Enumeration of Set Partitions Refined by Crossing and Nesting Numbers

by

Wei Chen
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Name: Wei Chen
Degree: Master of Science
Title: Enumeration of Set Partitions Refined by Crossing and Nesting Numbers

Examining Committee:

Chair: Dr. Nathan Ilten
Assistant Professor

Dr. Marni Mishna
Senior Supervisor
Associate Professor

Dr. Lily Yen
Co-Supervisor
Adjunct Professor

Dr. Matthew DeVos
Internal Examiner
Associate Professor

Date Approved: December 2nd, 2014
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Abstract

The standard representation of set partitions gives rise to two natural statistics: a crossing number and a nesting number. Chen, Deng, Du, Stanley, and Yan (2007) proved, via a non-trivial bijection involving sequences of Young tableaux that these statistics have a symmetric joint distribution. Recent results by Marberg (2013) has lead to algorithmic tools for the enumeration of set partitions with fixed crossing number and fixed nesting number. In this thesis we further consider set partitions refined by these two statistics. These subclasses can be recognized by finite automata, and consequently have rational generating functions. Our main contribution is an investigation into the structure of the automata, the corresponding adjacency matrices, and the generating functions.

Keywords: Set partitions; Crossings and nestings; Automata; Generating functions; Enumeration
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Chapter 1

Introduction

Set partitions are classical mathematical objects appearing in many combinatorial contexts. Two interesting combinatorial patterns that we investigate here are called crossings and nestings. These two patterns appear when set partitions are in their standard representation (see Figure 1.1). This representation consists of drawing the partition elements $1, 2, \ldots, n$ in a horizontal line with arcs drawn above connecting consecutive elements in the same block. Also associated with this representation are $j$-crossings and $k$-nestings for $j, k \geq 2$ in which $j$ arcs mutually cross, and $k$ arcs mutually nest respectively. This thesis is primarily interested in refining partition enumeration according to their maximal nesting and crossing patterns.

![Diagram of set partition]{1 2 3 4 5 6 7}

Figure 1.1: Standard representation of $\{\{1\}, \{2, 4, 6\}, \{3, 7\}, \{5\}\}$

This introduction begins with the motivation for discussing crossings and nestings followed by examples of their applications. Then we present the history of crossings and nestings on set partitions and related structures. This includes an important theorem of Chen et al. [14] which states these two parameters possess a symmetric joint distribution. We conclude the history by summarizing various major enumerative results. The introduction finishes with an outline of the thesis, and our main contributions.
1.1 Motivation

The long history of crossings and nestings has led to an increasingly refined decomposition of set partitions. Partitions are counted by the classical Bell numbers $B_n$. Examining the generating function for Bell numbers, we ask what properties $\sum_{n=0}^{\infty} B_n z^n$ possesses when we restrict the coefficients to relevant subsets. Bell numbers can be decomposed according to the maximum crossing number yielding the generating function

$$P(x; z) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} p_j(n) x^j z^n,$$  \hspace{1cm} (1.1)

where $p_j(n)$ is the number of partitions of $[n] = \{1, 2, \ldots, n\}$ that are maximally $j$-crossing. This refinement has been studied in the last decade. The analytic nature of the generating functions has been of particular interest in these studies. We can further decompose Bell numbers based on both maximum crossing number and maximum nesting number yielding

$$P(x, y; z) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{j,k}(n) x^j y^k z^n$$ \hspace{1cm} (1.2)

where $p_{j,k}(n)$ is the number of partitions of $[n] = \{1, 2, \ldots, n\}$ that are maximally $j$-crossing and maximally $k$-nesting. It turns out to be easier to enumerate according to the related property of noncrossing and nonnesting values. This thesis investigates properties pertaining to both decompositions.

Crossings and nestings have many applications to mathematics and beyond. For example, the noncrossing pattern can be defined in numerous other combinatorial structures such as trees (Exercise 5.46 of [54]) and graphs (Exercise 6.39p of [54]). In [43], Prodinger proves that the number of noncrossing partitions of $[n]$ into $k$ blocks is counted by the Narayana numbers $\frac{1}{n} \binom{n}{k} \binom{n}{k-1}$; these numbers count various other combinatorial objects. Yeats and Marie show in [39] that nestings have applications to Dyson-Schwinger equations in physics. In another application, Lorenz et al. in [35] show these parameters model constraints on RNA proteins in biology. This thesis focuses on refining by both nesting and crossing numbers without direct consideration for any immediate applications.
1.2 History

We provide a comprehensive history on the study of crossings and nestings for set partitions from their inception to modern results that are applicable to this thesis.

Surprisingly, the origin of noncrossing partitions is an English poetry and rhetoric book [44] by Puttenham. This book describes a rhyming pattern for the endings of lines in a poem. While analyzing these rhyming patterns in 1948, Becker [2] called them planar rhyme schemes. This rhyme scheme forms the same structure as what we now refer to as noncrossing partitions; a reference that clarifies this relationship is Rogers [47]. Noncrossing set partitions have no \( a < b < c < d \) pattern where both \( a \) and \( c \) are in a block and both \( b \) and \( d \) are in a different block. The similar nonnesting partitions have no \( a < b < c < d \) pattern where both \( a \) and \( d \) are in a block and both \( b \) and \( c \) are in a different block; these avoid 2-nestings. The 2-crossing and 2-nesting pattern to be avoided are shown in Figures 1.2 and 1.3 respectively.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

Figure 1.2: A 2-crossing

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

Figure 1.3: A 2-nesting

Initially, noncrossing set partitions and matchings dominated. Much work has been done on matchings which are set partitions where every block is of size 2. Indeed, they are easier than set partitions to analyze. There is the classical result that noncrossing or nonnesting matchings and noncrossing or nonnesting set partitions are all counted by the Catalan numbers \( C_n \) (see for example Exercises 6.19pp. and 6.19uu. of [54]). The distribution of the number of 2-crossings in matchings was first studied by Touchard [58] in 1952. Continuing Touchard’s work, Riordan in [46] found an explicit generating function for this distribution fixing the size of the matchings. In [20], de Sainte-Catherine showed that the distribution of the number of matchings that are 2-crossing is the same as the distribution of the number of matchings that are 2-nesting. We direct the reader to [50] by Simion for a survey on noncrossing partitions.
There exist further scattered results that paved the way for Chen et al.’s groundbreaking paper [14] that demonstrates how crossings and nestings are related. Standard Young Tableau (see Definition 3.2.4) with at most 2 rows and at most 3 rows were counted by Regev in [45]; these are an important component of the key bijection in [14]. Gouyou-Beauchamps in [24] further counted these tableau for 4 rows and 5 rows. He also gave a bijection from 3-nonnesting matchings to pairs of noncrossing Dyck paths (see Definition 3.3.1). Klazar in [33] first defined 3-crossings although the definition is slightly different from the standard definition that we adopt. Chen et al. were able to show a similar bijection from 3-noncrossing matchings to pairs of noncrossing Dyck paths. These small examples provided enough evidence for Chen et al. to define general $j$-crossing and $k$-nesting partitions as well as prove the symmetric joint distribution of these statistics across matchings and set partitions.

We briefly mention several references from representation theory which provided the general tools for Chen et al. to relate standard representations of set partitions to special sequences of Young diagrams. The papers [27] and [26] introduce the notion of vacillating tableaux although they are not defined explicitly. These tableaux are related to irreducible representations of a semisimple algebra called the partition algebra. This algebra is related to set partitions in a profound way. In a very similar manner, we have oscillating tableaux which are related to irreducible representations of a semisimple algebra called the Brauer algebra. They relate to matchings in a deep way (see Barcelo and Ram [1]). Oscillating tableaux first appeared but with a different name in [4] by Berele. In [56], Sundaram gave a bijection from oscillating tableaux to matchings (see also Exercise 7.24 of [54]).

Combining the knowledge obtained from numerous previous results, Chen et al. in 2007 published one of the most important papers in this field. Their work [14] showed that the statistics for the number of $j$-noncrossing and number of $k$-nonnesting of set partitions satisfied a symmetric joint distribution; this means that the number of partitions of a fixed size that are $j$-noncrossing and $k$-nonnesting is the same as the number of partitions which are $k$-noncrossing and $j$-nonnesting. They introduced new tools to study crossings and nestings in a systematic manner and opened the field to further research.

Some extensions to Chen et al. include showing the equidistribution of crossings and nestings to permutations and coloured versions of all combinatorial objects. With their origin as motivation, coloured set partitions were studied by Rogers [47] in 1981 under
the name of coloured rhyming schemes. We also note that coloured set partitions are related to the representation theory of certain upper triangular matrices (see Thiem [57]). Without delving deeply into the history, we mention that Chen and Guo [15] proved the equidistribution for coloured matchings; they found a bijection from oscillating rim hook tableaux to coloured matchings. It is natural to try to generalize this property to more complicated structures. In 2013, Marberg in [38] showed the symmetric joint distribution for coloured set partitions. Applying mainly the work of de Mier [19] and [18] on graphs, Burrill et al. were able to show the equidistribution for permutations in [11]. Yen, following the work of Chen and Guo [15] and Marberg [38], generalized this equidistribution to coloured permutations in [63].

We finish the history with major enumerative results on noncrossings and nonnestings. First there is the classical result that noncrossing partitions are counted by an algebraic generating function. Grabiner and Magyar in [25] show that generating functions counting matchings are D-finite (see Definition 2.1.4). This D-finiteness property is generalized to coloured matchings by Chen and Guo in [15]. Bousquet-Mélou and Xin in [9] further showed that the generating function representing 3-noncrossing partitions is also D-finite. Marberg in [38] showed that noncrossing 2-coloured set partitions also have a D-finite generating function. In the literature, there are several conjectures of more complicated structures having generating functions that are not D-finite but this has not been verified.

1.3 Outline of Thesis

Chapter 2 gives a background of basic definitions and results on generating functions. Beginning with combinatorial classes, we define generating functions focusing on rational, algebraic, and D-finite generating functions. We conclude with some tools for studying generating function asymptotics.

Chapter 3 provides recent, applicable results in detail. Arc diagrams and all related vocabulary to crossings and nestings are defined. Then we introduce tools associated to Young diagrams. Next, we prove an important component of the Chen et al. bijection of [14]. This is followed by enumerative and bijective results for matchings, 2-noncrossing and 3-noncrossing set partitions, and permutations. Then we go into detail describing Marberg’s work in [38]. We present his algorithm for computing the generating functions
for fixed noncrossing and nonnesting numbers. The final section discusses finite state machines, Marberg's algorithm, and the Transfer-Matrix Method for producing these generating functions.

Chapter 4 details our new work on set partitions. Several automata are presented. We prove several propositions in our analysis of transfer matrices. Then singularities of the characteristic polynomials for these matrices are analyzed. We study the associated rational generating functions and asymptotics. Finally, the chapter concludes with enumerative data.

Chapter 5 discusses asymptotics for multicoloured set partitions and permutations. Some enumerative data are presented.

Chapter 6 is the conclusion of the thesis. We repeat several conjectures for further exploration.

Note that there is a slight variant called enhanced crossing and nesting in the literature with respective results paired with the standard case; we do not discuss the enhanced case in this thesis.

1.4 Contribution to Thesis

Our contributions serve as a continuation to the paper [38] of Marberg with Chapters 4 and 5 being new work. We implement Marberg's construction in an algorithm presented in the Appendix. In the process we analyze the transition from set partitions to the generating function asymptotics; an example is outlined in Table 1.1. In doing so, many properties of matrices arise and are summarized in Section 4.3. Asymptotic results are in Section 4.4. Interesting patterns in the generating functions are then described in Section 4.5.

Some of our contributions include:

1. Finding the general form of the 2-noncrossing, \(k\)-nonnesting generating functions for set partitions in Proposition 4.5.1.

2. Determining a recursion for the structure behind the transfer matrices (see Proposition 4.3.5).
<table>
<thead>
<tr>
<th>Object</th>
<th>Notation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set partition (or a matching)</td>
<td>$P_{3,3}(6)$</td>
<td>${{1, 4}, {2, 6}, {3, 5}}$</td>
</tr>
<tr>
<td>Arc diagram</td>
<td>$P_{3,3}(6)$</td>
<td><img src="image" alt="Arc Diagram" /></td>
</tr>
<tr>
<td>Vacillating tableau</td>
<td>$V_{3,3}^{1,2}$</td>
<td><img src="image" alt="Vacillating Tableau" /></td>
</tr>
<tr>
<td>Class Level Object</td>
<td></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>Finite state machine</td>
<td>$G_{3,3}$</td>
<td>$\begin{pmatrix} 1 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 1 &amp; 2 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 1 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 2 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 1 &amp; 3 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 &amp; 2 \end{pmatrix}$</td>
</tr>
<tr>
<td>Adjacency matrix</td>
<td>$M_{3,3}$</td>
<td><img src="image" alt="Adjacency Matrix" /></td>
</tr>
<tr>
<td>Rational generating function</td>
<td>$R_{3,3}(z)$</td>
<td>$\frac{(3z - 1)(4z^3 - 11z^2 + 7z - 1)}{(z - 1)(z^4 - 30z^3 + 31z^2 - 10z + 1)}$</td>
</tr>
<tr>
<td>Coefficient asymptotics</td>
<td>$\lim_{n \to \infty} NCN_{3,3}(n, 1)$</td>
<td>$\approx 1.79 \cdot 4.97^n$</td>
</tr>
</tbody>
</table>

Table 1.1: Examples of each type of object in a 3-noncrossing, 3-nonnesting partition

3. Several results on adjacency matrices (e.g. Propositions 4.3.2 and 4.3.4) including relating these matrices to the finite state machines.
Chapter 2

Background

2.1 Generating Functions

2.1.1 Combinatorial Class

Definition 2.1.1 A combinatorial class \((\mathcal{C}, |\cdot|)\) is a countable set \(\mathcal{C}\) with a size function \(|\cdot|: \mathcal{C} \rightarrow \mathbb{N}\) such that \(\mathcal{C}_n := \{C \in \mathcal{C} : |C| = n\}\) is finite for all \(n \in \mathbb{N}\). We call the combinatorial class \(\mathcal{C}\) instead of \((\mathcal{C}, |\cdot|)\) when there is no confusion. Define \(c(n) = |\mathcal{C}_n|\) and call \(\{c(n)\}_{n=0}^{\infty} = \{c(1), c(2), \ldots\}\) the counting sequence. Given a combinatorial class \(\mathcal{C}\), we define the ordinary generating function as

\[
C(x) = \sum_{c \in \mathcal{C}} x^{|c|} = \sum_{n=0}^{\infty} c(n) x^n
\]

and denote \(c(n) = [x^n]C(x)\) as the coefficient of \(x^n\).

A generating function is a formal power series whose coefficients encode combinatorial information. First, fix some field \(K\) for the coefficients, typically the complex numbers \(\mathbb{C}\). For our purposes, the coefficients are natural numbers since they are counting objects. We may also consider multivariate generating functions. These functions track multiple parameters about the objects.

Definition 2.1.2 A multivariate ordinary generating function \(F\) in the variables \(z_1, \ldots, z_k\) over the field \(K\) is an expression in \(K[[z_1 \ldots, z_k]]\) of the form

\[
F(z_1, \ldots, z_k) = \sum_{n_1, \ldots, n_k} f(n_1, \ldots, n_k) z_1^{n_1} \cdots z_k^{n_k} \tag{2.1}
\]

where \(n_i \in \mathbb{N}\) for \(i = 1, \ldots, k\).
2.1.2 Algebraic Generating Functions

Recall the class of generating functions known as algebraic generating functions.

Definition 2.1.3 A formal power series \( f \in K[[z]] \) is **algebraic** if there exists polynomials \( P_0(z), P_1(z), \ldots, P_d(z) \in K[z] \), not all 0, such that

\[
P_0(z) + P_1(z)f(z) + P_2(z)f(z)^2 + \cdots P_d(z)f(z)^d = 0.
\] (2.2)

The degree of \( f \) is the smallest positive integer \( d \) for which Equation (2.2) holds.

Note that this definition is a natural generalization of algebraic numbers. Also note that if \( f \) has degree 1 then we can write

\[
f(z) = \frac{P(z)}{Q(z)}
\]

for \( P, Q \in K[z] \). These are **rational functions**.

**Example 2.1.1** Consider the generating function for the famous Catalan numbers

\[
C(z) = \sum_{n \geq 0} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}
\]

where \( C_n := \frac{1}{n+1} \binom{2n}{n} \). Since

\[
zC(z)^2 - C(z) + 1 = 0,
\]

\( C(z) \) is an algebraic generating function.

There exists a useful theorem from Stanley [55] relating rational generating functions to a recurrence relation which we present below.

**Proposition 2.1.1** [55, Theorem 4.1.1] Given complex numbers \( \alpha_1 \ldots, \alpha_d \neq 0 \) and a function \( f : \mathbb{N} \to \mathbb{C} \). Then the following are equivalent.

1. \( Q(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_d z^d \) and \( P(z) \) is a polynomial of degree less than \( d \) and

\[
F(z) = \sum_{n \geq 0} f(n) z^n = \frac{P(z)}{Q(z)}
\]

2. We have the recurrence

\[
f(n + d) + \alpha_1 f(n + d - 1) + \alpha_2 f(n + d - 2) + \cdots + \alpha_d f(n) = 0
\]

for all \( n \in \mathbb{N} \).
3. We have
\[ f(n) = \sum_{j=1}^{k} P_j(n) r_j^n \]
where the \( r_j \)'s are distinct values satisfying
\[ 1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_d z^d = \prod_{j=1}^{k} (1 - r_j z)^{d_j} \]
and \( P_j(n) \) are polynomials of degree less than \( d_j \) for all \( n \in \mathbb{N} \).

A natural question to ask is if we can generalize condition 2 of Proposition 2.1.1 to algebraic functions. Not directly but, there exists a generalization to another class which contains algebraic functions. This is the class of D-finite functions of the next section (see Definition 2.1.5 and Proposition 2.1.3).

### 2.1.3 D-finite Generating Functions

**Definition 2.1.4** We call a formal power series \( f \in K[[z]] \) **D-finite (differentiably finite)** if there exist polynomials \( p_0(z), p_1(z), \ldots, p_d(z) \neq 0, p(z) \in K[z] \) such that
\[
p_0 f + p_1 \frac{df}{dz} + p_2 \frac{d^2 f}{dz^2} + \cdots + p_d \frac{d^d f}{dz^d} = p(z). \tag{2.3}
\]

Often, we take \( p(z) = 0 \).

**Proposition 2.1.2** [53, Theorem 5.1] An algebraic generating function in \( \mathbb{C}[[z]] \) is D-finite.

There is another useful characterization of D-finiteness. First, we introduce the following definition.

**Definition 2.1.5** A function \( f : \mathbb{N} \to K \) is **P-recursive** (polynomially recursive) if there exist polynomials \( p_0(z), p_1(z), \ldots, p_d(z) \neq 0 \in K[z] \) such that
\[
p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_d(n)f(n+d) = 0 \tag{2.4}
\]
for all \( n \in \mathbb{N} \).

**Proposition 2.1.3** [54, Proposition 6.4.3] A formal power series \( F(z) = \sum_{n=0}^{\infty} f(n)z^n \in K[[z]] \) is D-finite if and only if \( f \) is P-recursive.

**Example 2.1.2** Consider again the Catalan numbers mentioned in Example 2.1.1. We already know its generating function is algebraic so it is also D-finite. This can be seen explicitly from the equation
\[ p_0(n)C_n + p_1(n)C_{n+1} = 0 \]
where \( p_0(n) = -2(2n + 1) \) and \( p_1(n) = n + 2 \). Thus \( C_n \) is P-recursive and we can apply Proposition 2.1.3. The related generating function \( \sum_{n=0}^{\infty} C_n^2 \) is D-finite but not algebraic.

