ASSET VALUATION OPERATORS
WITH DIFFUSION PROCESSES

by

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Asset Valuation Operators with Diffusion Processes

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ABSTRACT

Heaney and Garman develop a linear valuation operator which prices risky income streams when arbitrage profits are precluded. Both study the case where the states of nature are presumed to follow a diffusion process over the real line; each developing a differential equation involving the values (prices) of assets, as a function of the underlying states and time, dividends to these assets and the valuation operator.

It is shown that the differences in the developments of these two equations - arising partially from different definitions of diffusion processes - are more apparent than real. These differences in derivation are only changes in the order that the steps are performed, not the application of different assumptions.

Further, Heaney's differential equation, which governs the valuation operator for all times and states, is shown to hold only when a certain consistency condition is satisfied. Requiring this condition to be satisfied restricts the class of accepted no-arbitrage economies, but allows the valuation operator to be obtained from Heaney's equation.

Lastly the effect of barriers to the diffusion process is investigated.
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1. Introduction.

In this paper certain results of Heaney [6] are explored. These results are compared and contrasted with the work of Garman [3], [4], [5]. It is shown that in some sense Heaney's work is a generalization of Garman's and in another sense Heaney's work is a specialization of Garman's. It is also shown that this relationship is due to the methods of derivation of Heaney's and Garman's results. In fact the difference in derivation is more of the form of a difference in the order of application of formulae than a fundamental difference in approach.

The problem is considered by both Heaney and Garman is that of asset valuation when the underlying states of the world follow a diffusion process. Heaney defines a diffusion process as a process with probability density function satisfying the Fokker-Planck equation for particular drift and instantaneous variance. Garman, on the other hand, uses the Lindeberg type conditions to define the drift and variance of the diffusion process. Heaney thus deals with a differential equation; whereas Garman is working from integral expressions.

Before the nature of the process is considered, both Garman and Heaney consider an Arrow-Debrue state space setting, where no arbitrage opportunities exist. In the discrete state case the Farkas-Minkowski lemma gives the existence of an operator, say K, which relates future prices to current prices. The Farkas-Minkowski lemma and the state space are then extended to continuous states. The operator K is decomposed into the product of the probability density function of the underlying state transition
process and a second function, say $G$. $G$ is interpreted as the generalized discount operator relating current time and state to future times and states. That is $G$ is the marginal rate of substitution between the current state and some future state at a particular future time.

When the transition process for the underlying states is particularized to diffusion processes Heaney's and Garman's developments diverge. Heaney attempts to develop a differential equation governing the operator $G$, subject to the current prices of the primary assets. This equation is to hold for all future times and states. Garman develops an equation governing current asset values where derivatives of $G$ evaluated at the current time and state are found. Garman makes no attempt to define $G$ but rather works around it, using its existence to develop equations involving the asset values. In fact in some cases $G$ and its derivatives are eliminated entirely from the governing equations.
2. Derivation of Garman's "Universal Differential Equation":

From his "Intertemporal Parity Principle" Garman deduces that to ensure there are no riskless arbitrage possibilities the following equation must hold:

\[
V_j(p_0, t_0) = \int_{\{p\}} \int_{\tau \geq t_0} \left\{ h_t(\tau) d_j(\pi, \tau) + \delta_t(\tau) V_j(\pi, \tau) \right\} G(\pi, \tau, p_0, t_0) q(\pi, \tau; p_0, t_0) \, d\pi d\tau
\]

for all \( t \geq t_0 \). \hspace{1cm} (1)

where \( p_0 \) is the initial state at time \( t_0 \); \( p \) is the state at time \( t \);

- \( h_t(\tau) \) is the Heavyside Step Function, \( h_t(\tau) = \begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases} \),

- \( \delta_t(\tau) \) is the Dirac Delta function about \( \tau = t \),

- \( d_j(p, t) \) is the "natural payoff stream" or dividend stream of an asset \( A_j \).

- \( V_j(p, t) \) is the value of asset \( A_j \) at time \( t \) if the state \( p \) occurs.

- \( q(p, t; p_0, t_0) \) is the transition density function relating the probability of starting in state \( p_0 \) at time \( t_0 \) and arriving at state \( p \) at time \( t \).
- \( G(p,t; p_0,t_0) \) is the "observer specific quotient kernel";
that is \( G \) essentially represents an economy-wide marginal rate
of substitution between values in the current state of the world
at current times \([t_0]\) and a future state of the world at
state \([p]\) occurring at future time \([t]\).

\( q \) is the transition density function posited for the stochastic
process \( \tilde{p} \) by an observer. \( \tilde{p} \) is assumed to be some diffusion process
on the real line.

To derive his universal or fundamental differential equation
Garman utilized the Lindeberg type conditions\(^1\) governing or defining
the diffusion process. These are:

\[
\int_{|p-p_0| \geq \varepsilon} q(p,t;p_0,t_0) \, dp = o(t-t_0), \tag{2}
\]

\[
\int_{|p-p_0| < \varepsilon} (p-p_0)q(p,t;p_0,t_0) \, dp = \alpha(p_0,t_0)(t-t_0) + o(t-t_0), \tag{3}
\]

\[
\int_{|p-p_0| < \varepsilon} (p-p_0)^2 q(p,t;p_0,t_0) \, dp = \sigma^2(p_0,t_0)(t-t_0) + o(t-t_0). \tag{4}
\]

Equation (2) serves as a continuity condition on the process. Equations
(3) and (4) define the instantaneous drift of the process, \( \alpha \), and the
instantaneous volatility or variance, \( \sigma^2 \), of the process.

It is important to note that (2), (3) and (4) are conditions
which hold locally about \((p_0,t_0)\). That is they relate the present
state and time to close or nearby future states and times.

