graph G is maximum if and only if $\frac{|\delta_G(X)|}{|\delta_H(X)|} \leq 2$ for all $X \subseteq V(G)$.*

Proof. We have already remarked that $|\delta(A \triangle B)| = |\delta(A)| + |\delta(B) \setminus \delta(A)| - |\delta(B) \cap \delta(A)|$ implies $\frac{|\delta_G(X)|}{|\delta_H(X)|} \leq 2$ when $E(H) = \delta(A)$ is the edge-set of a maximum bipartite subgraph. Conversely, if we arrange for $\delta(A \triangle B)$ to be maximum and we assume $\frac{|\delta_G(X)|}{|\delta_H(X)|} \leq 2$, then the fact that $|\delta(B) \setminus \delta(A)| \leq |\delta(B) \cap \delta(A)|$ implies $|E(H)| = |\delta(A)| \geq |\delta(A \triangle B)|$. □

However, if we could quickly precompute a bipartite subgraph H containing a constant fraction of edges from each cut, we could greatly improve the runtime of our algorithms in

Section 2.3 and especially in Section 4.3.1 (at the cost of having to adjust the appropriate approximation ratios). It is well-known that a bipartite subgraph can easily be constructed with at least half the edges of G, and there is a polynomial-time algorithm to find a bipartite subgraph with at least $\approx 87.8\%$ of the maximum number of edges [47], but these approximations may not be uniform over all cuts in G.

Question 6.5. *Is there a constant c for which there exists a polynomial-time algorithm to find a bipartite subgraph H of G such that $\frac{|\delta_G(X)|}{|\delta_H(X)|} \leq c$ for all $X \subseteq V(G)$?*

6.2 Sufficient conditions for totally odd clique immersions

In Chapter 5, we proposed a totally odd immersion variant of Hadwiger's Conjecture:

Conjecture 6.6. *Every graph with chromatic number t admits a totally odd immersion of K_t.*

Although we have made some partial progress towards this conjecture (see Theorem 4.5, Theorem 5.3, and Figure 5.1), many interesting and realistically-answerable questions remain. Our study of totally odd clique immersions in random graphs uses very crude methods—for example, the immersions produced only use paths of length one or three—and if longer trails are taken into account, it is probable that sparse random graphs could admit immersions of larger cliques.

Question 6.7. *Is there a constant $\alpha \in (0,1)$ such that almost every graph in $\mathcal{G}_{n,p}$ admit a totally odd immersion of $K_{\alpha pn}$?*

A less greedy method of choosing short trails may also be possible. Indeed, the uniform edge-connectivity and "non-bipartiteness" of random graphs makes it plausible that they could contain totally odd immersions of any complete graph whose vertex degrees fit. If so, it may be possible to improve our result for $\mathcal{G}_{n,1/2}$.

Fact 6.8. *Let $\epsilon \in (0,1)$. Almost all graphs in $\mathcal{G}_{n,1/2}$ admits a totally odd strong immersion of a complete graph on $(1-\epsilon)\frac{n}{2(1+\sqrt{2})} \approx 0.2071n$ vertices.*

Proof. Let $\epsilon' \in (0,1)$ be such that $(1-\epsilon) \leq (1-\epsilon')^{3/2}$ Apply Theorem 5.3 with this ϵ'. Almost all graphs in $\mathcal{G}_{n,p}$ admit a totally odd strong immersion of a complete graph on γpn vertices where

$$\gamma pn = (1-\epsilon')^{3/2} \frac{np\sqrt{p}}{1+\sqrt{p(1-\epsilon)}} \geq (1-\epsilon)\frac{np\sqrt{p}}{1+\sqrt{p}}.$$

When $p = 1/2$, this simplifies to $\gamma pn \geq \frac{n}{2(1+\sqrt{2})}$ □

Question 6.9. *For every $\epsilon > 0$, does almost every graph in $\mathcal{G}_{n,1/2}$ admit a totally odd immersion of $K_{(\frac{1}{2}-\epsilon)n}$?*

If the answer to Question 6.9 is "yes", the result would be very tight: a totally odd immersion of $K_{n/2}$, if it exists, uses almost every edge on average. Since there are $\frac{1}{2}\binom{n/2}{2}$ expected non-edges among a vertex-set of size $n/2$, a totally odd clique immersion on those root vertices is expected to use at least $\left(\frac{1}{2}+\frac{3}{2}\right)\binom{n/2}{2} = \frac{n(n-2)}{4}$ edges. In comparison, the expected number of edges in $\mathcal{G}_{n,1/2}$ is $\frac{n(n-1)}{4}$.

Following the extensive literature on graph minors [26, 66], Vergara [114, 115] studied clique immersions in dense graphs and showed that every n-vertex graph with no independent set of size 3 admits an immersion of $K_{n/3}$. It follows from our Corollary 5.9 that such an immersion can be chosen to be totally odd. More recently, Gauthier and Wollan [42] have shown that these graphs admit complete graph immersions on two-fifths of their vertices.