**Example 2.1.3** Consider the generating function \( \sum_n B_n \frac{z^n}{n!} \), where \( B_n \) is the number of partitions of an \( n \)-set\(^1\). \( B_n \) are known as **Bell numbers**. Counting all partitions of a fixed size summing through all fixed crossing and nesting numbers should return the Bell numbers. It is well known that if \( S(n,k) \) (**Stirling numbers of the second kind**) counts the number of partitions of an \( n \)-set into \( k \) parts, then we can prove the classical formula \( S(n,k) = \frac{1}{k!} \sum_{j=1}^{k} \left( \begin{array}{c} k \cr j \end{array} \right) (-1)^{k-j} j^n \) using the Principle of Inclusion-Exclusion. Then

\[
\sum_{n \geq 0} S(n,k) \frac{z^n}{n!} = \sum_{n \geq 0} \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \left( \begin{array}{c} k \cr j \end{array} \right) j^n \frac{z^n}{n!} = \frac{1}{k!} \sum_{j=1}^{k} \left( \begin{array}{c} k \cr j \end{array} \right) (-1)^{k-j} \sum_{n \geq 0} \frac{(jz)^n}{n!} = \frac{1}{k!} \sum_{j=1}^{k} \left( \begin{array}{c} k \cr j \end{array} \right) (-1)^{k-j} e^{jz} = \frac{(e^z - 1)^k}{k!}
\]

where the last equality follows from the Binomial Theorem. The above function is D-finite. Since \( B_n = \sum_k S(n,k) \),

\[
B(z) := \sum_{n \geq 0} B_n \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{k=0}^{n} S(n,k) \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{(e^z - 1)^k}{k!} = \exp(e^z - 1).
\]

A reference can be found at [13, Proposition 5.4.1]. From [33], we deduce the function \( B(z) \) is not D-finite.

### 2.2 Asymptotic Enumeration

Sometimes, we are interested in the singularities of a generating function, which can be analyzed from the growth of the coefficients of the function. To study the coefficient asymptotics of a generating function \( f(z) = \sum a(n) z^n \), we want a simple function \( b(n) \) such that \( \lim_{n \to \infty} \frac{a(n)}{b(n)} = 1 \). Rather than determining the coefficients explicitly, which may

\(^{1}\)It turns out to be more convenient to use the exponential generating function for partitions since it is convergent.
be too cumbersome to deal with or difficult to extract, this provides an approximate value for higher degree coefficients. For example, here is the well known Stirling’s formula to approximate $n!$:

$$
\lim_{n \to \infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.
$$

We also present some results from the field of analytic combinatorics [22].

Many generating functions $f(z) = \sum_{n=0}^{\infty} a(n)z^n$ have coefficients with an asymptotic growth of the following form:

$$
[z^n]f(z) = A^n \theta(n),
$$

where $A$ is a positive constant and $\lim_{n \to \infty} \frac{\theta(n)}{a_n} = 0$ for all $a > 1$. We call $A^n$ the exponential growth factor and $\theta(n)$ the subexponential growth factor. The asymptotic growth of these two factors gives information on the type and number of singularities. There are two general principles from Chapter IV.1. of [22] to encapsulate this idea.

**Principles of Coefficient Asymptotics**

1. **First Principle** The location of a function’s singularities dictates the exponential growth ($A^n$) of its coefficients.

2. **Second Principle** The nature of a function’s singularities determines the associated subexponential factor ($\theta(n)$).

Before we expand on the first principle in a theorem, let us introduce the following definition and notation.

**Definition 2.2.1** A sequence $\{a(n)\}_{n=0}^{\infty}$ is of exponential order $\rho^n$ with the notation $a(n) \asymp \rho^n$ if $\rho = \limsup |a(n)|^{1/n}$. We can view $\frac{1}{\rho}$ as the radius of convergence of $\sum a(n)z^n$ about 0.

**Theorem 2.2.1** (Exponential Growth Formula [22, Theorem IV.7]) Let $f(z)$ be analytic at 0 and $\rho := \sup\{r \geq 0 : f \text{ is analytic in } |z| < r\}$ the radius of convergence of $f$ about 0. Then

$$
[z^n]f(z) \asymp \left(\frac{1}{\rho}\right)^n.
$$

For combinatorial generating functions with nonnegative real coefficients, it suffices to restrict $\rho$ to

$$
\rho := \sup\{r \geq 0 : f \text{ is analytic for all } 0 \leq z < r\}.
$$
We next discuss the second principles of coefficient asymptotics. Since we are dealing with rational functions in this thesis, we only present the results for rational functions. Essentially this principle states that the subexponential factors $\theta(n)$ grow polynomially.

**Theorem 2.2.2** *(Expansion of rational functions [22, Theorem IV.9]*) Let $f(z)$ be a rational function analytic at 0, with poles at $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then the coefficients of $f$ are sums of exponential functions multiplied by polynomials, called exponential-polynomials. More explicitly, for large enough $n$,

$$[z^n]f(z) = \sum_{j=1}^{m} \frac{\Pi_j(n)}{\alpha_j^n}$$

where $\Pi_j(z)$ are polynomials for $j = 1, \ldots, m$. Also, the degree of $\Pi_j$ equals the order of the pole of $f$ at $\alpha_j$ minus one.

Combining Theorems 2.2.1 and 2.2.2 after calculations, we obtain the following theorem pertaining to the dominant singularity.

**Proposition 2.2.1** Let $f(z) = \sum_n a(n)z^n$ be a generating function about 0. Let $\rho = \lim \sup_n |a(n)|^{1/n}$ be a dominant singularity of $f$ (closest to 0 for rational functions) of order $k$. Let $r = k - 1$ and $C = f(z) \left(1 - \frac{z}{\rho}\right)^k \bigg|_{z=\rho}$ be constants, then

$$\lim_{n \to \infty} \frac{a(n)}{C n^r \rho^{-n}} = 1. \quad (2.7)$$

Here, $\left(\frac{1}{\rho}\right)^n$ is the exponential growth factor and $Cn^r = \theta(n)$ is the subexponential growth factor.
This chapter aims to survey the literature on the symmetric joint distribution of crossings and nestings, and give numerous bijective and enumerative results emphasizing set partitions. We begin by defining the main concepts and ideas involving arc diagrams and Young diagrams. Next comes the clarification of the main bijection by Chen et al., which applies to both set partitions and matchings. Then, we specialize and consider results on simple matching structures. The discussion moves on to partitions followed by the more complicated permutation structures. Then we present the main tools in our approach to studying set partitions, which is an algorithm by Marberg built from the original Chen et al. bijection. For further references on applications of the Chen et al. bijection, see for instance Chen and Guo [15], Marberg [38], and Yen [63].

3.1 Arc Diagrams

We begin with main definitions related to crossings and nestings on arc diagrams and use the notation of Marberg [38], Chen et al. [14], and Yen [63].

3.1.1 Set Partitions

Definition 3.1.1 A set partition $P_n$ of the standard $n$-set $[n] := \{1, \ldots, n\}$ is a collection of subsets, called blocks, with empty intersection such that no block is empty and their union is the entire set. Let $P_n^\circ$ denote the set of partitions $P_n$ and $P$ be the set of all set partitions.
Definition 3.1.2 The arc diagram representation of a set partition \( P \) of \([n]\) is a simple, labelled graph of \([n]\) with edges only among consecutive elements in a block (i.e., for each block \( \{i_1 < i_2 < \ldots < i_m\} \) of \( P \), \( i_j \) is adjacent to \( i_k \) if and only if \( j \) and \( k \) are consecutive integers). We list each integer element from 1 to \( n \) sequentially on a horizontal line. Then if \( i < j \) are adjacent, we draw their edge as an arc \((i,j)\) above the horizontal line. Note that a block of size 1 has no arcs. Let \( \text{Arc}(P) \) denote the set of all arcs of a partition \( P \).

Figure 3.1: For \( P = \{\{1\}, \{2,4,6\}, \{3,7\}, \{5\}\} \), \( \text{Arc}(P) = \{(2,4), (3,7), (4,6)\} \)

This definition is also known as the standard representation of \( P \). For a partition \( P \), define \( \text{min}(P) \) (\( \text{max}(P) \)) to be the set of minimal (maximal) elements of \( P \). Thus, \( \text{min}(P) = \{1,2,3,5\} \) and \( \text{max}(P) = \{1,5,6,7\} \) for Figure 3.1. There has been much work done on the much simpler substructure of matchings in, for example, [14], [15], and [31].

Definition 3.1.3 A (complete) matching of \([2n]\) is a set partition of \([2n]\) into \( n \) subsets each of size 2. A partial matching of \([n]\) is a set partition of \([n]\) where every subset is of size 1 or 2.

Figure 3.2: For \( P = \{\{1,3\}, \{2,5\}, \{4,8\}, \{6,7\}\} \), \( \text{Arc}(P) = \{(1,3), (2,5), (4,8), (6,7)\} \)

Note that a partition \( P \) of \([n]\) is a partial matching if and only if

\[
\text{min}(P) \cup \text{max}(P) = [n] \tag{3.1}
\]

and a partition \( P \) of \([2n]\) is a (complete) matching if and only if Equation (3.1) and additionally

\[
\text{min}(P) \cap \text{max}(P) = \emptyset \tag{3.2}
\]

are both satisfied.

Two well studied combinatorial structures on the arc diagrams include crossings and nestings. If there exists a collection of arcs \( \{(i_1,j_1), \ldots, (i_k,j_k)\} \) for a partition \( P \) where
Figure 3.3: A 3-crossing

Figure 3.4: A 4-nesting

\[ i_1 < i_2 < \ldots < i_k < j_1 < j_2 < \ldots < j_k, \]
we say that \( P \) contains a \( k \)-crossing or that \( P \) is \( k \)-crossing (see Example 3.3 for a 3-crossing) and if \( i_1 < i_2 < \ldots < i_k < j_k < j_{k-1} < \ldots < j_1 \), we say that \( P \) contains a \( k \)-nesting or that \( P \) is \( k \)-nesting (see Example 3.4 for a 4-nesting).

Then define \( cr(P) \) to be the maximum \( k \) such that \( P \) is \( k \)-crossing and \( ne(P) \) to be the maximum \( k \) such that \( P \) is \( k \)-nesting. If \( P \) does not contain a \( k \)-crossing (\( k \)-nesting), we say that \( P \) is \( k \)-noncrossing (\( k \)-nonnesting). We can also say that \( P \) avoids the pattern \( 12 \cdots k12 \cdots k \) (\( 12 \cdots kk \cdots 21 \)). This notation is standard in permutation avoidances and follows from [37]. Partition \( P \) avoids \( 12 \cdots k12 \cdots k \) if there is no subsequence \( a_1 < a_2 < \ldots < a_k < b_1 < b_2 < \ldots < b_k \) of \( 1, \ldots, n \) where \( (a_i, b_i) \) is an arc for all \( i \in [k] \); avoiding \( 12 \cdots kk \cdots 21 \) is similarly defined. Also, we identify the term noncrossing (nonnesting) partition with 2-noncrossing (2-nonnesting). Let \( P_{j,k} \) be the set of \( j \)-noncrossing and \( k \)-nonnesting partitions and \( P_{j,k} \) be a specified element of this set. Also, let \( P_{j,k}(n) \subseteq P_{j,k} \) be the subset of partitions of \([n]\) and \( P_{j,k}(n) \) be a particular element of this set.

**Coloured Arcs**

We may also multi-colour the arcs following the notation of Marberg [38].

**Definition 3.1.4** Given a partition \( P \) and the surjective function \( \phi : Arc(P) \to [r] \), we call \((P, \phi)\) (or still \( P \) for shorthand) an \( r \)-coloured set partition. With a given \( r \), let \( P^n_r \) be the set of all \( r \)-coloured set partitions of \([n]\) and \( P^r \) the set of all \( r \)-coloured set partitions.

**Example 3.1.1** Consider \( P = \{\{1\to 5\to 6\}, \{2\}, \{3\to 8\}, \{4\to 7\}\} \in P^2_8 \). The standard representation is shown in Figure 3.5. Note that there is no 3-nesting but a 2-nesting and 2-crossing in red.

**Notation** All definitions on crossings and nestings extend to \((P, \phi)\) by taking a maximum over all colours (e.g., \( cr(P) = k \) means that \( P \) contains a maximum \( k \)-crossing with arcs
all of one colour). Let us denote by $NCN_{j,k}(n,r)$ to be the number of $r$-coloured arc diagrams on set partitions of $[n]$ that are $j$-noncrossing and $k$-nonnesting. Additionally, if the partitions $P$ have fixed $\min(P) = S$ and $\max(P) = T$, then let the number of such partitions be $NCN_{j,k}^{S,T}(n,r)$. We focus mainly on the one colouring case and we leave out the $r$ and write $NCN_{j,k}(n)$ and $NCN_{j,k}^{S,T}(n)$ instead. We also define $NC_j(n)$ and $NN_k(n)$ to be the number of $j$-noncrossing partitions of $[n]$ with any number of nestings and $k$-nonnesting partitions of $[n]$ with any number of crossings respectively.

The generating functions we study are of the form

$$R_{j,k}(z,r) := \sum_{n \geq 0} NCN_{j,k}(n,r)z^n,$$

but we mainly consider the case when $r = 1$ and study

$$R_{j,k}(z) := \sum_{n \geq 0} NCN_{j,k}(n)z^n \quad (3.4)$$

for various values of $j \geq 2$ and $k \geq 2$. We show in Section 3.7.4 that they are rational functions. Consequently, we define the following notation:

$$R_{j,k}(z) = \frac{f_{j,k}(z)}{g_{j,k}(z)}, \quad (3.5)$$

where $f_{j,k}, g_{j,k} \in \mathbb{Z}[z]$. We may wish to also fix $j$ in the $j$-noncrossing number and let the nesting number $k$ vary over all values and examine

$$R_{j}(z) := \sum_{n \geq 0} NC_j(n)z^n \quad (3.6)$$

$$= \lim_{k \to \infty} R_{j,k}(z)$$

for $j \geq 2$. 

Figure 3.5: A 2-coloured set partition
3.1.2 Permutations

Crossings and nestings also exist for permutations of a set, whose arc diagrams are generalizations of those of partitions. Crossings and nestings were first defined by Cor- 

teel [17]. Burrill et al. [11] defined $k$-crossings and $k$-nestings and were the first to extend 
the equidistribution property to permutations.

**Definition 3.1.5** The arc diagram of a permutation $\pi = \pi_1 \ldots \pi_n$ of $[n]$ is a graphical repre- 
sentation of $\pi$ with the integers from 1 to $n$ on a horizontal line. If $\pi_i = i$, then we draw an 
upper loop on $i$. If $i < \pi_i$, we draw an upper arc from $i$ to $\pi_i$, and if $i > \pi_i$, we draw a lower 
arc from $i$ to $\pi_i$.

**Example 3.1.2** Figure 3.6 is an example of an arc diagram of a permutation.

![Figure 3.6: Arc diagram of permutation 5, 1, 3, 2, 6, 4](image)

Fixed points for permutations drawn as an upper arc contribute only to top nestings. 

Crossings, nestings, and colourings for permutations include lower arcs. Similarly, we de-

define the notation $NCN_{j,k}^M(n, r)$ and $NCN_{j,k}^P(n, r)$ for the number of $r$-coloured arc diagrams 
for matchings and permutations of $[n]$ ($n$ even for matchings), which are $j$-noncrossing and 
$k$-nonnesting respectively. All other derived notations are similar to how they were defined 
for set partitions.

3.2 Bijection and Generalizations

Chen et al.’s paper [14] introduced the concept of $k$-crossings and $k$-nestings on match-

ings and set partitions. They constructed a bijection using vacillating tableaux (see Sec-

tion 3.2.1) to prove that crossings and nestings form a symmetric joint distribution and open 
the door to other combinatorial classes.

Following the notation of [14], fix $|S| = |T|$ for $S, T \subseteq [n]$. Let $P_n(S, T) = \{P \in P_n : 

\min(P) = S, \max(P) = T\}$ and $f_{n,S,T}(i, j) = |\{P \in P_n(S, T) : cr(P) = i, ne(P) = j\}|$

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be the number of partitions of \([n]\) with sets of minimal and maximal elements \(S\) and \(T\) respectively and fixed maximum crossing number and maximum nesting number \(i\) and \(j\) respectively.

**Theorem 3.2.1** (Chen et al. [14, Theorem 1.1]) The statistics \(cr(P)\) and \(ne(P)\) for set partitions have a symmetric joint distribution over each set \(P_n(S, T)\).

\[
f_{n,ST}(i, j) = f_{n,ST}(j, i)
\]

or in terms of generating functions

\[
\sum_{P \in P_n(S, T)} x^{cr(P)} y^{ne(P)} = \sum_{P \in P_n(S, T)} x^{ne(P)} y^{cr(P)}.
\]

A whole collection of useful corollaries follow from Theorem 3.2.1 and its bijective proof by summing up partitions over certain patterns. In light of Equations (3.2) and (3.1), the crossing and nesting results on set partitions with fixed maximal and minimal elements also apply to matchings and partial matchings. First recall the notation from Section 3.1 that \(NCN_{j,k}(n)\) is the number of \(j\)-noncrossing and \(k\)-nonnesting set partitions on \([n]\). The number of \(k\)-noncrossing partitions on \([n]\) and the number of \(k\)-nonnesting partitions on \([n]\) are respectively denoted \(NC_k(n)\) and \(NN_k(n)\). Also, \(NCN_{j,k}^M(2n)\) is the number of \(j\)-noncrossing and \(k\)-nonnesting matchings on \([2n]\).

**Corollary 3.2.1** (Chen et al. [14]) The following is true for all values of \(n, j, k\):

1. The statistics \(cr(P)\) and \(ne(P)\) have a symmetric joint distribution over all partitions of \([n]\);
2. the statistics \(cr(P)\) and \(ne(P)\) have a symmetric joint distribution over all matchings of \([2n]\);
3. \(NCN_{j,k}^M(2n) = NCN_{k,j}^M(2n)\);
4. \(NCN_{j,k}(n) = NCN_{k,j}(n)\);
5. \(NN_k(n) = NC_k(n)\).

We present the bijection of Chen et al. [14], not in full generality, but as required for our purposes. First, we take a slight detour to introduce the necessary definitions and tools. We begin by introducing Young diagrams and some related definitions.
3.2.1 Young Diagrams

**Definition 3.2.1** Let \( \mathbb{P} \) denote the set of positive integers. An **integer partition** of \( n \in \mathbb{P} \) is a sequence of nonincreasing positive integers \( \lambda = (\lambda_1, \ldots, \lambda_l) \) that sum to \( n \). Call \( l \) the number of parts of \( \lambda \) and write \( \lambda \vdash n \) or \( |\lambda| = n \). Define \( \lambda_i = 0 \) for \( i > l \) so \( \lambda_i \) is defined for all \( i \in \mathbb{P} \). Denote by \( \emptyset \) the empty Young diagram. A **Young diagram** corresponding to \( \lambda \) is a graphical representation of the form \( \{(x, y) \in \mathbb{P}^2 : x \leq \lambda_{l+1-y}, y \leq l\} \) with each entry consisting of squares (or dots).

**Example 3.2.1** Let \( \lambda = (4, 2, 2, 1) \) be an integer partition. Figure 3.7 is its Young diagram.

![Young diagram of (4, 2, 2, 1)](image)

Figure 3.7: Young diagram of \((4, 2, 2, 1)\)

For integer partitions \( \lambda, \mu \), we say \( \mu \leq \lambda \) if \( \mu_i \leq \lambda_i \) for all \( i \in \mathbb{P} \). Equivalently, the Young diagram of \( \lambda \) contains the Young diagram of \( \mu \). The operation \( \leq \) forms a partial ordering on the set of all integer partitions. In fact, this partially ordered set forms a lattice since we can define a join and meet with \( \lambda \vee \mu = (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \ldots) \) and \( \lambda \wedge \mu = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots) \).