By taking Taylor series expansions of $G$, $d$ and $V$ about $p = p_0$, $t = t_0$ in (1) and letting $(p, t)$ tend to $(p_0, t_0)$, making use of (2), (3) and (4), Garman obtains his "Fundamental Differential Equations":

\[
d_j(p_0', t_0') + \frac{\partial}{\partial t} V_j(p, t) \bigg|_{p_0, t_0} + \alpha(p_0, t_0) \frac{\partial}{\partial p} V_j(p, t) \bigg|_{p_0, t_0}
\]

\[+ \frac{1}{2} \sigma^2(p_0, t_0) \frac{\partial^2}{\partial p^2} V_j(p, t) \bigg|_{p_0, t_0} + V_j(p_0, t_0') \left[ \frac{\partial}{\partial t} G(p, t; p_0, t_0) \bigg|_{p_0, t_0} \right]
\]

\[+ \alpha(p_0, t_0) \frac{\partial}{\partial p} G(p, t; p_0, t_0) \bigg|_{p_0, t_0} + \frac{1}{2} \sigma^2(p_0, t_0) \frac{\partial^2}{\partial p^2} G(p, t; p_0, t_0) \bigg|_{p_0, t_0}
\]

\[+ \frac{\partial}{\partial p} V_j(p, t) \frac{\partial}{\partial p} G(p, t; p_0, t_0) \bigg|_{p_0, t_0} = 0 .
\]

(5)

Note $G(p_0', t_0'; p_0, t_0) \equiv 1$ has been used to obtain (5).

(5) is viewed as a differential equation defining $V(p, t)$ for a given diffusion process for $p$. $G_t(p_0, t_0')$, $G_p(p_0, t_0')$ and $G_{pp}(p_0, t_0')$ are considered as market wide parameters. That is $G$ is assumed to be known or have obtainable values at $(p_0, t_0')$. 
In [5] Garman defines the differential operator $L$

$$L \equiv \frac{3}{\partial t} + \alpha \frac{3}{\partial p} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial p^2}$$

Thus (5) may be rewritten in abbreviated form, using $L$, as:

$$L(V_j G) + d_j \bigg|_{P_0}^{t_0} = 0$$

or

$$V_j L(G) + L(V_j) + \sigma^2 G \cdot V_j \cdot d_j \bigg|_{P_0}^{t_0} = 0.$$  (5A)
3. **Derivation of Heaney's General Theory of the Discount factor** \( Z \):

By assuming no arbitrage possibilities exist in the market place and states \( p \) follow a diffusion process on the real line \(^3\)
Heaney obtains the equation:

\[
E(pz) = p(0),
\]

where \( z = z(p,t) \) is the "Discount Factor". The value of an asset in state \( p \) at time \( t \) is defined to be the value of the state at that time, i.e., \( V(p,t) \equiv p \). That is, the end of period (time \( t \)) states of the world are taken to be the possible prices of the asset.

(6) is then a particular case of (1), with

\[
V(p,t) = p, \quad d(p,t) = 0, \quad z = G \quad \text{and}
\]

\[
E(pz) = \int_{\{p\}} \pi z(\pi,t;q(\pi,t;p_0,t_0))d\pi = p(0).
\] \(\text{(6A)}\)

Heaney defines diffusion processes as "those for which the probability density function \( q(p,t;p_0,t_0) \) of asset prices at time \( t \), contingent on their prices of time zero, obeys the Fokker-Planck equation": That is, in this case \( q \) must satisfy:

\[
\frac{\partial}{\partial t} q + \frac{\partial}{\partial p} (aq) - \frac{1}{2} \frac{\partial^2}{\partial p^2} (a^2 q) = 0,
\]

subject to appropriate boundary conditions. \(^2\)
To obtain a differential equation for \( z \) the following procedure is used:

differentiate (6A) wrt \( t \), this gives

\[
\int_\{p\} \pi q \frac{\partial z}{\partial t} + \pi z \frac{\partial}{\partial t} q \, dq = 0 . \tag{8}
\]

The term \( \frac{\partial}{\partial t} q \) is eliminated from (8) using (7). This leaves \( \frac{\partial q}{\partial p} \)

and \( \frac{\partial^2 q}{\partial p^2} \) terms in (8) which can be "exchanged" for \( \frac{\partial}{\partial p} z \) and \( \frac{\partial^2}{\partial p^2} z \)
terms by integrating by parts. The following equation then results:

\[
\int_\{p\} q(p,t;p_0,0) \{ \pi \frac{\partial z}{\partial t} + \alpha(\pi,t)z + \alpha(\pi,t) \frac{\partial}{\partial \pi} z + \sigma^2(\pi,t) \frac{\partial}{\partial \pi} z \\
+ \frac{1}{2} \sigma^2(\pi,t) \pi \frac{\partial^2}{\partial \pi^2} z \} \, dq = 0 . \tag{9}
\]

It is then stated that (9) must hold for arbitrary choices of \( p_0 \).

Since different \( p_0 \) will change \( q \) it must be the case that the bracketted terms \( \{-\} \) are identically zero. Hence the equation

\[
p \frac{\partial z}{\partial t} + \alpha z + \alpha p \frac{\partial z}{\partial p} + \sigma^2 \frac{\partial z}{\partial p} + \frac{1}{2} \sigma^2 p \frac{\partial^2 z}{\partial p^2} = 0 , \tag{10}
\]

s.t. \( z(p_0,t_0) = 1 \). \tag{10A}

That this argument is not generally true will be dealt with shortly.

If equation (10) holds it holds for all \( t \geq t_0 \) and all \( p \). Thus (10)
is in some sense a generalization of (5) for a specific asset valuation function (and no dividends). Equation (10) is viewed as a differential equation defining $z$.

