Symmetric Differential Forms on the
Barth Sextic Surface

by

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Abstract

This thesis concerns the existence of regular symmetric differential 2-forms on the Barth sextic surface, here denoted by $X$. This surface has 65 nodes, the maximum possible for a sextic hypersurface in $\mathbb{P}^3$. This project is motivated by a recent work of Bogmolov and De Oliveira where it is shown that a hypersurface in $\mathbb{P}^3$ with many nodes compared to its degree contains only finitely many genus zero and one curves. We find that there are symmetric differential 2-forms on $X$ that are regular everywhere outside the nodes. We also find that none of these extend to a regular form on the minimal resolution of $X$. Using these forms we can prove that any genus 0 curve on $X$ must pass through at least one node, and we determine the curves passing through just a select set of nodes.

Keywords: differential forms; graded modules; coherent algebraic sheaves; genus zero curves.
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Chapter 1

Introduction

An important way to classify algebraic varieties is by describing the kinds of subvarieties that they admit. For instance, in the case of surfaces, one can try to describe the rational or elliptic curves that lie on them. In fact, surfaces that have only finitely many rational and elliptic curves are known in the literature as \textit{algebraically quasi-hyperbolic}. Of particular interest to us is a recent work of Bogomolov and De Oliveira [2] on hyperbolicity of nodal hypersurfaces where the authors employ the theory of symmetric differential forms to show that a hypersurface $X \subseteq \mathbb{P}^3$ of degree $d$ with $\ell$ nodal singularities (and smooth otherwise) satisfying $\ell > \frac{2}{3} (d^2 - \frac{5}{2}d)$ is algebraically quasi-hyperbolic, but do not provide a method for describing these curves.

An example of a hypersurface that satisfies this bound is the Barth sextic surface, a degree 6 hypersurface in $\mathbb{P}^3$ with the maximum possible number of nodal singularities. To be specific, let $\mathbb{P}^3_{\mathbb{C}}$ denote the complex projective three-space and let

$$\phi = \frac{1 + \sqrt{5}}{2},$$

denote the golden ratio. The hypersurface $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ given in homogeneous coordinates $X_0, X_1, X_2, X_3$ by the equation

$$4(\phi^2 X_0^2 - X_1^2)(\phi^2 X_1^2 - X_2^2)(\phi^2 X_2^2 - X_3^2) - (1 + 2\phi)(X_0^2 + X_1^2 + X_2^2 + X_3^2)^2 X_3^2 = 0,$$

is called Barth’s sextic surface. This surface was first constructed by the German mathematician Wolf Barth [1] who showed that $X$ has 65 isolated singularities (which happen to be ordinary double points), and that this number is bigger than those observed on other known surfaces of the same degree at the time. It was not until a few years later in 1997
that David B. Jaffe and Daniel Ruberman [9] proved that this number is in fact maximal among all normal sextic surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

In this thesis we investigate the explicit computation of regular symmetric differential $m$-forms on $X$. For $m = 2$, we find that there are forms that are regular outside the singularities of $X$, but none of them extend to all of the minimal resolution of $X$. These results are summarized as follows:

**Theorem 1.1 (Main Theorem).** Write $\varphi : Y \rightarrow X$ for the desingularization of $X$ that is obtained through blowing up the nodal singularities of $X$. Write $\text{Sym}^2 \Omega_Y$ for the sheaf of symmetric differential 2-forms on $Y$. Then $\Gamma(Y, \text{Sym}^2 \Omega_Y) = 0$; i.e., $\text{Sym}^2 \Omega_Y$ has no global sections. Moreover, if we let $\mathcal{S}$ denote the set of singular points of $X$ and set $E = \bigcup_{s \in \mathcal{S}} \varphi^{-1}(s)$, then $\dim_k \Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y) = 15$; i.e, there are non-trivial symmetric differential forms regular everywhere on $Y \setminus E$.

In order to get an idea of which genus zero curves one should expect to lie on $X$, we intersected $X$ with planes spanned by singularities on $X$ and determined the genus of the curves in these intersections. We found 27 genus zero curves on $X$ this way and we have listed them in Appendix A. Six of these curves lie in the plane $X_3 = 0$ and they pass through 5 singularities each. The other curves pass through 10 singularities each.

As it turns out, if we let $E' \subseteq E$ be the exceptional curves on $Y$ above the singularities of $X$ with $X_3 \neq 0$, then we find in Proposition 4.8 that $\dim_k \Gamma(Y \setminus E', \text{Sym}^2 \Omega_Y) \geq 3$. This allows us to prove the following result, which implies that there are no genus zero curves on $X$ that avoid all singularities.

**Theorem 1.2.** The only genus zero curves that avoid the singularities with $X_3 \neq 0$ are the curves $C_{22}, \ldots, C_{27}$ listed in Appendix A.

It is worth noting that while our computations are inspired by the work presented in [2], our results do not depend on it. The outline of this thesis is as follows.

We begin by providing a brief overview of some standard concepts from algebraic geometry and commutative algebra. These includes a brief review of affine and projective varieties, singular and non-singular varieties, the notion of blow-up of a variety at a point, and a brief introduction to Kähler differentials and sheaves. Since these topics play a central role in the development of this work, we shall provide examples along the way to better...
illustrate these ideas. Lastly, we will discuss how to interpret sections of the sheaf of symmetric differential 2-forms as elements of symmetric differential 2-forms of the function field of $X$. We will then describe what regularity of such a differential form means at a point of $X$ and consider a convenient method for detecting when a given differential form is regular at a non-singular point of $X$. This will be the focus of Chapter 2.

In Chapter 3 we discuss part of J. P. Serre’s seminal work (see [12]) on the correspondence between coherent algebraic sheaves and graded modules. Chapter 3 will serve as a basis for most of our computations in Chapter 4. We discuss two functors and state, without proofs, some of their properties: the functor $\mathscr{A}$ which associates to a graded module a coherent algebraic sheaf, and the functor $\Gamma_*$ taking a coherent algebraic sheaf to a graded module. We also discuss the “saturation” map taking a graded module $M$ to $\Gamma_*(\mathscr{A}(M))$. In particular, we shall use a few results from [12] to conclude

**Theorem 1.3.** If $k$ is a field, $S$ is graded $k$-algebra that, as a module over itself is saturated, and $M$ is a graded $S$-module, then $M^\vee \cong \Gamma_*(\mathscr{A}(M^\vee))$, where $M^\vee$ is the dual (graded) module associated to $M$.

This allows us to conclude that the homogeneous component of degree 0 of the double dual module $M^{\vee\vee}$ is, in fact, describing the global sections of the sheaf associated to $M^{\vee\vee}$.

In Chapter 4 we give a detailed description of our computations. We give an explicit description of the graded module representing the sheaf of symmetric differential 2-forms associated to $X$, and the double dual sheaf (often referred to as the reflexive hull), henceforth denoted by $\text{Sym}^2 DX$ and $(\text{Sym}^2 DX)^{\vee\vee}$ respectively. Moreover, we give an explicit description (on an affine chart) of the blow-up $Y$ of $X$ at a nodal singularity. We will then use the observations from Section 2.7 to write down Laurent series expansions for our differentials, which will, in return, aid in determining regularity. The proof of Theorem 1.1 will then follow as a corollary of our computations. Lastly, we use the modules $\text{Sym}^2 DX$ and $(\text{Sym}^2 DX)^{\vee\vee}$ to find the genus zero curves that pass through only a few nodal singularities. We also find all the curves that avoid the singularities with $X_3 \neq 0$ (see Section 4.7 and Appendix A).
Chapter 2

Background and Notation

In this chapter we develop the necessary algebro-geometric language for the rest of this thesis. We shall adopt, for the most part, the notation of Hartshorne [7], to which we also refer the reader for a more in depth treatment of the concepts.

We adopt the following notational conventions. We reserve boldface letters to denote vectors. Unless stated otherwise, $k$ refers to an arbitrary *algebraically closed* field. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R},$ and $\mathbb{C},$ and $\mathbb{F}_q$ will refer to the set of integers, rationals, reals, complex numbers, and the finite field with $q$ elements (where $q$ is a prime power) respectively. All rings are assumed to be commutative with unity.

2.1 Affine Varieties

Given a field $k$, the *affine* $n$-space over $k$, denoted by $\mathbb{A}^n_k$, is simply the set of all $n$-tuples of elements of $k$. Elements of $\mathbb{A}^n_k$ will be called *points*. We shall write $R = k[x_1, \ldots, x_n]$ to denote the polynomial ring over $k$ in indeterminates $x_1, \ldots, x_n$. It is then evident that every polynomial $f \in R$ defines a map (via evaluation) $\mathbb{A}^n_k \to k$. Hence, we may speak of the *zero set* of a polynomial $f \in R$ defines a map (via evaluation) $\mathbb{A}^n_k \to k$. Hence, we may speak of the zero set of $J$, denoted $\mathbb{V}(J)$, is $\{p \in \mathbb{A}^n_k : f(p) = 0 \text{ for all } f \in J\}$. By the Hilbert Basis Theorem every ideal $J$ of $R$ is finitely generated. Hence, given an ideal $J \subseteq R$, $\mathbb{V}(J) = \mathbb{V}(f_1, \ldots, f_m)$, where $f_1, \ldots, f_m \in R$ form a generating set for $J$. Similarly, if $\mathcal{X} \subseteq \mathbb{A}^n_k$, one can define the *ideal of* $\mathcal{X}$, denoted $\mathbb{I}(\mathcal{X})$, as the set $\{f \in R : f(p) = 0 \text{ for all } p \in \mathcal{X}\}$. 
If $X \subseteq \mathbb{A}^n_k$, then $X$ is said to be an algebraic set if there exists an ideal $J$ of $R$ satisfying $V(J) = X$. One can endow $\mathbb{A}^n_k$ with a topology; in which the open subsets of $\mathbb{A}^n_k$ are the complements of algebraic sets. This is the Zariski topology on $\mathbb{A}^n_k$.

An affine variety $X \subseteq \mathbb{A}^n_k$ is one where $I(X)$ is a prime ideal of $R$. An open subset of an affine variety is called a quasi-affine variety. If $f \in R$ is irreducible, then $V(f)$ is an affine variety since $R$ is a unique factorization domain. An affine variety defined by a single polynomial is an example of a hypersurface.

Given an affine variety $X \subseteq \mathbb{A}^n_k$, the affine coordinate ring of $X$ is the quotient $R/I(X)$, denoted $k[X]$. Intuitively, one can think of $k[X]$ as the ring of polynomial functions on $X$. As $I(X)$ is prime, $k[X]$ is a domain, and thus we can consider its field of fractions, namely $k(X) := \left\{ \frac{f}{g} : f, g \in k[X] \text{ and } g \neq 0 \right\}$. The field $k(X)$ is called the function field of $X$ and we define the dimension of $X$ to be trdeg $k(X)/k$; i.e., the transcendence degree of the field extension $k(X)/k$.

### 2.2 Projective Varieties

Let $V$ be a vector space over $k$. One can define an equivalence relation $\sim$ on $V \setminus \{0\}$ by declaring that two vectors $u$ and $v$ are equivalent, written $u \sim v$, if and only if $u = \lambda v$ for some $\lambda \in k$. The projective space of $V$, denoted $\mathbb{P}(V)$ is then $(V \setminus \{0\})/\sim$. In particular, taking $V = k^{n+1}$, we define the $n$-dimensional projective space as $\mathbb{P}^n_k := \mathbb{P}(V)$. Then $\mathbb{P}^n_k$ is the set of all lines through the origin in $V$.

We write $[a_0 : \cdots : a_n] = \{(\lambda a_0, \ldots, \lambda a_n) : \lambda \in k^*\}$ for the equivalence class of a point $p$. We call this the homogeneous coordinates for $p$.

Write $S = k[X_0, \ldots, X_n]$ for the polynomial ring over $k$ in indeterminates $X_0, \ldots, X_n$. Here we are using capital letters to distinguish between affine and projective coordinates. Unlike in the affine case, polynomials in $S$ do not, a priori, give rise to well-defined functions on $\mathbb{P}^n_k$; since any non-zero scalar multiple of a given point represents the same point of $\mathbb{P}^n_k$, an arbitrary polynomial function may very well take on different values for different representatives. To account for such behaviors, one needs to first turn $S$ into a graded ring.
**Definition 2.1.** A graded ring is a ring $R$ together with a decomposition into abelian groups: $R = \bigoplus_{d \geq 0} R_d$, where $R_i R_j \subseteq R_{i+j}$ for each $i, j \geq 0$. An ideal $J \subseteq R$ is said to be **graded** or homogeneous if $J = \bigoplus_{d \geq 0} (R_d \cap J)$. Furthermore, elements of $R_d$ are said to be **homogeneous of degree** $d$ and $R_d$ is called the homogeneous component of $R$ of degree $d$.

**Remark 2.2.** It follows from Definition 2.1 that an ideal $J$ of a graded ring $R$ is homogeneous if and only if each $f \in J$ can be written as $f = f_1 + \cdots + f_N$, where each $f_j$ is a homogeneous element of $J$ of degree $d_j$. It is then not hard to show that an ideal $J$ is homogeneous if and only if it can be generated by homogeneous elements. Furthermore, a homogeneous ideal $J$ is prime if and only if for any pair of homogeneous elements $f, g \in R$ with $fg \in J$, one has $f \in J$ or $g \in J$. Lastly, the property of being homogeneous is preserved under the usual ideal operations of sums, products, intersections, and taking radicals.

**Definition 2.3.** Let $R$ be a graded ring. A **graded** $R$-module is an $R$-module $M$ with a decomposition into a direct sum $\bigoplus_{n \in \mathbb{Z}} M_n$, where $M_n$ are subgroups of $M$ satisfying $R_d M_n \subseteq M_{d+n}$ for all integers $d$ and $n$. Elements of $M_n$ are said to be homogeneous of degree $n$. If $M$ and $M'$ are two graded $R$-modules, then an $R$-homomorphism of degree $r$ is any $R$-homomorphism $\phi: M \to M'$ satisfying $\phi(M_n) \subseteq M'_{n+r}$. Moreover, given graded $R$-modules $M$ and $M'$, we write $\text{Hom}_R(M, M')_d$ for the group of homogeneous $R$-homomorphisms of degree $d$. Similarly, we write $\text{Hom}_R(M, M')$ for the graded $R$-module $\bigoplus_{d \in \mathbb{Z}} \text{Hom}_R(M, M')_d$.

It is now evident how to turn $S$ into a graded ring; let $S_d$ be the set of all $k$-linear combinations of monomials of total degree $d$. If $f \in S$ is homogeneous of degree, say $d \geq 0$, then note that $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$, so one needs to consider quotients of homogeneous polynomials of the same degree. Indeed, if $f, g \in S$ are both homogeneous of degree $d \geq 0$, then

$$\frac{f(\lambda a_0, \ldots, \lambda a_n)}{g(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)},$$

so as long as $g(a_0, \ldots, a_n) \neq 0$, one obtains a well-defined function on $\mathbb{P}^n_k$.

Similar to affine algebraic sets, we say that a subset $\mathcal{V}$ of $\mathbb{P}^n_k$ is **algebraic** if it is the set of points where a set $T$ of homogeneous elements of $S$ vanishes. With this definition, one can show that a finite union of algebraic sets is again algebraic. Moreover, an arbitrary intersection of algebraic sets is also algebraic, and since the empty set and the whole space are algebraic, one can define the **Zariski topology** on $\mathbb{P}^n_k$ in an analogous fashion to the affine case; i.e., we take open sets to be the complements of algebraic sets.
A projective variety is an irreducible algebraic set in $\mathbb{P}^n_k$ and any open subset of such a set is called a quasi-projective variety. The homogeneous ideal of a subset $\mathcal{Y}$ of $\mathbb{P}^n_k$, denoted $\mathbb{I}(\mathcal{Y})$, is the ideal generated by $\{f \in S : f$ is homogeneous and $f(p) = 0$ for all $p \in \mathcal{Y}\}$. One defines the projective coordinate ring of $\mathcal{Y}$ as $S/\mathbb{I}(\mathcal{Y})$.