**Definition 3.2.2** A sequence of integer partitions \( S^n = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^n = \emptyset) \) is a **semi-oscillating tableau** of length \( n \) if for each \( i \in [n] \), the Young diagram of \( \lambda^i \) is constructed from the Young diagram of \( \lambda^{i-1} \) by either adding a square, removing a square, or staying the same.

We can think of a semi-oscillating tableau as a walk in the Young lattice from \( \emptyset \) to itself, in \( n \) steps.

**Definition 3.2.3** A **vacillating tableau** \( V^{2n} \) is a semi-oscillating tableau of length \( 2n \) with additional restrictions. When \( i \) is even, the Young diagram of \( \lambda^i \) is constructed from the Young diagram of \( \lambda^{i-1} \) by adding a square or by staying the same. When \( i \) is odd, the Young diagram of \( \lambda^i \) is constructed from the Young diagram of \( \lambda^{i-1} \) by removing a square or by staying the same.
We use the notation $V_{2n}^{j,k}$ to denote a vacillating tableau of length $2n$ that has every element with less than $j$ columns and less than $k$ rows.

**Example 3.2.2** Figure 3.8 is a vacillating tableau of length 12.

\[
(\emptyset, \emptyset, \square, \square, \square, \square, \square, \square, \square, \square, \emptyset, \emptyset)
\]

Figure 3.8: A vacillating tableau $V_{3,3}^{12}$

**Definition 3.2.4** A *standard Young tableau* (SYT) of shape $\lambda$ where $\lambda \vdash n$ is a filling of the Young diagram $\lambda$ with $\{1, 2, \ldots, n\}$ where every row and every column is strictly increasing.

**Example 3.2.3** Figure 3.9 is a SYT $\lambda$ of shape $(4, 2, 1)$ and size 7.

```
1 3 4 7
2 5
6
```

Figure 3.9: Standard Young tableau of shape $(4, 2, 1)$

A standard Young tableau with $n$ boxes can be viewed as a semi-oscillating tableau $S^n$ where boxes are only added at each step. The numbers in the SYT determine the order in which the boxes are added and it does not end with $\emptyset$ when $n > 0$.

### 3.2.2 Robinson-Schensted-Knuth (RSK) Algorithm

We present the necessary portions of the Robinson-Schensted-Knuth (RSK) algorithm, from the presentation in Chapter 7 of [54]. Originally this algorithm was developed as a step towards proving the Littlewood-Richardson Rule (see Appendix A1 of [54]). This is a useful algorithm for mapping a sequence of distinct integers to a Young diagram and it provides a bijection between permutations of $[n]$ and pairs of standard Young tableaux of $[n]$ of the same shape. We briefly outline the element insertion and deletion steps because they are needed for the Chen et al. bijection.

**RSK Algorithm (Insertion)**

Given a Young diagram $T$ with distinct entries and $i$ not in $T$. The procedure to insert $i$ into $T$ is as follows.
1. Begin with the current row set as the top row of $T$.

2. Find the smallest $j > i$ in the current row if it exists.

3. If no such $j$ exists, put a new square at the end of the current row and place $i$ inside. STOP.

4. Else put $i$ where $j$ was and continue with the next row down. Define the current row as this next row and go back to step 2 with $i$ replaced by $j$.

Example 3.2.4 Figure 3.10 inserts $i = 3$ into $T = \begin{array}{ccc} 2 & 5 & 6 \\ 7 & 8 \end{array}$.

Steps $T$ $i$
\begin{align*}
1 &\Rightarrow 2 & \begin{array}{ccc} 2 & 5 & 6 \\ 7 & 8 \end{array} & 5 \leftarrow 3 \\
\downarrow
\end{align*}

\begin{align*}
4 &\Rightarrow 2 & \begin{array}{ccc} 2 & 3 & 6 \\ 7 & 8 \end{array} & 7 \leftarrow 5 \\
\downarrow
\end{align*}

\begin{align*}
4 &\Rightarrow 2 & \begin{array}{ccc} 2 & 3 & 6 \\ 5 & 8 \end{array} & \leftarrow 7 \\
\downarrow
\end{align*}

\begin{align*}
2 &\Rightarrow 3 & \begin{array}{ccc} 2 & 3 & 6 \\ 5 & 8 & 7 \end{array}
\end{align*}

Figure 3.10: An RSK insertion step: 3 into $T$

RSK Algorithm (Deletion)

Given a Young tableau $T$ with distinct entries and $j$ in an outer corner box of $T$. The procedure to delete the box containing $j$ from $T$ is as follows.

1. Remove $j$ and its box.

2. In the row above find the largest entry $i$, $i < j$ (always exist if there is a row) and replace $i$ by $j$.

3. Repeat step 2 in the row above with $i$ in place of $j$ whenever possible.

4. When we process the top row, we remove that value from the tableau.
Example 3.2.5 Figure 3.11 deletes the box containing 7 and removes 3 from \( T = \begin{array}{ccc} 1 & 3 & 8 \\ 2 & 4 & \\ 5 & 6 \\ 7 & \end{array} \).

![Steps](image)

The contrapositive of Theorem 3.2.3 below helps us to determine a bound on the size of any Young diagram in a vacillating tableau corresponding to a partition that is \( j \)-noncrossing and \( k \)-nonnesting. We prove this theorem using the RSK algorithm. Note that this is not the original proof. First, we need the following result of Schensted as a lemma.

**Theorem 3.2.2** (Schensted [49]) Let \( \pi \) be a permutation that is mapped to \((A, B)\) by the RSK correspondence, where \( A \) and \( B \) are SYT’s of the same shape \( \lambda \). Then the length of the longest increasing subsequence of \( \pi \) is the number of columns of \( \lambda \) and the length of the longest decreasing subsequence of \( \pi \) is the number of rows of \( \lambda \).

**Theorem 3.2.3** (Erdős-Szekeres [21]) If \((j - 1)(k - 1) + 1\) distinct real numbers are in a sequence, then there is a monotonically increasing subsequence of length \( j \) or a monotonically decreasing subsequence of length \( k \).
Proof Let \(a_1, \ldots, a_{(j-1)(k-1)+1}\) be the sequence of real numbers. Consider the permutation \(\pi \in S_{(j-1)(k-1)+1}\) with \(\pi_i = |\{a_j : a_j \leq a_i\}|\). Essentially, this keeps the ordering of the original sequence but replaces them with the numbers from 1 to \((j-1)(k-1)+1\). The RSK correspondence maps \(\pi\) to a pair of standard Young tableaux of the same shape \(\lambda\) by inserting every entry of \(\pi\) in the order they are read. Applying Theorem 3.2.2, the length of the longest increasing (decreasing) subsequence of \(\pi\) is the number of columns (rows) of \(\lambda\). The size of \(\lambda\) is \((j-1)(k-1)+1\), so \(\lambda\) does not fit inside an \((k-1) \times (j-1)\) Young diagram. Hence, either the first row of \(\lambda\) is longer than \(j-1\) or the first column of \(\lambda\) is longer than \(k-1\). This yields an increasing subsequence of length at least \(j\) or a decreasing subsequence of length at least \(k\) for \(\pi\) and the original sequence.

Example 3.2.6 Consider the sequence of 12 numbers \(\pi = 10, 11, 12, 7, 8, 9, 4, 5, 6, 1, 2, 3\) with a maximum increasing sequence of length 3 (e.g. \(10, 11, 12\)) and a maximum decreasing sequence of length 4 (e.g. \(10, 7, 4, 1\)). This sequence \(\pi\) is mapped to the SYT of Figure 3.12 by RSK. From Theorem 3.2.2, there are 3 columns and 4 rows as expected. Let \(j = 4\) and \(k = 5\). If an additional distinct number is added to the sequence, we will have \((4-1)(5-1) + 1 = 13\) numbers. By the Erdős-Szekeres Theorem (Theorem 3.2.3), there will be an increasing sequence of length 4 or a decreasing sequence of length 5.

![Figure 3.12: RSK: SYT of 10, 11, 12, 7, 8, 9, 4, 5, 6, 1, 2, 3](image)

3.2.3 Set Partitions to Vacillating Tableaux

In this section we describe a bijection \(\phi_n : \mathcal{P}_n \mapsto \{V^{2n}\}\) from the standard representation of a set partition of \([n]\) to a vacillating tableau of length \(2n\). Note that this bijection has the same description as the Chen et al. bijection from [14] but in reverse order. This means that every Young diagram in the vacillating tableau is the transpose of the one from [14]. Also, note that our bijection is equivalent to Marberg's bijection in [38]. However, our presentation of the bijection is different since ours is derived from the Chen et al. bijection.
while Marberg’s bijection is based on two bijections involving sequences of integers and matchings.

**Theorem 3.2.4** (Marberg [38, Theorem 2.7]) Given a partition $P \in \mathcal{P}_n$, there exists a bijection $\phi_n : P \mapsto (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$ from the set of partitions of $[n]$ to the set of vacillating tableaux of length $2n$ where $cr(P)$ is the maximum number of columns of any $\lambda^i$ and $ne(P)$ is the maximum number of rows of any $\lambda^i$ for $i \in [n]$. Thus, there is a bijection from the set of $j$-noncrossing, $k$-nonnesting partitions $\mathcal{P}_{j,k}(n)$ of $[n]$ and the set of vacillating tableaux $V_{j,k}^{2n}$ of length $2n$ such that every Young diagram has fewer than $j$ columns and fewer than $k$ rows.

**Sketch of Bijection** Given $P \in \mathcal{P}_n$, we build a sequence of SYTs $T = (T_0, T_1, \ldots, T_{2n})$ of length $2n$ such that the corresponding Young diagrams in this sequence form the announced vacillating tableau. We begin by defining $T_0 = \emptyset$ to be the empty diagram. Then read the elements $j \in [n]$ in increasing order and for each $j$ define $T_{2j-1}$ and $T_{2j}$ from $T_{2j-2}$ as follows:

1. $(j \in \min(P) \setminus \max(P))$ If $j$ is the left endpoint of an arc $(j, i)$ but not a right endpoint of an arc, then $T_{2j-1} = T_{2j-2}$ and $T_{2j}$ is obtained from $T_{2j-1}$ by inserting $i$ according to the RSK algorithm;

2. $(j \in \max(P) \setminus \min(P))$ If $j$ is the right endpoint of an arc $(i, j)$ but not a left endpoint of an arc, then $T_{2j-1}$ is obtained from $T_{2j-2}$ by deleting $j$ according to the RSK algorithm and $T_{2j} = T_{2j-1}$;

3. $(j \in \min(P) \cap \max(P))$ If $j$ is not on any arc, then $T_{2j-1} = T_{2j-2} = T_{2j}$;

4. $(j \notin \min(P) \cup \max(P))$ If there exists arc $(i, j)$ and arc $(j, k)$, then $T_{2j-1}$ is obtained from $T_{2j-2}$ by deleting $j$ (by RSK) and $T_{2j}$ is obtained from $T_{2j-1}$ by inserting $k$ (by RSK).

**Example 3.2.7** Consider $P = \{\{1, 5\}, \{2, 6, 7\}, \{3\}, \{4, 8\}\} \in \mathcal{P}_8$. The standard representation, the sequence of SYTs constructed, and the image $\phi_8(P)$ are in Figure 3.13.

Vacillating tableaux have natural interpretations as lattice walks. The tableau restriction on height of Theorem 3.2.4 is encoded as a restricted region.
Corollary 3.2.2 (Chen et al. [14, Theorem 3.6]) Let $\varepsilon_i$ be the $i$-th standard unit vector in $\mathbb{R}^{j-1}$. The number of $j$-noncrossing partitions of $[n]$ equals the number of lattice walks from the origin to itself of length $2n$ in the region $\{(a_1, \ldots, a_{j-1}) : a_1 \geq a_2 \geq \ldots \geq a_{j-1} \geq 0, a_i \in \mathbb{Z}\}$. The steps take the form $\pm \varepsilon_i$ or the empty step, and the path takes an empty step or step of the form $-\varepsilon_i$ after an even number of steps, and the path takes an empty step or step of the form $\varepsilon_i$ after an odd number of steps.

Multi-coloured Bijection

The bijection of Chen et al., and the noncrossing and nonnesting symmetric joint distribution can be generalized to a multi-coloured version. This is done by Marberg [38] to $r$-coloured set partitions.

Theorem 3.2.5 (Marberg [38, Theorem 1.4]) For all $n, j, k, r$ and $S, T \subseteq [n], \quad NCN_{j,k}^{S,T}(n, r) = NCN_{k,j}^{S,T}(n, r). \quad (3.9)$

In particular,

$\quad NCN_{j,k}(n, r) = NCN_{k,j}(n, r) \quad (3.10)$

and

$\quad NC_{k}(n, r) = NN_{k}(n, r). \quad (3.11)$

In order to discuss the altered bijection with multi-coloured arcs, we need the following definition.

Definition 3.2.5 For $r$ a positive integer, an $r$-partite partition of $[n]$ is a sequence $\lambda = (\lambda^1, \ldots, \lambda^r)$ of Young tableaux such that $\sum_{i=1}^{r} |\lambda^i| \leq n.$
The bijection above is modified by allowing \( r \) colours. We use previously defined notation in this \( r \)-coloured context. Essentially a set partition \( P \) of \([n]\) can be decomposed as \( r \) partitions \( P = (P^1, \ldots, P^r) \) of \([n]\) such that \( P^i \) is \( P \) with only the \( i \)-th coloured arc appearing for \( i \in [n] \). Think of it as getting back the original \( P \) when we draw all \( rP^i \) on the same diagram. Then we reduce to the \( r = 1 \) case bijection on each of the \( r \) component partitions to get \( \phi_n(P^i) = \lambda_i = (\lambda^0_i, \lambda^1_i, \ldots, \lambda^{2n}_i) \) with corresponding vacillating standard Young tableau \( T_i = (T^0_i, T^1_i, \ldots, T^{2n}_i) \) for \( i \in [n] \). Let \( \lambda = (\lambda_1, \ldots, \lambda_r)^T \) and \( T = (T_1, \ldots, T_r)^T \) so the \( r \)-partite partitions are \( \lambda^i := (\lambda^0_i, \ldots, \lambda^i_i)^T \) for \( 0 \leq i \leq 2n \); \( \lambda = (\lambda^0_1, \lambda^1_1, \ldots, \lambda^{2n}_2) \) with each component as column vectors. **Vacillating \( r \)-partite tableau** \( V^{2n} \) of length \( 2n \) is defined the same as in Definition 3.2.3, but on \( r \)-partite partitions instead of single integer partitions. Also, **semi-oscillating \( r \)-partite tableau** is defined similarly. The vacillating \( r \)-partite tableau \( \lambda \) can be viewed as an \( r \times (2n+1) \) matrix with the \( ij \)-th entry as \( \lambda^j_{i-1} \); each row is a vacillating tableau \( \lambda_i \) and each column is an \( r \)-partite tableau \( \lambda^i \). Notice each column can differ from the previous column in at most one component. We then construct a bijection \( \phi^n : P \mapsto \lambda \) where \( \phi^n(P^1, \ldots, P^r) = (\phi_n(P^1), \ldots, \phi_n(P^r)) \). The following example should clarify the construction.

**Example 3.2.8** Consider \( P = \{1 \to 5 \to 6\}, \{2 \to 8\}, \{3 \to 7\}, \{4\} \in P^2_8 \). The arc diagram, the corresponding sequence of standard Young tableaux, and 2-partite tableau \( \phi^8(P) \) are in Figure 3.14. The image \( \phi^8(P) \) has rows \( \lambda^j_{i-1} \) (for red arcs) and \( \lambda^j_{2-1} \) (for blue arcs) for \( j = 0, 1, \ldots, 16 \).

![Figure 3.14: Arc diagram of \( P = \{1 \to 5 \to 6\}, \{2 \to 8\}, \{3 \to 7\}, \{4\} \) and 2-partite tableau \( \phi^8(P) \)](image-url)
Next, one last definition is given before we relate these new ideas to arc diagrams as a theorem. Theorem 3.2.6 extends Theorem 3.2.4 to the multi-coloured case for both set partitions and matchings.

**Definition 3.2.6** An **oscillating tableau** is a semi-oscillating tableau \((\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^n = \emptyset)\) where \(\lambda^i \neq \lambda^{i+1}\) for all \(i\). The definition for oscillating \(r\)-partite tableau extends naturally from Definition 3.2.5.

**Theorem 3.2.6** (Marberg [38, Theorem 3.3]) For all \(n, r\), we have the following three bijections \(\phi_1, \phi_2, \phi_3\):

1. \(\phi_1 : r\)-coloured set partitions of \([n]\) \(\mapsto\) vacillating \(r\)-partite tableaux of length \(2n\);
2. \(\phi_2 : r\)-coloured partial matchings of \([n]\) \(\mapsto\) semi-oscillating \(r\)-partite tableaux of length \(n\);
3. \(\phi_3 : r\)-coloured matchings of \([2n]\) \(\mapsto\) oscillating \(r\)-partite tableaux of length \(2n\).

Also, the maximum crossing (nesting) number is the maximum number of columns (rows) over all Young diagrams.

### 3.3 Results on Matchings

Matchings are a useful subset of partitions and are much easier to study. Many of the enumerative results we hope to prove for set partitions are already known for matchings. Restricted to matchings, Corollary 3.2.2 becomes the following:

**Corollary 3.3.1** (Chen et al. [14, Corollary 5.3]) The number of \(j\)-noncrossing matchings of \([2n]\) equals the number of lattice walks of length \([2n]\) from the origin to itself with steps of the form \(\pm \epsilon_i\) where \(\epsilon_i\) is the \(i\)-th standard unit vector in \(\mathbb{R}^{j-1}\) and restricted to the region \(\{(a_1, \ldots, a_{j-1}) : a_1 \geq a_2 \geq \ldots \geq a_{j-1} \geq 0, a_i \in \mathbb{Z}\}\).

A simple well-known result is that the number of noncrossing (or nonnesting) matchings of \([2n]\), \(NC_2^M(n)\), is counted by the \(n\)-th Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\). The \(n\)-th Catalan number also counts the number of Dyck paths of length \(2n\).
Definition 3.3.1 A Dyck path of length $2n$ is a lattice walk from $(0, 0)$ to $(2n, 0)$ taking steps of the form $(1, 1)$ and $(1, -1)$ and never passing below the $x$-axis. A pair of Dyck paths are called noncrossing if they are the same length and one never goes below the other.

Then another proposition that follows from Corollary 3.3.1 as shown in [14] is that the number of 3-noncrossing matchings is in bijection with pairs of noncrossing Dyck paths.

Although Theorem 3.2.6 is cited by Marberg [38], bijections on multi-coloured matchings were first studied by Chen and Guo in [15]. We do not explore their bijection in detail but mention an enumerative result they proved.

Proposition 3.3.1 (Chen, Guo [15, Theorem 4.1]) The number of noncrossing 2-coloured matchings of $[2n]$ is counted by $C_n \cdot C_{n+1}$.

We also mention two other results on properties of generating functions for matchings. D-finiteness is easier to determine for matchings as we have the following results from [38]; compare this with Corollary 3.7.2.

Corollary 3.3.2 (Marberg [38, Corollary 5.4]) For all $j, k$ and $r$ colours, the generating function $R_{j,k}^M(z, r) := \sum_{n=0}^{\infty} NCN_{j,k}^M(n, r)z^n$ is a rational function in $z$.