Question 6.10. *Does every n-vertex graph with no independent set of size 3 admit a (totally odd) immersion of $K_{n/2}$?*

Finally, it remains to be seen if a totally odd immersion of K_t is guaranteed by some variant of the obvious necessary conditions for its existence. This question justifies further exploration of odd edge-connectivity and the world of edge-disjoint odd trails.

Question 6.11. *If a graph has a set of t vertices R such that $\lambda_o(S, R\setminus S) \geq s(t-s)$ for every s-vertex subset $S \subseteq R$, must there be a totally odd immersion of K_t with root vertices R?*

6.3 Further questions

Let G be an n-vertex graph, possibly with parallel edges. Although there are $\binom{n}{2}$ distinct pairs of vertices in G, the Gomory–Hu Theorem [48] implies that there are at most $n-1$ distinct sizes among minimum (u,v)-cuts—and therefore at most $n-1$ distinct edge-connectivities. In comparison, directed graphs may have up to $\frac{(n+2)(n-1)}{2}$ distinct minimum cut values [40, 50]. Although we proved an odd trails variant of the Gomory–Hu Theorem in Chapter 4, we do not have as precise control over the exact odd edge-connectivities.

Question 6.12. *Among the $\binom{n}{2}$ pairs of vertices in a graph G, how many distinct values can $\lambda_o(u,v)$ take on?*

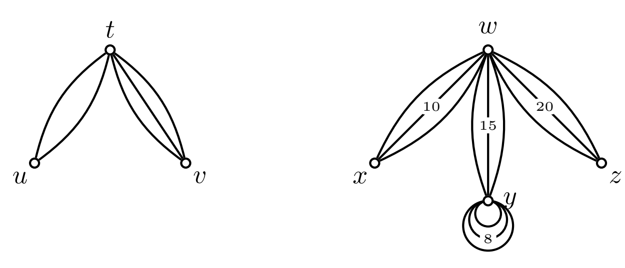

Figure 6.2: Graphs with more than $n-1$ distinct odd edge-connectivities; labels denote the number of parallel edges.

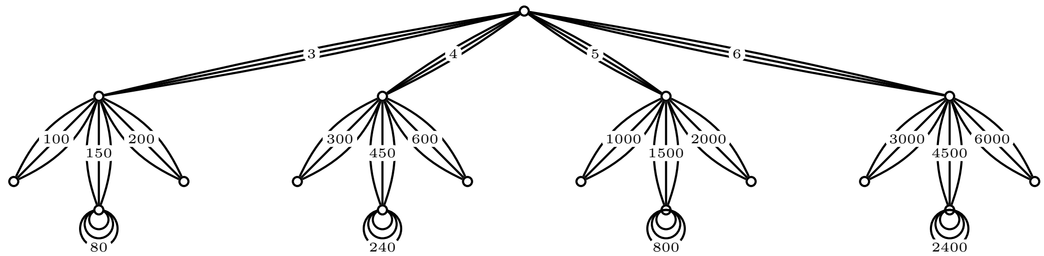

Figure 6.3: A graph with $1.5(n-1)$ distinct pairwise odd edge-connectivities.

Since a connected bipartite graph can have vertices with zero odd edge-connectivity, it is obvious that a graph can have more distinct odd edge-connectivities than edge-connectivities. The 3-vertex example on the left of Figure 6.2, for instance, has three distinct odd edge-connectivities: $\lambda_o(t,u) = 2$, $\lambda_o(t,v) = 3$, and $\lambda_o(u,v) = 0$. It is perhaps not so obvious that an n-vertex graph can have more than n distinct odd edge-connectivities, but this also true: the 4-vertex graph on the right of Figure 6.2 has $\lambda_o(w,x) = 10$, $\lambda_o(w,y) = 15$, $\lambda_o(w,z) = 20$, $\lambda_o(x,y) = \lambda_o(y,z) = 8$, and $\lambda_o(x,z) = 7$, for a total of five different values. Generalizing to an infinite family of graphs, a typical example of which is illustrated in Figure 6.3, we find that a graph can have almost three-halves as many odd edge-connectivities as vertices.

6.4 Concluding remarks

In this thesis, we introduced the odd edge-connectivity $\lambda_o(u,v)$ between vertices of a graph. We proved an approximate packing–covering duality for odd (u,v)-trails, achieved a better approximation ratio in the case of Eulerian graphs, and described several polynomial-time approximation algorithms for $\lambda_o(u,v)$.

We also experimented with a totally odd immersion version of Hadwiger's Conjecture. We showed that random graphs admit totally odd immersions of a clique of linear size, and presented several sufficient conditions for a graph to have a totally odd K_t-immersion. Our unsophisticated bounds are tight in the case of $\overline{K_3}$-free graphs if we want to be able to choose the root vertices of the immersion.

Our most important contribution is the definition of perimeter, a submodular function that is closely related to the odd edge-connectivity. The perimeter is at the heart of both of our approximate duality theorems, our rough structure theorem for totally odd clique immersion, and all of our algorithmic results. We expect many more interesting structural and algorithmic applications of the perimeter measure and its submodularity.

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