We end this section with a brief discussion concerning the covering of $\mathbb{P}^n_k$ with affine $n$-spaces. Write, for each $0 \leq j \leq n$, $H_j$ for the zero set of $X_j$; i.e., $H_j = \mathbb{V}(X_j)$. The set $H_j$ is called the hyperplane (relative to $X_j$) at infinity. Set $U_j := \mathbb{P}^n_k \setminus H_j$. By definition, each $U_j$ is open and $\mathbb{P}^n_k = \bigcup_{j=0}^{n} U_j$ since if $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n_k$, then $a_i \neq 0$ for some $0 \leq i \leq n$, and so $p \in U_i$. There is a natural bijection, for each $0 \leq j \leq n$,

$$\phi_j : U_j \longrightarrow \mathbb{A}^n_k$$

$$[a_0 : \cdots : a_n] \mapsto \left(\frac{a_0}{a_j}, \ldots, \frac{a_j-1}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right).$$

**Proposition 2.4.** The map $\phi_j$ of (2.1) is a homeomorphism.

**Proof.** See [7, Chapter I, Proposition 2.2].

In light of Proposition 2.4, we say that $\mathbb{P}^n_k$ has a covering by affine patches and we call $U_j$ the $j^{th}$ affine piece of $\mathbb{P}^n_k$. In particular, if $\mathcal{Y} \subset \mathbb{P}^n_k$ is a projective variety, then the above covering induces an obvious affine covering of $\mathcal{Y}$ via $\mathcal{Y} = \bigcup_{j=0}^{n} (\mathcal{Y} \cap U_j)$. Moreover, every affine variety $\mathcal{X}$ can be completed to a projective variety via homogenizing the polynomials in $\mathbb{I}(\mathcal{X})$; given $f(x_1, \ldots, x_n) \in \mathbb{I}(\mathcal{X})$ with $\deg f = m$, then $\overline{f} := X_0^m f \left(\frac{x_1}{X_0}, \ldots, \frac{x_n}{X_0}\right)$ is the homogenization of $f$. In a similar fashion, one can dehomogenize a homogeneous element in $S$ by simply evaluating at $(1, x_1, \ldots, x_n)$ to obtain an element of $R = k[x_1, \ldots, x_n]$. The above discussion allows us to view affine varieties as being embedded (in the natural way) in $\mathbb{P}^n_k$. Hence, we can define the closure of an affine variety $\mathcal{X}$ via the following: the closure of $\mathcal{X}$, denoted $\overline{\mathcal{X}}$, is the projective variety whose ideal is obtained by homogenizing the elements of $\mathbb{I}(\mathcal{X})$. Henceforth, a variety will refer to one of affine, projective, quasi-affine, or quasi-projective varieties.

### 2.3 Singular and Non-singular Varieties

One notion that we will be dealing with throughout Chapter 4 is that of a blow-up (see Section 2.5) of a variety at a point. As we will shortly see, blowing up points is one of the
most important tools used to resolve singularities of a given variety. Here, we define the notion of a singularity of a variety and give a convenient method that allows us to determine when a variety is singular.

**Definition 2.5.** Let $\mathcal{X} \subseteq \mathbb{P}^n_k$ be a variety of dimension $d$ with ideal $\mathbb{I}(\mathcal{X}) = \langle f_1, \ldots, f_r \rangle$. Let $p \in \mathcal{X}$. We say that $\mathcal{X}$ is nonsingular (or smooth) at $p$ if and only if the Jacobian matrix

$$
\begin{bmatrix}
\frac{\partial f_1}{\partial X_0}(p) & \frac{\partial f_1}{\partial X_1}(p) & \cdots & \frac{\partial f_1}{\partial X_n}(p) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_r}{\partial X_0}(p) & \frac{\partial f_r}{\partial X_1}(p) & \cdots & \frac{\partial f_r}{\partial X_n}(p)
\end{bmatrix},
$$

has rank $n - d$. Otherwise, $\mathcal{X}$ is said to be singular at $p$.

Observe that in the case where $\mathcal{X}$ is a hypersurface (i.e., $\mathbb{I}(\mathcal{X}) = \langle f \rangle$), this tells us that $\mathcal{X}$ is singular at $p$ if and only if all partial derivatives of $f$ vanish simultaneously at $p$.

**Theorem 2.6.** Let $\mathcal{X}$ be a variety. The set of all singular points of $\mathcal{X}$, denoted by $\text{Sing } \mathcal{X}$ is a proper closed subset of $\mathcal{X}$.

**Proof.** See [7, Chapter I, Theorem 5.3].

**Example 2.7.** Consider the variety $\mathcal{X} := \mathbb{V}(y^2 - x^3 - x^2) \subset \mathbb{A}^2_k$. Then $\frac{\partial f}{\partial x} = -3x^2 - 2x$ and $\frac{\partial f}{\partial y} = 2y$. It is evident that both partial derivatives vanish at $(0,0)$, and so $\mathcal{X}$ is singular at the origin. Similarly, when $\mathcal{X} := \mathbb{V}(xz - y^2) \subset \mathbb{A}^3_k$, then one easily sees that $(0,0,0)$ is again a singularity (in fact the only singularity) of $\mathcal{X}$.

### 2.4 Morphisms and Rational Functions

Our main goal in this section is to develop the necessary tools in order to discuss blow-ups. So let us begin with an overview of morphisms.

#### 2.4.1 Morphisms

**Definition 2.8.** Let $\mathcal{Y} \subseteq \mathbb{P}^n_k$ be a quasi-projective variety. A function $f: \mathcal{Y} \to k$ is said to be regular at a point $p \in \mathcal{Y}$ if there exists an open subset $U \subseteq \mathcal{Y}$ containing $p$ and homogeneous polynomials of the same degree $f, g \in S$ with $h \neq 0$ on $U$ such that $f = \frac{g}{h}$ on all of $U$. 
Lemma 2.9. A regular function is necessarily continuous.

Proof. See [7, Chapter I, Lemma 3.1]. □

Definition 2.10. Given two varieties $\mathcal{X}$ and $\mathcal{Y}$, a map $\phi: \mathcal{X} \to \mathcal{Y}$ is a morphism if $\phi$ is continuous and for each open subset $V$ of $\mathcal{Y}$ and every regular function $f: V \to k$, the map $f \circ \phi: \phi^{-1}(V) \to k$ is again regular.

We say that two varieties $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic if there exists a morphism $\phi: \mathcal{X} \to \mathcal{Y}$ admitting an inverse morphism $\psi: \mathcal{Y} \to \mathcal{X}$ with $\phi \circ \psi = \text{id}_Y$ and $\psi \circ \phi = \text{id}_X$. Before we define the ring of regular functions on a variety, we define a local ring.

Definition 2.11. A ring $R$ is called a local ring if $R$ has a unique maximal ideal $m$. Equivalently, a local ring is one where $R \setminus m$ is the set of all units in $R$.

Definition 2.12. Let $\mathcal{Y}$ be a variety. For $p \in \mathcal{Y}$, we write $\mathcal{O}_{\mathcal{Y},p}$ for the ring of regular functions at $p$; i.e.,

$$\mathcal{O}_{\mathcal{Y},p} := \{(f,U): U \subset \mathcal{Y} \text{ is open, } p \in U \text{ and } f: U \to k \text{ is regular}\}/\sim,$$

where $(f,U) \sim (g,V)$ if and only if $f|_{U \cap V} = g|_{U \cap V}$. Moreover, given an open $U$ in $\mathcal{Y}$, we define $\mathcal{O}_\mathcal{Y}(U) = \bigcap_{p \in U} \mathcal{O}_{\mathcal{Y},p}$.

Remark 2.13. Note that $\mathcal{O}_{\mathcal{Y},p}$ is in fact a local ring. Indeed, the ideal $m_p$ of all regular functions that vanish at $p$ is a maximal ideal as $\mathcal{O}_{\mathcal{Y},p}/m_p \cong k$ via the evaluation map and the first isomorphism theorem. If $f \notin m_p$, then $1/f$ is clearly regular at $p$ and is the inverse of $f$. So $\mathcal{O}_{\mathcal{Y},p}$ is a local ring with maximal ideal $m_p$ by Definition 2.11.

It is worth mentioning that a morphism $\phi: \mathcal{X} \to \mathcal{Y}$ pulls back regular functions on $\mathcal{Y}$ to regular functions on $\mathcal{X}$.

2.4.2 Rational and Birational Maps

We begin with the notion of rational maps.

Definition 2.14. Let $\mathcal{X}$ and $\mathcal{Y}$ be two varieties. A rational map $\phi: \mathcal{X} \to \mathcal{Y}$ consists of the data of an equivalence class of pairs $(U,\phi_U)$ such that $U \subset \mathcal{X}$ is open and $\phi_U: U \to \mathcal{Y}$ is a morphism. Two pairs are considered equivalent if they agree on intersections.
However, more care is required to define the composition of rational functions. Indeed, suppose \( \phi: X \to Y \) and \( \phi': Y \to Z \) are rational maps of varieties. If \( V \) and \( U \) are open in \( Y \) and \( X \) respectively, \( \phi_U: U \to Y \) and \( \phi'_V: V \to Z \) are morphisms with \( \text{im } \phi_U \) dense in \( Y \), then \( \phi_U^{-1}(V) \) is again open in \( X \) so that \( (\phi^{-1}(V), \phi'_V \circ \phi_U) \) defines a rational map \( X \to Z \), which we denote by \( \phi' \circ \phi \).

**Definition 2.15.** A birational map \( \phi: X \to Y \) is a rational map admitting a rational inverse; i.e., a rational map \( \psi: Y \to X \) with \( \phi \circ \psi = \text{id}_Y \) and \( \psi \circ \phi = \text{id}_X \). In such a case, we say that \( X \) and \( Y \) are birationally equivalent.

Before we introduce blow-ups, we note the following important result concerning birational equivalence of two varieties.

**Lemma 2.16.** For any two varieties \( X \) and \( Y \), the following are equivalent:

- \( X \) and \( Y \) are birationally equivalent.
- \( X \) and \( Y \) contain isomorphic open subsets.
- \( k(X) \cong k(Y) \) as \( k \)-algebras.

**Proof.** See [7, Chapter I, Corollary 4.5].

### 2.5 Blowing Up

In this section we discuss an important example of birational equivalence in algebraic geometry; namely, the notion of blow-up of a point in \( \mathbb{A}^n_k \). Recall that if \( X \subseteq \mathbb{P}^n_k \) and \( Y \subseteq \mathbb{P}^m_k \) are any two varieties, their product \( X \times Y \subseteq \mathbb{P}^n_k \times \mathbb{P}^m_k \) is again a variety via the well-known Segre embedding (see [13, Chapter I, Section 5] for a complete discussion of products of varieties).

**Lemma 2.17.** Every closed subvariety of \( \mathbb{A}^n_k \times \mathbb{P}^m_k \) is given by a system of equations \( f_j(x_1, \ldots, x_n; y_0: \ldots: y_m) = 0 \) for \( 1 \leq j \leq r \) such that each \( f_j \) is homogeneous in \( y_0, \ldots, y_m \).

**Proof.** See [13, Chapter 1, Section 5, Theorem 1].

Suppose \( p = (a_1, \ldots, a_n) \in \mathbb{A}^n_k \) is given. Since there exists a change of coordinates bringing \( p \) to the origin, it is no loss of generality to assume \( p = (0, \ldots, 0) \). Write \( O \) for
the origin of \( \mathbb{A}^n_k \) and let \( x_1, \ldots, x_n \) and \( Y_1, \ldots, Y_n \) denote the coordinates of \( \mathbb{A}^n_k \) and \( \mathbb{P}^{n-1}_k \) respectively. Note \( Y_1, \ldots, Y_n \) represent homogeneous coordinates. Define the blow-up of \( \mathbb{A}^n_k \) at \( O \), denoted \( \mathcal{B} \), as the set of all pairs \((q, \ell_q)\), where \( q \in \mathbb{A}^n_k \) and \( \ell_q \in \mathbb{P}^{n-1}_k \) is the line passing through \( q \) and \( O \) in \( \mathbb{A}^n_k \). That is

\[
\mathcal{B} := \{(q, \ell_q) : q \in \mathbb{A}^n_k \} \subseteq \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k.
\]

It follows that \( \mathcal{B} \) is the quasi-projective variety given by the vanishing set of \( x_1Y_j - x_jY_i \) for all \( 1 \leq i, j \leq n \). Indeed, \( q = (x_1, \ldots, x_n) \) lying on the line \( \ell = [Y_1 : \cdots : Y_n] \) is equivalent to \( q \) being a scalar multiple of \((Y_1, \ldots, Y_n)\). This is equivalent to requiring that all \( 2 \times 2 \) minors of the matrix

\[
\begin{bmatrix}
x_1 & \cdots & x_n \\
Y_1 & \cdots & Y_n
\end{bmatrix}
\]

vanish. This means \( x_i = (x_jY_j^{-1})Y_i \) for all \( 1 \leq i \leq n \) and a suitable \( j \) for which \( Y_j \neq 0 \). As \( Y_1, \ldots, Y_n \) are homogeneous coordinates, \( Y_j \neq 0 \) for some \( j \), so that \( x_jY_j^{-1} \) is the desired scalar we were after. Restricting the projection morphism \( \mathbb{A}^n_k \times \mathbb{P}^n_k \to \mathbb{A}^n_k \) gives a natural morphism \( \phi: \mathcal{B} \to \mathbb{A}^n_k \).

**Theorem 2.18.** Let \( \phi: \mathcal{B} \to \mathbb{A}^n_k \) be as above. Then

1. the map \( \phi \) above induces an isomorphism \( \mathcal{B} \setminus \phi^{-1}(O) \to \mathbb{A}^n_k \setminus O \);
2. \( \phi^{-1}(O) \cong \mathbb{P}^{n-1}_k \);
3. the map \( \ell \mapsto \phi^{-1}(\ell \setminus O) \cap \phi^{-1}(O) \) gives a one-to-one correspondence between lines \( \ell \) through \( O \) and \( \phi^{-1}(O) \); and
4. \( \mathcal{B} \) is irreducible. That is, \( \mathcal{B} \) cannot be written as a union two proper closed subsets.

**Proof.** For (1), suppose \( p = (a_1, \ldots, a_n) \in \mathbb{A}^n_k \) is non-zero. Then \( a_i \neq 0 \) for some \( i \). If \( p \times [Y_1 : \cdots : Y_n] \in \mathbb{B} \setminus \phi^{-1}(p) \), then \( Y_j = (a_ja_i^{-1})Y_i \) for each \( 1 \leq j \leq n \), thus uniquely determining \([Y_1 : \cdots : Y_n] \). Hence, \( p \) pulls back to a unique point in \( \mathcal{B} \). The map \( \psi: \mathbb{A}^n_k \setminus O \to \mathcal{B} \setminus \phi^{-1}(O) \), where \( p \mapsto p \times [a_1 : \cdots : a_n] \), is immediately seen to be the inverse morphism to \( \phi \). The second assertion is immediate as any point in the pull back of the origin must be of the form \( O \times [Y_1 : \cdots : Y_n] \) with no restrictions on \([Y_1 : \cdots : Y_n] \). To establish the third assertion, begin by letting \( \ell \) denote a line through \( O \) in \( \mathbb{A}^n_k \). Consider the pull back (along \( \phi \)) of \( \ell \setminus O \) and write \( L = \phi^{-1}(\ell \setminus O) \). Observe that \( \ell \) has a parametrization
given by \( x_i = a_i t \) for some \( a_1, \ldots, a_n, t \in k \). The first part of the Theorem tells us that every point on \( \ell \setminus O \) pulls back to a unique point on \( L \). So we also have a parametrization of \( L \) given by \( x_i = a_i t \) and \( Y_i = a_i t \) with \( t \neq 0 \) and \( a_i \) as before. But the \( Y_i \) are homogeneous coordinates and so scaling is irrelevant. Hence, the above parametrization is equivalent to \( x_i = a_i t \) and \( Y_i = a_i \), where \( t \) is now allowed to be zero. These equations describe the closure \( L \) of \( L \) in \( \mathcal{B} \). Consider \( L \cap \phi^{-1}(O) \). This is clearly the point \([a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}_k \). So the map sending \( \ell \) to \([a_1 : \cdots : a_n] \) is the desired correspondence. Lastly, note that \( \mathcal{B} \setminus \phi^{-1}(O) \cong \mathbb{A}^n_k \setminus O \) and the latter is irreducible. By (3), every point of \( \phi^{-1}(O) \) is in the closure of a line (subset) in \( \mathcal{B} \setminus \phi^{-1}(O) \), thus proving that \( \mathcal{B} \setminus \phi^{-1}(O) \) is dense in \( \mathcal{B} \). It is a well-known fact in topology that a subset \( U \) of a topological space \( T \) is irreducible if and only if its closure \( \overline{U} \) is irreducible. As \( \mathcal{B} \setminus \phi^{-1}(O) = \mathcal{B} \) and \( \mathcal{B} \setminus \phi^{-1}(O) \) is irreducible, (4) follows. 

Suppose \( \mathcal{X} \subseteq \mathbb{A}^n_k \) is variety containing \( O \). We define the blow-up of \( \mathcal{X} \) at \( O \), denoted by \( \mathcal{B}_X \), to be the closure (in \( \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \)) of \( \phi^{-1}(\mathcal{X} \setminus O) \). We call this the strict transform of \( \mathcal{X} \). The set \( E_O := \mathcal{B}_X \cap \phi^{-1}(O) \) is referred to as the exceptional subvariety of \( \mathcal{B}_X \). Furthermore, \( \mathcal{B}_X \) is now easily seen to be birationally equivalent to \( \mathcal{X} \setminus O \) via part (1) of Theorem 2.18. To illustrate these notions, we give the following examples.