Theorem 3.3.1 (Marberg [38, Corollary 5.5]) For all $j$ and $r$ colours, the generating function $R_j^M(z, r) := \sum_{n=0}^{\infty} NC_j^M(n, r)z^n$ is D-finite.

3.4 Enumeration of 2-noncrossing Set Partitions

Similar to noncrossing matchings, the number of noncrossing (or nonnesting) partitions of $[n]$, $NC_2(n)$, is counted by the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$. Another simple property is that $NC_{2,2}(n)$ is counted by the odd numbered Fibonacci numbers $f_{2n-1}$ where $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_n + f_{n+1}$ for $n \geq 0$. We show in Section 4.5.1 a way to find the generating functions $\sum_n NC_{2,2,k}(n)z^n$ for all $k \geq 2$ and we analyze the $r$-colouring cases in Section 5.1. To aid us, we have some useful results by Marberg from [38].

Proposition 3.4.1 (Marberg [38, Theorem 1.6]) Let $e_i$ be the $i$-th standard unit vector in $\mathbb{R}^r$. The value $NC_2(n, r)$ equals the number of $n-1$-step walks from the origin to itself in $\mathbb{N}^r$ using steps given by $\pm e_i, e_i - e_j$ for $i \neq j$, and $r + 1$ distinct zero steps.
Theorem 3.4.1 (Marberg [38, Theorem 1.7]) The \([x^0 y^0]-th coefficient of the expression\)
\[(1 - x^2 y^{-1} + x^3 - x^2 y^2 + y^3 - x^{-1} y^2)(3 + x + y + x^{-1} + y^{-1} + x y^{-1} + x^{-1} y)^{n-1}\]
is \(NC_2(n, 2)\). For positive \(n\),
\[9n(n + 3)NC_2(n, 2) - 2(5n^2 + 26n + 30)NC_2(n + 1, 2)\]
\[+ (n + 4)(n + 5)NC_2(n + 2, 2) = 0.\]

Thus \(C(z) := \sum_n NC_2(n, 2) z^n\) is D-finite. Equivalently,
\[z^2(9z - 1)(z - 1) \frac{d^2}{dz^2} C(z) + 6z(6z - 1)(z - 1) \frac{d}{dz} C(z)\]
\[+ 6(1 - 3z)C(z) = 6(1 - z)\]
with \(C(0) = 1\) and \(C'(0) = 1\). The first few terms of \(NC_2(n, 2)\) are
\[1, 1, 3, 11, 47, 225, 1173, 6529, 38265, 233795, 1478265, 9619065, \ldots\]
and as \(n \to \infty\),
\[NC_2(n, 2) \sim \frac{3^5 \sqrt{3} 9^n}{24^4 \pi n^4}. \quad (3.12)\]

The proof of the D-finiteness of \(\sum_n NC_2(n, 2) z^n\) begins by using Proposition 3.4.1 to
equivalently counts walks in a different region called a Weyl chamber. This leads to reflection arguments that allow the use of a Kernel method (see Bousquet-Mélou and Mishna [7])
and finishes using creative telescoping (see Petkovšek et al. [42]). Asymptotics are derived using Maple packages (see Zeilberger [65] and [66]).

However, due to the inability to detect a recurrence from data, Marberg makes the
conjecture in [38] that \(R_j(z, r)\) is not D-finite for \(j \geq 3, r \geq 2\).

3.5 Enumeration of 3-noncrossing Set Partitions

It has already been shown that \(R_3(z)\) is D-finite by Bousquet-Mélou and Xin.

Theorem 3.5.1 (Bousquet-Mélou, Xin [9, Proposition 1]) The number of 3-noncrossing
partitions, \(NC_3(n)\), satisfies the following recursion:
\[9n(n + 3)NC_3(n) - 2(5n^2 + 32n + 42)NC_3(n + 1)\]
\[+ (n + 7)(n + 6)NC_3(n + 2) = 0 \quad (3.13)\]
for \( n \geq 0 \) with the initial conditions \( NC_3(0) = 1 = NC_3(1) \).

Thus, \( R_3(z) = \sum_{n \geq 0} NC_3(n) z^n \) is D-finite and we have equivalently

\[
\begin{align*}
&z^2(1-9z)(1-z) \frac{d^2}{dz^2} R_3(z) + 2z(5-27z + 18z^2) \frac{d}{dz} R_3(z) \\
+&10(2-3z)R_3(z) = 20 \\
\end{align*}
\]

(3.14)

with \( R_3(0) = 1 \) and \( R'_3(0) = 1 \). The first few terms of \( NC_3(n) \) are

\[
1, 1, 2, 5, 15, 52, 202, 859, 3930, 19095, 97566, \ldots
\]

with asymptotics

\[
NC_3(n) \sim \frac{3^9 \cdot 5 \cdot 9^n}{25} \sqrt{3} \pi n^7.
\]

(3.15)

**Theorem 3.5.2** (Bousquet-Mélou, Xin [9, Proposition 7]) For all positive \( n \), we have the explicit formula

\[
NC_3(n) = \sum_{j=1}^{n} \frac{4(n-1)!n!(2j)!}{(j-1)!j!(j+1)!(j+4)!n!(n-j)!(n-j+2)!} Q(j, n)
\]

where

\[
Q(j, n) = 24 + 18n + (5 - 13n)j + (11n + 20)j^2 \\
+ (10n - 2)j^3 + (4n - 11)j^4 - 6j^5.
\]

Their proof uses the idea of equivalently counting walks in a region called the Weyl chamber. They used a clever reflection principle argument. This resulted in finding solutions to functional equations using a well known technique called the Kernel method. In their paper [9], they pose the conjecture that \( R_j(n) \) is not D-finite for \( j \geq 4 \). Two reasons for this conjecture include the inability to detect a recurrence from data and that their symmetry method no longer works for \( j \geq 4 \).

An explicit generating function for \( R_3(z) \) was later found by van Hoeji at sequence A108304 in the OEIS (see [52]).

**Theorem 3.5.3** (van Hoeji [52, A108304]) An explicit form of the generating function \( R_3(z) \) for the number of 3-noncrossing set partitions is given by

\[
R_3(z) = \frac{1 + 7z - 20z^2 + 30z^3 - 18z^4 - (3z + 1)^2(z - 1)^2 \sum_{n=0}^{\infty} \frac{(x^2)^n}{(2)_n} \frac{z^n}{n!}}{6z^4} \\
\]

(3.16)

where \( x = \frac{27z(z - 1)^2}{(3z + 1)^3} \) and \( (q)_n = \begin{cases} q(q + 1) \cdots (q + n - 1) & n > 0; \\
1 & n = 0. \end{cases} \)
3.6 Permutation Enumeration

The symmetric joint distribution seen in Section 3.2 also applies to permutations but was shown by Burrill et al. [11]. They apply results shown by de Mier in [19] and [18] on graphs.

\textbf{Theorem 3.6.1 (Burrill et al. [11, Theorem 1])} The statistics $cr(P)$ and $ne(P)$ have a symmetric joint distribution over all permutations of $[n]$ for all $n, j, k$.

Yen further generalized the symmetric joint distribution in [63] to coloured permutations.

\textbf{Theorem 3.6.2 (Yen [63, Theorem 1])} Under the same conditions as in Theorem 3.2.5, $NCN^S_{j,k}(n, r) = NCN^{STP}_{k,j}(n, r)$. Thus, $NCN^P_{j,k}(n, r) = NCN^{TP}_{k,j}(n, r)$ and $NC^P_k(n, r) = NN^P_k(n, r)$.

Similar to both Corollaries 3.7.2 and 3.3.2, Yen has shown in [63] a similar result on permutations. She programmed algorithmic codes in [64] for permutations with $j$-noncrossing and $k$-nonnesting and $r$-colourings similar to what is shown in Section 3.7.2.

\textbf{Corollary 3.6.1 (Yen [63])} For all $j, k$ and $r$ colours, the generating function $R^{P}_{j,k}(z, r) := \sum_{n=0}^{\infty} NCN^P_{j,k}(n, r) z^n$ is a rational function in $z$.

Unfortunately, the situation for permutations is even more complicated than for set partitions and not many enumerative results have been discovered. It can be seen from [11] that $NC^P_2(n, 1) = C_n$, the $n$-th Catalan numbers, but it is unknown what patterns exist for higher values of $j, k, r$. D-finiteness of the generating functions $R^P_j(n, r)$ for $j \geq 3$ are also open questions.

3.7 Marberg’s Construction

This section describes how to build an automata, that given a sequence of tableaux of length $2n$ decides if it is in $V^2_{j,k}$. We then use the adjacency matrix to determine a rational generating function for the tableau class, and consequently for $P_{j,k}$. We reproduce the relevant portions of Table 1.1 here as a refresher.
### 3.7.1 Finite State Machines

A finite state machine is a way to describe a combinatorial class. Input is parsed and elements that are in the class are accepted. We define finite state machines following the definition in Sipser’s book [51] with a slight modification.

**Definition 3.7.1** A finite state machine (FSM) is a 5-tuple \((S, A, \tau, s_1, F)\) where

1. \(S\) is a finite set of states;
2. \(A\) is a finite set called the alphabet, and labels the ways to transition from state to state;
3. \(\tau : S \times A \rightarrow S \cup \{\text{null}\}\) is a transition function which maps a state and an operation from the alphabet to another state. If a pair cannot move to any state, we instead conveniently map it to \(\text{null} \notin S\);
4. \(s_1 \in S\) is the starting state;
5. \(F \subseteq S\) is the set of final states that are accepted.

<table>
<thead>
<tr>
<th>Class Level Object</th>
<th>Notation</th>
<th>Example for (P_{3,3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite state machine accepting (V_{3,3}^{2n})</td>
<td>(G_{3,3})</td>
<td><img src="image" alt="Finite state machine" /></td>
</tr>
</tbody>
</table>
| Adjacency matrix of \(G_{3,3}\)                  | \(M_{3,3}\) | \[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 1 & 3 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
\] |
| Rational generating function of \(P_{3,3}(n)\)  | \(R_{3,3}(z)\) | \[- \frac{(3z - 1)(4z^3 - 11z^2 + 7z - 1)}{(z - 1)(z^4 - 30z^3 + 31z^2 - 10z + 1)}\] |

Table 3.1: From automata accepting the class to generating function
A FSM is a finite, labelled, directed multigraph on \( S \cup \{null\} \) with edges labelled by elements of the alphabet \( \mathcal{A} \). There are exactly \(|\mathcal{A}| \cdot |S \cup \{null\}| = |\mathcal{A}| \cdot |S + 1| \) directed edges with \(|\mathcal{A}| \) edges directed outward from each node labelled by a unique alphabet element.

This FSM can parse any string \( t = t_1, t_2, \ldots, t_n \) where \( t_i \in \mathcal{A} \) for all \( i \) as follows. We begin at the starting state \( s_1 \) and travel along the edge \( t_1 \) to arrive at \( \tau(s_1, t_1) = s_2 \) for some state \( s_2 \). Next, we continue along the edges labelled by this string until the end. Therefore, we define \( \tau(s_k, t_k) = s_{k+1} \) for some state \( s_{k+1} \) for \( k = 2, \ldots, n - 1 \). The string \( t \) is accepted by the FSM if \( s_n \in \mathcal{F} \), and rejected otherwise.

Every possible string in the alphabet is readable but if some intermediate operation is not applicable, we can stop prematurely by mapping to \( null \). We treat \( null \) as an extra state with \( \tau(null, a) = null \) for all \( a \in \mathcal{A} \). So it makes sense that once we enter \( null \), we should remain there.

**Finite state machine \( G_{j,k} \)**

We now describe the automaton \( G_{j,k} \) which accepts the \( j \)-noncrossing and \( k \)-nonnesting set partitions for any \( j, k \). The set of states are the elements in a vacillating tableau \( V_{j,k}^{2n} \). We define \( S = \{ \text{Young diagram with less than } j \text{ columns and less than } k \text{ rows} \} \).

Following Marberg’s convention, we only consider the odd numbered elements, so that a \( V_{j,k}^{2n} \) is a walk of length \( n \) in \( G_{j,k} \). It pauses on elements numbered 1, 3, 5, \ldots, 2n - 1, and begins and ends at the empty Young diagram. To transition from an odd numbered state to an even numbered state, we add a box or do nothing. There are \((j - 1)(k - 1)\) places to add a box yielding \((j - 1)(k - 1) + 1\) total steps. To then go to an odd numbered state from an even numbered state, we remove a box or do nothing. There are again \((j - 1)(k - 1) + 1\) total steps. Many of these combinations are impossible and we omit illustrating them when it is clear. We have \( \mathcal{A} = \{ \text{add a box or do nothing then remove a box or do nothing} \} \).

The transition function \( \tau \) only makes sense when the output is a a Young diagram. For \( s \in S \) and \( s \in \mathcal{A} \), define \( \tau(s, a) \) to be the Young diagram obtained by applying \( a \) on \( s \) and \( null \) when it is not a Young diagram. The starting state \( s_1 \) is \( \emptyset \) and the set of final states is \( \{ \emptyset \} \).

The number of strings of length \( n - 1 \) from the alphabet that leads to the single final state \( \emptyset \) corresponds to the number of \( j \)-noncrossing and \( k \)-nonnesting partitions of \([n]\).
Alternatively, $G_{j,k}$ can be viewed as taking a sequence of Young diagrams (the states and not alphabet) as input and only accepting sequences in $V_{j,k}^{2n}$.

Considering the bijection of Theorem 3.2.4, if $\tau(s, a) = t$ for any states $s \neq t$ and $a$ in the alphabet, then there exists $b \in \mathcal{A}$ such that $\tau(t, b) = s$. Also, for $s \neq t$, if $\tau(s, a) = t$ then $\tau(s, b) \neq t$ for any $b \in \mathcal{A}$. Essentially, all the edges turn out to be bidirectional and we have multiple loops but no multi-edge.

Since all Young diagrams are contained in a $(k - 1) \times (j - 1)$ box, the number of subsets of blocks is $2^{(j-1)(k-1)}$. This is a simple expression for a crude upper bound on the number of states showing there are only a finite number of states (see Proposition 4.2.1 for an exact value).

The following example illustrates the construction.

**Example 3.7.1** We construct $G_{2,3,1}$. For a partition of $[n]$, we have $n$ points in a row. Consider drawing the arc diagram by moving from left to right, each step from node to node, then there is a total of $n - 1$ steps. Each step can take one of the five forms as shown in Table 3.2. Note that for $x$ and $c$, there may be more than one type depending on which arc we are closing.

<table>
<thead>
<tr>
<th>One step</th>
<th>Types of arcs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e$ (no arcs)</td>
</tr>
<tr>
<td></td>
<td>$a$ (one arc)</td>
</tr>
<tr>
<td></td>
<td>$o$ (an opener)</td>
</tr>
<tr>
<td></td>
<td>$c$ (a closer)</td>
</tr>
<tr>
<td></td>
<td>$x$ (arcs that cross)</td>
</tr>
</tbody>
</table>

Table 3.2: Arc steps

There are no crossings in this arc diagram so $x$ does not appear as an arc step. Also, note that by the Erdős-Szekeres Theorem (Theorem 3.2.3) a sequence with three distinct numbers have either a subsequence of two that is increasing or a subsequence of three that is decreasing. If these three numbers correspond to the closer elements of an arc diagram, then we cannot have 2-noncrossing and 3-nonnesting. Thus, there are at most two open arcs at one time. Using this information we can define the FSM. The set of states
is
\[ S = \{ \emptyset, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array} \}, \]
where \( \emptyset \) is the starting state with no openers, \( \begin{array}{c}
\end{array} \) has one opener, and \( \begin{array}{c}
\end{array} \) has two openers. The alphabet is the set of transition steps between nodes
\[ \mathcal{A} = \{ e, a, o, c \}. \]

The transition function \( \tau \) can be represented by the following table where it maps a row element from \( S \) and column element from \( \mathcal{A} \).

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>o</th>
<th>c</th>
</tr>
</thead>
</table>
| \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \begin{array}{c}
\end{array} \) | \( \text{null} \) |
| \( \begin{array}{c}
\end{array} \) | \( \begin{array}{c}
\end{array} \) | \( \begin{array}{c}
\end{array} \) | \( \begin{array}{c}
\end{array} \) | \( \emptyset \) |
| \( \begin{array}{c}
\end{array} \) | \( \text{null} \) | \( \text{null} \) | \( \begin{array}{c}
\end{array} \) |

Table 3.3: Transition function

Finally, the starting state and final states is the same set
\[ \mathcal{F} = \{ \emptyset \}. \]

This leads to the following finite state machine in Figure 3.15. Since \( \text{null} \) is just a conven-

![Figure 3.15: Complete finite state machine \( \mathcal{G}_{2,3,1} \)](image)

...
arcs and another for an open arc that closes in one step. State $\emptyset$ has one loop since the only way to transition to itself is to have no new open arcs. We can transition from $\emptyset$ to $\square$ by opening a new arc or the other way by closing an existing arc, and we can transition from $\square$ to $\blacksquare$ the same way by opening up to two arcs and the other way by closing one of the two existing arcs. Since we always have only one bidirectional path to transition between separate states, we have no multiple edges and can use one bidirectional edge instead of directed edges. Note that Figure 3.16 in the example above captures all the information of the finite state machine but can be represented as a simpler finite undirected graph.

### 3.7.2 Marberg’s Algorithm

The strategy presented in Section 3.2.3 is altered to a programmable form. The Maple code for the program is reproduced in the appendix and it computes $R_{j,k}(z,r)$ as a rational function in $z$ using a standard technique called the Transfer-Matrix Method (see Section 3.7.4) along with a corresponding adjacency matrix (see Definition 3.7.2). The program applies to any $j$-noncrossing, $k$-nonnesting, $r$-coloured set partition with $j, k \geq 2$.

### 3.7.3 Finite state machine $G_{j,k,r}$

Before applying Marberg’s algorithm, we first translate Young diagrams into a simpler form of sequences of integers. We rewrite each Young diagram as a horizontal vector. Since each Young diagram is $j$-noncrossing and $k$-nonnesting, by Theorem 3.2.4 it can be written as a nonincreasing, nonnegative integer vector of length $k - 1$ with each component less than $j$; each Young diagram is an integer partition (with trailing zeros).

**Example 3.7.2** Consider the Young diagram of Figure 3.7 corresponding to a set partition that is 6-noncrossing and 7-nonnesting. Then in matrix form, this Young diagram is the row
To deal with the $r$-colouring case, we have $r$ separate Young diagrams or an $r$-partite partition. Each $r$-partite partition is written as $r$ horizontal vectors, one row for each of the $r$ parts. Thus, each $r$-partite partition is translated into an $r \times (k - 1)$ matrix. The automata for $r$-coloured partitions with these $r \times (k - 1)$ matrix as states are called $G_{j,k,r}$ and $G_{j,k,1} = G_{j,k}$.

Interpreting $r$-partite partitions as matrices, we have an alternate characterization of $G_{j,k,r}$ (and of $G_{j,k}$) by Marberg [38]. Let $G_{j,k,r}$ be the multi-graph with vertices as $r \times (k - 1)$ integer matrices $A = \{a_{i,l}\}$ with nonnegative entries nonincreasing along each row and strictly less than $j$.

For states $A$ and $A'$, denote $e(A, A')$ by the number of bidirectional edges between these two states. Let $E_{i,l}$ be the $r \times (k - 1)$ matrix with 1 in position $(i, l)$ and 0 everywhere else. Then states $A \neq A'$ satisfy $e(A, A') = 1$ if $A - A' = \pm E_{i,l}$ or $A - A' = E_{i,l} - E_{i',l'}$ for some pairs $(i, l) \neq (i', l')$. Otherwise, $e(A, A') = 0$.