Thus (10) is the complement of (5). In (10) the asset valuation function $V(p,t)$ is given and $Z$ is the function to be determined in (5) values of $G_t$, $G_p$, $G_{pp}$ at $(p_0,t_0)$ are assumed to be known and the function $V(p_0,t_0)$ is to be determined.
4. **Relating Heaney's and Garman's Derivations.**

As we observed above (6) is a particular case of (1) where no dividends are paid and the asset value $V$ as a function of the state is prescribed. In fact by evaluating (9) or (10) at $(p,t) = (p_0,t_0)$ we obtain equation (5) for the particular choices of $V$ and $d$.

This suggests that (5) could be obtained in the "manner of Heaney" by use of the Fokker-Planck equation rather than by use of the Lindeberg type conditions to define the diffusion process. That is, we could assume the state variable $p$ follows a diffusion process where the p.d.f. for $p$ is governed by (7).

Differentiating (1) wrt $t$, substituting for $\frac{\partial q}{\partial t}$ and integrating by parts gives:

$$
0 = \int_{\{p\}} q(\pi,t;t_0,p_0) \left\{ Gd_j + G \frac{\partial}{\partial t} v_j + \alpha G \frac{\partial}{\partial p} v_j + \frac{1}{2} \sigma^2 G \frac{\partial^2}{\partial p^2} v_j \right. \\
+ \left. v_j \frac{\partial}{\partial t} G + \alpha v_j \frac{\partial}{\partial p} G + \frac{1}{2} \sigma^2 v_j \frac{\partial^2}{\partial p^2} G + \sigma^2 \frac{\partial}{\partial p} G \frac{\partial}{\partial p} v_j \right\} d\pi .
$$

(11)

Since $q(p,t_0;P_0,t_0) = \delta_{p_0}(p)$, (11) evaluated at $t = t_0$ gives (5), Garman's "Fundamental Differential Equation".

However (11) is formally similar to (9) which suggests Heaney's argument that the bracketted term $\{\ldots\}$ can be taken as zero again. Doing so gives:
where Garman's $L$ operator has been used for brevity. (12) evaluated at $t = t_0$, $p = p_0$ gives (5).

The reason that (5) is obtainable by defining the diffusion process either with the Lindeberg type conditions or with the Fokker-Planck equation is that the Fokker-Planck equation itself is obtainable (by one method) from the Lindeberg type conditions [see Feller Vol. II, pp. 320]. The Lindeberg type conditions, as noted before, hold locally about some point $(p, t)$. When used to derive the Fokker-Planck equation they are taken to hold about arbitrary points $(p, t)$, $t \geq t_0$. Thus the Fokker-Planck equation holds for all $(p, t)$, $t \geq t_0$.

By using the Fokker-Planck equation to define the diffusion process Heaney is directly using the Lindeberg conditions for $t \geq t_0$. By letting $t \rightarrow t_0$ Heaney's method collapses to Garman's method since now the Lindeberg type conditions are being applied about $t = t_0$, exactly as Garman has applied them.

Thus using Heaney's method and taking $t$ to $t_0$ yields the same results as Garman. The only difference between the two methods is the order of using the Lindeberg type condition and letting $t$ tend to $t_0$. 

\[
G(p, t; p_0, t_0) d_j(p, t) + L(G(p, t; p_0, t_0) V_j(p, t) = 0 ,
\]

It was seen above that Heaney obtained his fundamental differential equation (10) from equation (9). By noting that the p.d.f. \( q(p, t; p_0, t_0) \) was a function of \( p_0 \) and that the equality in (9) must hold for arbitrary values of \( p_0 \), it was argued that the term \{...\} within the integral in (9) must itself be identically zero - that is that equation (10) must hold. This however is true in general only if the \{...\} term is free from any dependency on \( p_0 \). But \( z \) must have \( p_0 \) as a parameter. The reason is that (10) does not uniquely specify \( z \). Even with the addition of a parallel equation for a risk free asset there is a degree of freedom in the choice of \( z \). Thus the initial condition

\[
Z(p_0, t_0) = 1
\]

(10A)

is needed to define \( z \) uniquely. Hence \( z \) may be a different function for different values of \( p_0 \). (Alternatively if \( z(p, t) \) did not have \( p_0 \) as a parameter then (10A) would hold for arbitrary \( p_0 \) and hence

\[
Z(p, t_0) = 1, \text{ for arbitrary } p.
\]

In this case equation (10) would reduce to

\[
p \frac{\partial z}{\partial t} \bigg|_{t_0} + \alpha(p, t_0) = 0.
\]

(13)

Taking \( p = p_0 \), (13) is clearly not a special case of (5) unless \( \frac{\partial}{\partial p} G = 0 \) is assumed.)
Hence, in general, $Z$ should be written $Z = Z(p, t; p_0, t_0)$. From now on we will use Garman's notation $G$ for $Z$, that is $Z(p, t; p_0, t_0) \equiv G(p, t; P_0, t_0)$. Unless it becomes ambiguous, however, the dependency on parameters $p_0, t_0$ will be suppressed. Although it is not necessary to have $\{ \ldots \}$ in (9) (or $\{ \ldots \}$ in (11)) identically zero, it is sufficient to obtain a function $Z(G)$ which satisfies the equation. Assuming $\{ \ldots \} \equiv 0$ allows $Z(G)$ to be found, but predictably this assumption restricts the class of allowable $Z(G)$ functions.

Consider equation (12) for a risky asset with value $V(p, t)$ and dividend stream $d(p, t)$, consider (12) also for a riskless asset with value $R(p, t)$ and dividend stream $r(p, t)$. That is

$$G(p, t)d(p, t) + L[G(p, t)V(p, t)] = 0$$

(14A)

$$G(p, t)r(p, t) + L[G(p, t)R(p, t)] = 0$$

(14B)

s.t. $G(p_0, t_0; P_0, t_0) = 1$.  