**Example 2.19.** Let us compute the blow-up of \( \mathcal{X} := \mathbb{V}(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2_k \) at \( O \). Recall from Example 2.7 in Section 2.3 that \( \mathcal{X} \) has a singularity at the origin, so let us use the homogeneous machinery we have developed so far to resolve this singularity. Let \( U, V \) denote the homogeneous coordinates of \( \mathbb{P}^1_k \). Then we know that in order to determine \( \mathcal{B}_X \), we must consider the equations \( y^2 = x^3 + x^2 \) and \( xV = yU \). As \( \mathcal{B}_X \subseteq \mathbb{A}^2_k \times \mathbb{P}^1_k \) and \( \mathbb{P}^1_k \) is covered by the two affine pieces \( U \neq 0 \) and \( V \neq 0 \), we study the behavior of \( \mathcal{B}_X \) on each of these affine pieces separately. Suppose \( U \neq 0 \). Then we may work with the affine coordinates \( v = V/U, x, \) and \( y \). The defining equations then become \( y^2 = x^3 + x^2 \) and \( y = xv \). This gives \( x^2v^2 - x^3 - x^2 = x^2(v^2 - x - 1) = 0 \). There are two possibilities. If \( x = 0 \), this forces \( y = 0 \) and we have no restriction on \( v \). This is precisely the exceptional component of \( \mathcal{B}_X \). Otherwise, \( x = v^2 - 1 \) which immediately implies \( y^2 = (v^2 - 1)^2 + (v^2 - 1)^3 = v^2(v^2 - 1)^2 \). Hence, we have obtained a parametrization (in \( \mathbb{A}^2_k \)) of \( \mathcal{B}_X \setminus \phi^{-1}(O) \) on the chart \( U \neq 0 \). Note \( E_O \) is just the set \( \{u = \pm 1\} \) which consists of precisely the slopes of the branches of \( \mathcal{X} \) at the origin.
Example 2.20. Let us consider the cone $X := \mathbb{V}(xz - y^2) \subseteq \mathbb{A}^3_k$, which we know has an ordinary double point at the origin. Write $U, V, T$ for the homogeneous coordinates of $\mathbb{P}^2_k$. To determine $B_X$, we consider the equations $xz = y^2, xV = yU, xT = zU$, and $yT = zV$. To obtain a parametrization of $B_X$ we must first pick an affine chart. So suppose $T \neq 0$ and let $u = U/T$ and $v = V/T$ denote affine coordinates. We then have $xz = y^2, xv = yu, x = zu$, and $y = zv$. Note $xv = yu$ is superfluous at this point so we may disregard this equation. Making the appropriate substitution gives $z^2u - z^2v^2 = z^2(u - v^2) = 0$. If $z = 0$, this forces $x = y = 0$ and no restrictions on $u, v$. Otherwise, $u = v^2$ and one obtains a parametrization of the strict transform of $X$ (on the chart $T \neq 0$).

Observation 2.21. We see in the previous two examples that blowing up resolves singularities in the sense that $B_X$ is non-singular and admits a surjective birational morphism to $X$.

2.6 Sheaves and Differential Forms

In this section we recall the definition of a sheaf and give some examples. We will then recall the notion of differentials, which are a main component of this thesis. We first present a completely algebraic definition of differentials and then proceed to define the sheaf of differential forms. For the remainder of this section, we assume $R$ and $S$ are commutative rings with unity.

2.6.1 Sheaves

Definition 2.22. A presheaf $\mathcal{F}$ of abelian groups on a topological space $\mathcal{X}$ is the following data:

1. to each open subset $U \subseteq \mathcal{X}$ we assign an abelian group $\mathcal{F}(U)$;

2. for every inclusion of open subsets $U \subseteq V$ of $\mathcal{X}$ we have a map of abelian groups $\rho^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$, which we call the restriction map;

such that $\mathcal{F}(\emptyset) = 0$, for each open $U \subseteq \mathcal{X}$ we have $\rho^U_U = \text{id}_\mathcal{F}(U)$, and for every inclusion of open subsets $U \subseteq V \subseteq W$ of $\mathcal{X}$, the restriction maps commute; i.e., $\rho^W_U = \rho^V_U \circ \rho^W_V$. 

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Definition 2.23. A presheaf $\mathcal{F}$ of abelian groups on a topological space $\mathcal{X}$ is called a sheaf if it satisfies the following two axioms.

$A_1$ Identity Axiom: Given an open subset $U$ of $\mathcal{X}$ and an open cover $\{U_\alpha\}_{\alpha \in I}$ of $U$ and $f, g \in \mathcal{F}(U)$ such that $\rho_{U \cap U_\alpha}^U(f) = \rho_{U \cap U_\alpha}^U(g)$ for all $i$, then $f = g$.

$A_2$ Gluing Axiom: Let $U$ and $\{U_\alpha\}_{\alpha \in I}$ be as above and let $f_\alpha \in \mathcal{F}(U_\alpha)$ for all $\alpha$ be such that $\rho_{U \cap U_\alpha}^U(f_\alpha) = \rho_{U \cap U_\beta}^U(f_\beta)$ for all distinct $\alpha, \beta \in I$, then there exists an $f \in \mathcal{F}(U)$ such that $\rho_{U \cap U_\alpha}^U(f) = f_\alpha$.

The elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over $U$. The set of all such sections is denoted by $\Gamma(U, \mathcal{F})$. In particular, when $U = \mathcal{X}$, we refer to the elements of $\Gamma(\mathcal{X}, \mathcal{F})$ as global sections of the sheaf $\mathcal{F}$. It is worth noting that the identity axiom implies that there is at most one way to glue sections, and the gluing axiom implies that there is at least one way to glue; i.e., sections glue together in a unique manner.

Definition 2.24. Let $\mathcal{X}$ be a variety. For every affine open subset $U$ of $\mathcal{X}$, we have the coordinate ring $k[U]$. Since the affine opens form a basis for the Zariski topology, we can use them to define a sheaf [14, Theorem 2.7.1], henceforth denoted by $\mathcal{O}_\mathcal{X}$, such that $\Gamma(U, \mathcal{O}_\mathcal{X}) = k[U]$. This is a sheaf of rings on $\mathcal{X}$ and is referred to as the structure sheaf of $\mathcal{X}$.

Definition 2.25. Let $\mathcal{X}$ be a topological space and let $\mathcal{O}_\mathcal{X}$ be a sheaf of rings on $\mathcal{X}$. The pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is called a ringed space.

If $\mathcal{X}$ is a variety with the structure sheaf $\mathcal{O}_\mathcal{X}$, then the pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an example of a ringed space. A sheaf $\mathcal{F}$ on $\mathcal{X}$ is said to be a sheaf of $\mathcal{O}_\mathcal{X}$-modules if for each open subset $U$ of $\mathcal{X}$, the set $\mathcal{F}(U)$ is an $\mathcal{O}_\mathcal{X}(U)$-module and the obvious compatibility conditions regarding restrictions hold. According to the terminology introduced in Serre’s work on coherent sheaves and graded modules [12] (see also Chapter 3), sheaves of $\mathcal{O}_\mathcal{X}$-modules are referred to as algebraic sheaves.

Definition 2.26. Let $\mathcal{X}$ be a variety and suppose $\mathcal{F}$ is an algebraic sheaf. We say that $\mathcal{F}$ is free of rank $n$ if $\mathcal{F} \cong \mathcal{O}_\mathcal{X}^n$. It is locally free if there is an open covering of $\{U_j\}$ for $\mathcal{X}$ for which $\mathcal{F}(U_j)$ is a free $\mathcal{O}_\mathcal{X}(U_j)$-module.
2.6.2 Kähler Differentials

**Definition 2.27.** Let \( R \) be a commutative ring and suppose \( M \) is an \( R \)-module. A *derivation* from \( R \) to \( M \) is a map \( d: R \to M \) satisfying \( d(f + g) = d(f) + d(g) \) and \( d(fg) = gd(f) + fd(g) \) for all \( f, g \in R \). The second property is usually referred to as the Leibniz rule.

**Definition 2.28.** If \( R \) is an \( S \)-algebra via \( f: S \to R \) and \( M \) is an \( R \)-module, then an \( S \)-linear derivation \( R \to M \) is characterized by the property that \( d(f(s)) = 0 \) for all \( s \in S \).

**Lemma 2.29.** Consider the same setup as in Definition 2.28. The \( S \)-linear derivations from \( R \) to \( M \) form an \( R \)-module \( \text{Der}_S(R, M) \).

**Proof.** Given an \( S \)-linear derivation \( d: R \to M \) and \( r \in R \), define \( r \cdot d: R \to M \) via \( r' \mapsto r(d(r')) \in M \).

**Definition 2.30.** The module of *relative Kähler differential forms* of \( R \) (over \( S \)) is an \( R \)-module \( \Omega_{R/S} \), together with an \( S \)-linear derivation \( D: R \to \Omega_{R/S} \) such that the map \( \text{Hom}_R(\Omega_{R/S}, M) \to \text{Der}_S(R, M) \) (for every \( R \)-module \( M \)) defined by \( \phi \mapsto \phi \circ D \) is an isomorphism.

**Example 2.31.** Let \( k = \mathbb{R} \) and consider the polynomial ring \( R = k[x, y] \). Then the usual partial differentiation operator \( \frac{\partial}{\partial x} \) from calculus is a \( k[y] \)-linear derivation from \( R \) to itself.

**Proposition 2.32.** Consider the polynomial ring \( R = S[x_1, \ldots, x_n] \). Then

\[
\Omega_{R/S} = \bigoplus_{j=1}^n Rdx_j.
\]

**Proof.** Consider the map \( R \to R^n \) given by \( f \mapsto \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \). Note this is an \( S \)-linear derivation. By the universal property, we have a map \( \phi: \Omega_{R/S} \to R^n \) given by \( df \mapsto \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \). We claim that \( \phi \) is an isomorphism. Indeed, note \( \phi \) is surjective since \( dx_j \mapsto (0, \ldots, 0, 1, 0, \ldots, 0) \), where there is a 1 in the \( j \)th position. Consider the map \( \psi: R^n \to \Omega_{R/S} \) given by \( (a_1, \ldots, a_n) \mapsto \sum_{j=1}^n a_j dx_j \). We wish to show that \( \psi \) is the desired inverse to \( \phi \). Indeed, note

\[
\phi(\psi((a_1, \ldots, a_n))) = \phi \left( \sum_{j=1}^n a_j dx_j \right) = \phi(a_1 dx_1) + \cdots + \phi(a_n dx_n) = (a_1, \ldots, a_n).
\]
This shows that $\phi \circ \psi = \text{id}_{\mathbb{R}^n}$. Conversely, we wish to show that $\psi \circ \phi = \text{id}_{\Omega_{\mathbb{R}/\mathbb{S}}}$. That is, we need $\psi(\phi(df)) = df$ for all $f \in \mathbb{R}$. But this is equivalent to showing that $df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$.

Since both sides respect addition and agree on monomials of degree 1, the result follows.

The following lemma shall provide us with a convenient computational tool.

**Lemma 2.33.** [Second Exact Sequence] Let $R$ be a ring and suppose $R$ is an $S$-algebra. Let $I$ be an ideal of $R$ and set $T = R/I$. The sequence of $T$-modules

$$I/I^2 \xrightarrow{d} T \otimes_S \Omega_{\mathbb{R}/\mathbb{S}} \xrightarrow{D} \Omega_{T/S} \xrightarrow{} 0,$$

where $D(c \otimes db) = cdb$ and $d(f + I^2) = 1 \otimes df$, is exact.

**Proof.** See [7, Chapter II, Theorem 8.4A]. However, since there is only a reference to the proof of this result in [7], we shall provide all the details here. First we check that the map $d$ is well-defined. For simplicity, let us write $f$ for $f + I^2$. Suppose we have two coset representatives $f, g \in I$ such that $f - g \in I^2$. Then we may find $h_1, h_2 \in I$ such that $f - g = h_1h_2$. Applying $d$ gives

$$d(f - g) = 1 \otimes d(h_1h_2) = 1 \otimes (h_2dh_1 + h_1dh_1). \quad (2.2)$$

Recall that if $R'$ is any commutative ring with unity, $I'$ is an ideal of $R$, and $N'$ is an $R'$-module, then $R'/I' \otimes N' \cong N'/I'N'$ as $R'$-modules. Thus we have an isomorphism

$$T \otimes_R \Omega_{R/S} = R/I \otimes_R \Omega_{R/S} \cong \Omega_{R/S} / I\Omega_{R/S},$$

given by $(r + I \otimes dr') \mapsto rdr' + I\Omega_{R/S}$. Hence, the right hand side of (2.2) is zero, and we have $d(f - g) = d(f) - d(g) = 0$, proving that $d$ is well-defined.

Taking $N' = I$ in above we also have an isomorphism $R/I \otimes_R I \cong I/I^2$. As $R$ is an $S$-algebra, we have the universal derivation $\delta: R \rightarrow \Omega_{R/S}$. Consider the restriction of $\delta$ to $I$.

Let $\nu := \delta|_I$. It then follows that the map $d$ in consideration is, in fact, the map

$$\text{id}_{R/I} \otimes \nu : R/I \otimes_R I \rightarrow R/I \otimes_R \Omega_{R/S}.$$

Moreover, $I/I^2$ is clearly a $T$-module. Also, $d$ is a map of $T$-modules. Indeed, let $a \in I$ and let $t \in T$ be given. Then

$$d(at) = 1 \otimes (tda + adt) = (1 \otimes tda) + (1 \otimes adt) = 1 \otimes tda = t(1 \otimes da) = td(a),$$
where the third equality follows from a similar argument to the one given in the first paragraph.

Note that $D$ is surjective by definition of $T$. So it remains to verify that $\Omega_{T/S}$ is the cokernel of $d$. Indeed, consider $T \otimes_S \Omega_{R/S}$ as a $T$-module. It is generated by elements of the form $dr$ for $r \in R$ subject to linearity, the Leibniz rule, and $ds = 0$ for all $s \in \text{im } f$, where $f : S \to R$ exhibits $R$ as an $S$-algebra. Similarly, $\Omega_{T/S} = \Omega_{(R/I)/S}$ is also generated by the same elements, except there are extra relations; namely, $da = 0$ for each $a \in I$. But these are precisely the images of $d : I/I^2 \to T \otimes_S \Omega_{R/S}$.

These results allow us to give an explicit construction of the module of Kähler differentials of the coordinate ring of an affine variety. We compute $\Omega_{R/k}$ in the case where $R = k[x_1, \ldots, x_n]/I$ where $I = (f_1, \ldots, f_m)$ represents the ideal of some affine variety $\mathcal{X}$. By Lemma 2.33, we know $\Omega_{R/k}$ is the cokernel of the map

$$\delta : I/I^2 \to R^n,$$

where $\delta$ is the map $d$ from Lemma 2.33. Since every $R$-module is a homomorphic image of a free $R$-module, we may find a free $R$-module $R^m = \bigoplus_{i=1}^m R e_i$ together with a surjection $\phi : R^m \to I/I^2$, where $e_1, \ldots, e_m$ form a basis for $R^m$ and $\phi(e_i) = f_i + I^2$. Consider the composition

$$R^m \xrightarrow{\phi} I/I^2 \xrightarrow{\delta} R^n, \quad e_i \mapsto f_i + I^2 \mapsto df_i.$$

This composition is a map of free $R$-modules and by the Leibniz rule, we know that $df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$ for each $i = 1, \ldots, m$. In other words, we have identified $\Omega_{R/k}$ as the cokernel of the Jacobian matrix

$$J := \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}.$$

**Example 2.34.** As an easy, yet important, first example, let us consider the case when $\mathcal{X} = \mathbb{A}^n_k$. Then $k[\mathcal{X}] = k[x_1, \ldots, x_n]$ and there are no relations to account for. Hence, it follows from Proposition 2.32 our above discussion that the module of Kähler differentials associated to $\mathcal{X}$ (over $k$), denote also by $\Omega_{\mathbb{A}^n_k}$, is given by

$$\Omega_{k[x_1, \ldots, x_n]/k} = \bigoplus_{j=1}^n k[x_1, \ldots, x_n] dx_j.$$
We are now ready to define the sheaf of differentials of a variety \(X\), and its symmetric powers.

**Definition 2.35.** Let \(X\) be a variety defined over a field \(k\). We define the sheaf of differentials (1-forms) on \(X\), denoted by \(\Omega_X\), to be the sheaf such that for any affine open \(U\) of \(X\) we have \(\Gamma(U, \Omega_X) = \Omega_k[U]/k\) (see [10, Chapter 6, Proposition 1.17] or [7, Chapter 8, Remark 8.9.2]).