The number of loops on the state $A$ is $e(A, A) = 1 + d_1 + \cdots + d_r$ where $d_i$ is the number of distinct entries in the $i$-th row of $A$ less than $j - 1$. These $d_i$ count the number of corner locations where one may add and remove the same block to the Young diagram form of $A$ without leaving the $(k - 1) \times (j - 1)$ block. There are no other edges or loops.

We then define $M_{j,k,r}$ as the adjacency matrix (see Definition 3.7.2) corresponding to $G_{j,k,r}$. Also, let $M_{j,k}$ correspond to $G_{j,k}$.

**Theorem 3.7.1** (Marberg [38, Theorem 3.5]) For $j, k \geq 2$ and $n, r$ positive integers, the value $NCN_{j,k}(n, r)$ coincides with the number of $(n - 1)$-step walks from the empty set to itself on the multigraph $G_{j,k,r}$.

**Proof** The first bijection of Theorem 3.2.6 shows that $NCN_{j,k}(n, r)$ counts the number of vacillating $r$-partite tableaux of length $2n$. From the discussion at the beginning of this section, it is easier to view the states as matrices instead of as $r$-partite partitions. Thus, it suffices to show that each vacillating $r$-partite tableau of length $2n$ corresponds to a walk in $G_{j,k,r}$ with $n - 1$ steps if we only stop at odd numbered elements.

After we characterize all the $r$-partite partitions as matrices by the description above, we want to specify how to transition from one $r$-partite partition to another. Each partition $P$
of $[n]$ is mapped to a vacillating $r$-partite tableau $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ composed of a sequence of $2n + 1$ $r$-partite partitions beginning and ending with a zero matrix. Let $m^i$ be the matrix form of $\lambda^i$ for all $i$. Since $\lambda^0 = \lambda^1$ and $\lambda^{2n-1} = \lambda^{2n}$, to specify the transitions from $\lambda^0$ to $\lambda^{2n}$, it suffices to understand how to transition from consecutive odd numbered states. Then there are $n - 1$ steps from $\lambda^1$ to $\lambda^{2n-1}$ only stopping at $\lambda^{2i-1}$ for $i = 1, \ldots, n - 1$. To move from $\lambda^{2i-1}$ to $\lambda^{2i+1}$, there are four possibilities: add one box, remove one box, add then remove a different box, and staying the same.

Let $E_{xy}$ denote the $r \times (k - 1)$ matrix with 0 everywhere except with a 1 in the $xy$-th entry. Then to add or remove a box corresponds to $m^{2i+1} = m^{2i-1} \pm E_{xy}$ for some $x, y$. To add and remove a different box corresponds to $m^{2i+1} = m^{2i-1} + E_{xy} - E_{x'y'}$ for some $(x, y) \neq (x', y')$. All these cases must preserve tableau properties and yield one edge between these states. In the last case, either we do nothing twice or we add and remove the same box. There are $d := 1 + d_1 + d_2 + \cdots + d_r$ placement of this fleeting box where $d_x$ is the number of distinct entries less than $j - 1$ in the $x$-th row of $m^{2i-1}$. This follows since each distinct entry less than $j - 1$ contribute to a corner spot adding a box to maintain a Young diagram. See Figure 3.17 for an example. Thus, there are $d$ loops on this state when $m^{2i-1} = m^{2i+1}$. These are the only possible transitions and all other states have no edges between them. This description is precise the characterization of the multigraph $G_{j,k,r}$ at the beginning of this section. The number of partitions of $[n]$ (with crossing, nesting, and colouring conditions) is then the number of paths among these matrices from the zero matrix back to itself of length $n - 1$.

\[\begin{array}{c|c|c|c|c|c}
\hline
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\hline
\end{array}\]

Figure 3.17: $G_{6,7}$ loops: (4, 2, 2, 1, 0, 0) has 4 locations

**Example 3.7.3** We rewrite parts of $\phi^8(P)$ in Example 3.2.8 in terms of matrices. Note that $P$ is maximum 2-crossing and 2-nesting so that it is 3-noncrossing and 3-nonnesting. Thus, $j = k = 3$ and $r = 2$. Each 2-partite partition is represented as a $2 \times 2$ integer matrix with nonincreasing, nonnegative entries less than 3. Thus, the first column of $\phi^8(P)$
is the zero matrix \( m^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). As another illustration, The eighth column is \( m^8 = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \) which subtracts \( E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) to reach the 9th column and then adds \( E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) to arrive at the 10th column \( m^{10} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \).

### 3.7.4 Transfer-Matrix Method

The Transfer-Matrix method introduced in Section 4.7 of [55] is the main tool for deriving the generating functions from the matrices. First, we introduce adjacency matrices and recall a few preliminary results. Adjacency matrices allow us to convert the information contained in a FSM into a form where we can convert into a generating function.

**Definition 3.7.2** Given a finite graph \( G \) with vertices \( \{s_1, \ldots, s_n\} \), an associated adjacency matrix \( A_G \) is a \( n \times n \) matrix with the \((i,j)\)-th entry being the number of edges between \( s_i \) and \( s_j \).

Note that all matrices \( M_{j,k,r} \) are symmetric for undirected graphs and we can order the vertices in any manner as long as we keep track of which row (column) corresponds to which vertex. The diagonal entries correspond to the number of loops.

Another way to look at adjacency matrix \( A \) is that \( A_{i,j} \) is the number of edges (paths of length 1) from \( s_i \) to \( s_j \). A classical result in graph theory is that the \((i,j)\)-entry of \( A^n \) is the number of paths of length \( n \) from \( i \) to \( j \). Example 3.7.4 is an adjacency matrix for the FSM in the earlier Example 3.7.1.

**Example 3.7.4** Here is the matrix \( M_{2,3,1} \) when row \( i \) corresponds to \( s_{4-i} \) for \( i = 1, 2, 3 \):

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]

Using a known technique called the Transfer-Matrix Method we can find the ordinary generating function \( \sum_{n=0}^{\infty} c(n)z^n \) from the adjacency matrix where \( c(n) \) is the number of ways to transition from a fixed state (the starting state) to itself in \( n \) steps. This proof uses Cramer’s Rule from linear algebra which we first refresh.
Proposition 3.7.1 (Cramer’s Rule) The solution to the matrix equation $Ax = b$ where $A$ is an $n \times n$ invertible matrix is

$$x_i = \frac{\det A_i}{\det A}$$

where $A_i$ is the matrix $A$ with column $i$ replaced by $b$ for $i = 1, \ldots, n$.

From Cramer’s Rule, we get this immediate corollary.

Corollary 3.7.1 Let $A$ be an invertible matrix and $M$ the same size matrix with $M_{i,j}$ the $(i, j)$-th minor of $A$. Then the inverse of $A$ is

$$A^{-1} = \frac{M}{\det A}.$$

Let $\det(M : i, i)$ denote the $(i, i)$-th minor of matrix $M$ and $I$ the identity matrix when the size is clear.

Theorem 3.7.2 (Transfer-Matrix [55, Theorem 4.7.2]) Given an adjacency matrix $A$ with starting state corresponding to row $i$,

$$1 + z \frac{\det(I - zA : i, i)}{\det(I - zA)}$$

(3.17)

is a closed form for the ordinary generating function in $z$ with the $n$-th coefficient representing the number of paths of length $n - 1$ from state $s_i$ to itself.

Proof It suffices to show that the ordinary generating function for the number of paths of length $n$ from $s_i$ to itself is $\frac{\det(I - zA : i, i)}{\det(I - zA)}$ and then Equation (3.17) shifts this by one place. The generating function for paths of length $n$ from $s_i$ to itself can be expressed as the $(i, i)$-entry of

$$\sum_{z \geq 0} A^n z^n = (I - zA)^{-1}.$$

We then apply Corollary 3.7.1 to find $(I - zA)^{-1}$.

Since a path of length $n - 1$ corresponds to a partition of $[n]$, Equation (3.17) is the generating function for the number of partitions of $[n]$ with some fixed crossing and nesting conditions. The following example illustrates the computation of the generating function for the adjacency matrix of Example 3.7.4.
Example 3.7.5 We have

\[
I-zA = \begin{bmatrix}
1-2z & -1 & 0 \\
-1 & 1-2z & -1 \\
0 & -1 & 1-z
\end{bmatrix}
\]

so \(\det(I-zA) = -z^3 + 6z^2 - 5z + 1\) and \(\det(I-zA : 1,1) = z^2 - 3z + 1\). Thus, the generating function is

\[
1 + z \frac{z^2 - 3z + 1}{-z^3 + 6z^2 - 5z + 1} = -\frac{(3z-1)(z-1)}{z^3 - 6z^2 + 5z - 1}.
\]

The next important corollary is a direct consequence of The Transfer-Matrix Theorem.

Corollary 3.7.2 (Marberg [38, Corollary 4.1]) For all \(j, k\) and \(r\) colours, the generating function \(R_{j,k}(z,r) = \sum_{n=0}^{\infty} NCN_{j,k}(n,r)z^n\) is a rational function in \(z\).

This provides a useful tool for studying the more complicated limits of these functions, \(R_{j}(z)\). For \(j \geq 4\), it is speculated that \(R_{j}(z)\) is not even D-finite, although they are the limits of rational functions. We next present an example of Marberg’s algorithm from [38] that we outline in Section 3.7.2 which uses the transfer-matrix method.

Example 3.7.6 We apply Marberg’s algorithm for \(j = 3\) and \(k = 4\). The states are \(1 \times 3\) integer matrices \(A = (A_{i,l})\) with \(3 > A_{1,1} \geq A_{1,2} \geq A_{1,3} \geq 0\). The states are listed in lexicographic order as integer partitions:

\[
\emptyset, [\begin{array}{c}
1
\end{array}], [\begin{array}{cc}
1 & 0
\end{array}], [\begin{array}{ccc}
1 & 0 & 0
\end{array}], [\begin{array}{c}
2
\end{array}], [\begin{array}{cc}
2 & 0
\end{array}], [\begin{array}{ccc}
2 & 0 & 0
\end{array}], [\begin{array}{ccc}
2 & 0 & 0
\end{array}], [\begin{array}{cc}
2 & 0 & 1
\end{array}], [\begin{array}{ccc}
2 & 0 & 1
\end{array}], [\begin{array}{cc}
2 & 1 & 1
\end{array}], [\begin{array}{ccc}
2 & 1 & 1
\end{array}], [\begin{array}{ccc}
2 & 1 & 1
\end{array}], [\begin{array}{ccc}
2 & 1 & 1
\end{array}, [\begin{array}{ccc}
2 & 1 & 1
\end{array}].
\]

The corresponding finite state machine is depicted in Figure 3.18.

As an example, the state \(\emptyset\) (or \(\begin{array}{c}
0
\end{array}\) in matrix form) has two loops since there is only one row and the number of distinct entries in that row which are less than \(j-1 = 2\) is 1. Thus, \(\emptyset\) has \(e(\emptyset, \emptyset) = 1 + 1 = 2\) loops. The only state adjacent to \(\emptyset\) is \(\begin{array}{c}
1
\end{array}\), thus \(e(\emptyset, \begin{array}{c}
1
\end{array}) = 1\). We continue by the same reasoning to draw in all edges and loops following Marberg’s algorithm.

The adjacency matrix is shown in Figure 3.19 with states ordered in reverse lexicographic order (from right to left of what is displayed above).

Applying the transfer-matrix method,

\[
\sum_{n \geq 0} NCN_{3,4}(n)z^n = 1 + z \frac{\det(I_{10} - zM : 10, 10)}{\det(I_{10} - zM)}
\]
Figure 3.18: Finite state machine $G_{3,4,1}

\[
M = M_{3,4} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]

Figure 3.19: Adjacency matrix $M_{3,4}$

which equals

\[
\frac{-(z - 1)(2z - 1)(36z^7 - 258z^6 + 659z^5 - 750z^4 + 426z^3 - 125z^2 + 18z - 1)}{(z^{10} - 205z^9 + 1452z^8 - 4117z^7 + 6177z^6 - 5460z^5 + 2970z^4 - 999z^3 + 201z^2 - 22z + 1)}
\]

in factored form.
Chapter 4

Refined Enumeration of $k$-noncrossing Set Partitions

4.1 Introduction

Using the machinery of the previous chapter, specifically the monochromatic case of Marberg’s algorithm (see Section 3.7.2), we enumerate and analyze the key components of the construction leading up to the generating functions with fixed noncrossing and nonnesting numbers. We begin by describing patterns in the enumerated automata structures $G_{j,k,r}$ defined in Section 3.7.1. This leads to patterns in the adjacency matrices $M_{j,k,r}$. From here we derive some $j$-noncrossing and $k$-nonnesting set partition generating functions $R_{j,k}(z)$ for $j = 2, 3, 4$ and study their singularities. Then we analyze the limits of these generating functions $R_j(z)$. We finish by analyzing series data of $j$-noncrossing, $k$-nonnesting set partitions on $[n]$, namely the $NCN_{j,k}(n)$ values.

4.2 Automata Analysis

Marberg’s algorithm converts sequences of Young diagrams to sequences accepted by automata which are then converted to adjacency matrices. Since the states of the automata are quite easily specified, it is possible to go directly to matrices bypassing automata completely. Thus, the propositions we state are presented in Section 4.3 in the language of matrices. However, it is still relevant to study the automata directly to clarify
the structure and patterns we observe. In particular, the automata can give us a regular language for random generation.

Before constructing any finite state machines, we show that they have only a finite number of states. This is important because a finite number of states implies that the generating function is rational. Recall from Section 3.7.1 that $G_{j,k,r}$ is the finite state machine for set partitions that are $j$-noncrossing, $k$-nonnesting, and $r$-coloured. In our notation, we also define $G_{j,k} = G_{j,k,1}$.

**Proposition 4.2.1** The automaton $G_{j,k,r}$ has $(\frac{j+k-2}{k-1})^r$ states for all $j, k, r$.

**Proof** We first compute the number of states of $G_{j,k,1}$. Recall from Marberg’s algorithm that each state corresponds to a Young diagram. Also, recall from Theorem 3.2.4 that there is a maximum of $k-1$ rows and $j-1$ columns in each state, so that each Young diagram is contained in a $(k-1) \times (j-1)$ block. Every such Young diagram contained in this block is a state of $G_{j,k,1}$.

View this $(k-1) \times (j-1)$ block as a lattice grid in the plane with lower left hand corner at $(0,0)$ and upper right hand corner at $(j-1, k-1)$. Each Young diagram corresponds precisely to the blocks on the left of a lattice path from $(0,0)$ to $(j-1, k-1)$ with steps going up and right. There are $\binom{(j-1)+(k-1)}{k-1}$ such lattice paths giving the number of states of $G_{j,k,1}$.

Note that the number of states for $G_{j,k,r}$ is the $r$-th power of the number of states of $G_{j,k,1}$, since there are $r$ independent rows. Thus, the number of states of $G_{j,k,r}$ is $(\frac{j+k-2}{k-1})^r$.

Figure 4.1 illustrates the proof of Proposition 4.2.1 with the state $(5, 3, 2, 1)$ of $G_{6,5,1}$.

![Figure 4.1: $G_{6,5,1}$: state (5, 3, 2, 1) and lattice path](image-url)
4.2.1 Automata for $j = 2$

Next, we construct several finite state machines for analysis. First, we describe explicitly how $G_{2,k}$ is obtained from the Young diagrams. The general automata $G_{2,k}$ structure is easy to describe and $G_{2,5}$ is shown in Figure 4.2. Recall the states are labelled with $1 \times (k-1)$ matrices. They are of the matrix form $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$ with $i$ 1’s for $i = 0, 1, \ldots, k – 1$. The starting and final state is the zero vector. The specifics of the construction are found in the proof of Proposition 4.3.1. Note that for $G_{k,2}$, the states are $(k-1) \times 1$ matrices of the form $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T$ which are the transpose of the states for $G_{2,k}$; the relationships to transition between states are also preserved. This yields the same finite state machine, $G_{k,2} = G_{2,k}$.

As shown in Proposition 4.3.5 of Section 4.3, the submatrix structure of the adjacency matrices implies the automata are nested. Since $G_{2,k}$ contains one more state than $G_{2,k-1}$, we see that $G_{2,k-1}$ is contained in Figure 4.2 with one fewer state. We colour red the states which form this sub-automaton for clarity. Since the states are listed in reverse order in the adjacency matrices, $s_1$ is not contained in $G_{2,k-1,1}$ and $s_2$ becomes the starting and final state in the sub-automaton of $G_{2,k}$ that is equivalent to $G_{2,k-1}$.

![Figure 4.2: Finite state machine $G_{2,5}$ with sub-automaton $G_{2,4}$](image)

4.2.2 Automata for $j = 3$

We draw the automata $G_{3,k}$ for $k = 2, 3, 4, 5$ in Figures 4.3, 4.4, 4.5, and 4.6 to examine similar nesting structures as seen in the 2-noncrossing scenario. The red states for $G_{3,k+1}$ form a sub-automaton with the same structure as $G_{3,k}$ for all $k$ shown; the ordering of the states is also preserved.

The finite state machines for 4-noncrossings are too complicated to interpret easily from a drawing. Also, they are no longer planar graphs.
Figure 4.3: Finite state machine $G_{3,2}$

Figure 4.4: Finite state machine $G_{3,3}$

Figure 4.5: Finite state machine $G_{3,4}$

Figure 4.6: Finite state machine $G_{3,5}$
4.3 Structure of Transfer Matrices

Recall from Section 3.7.2 that $M_{j,k,r}$ denotes the corresponding adjacency matrices for $G_{j,k,r}$. We also abbreviate $M_{j,k,1}$ by $M_{j,k}$. These matrices exhibit a lot of structure, and the goal of this section is to describe the patterns we discovered for $j = 2, 3, 4$. Keep in mind by convention that the first row corresponds to the largest possible $(k−1) \times (j−1)$ Young diagram shape while the last row/column of the matrix corresponds to the smallest possible empty Young diagram which is also the initial and final state of the vacillating tableau. Essentially, the states as matrices are in reverse lexicographical order. This is the ordering of the states in the adjacency matrix $M_{j,k,r}$ that we choose to study although a different ordering also works. All our work will be in terms of this ordering only.

4.3.1 Adjacency Matrices $M_{2,k}$

We describe the $k \times k$ matrix $M_{2,k}$ in Proposition 4.3.1. The form is shown in Figure 4.7 and is derived from the finite state machine of Figure 4.2. Note that for each successive $k$, $M_{2,k}$ has one extra row and column and contains the previous matrix in the upper left corner. This allows an easy computation of its characteristic polynomial as we see in Section 4.5.1.

$$
M_{2,k} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & \ddots & 0 \\
0 & 0 & \ddots & \ddots & 1 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}.
$$

Figure 4.7: Adjacency matrix $M_{2,k}$

Proposition 4.3.1 The matrix $M_{2,k,1} = [m_{ij}]_{1 \leq i,j \leq k}$ is given by

$$
m_{ij} = \begin{cases}
1 & \text{if } |i − j| = 1 \text{ or } i = 1 = j; \\
2 & \text{if } i = j \neq 1; \\
0 & \text{otherwise}.
\end{cases}
$$

Proof Proposition 4.2.1 implies that $M_{2,k}$ is of size $k \times k$. Recall from Marberg’s algorithm of Section 3.7.2 that a state $s = [s_1, s_2, \ldots, s_{k−1}]$ is a $1 \times (k−1)$ matrix such that
2 > s_1 \geq s_2 \geq \ldots \geq s_{k-1} \geq 0. By our ordering of the states, we see that the state corresponding to the i-th row is the 1 \times (k-1) matrix state \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{bmatrix} with k - i 1's followed by i - 1 0's. From Marberg's algorithm, a state has 1 + d_1 loops to itself where d_1 is the number of distinct entries less than 1. There is no entry in [1,1,...,1] less than 1 so this state has one edge to itself. For all other states, there is at least one 0, so they all have 2 loops. This describes \( m_{ii} \) completely for all i. It is only possible to transition between different states if their Young diagrams differ by one block. Thus, [1,1,...,1] is only adjacent to [1,1,...,1,0] so \( m_{12} = 1 = m_{21} \) and [0,0,...,0] is only adjacent to [1,0,...,0] so \( m_{k,k-1} = 1 = m_{k-1,k} \). If a state s has 0 < i < k 1's followed by k - 1 0's, then s is adjacent to the state with i - 1 and i + 1 1's. This gives \( |i-j| = 1 \) for for all i, j. Lastly, the remaining cases all differ by more than two entries, so that there is 0 way to transition between them in one step. This describes \( M_{2,k} \) completely. \[ \square \]

Recall from Section 4.2.1 that the finite state machine \( G_{k,2} \) has the same structure as \( G_{2,k} \). Since adjacency matrices are equivalent to finite state machines but in a different representation, this implies the following Corollary 4.3.1.