(14C)

Before proceeding further the definition of $R$ and $r$ for the riskless asset should be resolved. Garman defines the riskless asset as $R(p, t) \equiv 1$, $r(p, t)$ not specified. Substituting into (14B) gives

$$G(p, t)r(p, t) = - L(G(p, t)).$$

(15)

Heaney defines $R(p, t) = R(t)$ (where $t_0 = 0$). That is $R(t)$ is the value at time $t$ of $\$1$ invested risklessly at time $t = 0$. Allowing the initial time to be $t_0$, $R(p, t) = R(t-t_0)$. The dividend
(interest) stream is taken as identically zero \(- r(p,t) \equiv 0\). Equation (14B) in this case becomes:

\[
0 = L(R(t-t_0)G(p,t)) = G(p,t) \frac{\partial}{\partial t} R(t-t_0) + R(t-t_0)L(G(p,t))
\]

or

\[
\frac{R'(t-t_0)}{T(t-t_0)} = - L(G(p,t)). \tag{16}
\]

Clearly (15) and (16) are identical if \(\frac{R'(t-t_0)}{R(t-t_0)} \equiv r(t) = r(p,t)\). That is if

\[
R(t-t_0) = \exp\left\{ \int_{t_0}^{t} r(T) \, dT \right\}.
\]

Thus Heaney's and Garman's specifications of the riskless asset are very similar, but with Garman's definition allowing slightly more generality. We will thus use Garman's definition of the risk free asset; namely \(R(p,t) \equiv 1\) \((r(p,t)\) to be determined endogenously).

Expanding (14A) slightly,

\[
G \sigma^2 + V L(G) + GL(V) + \sigma^2 \frac{\partial}{\partial p} G \frac{\partial}{\partial p} V = 0
\]

and using (15) to eliminate \(L(G)\) we have:

\[
\frac{\partial}{\partial p} G = \frac{-1}{\sigma^2} \frac{\partial}{\partial p} \left( [d - Vr + L(V)]G \right). \tag{17}
\]

Integrating through (17) wrt \(p\) gives

\[
G(p,t) = A(t) e^{- \int_{p}^{\overline{p}} \frac{1}{\sigma^2} \frac{\partial}{\partial \pi} \left( [d - Vr + L(V)]d\pi \right)}, \tag{18}
\]

where \(A(t)\) is a function purely of \(t\), and \(\overline{p}\) is some arbitrary value of \(p\) used for concreteness.
A(t) is determined by substituting (18) into (15). After rearranging we have:

\[
\frac{A'(t)}{A(t)} = -r(t) - \exp\left\{ \int_{p}^{P} \frac{1}{\sigma^2} \frac{\partial}{\partial \pi} \left[ d - vr + L(V) \right] \, d\pi \right\}
\]

or

\[
\frac{A'(t)}{A(t)} = [ - r + \int_{p}^{P} \frac{\partial}{\partial t} \left( \frac{d-vr+L(V)}{\sigma^2} \right) \, d\pi + \alpha \left( \frac{d-vr+L(V)}{\sigma^2} \right) ]
\]

\[
- \frac{1}{2} \alpha^2 \left( \frac{d-vr+L(V)}{\sigma^2} \right)^2 - \frac{\partial}{\partial p} \left( \frac{d-vr+L(V)}{\sigma^2} \right) \right] .
\]

The left hand side of (19) is a function of \( t \) alone. Therefore the right hand side of (19) must also be a function of \( t \) only. That is \( V(p,t), d(p,t), r(t), \alpha(p,t) \) and \( \sigma^2(p,t) \) must such that

\[
\frac{\partial}{\partial p} [ \ldots ] \equiv 0 .
\]

Garman has dealt with similar questions in [5] and we will relate this to (19) shortly.

Meanwhile assume that \([\ldots] \) is indeed independent of \( p \).

Then \( A(t) \) is given by:
\[ A(t) = A_0 \, e^{\int_\tau^t \ldots \, d\tau} \]

where \( A_0 \) is a constant and \( \tau \) is an arbitrary value of \( t \) included for concreteness. This gives \( G(p,t) \) as:

\[ G(p,t; p_0, t_0) = A_0 \, e^{-\int_p^\tau \frac{1}{\sigma^2} \, [d-V_r+L(V)] \, d\pi} \]

Thus:

\[ G(p,t; p_0, t_0) = A_0 \, e^{-\int_\tau^{t_0} \ldots \, d\tau} \, e^{-\int_p^\tau \frac{1}{\sigma^2} \, [d-V_r+L(V)] \, d\pi} \]

\[ = A_0 \, e^{-\int_\tau^{t_0} \ldots \, d\tau} \, e^{-\int_p^\tau \frac{1}{\sigma^2} \, [d-V_r+L(V)] \, d\pi} = 1 \]

To summarize: if it is assumed that equation (12) holds for some \( G(p,t; p_0, t_0) \) then either

\[ G(p,t; p_0, t_0) = \frac{1}{g(p_0,t_0)} \, g(p,t) \]

where

\[ g(p,t) = e^{-\int_p^\tau \frac{1}{\sigma^2} \, [d-V_r+LV] \, d\pi} \]

\[ = e^{-\int_\tau^{t_0} \ldots \, d\tau} \, e^{-\int_p^\tau \frac{1}{\sigma^2} \, [d-V_r+LV] \, d\pi} \]
\[ G(p,t;p_0,t_0) = \frac{g(p,t)}{g(p_0,t_0)}, \text{ where } g(p,t) \text{ is given in (21)}, \]

when

\[ \frac{\partial}{\partial p} \left[ - r + \int \frac{\partial}{\partial t} \left( \frac{d}{\partial \pi} g(p,t) \right) d\pi + \alpha \left( \frac{d}{\partial \pi} g(p,t) \right) \right] = 0; \tag{22} \]

or

\[ G(p,t;p_0,t_0) \text{ does not exist, when } \frac{\partial}{\partial p} [...] \neq 0. \]