This leads to the following criterion for smoothness.

**Proposition 2.36.** A variety \(X\) is smooth at a point \(p\) if and only if the sheaf \(\Omega_X\) is a locally free algebraic sheaf of finite rank at \(p\), and singular otherwise.

**Proof.** See [11, Chapter III, §4, Proposition 3]. \(\square\)

**Definition 2.37.** If \(R\) is a ring and \(N\) is an \(R\)-module, then the tensor algebra of \(N\) is the graded algebra \(R \oplus N \oplus (N \otimes R N) \oplus \cdots\). The symmetric algebra of \(N\) is then obtained from its tensor algebra by imposing commutativity; i.e., by quotienting out by the ideal generated by all elements of the form \(n_1 \otimes n_2 - n_2 \otimes n_1\). Similarly, the symmetric \(m^{th}\) power of \(N\) is given by \(N^{\otimes m}/\langle n_1 \otimes \cdots \otimes n_m - n_{\sigma(1)} \otimes \cdots \otimes n_{\sigma(m)} \rangle\) for all permutations \(\sigma \in \text{Sym}\{1, \ldots, m\}\).

**Definition 2.38.** The sheaf of symmetric differential \(m\)-forms on \(X\), denoted by \(\text{Sym}^m \Omega_X\), is the sheaf defined by \(\Gamma(U, \text{Sym}^m \Omega_X) = \text{Sym}^m \Omega_k[U]/k\), where \(\text{Sym}^m\) is the \(m^{th}\) symmetric power as a \(k[U]\)-module.

Let us revisit the previous example with Definition 2.35 at our disposal.

**Example 2.39.** We determine \(\Omega_{\mathbb{A}^n_k}\). Write \(X = \mathbb{A}^n_k\) and note since \(X\) is affine, we immediately have \(\Gamma(X, \Omega_X) = \Omega_{k[X]}/k = \bigoplus_{j=1}^n k[x_1, \ldots, x_n]dx_j\).

The next two examples illustrate how to compute with differential forms on projective varieties.

**Example 2.40.** In this example we show that there are no regular differential forms on \(\mathbb{P}^1_k\); i.e., \(\Gamma\left(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k}\right) = 0\). Let \(X = \mathbb{P}^1_k\) and let \(X_0\) and \(X_1\) denote the homogeneous coordinates for \(\mathbb{P}^1_k\). Note \(X\) is covered by the two standard affine patches \(U_0\) and \(U_1\) both of which are
isomorphic to $\mathbb{A}^1_k$. Write $u$ and $v$ for the affine coordinate of $U_0$ and $U_1$ respectively and note that $v = \frac{1}{u}$. By the previous example we know that if $\varphi \in \Omega_X(X)$ then

$$\varphi = \begin{cases} f(u)du & \text{on } U_0 \\ g(v)dv & \text{on } U_1 \end{cases}$$

Using the relation $v = \frac{1}{u}$, it follows that $dv = -\frac{du}{u^2}$. Suppose $\deg g(v) = n$. Then the following holds on $U_0 \cap U_1$

$$f(u)du = -\frac{g(1/u)}{u^2}du.$$ 

Since $f$ and $g$ are polynomials, such a relation is only possible if $f = g = 0$. This immediately gives $\Gamma (X, \Omega_X) = 0$.

**Example 2.41.** Let $k$ be any field of characteristic other than 3 and suppose $X \subseteq \mathbb{P}^2_k$ is the variety cut out by $X_0^3 + X_1^3 + X_2^3 = 0$, where $X_0, X_1,$ and $X_2$ denote the homogeneous coordinates of $\mathbb{P}^2_k$. Consider the affine charts $U_{ij}$ corresponding to the set of points with $X_iX_j \neq 0$. We then have $X = U_{10} \cup U_{12} \cup U_{20}$. Consider the affine coordinates $x = \frac{X_1}{X_0}$ and $y = \frac{X_2}{X_0}$ of $U_{10}$, $u = \frac{X_2}{X_1}$ of $U_{12}$, and $s = \frac{X_0}{X_2}$ and $t = \frac{X_1}{X_2}$ of $U_{20}$. The claim is that the differentials $\omega_1 = \frac{dy}{x^2}, \omega_2 = \frac{du}{u^2},$ and $\omega_3 = \frac{ds}{s^2}$ corresponding to the three affine patches $U_{10}, U_{12},$ and $U_{20}$ respectively, give a global differential form on all of $X$; i.e., they define an element of $\Omega_X(X)$. To see this, note it is evident that $\omega_1 \in \Omega_X(U_{10}), \omega_2 \in \Omega_X(U_{12}),$ and $\omega_3 \in \Omega_X(U_{20})$. So it suffices to show that these differentials agree on pairwise intersections. We show that $\omega_1 = \omega_2$ on $U_{10} \cap U_{12}$. On this intersection, note $v = \frac{1}{x}$ and $u = \frac{y}{x}$. This gives

$$\frac{dv}{u^2} = \frac{-\frac{dx}{x^2}}{\frac{y^2}{x^2}} = -\frac{dx}{y^2}.$$ 

Moreover, the affine model for $X$ on $U_{10}$ is given by $1 + x^3 + y^3 = 0$. This gives the relation $3x^2dx + 3y^2dy = 0$. Since the characteristic of the base field is not 3, we obtain $-x^2dx = y^2dy$. Hence, $-\frac{dx}{y^2} = \frac{dy}{x^2}$, thus verifying that $\omega_1 = \omega_2$ on $U_{10} \cap U_{12}$. The other cases are similar.

**Remark 2.42.** It is important to note the significance of the previous two examples; even though the only functions that are everywhere regular on a projective variety are the constants, it is possible for a projective variety $X$ to have non-constant differential forms that are regular everywhere on $X$. Moreover, if a variety $X \cong \mathbb{P}^1_k$, then it follows from Example 2.40 that there are no non-zero differentials that are regular everywhere on $X$. 

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2.7 Interpreting $\Gamma(U, \text{Sym}^2 \Omega_\mathcal{X})$ as Differentials Over $k(\mathcal{X})$

In this section we are going to discuss how to interpret sections of the sheaf $\text{Sym}^2 \Omega_\mathcal{X}$ as symmetric differential forms over the function field $k(\mathcal{X})$; i.e., as elements of $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$. We will use this interpretation later on in Chapter 4 when we discuss the sheaf of symmetric differential 2-forms on the Barth sextic surface. First, we recall some facts concerning $\Omega_{k(\mathcal{X})/k}$. While the discussion here concerns a variety $\mathcal{X} \subseteq \mathbb{P}^3_k$ of dimension 2, we remark that one can generalize these ideas to any variety of dimension $d$.

It is a standard result that if $u, v \in k(\mathcal{X})$ are such that the field extension $k(\mathcal{X})/k(u,v)$ is finite separable, then

$$\Omega_{k(\mathcal{X})/k} = k(\mathcal{X})du \oplus k(\mathcal{X})dv.$$  \hfill (2.3)

Moreover, we know from linear algebra [4, Section 11.5, Corollary 35] that $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$ is a $(\binom{2+2-1}{2-1}) = 3$-dimensional $k(\mathcal{X})$-vector space. Moreover, there is an isomorphism between the symmetric tensor algebra of $\Omega_{k(\mathcal{X})/k}$ as a graded $k(\mathcal{X})$-algebra and the ring of polynomials in 2 variable over $k(\mathcal{X})$. In particular, this isomorphism induces a vector space isomorphism from $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$ onto the space of homogeneous polynomials of degree 2. Hence, we may write

$$\text{Sym}^2 \Omega_{k(\mathcal{X})/k} = k(\mathcal{X})(du \cdot du) \oplus k(\mathcal{X})(du \cdot dv) \oplus k(\mathcal{X})(dv \cdot dv).$$

2.7.1 Regularity of a Differential at a Non-singular Point

We have described a way to interpret sections of the sheaf $\text{Sym}^2 \Omega_\mathcal{X}$ as elements of $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$. We would like to be able to recognize when a given section is regular at a non-singular point $p \in \mathcal{X}$. Let $p \in \mathcal{X}$ be a non-singular point and consider the local ring $\mathcal{O}_{\mathcal{X}, p}$, which for ease of notation will be denoted here by $\mathcal{O}$. It follows from Definition 2.5 that $\text{Sym}^2 \Omega_{\mathcal{O}/k}$ is free of rank 3 and we may write

$$\text{Sym}^2 \Omega_{\mathcal{O}/k} = \mathcal{O}(du \cdot du) \oplus \mathcal{O}(du \cdot dv) \oplus \mathcal{O}(dv \cdot dv),$$

so that we have an injection $\text{Sym}^2 \Omega_{\mathcal{O}/k} \hookrightarrow \text{Sym}^2 \Omega_{k(\mathcal{X})/k}$. Hence, if $\omega \in \Gamma(\mathcal{X}, \text{Sym}^2 \Omega_\mathcal{X})$, we may recognize regularity at $p$ from the image of $\omega$ in $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$; it must lie in the image...
of $\iota$: $\text{Sym}^2 \Omega_{\mathcal{O}/k} \hookrightarrow \text{Sym}^2 \Omega_{k(\mathcal{X})/k}$ for all $p \in \mathcal{X}$. We are therefore interested in recognizing when a given element of $k(\mathcal{X})$ lies in $\text{im} \; \iota$. To accomplish this task (for a given non-singular point $p$) from a computational point of view, we use completions. To be precise, we use the fact that completion of $\mathcal{O}$ at the maximal ideal $\mathfrak{m}_p$, denoted $\hat{\mathcal{O}}$, is isomorphic to $k[[u,v]]$ (i.e., the ring of power series in the variables $u$ and $v$). Since $k(\mathcal{X})$ is the field of fractions of $\mathcal{O}$, we get a map $k(\mathcal{X}) \hookrightarrow k((u,v))$. Writing $\overline{k(\mathcal{X})}$ for the field of fractions of $\hat{\mathcal{O}}$, we may summarize the above discussion in the following diagram

That is, $\omega$ is regular at $p$ if its image in $\text{Sym}^2 \Omega_{k(\mathcal{X})/k}$, in fact lies inside $\text{Sym}^2 \Omega_{\hat{\mathcal{O}}/k}$. In other words, we require that the Laurent series expansion for $\omega$ around $p$ has no negative powers and it therefore is a power series expansion around $p$. 
Chapter 3

Graded Modules and Coherent Sheaves

Let $X$ be the Barth sextic and let $\varphi: Y \to X$ be the (resolution of $X$) variety obtained from blowing up each nodal singularity of $X$. Using computations similar to Example 2.20 one can show that, since the singularities on $X$ are ordinary double points, the variety $Y$ is non-singular and birational to $X$. Furthermore, each of the 65 singularities $s \in S = \text{Sing } X$, pulls back, via $\varphi$, to an exceptional component $E_s \cong \mathbb{P}^1$, giving a genus zero curve on $Y$. By setting $E = \bigcup_{s \in S} E_s$, it follows that $X \setminus S \cong Y \setminus E$. As mentioned in Chapter 1, we are ultimately interested in the global sections of the sheaf $\text{Sym}^2 \Omega_Y$. We would like to identify elements of $\Gamma(Y, \text{Sym}^2 \Omega_Y)$ with global sections of some sheaf on $X$. So we appeal to the work of Serre [12] to obtain an explicit correspondence between graded modules and coherent algebraic sheaves. As we shall see, every sheaf has a natural graded module connected to it, for which the homogeneous component of degree 0 gives exactly the global sections. However, computing this module is often a challenge. Instead, we show that the dual of a module over a projectively normal coordinate ring always has this property. This allows us to determine the global sections of a closely related sheaf, namely the double dual sheaf (or the reflexive hull). Applied to $\Omega_Y$, this allows us to determine $\Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y)$. In Sections 4.4, 4.5, and 4.6 we describe the additional computations one has to do to determine the conditions for such a section to extend to all of $Y$. Since our goal is to focus mainly on the computational aspects of these concepts, we shall omit most of the details (and refer the reader to appropriate references) and thus focus on key aspects of these arguments instead.
We begin with a definition:

**Definition 3.1.** Given a graded ring $R$ and a graded $R$-module $M$, we denote by

$$M^\vee = \text{Hom}_R(M, R) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_R(M, R)_d,$$

the *dual* module associated to $M$.

Recall that in case of finite-dimensional vector spaces, there is a canonical isomorphism between a vector space and its double dual. This need not be true in the case of an arbitrary module; i.e., $M^{\vee \vee}$ need not be isomorphic to $M$.

**Example 3.2.** Let $R = \mathbb{Z}[x]$ and consider the maximal ideal $M = \langle 2, x \rangle$. Then one can show that $M^\vee = R$ so that $M^{\vee \vee} = R^{\vee} = R$, which is not the same as $M$.

### 3.1 Graded Modules and Algebraic Sheaves

#### 3.1.1 The Sheaf Associated to a Graded Module

Let $\mathcal{X}$ be a projective variety and let $S = k[\mathcal{X}]$ be its projective coordinate ring. We have $S = k[t_0, \ldots, t_n] = k[X_0, \ldots, X_n]/I$ for some homogeneous ideal $I$, where the $t_j$ are the images of $X_j$ in $S$. In the same way that $S$ determines the structure sheaf $\mathcal{O}_\mathcal{X}$, we wish to show that every graded $S$-module $M$ also determines a sheaf on $\mathcal{X}$. If $U$ is an open subset of $\mathcal{X}$, we let $S(U)$ denote the set of homogeneous polynomials that do not vanish on $U$. Write $M_U$ for the set of quotients of the form $\frac{m}{Q}$ with $m \in M$ and $Q \in S(U)$ and $m$ and $Q$ homogeneous of the same degree. One can define an equivalence relation on $M_U$ in the obvious way by saying $\frac{m}{Q} = \frac{m'}{Q'}$ if and only if there exists $Q'' \in S(U)$ with $Q''(mQ' - m'Q) = 0$. Furthermore, we can equip $M_U$ with the structure of an $S_U$-module in a natural way. This observation, together with the fact that $U \subseteq V$ implies $S(V) \subseteq S(U)$ gives a natural map $\rho_U^V : M_V \rightarrow M_U$. One can then verify that as $U$ and $V$ run over all non-empty open subsets of $\mathcal{X}$, the system $(M_U, \rho_U^V)$ defines a sheaf, which we denote by $\mathcal{A}(M)$. In particular, we have that $\mathcal{A}(S) = \mathcal{O}_\mathcal{X}$. As Serre shows [12, Chapter III, §2, Section 57], the map $\mathcal{A}$ defines a functor from the category of graded modules over $k[\mathcal{X}]$, to the category of algebraic sheaves on $\mathcal{X}$. 

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3.1.2 The Graded Module Associated to an Algebraic Sheaf

Let $S$ and $A$ be as in Section 3.1.1.

**Definition 3.3.** Let $X$ be a projective variety with projective coordinate ring $S$. Let $O_X = A(S)$ be the structure sheaf of $X$. For each $n \in \mathbb{Z}$, write $S^{(n)}$ for the twisted module defined by shifting the grading; i.e., $S^{(n)} = S^{d+n}$. We define the $n^{th}$ twisting sheaf of $O_X$, denoted $O_X^{(n)}$, as $A(S^{(n)})$. More generally, if $F$ is any algebraic sheaf on $X$, we define $F^{(n)} := F \otimes_{O_X} O_X^{(n)}$, where the tensor product sheaf is defined via sections; i.e., for each affine open subset $U$ of $X$, we have $F^{(n)}(U) = F(U) \otimes_{O_X(U)} O_X^{(n)}(U)$.

It is understood that $O_X^{(0)} = O_X$ and intuitively, one should think of $O_X^{(n)}$ as the sheaf whose sections over a given open subset $U$ are just quotients of homogeneous polynomials of degree $n+d$ by homogeneous polynomials of degree $d$. Suppose $F$ is a sheaf of $O_X$-modules. Then the graded module associated to $F$ is defined via

$$\Gamma^* F = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, F^{(d)}).$$

Note we may equip $\Gamma^* F$ with a structure of an $S$-module. Indeed, we have a canonical map $S \to \Gamma^* (O_X) = \Gamma^* (A(S))$ (see [12, Chapter III, §2, Section 59]) so an element of $S_d$ determines a section of $A(S^{(d)}) = O_X^{(d)}$. Hence, if $f \in S_d$ and $g \in \Gamma(X, F^{(n)})$ are given, it follows that $f \otimes g$ is a section of $O_X^{(d)} \otimes F^{(n)} = F^{(n)}(d) = F^{(n+d)}$. Thus $(f,g) \mapsto f \otimes g$ defines the desired multiplication by elements of $S$.

**Note.** If $S$ is not a graded polynomial ring, then it is in general not true that $S = \Gamma^* (O_X)$ (see [7, Caution 5.13.1] and [7, Chapter 2, Exercise 5.14]).