**Corollary 4.3.1** We have equivalence of adjacency matrices \( M_{k,2} = M_{2,k} \) for all k.

A few easy remarks follow:

1. Matrix \( M_{2,k} \) is a tridiagonal matrix. A tridiagonal matrix is one where the only nonzero entries are on the main diagonal or the diagonal directly above and below the main diagonal.

2. Matrix \( M_{2,k+1} \) contains \( M_{2,k} \) as a submatrix formed by the first k rows and columns.

Another easy proposition follows.

**Proposition 4.3.2** The determinant \( \det(M_{2,k}) = 1 \) for all k.

**Proof** We apply Laplace expansion with induction. When \( k = 2 \), the determinant of \( \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \) is clearly 1. Suppose \( \det(M_{2,i}) = 1 \) for \( i < k \). Then computing the determinant of \( M_{2,k} \) by Laplace expansion on the last row yields \( \det(M_{2,k}) = 2 \det(M_{2,k-1}) - 1 \cdot D = 2 - D \) where
$D$ is the determinant of the $(k - 1) \times (k - 1)$ matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & \ddots & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

from removing the last row and second to last column. Applying Laplace expansion on the last column yields $D = 1 \cdot \det(M_{2,k-2}) = 1$. Thus, $\det(M_{2,k}) = 2 - 1 = 1$ as desired. □

We state the following lemma from [23] used in the next proof. Define the diameter of a graph as the maximum distance between two pairs of nodes.

**Proposition 4.3.3** If a graph has diameter $d$, then its corresponding adjacency matrix has at least $d + 1$ distinct eigenvalues.

**Proposition 4.3.4** All eigenvalues of $M_{2,k}$ are distinct.

**Proof** Viewing $G_{2,k}$ as a graph, state $s_i$ is adjacent to $s_j$ if and only if $i$ and $j$ are consecutive integers or the same integer. Thus, the farthest two nodes are $s_1$ and $s_k$ with a distance of $k - 1$. The diameter of $G_{2,k}$ is $k - 1$, so $M_{2,k}$ has at least $k$ distinct eigenvalues. Since $M_{2,k}$ has $k$ rows, this matrix has exactly $k$ eigenvalues, so they are all distinct. □

Before moving on to the next section, we present a recursion in Proposition 4.3.5 for computing larger adjacency matrices from smaller ones with $M_{2,k}$ and $M_{k,2}$ as the basis.

First let $I_{x,y} = [(I_{x,y})_{i,j}]_{1 \leq i,j \leq x+y}$ denote the $(x+y) \times (x+y)$ matrix defined by

$$(I_{x,y})_{i,j} = \begin{cases} 1 & \text{if } 1 \leq i = j \leq x; \\
0 & \text{otherwise} \end{cases}$$

with a block matrix example shown in Figure 4.8. The notation $I_x$ is the $x \times x$ identity matrix and $0_y$ is the $y \times y$ zero matrix.

$$
\begin{bmatrix}
I_x & 0 \\
0 & 0_y
\end{bmatrix}
$$

Figure 4.8: An $I_{x,y}$ matrix
Proposition 4.3.5 We can write in block matrix form

\[ M_{j,k} = \begin{bmatrix} M_{j,k-1} & N_{j,k}^T \\ N_{j,k} & M_{j-1,k} + I^{(j+k-4)_{j-k-2}} - \binom{j+k-4}{j-k-2} \\ \end{bmatrix} \]

where \( N_{j,k} \) is some \((0,1)\)-matrix of the appropriate size and \( N_{j,k}^T \) is its transpose for all \( j, k \geq 3 \).

Proof Essentially the proof takes a careful look at precisely how the states in Marberg’s algorithm are defined. Recall that the states of \( M_{j,k} \) are all partitions between \([0,0,\ldots,0]\) and \([j-1,j-1,\ldots,j-1]\) of length at most \( k-1 \). The states of \( M_{j,k-1} \) are similarly defined but of length \( k-2 \). For the first \( \binom{j+k-3}{k-2} \) (this is the number of rows of \( M_{j,k-1} \)) states of \( M_{j,k} \), the first entry of each partition is \( j-1 \). Thus, the structure of the first \( \binom{j+k-3}{k-2} \) states of \( M_{j,k} \) is the same structure as the states of \( M_{j,k-1} \). This determines the upper left corner of \( M_{j,k} \) in our block matrix decomposition.

The remaining states form all partitions from \([0,0,\ldots,0]\) to \([j-2,j-2,\ldots,j-2]\) of length \( k-1 \). This structure is precisely the same structure as \( M_{j-1,k} \) with \( \binom{j+k-3}{k-1} \) states. These states have the first entry of \( M_{j,k} \) as \( j-2 \) or less than \( j-2 \). The states with first entry as \( j-2 \) has the same structure as the states of the matrix \( M_{j-1,k-1} \) since the last \( k-2 \) entries are not fixed. This matrix \( M_{j-1,k-1} \) has \( \binom{j+k-4}{k-2} \) states. The bottom right corner structure is \( M_{j-1,k} \) but Marberg’s algorithm states that the number of loops of a state is one more than the number of distinct entries less than \( j-1 \). Thus, the bottom right corner is \( M_{j-1,k} \) except the states with first entry \( j-2 \) have an extra loop. There are \( \binom{j+k-3}{k-1} - \binom{j+k-4}{k-2} \) remaining states with first entry less than \( j-2 \). These values form the remaining \( I_{x,y} \) portion of the lower right corner.

Since \( M_{j,k} \) is symmetric, the upper right corner is the transpose of the lower left corner \( N_{j,k} \). This region only captures information about whether there exists a transition or not from different states. Thus, all entries in \( N_{j,k} \) are 0 or 1. This describes all 4 blocks. \( \Box \)

The structure of \( N_{j,k} \) is not clear for \( j > 5 \) but the nesting structure of the main diagonal blocks provide a systematic way to build matrices with higher \( j, k \) values. It turns out all matrices we consider are block tridiagonal matrices. We introduce the following notation in Figure 4.9 to represent individual blocks in a matrix.

For \( M_{2,k} \), all block matrices are \( 1 \times 1 \). In fact, \( E_{2,i} = 1 \) for all \( i \), \( D_{2,1} = 1 \), and \( D_{2,i} = 2 \) for \( i > 1 \).
The matrix $M_{j,k}$ is a square matrix with $(3 + k - 2) = (k + 1 - 1) = k(k+1) / 2$ rows. We draw $M_{3,5}$ in Figure 4.10 and then describe the general pattern. Additional horizontal and vertical lines are for clarification to illustrate the partitioning block matrix form. All $E_{3,i}$ are $(i + 1) \times (i + 1)$ matrices as shown in Figure 4.11. We have $D_{3,1} = M_{2,3}$. For $D_{3,i}$ with $i \neq 1$, consider the $(i + 1) \times (i + 1)$ matrix in Figure 4.12.
4.3.3 Adjacency Matrices $M_{4,k}$

The matrices, $M_{4,k}$, have a much more complicated structure and a template example is built from smaller components. Instead, we present portions of several matrices. $M_{4,k}$ is square with $\frac{k(k+1)(k+2)}{6}$ rows. Figure 4.13 presents $M_{4,4}$ with three blocks composing the main diagonal. We will give more details about the off-diagonal blocks as they are not understood from Proposition 4.3.5.

\[
D_{3,3} = \begin{bmatrix}
2 & 1 \\
1 & 3 & 1 \\
& 1 & 3 & \ddots \\
& & \ddots & 1 \\
& & & 1 & 3 & 1 \\
& & & & 1 & 2
\end{bmatrix}
\]

Figure 4.12: Block matrix $D_{3,3}$

The next additional main diagonal block $D_{4,4}$ for $M_{4,5}$ is in Figure 4.14 with extra lines for clarification.

Figure 4.13: Adjacency matrix $M_{4,4}$ with submatrices
It remains to consider the off-diagonal blocks $E_{4,i}$ of 1's and 0's. Figure 4.15 reveals $E_{4,5}$ as the bottommost one for the lower main diagonal. Lines are drawn for visualization. They are made of increasing pairs of squares with the $E_{3,i}$ pattern in one paired with an identity matrix of the same size.

4.3.4 Adjacency Matrices $M_{5,k}$

Since we have a recursion for the main diagonal blocks $D_{j,i}$, we analyze $E_{5,i}$ for a pattern of the off-diagonal blocks. An off-diagonal of sufficient information is displayed in three pictures: Figures 4.16, 4.17, and 4.18. These three figures form a diagonal with the middle figure under the bottom right block of the top figure; the bottom figure is under the bottom right block of the middle figure. Although a general pattern is still not explicit for $E_{j,i}$, we see another recursive pattern since $E_{j',i}$ is contained inside $E_{j,i}$ for $j' < j$.

4.3.5 Conjectures on Adjacency Matrices

We make several observations about these matrices and conjecture that they hold for all $M_{j,k,1}$ matrices for all $j, k$. Recall from the beginning of Section 4.3 that we have defined $M_{j,k,r}$ with states in reverse lexicographical order, and we only study matrices of this ordering. By looking at the data for $j = 2, 3, 4$ and $k \leq 10$, we are led to the following conjectures.
Figure 4.15: $E_{4,5}$: pairs of increasing jordan forms paired with identity matrices of the same size

Figure 4.16: Top part of an $E_{5,i}$
Figure 4.17: Middle part of an $E_{5,i}$

Figure 4.18: Bottom part of an $E_{5,i}$
Conjecture 4.3.1  All $M_{j,k}$ are tridiagonal block matrices for all $j, k$.

This conjecture may be proved through tracing back to the original construction of the matrices. They have been shown for $M_{2,k}$. They also may give a recursive structure on the matrices to decompose and analyze them.

Conjecture 4.3.2  All leading principal minors of $M_{j,k}$ for all $j, k$ are exactly 1.

We apply a linear algebra proposition from Theorem 7.2.5 of [28]:

Proposition 4.3.6  (Sylvester’s Criterion [28, Theorem 7.2.5]) If every leading principal minor of a symmetric matrix $M$ is positive, then $M$ is positive definite.

Thus, Conjecture 4.3.2 would imply that $M_{j,k}$ is also positive definite. This also implies 0 is not an eigenvalue.

Conjecture 4.3.3  The coefficient of every term of the characteristic polynomial $g_{j,k}(z)$ of $M_{j,k}$ is nonzero for all $j, k$. Also, all consecutive terms alternate in sign.

By Descartes’ Rule of Signs, this implies that there are no negative roots. It is well-known that symmetric matrices only have real eigenvalues, thus combined with Conjecture 4.3.2, all eigenvalues are real positive values. This may give information for the roots of the characteristic polynomials.

Conjecture 4.3.4  All eigenvalues for each matrix $M_{j,k}$ are distinct for all $j, k$.

This means all eigenvalues have multiplicity one and that all eigenvectors are orthogonal. This conjecture with Conjecture 4.3.3 provides further information for the roots of the characteristic polynomials.

4.4 Characteristic Polynomials

We first compute several characteristic polynomials to analyze their structure. Recall that the rational generating functions that we finally compute from Marberg’s algorithm are denoted by $R_{j,k}(z) = \frac{f_{j,k}}{g_{j,k}}$ where $g_{j,k}$ is also the characteristic polynomial for $M_{j,k}$. Since the zeroes of the characteristic polynomials (eigenvalues of adjacency matrices) are the singularities of the generating functions, this provides us with information on the
asymptotics of the generating functions. Note that the degree of $g_{j,k}$ is $(j+k-2)_{k-1}$ as seen in Theorem 4.2.1 and $\text{deg}(f_{j,k}) = \text{deg}(g_{j,k}) - 1$. There are some cancellation of terms between $f_{j,k}$ and $g_{j,k}$ for $j \geq 3$. Thus, the degrees of $g_{j,k}$ will not follow the binomial coefficient $(j+k-2)_{k-1}$ pattern when we exclude common factors. We then analyze the eigenvalues of $M_{j,k}$ with a particular interest towards their distribution and growth. Finally, plots of a range of singularities, their maximum values, and their minimum values are displayed.

4.4.1 Singularity Analysis for 2-noncrossing Partitions

We analyze the terms of $R_{2,k}(z)$ for prominent patterns. Note that $\text{deg}(g_{2,k}) = k$ and there are no common factors between the numerators and denominators for all $k$. Based on our observations on the initial terms, we make the following Conjecture 4.4.1. The first few initial terms are in Table 4.1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$R_{2,k}(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1-2z}{z^2-3z+1}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{(3z-1)(z-1)}{-z^3+4z^2-5z+1}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{(1-2z)(2z^2-4z+1)}{(z-1)(z^3-9z^2+6z-1)}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{(z^2-3z+1)(5z^2-5z+1)}{-z^5+15z^4-35z^3+28z^2-9z+1}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{(z-1)(3z-1)(1-2z)(z^2-4z+1)}{z^6-21z^5+70z^4-84z^3+45z^2-11z+1}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{(7z^3-14z^2+7z-1)(z^2-6z^2+5z-1)}{(1-z)(z^2-3z+1)(z^4-24z^3+26z^2-9z+1)}$</td>
</tr>
</tbody>
</table>

Table 4.1: First few generating functions $R_{2,k}(z)$

Conjecture 4.4.1 The patterns in the denominator and numerator of $R_{2,k}(z)$ are characterized as follows.

1. The denominator $g_{2,k}$ is the product of several factors. If one such factor appears for the first time (smallest $k$ of all $g_{2,k}$) in $R_{2,k}$, then it also appears exactly in every denominator of the terms $R_{2,k+mt}$ for all $t \in \mathbb{N}$ and $m = 2k + 1$. For all $k$, every $g_{2,k}$ has a new factor that appears for the first time.

2. The numerator $f_{2,k}$ is also the product of several factors with new factors appearing for all $k$. There are exactly two new factors of the same degree when $k$ is odd and
exactly one new factor when \( k \) is even. Exactly one of the two new factors when \( k = 2k' + 1 \) is odd also appears in \( g_{2,k'} \). If a term appears for the first time in \( f_{2,k} \), then it also appears exactly in the numerator for terms \( f_{2,kt} \) for all positive integers \( t \).

Consider Table 4.2 with the first few denominator values. For example, \( z^2 - 3z + 1 \) only appears as a factor in \( g_{2,k} \) for \( k = 2, 7, 12, 17, \ldots \).

Table 4.3 shows the new numerator terms. For example, \( z^2 - 3z + 1 \) appears in \( f_{2,k} \) for \( k = 5, 10, 15, 20, \ldots \) which has the same spacing of five as seen when it appears in \( g_{2,k} \) but among different terms; \( z^2 - 3z + 1 \) belongs to a different congruence class modulo 5.

\[
\begin{array}{ccc}
 k & m & \text{New Denominator Term} \\
1 & 3 & z - 1 \\
2 & 5 & z^2 - 3z + 1 \\
3 & 7 & -z^3 + 6z^2 - 5z + 1 \\
4 & 9 & z^3 - 9z^2 + 6z - 1 \\
5 & 11 & -z^5 + 15z^4 - 35z^3 + 28z^2 - 9z + 1 \\
\end{array}
\]

Table 4.2: Denominator terms for \( R_{2,k} \)

\[
\begin{array}{ccc}
 k = m & \text{New Numerator Term} \\
2 & 1 - 2z \\
3 & (3z - 1)(z - 1) \\
4 & 2z^2 - 4z + 1 \\
5 & (z^2 - 3z + 1)(5z^2 - 5z + 1) \\
6 & z^2 - 4z + 1 \\
7 & (7z^3 - 14z^2 + 7z - 1)(z^3 - 6z^2 + 5z - 1) \\
\end{array}
\]

Table 4.3: Numerator terms for \( R_{2,k} \)

Since \( R_2(z) \) only has one singularity, these observations on the numerator and denominator of \( R_{2,k}(z) \) leads to the hypothesis that the cancellations eventually occur in the limit among the infinitely many terms in which they appear. However, we manage to derive \( R_2(z) \) by other means in Section 4.5.1.

We next analyze the singularities of \( R_{2,k} \) by plotting their distribution and extrema values. As an example to illustrate the general distribution pattern, consider the singularities of \( R_{2,100} \) for analysis. The greatest singularities are spread far wider than the remaining
ones to show reasonably in a graph. If we remove the 40 largest singularities we get Figure 4.19.

Next, consider the growth of the maximum and minimum singularities. The graph of the maximum singularities of $R_{2,k}$ for $k = 2, \ldots, 100$ are in Figure 4.21, and the minimum singularities are in Figure 4.22. Recall that $\lim_{k \to \infty} R_{2,k}(z) = R_2(z) = \sum C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}$ which only has the singularity at $\frac{1}{4}$. From Figure 4.22, we see the roots approaching the unique singularity of $R_2(z)$, $\frac{1}{4}$, as a limit point. The maximum singularities seem to be growing exponentially. We verify this is the case. Let us denote $\max_{j,k}$ for the maximum singularity of $R_{j,k}$. Then consider the ratio of change of the maximum singularities in $k$, $\max_{j,k} := \frac{\max_{j,k+1}}{\max_{j,k}}$, to analyze the growth rate of the singularities. Figure 4.22 plots this ratio $\max_{2,k}$. Experimentally, it seems this ratio approaches about 1.02 and is decreasing. So the growth of the maximum singularity is decreasing but approximately modeled by an exponential function. We summarize these results in Conjectures 4.4.5 and 4.4.6 at the end of Section 4.4.3.

Asymptotics of $NCN_{2,k}(n, 1)$

We apply Proposition 2.2.1 on Equation 4.1 of Section 4.5.1 for each $k$. Even though we possess the general formulas, their dominant singularities are not simple. Thus we only
Figure 4.22: Selection of minimum singularities of $R_{j,k}$ and ratio of maximum singularities $mx_{j,k}$ for $j = 2, 3, 4$.
produce approximate, asymptotic results in Table 4.4. We also include the asymptotics of the Catalan numbers, which is reached as $k$ approaches infinity.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Asymptotics of $NCN_{2,k}(n, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$0.03313(3.247)^n$</td>
</tr>
<tr>
<td>4</td>
<td>$0.0147(3.532)^n$</td>
</tr>
<tr>
<td>5</td>
<td>$0.00784(3.6825)^n$</td>
</tr>
<tr>
<td>6</td>
<td>$0.00467(3.7709)^n$</td>
</tr>
<tr>
<td>7</td>
<td>$0.003012(3.827)^n$</td>
</tr>
<tr>
<td>100</td>
<td>$5.069 \cdot 10^{-21} \cdot (3.999)^n$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\frac{1}{\sqrt{\pi}} \cdot n^{-\frac{3}{2}} 4^n$</td>
</tr>
</tbody>
</table>

Table 4.4: Approximate asymptotics of several $NCN_{2,k}(n, 1)$

Recall that

$$\lim_{k \to \infty} NCN_{2,k}(n, 1) = C_n.$$ 

By Stirling’s formula, we get

$$C_n \sim \frac{4^n}{\sqrt{\pi n^{3/2}}}.$$ 

This accounts for the data as the subexponential factor approaches 0 and the dominant singularity approaches $1/4$.