On the other hand, if \( G \) is assumed to be of the form (20A), that is \( G \) is multiplicatively separable in the initial state and time \((p_0,t_0)\), then equation (12) must hold. To see this substitute (20A) into (11):

\[
\int q(\pi,t;p_0,t_0) \left\{ \frac{g(\pi,t)}{g(p_0,t_0)} d_j(\pi,t) + L \left( \frac{g(\pi,t)}{g(p_0,t_0)} \right) v_j(\pi,t) \right\} d\pi = 0.
\]

Multiplying through by \( g(p_0,t_0) \) leaves the bracketed quantity \( \{g(\pi,t)d_j(\pi,t) + L(g(\pi,t)v_j(\pi,t))\} \) free of a dependence on \((p_0,t_0)\). Thus \( \{ \} \equiv 0 \) must hold since (11) holds for all \( p_0 \); that is (12) now holds. From (12) we could again develop the consistency condition
That is, if $G$ is to exist then (22) must hold.

Hence if (22) holds then equation (12) holds if and only if $G$ is of the form in equation (20A). It has been assumed throughout this last discussion that $V(p,t)$ is some known function of $p$ and $t$. That is, we have been solving the "Heaney Problem" of finding $G$.

If $V(p,t) = p$ and $d(p,t) = 0$ we have the model Heaney investigated.

It is possible to link (12) and (20A), formally only, in yet another way. If we let $(p_0, t_0)$ tend to $(p, t)$ in (12), we obtain equation (5), as was observed before. If in equation (5) $G$ is assumed to have the form (20A) we obtain:

$$L(V_j(p, t) \frac{g(p, t)}{g(p_0, t_0)}) \bigg|_{p_0, t_0} + d_j(p_0, t_0) = 0$$

or

$$[L(V_j(p, t)g(p, t)) + g(p, t)d_j(p, t)] \bigg|_{p_0, t_0} = 0 \tag{23}$$

But (23) must hold for any $(p_0, t_0)$ hence we may write (23) as either

$$L_0(V_j(p_0, t_0)g(p_0, t_0)) + g(p_0, t_0)d_j(p_0, t_0) = 0,$$

where

$$L_0 = \frac{\partial}{\partial t_0} + \alpha(p_0, t_0) \frac{\partial}{\partial p_0} + \frac{1}{2} \sigma^2(p_0, t_0) \frac{\partial^2}{\partial p_0^2}.$$
or

\[ L(V_j(p,t)g(p,t)) + g(p,t)d_j(p,t) = 0. \] (23A)

Take another point \((p_1, t_1)\) s.t. \(t_1 \geq t_0\). Divide through (23A) by \(g(p_1,t_1)\). This gives

\[ L(V_j(p,t)G(p,t;p_1,t_1)) + G(p,t;p_1,t_1)d_j(p,t) = 0, \]

which is just equation (12) with \((p_0, t_0) \rightarrow (p_1, t_1)\). This gives us a second (Heuristic) method of linking (12) and (20A). (23) was developed because it shows more clearly the relationship between Heaney's and Garman's equations and the restrictions on the form of \(G\).

Garman [5] gives the following proposition, called the "Exclusion Rule", for the discrete state model:

**Proposition:** Admissible models of capital markets may not exogenously specify all of the quantities involved in any equation derived via the pure theory of arbitrage; at least one quantity must be excluded from such specification and instead endogenously derived. Violation of this rule may lead to internal inconsistency with respect to the arbitrage equilibrium of the economy.

This proposition explains the existence of the consistency condition, equation (22). Equations (14) define \(G\) uniquely in terms of the functions: \(V(p,t)\), \(d(p,t)\), describing the risky asset; \(r(p,t)\) describing the riskless asset; and \(\alpha(p,t)\), \(\sigma^2(p,t)\) describing the diffusion process for the state variable \(p\). By insisting that \(G\) satisfy (12) ((14A) and (14B)) an exogenous constraint has been placed
on \( G \), namely \( G \) must be of the form \((20A)\). Thus if it is the case that \( V(p,t), d(p,t) \) and \( \alpha(p,t), \sigma^2(p,t) \) are exogenously specified, as in Heaney's examples of Arithmetic and Geometric Brownian Motion, then the only remaining function \( r(p,t) \) must be endogenously derived to be compatible with \( G, V, d, \alpha \) and \( \sigma^2 \). The derivation of \( r \) in this case is from the consistency condition, equation \((22)\).

In general \( G \) is to be determined from \((14A)\) and \((15)\) (rather than \((14B)\)) and the initial condition \((14C)\). Garman's proposition must hold for each equation. Thus one of \( G, \alpha, \sigma^2 \) or \( r \) must be, in \((15)\), endogenously derived. Similarly one of \( G, \alpha, \sigma^2, d \) or \( V \) must be endogenously derived in \((14A)\). Neither \((14A)\) nor \((15)\) alone defines \( G \) uniquely. But requiring \( G \) to satisfy either equation places an exogenous constraint on \( G \) that it should be of the form \((20A)\). Thus although it would appear we should be able to take \((14A)\) and \((15)\) and determine \( G \) for exogenously defined \( \alpha, \sigma^2, r, d \) and \( V \), we can't in fact because there has been an exogenously placed constraint on \( G \). Thus equation \((22)\) can be viewed as the endogenously derived condition that \( \alpha, \sigma^2, r, d \) and \( V \) must satisfy together once \( G \) has been chosen to be of the form \((20A)\).
6. Comments.