### 3.2 Dualizing and Saturation

Henceforth, we shall write $M^2 := \Gamma^* (A(M))$. If $M$ is a finitely generated graded $S$-module, the map $M \to M^2$ is called the saturation map. Note that $A(\Gamma^* (A(M))) = A(M)$ [12, Chapter III, §2, Proposition 7], so we see that $(M^2)^2 = M^2$. As it turns out, computing $M^2$ directly is rather involved, so instead we will prove that if $S$ satisfies certain properties, then $M^\vee = (M^\vee)^2$.

**Remark 3.4.** The saturation map is defined as follows. Given an element $m \in M_n$, note that $m \in M(n)_0$. It therefore defines a section of $A(M(n)) = A(M)(m)$, where the equality here follows from [12, Chapter III, §2, Proposition 4].
**Definition 3.5.** If $\mathcal{F}$ is an algebraic sheaf on a variety $\mathcal{X}$, then the dual sheaf of $\mathcal{F}$, denoted by $\mathcal{F}^\vee := \text{Hom}(\mathcal{F}, \mathcal{O}_\mathcal{X})$, is defined, for each open subset $U$ of $\mathcal{X}$, in the natural way via $\mathcal{F}^\vee(U) := \text{Hom}_{\mathcal{O}_\mathcal{X}(U)}(\mathcal{F}(U), \mathcal{O}_\mathcal{X}(U))$.

**Proposition 3.6.** Let $M$ be a finitely generated $S$ module. The functor $\mathcal{A}$ described above commutes with taking duals; i.e., $\mathcal{A}(M^\vee) = \mathcal{A}(M)^\vee$.

*Proof.* See [12, Chapter III, Section 5, Proposition 1].

**Proposition 3.7.** Let $S = k[t_0, \ldots, t_n]$ be a $k$-algebra as in Section 3.1 and suppose $M$ is an $S$-module. Then $M = M^\natural$ as $S$-modules if and only if

(a) If $m \in M$ is such that $t_jm = 0$ for all $1 \leq j \leq n$, then $m = 0$.

(b) If $m_1, \ldots, m_n$ is any collection of homogeneous elements of $M$ of the same degree satisfying $t_jm_i = t_im_j$ for all $1 \leq i, j \leq n$, then there exists $m \in M$ with $m_j = t_jm$ for each $1 \leq j \leq n$.

*Proof.* See [12, Chapter III, Section 3, Proposition 9].

**Definition 3.8.** Let $\mathcal{X} \subseteq \mathbb{P}^n$ be a variety and write $S$ for its homogeneous coordinate ring. Then $\mathcal{X}$ is said to be projectively normal if $S = S^\natural$; i.e., if $S$ is saturated as an $S$-module.

**Definition 3.9.** A variety $\mathcal{X}$ is said to be normal if the local ring at every point is an integrally closed domain.

**Theorem 3.10.** If $\mathcal{X} \subseteq \mathbb{P}^n$ is a complete intersection (i.e., if its ideal is generated by codim $\mathcal{X}$ elements) of dimension at least 1 and $\mathcal{X}$ is normal, then $\mathcal{X}$ is projectively normal.

*Proof.* See [7, Chapter II, Exercise 8.4].

**Corollary 3.11.** The Barth sextic surface is projectively normal and so its projective coordinate ring is saturated.

*Proof.* Since $\dim X = 2$ and $X \subseteq \mathbb{P}^2$, it follows that $X$ is a complete intersection. Together with codim $\mathcal{G} = 2$, this implies that $X$ is also normal (see [7, Chapter II, Proposition 8.23]). Applying Theorem 3.10 we see that $X$ is projectively normal. The result now follows.
Theorem 3.12. If $S$ is a graded k-algebra as in Proposition 3.7 with the property that $S$ is saturated in the sense of Proposition 3.7 and $M$ is a graded $S$-module, then $M' = (M')^\sharp$.

Proof. It will be sufficient to verify conditions (a) and (b) of Proposition 3.7. First, recall that

$$M' = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_S(M, S)_{d},$$

so an element of $(M')_d$ is a homomorphism of degree $d$ mapping, for each $\ell \in \mathbb{Z}$, the $S$-module $M_\ell$ to $S_{\ell+d}$. So pick any $\psi \in (M')_d$ and suppose $t_j \psi = 0$ for each $1 \leq j \leq n$. Then $t_j \psi(m) = 0$ for all $m \in M$ and since $\psi(m) \in S$, it is sufficient to check this property on $S$ instead. But $S$ is a domain and $t_j \neq 0$ for some $j$ so we immediately have (a). To verify (b), suppose we have a family $\{\psi_1, \ldots, \psi_n\}$ of homogeneous elements of $M'$ of the same degree, say $d$, such that $t_i \psi_j = t_j \psi_i$ for all $1 \leq i, j \leq n$. So we have $t_i \psi_j(m) = t_j \psi_i(m)$ for each $m \in M$. Write $s_i$ for the image of $m$ under $\psi_i$. We then have a collection $s_1, \ldots, s_n \in S_d$ with $t_i s_j = t_j s_i$ for all pairs $(i, j)$. Since $S$ is saturated, there exists $s \in S$ with $s_j = s_j$ for all $j$. First, note that this $s$ must be unique, for if there exists $s' \in S$ satisfying $t_j s = s_j = t_j s'$, then $t_j (s - s') = 0$ and so $s = s'$. We need to exhibit a homomorphism $\psi: M \to S$ satisfying $\psi_j = t_j \psi$. Define $\psi$ via $m \mapsto s$, where $m$ and $s$ are as above. This map is obviously well-defined (by uniqueness of $s$). Next, if $m_1, m_2 \in M$, then $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2) = 0$ if and only if $t_j \psi(m_1 + m_2) = t_j \psi(m_1) + t_j \psi(m_2) = 0$ for every $j$. But observe that $t_j \psi(m_1 + m_2)$ is, by construction, $\phi_j(m_1 + m_2)$. Similarly, $t_j \psi(m_i) = \psi_j(m_i)$ and each $\psi_j$ is a homomorphism. Lastly, if $s' \in S$, then $\psi(s'm_1) - s' \psi(m_1) = 0$ if and only if $t_j \psi(s'm_1) - s't_j \psi(m_1) = \psi_j(s'm_1) - s' \psi_j(m_1)$. Again, since the $\psi_j$ are homomorphisms, so is $\psi$. This finishes the proof. □

Theorem 3.13. Consider the setup of Theorem 3.12. Then $(M^{\vee\vee})_0 = \Gamma(\mathcal{X}, \mathcal{A}(M^{\vee\vee})).$

Proof. It follows from Theorem 3.12 that

$$M^{\vee\vee} = (M^{\vee\vee})^\sharp = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathcal{X}, \mathcal{A}(M^{\vee\vee})(d)),$$

which proves the claim. □

In other words, the homogeneous component of degree 0 of $M^{\vee\vee}$ is indeed describing the global sections of the sheaf represented by the double dual module of $M$.  

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Definition 3.14. If \( R \) is a graded ring and \( M \) is a graded \( R \)-module, we say that \( M \) is reflexive if \( M^\vee\vee = M \) (as graded modules of course). Similarly, a sheaf \( \mathcal{F} \) on a topological space \( X \) is said to be reflexive if it is isomorphic to its double dual.

Definition 3.15. Suppose \( \mathcal{F} \) is a sheaf of \( \mathcal{O}_X \)-modules and let \( \{U_j\} \) be an affine open covering of a variety \( X \). We say that \( \mathcal{F} \) is coherent if there exists finitely generated \( k[U_j]\)-modules \( M_j \) such that \( \mathcal{F}|_{U_j} \cong \mathcal{A}(M_j) \).

Lemma 3.16. The dual of any coherent sheaf on a normal variety is reflexive.

Proof. See [8, Corollary 1.2].

Recall from the introduction to this chapter that we have an isomorphism \( Y \setminus E \cong X \setminus \mathcal{G} \).

Hence, it follows that

\[
\Gamma(X \setminus \mathcal{G}, \text{Sym}^2 \Omega_X) \cong \Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y).
\] (3.1)

Proposition 3.17. The sheaf \( \text{Sym}^2 \Omega_X \) is reflexive on \( X \setminus \mathcal{G} \).

Proof. Note \( X \setminus \mathcal{G} \) is a smooth quasi-projective variety and hence, \( \text{Sym}^2 \Omega_X \) is a coherent locally free sheaf of finite rank on \( X \setminus \mathcal{G} \). It follows from [7, Chapter II, Exercise 5.1] that \( \text{Sym}^2 \Omega_X \) is reflexive on \( X \setminus \mathcal{G} \). \( \square \)

Definition 3.18. A coherent sheaf \( \mathcal{F} \) on a normal variety \( V \) is said to be normal if for every open subset \( U \subseteq V \) and every closed subset \( Z \subseteq U \) of codimension at least 2, the restriction map \( \mathcal{F}(U) \to \mathcal{F}(U \setminus Z) \) is bijective.

The following result due to Hartshorne gives a useful characterization of reflexive sheaves.

Proposition 3.19. Let \( \mathcal{F} \) be a coherent sheaf on a normal variety \( V \). Then \( \mathcal{F} \) is reflexive if and only if \( \mathcal{F} \) is normal in the sense of Definition 3.18.

Proof. See [8, Proposition 1.6]. \( \square \)

Since \( \dim X = 2 \), it follows that \( \mathcal{G} \) is indeed of codimension 2 so we may apply Proposition 3.19 to \( U = X \) and \( Z = \mathcal{G} \) to obtain

\[
\Gamma(X, (\text{Sym}^2 \Omega_X)^\vee\vee) \cong \Gamma(X \setminus \mathcal{G}, (\text{Sym}^2 \Omega_X)^\vee\vee).
\] (3.2)
Moreover, as $Y$ is smooth, the sheaf $\text{Sym}^2 \Omega_Y$ is locally free of rank 2 and is therefore reflexive by Proposition 3.17 so that

$$\Gamma(Y, \text{Sym}^2 \Omega_Y) \cong \Gamma(Y, (\text{Sym}^2 \Omega_Y)^{\vee\vee}). \tag{3.3}$$

Consider the restriction maps

$$\rho_Y : \Gamma(Y, (\text{Sym}^2 \Omega_Y)^{\vee\vee}) \to \Gamma(Y \setminus E, (\text{Sym}^2 \Omega_Y)^{\vee\vee})$$

and

$$\rho_X : \Gamma(X, \text{Sym}^2 \Omega_X) \to \Gamma(X \setminus S, \text{Sym}^2 \Omega_X).$$

We have the following diagram

\[
\begin{array}{ccc}
\Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y) & \cong & \Gamma(Y, (\text{Sym}^2 \Omega_Y)^{\vee\vee}) \\
\downarrow & & \downarrow \\
\Gamma(Y, \text{Sym}^2 \Omega_Y) & \xrightarrow{(3.3)} & \Gamma(Y \setminus E, (\text{Sym}^2 \Omega_Y)^{\vee\vee}) \\
\downarrow & & \downarrow \\
\Gamma(X, \text{Sym}^2 \Omega_X) & \xrightarrow{\rho_X} & \Gamma(X \setminus S, \text{Sym}^2 \Omega_X) \\
\downarrow & & \downarrow \\
\Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee\vee}) & \cong & \Gamma(X \setminus S, (\text{Sym}^2 \Omega_X)^{\vee\vee}) \\
\end{array}
\]

The above diagram indicates that we can determine $\Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y)$ by determining $\Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee\vee})$. We will carry out these computations in Section 4.3. So it remains to determine which elements of $\Gamma(Y \setminus E, \text{Sym}^2 \Omega_Y)$ extend to $\Gamma(Y, \text{Sym}^2 \Omega_Y)$; i.e., which elements lie in the image of $\rho_Y$. This will be discussed in Section 4.4 and 4.6.
Chapter 4

Computations and Results

In this chapter, we present the core ideas at play behind our computations. These include a complete discussion of our computation of the graded module representing the sheaf of symmetric differential 2-forms and an explicit description of the blow-up map \( \varphi: Y \to X \) on an affine chart. In particular, we prove Theorem 1.1 in Section 4.4. In Section 4.7 we describe how to use the global sections of \( \text{Sym}^2 \Omega_X \setminus \mathfrak{S} \) to find genus zero curves that pass through at most the 15 nodal singularities with \( X_3 = 0 \). All computations are done using the computer algebra system Magma [3].

Note. Although most of the objects we are constructing are quite standard, the methods which are employed to construct them may seem convoluted. This is due to the fact that in order to work with a graded module, say \( M \), in Magma [3], one needs to express \( M \) as a quotient of a free module over a polynomial ring by a graded submodule. As such, some extra care is required to construct these otherwise standard structures.

4.1 The Graded Module Representing \( \Omega_X \)

For computational efficiency purposes, all our computations are done over \( k = \mathbb{F}_{50021} \). We note that there is not much special about our choice of the prime 50021; we simply require a large enough prime \( p \) for which \( \mathbb{F}_p \) contains \( \sqrt{5} \). This is due to the observation that the standard equation for the Barth sextic surface requires the golden ratio and therefore, one needs \( \sqrt{5} \in k \) for the Barth sextic to be defined over \( k \). To ensure that this polynomial is defined over \( \mathbb{F}_p \), we must ensure that 5 is a quadratic residue modulo \( p \).
We write $\mathbb{P}^3_k$ for the projective 3-space over $k$ and let $R = k[X_0, X_1, X_2, X_3]$ denote the homogeneous coordinate ring for $\mathbb{P}^3_k$. Write
\[ F := 4(\phi^2 X_0^2 - X_1^2)(\phi^2 X_1^2 - X_2^2)(\phi^2 X_2^2 - X_0^2) - (1 + 2\phi)(X_0^3 + X_1^3 + X_2^3 + X_3^2)X_3, \]
where $\phi$ denotes the golden ratio. Barth’s sextic surface is the projective variety $X := \mathbb{V}(F)$ cut out by the equation $F = 0$. Let $k[X]$ denote the homogeneous coordinate ring of $X$ and note that $k[X]$ is saturated (as a module over itself) by Theorem 3.10. This is the quotient $R/\langle F \rangle$. Write $R(\omega_1, \ldots, \omega_n)$ for the free graded $R$-module of rank $n$ with grading $-\omega_j$ on the $j$th basis element. If all the gradings are equal to $\omega$, we shall write $R(\omega)^n$ in place of $R(\omega_1, \ldots, \omega_n)$. Note that this is consistent with the notation of Definition 3.3. Unless stated otherwise, it is understood that $R^n$ comes equipped with the standard basis vectors $e_j = [0, \ldots, 0, 1, 0, \ldots, 0]$ for each $1 \leq j \leq n$, where there is a 1 in the $j$th position. An element $r$ of $R^n$ is therefore given by a list, $[a_1, \ldots, a_n]$, of elements of $R$. In this notation we see that $k[X]$, as a graded module, is $R^1(0)/[F]$. We begin by recalling some facts concerning the sheaf of differentials on $\mathbb{P}^3_k$. First, we have the Euler exact sequence which tells us how the sheaf of differentials of $\mathbb{P}^3_k$ is related to sheaves that we have already seen.

**Theorem 4.1.** [7, Chapter 2, Theorem 8.13] The sheaf $\Omega_{\mathbb{P}^3_k}$ satisfies the Euler exact sequence
\[ 0 \rightarrow \Omega_{\mathbb{P}^3_k} \rightarrow \mathcal{O}_{\mathbb{P}^3_k}(-1)^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^3_k} \rightarrow 0. \]

Upon examining the proof, one can see that the idea is to interpret the above sequence in terms of graded modules and work with them instead. In fact, if we write $S$ for the homogeneous coordinate ring of $\mathbb{P}^3_k$ and consider the graded $S$-module $S(-1)^4$ with basis $e_0, \ldots, e_3$ (each of degree 1), then one defines a graded homomorphism $S(-1)^4 \rightarrow S$ via $e_i \mapsto X_i$ and seeks to identify the kernel $K$ of this map as a graded module for which $\mathcal{A}(K) = \Omega_{\mathbb{P}^3_k}$. It can be shown that this kernel is generated by elements of the form $X_i e_j - X_j e_i$ (note there are 6 such elements). So we proceed to give an explicit description of these generators.

Let $U_{X_0}, U_{X_1}, U_{X_2},$ and $U_{X_3}$ denote the affine patches of $\mathbb{P}^3_k$ corresponding to $X_0 \neq 0$, $X_1 \neq 0$, $X_2 \neq 0$, and $X_3 \neq 0$ respectively. We consider each affine patch separately.