### 4.4.2 Singularity Analysis for 3-noncrossing Partitions

The clear pattern seen in $R_{2,k}(z)$ is not apparent, as $R_{3,k}(z)$ does not yield any obvious patterns through straightforward enumeration. Conjecture 4.4.2 sums up our observations.

**Conjecture 4.4.2** Between $f_{3,k}$ and $g_{3,k}$, there is a common factor of $(z-1)^{k'}$ for $k = 2k' + 1$ for all $k$. After such a cancellation, there is an extra $z = 1$ term in the numerator $f_{3,k}$ when $k$ is even and an extra $z = 1$ term in the denominator $g_{3,k}$ when $k$ is odd. Besides these simple factors, each $f_{3,k}$ and $g_{3,k}$ has one large irreducible factor that has not appeared before. Symbolically, if $N$ and $D$ are both one irreducible factors, then $R_{3,k}$ is of the form $\frac{(z-1)^N}{D}$ when $k$ is even and of the form $\frac{N}{(z-1)^D}$ when $k$ is odd.

After cancellation of common factors, the degree of $g_{3,k}$ no longer follows Proposition 4.2.1. When $k$ is odd, the degree is $(\frac{k+1}{k-1}) - \frac{k-1}{2} = \frac{k^2+1}{2}$.

**Conjecture 4.4.3** After cancellation of common factors in the rational functions $R_{3,k}(z)$, the degree of its denominator is $\frac{k^2+1}{2}$ if $k$ is odd and unchanged as $\frac{k(k+1)}{2}$ if $k$ is even.
Similar to the case with plotting singularities for $R_{2,k}$, the largest singularities of $R_{3,k}$ are also much greater than the remaining ones. We analyze the singularities of $R_{3,81}$, as 81 is the largest $k$ value computed. Figure 4.20 plots the singularities of $R_{3,81}$ ignoring the largest 55 roots. Graphically it seems denser for smaller values compared to the $j = 2$ cases and along with less smooth gaps between singularities. For another perspective, see Figure 4.23 for the natural logarithm of all the singularities of $R_{3,81}$. This logarithmic plot reveals a much smoother distribution.

The minimum singularities of $R_{3,k}$ plotted in Figure 4.22 seemingly approach a limit of $\frac{1}{9}$. The maximum singularities expand too rapidly for a plot to be meaningful. However, comparing the ratio of maximum singularities $m_{x3,k}$ in Figure 4.22 just as in the 2-noncrossing case, shows the difference that the maximum singularity is growing over time for 3-noncrossing; this is counter to the 2-noncrossing case. This growth ratio seems to be approaching a limit of about 15.5 from our limited data.

**Figure 4.23**: Natural logarithm of singularities of $R_{3,81}$

**Figure 4.24**: Natural logarithm of singularities of $R_{4,29}$

**Asymptotics of $NCN_{3,k}(n, 1)$**

Applying Proposition 2.2.1 on $R_{3,k}(z)$ yields the values of Table 4.5. Also included is the asymptotics as $k$ goes to infinity by Bousquet-Mélou and Xin [9].

In light of Equation 3.15, as $k \to \infty$, the dominant singularity approaches $1/9$ and the subexponential factor approaches 0.

**4.4.3 Singularity Analysis for 4-noncrossing Partitions**

Again we apply similar analysis as before. Note that we may not have clear information due to limited data. After cancelling common factors in $R_{4,k}$, every term is of the form $N/D$ or $\frac{N}{(z-1)D}$ where $N$ and $D$ are single irreducible factors. The pattern for the occurrence
Asymptotics of $NCN_{3,k}(n,1)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\text{Asymptotics of } NCN_{3,k}(n,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$4.186 \cdot 3.247^n$</td>
</tr>
<tr>
<td>3</td>
<td>$1.793 \cdot 4.965^n$</td>
</tr>
<tr>
<td>4</td>
<td>$1.590 \cdot 6.079^n$</td>
</tr>
<tr>
<td>5</td>
<td>$1.858 \cdot 6.807^n$</td>
</tr>
<tr>
<td>6</td>
<td>$1.877 \cdot 7.302^n$</td>
</tr>
<tr>
<td>7</td>
<td>$1.958 \cdot 7.649^n$</td>
</tr>
<tr>
<td>8</td>
<td>$0.03527 \cdot 7.902^n$</td>
</tr>
<tr>
<td>9</td>
<td>$0.1059 \cdot 8.090^n$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\frac{3^{9.5} \sqrt{3}}{\pi} \cdot n^{-79^n}$</td>
</tr>
</tbody>
</table>

Table 4.5: Approximate asymptotics of several $NCN_{3,k}(n,1)$

of the extra $z - 1$ factor in the denominator is not well established based on limited data; this factor appears for $k = 2, 5, 6, 8, 11, 14, 16, 17, 20, 21, 23, 26, 29$. Also, note that there are never any repeats of $N$ or $D$ for any $k$ in $R_{4,k}$. The common factors that are cancelled out are more interesting and we list the first few terms in Table 4.6.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\gcd(f_{4,k}, g_{4,k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$g_{2,6}$</td>
</tr>
<tr>
<td>5</td>
<td>$z - 1$</td>
</tr>
<tr>
<td>6</td>
<td>$g_{2,8}$</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>$(z - 1)^4 g_{2,10}$</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>$g_{2,12}$</td>
</tr>
<tr>
<td>11</td>
<td>$(z - 1)^4$</td>
</tr>
<tr>
<td>12</td>
<td>$g_{2,14}$</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>$(z - 1)^9 g_{2,16}$</td>
</tr>
</tbody>
</table>

Table 4.6: Common factors of $R_{4,k}$

Although a precise description of the common factors is unavailable due to insufficient data, Conjecture 4.4.4 contains most of the observations.

**Conjecture 4.4.4** The common factors for $R_{4,k}$, $c_k := \gcd(f_{4,k}, g_{4,k})$, satisfy

1. $c_2 = 1$;

2. $c_k = 1$ when $k \equiv 1, 3 \pmod{6}$;
3. $g_{2,2k}$ is a factor of $c_{2k-2}$ for all $k$ with this being the only factor when $2k - 2 \equiv 0, 4 \pmod{6}$;

4. $c_k$ is some power of $z - 1$ for $k \equiv 5 \pmod{6}$;

5. $c_{2k-2}$ is $g_{2,2k}$ times some power of $z - 1$ for $k \equiv 2 \pmod{6}$.

Figure 4.24 plots the natural logarithm of the singularities of $R_{4,29}$ after removing the largest 10 singularities. Comparing this graph to the natural logarithm graph for the $j = 3$ case, this graph is not as evenly distributed. The singularities for $R_{4,k}$ are similar to those of $R_{3,k}$ in that the maximum one for a fixed $k$ is magnitudes greater than even the second largest singularity. The minimum singularities of $R_{4,k}$ from Figure 4.22 seems to approach a lower bound of about $\frac{1}{16}$. Combining with the data for $j = 2, 3$, we formalize the observations with Conjecture 4.4.5. Computing the ratio of maximum singularities in Figure 4.22 yields that $mx_{4,k}$ is increasing. It seems to be approaching an unidentified limit, but it may also be increasing without bound. We form Conjecture 4.4.6 based on this graph along with the $j = 2, 3$ graphs.

**Conjecture 4.4.5** The singularities of $g_{j,k,1}$ have $\frac{1}{j^2}$ as a lower bound and as an infimum as $k \to \infty$.

Note that Burrill [12] proves the first part of Conjecture 4.4.5 in Corollary 7.2.3. She also makes Conjecture 7.2.6, which is the same as the second part of our Conjecture 4.4.5.

**Conjecture 4.4.6** For all $j$, the ratio of consecutive maximum singularities of $R_{j,k}(z)$, $mx_{j,k}$, approaches a finite limit as $k$ increases and $j$ fixed. The ratio $mx_{2,k}$ is decreasing while $mx_{j,k}$ is increasing for all $j \geq 3$.

**Asymptotics of $NCN_{4,k}(n, 1)$**

We perform the same analysis on the asymptotics for $R_{4,k}(z)$ as shown for $R_{2,k}(z)$ and $R_{3,k}(z)$. Proposition 2.2.1 is used and data is shown in Table 4.7.

**4.5 The Limit $R_j(n)$**

A goal is to make advancements to a conjecture posed by Bousquet-Mélou and Xin in [9], that $R_j(n)$ is not D-finite for $j \geq 4$. It has been shown that $R_2(n)$ and $R_3(n)$ are
Asymptotics of $NCN_{4,k}(n,1)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>Asymptotics of $NCN_{4,k}(n,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.05199 \cdot 3.532^n$</td>
</tr>
<tr>
<td>3</td>
<td>$7.895 \cdot 10^{-4} \cdot 6.079^n$</td>
</tr>
<tr>
<td>4</td>
<td>$1.170 \cdot 10^{-5} \cdot 8.115^n$</td>
</tr>
<tr>
<td>5</td>
<td>$2.249 \cdot 10^{-11} \cdot 9.660^n$</td>
</tr>
<tr>
<td>6</td>
<td>$2.050 \cdot 10^{-9} \cdot 10.83^n$</td>
</tr>
<tr>
<td>7</td>
<td>$2.946 \cdot 10^{-7} \cdot 11.72^n$</td>
</tr>
<tr>
<td>8</td>
<td>$2.262 \cdot 10^{-3} \cdot 12.40^n$</td>
</tr>
</tbody>
</table>

Table 4.7: Approximate asymptotics of several $NCN_{4,k}(n,1)$

D-finite, and we first consider those two cases for potential hints. Using our methods, we derive $R_{2,k}(n)$ and $R_2(n)$. Then we discuss our approach to the $j \geq 3$ cases.

4.5.1 Analysis for 2-noncrossing Partitions

We already know that $R_2(z) = \lim_{k \to \infty} \sum_{n \geq 0} NCN_{2,k}(n)z^n = \frac{1 - \sqrt{1 - 4z}}{2z}$ is the generating function counting noncrossing partitions. This is D-finite and converges to the generating function which counts the Catalan numbers (see Exercises 6.19 pp. of [54]) and Example 2.1.1). We rederive this using our own methods.

Proposition 4.5.1 We have

$$R_{2,k}(z) = \frac{\sum_{i=1}^{k} \binom{k+i-1}{2i-1} (-z)^{k-i}}{\sum_{i=0}^{k} \binom{k+i}{2i} (-z)^{k-i}}$$ (4.1)

and

$$R_2(z) = \lim_{k \to \infty} R_{2,k} = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Proof Idea We make Equation (4.1) as a claim from our data and proceed to prove it. We compute the characteristic polynomial of $M_{2,2}$ and find a recursion for the characteristic polynomial of $M_{2,k}$. The numerator and denominator of $R_{2,k}$ can both be written in terms of these polynomials. Next, we show that our guessed expression satisfies this recursion and the initial conditions. By uniqueness, the conjectured form is the expression we desire.

Proof The right side of Equation (4.1) is equivalent to

$$\frac{\sum_{i=1}^{k} \binom{k+i-1}{2i-1} (-1)^i z^{k-i}}{\sum_{i=0}^{k} \binom{k+i}{2i} (-1)^i z^{k-i}}$$ (4.2)
where the second expression is just shifting a negative sign from the numerator to the denominator for odd numbered terms. We do this because Equation (4.2) produces a more convenient expression through creative telescoping from the SumTools package in Maple. We then verify that this expression is indeed correct by computing determinants. (Note the similarity to Theorem 1.1 of [37] which finds the generating functions to set partitions avoiding the pattern 12 · · · k12.) If we denote $M_k$ as the $k$ by $k$ matrix $M_{2,k}$ for convenience, then

$$I - zM_k = \begin{bmatrix} 1 - z & -z & 0 & \cdots & 0 & 0 \\ -z & 1 - 2z & -z & \cdots & 0 & 0 \\ 0 & -z & 1 - 2z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 - 2z & -z \\ 0 & 0 & 0 & \cdots & -z & 1 - 2z \end{bmatrix}.$$ 

Since $R_{2,k}(z) = \frac{f_{2,k}(z)}{g_{2,k}(z)}$ by Equation (3.5), then the expression we want is

$$f_{2,k}(z) = \frac{\det(I_k - zM_k) + z \det(I - z(M_k)_{k,k})}{\det(I_k - zM_k)} = g_{2,k}(z) + zg_{2,k-1}(z) \cdot \frac{g_{2,k}(z)}{g_{2,k}(z)}.$$

Clearly,

$$\det(I_1 - zM_1) = 1 - z$$

and applying the Laplace expansion on the last row produces the recursion

$$g_{2,k}(z) = (1 - 2z)g_{2,k-1}(z) - z^2g_{2,k-2}(z).$$

This with the initial condition uniquely determines $g_{2,k}(z)$. It remains to check that the denominator of Equation (4.2) satisfies this recursion and also that $f_{2,k}(z) = g_{2,k}(z) + zg_{2,k-1}(z)$. The second condition is easy to check using Equation (4.1) since

$$g_{2,k}(z) + zg_{2,k-1}(z) = \sum_{i=0}^{k} \binom{k + i}{2i} (-z)^{k-i} + z \sum_{i=0}^{k-1} \binom{k + i - 1}{2i} (-z)^{k-i-1}
= 1 + \sum_{i=0}^{k-1} \left(\binom{k + i}{2i} (-1)^{k-i} + \binom{k + i - 1}{2i} (-1)^{k-i-1}\right) z^{k-i}
= f_{2,k}(z).$$
To prove the first part we consider cases. Define $D = (1 - 2z)g_{2,k-1}(z) - z^2g_{2,k-2}(z)$.

For degrees $0, 1, k$ in $z$ we have $[z^k]D = 2(-1)^k - (-1)^k = (-1)^k$, $[z^0]D = 1$, and $[z^1]D = \left(\frac{2k-3}{2k-2}\right)(-1) + 2\left(\frac{2k-2}{2k-2}\right)(-1) = 1 - 2k$ and all three terms agree with the respective coefficients for $g_{2,k}(z)$. All we need to do is to show that the coefficients also match for the degree from $2$ to $k - 1$. We apply the well-known identity $\binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}$ numerous times. We can rewrite the first term of $D$ as

$$\sum_{i=0}^{k-3} \binom{k+i-1}{2i} (-z)^{k-i-1} = \sum_{i=1}^{k-2} \binom{k+i-2}{2(i-1)} (-z)^{k-i}.$$  

Extracting coefficients we obtain

$$[z^{k-i}]D = \left(\frac{k+i-2}{2(i-1)}\right) (-1)^{k-i} - 2\left(\frac{k+i-1}{2i}\right) (-1)^{k-i-1} - \left(\frac{k+i-1}{2i}\right) (-1)^{k-i}.$$  

Matching coefficients from both sides, we just need to show that

$$\binom{k+i}{2i} = \binom{k+i-1}{2i} + 2\binom{k+i-1}{2i} - \binom{k+i-2}{2i}$$


to match the coefficients to that of $g_{2,k}(z)$. However,

$$\binom{k+i}{2i} = \binom{k+i-1}{2i} + \binom{k+i-1}{2i-1}$$

$$= \binom{k+i-1}{2i} + \left(\binom{k+i-2}{2i-1} + \binom{k+i-2}{2i-2}\right)$$

$$+ \left(\binom{k+i-1}{2i} - \binom{k+i-2}{2i} - \binom{k+i-2}{2i-1}\right)$$

$$= \binom{k+i-2}{2i-2} + 2\binom{k+i-1}{2i} - \binom{k+i-2}{2i}.$$  

Thus, our guessed rational function for $R_{2,k}(z)$ was correct. Through creative telescoping done in Maple 17, we obtain that the numerator and denominator of Equation (4.2) becomes, respectively,

$$\sqrt{x}(y - \sqrt{\frac{x}{2}})^k - \sqrt{x}(y + \sqrt{\frac{x}{2}})^k$$

and

$$\frac{1}{2} \left(\frac{-x - \sqrt{x}(y + \sqrt{\frac{x}{2}})^k}{-x} - \frac{1}{2} \left(-x + \sqrt{x}(y - \sqrt{\frac{x}{2}})^k\right)ight.$$

where $x = 1 - 4z$ and $y = -z + \frac{1}{2}$. If we then divide the numerator by the denominator we obtain

$$2\sqrt{x} \left(\frac{(y + \sqrt{\frac{x}{2}})^k - (y - \sqrt{\frac{x}{2}})^k}{(-x - \sqrt{x})(y - \sqrt{\frac{x}{2}})^k + (-x + \sqrt{x})(y + \sqrt{\frac{x}{2}})^k}\right).$$  

(4.3)
We next proceed to make the following simplification. We assume that \( z \in (0, \frac{1}{4}) \) since the singularities occur at those boundaries. This in turn implies that \( y < 0 < x \). Hence, \(|y - \sqrt{z}| > |y + \sqrt{z}| \) since the first term is a sum of two negative terms while the second is a sum of a negative term and a positive term. By the same reasoning, \( |x - \sqrt{z}| > |x + \sqrt{z}| \).

This implies that Equation (4.3) can be simplified in the limit as \( k \rightarrow \infty \) to

\[
\lim_{k \rightarrow \infty} R_{2,k}(z) = \frac{-2\sqrt{x}}{-x - \sqrt{x}} = \frac{-2\sqrt{1 - 4z}}{-1 + 4z + \sqrt{1 - 4z}} - \frac{1 + 4z + \sqrt{1 - 4z}}{2\sqrt{1 - 4z} - 8z\sqrt{1 - 4z} - 2(1 - 4z)} = \frac{(2 - 8z)\sqrt{1 - 4z} - 2 + 8z}{(1 - 4z)(1 - 4z - 1)} = \frac{2\sqrt{1 - 4z} - 2}{-4z} = \frac{1 - \sqrt{1 - 4z}}{2z}
\]
as desired. \( \square \)

We mention that the above generating functions \( R_{2,k}(z) \) are new. Marberg [38] incorrectly states on page 13 that such computations have already been shown by Mansour and Severini in [37, Theorem 1.1]. However, we compute set partitions generating functions avoiding the patterns 1212 and 12...k1212...21 while their functions avoid the pattern 12...k12 (recall this notation from page 16). We can also justify this from their series data in Table 1, which does not match with our series data in Table 4.8 of Section 4.6.

4.5.2 Analysis for Higher Noncrossing Partitions

In order to analyze the 3-noncrossing partitions case, note from the matrices \( M_{3,k} \) that we cannot apply a straightforward determinant calculation like we did with \( M_{2,k} \). We saw in Section 3.5 that Bousquet-Mélou and Xin have computed a recursion and explicit form for \( NC_3(n) \) and derived a differential equation satisfied by \( R_3(z) \). However, we did not make progress through considering limits of the rational functions \( R_{3,k}(z) \).

From all our work in this chapter, 4-noncrossing partitions differs from 3-noncrossing partitions in being a larger version but with similar structures. The only structure that shows an increase in complexity is the comparison of the diagonal blocks of Figure 4.15 for 4-noncrossings to Figure 4.10 for 3-noncrossings. For the 3-noncrossing case, the diagonal
blocks are pairs of diagonals of 1’s. However, 4-noncrossing diagonal blocks have increasing squares each made up of the entire diagonal block for 3-noncrossing. Further analysis of these diagonal blocks is needed to determine the effects of the increased complexity.