Heaney [6], pp. 37-38, compares his method with that of Garman. Garman [3], p. 6, notes: "One convenience of the diffusion assumption turns out to be that [given diffusion belief] exactly three quantities completely determine all asset prices. These quantities are seen to be identifiable via simple linear regression against the current interest rate". Heaney replies that "The three determinants of prices referred to by Garman are \( \frac{\partial G}{\partial t} \), \( \frac{\partial G}{\partial p} \) and \( \frac{\partial^2 G}{\partial p^2} \), all evaluated at prevailing market prices \((t = t_0)\). We have shown however that whenever the riskless term structure is given, specification of diffusion beliefs determine \( G \) and hence the three quantities."

As we have seen, Heaney and Garman are working with different functions exogenously or endogenously specified. Heaney takes \( \alpha, \sigma^2, d(d = 0) \) and \( r \) \((R'(t)/R(t) = r(p,t))\) to be known and specifies \( V(p,t) = p \). With these functions given he solves for \( G \). Garman on the other hand is usually seeking \( V \) and takes \( \alpha, \sigma^2, d \) and \( r \) to be known. Since he is dealing with current price(s) and time \((p_0,t_0)\) he is left to find the values of the three derivatives of \( G \) at \((p_0,t_0)\). Thus in some sense both Garman and Heaney are correct.

Considering Garman in more detail:
Garman's approach has been to take \( G \) as exogenous (known) and seek \( V \), given \( \alpha, \sigma^2, r \) and \( d \) specified exogenously. In his model of a riskless asset and a risky asset existing in an economy
two equations must hold simultaneously:

\[ r(p_0, t_0) = -L(G(p,t)) \bigg|_{p_0 \atop t_0} \]  

(24A)

and

\[ d(p_0, t_0) + V(p_0, t_0) L(G(p,t)) \bigg|_{p_0 \atop t_0} + L(V(p,t)) \bigg|_{p_0 \atop t_0} \]

\[ + \sigma^2(p_0, t_0) \frac{\partial}{\partial t} V(p,t) \frac{\partial}{\partial t} G(p,t) \bigg|_{p_0 \atop t_0} = 0. \]  

(24B)

Indeed after \( \alpha \) and \( \sigma^2 \) are specified, and if \( d \) is known, only \( \frac{\partial}{\partial p} G, \frac{\partial^2}{\partial p^2} G \) and \( \frac{\partial}{\partial t} G \) remain to determine \( V \) in (24B). This is as true for \( n \) risky assets as it is of the one risky asset.

Equation (24B) is consistent with Garman's Exclusion Rule: given \( \alpha, \sigma^2, d \) and \( G \) exogenously \( V \) is to be determined endogenously.

(24A) is used to determine the three quantities:

\[ \frac{\partial G}{\partial t} \bigg|_{p_0 \atop t_0}, \frac{\partial G}{\partial p} \bigg|_{p_0 \atop t_0} \quad \text{and} \quad \frac{\partial^2 G}{\partial p^2} \bigg|_{p_0 \atop t_0} \quad . \]

Following Garman's Exclusion Rule, in equation (24A) \( r, \alpha \)
and $\sigma^2$ are exogenous and $G$ is endogenous. But (24A) is not a differential equation defining $G$. Rather it is the evaluation of three derivatives of $G$ at the initial point $(p_0,t_0)$. $r$, $\alpha$ and $\sigma^2$ are known at $(p_0,t_0)$. Thus in some manner (e.g., Least Squares Regression) the values of the three derivatives must be obtained. (If it is possible to obtain them at all.) Thus Garman has a two step evaluation process. Find the values of the three derivatives of $G$; then with these known evaluate $V$. In certain models Garman takes $V(p,t) = p$, as Heaney has. For example see Garman's CAPM. Here however Garman is more interested in relating the return on the risky assets to the "market" return. Thus $G$, and hence its three derivatives, are known in this case. But this is of secondary importance to determining the general asset return - market return relationship.
7. **Barriers.**

The diffusion processes considered thus far have been over the unbounded real line (or n-dimensional Euclidean space). With diffusion processes it is possible to place barriers – reflecting or absorbing, above or below a particle. For example an absorbing barrier might be placed below at \( p = 0 \), where \( p \) is the price of some asset. This would model bankruptcy with limited liability. A reflecting barrier at \( p = 0 \) could be interpreted as a firm guaranteed against bankruptcy. These barriers can be added to any diffusion process and may add to the realism of the model. The cost is the increased difficulty of solution – with either Garman's or Heaney's problems.

Barriers will be introduced in both Garman's and Heaney's models. Garman's differential equation will be shown not to be directly affected by the introduction of barriers. The analogue with barriers, to Heaney's differential equation will then be developed and the problems solving it indicated.

Garman's Fundamental Differential Equation holds for arbitrary diffusion processes. This means that it must hold for processes with barriers, absorbing or reflecting. This isn't immediately obvious, but seems reasonable intuitively since Garman's equation holds "locally". That is (5) holds for states and times very close to the current state and time. Thus, from the point of view of equation (5) any barrier is a very great distance away, no matter how close \( p_0 \) is to the barrier (as long as \( p_0 \) is not on the barrier). However another way to show that barriers do not change the equation but only affect the
valuation thru G and boundary conditions is by using "Heaney's Method" of developing Garman's equation.

When no barriers are present \( q(p,t) \) and \( \frac{\partial}{\partial p} q(p,t) \) are both assumed to disappear at infinity and negative infinity. When an absorbing barrier is placed at \( p = a \) \( q(p,t) \) must instead satisfy the boundary condition:  

\[
q(a,t) = 0 \quad t > t_0 . \tag{25}
\]

For a reflecting barrier at \( p = a \) \( q \) must satisfy the boundary conditions:  

\[
- \alpha(p,t)q(p,t) + \frac{1}{2} \sigma^2(p,t) \frac{\partial}{\partial p} q(p,t) \bigg|_{p=q} = 0 , \quad t > t_0 . \tag{26}
\]

The development of (5) from (1) by Heaney's Method when barriers are present is exactly analogous to the no barrier case except for boundary terms in intermediate expressions due to the presence of the barriers (before the limit \( (p,t) \) to \( (p_0,t_0) \) is taken). We will develop (5) in the case of a reflecting barrier below at \( p = a \).

Equations (1) and (7) hold subject to (26) and

\[
q(p,t_0) = \delta_{p_0}(p) .
\]

Differentiating (1) wrt \( t \) and using (7), as before, to integrate by parts gives:
\[
\int_{\alpha < p < \infty} q(\pi, t; p_0, t_0) \{ G(\pi, t) d_j(\pi, t) + L(V_j(\pi, t) G(\pi, t)) \} d\pi \\
+ \frac{1}{2} \frac{\partial}{\partial p} \left( G(p, t) V_j(p, t) \right) \sigma^2(p, t) q(p, t) \bigg|_{p=\alpha} = 0.
\] (27)

As \( t + t_0 \) \( q(p, t) \to \delta_{p_0}(p) \) and (27) reduces to (5), as long as \( \alpha \neq p_0 \).

Similarly in the case of an absorbing barrier, (1) and (7) subject to (25) yield:

\[
\int_{\alpha < p < \infty} q(\pi, t; p_0, t_0) \{ G(\pi, t) d_j(\pi, t) + L(V_j(\pi, t) G(\pi, t)) \} d\pi \\
+ \frac{1}{2} G(p, t) V_j(p, t) \frac{\partial}{\partial p} \left( \sigma^2(p, t) q(p, t) \right) \bigg|_{p=\alpha} = 0.
\] (28)

Assuming sufficient regularity to allow \( t + t_0 \) to be interchanged with differentiation wrt \( p \) in (28), then as \( t + t_0 \) (28) reduces to (5) again. 7

All combinations of absorbing or reflecting barriers, above and below, will give equations similar to (27) and (28) but with a linear combination of the extra barrier induced boundary terms as in (27) and (28).

Thus (5) the P.D.E. determining \( V_j(p, t) \) in terms of \( \frac{\partial}{\partial t} G \), \( \frac{\partial}{\partial p} G \) and \( \frac{\partial^2}{\partial p^2} G \) at \((p, t) = (p_0, t_0)\) is not directly affected by the presence of barriers. However the appropriate boundary conditions
will change with the introduction of barriers. For example without barriers there is no need of boundary conditions in $p$. The equation to be solved is a parabolic P.D.E. on the infinite line. Introduce for example a reflecting barrier below at $p = a$ and the problem changes to a parabolic P.D.E. on the semi infinite line, $p > a$. This problem requires the specification of some boundary condition at $p = a$; that is, requires that:

$$V_j(a, t) = \psi(t), \text{ for some function } \psi.$$ 

Thus the introduction of barriers affects both $V_j$ and $G$ thru their respective equations.

$\psi$ must be specified exogenously. Just what this specification should be is a good question - left open to the reader.

In the case of the Option Pricing Model the introduction of a barrier does not seriously affect the difficulty of solving for $V$ from the differential equation since the solution to this type of equation on the semi infinite line is well known.

Consider equation (27), the analogue of (11) in the case of a reflecting barrier at $p = a$. (28) may be rewritten as:

$$\int_{\pi < p < \infty} q(\pi, t; p_0, t_0) \left\{ G(\pi, t) d_j(\pi, t) + L(V_j(\pi, t)G(\pi, t)) \right\} d\pi = 0, \quad (30)$$

since

$$\int_{\pi < p < \infty} q(\pi, t; p_0, t_0) \, d\pi = 1.$$
Following Heaney's method assume the bracketted quantity of (30) to be equal to zero, which gives the analogue of (12):

\[
G(p,t)d_j(p,t) + \left[V_j(p,t) G(p,t) +
\right.
\]
\[
+ \frac{1}{2} \frac{\partial}{\partial p} \left( G(p,t) V_j(p,t) \right) \sigma^2(p,t) q(p,t) \bigg|_{p=a} = 0 .
\]  

(31)

Now in section 4, above, we considered one risky and one riskless asset. (31) in this case gives:

\[
0 = G(p,t)d(p,t) + \left[ V(p,t) G(p,t) + \right.
\]
\[
+ \frac{1}{2} \frac{\partial}{\partial p} \left( G(p,t) V(p,t) \right) \sigma^2(p,t) q(p,t) \bigg|_{p=a}
\]  

(32A)

\[
0 = G(p,t)r(p,t) + \left[ V(p,t) G(p,t) + \right.
\]
\[
+ \frac{1}{2} \frac{\partial}{\partial p} \left( G(p,t) \right) \sigma^2(p,t) q(p,t) \bigg|_{p=a}
\]  

(32B)

where the notation is the same as before.

Combining (32A) and (32B) gives:

\[
\frac{\partial}{\partial p} G = \frac{-1}{\sigma^2(p,t) \frac{\partial}{\partial p} V(p,t)} [d(p,t) - V(p,t) r(p,t) + L(V(p,t))] G(p,t) +
\]
\[
+ \frac{1}{2} \sigma^2(a,t) q(a,t) (G_p(a,t) G_p(a,t) V(a,t) + G(a,t) V_p(a,t)
\]
\[
- G_p(a,t) V(p,t)) .
\]  

(33)
(33) is generally of the form:

\[
\frac{\partial G(p,t)}{\partial p} = p(p,t)G(p,t) + Q(p,t) \tag{33G}
\]

\(G\) and \(G_p\), evaluated at \(p = a\), are treated as known quantities. It is possible to unravel this recursive relationship.

\(G\) will have the form

\[
G(p,t) = [A(t) + \int_0^\pi Q(\pi,t) \text{e}^{-\int_0^p \frac{p(\xi,t)\text{d}\xi}{\text{d}\pi}} \text{d}\pi] \text{e}^{-\int_0^p \frac{p(\pi,t)\text{d}\pi}{\text{d}p}} \tag{34}
\]

Substituting into (32B) gives the equation \(A(t)\) must satisfy:

\[
-A'(t) = + A(t) [r(p,t) + \frac{\partial}{\partial t} \int_0^p p(\pi,t) \text{d}\pi + \alpha p(p,t) + \frac{1}{2} \sigma^2 (p(p,t))^2 \\
+ \frac{1}{2} \sigma^2 \frac{\partial}{\partial p} p(p,t) + \frac{1}{2} \sigma^2 (a(t)q(a,t)P(a,t)] \\
+ r(p,t) \int_0^p Q(\pi,t) \text{e}^{-\int_0^\pi p(\xi,t)\text{d}\xi} \text{d}\pi \\
+ \frac{\partial}{\partial t} \int_0^p p(\pi,t) \text{d}\pi \int_0^\pi Q(\pi,t) \text{e}^{-\int_0^\pi p(\xi,t)\text{d}\xi} \text{d}\pi \\
+ \frac{\partial}{\partial t} \int_0^\pi Q(\pi,t) \text{e}^{-\int_0^\pi p(\xi,t)\text{d}\xi} \text{d}\pi \\
+ \alpha p(p,t) [P(p,t) \int_0^\pi Q(\pi,t) e^{-\int_0^\pi p(\xi,t)\text{d}\xi} \text{d}\pi + Q(p,t) e^{-\int_0^\pi p(\pi,t)\text{d}\pi}]
\]
\[
\frac{1}{2} \sigma^2(p,t) \left[ \frac{\partial}{\partial p} Q(p,t) + P(p,t) Q(p,t) \right] e^{P(p,t) - \int_0^\pi P(\pi,t) \, d\pi} \\
- \int_0^\pi P(\pi,t) \, d\pi \\
+ (p^2(p,t) + \frac{3}{2} P(p,t) \int_0^p Q(\pi,t) \, e^{P(\pi,t) - \int_0^\pi P(\pi,t) \, d\pi}) \\
+ \frac{1}{2} \sigma^2(a,t) q(a,t) (A(a,t) e^{-P(a,t) - \int_0^a Q(\pi,t) \, e^{P(\pi,t) - \int_0^\pi P(\pi,t) \, d\pi})}.
\]

(35) is of the form:

\[
A'(t) = A(t) P(p,t) + Q(p,t)
\]

(35G)

Solving for \(A(t)\)

\[
A(t) = A_0 + \int_{t_0}^t Q(p,\tau) \, e^{-P(p,\tau) - \int_{t_0}^\tau P(p,\tau) \, d\tau} \, e^{P(p,\tau) - \int_{t_0}^\tau P(p,\tau) \, d\tau}.
\]

(36)

It can be shown that for (35G) to have a solution, i.e., for \(A(t)\) to be as in (36), that \(P(p,t)\) and \(Q(p,t)\) must be of the form:

\[
P(p,t) = p(t) F(p,t) + g(t)
\]

(37A)
\[ Q(p,t) = q(t) F(p,t) + h(t), \]

where

\[ q(t) = -p(t)(A_0 + \int_{t_0}^{t} h(\tau) e^{\int_{t_0}^{\tau} \dot{g}(\tau) d\tau}) e^{-\int_{t_0}^{t} g(\xi) d\xi}. \]

This just causes (35G) to reduce to

\[ A'(t) = A(t) g(t) + h(t). \]

Equations (37) are the analogue of the consistency conditions (22).

Thus Heaney's method will succeed in finding \( G \) when (37) are satisfied.

Examining \( P \) and \( Q \) in (35) to see if they are of the form (37) appears to be a difficult task.

\( Q \) is of the form

\[ Q(p,t) = (P(p,t) - \frac{1}{2} \sigma^2(a,t)q(a,t)p(a,t) \int_{P_0}^{P} Q(\Pi,t) e^{\int_{P_0}^{\Pi} \dot{Q}(\eta,t) d\eta} d\Pi + \ldots. \]

It would appear that \( Q \) and \( P \) do not satisfy (37). An exact proof that \( Q \) and \( P \) do not in general satisfy (37) has not been developed.

Nor has a counter example of a \( P \) and \( Q \) that do satisfy (37) been found.

For the purposes of this project the question is left open - with the expectation that \( Q \) and \( P \) cannot be found to satisfy (37).
Notes:

1 Actually only (2) is the Lindelberg type condition. (3) and (4) are definitions of $\alpha$ and $\sigma^2$. (In fact (3) and (4) are a consequence of (2) - Feller, [2]).

2 These are generally $\lim_{p \to \pm \infty} q(p,t) = 0$, $\lim_{p \to \pm \infty} q_p(p,t) = 0$ and

$q(p,t_0) = \delta_{p_0}(p) = \delta(p-p_0)$. However other B.C.'s modelling different processes will be discussed later.

3 Heaney considered the case of an n-dimensional state vector $\mathbf{p}$. The one dimensional case is considered here for simplicity and ease of comparison with Garman's work. There is no ultimate loss of generality on the n-dimensional case follows in a parallel manner.

4 An absorbing barrier allows a particle to pass the barrier in one direction, but having passed the particle can never return to the other side of the barrier. A reflecting barrier, on the other hand, will not allow the particle to pass, "bouncing" it back from the barrier.

5 See Cox and Miller, p. 220.

6 Ibid. p. 224.

7 $\lim_{t \to t_0} \frac{\partial}{\partial p} q(p,t) \bigg|_{p=a} = \frac{\partial}{\partial p} \delta_{p_0}(p) \bigg|_{p=a} = \frac{\delta_{p_0}(p)}{p-p_0} \bigg|_{p=a} = 0$.

See Sneddon, [7].
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