Consider $U_{X_3}$ with affine coordinates $\frac{X_0}{X_3}$, $\frac{X_1}{X_3}$, and $\frac{X_2}{X_3}$. Since $U_{X_3} \cong \mathbb{A}^3_k$, we know from Examples 2.34 and 2.39 that the module (of sections) of Kähler differentials on $\mathbb{A}^3_k$
is generated by $d\frac{X_0}{X_3}, d\frac{X_1}{X_3}$, and $d\frac{X_2}{X_3}$. Similarly, $d\frac{X_0}{X_0}, d\frac{X_1}{X_0}$, $d\frac{X_2}{X_0}$ generate $\Omega_{U_{X_0}}$, $d\frac{X_0}{X_1}, d\frac{X_1}{X_1}$, $d\frac{X_2}{X_1}$ generate $\Omega_{U_{X_1}}$, and $d\frac{X_0}{X_2}, d\frac{X_1}{X_2}$, $d\frac{X_2}{X_2}$ generate $\Omega_{U_{X_2}}$. Note that relations of the form $d \frac{X_1}{X_0} = -\frac{X_2}{X_1} d \frac{X_0}{X_0}$ mean it is sufficient to consider only the following differentials

$$d\frac{X_0}{X_1}, d\frac{X_0}{X_2}, d\frac{X_0}{X_3}, d\frac{X_1}{X_3}, d\frac{X_2}{X_3}.$$ (4.1)

Moreover, the differentials $dX_0, dX_1, dX_2$, and $dX_3$ form a basis for $\Omega_{R/k}$. Using the Leibniz rule we can express the differentials in (4.1) as

$$d\frac{X_0}{X_1} = \frac{X_1 dX_0 - X_0 dX_1}{X_1^2}, \quad d\frac{X_0}{X_2} = \frac{X_2 dX_0 - X_0 dX_2}{X_2^2}$$

$$d\frac{X_0}{X_3} = \frac{X_3 dX_0 - X_0 dX_3}{X_3^2}, \quad d\frac{X_1}{X_2} = \frac{X_2 dX_1 - X_1 dX_2}{X_2^2}$$

$$d\frac{X_1}{X_3} = \frac{X_3 dX_1 - X_1 dX_3}{X_3^2}, \quad d\frac{X_2}{X_3} = \frac{X_3 dX_2 - X_2 dX_3}{X_3^2}.$$ (4.2)

Clearing denominators in (4.2) we obtain the following set of generators each having weight $w = 2$.

$$\omega_1 = X_1 dX_0 - X_0 dX_1, \quad \omega_4 = X_2 dX_0 - X_0 dX_2$$

$$\omega_2 = X_3 dX_0 - X_0 dX_3, \quad \omega_5 = X_2 dX_1 - X_1 dX_2$$

$$\omega_3 = X_3 dX_1 - X_1 dX_3, \quad \omega_6 = X_3 dX_2 - X_2 dX_3.$$ (4.3)

These are precisely the generators we sought. Next, we would like to know how these generators behave on $X$; i.e., we need to take into account the relations coming from $X$, and therefore determine the sheaf $\Omega_X$. Once again, we recall another fact regarding sheaves of differentials; namely the conormal exact sequence (see [7, Chapter 2, Proposition 8.12]) which can be regarded as the sheaf analogue of the Second Exact Sequence (see Lemma 2.33). This result together with [7, Chapter 2, Remark 8.9.2] allow us to conclude that the module describing the sheaf $\Omega_X$ is in fact obtained by taking the free $R$-module $R(-2)^6$ generated by $\omega_1, \ldots, \omega_6$ and dividing out by appropriate relations induced by $F$. So it remains to determine these relations.

Consider the free $R$-module

$$R(-1)^4 = RdX_0 \oplus RdX_1 \oplus RdX_2 \oplus RdX_3.$$ (4.4)

We have the relation

$$\frac{\partial F}{\partial X_0} dX_0 + \frac{\partial F}{\partial X_1} dX_1 + \frac{\partial F}{\partial X_2} dX_2 + \frac{\partial F}{\partial X_3} dX_3 = 0.$$
In $R(-1)^4$, the left hand side of (4.4) is $DF := \left[ \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2}, \frac{\partial F}{\partial X_3} \right]$. Furthermore, note that all multiples of $F$ are by definition zero on the coordinate ring of $X$. So to ensure that this still holds in the case of differentials, we also require the following four relations

$$FdX_0 = 0, \quad FdX_1 = 0, \quad FdX_2 = 0, \quad FdX_3 = 0. \quad (4.5)$$

Again, the left hand sides of each of the four equations in (4.5) are written as $[F, 0, 0, 0], [0, F, 0, 0], [0, 0, F, 0], \text{ and } [0, 0, 0, F]$. Write $\mathcal{M}$ for the submodule generated by $DF, FdX_0, FdX_1, FdX_2,$ and $FdX_3$. The module of Kähler differentials of the projective coordinate ring $k[X]$ is now

$$\Omega_{k[X]/k} = R(-1)^4/\mathcal{M}.$$ 

We have an obvious module homomorphism $\phi: R(-2)^6 \to \Omega_{k[X]/k}$ induced by the elements in (4.3). Observe that the matrix of this homomorphism is then given by

$$M_\phi = \begin{bmatrix}
-X_1 & X_0 & 0 & 0 \\
-X_2 & 0 & X_0 & 0 \\
-X_3 & 0 & 0 & X_0 \\
0 & -X_2 & X_1 & 0 \\
0 & -X_3 & 0 & X_1 \\
0 & 0 & -X_3 & X_2
\end{bmatrix},$$

where it is understood that each row $r_j = \left[ a_{j1}, a_{j2}, a_{j3}, a_{j4} \right]$ of $M_\phi$ corresponds to the element $[a_{j1}, a_{j2}, a_{j3}, a_{j4}]$ of $\Omega_{k[X]/k}$. Lastly, consider the kernel of the map $\phi$. We now set

$$DX := R(-2)^6/\ker \phi.$$ 

Observe that $DX$ contains no degree zero elements since it is generated (over $R$) in degree 2. It follows from our discussion in this section that this is a graded module for which $\mathcal{A}(DX) = \Omega_X$; i.e., the sheaf of differential forms on $X$. One can check, for instance with Magma [3], that $DX^{\vee\vee}$ has a trivial degree 0 component. It follows that $X \setminus \mathcal{G}$ has no regular 1-forms.

### 4.2 Symmetric Differential 2-forms on $X$

We proceed to construct the graded module representing the sheaf of symmetric differential 2-forms on $X$, denoted by $\text{Sym}^2 \Omega_X$. First, we recall some facts from linear algebra.
The $n^{th}$ symmetric power of $N$, denoted $S^n(N)$, is equal to $N^\otimes n$ modulo the submodule generated by all elements of the form $n_1' \otimes \cdots \otimes n_n' - n_\sigma(1) \otimes \cdots \otimes n_\sigma(n)$ for all permutations $\sigma$ of $\{1, 2, \ldots, n\}$. It is well-known [4, Chapter 11, Proposition 40] that when $n!$ is a unit in $S$, symmetric powers of $N$ coincide with symmetric tensors; i.e., there exists an $S$-module isomorphism between symmetric $n^{th}$ powers of $N$ (which is realized as a quotient) and the $S$-submodule of symmetric $n$-tensors. For a more thorough look at symmetric algebras we refer the reader to [4].

Write $DX^\otimes 2$ for $DX \otimes DX$. We have the canonical map $DX \times DX \to DX^\otimes 2$ taking an element $(a, b) \in DX \times DX$ to $a \otimes b \in DX^\otimes 2$. Write $G$ for the set of generators of $DX$:

$$G = [\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6].$$

As $DX$ is generated by the elements of $G$, we have that $DX^\otimes 2$ is generated by $a \otimes b$ for all $a, b \in G$ with $a$ and $b$ not necessarily distinct. We write $\omega_{i,j}$ to denote $\omega_i \otimes \omega_j$ and set

$$J := [\omega_{1,1}, \ldots, \omega_{6,1}, \omega_{2,2}, \ldots, \omega_{6,2}, \ldots, \omega_{5,5}, \omega_{6,5}, \omega_{6,6}].$$

Let $I$ be the submodule of $DX$ generated by all elements of the form

$$\omega_{i,j} - \omega_{j,i}, \quad \text{for each } 1 \leq i < j \leq 6,$$

where $G[i]$ is the $i^{th}$ element of $G$. We then have

$$\text{Sym}^2 DX := DX^\otimes 2 / I,$$

and we write, for each $1 \leq \ell \leq 21$, $s_\ell$ for the image of the $\ell^{th}$ element of $J$ in $\text{Sym}^2 DX$. A direct computation shows that $\text{Sym}^2 \mathcal{A}(M') = \mathcal{A}(\text{Sym}^2 M')$ for a graded module $M'$, so that $\text{Sym}^2 \Omega_X = \mathcal{A}(\text{Sym}^2 DX)$.

4.3 Double Dual Construction

Using our presentation of $\text{Sym}^2 DX$ from Section 4.2, we can let Magma [3] compute $\text{Sym}^2 DX$ as a finitely generated $k[X]$-module with 21 generators, each of weight 4 which we denote here by $s_1, \ldots, s_{21}$ (corresponding to the ordering of $J$ in Section 4.2). To ease the notation, write $M = \text{Sym}^2 DX$. Calling the function $\text{Hom}(M, k[X])$ in Magma [3] creates a finitely generated “abstract module” ($k[X]$-module), denoted here by $M^\vee$, together with a
“transfer map” $\tau: M^\vee \to \text{Hom}_{k[X]}(M, k[X])$; i.e., $M^\vee$ represents the module corresponding to the set of homomorphisms from $M$ to $k[X]$ whereas $\tau$ maps elements of $M^\vee$ to actual homomorphisms in $\text{Hom}_{k[X]}(M, k[X])$. Since we have represented $M$ using 21 generators, any such homomorphism is naturally represented by a $21 \times 1$ matrix, although not uniquely so. We define $M^{\vee\vee}$ in the same fashion and write $M^{\vee\vee}$ and $\sigma$ to denote the abstract module representing $\text{Hom}_{k[X]}(M^\vee, k[X])$ and the transfer map $M^{\vee\vee} \to \text{Hom}_{k[X]}(M^\vee, k[X])$ respectively.

Magma [3] finds presentations of $M^\vee$ and $M^{\vee\vee}$ generated by 6 generators, which we shall denote by $\gamma_1, \ldots, \gamma_6$ and $\eta_1, \ldots, \eta_6$ respectively with

$$\deg(\gamma_j) = \begin{cases} 
4 & \text{for } 1 \leq j \leq 3 \\
3 & \text{for } 4 \leq j \leq 6 
\end{cases}$$

and

$$\deg(\eta_j) = \begin{cases} 
0 & \text{for } 1 \leq j \leq 3 \\
-1 & \text{for } 4 \leq j \leq 6 
\end{cases}$$

Since the $\gamma_j$ generate all of $M^\vee$, it follows that $\tau(\gamma_1), \ldots, \tau(\gamma_6)$ generate all of $\text{Hom}_{k[X]}(M, k[X])$. Consider the $21 \times 6$ matrix $A$ whose $(i, j)^{\text{th}}$-entry is given by $\tau(\gamma_j)(s_i)$; that is,

$$A = \begin{bmatrix} 
\tau(\gamma_1)(s_1) & \cdots & \tau(\gamma_6)(s_1) \\
\vdots & \ddots & \vdots \\
\tau(\gamma_1)(s_{21}) & \cdots & \tau(\gamma_6)(s_{21}) 
\end{bmatrix}.$$ 

Let $s = r_1s_1 + \cdots + r_{21}s_{21} = [r_1, \ldots, r_{21}] \in M$ be given. Then multiplying $A$ on the left by $s$ gives a $1 \times 6$ matrix

$$s \cdot A = \begin{bmatrix} 
\tau(\gamma_1)(s) & \cdots & \tau(\gamma_6)(s) 
\end{bmatrix}.$$ 

Note we may now multiply $s \cdot A$ on the right by an element of $M^\vee$ (which is a $6 \times 1$ column vector) to obtain an element in $k[X]$; i.e., we have an element of the double dual of $M$. Hence, the matrix $A$ gives a map

$$M \xrightarrow{A} \text{Hom}_k[M^\vee, k[X]] \xleftarrow{B} M^{\vee\vee}$$
where $B$ is the $6 \times 6$ matrix whose rows represent the homomorphisms $\sigma(\eta_j)$:

$$B = \begin{bmatrix}
\sigma(\eta_1)(\gamma_1) & \cdots & \sigma(\eta_1)(\gamma_6) \\
\vdots & & \vdots \\
\sigma(\eta_6)(\gamma_1) & \cdots & \sigma(\eta_6)(\gamma_6)
\end{bmatrix}.$$  

From Theorem 3.13 we know that the degree zero elements of $M^\vee\vee$ coincide with the global sections of the algebraic sheaf associated to $M^\vee\vee$. The 3 degree zero generators give us three global sections. Moreover, since the remaining three generators all have degree $-1$, we may multiply each of these with $X_0, X_1, X_2$, and $X_3$ to obtain 15 degree zero element in total. Magma [3] confirms that these are in fact linearly independent. Thus, we see that $\Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee\vee})$ is a 15-dimensional $k$-vector space.

4.3.1 Global Sections of $\text{Sym}^2 \Omega_Y$

Recall the following useful facts concerning localization and the symmetric algebras.

**Lemma 4.2.** If $S$ is any multiplicative set in a (commutative) ring $R$ and $M$ is an $R$-module, then $S^{-1}R \otimes_R M \cong S^{-1}M$.

*Proof.* See [5, Lemma 2.4].

** Proposition 4.3.** Let $R$ be a ring and suppose $R'$ is an $R$-algebra. If $M$ is any $R$-module, then $S^2( R' \otimes_R M ) \cong R' \otimes_R S^2(M)$.

*Proof.* See [5, Proposition A2.2].

In light of our discussion in Section 2.7 with $X$ replaced by $X$, we shift our focus to representation of elements in $\Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee\vee})$ as symmetric differential 2-forms over the function field. In fact, since we are ultimately interested in $\Gamma(Y, \text{Sym}^2 \Omega_Y)$, we aim to represent these global sections as symmetric differentials over the function field $k(Y)$, which from Lemma 2.16, is isomorphic to $k(X)$. We proceed as follows. Fix an affine chart $U$ of $Y$ (note if $U$ is chosen so that it avoids exceptional components, then $U$ defines an affine chart of $X$ so we may freely go back and forth between $X$ and $Y$), write $k[U]$ for its affine coordinate ring, and consider $\Omega_{k[U]/k}$. Combining Lemma 4.2 and Proposition 4.3 together with the fact that $k(Y) \cong k(U)$ is simply the localization of $k[U]$ at all non-zero elements,
we have

\[ \text{Sym}^2 \Omega_{k[U]/k} \otimes_{k[U]} k(Y) = \text{Sym}^2(\Omega_{k[U]/k} \otimes_{k[U]} k(Y)) = \text{Sym}^2 \Omega_{k(Y)/k} \cong k(Y)^3 \cong k(X)^3. \]

(4.6)

As \( Y \) is non-singular, \( \text{Sym}^2 \Omega_Y \) is reflexive so the same result holds for \( (\text{Sym}^2 \Omega_Y)^{\vee \vee} \). Recall the maps

\[ M \xrightarrow{A} \text{Hom}_{k[X]}(M^{\vee}, k[X]) \xleftarrow{B \cong} M^{\vee \vee} \]

defined in Section 4.3, where multiplication by (the \( 21 \times 6 \) matrix) \( A \) defines a map from a finitely generated module on \( 21 \) generators to a finitely generated module on \( 6 \) generators.

Recall also that the sheaf \( \text{Sym}^2 \Omega_X \) is defined, on each open \( U \) of \( X \) as \( \text{Sym}^2 \Omega_{k[U]/k} \). We have

\[ \Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee \vee}) \rightarrow \Gamma(U, (\text{Sym}^2 \Omega_X)^{\vee \vee}) = (\text{Sym}^2 \Omega_{k[U]/k})^{\vee \vee}. \]

(4.7)

But the right hand side of (4.7) maps to \( (\text{Sym}^2 \Omega_{k(X)/k})^{\vee \vee} \cong \text{Sym}^2 \Omega_{k(X)/k} \) which we know is a 3-dimensional \( k(X) \)-vector space. We therefore have a map

\[ \Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee \vee}) \rightarrow \text{Sym}^2 \Omega_{k(X)/k}. \]

Since \( M \) and \( M^{\vee \vee} \) are both finitely generated \( k[X] \)-modules, we have the following diagram

\[ \begin{array}{ccc}
R(-4)^{21} & \xrightarrow{A} & \text{Hom}(R(-4, -4, -4, -3, -3, -3), k[X]) \xleftarrow{B \cong} R(0, 0, 0, 1, 1, 1) \\
\downarrow & & \downarrow \\
\text{Sym}^2 DX & \xrightarrow{A} & \text{Hom}_{k[X]}(M^{\vee}, k[X]) \xleftarrow{B \cong} M^{\vee \vee} \\
\downarrow & & \downarrow \\
\Gamma(U, \text{Sym}^2 \Omega_X) & \rightarrow & \text{Sym}^2 \Omega_{k(X)/k} \\
\downarrow & & \downarrow \\
\Gamma(X, (\text{Sym}^2 \Omega_X)^{\vee \vee}) & \rightarrow & \Gamma(U, (\text{Sym}^2 \Omega_X)^{\vee \vee}) \\
\end{array} \]

where the top row corresponds to our explicit representations in Magma [3]. Note that \( R(-4)^{21} \) contains a free submodule of rank 3 whose generators correspond to the generators of \( \text{Sym}^2 \Omega_{k(X)/k} \). So restricting the map induced by multiplication by \( A \) to this free submodule amounts to choosing a suitable \( 3 \times 3 \) submatrix of \( A \), say \( A_U \), whose rows correspond, over the chosen affine chart, to the generators of \( \text{Sym}^2 \Omega_{k(X)/k} \). Therefore, choosing an
affine patch and tensoring $M$ and $M^{\vee\vee}$ with $k(X)$ induces an isomorphism of $k(X)$-vector spaces

$$k(X)^3 \xrightarrow{A_U} k(X)^3.$$  

Writing $B'$ for the $15 \times 3$ matrix (where the three columns are chosen so that the image of $B$ lands inside the image of $A_U$) representing the global sections of the double dual sheaf, we see that $B'A_U^{-1}$ describes these global sections in terms of the generators of $k(X)$ corresponding to the rows of $A_U$. For instance, on the chart $U$ corresponding to $X_3 = 1$, we have the differentials $d\frac{X_0}{X_3}, d\frac{X_1}{X_3},$ and $d\frac{X_2}{X_3}$ so we may choose $d\frac{X_0}{X_3}$ and $d\frac{X_1}{X_3}$ as our generating elements so that a basis is given by

$$d\frac{X_0}{X_3}d\frac{X_0}{X_3}, \quad d\frac{X_0}{X_3}d\frac{X_1}{X_3}, \quad d\frac{X_1}{X_3}d\frac{X_1}{X_3}.$$  

In terms of our labeling of these generators from Section 4.2, these are precisely corresponding to the generators $s_{12}, s_{14},$ and $s_{19}$ (i.e., rows 12, 14, and 19 of $A$).

### 4.4 Toward Blow-Ups

For the remainder of this section we shall write $P$ to denote an arbitrary element of $\mathcal{S}$.

#### 4.4.1 The Setup

We begin by choosing an appropriate affine patch $U$ where $P$ is the origin. We do this explicitly as follows: since every singular point has at least one non-zero homogeneous coordinate, we may assume, for ease of exposition, that $P = [p_1 : p_2 : p_3 : 1]$. Let $x_0, y_0,$ and $z_0$ denote the affine coordinates of $U$ (where it is understood that $x_0 = \frac{X_0}{X_3}, y_0 = \frac{X_1}{X_3},$ and $z_0 = \frac{X_2}{X_3}$). The change of coordinate

$$(x_0, y_0, z_0, 1) \mapsto (x_0 + p_1, y_0 + p_2, z_0 + p_3, 1), \quad (4.8)$$

will certainly satisfy $(0, 0, 0, 1) \mapsto P$.

**Definition 4.4.** If $\mathcal{X}$ is an affine variety containing the point $p = (0, \ldots, 0)$, then the tangent cone of $\mathcal{X}$ at $p$ is defined as follows: if $\mathcal{I}(\mathcal{X})$ is the ideal of $\mathcal{X}$ and $f \in \mathcal{I}(\mathcal{X})$, then write $f_{in}$ for the initial part of $f$; i.e., $f_{in}$ is the non-zero homogeneous component of lowest degree of $f$. Then the tangent cone of $\mathcal{V}$ is the zero locus of the ideal generated by the $f_{in}$. One could think of the tangent cone as an approximation of $\mathcal{I}(\mathcal{X})$ around the origin.
In the case of the hypersurface $X$, we have that the tangent cone of $X$ is the vanishing set of the smallest homogeneous component of $F$. Let $F_{\text{aff}} = F(x_0 + p_1, y_0 + p_2, z_0 + p_3, 1)$ and write $Q$ for its homogeneous component of degree 2; i.e., $Q$ describes the tangent cone of $F_{\text{aff}}$. We find that if $P$ is a singularity of $X$, then the lowest degree non-zero homogeneous component $Q$ of $F_{\text{aff}}$ is of degree 2. So $Q$ is given by a quadratic form; that is,

$$a_{11}x_0^2 + a_{12}x_0y_0 + a_{13}x_0z_0 + a_{21}x_0y_0^2 + a_{22}y_0^2 + a_{23}y_0z_0 + a_{31}x_0z_0 + a_{32}y_0z_0 + a_{33}z_0^2,$$

for some $a_{ij} \in k$. As we shall see, we can bring $Q$ into the form $Q^* = Axz + By^2$, where $A$ and $B$ are some constants (not both zero), whose resolution was discussed in Example 2.20 in Chapter 2.

### 4.4.2 Ordinary Double Points as Quotient Singularities

Let $\mathcal{V}$ be a variety and let $G$ be a finite group acting on $\mathcal{V}$. Write $k[\mathcal{V}]^G$ for the ring of invariant functions on $\mathcal{V}$.

**Theorem 4.5.** Given an affine variety $\mathcal{V}$ and a finite group $G$, we have $k[\mathcal{V}]^G \subseteq k[\mathcal{V}]$ is a finitely generated $k$-algebra; so it is the coordinate ring of an affine variety $X$. The points of $X$ are in a one-to-one correspondence with the orbits of $\mathcal{V}/G$.

**Proof.** See [6, Chapter 10].

Consider the affine plane $\mathbb{A}_k^2$ (with coordinates $s$ and $t$) and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{A}_k^2$ via involution $(s, t) \mapsto (-s, -t)$. Then $k[\mathbb{A}_k^2] = k[s, t]$ and it is not hard to see that $k[s, t]^G = k[s^2, st, t^2]$. Since the generators $x = s^2, y = st,$ and $z = t^2$ satisfy the relation $xz = y^2$, we see that $\mathbb{A}_k^2/G$ is in fact the cone in $\mathbb{A}_k^3$ cut out by the cone $xz = y^2$, which we denote here by $Q^*$. Hence, we have realized the tangent cone of $F_{\text{aff}}$ as a quotient variety and the singularity at the origin as a quotient singularity.

Next, consider the blow-up of $\mathbb{A}_k^2$ at the origin. If we use $s$ and $t$ to denote the affine coordinates of $\mathbb{A}_k^2$ and use $U, V$ to denote homogeneous coordinates for $\mathbb{P}_k^1$, then the equation for $B_{\mathbb{A}_k^2}$ is given by $sV = tU$. Taking the affine patch where $V \neq 0$, the defining equations become $u = \frac{t}{s}$ (where $u = U/V$) so that if $s \neq 0$ (and therefore $t \neq 0$), then $B_{\mathbb{A}_k^2}$ is parametrized via $u = t/s$. Let us write $a$ and $b$ for the coordinates of the blow-up where we have the relation $(s, t) = (a, ab)$ so that $b = \frac{t}{s}$. Putting everything together we obtain
the following commutative diagram

\[
\begin{array}{ccc}
B_{k^2}^2, (a, b) & \xrightarrow{\kappa} & A_{k^2}^2, (s, t) \\
& \searrow{\lambda} & \swarrow{\mu} \\
& \swarrow{\nu} \quad \mathbb{V}(Q^*), (x, y, z) & \\
\end{array}
\]

where we have the following relations among various coordinates charts

\[
\begin{array}{ccc}
x & y & z \\
\hline
s^2 & st & t^2 \\
p & pq & pq^2 \\
a^2 & a^2b & a^2b^2 \\
\end{array}
\]

The relations between the differentials (via pull-backs along \(\kappa\) and \(\lambda\)) in the different coordinate charts are

\[
dp = 2ada, \quad dq = db, \quad ds = da \quad dt = bda + adb.
\]

Hence, the pull-back of the symmetric differential 2-form monomials \(ds^2\), \(dsdt\), and \(dt^2\) are

\[
(da)^2, \quad a(da)^2 + adadb, \quad b^2(da)^2 + (2ab)dad + a^2(db)^2,
\]
respectively. Using the above table we see that

\[
(ds)^2 = \frac{1}{4p} \cdot (dp)^2, \quad dsdt = \frac{1}{2} \cdot dpdq + \frac{q}{4p} (dp)^2, \quad (dt)^2 = qdpdq + p(dq)^2 + \frac{q^2}{4p} (dp)^2. \tag{4.9}
\]

Suppose we have a differential \(\omega\) that is regular in a neighbourhood of \(P\), but not necessarily at \(P\). We want to examine how the pull-back of \(\omega\) (on the blow-up) behaves on the exceptional component in \(B_{Q^*}\). By Proposition 3.19, the sheaf of symmetric differential 2-forms on \(A^2_{k^2}\) is normal. As such the pull-back of \(\omega\) along \(\mu\), denoted by \(\mu^*(\omega)\), extends to \(\mu^{-1}(P)\) so that \(\mu^*(\omega)\) is regular on a neighbourhood of \(\mu^{-1}(P)\). Moreover, the pull-back of \(\omega\) along \(\kappa\) also does not disturb regularity. We may write

\[
\mu^*(\omega) = c_1 \cdot (ds)^2 + c_2 \cdot dsdt + c_3 \cdot (dt)^2
\]

for some \(c_1, c_2, \) and \(c_3\) regular on an open containing \(\kappa^{-1}(\mu^{-1}(P))\). Using Equations (4.9) we see that the image of such a differential under \(\lambda\) (after pulling-back along \(\kappa\)) is of the form

\[
\frac{1}{4p} \left( c_1' + c_2'q + c_3'q^2 \right) (dp)^2 + \text{other terms with no } p \text{ in the denominator},
\]

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for some $c_1', c_2'$, and $c_3'$ regular on an open containing $\nu^{-1}(P)$, which is the set of points corresponding to $p = 0$. Thus, we see that $\nu^*(\omega)$ is regular at all points of $\nu^{-1}(P)$ if $c_1' + c_2'q + c_3'q^2$ vanishes on $\nu^{-1}(P)$.

### 4.4.3 The Setup (Continued)

So it now remains to bring the tangent cone into the form $Q^*$. To do this, we first note that since $Q$ and $Q^*$ both define a quadratic form, there exist symmetric matrices associated to them; namely

$$M_Q := \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{bmatrix}, \quad \text{and} \quad M_{Q^*} := \begin{bmatrix} 0 & 0 & \frac{1}{2}A \\ 0 & B & 0 \end{bmatrix}. $$

From linear algebra we know that bringing $Q$ into the form $Q^*$ is equivalent to finding an invertible linear change of coordinate matrix $T$ such that

$$Q\left(T\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}\right) = Ax_0z_0 + By_0.$$ 

We therefore require that $T$ satisfies

$$TM_QT^{-1} = M_{Q^*}. \quad (4.10)$$

Write $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, and $r = (r_1, r_2, r_3)$ for the columns of $T$. A quadratic form defines a bilinear form, here denoted by $\langle \cdot, \cdot \rangle$, and so the right hand side of (4.10) is given by the Gram matrix

$$\begin{bmatrix} \langle p, p \rangle & \langle q, p \rangle & \langle r, p \rangle \\ \langle p, q \rangle & \langle q, q \rangle & \langle r, q \rangle \\ \langle p, r \rangle & \langle q, r \rangle & \langle r, r \rangle \end{bmatrix}.$$ 

It follows that $\langle p, p \rangle = \langle r, r \rangle = 0$ and $\langle p, r \rangle \neq 0$. To satisfy the first two conditions, it is sufficient to take $p$ and $r$ to represent any two distinct points on the projective conic defined by $Q = 0$. To ensure $\langle q, p \rangle$ and $\langle q, q \rangle$ (and therefore $\langle p, q \rangle$ and $\langle q, r \rangle$) are zero, note it is sufficient to choose $q$ orthogonal to both $p$ and $r$; i.e., we take

$$q \in \ker M_Q[ p \quad r ].$$
Remark 4.6. We remark that choosing two distinct points on $Q$ is in fact very easy. Indeed, fixing any two coordinates, say $y_0$ and $z_0$, one reduces the problem into solving a degree two polynomial in $x_0$ to obtain the third coordinate, and thus completely determining a point on $Q$.

It follows that the change of coordinates

$$\begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \mapsto \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}, \quad (4.11)$$

transforms $Q$ into $Q^*$. Combining the two transformations in (4.8) and (4.11) we obtain a linear transformation that brings $P$ into the origin (on an appropriate affine chart) and further transforms the tangent cone (on the prescribed affine chart) into the form $Q^*$; this is given by

$$\begin{bmatrix} x_0 & y_0 & z_0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} x_0p_1 + y_0q_1 + z_0r_1 + p_1 \\ x_0p_2 + y_0q_2 + z_0r_2 + p_2 \\ x_0p_3 + y_0q_3 + z_0r_3 + p_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0^* \\ y_0^* \\ z_0^* \\ 1 \end{bmatrix}.$$

4.5 Blow-Up of $X$

We now consider the blow-up of $X$ (on an affine chart). Since the affine model for $X$ resides in $\mathbb{A}^3_k$, we must first compute the blowing-up of $\mathbb{A}^3_k$ at the origin.

Let $x, y, z$ denote the affine coordinates for $\mathbb{A}^3_k$ and write $U, V, T$ for the homogeneous coordinates of $\mathbb{P}^2_k$. The defining equations for the blow-up in $\mathbb{A}^3 \times \mathbb{P}^2$ is then given by

$$xV = yU, \quad xT = zU, \quad yT = zV. \quad (4.12)$$

Taking the chart where $T = 1$, we see that the first equation in (4.12) becomes superfluous and the remaining two equations become $x = uz$ and $y = vz$ (where $u = \frac{U}{T}$ and $v = \frac{V}{T}$). Hence, relabeling $z$ as $t$, we see that a chart of the blow-up of $\mathbb{A}^3_k$ is parametrized by $u, v,$ and $t$ as $(ut, vt, t)$. Writing $\tilde{x}, \tilde{y},$ and $\tilde{z}$ for $x_0^*(ut, vt, t), y_0^*(ut, vt, t),$ and $z_0^*(ut, vt, t)$ respectively, it follows that the vanishing of the blow-up is given by

$$H(u, v, t) = F(\tilde{x}, \tilde{y}, \tilde{z}, 1) \in k[u, v, t]. \quad (4.13)$$
We know from Section 4.4.3 that
\[ F(x^*, y^*, z^*, 1) = Ax_0z_0 + By_0^2 + g(x_0, y_0, z_0), \]
where \( g \) has no homogeneous component of degree less than three. In fact, it is no loss of
generality to assume \( F(x^*, y^*, z^*, 1) \) is of the form
\[ x_0z_0 + Cy_0^2 + g(x_0, y_0, z_0), \]
where \( C = \frac{B}{A}. \)
Hence,
\[ H(u, v, t) = ut^2 + Cv^2t^2 + t^3g(u, v), \]
so that by dividing out the exceptional component corresponding to \( t = 0 \), we obtain the
strict transform of the blow-up as the vanishing of
\[ G(u, v, t) = u + Cv^2 + t\tilde{g}(u, v). \]

4.6 Power/Laurent Series Computations and \( \Gamma(Y, \text{Sym}^2 \Omega_Y) \)

In this section we carry out our Laurent series computations. To be specific, we describe
how to explicitly compute the Laurent series expansion at each of the 65 singularity for each
of the 15 global sections of \( M^\vee \vee \) and check for poles. This enables us to write down linear
conditions that must be met at each singularity in order to guarantee regularity. As we
shall see, satisfying all these linear conditions at once is impossible; i.e., we are able to prove
Theorem 1.1. We begin with a “Henselian”-type lemma (the proof for which is constructive)
concerning the existence of a power series solution \( \Psi \in k[v][[t]] \) for \( u \) so that \( G(\Psi, v, t) = 0. \)

Set \( \Delta(u, v, t) = -Cv^2 - t\tilde{g}(u, v, t) \) so that \( G(u, v, t) = u - \Delta(u, v, t), \) where it is worth
noting that every occurrence of \( u \) in \( \Delta \) has a factor of \( t \) in front.

**Lemma 4.7.** There exists \( \Psi \in k[v][[t]] \) such that \( G(\Psi, v, t) = 0. \) That is, there is a power
series solution for \( u \) in \( t \) with coefficients in \( k[v]. \)

**Proof.** The proof is similar to that of Hensel’s Lemma. Indeed, \( G(\Psi, v, t) = 0 \) if and
only if \( \Delta(\Psi, v, t) = \Psi. \) First we show that if there exists \( \Psi_0 \in \mathcal{R} = k[v][[t]] \) such that
\( G(\Psi_0, v, t) \in t\mathcal{R}, \) then the sequence
\[ \Psi_j := \Delta(\Psi_{j-1}, v, t), \quad \text{for all } j \geq 1 \]
satisfies $\Psi_j - \Psi_{j-1} \in t^j \mathcal{R}$ and $G(\Psi_j, v, t) \in t^{j+1} \mathcal{R}$. We proceed by induction on $j$. When $j = 1$, we have $\Psi_1 = \Delta(\Psi_0, v, t)$ so that

$$\Psi_1 - \Psi_0 = -\Psi_0 - Cv^2 - t\tilde{g}(\Psi_0, v, t) = -G(\Psi_0, v, t) \in t\mathcal{R},$$

where the second equality follows from the assumption on $\Psi_0$. Suppose this holds for all positive integers up to $\ell$ and consider $\Psi_\ell - \Psi_{\ell-1} = -t\tilde{g}(\Psi_{\ell-1}, v, t) + t\tilde{g}(\Psi_{\ell-2}, v, t)$. So it is sufficient to show that $-\tilde{g}(\Psi_{\ell-1}, v, t) + \tilde{g}(\Psi_{\ell-2}, v, t) \in t^{\ell-1} \mathcal{R}$. Since polynomial evaluation is a homomorphism and $\Psi_{\ell-1} - \Psi_{\ell-2} \in t^{\ell-1} \mathcal{R}$ by our induction hypothesis, we immediately obtain the desired result. Hence, we see that the sequence $\{\Psi_j\}_{j \geq 0}$ does in fact converge in $k[[v]][[t]]$. Lastly, note that the the second assertion concerning $G(\Psi_j, v, t)$ being in $t^{j+1} \mathcal{R}$ (for all $j \geq 1$) follows by a similar inductive argument, the details of which we omit here. To complete the proof, note that $\Psi_0 = -Cv^2$ satisfies $G(\Psi_0, v, t) \in t\mathcal{R}$. 

We have therefore expressed $u$ as an element in $k[[v]][[t]]$. Consider the field of fractions of $k[[v]][[t]]$ which is certainly contained in $k(v)((t))$. Recall that our goal is to show that $\text{Sym}^2 \Omega_Y$ has no non-trivial global sections. To do so, we must find appropriate change of variables that allows us to carry out our computations over $k(v)((t))$. Set

$$\hat{x} := \tilde{x}(\Psi, v, t), \quad \hat{y} := \tilde{y}(\Psi, v, t), \quad \hat{z} := \tilde{z}(\Psi, v, t).$$

Moreover, to express our differentials over $k(v)((t))$, we use the relation

$$\frac{\partial G}{\partial u} du + \frac{\partial G}{\partial v} dv + \frac{\partial G}{\partial t} dt = 0$$

to obtain

$$\overline{du} = -\frac{\overline{\partial G(v, t)}}{\partial u}(\Psi, v, t) dv + \frac{\partial G}{\partial t}(\Psi, v, t) dt.$$  \hfill (4.14)

We also have, for each $\hat{x}, \hat{y}, \hat{z}$, the differentials

$$d\hat{x} = \frac{\partial \hat{x}}{\partial u}(\Psi, v, t) \overline{du} + \frac{\partial \hat{x}}{\partial v}(\Psi, v, t) dv + \frac{\partial \hat{x}}{\partial t}(\Psi, v, t) dt,$$
$$d\hat{y} = \frac{\partial \hat{y}}{\partial u}(\Psi, v, t) \overline{du} + \frac{\partial \hat{y}}{\partial v}(\Psi, v, t) dv + \frac{\partial \hat{y}}{\partial t}(\Psi, v, t) dt,$$
$$d\hat{z} = \frac{\partial \hat{z}}{\partial u}(\Psi, v, t) \overline{du} + \frac{\partial \hat{z}}{\partial v}(\Psi, v, t) dv + \frac{\partial \hat{z}}{\partial t}(\Psi, v, t) dt,$$
which are, after using Equation (4.14), differentials in $dv$ and $dt$. Base changing our matrices $A$ and $B'$ amounts to evaluating the entries at $x = \hat{x}$, $y = \hat{y}$, $z = \hat{z}$, and $w = 1$. So let $\overline{A}$ and $\overline{B'}$ denote the resulting matrices after the aforementioned change of base. To achieve our goal, it remains to find a suitable basis for our symmetric differential forms (over the chart chosen in Section 4.4; i.e., $X_3 \neq 1$) over the function field. It follows that

$$\overline{B'}_U \cdot \overline{A}_U \begin{bmatrix} d\hat{x}^2 \\ d\hat{x}d\hat{y} \\ d\hat{y}^2 \end{bmatrix},$$

is the $15 \times 3$ matrix whose row $j$ is describing the Laurent series expansion of the global section $\omega_j$ of $\text{Sym}^2 \Omega_Y$.

### 4.6.1 Proof of Theorem 1.1

Carrying out the computations described in Section 4.6 for every point $P \in \text{Sing} X$, we find that an element of $\Gamma(Y, \text{Sym}^2 \Omega_Y)$ has a Laurent series expansion of the form

$$\omega = \left(\frac{\alpha v^2 + \beta v + \gamma}{t}\right) \cdot dt^2 + \left(\sum_{i \geq 0} f_i^{(1)}(v)t^i\right) \cdot dv^2 + \left(\sum_{j \geq 0} f_j^{(2)}(v)t^j\right) \cdot dvdt,$$

for some polynomials $f_i^{(1)}(v), f_j^{(2)}(v), f_\ell^{(3)}(v)$ and some $\alpha, \beta, \gamma \in k$, thus confirming our analysis from Section 4.4. We see that in order to ensure regularity at a singularity $P$ (i.e., where $t = 0$, since this corresponds to the exceptional component of $Y$), the term $\alpha v^2 + \beta v + \gamma$ must vanish. Since three points uniquely determine a quadratic polynomial, we then require three linear conditions at $P$ to ensure regularity there. Observe that the fifteen global sections computed above form a 15 dimensional $k$-vector space. Hence, imposing regularity at all 65 nodes amounts to imposing $3 \cdot 65 = 195$ linear conditions on this 15 dimensional vector space. Computation shows that this cuts out a 0 dimensional subspace, thus proving that $\Gamma(Y, \text{Sym}^2 \Omega_Y) = 0$. This proves Theorem 1.1. Moreover, our computations show the following:

**Proposition 4.8.** The differentials $w_\eta_4, w_\eta_5,$ and $w_\eta_6$ are regular at all the 15 nodes with $X_3 = 0$. 
4.7 Theorem 1.2 and Genus Zero Curves

Recall from Example 2.40 in Chapter 2 that there are no regular symmetric differential forms on $\mathbb{P}^1$. In fact, we saw that $\Gamma_*(\Omega_{\mathbb{P}^1}) = k[\mathbb{P}^1](-2)$. Hence, $\Gamma_*(\text{Sym}^2\Omega_{\mathbb{P}^1}) = k[\mathbb{P}^1](-4)$. So once again, there are no regular differential 2-forms on $\mathbb{P}^1$ either. Since every projective genus zero curve $C$ admits a morphism to $\mathbb{P}^1$, it follows that there are also no regular differential (1 or 2) forms on $C$. If $\omega$ is any differential form that is regular on all of $Y$, then the restriction of $\omega$ to $C$, denoted $\omega|_C$, must also be regular, and more importantly, we must have $\omega|_C = 0$. Since any genus zero curve on $Y$ is a 1 dimensional subvariety of $Y$, then $\text{Sym}^2\Omega_Y$ restricts to a 1 dimensional $k(C)$-vector space on $C$. Suppose we have fixed a chart, which for convention and consistency purposes is the chart where $X_3 \neq 0$. We know that $d\frac{X_0}{X_3}d\frac{X_0}{X_3}, d\frac{X_1}{X_3}d\frac{X_1}{X_3},$ and $d\frac{X_0}{X_3}d\frac{X_1}{X_3}$ form a basis for $\text{Sym}^2\Omega_{k(Y)/k}$ as a $k(Y)$-vector space. We also know how to express all fifteen global sections of $(\text{Sym}^2\Omega_Y)^\vee$ as elements of $\text{Sym}^2\Omega_{O_Y/k}$. Take any two of these global sections:

$$\omega_1 = A_1 dx'^2 + B_1 dx'dy' + C_1 dy'^2,$$
$$\omega_2 = A_2 dx'^2 + B_2 dx'dy' + C_2 dy'^2,$$

for some $A_i, B_i, C_i$ in $k(Y)$, where we are writing $x'$ and $y'$ to denote $\frac{X_0}{X_3}$ and $\frac{X_1}{X_3}$ respectively. Suppose $p \in C \subseteq X$ with $\frac{X_0}{X_3}$ and $\frac{X_1}{X_3}$ uniformizers at $p$ on $X$. We have the restriction map $\Gamma(Y,\text{Sym}^2\Omega_Y) \to \Gamma(C,\text{Sym}^2\Omega_C)$ and furthermore, any global section of $\text{Sym}^2\Omega_Y$ describes an element of

$$\text{Sym}^2\Omega_{O_{Y,p}/k} = \mathcal{O}_{Y,p}dx'^2 + \mathcal{O}_{Y,p}dx'dy' + \mathcal{O}_{Y,p}dy'^2.$$  

Any such element will then restrict to an element in $\text{Sym}^2\Omega_{O_{C,p}/k} = \mathcal{O}_{C,p}dt^2$, where $t$ is a local parameter at $p$ on $C$. It follows that $dx' \mapsto \alpha dt$ and $dy' \mapsto \beta dt$ for some $\alpha, \beta$ not both zero. Since any regular differential on $C$ must restrict to 0, we see that the following two conditions must hold:

$$A_1\alpha^2 + B_1\alpha\beta + C_1\beta^2 = 0, \quad A_2\alpha^2 + B_2\alpha\beta + C_2\beta^2 = 0.$$  

Consider

$$\Lambda = \text{det} \begin{bmatrix} A_1 & B_1 & C_1 & 0 \\ 0 & A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 & 0 \\ 0 & A_2 & B_2 & C_2 \end{bmatrix}.$$  

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Hence, we require that $\Lambda\big|_p = 0$. We are then interested in $X \cap V(\Lambda)$. This intersection, henceforth denoted by $Z$, is cut out by the (homogeneous) equations

$$F = \Lambda = 0.$$ 

We must therefore look for the irreducible components of $Z$ that describe a 1-dimensional variety in $\mathbb{P}^3_k$. This is equivalent to looking for the primary decomposition of the ideal $I_Z = \langle F, \Lambda \rangle \subset R$. We begin by finding the factorization of $\Lambda$ over $k$. Suppose $\Lambda = f_1^{a_1} \cdots f_m^{a_m}$ is the factorization of $\Lambda$. Since $F$ is irreducible, we may consider the primary decomposition of $\langle F, f_j \rangle$ instead. Having computed these decompositions for every (unordered) pair $(i, j)$ (as we run through various affine charts and bases for $\text{Sym}^2 \Omega_Y$, of course), it is then a matter of a simple check to determine if a given component defines a genus zero curve. In fact, all curves obtained in such a manner fall in the set of planar curves determined by cutting $X$ with planes spanned by three nodes (see the Appendix A for the list of these curves). In fact, going back to Proposition 4.8 from Section 4.6.1, we are now able to find the curves that avoid all the nodes with $X_3 \neq 0$. 

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Bibliography


Appendix A

Appendix

Here we give a list of genus zero curves that pass through at least 5 nodes on $X$. All of our computations are over $\mathbb{Q}(\sqrt{5})$. We first construct a list of genus zero curves on $X$ obtained by cutting $X$ with planes spanned by three nodes. Given any set of three distinct nodes on $X$, consider the plane $\mathcal{P}$ spanned by these points. We then proceed as in Section 4.7; we seek to find the irreducible components of $X \cap \mathcal{P}$. It turns out that there are only 27 such curves; 15 degree 6 curves (each passing through 10 nodes), 6 degree 2 curves (each passing through 10 nodes), and 6 degree 1 curves (each passing through 5 nodes). These degree 1 curves are all the curves that avoid the singularities with $X_3 \neq 0$. The defining equations for these curves in $\mathbb{P}^3_{\mathbb{Q}(\sqrt{5})}$ are as follows. First, note that the expression for $F$ over $\mathbb{Q}(\sqrt{5})$ is given by (here we are using $r$ to denote $\sqrt{5}$)

$$
(-6r - 14)X^4Y^2 + (2r + 6)X^2Y^4 + (2r + 6)X^4Z^2 + (16r + 32)X^2Y^2Z^2 + (-6r - 14)Y^4Z^2 + (-6r - 14)X^2Z^4 + (2r + 6)Y^2Z^4 + (-r - 2)X^4W^2 + (-2r - 4)X^2Y^2W^2 + (-r - 2)Y^4W^2 + (-2r - 4)X^2Z^2W^2 + (-2r - 4)Y^2Z^2W^2 + (-r - 2)Z^4W^2 + (2r + 4)X^2W^4 + (2r + 4)Y^2W^4 + (2r + 4)Z^2W^4 + (-r - 2)W^6.
$$

Then the 15 degree 6 curves are given by the vanishing of the following pairs of homogeneous polynomials

$$
\begin{align*}
C_1 & := (F, X + 1/2*(-r - 1)*Y + 1/2*(-r + 1)*Z) \\
C_2 & := (F, X + 1/2*(-r + 3)*Y + 1/2*(-r + 1)*Z) \\
C_3 & := (F, X + 1/2*(r - 3)*Y + 1/2*(-r + 1)*Z) \\
C_4 & := (F, X + 1/2*(-r + 3)*Y + 1/2*(r - 1)*Z) \\
C_5 & := (F, X + 1/2*(r + 1)*Y + 1/2*(r - 1)*Z) \\
C_6 & := (F, X + 1/2*(-r - 1)*Y + 1/2*(r + 3)*Z) \\
C_7 & := (F, X + 1/2*(-r - 1)*Y + 1/2*(r - 1)*Z) \\
C_8 & := (F, X + 1/2*(r + 1)*Y + 1/2*(r + 3)*Z) \\
C_9 & := (F, Y) \\
C_{10} & := (F, X + 1/2*(r + 1)*Y + 1/2*(-r - 3)*Z)
\end{align*}
$$

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\[
C_{11} := (F, X + 1/2*(-r - 1)*Y + 1/2*(-r - 3)*Z)
\]
\[
C_{12} := (F, X + 1/2*(r + 1)*Y + 1/2*(-r + 1)*Z)
\]
\[
C_{13} := (F, Z)
\]
\[
C_{14} := (F, X)
\]
\[
C_{15} := (F, X + 1/2*(r - 3)*Y + 1/2*(r - 1)*Z)
\]

The 6 degree 2 curves are given by the vanishing of the following pairs:

\[
C_{16} := (Y^2 + 1/10*(r + 5)*Z^2 + 1/10*(-r - 5)*W^2, X + 1/2*(-r + 1)*Y)
\]
\[
C_{17} := (X^2 + 1/2*(-r + 5)*Z^2 - W^2, Y + 1/2*(r - 1)*Z)
\]
\[
C_{18} := (Y^2 + 1/2*(r + 5)*Z^2 - W^2, X + 1/2*(r + 1)*Z)
\]
\[
C_{19} := (X^2 + 1/2*(-r + 5)*Z^2 - W^2, Y + 1/2*(-r + 1)*Z)
\]
\[
C_{20} := (Y^2 + 1/10*(r + 5)*Z^2 + 1/10*(-r - 5)*W^2, X + 1/2*(r - 1)*Y)
\]
\[
C_{21} := (Y^2 + 1/2*(r + 5)*Z^2 - W^2, X + 1/2*(-r - 1)*Z).
\]

The 6 degree 1 curves are given by the vanishing of the of pairs:

\[
C_{22} := (Y + 1/2*(-r + 1)*Z, W)
\]
\[
C_{23} := (X + 1/2*(-r + 1)*Y, W)
\]
\[
C_{24} := (X + 1/2*(r - 1)*Y, W)
\]
\[
C_{25} := (X + 1/2*(-r - 1)*Z, W)
\]
\[
C_{26} := (Y + 1/2*(r - 1)*Z, W)
\]
\[
C_{27} := (X + 1/2*(r + 1)*Z, W).
\]

The only curves that are obtained via our resultant computations from Section 4.7 are
given by \(C_9, C_{13}, C_{14}, \) and \(C_{16}, \ldots, C_{27}\). We also have the 65 genus zero curves arising from
blowing up the singularities; recall that blowing up a single node gives rise to a copy of \(\mathbb{P}^1_k\)
on \(Y\). Since \(X\) and \(Y\) are isomorphic outside \(\mathfrak{S}\) (recall Theorem 2.18), each of the curves
described above does in fact pull back to a genus zero curve on \(Y\) too.