4.6 Enumerative Data

There exists previous Maple code by Jones [30] on listing systematically set partitions with their standard representations along with their maximum crossing and nesting numbers. However, his direct method becomes too inefficient when the partition’s size reaches about 20. For monochromatic partitions, we compare computed statistics with those found in Jones’s code to double check our series.

We construct a table of values in Table 4.8 of statistics for one colour, \( j \)-noncrossing, and \( k \)-nonnesting partitions of size \( n \) as a first step. First recall that the number of partitions of \([n]\) is the \( n \)-th Bell number \( B_n \) or the counting sequence to Example 2.1.3 whose first few terms are in the last row of Table 4.8. It is easy to see the symmetric joint distribution in the table along with the numbers converging to the Bell numbers.

<table>
<thead>
<tr>
<th>((j, k))( \backslash )n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 2)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>13</td>
<td>34</td>
<td>89</td>
<td>233</td>
<td>610</td>
<td>1597</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>131</td>
<td>417</td>
<td>1341</td>
<td>4334</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>131</td>
<td>417</td>
<td>1341</td>
<td>4334</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1429</td>
<td>4846</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1429</td>
<td>4846</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>201</td>
<td>841</td>
<td>3720</td>
<td>17125</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>202</td>
<td>859</td>
<td>3929</td>
<td>19067</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>202</td>
<td>859</td>
<td>3929</td>
<td>19067</td>
</tr>
<tr>
<td>(B_n)</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
<td>4140</td>
<td>21147</td>
</tr>
</tbody>
</table>

Table 4.8: Distribution of \( j \)-noncrossing and \( k \)-nonnesting partitions of \([n]\)

We may instead construct a table of statistics for the number of partitions of \([n]\) fixing the maximum \( j \)-crossing and \( k \)-nesting numbers; for a fixed \( n \), we represent the statistic with the notation \([j, k]\). Note that 0 means no crossing (or nesting) and 2 means 2-crossing (or 2-nesting); we do not define 1 in \([j, k]\) notation. These values are calculated in Table 4.9 from recursively looking at differences of values in Table 4.8. In Table 4.9, summing over all possible maximum crossing and nesting statistics returns the Bell numbers. The symmetric
joint distribution is also evident. We can also see the distribution of crossing and nesting
numbers and density from the data.

\[
\begin{array}{|c|cccccc|}
\hline
[j, k] \backslash n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
[0,0] & 1 & 2 & 5 & 13 & 38 & 89 & 233 \\
[2,0] & 0 & 0 & 0 & 1 & 6 & 42 & 184 \\
[0,2] & 0 & 0 & 0 & 1 & 6 & 42 & 184 \\
[2,2] & 0 & 0 & 0 & 0 & 2 & 28 & 240 \\
[0,3] & 0 & 0 & 0 & 0 & 0 & 1 & 12 \\
[3,0] & 0 & 0 & 0 & 0 & 0 & 1 & 12 \\
[3,2] & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
[2,3] & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
\hline
B_n & 1 & 2 & 5 & 15 & 52 & 203 & 877 \\
\hline
\end{array}
\]

Table 4.9: Distribution of maximum $j$-crossing and $k$-nesting partitions of $[n]$

All our data have been computed using Marberg’s algorithm running on Maple 17/18. The code in the appendix is an implementation of his proof. We managed to compute monochromatic $R_{2,k}$ as far as desired; $R_{3,k}$ for $k \leq 81$; $R_{4,k}$ for $k \leq 29$; $R_{5,k}$ for $k \leq 17$; $R_{2,k}(z, 2)$ for $k \leq 69$; $R_{2,k}(z, 3)$ for $k \leq 18$; $R_{2,k}(z, 4)$ for $k \leq 9$; and $R_{3,k}(z, 2)$ for $k \leq 12$. Some of the above computations took several months which implies that perhaps a more efficient algorithm should be used for higher crossing, nesting, and colouring values.
Chapter 5

Extensions to Other Combinatorial Objects

5.1 Multi-coloured Set Partitions

Asymptotics for $NCN_{2,k}(n, 2)$

Based on our current algorithm, it seems hopeless to study higher noncrossing and nonnesting generating functions through pure computation as it is too inefficient to compute enough meaningful data. Luckily, the asymptotics of $\{NCN_{2,k}(n, 2)\}_k$ as $n \to \infty$ are still manageable enough to analyze. A selection is computed in Table 5.1. Based on the asymptotics of $NC_2(n, 2)$ from Equation 3.12, the dominant singularities of $NCN_{2,k}(n, 2)$ approach $1/9$ and the subexponential factor approaches $0$ as $k \to \infty$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Asymptotics of $NCN_{2,k}(n, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.01689 \cdot 4.709^n$</td>
</tr>
<tr>
<td>3</td>
<td>$0.001366 \cdot 6.505^n$</td>
</tr>
<tr>
<td>4</td>
<td>$0.0002027 \cdot 7.396^n$</td>
</tr>
<tr>
<td>5</td>
<td>$0.00004348 \cdot 7.889^n$</td>
</tr>
<tr>
<td>6</td>
<td>$1.490 \cdot 10^{-10} \cdot 8.187^n$</td>
</tr>
<tr>
<td>7</td>
<td>$4.268 \cdot 10^{-11} \cdot 8.380^n$</td>
</tr>
<tr>
<td>8</td>
<td>$1.029 \cdot 10^{-11} \cdot 8.512^n$</td>
</tr>
<tr>
<td>9</td>
<td>$8.212 \cdot 10^{-12} \cdot 8.607^n$</td>
</tr>
</tbody>
</table>

Table 5.1: Approximate asymptotics of several $NCN_{2,k}(n, 2)$

5.2 Permutations

Asymptotics for $NCN^P_{2,k}(n, k)$

Yen in [63] computed the general form for many of the permutation cases of the generating
functions $R_{2,2}^P(z, r) = \sum_n NCN_{2,2}^P(n, r)z^n$. They are rational functions and we can use Proposition 2.2.1 to determine their asymptotics. By analyzing the asymptotics for the terms $r = 1, \ldots, 7$, I make the following conjecture based on experimental data.

**Conjecture 5.2.1** The asymptotics to $NCN_{2,2}^P(n, r)$ for fixed $r$ as $n \to \infty$ is

$$NCN_{2,2}^P(n, r) \sim \frac{1}{(2r)^n}(r^2 + r)^n.$$  

By Stirling's formula, this last term is asymptotic to $\frac{\sqrt{\pi r}}{4^n}(r^2 + r)^n$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$R_{2,2}^P(z, r)$</th>
<th>Asymptotics</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1 - 6z + 4z^2}{(1 - 2z)(1 - 6z)}$</td>
<td>$6^n$</td>
</tr>
<tr>
<td></td>
<td>$1 - 17z + 66z^2 - 36z^3$</td>
<td>$6^n$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1 - 36z + 380z^2 - 1200z^3 + 576z^4}{(1 - 2z)(1 - 6z)(1 - 12z)(1 - 20z)}$</td>
<td>$12^n$</td>
</tr>
<tr>
<td></td>
<td>$1 - 65z + 1408z^2 - 11804z^3 + 32880z^4 - 14400z^5$</td>
<td>$20^n$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1 - 65z + 1408z^2 - 11804z^3 + 32880z^4 - 14400z^5}{(1 - 2z)(1 - 6z)(1 - 12z)(1 - 20z)(1 - 30z)}$</td>
<td>$70^n$</td>
</tr>
<tr>
<td></td>
<td>$1 - 106z + 4048z^2 - 68232z^3 + 496944z^4 - 1270080z^5 + 518400z^6$</td>
<td>$30^n$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1 - 106z + 4048z^2 - 68232z^3 + 496944z^4 - 1270080z^5 + 518400z^6}{(1 - 2z)(1 - 6z)(1 - 12z)(1 - 20z)(1 - 30z)(1 - 42z)}$</td>
<td>$252^n$</td>
</tr>
<tr>
<td>5</td>
<td>$1 - 106z + 4048z^2 - 68232z^3 + 496944z^4 - 1270080z^5 + 518400z^6$</td>
<td>$924^n$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1 - 106z + 4048z^2 - 68232z^3 + 496944z^4 - 1270080z^5 + 518400z^6}{(1 - 2z)(1 - 6z)(1 - 12z)(1 - 20z)(1 - 30z)(1 - 42z)}$</td>
<td>$924^n$</td>
</tr>
</tbody>
</table>

| Table 5.2: First few generating functions $R_{2,2}^P(z, r)$ and asymptotics |
Chapter 6

Conclusion

We conclude this thesis with a brief restatement of our main work, some conjectures of interest, some ideas to tackle these conjectures based on our work, and insights by our examiner DeVos.

The bulk of novel work comes from Chapter 4 on analyzing data for patterns in the automata structures, transfer matrices, generating function singularities, and generating functions. Our approach focuses on the enumerative aspects of Marberg’s algorithm of Section 3.7.2. We also provide a survey of crossings and nestings for set partitions up to present day.

6.1 Open Questions

We mention some conjectures stated formally earlier in this thesis and some questions for further exploration.

1. Is there a systematic way to determine the off-diagonal blocks $N_{j,k}$ mentioned in Proposition 4.3.5? How do they grow in complexity for larger $j$ values? Is this increase in complexity related to the D-finiteness of their respective $j$-noncrossing generating functions $R_j(z)$? The first part may be easy through analysis of more data. The second part is harder as we would need to already know the D-finiteness of the generating functions.
2. Conjecture 4.3.2 states that the leading principle minors of $M_{j,k}$ all are 1. Are there deep algebraic structures underneath these matrices to yield this conclusion? This may not be hard but may reveal interesting properties.

3. Conjecture 4.3.4 states that all singularities are distinct for $R_{j,k}$. Why is this the case? This might not be hard and may follow from analytic combinatorics.

4. We return to the conjecture by Bousquet-Mélou and Xin [9] that $R_{4}(z)$ is not D-finite. Does the answer depend on the containment structure of the adjacency matrices in any deep ways? This is interesting as no generating function involving crossings and nestings has yet been shown to not be D-finite. If their conjecture holds, then it would be the simplest example of a relevant generating function that is not D-finite.

6.2 Further Work

Some ideas which may be attempted to expand on this thesis are as follows.

1. We can compute more data to visualize matrices $M_{j,k}$ for $j \geq 5$. Then try to understand the diagonal structures and relate them to diagonal blocks of other $j$ values. It seems with enough data on sufficiently large $j$, a pattern may be more easily identified.

2. Perhaps a more creative form of computing determinants than what is done in Section 4.5.1 is needed to find analytically the exact form of $R_{j,k}$ for $j \geq 3$. If this is possible, the formula will be messy and not in any simple closed form.

3. Can we try using some statistical tools to understand the distribution of singularities for 3-noncrossing and 4-noncrossing set partitions? Perhaps there are more limit points of these singularities that we have not discovered although this is doubtful.

4. Burrill [12] asks similar questions to Conjecture 4.4.5, although she approached the subject with a heavier emphasis on tableaux structures. Can we combine our knowledge to arrive at new insights?

5. We mentioned how all transfer matrices can be written in block tridiagonal form. Maybe we can modify the Tridiagonal Matrix Algorithm from linear algebra by Llewellyn
Thomas for block matrix forms to our matrices. See [16] for a reference of this algorithm.

6. We can try to make the transfer matrices play a bigger role. One idea is to try some spectral graph theory techniques to the automata when viewed as a graph.

7. A technique used in [40] is to perform a transformation and map the singularities onto a unit circle. We could try a similar technique for our singularities.

8. We can do everything in this thesis for the enhanced crossings and nestings case. This is also necessary to study permutations. Most likely, everything will be similar in difficulty to the standard non-enhanced case.

9. We can study the distribution of the number of nestings fixing $2$-noncrossing in $G_{2,k}$ by considering an infinite automaton as $k \to \infty$. Then we use a random walk on this automaton to calculate the probability of the maximum distance away we walk from the starting state.

6.3 A Modified Automata Construction

The thesis defence allowed DeVos to share several key insights about the adjacency matrices:

1. Instead of a path in the automaton of length $n$ only stopping at the odd-numbered elements of a vacillating tableau $V^{2n}$ as the states, we can use all the $2n + 1$ elements as states and consider paths of length $2n$. In this case, each state also keeps track of whether it is currently at an odd step or even step. This leads to simpler automaton and consequently, simpler adjacency matrices to study.

2. DeVos' construction then leads to finite state machines that are paths with no loops for $2$-noncrossing set partitions.

3. Instead of a reverse lexicographical order for the states in the adjacency matrices, it may be useful to consider the states listed by increasing size of the Young diagram (Young diagrams of the same size can be sorted arbitrarily). This leads to a simpler
adjacency matrix of the block form
\[
\begin{bmatrix}
0 & D^T \\
D & 0
\end{bmatrix}
\]
where \(D\) is a triangular block matrix.

Essentially, the product \(DD^T\) becomes the original corresponding matrix that we study.

4. Listing the states by DeVos’ ordering with all vacillating tableau elements as states leads to simpler adjacency matrices that allows us to prove Conjecture 4.3.2. Using positive-definiteness of these matrices, one can then prove Conjecture 4.3.3.
Bibliography


[64] Lily Yen. unpublished. personal communication, 2014. 32

[65] Doron Zeilberger. Asyrec, a package for maple, available online at http://www.math.rutgers.edu/~zeilberg/programs.html. 30


Appendix A

Maple Code to Compute Adjacency Matrices and Generating Functions

with(linalg):
# Implements Marberg’s algorithm.
# input: j(>=2) noncrossing, k(>=2) nonnesting, r colouring.
# Ignore trivial partitions.
# n step walks for nth coefficient
# Output: M_{j,k,r} and R_{j,k}(z,r)
MarbAlg:=proc(j,k,r)
local size,states,temp,index,i,l,x,y,adjmat;
local L,Mat,loops,d,noedge,Edge,f,num,den,ser,comm,Mat2;
# states(j,k,r):= all r by (k-1) matrices with nonincreasing integer
# entries from 0 to j-1.
# states in reverse lex order
# size(states):= current size of states
states:=Array(1..2+binomial(j+k-2,k-1)):
states[1]:= [j-1$k-1]:
temp:=states[1];
index:=k-1;
size:=2;
while temp <> [0$k-1] do
if temp[index] <> 0 then
    temp[index]:=temp[index]-1;
    states[size]:=temp;
    size:=size+1;
else
    while temp[index] = 0 do
        index:=index-1;
end if;
extend do;

...
temp[index]:=temp[index]-1;
for i from index+1 to k-1 do
  temp[i]:=temp[index];
  od;
index:=k-1;
states[size]:=temp;
size:=size+1;
fi;
od;
size:=size-1;
#adjmat(states):= adjacency matrix indexed by states.
#loops(adjmat):= edges from a state to itself
#We next compute number of loops for each vertex:
#loops(S,S)=1+d1+...+dr where di is number of distinct entries
#in the ith row of S which are less than j-1 for i=1,...,r.
#Also compute existence of edges between vertices:
#loops(S,S')=1 if S-S'=\pm E_{iil} or S-S'=E_{iil}-E_{i'l'}
#for some (i,l)\neq(i',l') where E_{iil} is the rx(k-1)
#matrix with 1 in position (i,l) and 0 elsewhere.
#There are no edges in all other cases.
adjmat:=Array(1..size^r):
for i from 0 to size^r-1 do
  L:=convert(i,base,size):
  adjmat[i+1]:=Matrix([[seq(states[L[l]+1],l=1..nops(L)),
    seq(states[1],l=1..r-nops(L))]]):
  od;
#Mat(adjmat,loops):=adjacency matrix
Mat:=Matrix(size^r):
for i from 1 to size^r do
  loops:=1;
  for l from 1 to r do
    d:=convert(row(adjmat[i],l),set) minus {j-1};
    loops:=loops+nops(d);
    od;
  Mat[i,i]:=loops;
  od;

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for i from 1 to size^r-1 do
    for l from i+1 to size^r do
        #noedge:=whether (i,l) are adjacent
        noedge:=false;
        Edge:=[0,0];#number of 1,-1 in A-A'
        for x from r to 1 by -1 do
            if noedge=true then
                break;
            fi;
            for y from k-1 to 1 by -1 do
                if abs(adjmat[i][x,y]-adjmat[l][x,y]) > 1 then
                    noedge:=true;
                    break;
                elif adjmat[i][x,y]-adjmat[l][x,y]=1 then
                    Edge[1]:=Edge[1]+1;
                    elif adjmat[i][x,y]-adjmat[l][x,y]=-1 then
                        Edge[2]:=Edge[2]+1;
                    fi;
                fi;
            od;
        if noedge=false then
            Mat[i,l]:=1;
            Mat[l,i]:=1;
        fi;
    od;
#We have the nxn adjacency matrix Mat and compute:
#det(I-zMat)+zdet(I-zMat_{nn}) / det(I-zMat)
#then we can apply transfer matrix method.
#To take advantage of Maple coding, we are computing
#characteristic polynomial det(zI-Mat) instead.
#Mat2:= submatrix removing last row and column to compute minor
Mat2:=LinearAlgebra[SubMatrix](Mat,1..size^r-1,1..size^r-1):

#den:= denominator of $R_{j,k}(z,r)$
\begin{align*}
\text{den} & := \text{LinearAlgebra}:-\text{Modular}:-\text{IntegerCharacteristicPolynomial}(\text{Mat}, z); \\
\text{num} & := \text{numerator of } R_{j,k}(z,r)
\end{align*}
\begin{align*}
\text{num} & := \text{LinearAlgebra}:-\text{Modular}:-\text{IntegerCharacteristicPolynomial}(\text{Mat2}, z); \\
\text{den} & := z^{\text{size}^r}\text{subs}(z=1/z, \text{den}); \\
\text{num} & := \text{den} + z^{\text{size}^r}\text{subs}(z=1/z, \text{num}); \\
\text{comm} & := \text{common factors of numerator and denominator}
\end{align*}
\begin{align*}
\text{comm} & := \text{factor}(\text{gcd}(\text{den}, \text{num})); \\
\text{den} & := \text{factor}(\text{den} / \text{comm}); \\
\text{num} & := \text{factor}(\text{num} / \text{comm}); \\
\text{f} & := R_{j,k}(z,r) \\
\text{f} & := \text{num} / \text{den}; \\
\text{ser} & := \text{first few terms of series expansion}
\end{align*}
\begin{align*}
\text{ser} & := \text{series}(\text{f}, z, 30); \\
\text{f} & := \text{R}_{j,k}(z,r) \\
\text{f} & := \text{num} / \text{den}; \\
\text{ser} & := \text{series}(\text{f}, z, 30); \\
\text{f} & := \text{R}_{j,k}(z,r)
\end{align*}
\begin{align*}
\text{f} & := \text{num} / \text{den}; \\
\text{ser} & := \text{series}(\text{f}, z, 30);
\end{align*}
# Outputs $(j,k,r)$, function, series, adjacency matrix, common factors, numerator and denominator after cancelling
\begin{align*}
\text{print}(j, k, r); \\
\text{print}(& \text{Mat}); \\
\text{lprint}(& \text{f}); \\
\text{print}("\text{function}"); \\
\text{lprint}(& \text{ser}); \\
\text{print}("\text{series}"); \\
\text{lprint}(& \text{comm}); \\
\text{print}("\text{common factors}"); \\
\text{lprint}(& \text{num}); \\
\text{print}("\text{numerator}"); \\
\text{lprint}(& \text{den}); \\
\text{print}("\text{denominator}"); \\
\text{lprint}(& \text{den}); \\
\text{end}: \\
\text{end}: