An Agent-Based Approach to Modelling Chronic Offenders

by

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Abstract

Police departments are required to ensure the safety of the general population using limited resources. Chronic offenders are a major drain on police resources. The precise definition of a chronic offender may change between jurisdictions but the general concept refers to a class of offenders who commit many crimes in short intervals. Unlike other offenders who typically stop committing crimes early in life, chronic offenders continue committing crimes late in life. Understanding this class of offenders allows police departments to modify their operational strategies and make better use of their resources. We develop an agent-based model of how chronic offenders move between incarceration and freedom. Under appropriate limits we convert the agent-based model to a system of differential equations which is analyzed using established differential equation methods.

Keywords: Chronic offenders; high frequency offenders; mathematical modelling in criminology; quantitative criminology; agent-based modelling
To my amazing parents
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Chapter 1

Introduction

1.1 Background

Chronic offenders are distinct from typical offenders in both the frequency at which they commit crimes and the length of their careers. A typical offender has a short career, with few offences, while a chronic offender has a long career, often with shorter breaks between crimes [9]. The majority of chronic offenders commit several crimes before being caught, serve their time, and quickly return back to committing crimes with the cycle repeating. Chronic offenders primarily commit drug and property crimes with the odd more serious offence [19]. These offenders commit upwards of fifty crimes between arrests and are referred to as generalists [19]. In 2004 the Vancouver Police Department (VPD) identified 379 chronic offenders who, between 2001 and 2006, accounted for 26,755 police contacts (positive or negative interactions between police and offenders) and 12,418 charges on the Police Records Information Management Environment (PRIME) database [19]. The number of charges and police contacts demonstrates the drain chronic offenders place on police resources.

A mathematical model that accurately describes the movement of chronic offenders between offending and serving either a jail or community sentence will help police assess their tactics to reduce the resources required to maintain the safety of the general public. The effects of policy changes can be estimated using the mathematical model, avoiding the costs associated with implementing various pilot projects. Policy changes include modifying the frequency or length of jail and community sentences. More specifically, police could employ tactical
teams focussing on subpopulations of the chronic offenders determined by age, crime type or offence rate. Employing tactical teams reduces the number of offences between arrests as the chronic offenders are monitored much more closely.

Chronic offenders are known to constantly commit crimes until arrested. Once arrested, the chronic offender enters the criminal justice system. Chronic offenders are a subset of the general offending population where past offending influences how they move through the criminal justice system. Specifically, chronic offenders have disproportionate numbers of street and administrative convictions compared to typical offenders [3]. Examples of administrative offences include failure to appear in court and failure to comply with court orders. As a result of their past offending, chronic offenders are monitored more closely, increasing the likelihood of being arrested and detained by police. While an offender is in custody, they are given priority in the court system and move more quickly through the court process. If Crown decides to proceed with the charges and the chronic offender is in police detention, they appear at a judicial interim release (JIR) hearing to determine their bail or remand status. As a result of the extensive administrative offences, a chronic offender is more likely to be remanded (held in custody). If they are remanded, their case is given priority until it has concluded with a plea or a finding. At any point during the process, a chronic offender may choose to plead guilty. Assuming a finding of guilt, via guilty plea or trial, the chronic offender is sentenced. We denote the above as the court process.

The sentencing outcome for chronic offenders is unlikely to result in an absolute discharge, conditional discharge, a suspended sentence or a probation only sentence. The majority of chronic offenders receive custody sentences as their past record is an aggravating factor at sentencing. For rehabilitation and supervision considerations, it is likely that a chronic offender has a probation sentence attached to a custody sentence or a conditional sentence. Conditional sentences and probation are sentences that are served in the community with stringent conditions. A chronic offender cannot be given a jail sentence and a conditional sentence for a single crime. However, probation can be given as an additional sentence with a jail or conditional sentence. If a chronic offender breaches (fails) a conditional sentence, they have to appear in court where one possible outcome is the termination of the sentence with the remainder of the sentence being served in a custodial centre. In contrast, the violation of a probation sentence results in new charges.
Custody sentences of less than two years are served in a provincial custody centre whereas custody sentences of two years or more are served in a federal penitentiary. For most custody sentences, the offender only serves 2/3 of their sentence. In provincial custody centres offenders are released after 2/3 of their sentence for remission (i.e., good behaviour), while in federal penitentiaries they have their first statutory release hearing after 2/3 of their sentence is complete. For a complete overview of the criminal justice system refer to Learning Canadian Criminal Procedure [17].

We focus on modelling chronic offenders because they behave in a simple manner. Members of the Chronic Offending Unit (COU) of the VPD assert that this small class of offenders commits crimes until convicted and they immediately return to committing crimes once released. The COU believes the chronic offending population commits the majority of property and drug offences in Vancouver. Understanding the behaviour of chronic offenders allows large municipalities to reduce the resources required to monitor the chronic offenders, improving the overall safety of the public.

1.2 Mathematical Literature Review

We first summarize select papers giving a brief overview of various agent-based models in the current literature on mathematical modelling in criminology. These papers influence the various modelling approaches currently being employed. The mathematical models in the literature have many similarities, yet each has different aspects which often improve the model’s applicability at the cost of complexity and computational expense. First we summarize the work by Short et al. [15], the motivation behind several new models, describing a simple model of spatially distributed residential burglaries. We then discuss two papers, Jones et al. [7] and Pitcher [10], which expand on [15] followed by two papers, Malleson et al. [8] and Groff [4], that introduce different mathematical approaches to modelling similar behaviours. Lastly, we give a brief explanation of the approach we take and how it differs from the existing work.

Current research is aimed at understanding why specific locations within a city or neighbourhood are more susceptible to criminal activity than others. The work of Short et al. [15]
has been highly influential in mathematical modelling of crime hotspots, locations with high crime rates, and their dynamics. Short et al. allows criminals to travel throughout the city where their movement is biased towards high crime locations. The choice for a criminal to burglarize the house is influenced by the attractiveness. Criminals take breaks between burglaries removing the possibility of immediate repeat offending. Short et al. accomplishes this by constructing a simple agent-based model where crime hotspots evolve over time.

The agent-based model incorporates a finite set of agents (criminals) with minimal attributes distributed on a two-dimensional equispaced lattice shown in Figure 1.1. Each point on the lattice is a house which can be burglarized. The work [15] allows the criminals to travel randomly on the lattice with a bias towards attractive houses. At each time step, each offender has the choice to commit a burglary. Their choice is influenced by their current location. If they choose not to burglarize a house then they move to a neighbouring house. The probability of committing a burglary at location \( s \) is

\[
p_s(t) = 1 - e^{-A_s(t)\delta t}
\]

which corresponds to a standard Poisson process with an expected number of crimes \( A_s(t)\delta t \) in a time interval \( \delta t \). Short et al. defines the attractiveness as

\[
A_s(t) = A_0 + B_s(t)
\]

where \( A_0 \) is the constant base attractiveness and \( B_s(t) \) is the dynamical contribution from nearby recent events. The dynamical attractiveness is discrete in both time and space, defined as

\[
B_s(t + \delta t) = \left( 1 - \nu \right) B_s(t) + \frac{\nu}{4} \sum_{s' \sim s} B_{s'}(t) \left( 1 - \omega_1 \delta t \right) + \theta E_s(t),
\]

where \( \nu \in [0,1] \) measures the strength of the neighbourhood effects, \( \omega_1 \) is the time scale on which the effects decay and \( \theta E_s(t) \) is a weighted number of burglaries at location \( s \) at time \( t \). They also denote “\( s' \) is a neighbour of \( s \)” by \( s' \sim s \). The sum over neighbours is based on the broken windows theory which assumes that evidence, or knowledge, of past crimes generates new crimes [20]. Once an offender commits a crime, they are removed from the system, an assumption that cannot be extended to modelling chronic offenders. Chronic offenders typically commit crimes until they have reached a specific threshold, which often
requires multiple offences before leaving an area [19]. If an offender chooses not to commit a
crime, they move from their current location $s$ to a neighbouring house $n$ with a probability

$$q_{s \rightarrow n}(t) = \frac{A_n(t)}{\sum_{s' \sim s} A_{s'}(t)}.$$ 

New agents are uniformly distributed on the lattice at a rate of $\Gamma$ to simulate criminals
returning to the system. Short et al. convert the agent-based model to a system of par-
tial differential equations (PDEs) and perform linear stability analysis to understand the
circumstances under which hotspots appear. One major simplification in the model is the
exclusion of police officers or other deterrents.

![Diagram of an equispaced two-dimensional lattice with grid spacing $\ell$. The point labelled $s$ has
neighbours labelled $s'$.](image)

Figure 1.1: An equispaced two-dimensional lattice with grid spacing $\ell$. The point labelled $s$ has
neighbours labelled $s'$.

Subsequent research, such as the work by Jones et al. [7] or Pitcher [10], adds the effects
of deterrence to the Short et al. agent-based model. Jones et al. investigates the effects
deterrence has on hotspot formation [7]. Criminals continue travelling throughout the city
committing residential burglaries. Police officers move throughout the city deterring crim-
inals. The offenders are only deterred when they are sufficiently close to the police. All
burglaries are assumed to be successful as police do not apprehend offenders.

The agent-based model in [7] has both offenders and police officers travelling about the
lattice. The strategy of how police officers move is a key tool in understanding how hotspots
react to deterrence. The structure of the model is similar to [15] with the addition of police
officers and offenders being able to leave the system without committing a crime. With a probability rate of $\Lambda$ per day, fixed in time and space, an offender leaves the system without committing a burglary. Deterrence from police is modelled in two ways: reducing the attractiveness of homes and persuading criminals to leave the system. The presence of a police officer decreases the attractiveness of their current location resulting in a reduced likelihood of a criminal committing a burglary. The attractiveness of a house with deterrence included is defined as

$$\tilde{A}_s(t) = e^{-\chi\kappa_s(t)}A_s(t), \quad (1.4)$$

where $\kappa_s(t)$ is the number of police officers at location $s$, $\chi \geq 0$ is the effectiveness of the deterrence and $A_s(t)$ is the attractiveness introduced in [15]. The probability of a criminal committing a burglary at location $s$ at time $t$ is

$$p_s(t) = 1 - e^{-\tilde{A}_s(t)\delta t}. \quad (1.5)$$

The second form of deterrence is behaviour modification of criminals. When an offender and a police officer are at the same location, Jones et al. proposes that the offender leaves the system with probability

$$J\kappa_s(t) \over 1 + J\kappa_s(t), \quad J \geq 0. \quad (1.6)$$

The agent-based model in [7] is simulated as in Short et al. with the exception that $A_s(t)$ is replaced with $\tilde{A}_s(t)$ and criminals have the option to leave the system prior to committing a burglary.

This paper studies the effects of deterrence on hotspot pattern formation. Jones et al. investigates three possible policing strategies. The first is a random walk. This describes patrol officers that are not specifically looking for criminals. At each time step, police officers chose a neighbouring house to travel to, without any bias towards a specific location. This strategy is ineffective as police officers are equally likely to travel to low crime areas having little benefit to reducing crime hotspots. The second strategy is having police officers remain on the perimeters of hot spots. This strategy is unrealistic as it requires coordination among many police officers, possibly across multiple jurisdictions and real time information about active crimes. They found this strategy to be effective for reducing large hot spots. The third strategy is to have the officers move in a similar biased walk as the offenders.
This is a more realistic version of the second strategy. It is less effective for large hotspots than for smaller hotspots. The model only incorporates local effects. If an offender is at a neighbouring house as a police officer, we expect that the offender’s decision will be affected. This can not be captured with purely local deterrence.

The work by Pitcher [10], alluded to earlier, is another example of an agent-based model with deterrence and the structure of [15]. Criminals and police continue to travel throughout the city. A criminal commits a burglary if a house is attractive and they are unlikely to be caught. A single burglary may not yield enough wealth for the criminal allowing offenders to continue burglarizing houses immediately following a burglary. Police officers are able to influence a criminal’s behaviour at their location as well as neighbouring locations. In addition to non-local deterrence and repeat offending, Pitcher limits the attractiveness a house can achieve. If a single home has been burglarized repeatedly, it is reasonable to assume all valuables have been removed with insufficient time for the goods to be replaced.

To formulate the above mathematically, Pitcher constructs an agent-based model using the work of [15] with three additions: a fatigue factor, maximum attractiveness and non-local deterrence. The fatigue factor is the time scale of how long an offender remains in the system without finding a suitable target. This replaces the fixed probability rate, $\Lambda$, for an offender to leave the system prior to committing a crime in [10]. Allowing offenders to leave the system only as a result of fatigue allows for repeat offending. Pitcher incorporates a maximum attractiveness by replacing $\theta E_s(t)$ in Equation 1.3 with $\theta E_s(t) \left(1 - \frac{A_s(t)}{k}\right)$, where $k$ is the maximum attractiveness for a house. Similar to [7], Pitcher investigates the effects of deterrence on hotspot formation using a second agent type, police officers. Pitcher incorporates non-local deterrence by allowing the effects of police officers to diffuse spatially to neighbouring locations. Deterrence at location $s$ is updated discretely in time as

$$D_s(t + \delta t) = \left((1 - \zeta) D_s(t) + \frac{\zeta}{4} \sum_{s' \sim s} D_{s'}(t)\right) (1 - \omega_2 \delta t) + \xi U_s(t).$$

(1.7)

Pitcher defines $\zeta \in [0, 1]$ as the strength of the diffusing deterrence, $\omega_2$ as the time scale on which the effects decay and $\xi U_s(t)$ as the weighted number of officers at location $s$ at time $t$. The probability of committing a burglary becomes

$$p_s(t) = 1 - e^{-A_s(t)(1-D_s(t))^{+} \delta t},$$

(1.8)
where \((1 - D_s(t))^+\) is defined as

\[
(1 - D_s(t))^+ = \begin{cases} 
1 - D_s(t) & \text{if } 1 - D_s(t) \geq 0, \\
0 & \text{if } 1 - D_s(t) < 0.
\end{cases}
\]

This implies for sufficiently high deterrence, \(D_s(t) \geq 1\), offenders will not commit burglaries \((p_s(t) = 0)\). This differs from Jones et al. where \(p_s(t) > 0\) for all \(t\). Pitcher assumes the total number of officers is fixed and the number of officers at each location is given by the function \(U_s(t)\). It is determined by the strategy of police reaction to crime patterns. The function is non-negative and satisfies the conservation of officers constraint. Regardless of whether an offender commits a crime, they move to a corresponding neighbour or leave the system if fatigued. As in Short et al. offenders are biased towards attractive locations. Similar to [7], Pitcher examines the impact modifying the police strategies has on crime pattern formation.

Using \(U_s(t)\), Pitcher investigates two different policing strategies. The first is to place the \(M\) police officers at the \(M\) most attractive locations at the time the police officers are incorporated into the simulation. Police officers do not move in time resulting in being effective for short times only. Police officers are no longer in optimal locations after the hotspots diffuse. The second strategy allows the police officers to move one grid point per time unit. Not surprisingly, the two strategies are equally effective for short simulations, while the second strategy is more effective over longer simulations.

We now summarize two papers which take differing approaches to modelling burglaries. Improvements include allowing for diverse agents distributed on domains which better approximate true cities. The following papers are not direct extensions of [15], but aim to model similar behaviours. The first of the two papers, by Malleson et al., models residential burglaries [8] whereas the second paper, by Groff, models street robberies [4].

Malleson et al. studies how housing security and the layout of a city affect residential burglaries. The work [8] assumes burglaries are committed by people who are unable to reach their financial requirements legally. In addition, criminals balance their financial requirements with their need for rest. Malleson et al. considers a city containing residential properties, industrial properties and roads. Once criminals are rested, they travel throughout the city
towards either legal employment or attractive houses to burglarize. Formulating a mathe-
matical model with the above characteristics allows for measuring effects caused by modify-
ing the layout of a city (locations of roads, residential and industrial properties) and security.

The mathematical model is agent-based with two agent types (criminals and citizens) dis-
tributed on a two-dimensional equally spaced lattice where each point is either a road, a
residential property or an industrial property. Agents are either criminals or citizens and
are unable to change their classification. Each agent is linked to a residential property and
a commercial property. The residential property is the agent’s home with one agent per
home. The commercial location is the agent’s work and multiple agents are allowed to share
a single workplace. Each citizen has a job that is fixed and provides sufficient wealth. Each
criminal receives a random amount of wealth each day that is not guaranteed to satisfy
their wealth requirement. A criminal’s commercial location can change daily representing a
temporary job position. Agents travel between their house and work following the shortest
path, constrained to roads, moving one discrete block per time step. Criminals remember
houses they pass including the security level and the attractiveness. The security and at-
tractiveness of a house change over time. If an agent is in the house, the house is secure and
cannot be burglarized. Both housing attractiveness and security increase after a burglary
while decaying in time and diffusing in space post-burglary. The change in attractiveness is
a result of the broken windows effect [20] and the change in security is a result of the victim
securing their house to deter future burglaries. Each morning, the agents are rested and
receive a single day’s pay. As the day progresses, their wealth and energy decay. Civilians
travel to work, work until they need rest, then return home. When a criminal lacks sufficient
wealth, they choose a residential property from their memory to burglarize. The criminal
follows the shortest path. If each residential property on the path to the chosen location is
secure, they choose an alternate property from their memory. Once they find an acceptable
property, they commit the burglary and receive sufficient wealth for the day. Malleson et
al. assumes all burglaries are successful.

Malleson et al. studies the impact on residential burglaries by modifying housing security
and the layout of the city. They found that securing houses individually is insufficient as
criminals choose neighbouring properties. In order to reduce crimes, entire blocks of houses
need increases in security. Preliminary results led the authors to conclude that the layout of
the city is less important as offenders travel further distances to reach desirable properties. Unfortunately, [8] does not explicitly state how the elements of the model are defined and simulated.

Groff incorporates a real map of Seattle, Washington as the spatial domain [4]. Groff studies the effects of people spending more time away from their homes on the frequency of street robberies. For a robbery to take place, multiple people must come in contact with each other and one must be motivated to commit a robbery. In addition, the robbery can only take place if there is a suitable target and the absence of an intervening bystander. Victims are not fixed in space or time. The following outlines how the mathematical model is formulated.

Groff constructs a grid from a map of Seattle, Washington where each point is an intersection. The agent-based model has three agent types: police, civilians and criminals. Each non-police agent is given a percentage of a day spent in their home. Each day, the non-police agents spend the first part of the day in their home. Once they leave their home, they move at random throughout the city. Once the day ends, they return home to rest. The police agents also move at random, initially distributed throughout the city. As in Short et al., at each time step a criminal either commits a crime or moves to a neighbouring intersection. If there is an officer at the same intersection, the offender cannot commit a crime, otherwise

\[ G = N_s - 2 + P_G \]  

(1.9)
determines the deterrence. The number of agents at position \( s \) is \( N_s \) and \( P_G \) is random noise uniformly distributed on \([-2, 2]\) describing the capability of the remaining agents to deter a robbery. Two is subtracted as the offender and target are not counted. The decision to commit a robbery is: if \( G < 1 \) then rob, if \( G = 1 \) then make a random choice with equal probability or if \( G > 1 \) then do not rob. If there are multiple civilians at the location, the choice of which civilian to victimize is the maximum of

\[ S = W_T - W_A + P_S \]  

(1.10)
over all suitable targets. The wealth of the target is \( W_T \), \( W_A \) is the wealth of the active criminal and \( P_S \) is random noise uniformly distributed on \([-1, 1]\). One unit of wealth is transferred from the chosen target to the criminal. The goal of the paper is to understand the relationship between the time spent at home and the amount of crime in the city. Since
civilians are not victimized in their home, it is not surprising that the number of offences decreases as the average time civilians spend in their home increases. In addition, as civilians spend more time away from home, the number of locations with at least one robbery increases.

Much of the agent-based modelling focusses primarily on a single crime type committed by a single class of offenders. For many models, different classes of offenders are distinguished by the choice of model parameters. We focus on the class of offenders termed chronic offenders. The models discussed above are not suitable for modelling chronic offenders as they do not include any form of punishment. Since chronic offenders are involved in a high number of crimes and are monitored closely, they are more likely to be caught and incarcerated than a typical offender. Punishments must be incorporated as the behaviour of a chronic offender changes substantially when they are in jail versus on the street. While the majority of chronic offenders commit property crimes [19], current models of residential burglaries assume offenders will travel far distances to find a more suitable target. This is not true for chronic offenders as they are likely to choose a target nearby even if a more profitable target is available. From conversations with the COU, the work by Groff [4] is not applicable as chronic offenders rarely commit street robberies. Although the current literature is not appropriate, we will use many of the modelling concepts in constructing a more suitable model of chronic offenders.

Chronic offenders are thought to behave in a simple manner as many are unemployed and homeless. These offenders repeatedly commit crimes until convicted, serve their sentence then immediately begin committing crimes again [9, 19]. Chronic offenders are less likely to be influenced by spatial properties than a typical offender. They commit crimes regardless of their current location. As a result, we do not incorporate a spatial domain. Ignoring spatial properties does not prevent the model from measuring the impact on the total crimes committed by chronic offenders as a result of operational policy changes. Also, since we hope to validate the model to police data, incorporating a realistic physical domain would result in a complex model that would be difficult to validate and analyze. We hope that our choice of modelling chronic offenders will allow our initial model to accurately describe their behaviour. In addition, understanding a simple model is necessary before attempting to analyze a more complex model.
Chapter 2

Model Formulation

Chapter 2 describes the formulation of a mathematical model of how chronic offenders change states between freedom and incarceration. A brief overview of agent-based models is presented, followed by the formulation of the chronic offender agent-based model. An outline of the algorithm for the agent-based model is presented to explain the implementation of the agent and model properties. Lastly, we show under appropriate assumptions and limits that the expected output of our agent-based model is a solution of a continuous-time differential equation model.

2.1 Agent-Based Models

An agent-based model has two layers: microscopic and macroscopic. The microscopic layer is the finite set of autonomous agents. Agents are autonomous in the sense that each agent has the power to make decisions. The decisions may be influenced by external properties, but the final decision is left to the agent. Each agent performs various tasks based on a set of predetermined rules. The macroscopic level is the aggregate behaviour of the set of agents. The macroscopic layer is the real life behaviour being modelled. Agents represent the people or objects and have various characteristics that influence their decision processes. In reality, factors behind a decision depend on the interaction between the individual and the situation. For example, a 25-year-old male with a drug addiction is more likely to burglarize a house than a 75-year-old female without a drug addiction. Likewise, agent characteristics relevant for modelling residential burglaries are radically different from relevant characteristics for modelling homicides. Agent-based models are not restricted to humans and can be used to
model any entity with decision-making capabilities.

To understand the behaviour of an agent-based model, simulations are computed numerically. For the agent-based models we discuss, the domains (temporal and/or spatial) are discretized. At each discrete point in time, an agent completes a task that depends on individual properties and external influences. External influences are caused by other agents or the domain. A simulation starts with given initial data (time $t_0$). A small time step forward is taken, say from $t_0$ to $t_0 + \delta t$, and the agents’ characteristics are updated. The agents’ characteristics are updated by looping through all agents with each agent having an opportunity to perform a task. If agents interact, the ordering of the agents impacts the simulation. To avoid this, update all the agents simultaneously through synchronous updates by updating agents at time $t + \delta t$ using the solution at time $t$. After updating each agent, a time step forward is taken. This process is repeated until the final time is reached. When a task an agent performs is chosen at random with a given probability, the agent-based model is not deterministic. Averaging the output of several model simulations gives the expected behaviour of the model.

### 2.1.1 The Chronic Offender Agent-Based Model

The agent-based model has $N$ identical agents (chronic offenders) and three subpopulations. At each point in time, each chronic offender is in one of the three subpopulations:

- free on the street, or
- in jail (serving 2/3 of a custody sentence), or
- serving a community sentence (probation or conditional sentence).

We implement a time-only model. By incorporating time only, we are able to investigate how changes in resources impact subpopulation sizes. Moreover, it allows for estimating the impact of operational policy changes on the total number of crimes committed. Agent characteristics outlined in Table 2.1 include: average time between arrests while on the street, average jail sentence, average community sentence, average time until arrest while serving a community sentence and the probability of receiving a jail sentence. While agents are on the street, they are free to commit crimes until arrested.
CHAPTER 2. MODEL FORMULATION

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_j \in [0, 1]$</td>
<td>Fraction of street arrests resulting in a jail sentence</td>
<td>none</td>
</tr>
<tr>
<td>$\mu_s &gt; 0$</td>
<td>Expected time between arrests on street</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_{c,s} &gt; 0$</td>
<td>Expected community sentence length</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_{c,f} &gt; 0$</td>
<td>Expected time before failure of a community sentence</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_j &gt; 0$</td>
<td>Expected jail sentence length</td>
<td>days</td>
</tr>
<tr>
<td>$\delta t &gt; 0$</td>
<td>Agent-based model time step</td>
<td>days</td>
</tr>
<tr>
<td>$T &gt; 0$</td>
<td>Final time</td>
<td>days</td>
</tr>
</tbody>
</table>

Table 2.1: The model parameters.

We do not explicitly model the court process. We assume all chronic offenders who are arrested are convicted since most cases in Canada conclude in a guilty finding [18]. Once convicted, we model the sentencing process by giving the chronic offender either a jail or a community sentence. A fixed proportion of all sentences are served in jail, while the remaining are served in the community. The length of the sentence depends on whether the sentence is served in jail or in the community. If a chronic offender is given a jail sentence, they are unable to commit crimes until their sentence expires. We assume that all chronic offenders return to the offending population on the street at the conclusion of their jail sentence. While serving a community sentence, the agents are able to commit crimes. However, the punishment for being arrested for committing a crime during their community sentence is jail. If a chronic offender successfully completes their community sentence, they return back to the offending population on the street.

We assume chronic offenders only receive one type of sentence per crime. That is, a chronic offender cannot receive a jail sentence followed by a community sentence. As well, we assume a sentence does not carry over if a chronic offender is arrested of a new crime. In this case, they are given a new sentence and do not serve the remainder of the original sentence.

Computing Simulations

In this section, we outline how the agent-based model characteristics are defined numerically. The structure of the agent-based model has each agent serving time in a subpopulation for a predetermined length. This is achieved numerically by assuming the time spent in a subpopulation is sampled from a known distribution. We assume that both jail sentences
and the time between arrests are single variable or univariate distributions with non-negative support known as survival distributions. This means only non-negative sentences have a non-zero probability. If the probability of the chronic offender receiving a sentence of length \( x \leq a \) is

\[
Pr \{ x \leq a \} = F(a),
\]

then \( F(x) \) is the cumulative density function (CDF) of the distribution. We use a bivariate or joint distribution for community sentences and time between arrests while serving a community sentence. Using a bivariate distribution allows for the time to fail a community sentence to depend on the total length of the sentence. We say \( F(x, y) \) is the CDF of a bivariate distribution if

\[
Pr \{ x \leq a, \ y \leq b \} = F(a, b).
\]

We assume that a community sentence and the time between arrests while serving a community sentence are independent. Thus the CDF for the bivariate distribution is the product of the CDFs of the independent distributions, i.e. \( F(a, b) = F_x(a)F_y(b) \). The notation for the distributions is outlined in Table 2.2.

<table>
<thead>
<tr>
<th>Description</th>
<th>Distribution</th>
<th>CDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution of time between street arrests</td>
<td>( D_s )</td>
<td>( F_s(a) )</td>
</tr>
<tr>
<td>Distribution of jail sentences</td>
<td>( D_j )</td>
<td>( F_j(a) )</td>
</tr>
<tr>
<td>Joint distribution of time to pass/fail a community sentence</td>
<td>( D_c )</td>
<td>( F_c(a, b) )</td>
</tr>
<tr>
<td>Independent distribution of community sentences</td>
<td>( D_{c,s} )</td>
<td>( F_{c,s}(a) )</td>
</tr>
<tr>
<td>Independent distribution of time to fail a community sentence</td>
<td>( D_{c,f} )</td>
<td>( F_{c,f}(b) )</td>
</tr>
</tbody>
</table>

Table 2.2: The notation for the distributions in the agent-based model. Both \( a \) and \( b \) are numbers of days and are assumed positive.

The initial data includes the initial subpopulation and the first transition time for each chronic offender \( (N \text{ total}) \). We step forward in time, updating the chronic offenders’ states at discrete times until the final time is achieved. The agent-based model does not estimate the physical location of each chronic offender, but simply their subpopulation affiliation. This reduces the computational time required for each simulation while predicting the behaviour of chronic offenders. To describe the rules on how chronic offenders move between
subpopulations, we first describe how chronic offenders behave once released.

When a chronic offender is released, we sample $D_s$ to give a time until the next arrest. The chronic offender is allowed to commit a crime at each time step until this time has elapsed. At this time, the chronic offender is removed from the street and given their sentence. If they are given a jail sentence, we sample $D_j$ and the chronic offender is unable to commit crimes until they are released. If the chronic offender is given a community sentence we sample $D_c$. Sampling $D_c$ produces two values, one for a community sentence and one for the time until their next arrest. We fix the time the chronic offender is released with the minimum of the two values and ignore the larger value. If the minimum value corresponds to completing the community sentence, the chronic offender returns to the street. Otherwise, it corresponds to failing the community sentence and the chronic offender is placed in jail. We choose the minimum time as this corresponds to the event that first takes place. This assumes every chronic offender fails their community sentence if given a long enough sentence. In addition, we classify the unlikely event that the two values are the same as a failed community sentence.

Figure 2.1 is a Unified Modelling Language (UML) activity diagram of the numerical algorithm we use for a single simulation. Activity diagrams are a visual representation of the logic and flow of a process [12]. The choice of the various distributions is derived from data or approximated from literature, making the agent-based model both adaptable and flexible.

### 2.2 Continuous Approximation

We are interested in the expected number of chronic offenders in each subpopulation. This is obtained by averaging several simulations of the agent-based model. As a result, simulating the agent-based model is computationally expensive. To avoid this, we make assumptions that lead to a continuous approximation of the agent-based model. The approximation is a system of differential equations that can be solved either numerically or analytically. We define $s(t)$, $j(t)$, $c(t)$ as the fraction of chronic offenders on the street, in jail and serving a community sentence, respectively. The continuous model comes from taking the following limits: the time step, $\delta t \to 0$ and the number of agents, $N \to \infty$. We first study the case
CHAPTER 2. MODEL FORMULATION

For Each Chronic Offender
Check Street Status
Check Jail Status
Check Community Status
Update Time Left
Update Time Left
Update Time Left
Receive Sentence
Set Jail Sentence
Set Community Sentence
Set Time to Next Conviction
Check if Failed Community Sentence
Set Jail Sentence
Set Time To Next Conviction

Visual Paradigm for UML Community Edition [not for commercial use]

Figure 2.1: The UML activity diagram for the agent-based model. The diagram describes the process for computing a single agent-based model simulation. The green box is computed for each chronic offender simultaneously.
CHAPTER 2. MODEL FORMULATION

when the times spent in each state follow Poisson processes with differing rates between subpopulations. We then investigate how the model and end behaviour change by changing the distribution for jail sentences to the Delta distribution.

2.2.1 Using Poisson Processes

We assume chronic offenders leave each state following a Poisson process with rates

\[ \frac{1}{\mu_s}, \frac{1}{\mu_j}, \frac{1}{\mu_{c,s}}, \frac{1}{\mu_{c,f}}, \]

as described in Table 2.1. The Poisson process \( n(t) \) with rate \( \mu > 0, \ t \geq 0, \) is a process satisfying

1. \( n(0) = 0, \)

2. the process has stationary and independent increments,

3. \( P \{ n(\delta t) = 1 \} = \mu \delta t + o(\delta t), \)

4. \( P \{ n(\delta t) \geq 2 \} = o(\delta t), \)

where \( n(t) \) represents the total number of events that occur before time \( t \) [11, p. 305]. A function, \( f(\delta t) \), is on the order of “little-o of \( \delta t \)”, \( o(\delta t) \), if

\[ \lim_{\delta t \to 0} \frac{f(\delta t)}{\delta t} = 0, \]

which intuitively means \( f(\delta t) \) tends to zero faster than \( \delta t \) [11]. A Poisson process with rate \( \mu \) has events (agents leaving) exponentially distributed in time with mean \( \frac{1}{\mu} \) [11]. Table 2.3 describes the CDFs used for each subpopulation. For community sentences, we assume independent exponential distributions.

We use the strategy outlined in [5] to show that the continuous approximation to the agent-based model using Poisson processes is a system of ODEs. We denote \( X_t = (s_t, j_t, c_t) \), as a vector of the number of chronic offenders in each state at time \( t \). We write the probability
Table 2.3: The density functions for the agent-based model with Poisson processes. Both \( a \) and \( b \) are numbers of days and are assumed positive.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Cumulative Density Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_s )</td>
<td>( F_s(a) = 1 - e^{-\frac{a}{\mu_s}} )</td>
</tr>
<tr>
<td>( D_j )</td>
<td>( F_j(a) = 1 - e^{-\frac{a}{\mu_j}} )</td>
</tr>
<tr>
<td>( D_c )</td>
<td>( F_c(a, b) = F_{c,s}(a) F_{c,f}(b) )</td>
</tr>
<tr>
<td>( D_{c,s} )</td>
<td>( F_{c,s}(a) = 1 - e^{-\frac{a}{\mu_{c,s}}} )</td>
</tr>
<tr>
<td>( D_{c,f} )</td>
<td>( F_{c,f}(b) = 1 - e^{-\frac{b}{\mu_{c,f}}} )</td>
</tr>
</tbody>
</table>

From the definition of a Poisson process [11], term (a) can be written as

\[
\Pr\{X_{t+\delta t} = x\} = \left(\Pr\{X_t = x\} \text{ and no agents move in } (t, t+\delta t)\right) \\
+ \left(\Pr\{X_{t+\delta t} = x\} \text{ and one agent moves in } (t, t+\delta t)\right) \\
+ \left(\Pr\{X_{t+\delta t} = x\} \text{ and multiple agents move in } (t, t+\delta t)\right), \tag{2.1}
\]

and term (c) is \( o(\delta t) \). The expansion of terms (a) and (c) follows directly from the definition of a Poisson process. The expansion of term (b) uses the conservation of chronic offenders, \( (s + j + c = 1) \), along with the definition of a Poisson process. If we are at state \( \langle s, j, c \rangle \) at time \( t + \delta t \), and one chronic offender moves, they must arrive from one of the other two
states. For example,
\[
\Pr \{ X_t = (s + 1, j, c - 1) \} = \frac{1 - \chi_j}{\mu_s} (s + 1)
\]
states that at time \( t \), we have \( s + 1 \) chronic offenders on the street, \( j \) chronic offenders in jail, and \( c - 1 \) serving a community sentence. In order to be at state \( (s, j, c) \) at time \( t + \delta t \), we need one chronic offender to be arrested and given a community sentence. The rate chronic offenders leave the street is \( \frac{1}{\mu_s} \) with the proportion of \( 1 - \chi_j \) receiving a community sentence. Since there are \( s + 1 \) chronic offenders on the street, there are \( s + 1 \) candidates to move from the street to community. Term (b) is the sum of all possible combinations.

To calculate the ODE corresponding to the expected size of subpopulation \( x_i \), we multiply Equation 2.1 using Equations 2.2 and 2.3 by \( x_i \) and sum over \( x \). For example, the expected number of chronic offenders on the street satisfies,
\[
\frac{E[s_{t+\delta t}] - E[s_t]}{\delta t} = \frac{1 - \chi_j}{\mu_s} E[s_t] (s_t - 1) + \frac{\chi_j}{\mu_s} E[s_t] (s_t - 1) + \frac{1}{\mu_j} E[j_t] (s_t + 1) \\
+ \frac{1}{\mu_{c,s}} E[c_t] (s_t + 1) + \frac{1}{\mu_{c,f}} E[c_t] - \frac{1 - \chi_j}{\mu_s} E[s_t^2] - \frac{\chi_j}{\mu_s} E[s_t^2] \\
- \frac{1}{\mu_j} E[j_t s_t] - \frac{1}{\mu_{c,s}} E[c_t s_t] - \frac{1}{\mu_{c,f}} E[c_t s_t]. \tag{2.4}
\]
Using the identity \( E[A + B] = E[A] + E[B] \), Equation 2.4 simplifies to the Forward Euler numerical scheme for a first order ODE. We repeat the same calculations for both chronic offenders in jail and serving community sentences. If we rescale the variables by the total number of chronic offenders \( (N) \),
\[
s(t) = \frac{E[s_t]}{N}, \quad j(t) = \frac{E[j_t]}{N}, \quad c(t) = \frac{E[c_t]}{N}
\]
and take the appropriate limits \( (\delta t \to 0 \text{ and } N \to \infty) \) we attain the following system of ODEs:
\[
\frac{ds}{dt} = -\frac{1}{\mu_s} s(t) + \frac{1}{\mu_j} j(t) + \frac{1}{\mu_{c,s}} c(t), \tag{2.5}
\]
\[
\frac{dj}{dt} = \frac{\chi_j}{\mu_s} s(t) - \frac{1}{\mu_j} j(t) + \frac{1}{\mu_{c,f}} c(t), \tag{2.6}
\]
\[
\frac{dc}{dt} = \frac{1 - \chi_j}{\mu_s} s(t) - \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right) c(t). \tag{2.7}
\]
The system of ODEs (Equations 2.5, 2.6 and 2.7) describes flows between subpopulations, each with some given rate shown in Figure 2.2.

We note

\[
\frac{ds}{dt} + \frac{dj}{dt} + \frac{dc}{dt} = 0 \implies s(t) + j(t) + c(t) = s(0) + j(0) + r(0) = 1, \quad \forall t > 0,
\]

showing conservation of chronic offenders. The restriction on the initial data is from the assumption that all chronic offenders start in one of the three locations. The limit \( N \to \infty \), where \( N \) is the number of chronic offenders, ensures the output of the continuous model is a continuous density.

### 2.2.2 Using Deterministic Jail Sentences

We now assume jail sentences are no longer exponentially distributed, but instead that all jail sentences are \( \mu_j \) days. Many terms remain the same as in the ODE model except for the terms related to releasing chronic offenders from jail. These terms become delayed terms corresponding to releasing the fraction of chronic offenders expected to be given a jail sentence \( \mu_j \) days ago. The expected fraction of chronic offenders arrested on the street and given a jail sentence at time \( t \) is \( \frac{\chi_j}{\mu_s} s(t) \). Since jail sentences are exactly \( \mu_j \) days long, at time \( t + \mu_j \) these chronic offenders will be released back to the street. Therefore at time \( t \), the expected fraction of chronic offenders that are released to the street from sentences for crimes on the street is \( \frac{\chi_j}{\mu_s} s(t - \mu_j) \). The expected fraction of offenders being released from
jail for crimes while serving a community sentence is constructed in a similar fashion. The expected fraction of chronic offenders in each state becomes

\[
\frac{ds}{dt} = -\frac{1}{\mu_s} s(t) + \frac{\chi_j}{\mu_s} s(t - \mu_j) + \frac{1}{\mu_{c,s}} c(t) + \frac{1}{\mu_{c,f}} c(t - \mu_j),
\]

(2.8)

\[
\frac{dj}{dt} = \frac{\chi_j}{\mu_s} s(t) - \frac{\chi_j}{\mu_s} s(t - \mu_j) + \frac{1}{\mu_{c,f}} c(t) - \frac{1}{\mu_{c,f}} c(t - \mu_j),
\]

(2.9)

\[
\frac{dc}{dt} = 1 - \frac{\chi_j}{\mu_s} s(t) - \left(\frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}}\right) c(t).
\]

(2.10)

We note that conservation of chronic offenders remains \((s(t) + j(t) + c(t) = 1)\). Since the continuous model is now a system of delayed differential equations (DDE), the initial data required is on an interval of length \(\mu_j\), say \([t_0 - \mu_j, t_0]\), and must satisfy

\[s(t) + j(t) + r(t) = 1, \quad \forall t \in [t_0 - \mu_j, t_0].\]

Data is required on an interval of length \(\mu_j\) as the system of DDEs references the solution up to \(\mu_j\) days in the past.

### 2.2.3 Model Parameters

The agent characteristics, the expected time in each state and the probability of receiving a jail sentence, correspond to parameters in the continuous models. Table 2.1 outlines the model parameters. We estimate the parameter values utilizing current literature. The estimates for \(\mu_s\) and \(\mu_j\) are from [19]. Although \(\mu_s\) is not explicitly given in [19], we estimate it by

\[
\mu_s = \frac{\text{Length of fixed time interval (days)}}{\text{Average number of convictions during the fixed time interval}}.
\]

This is a common approximation in the literature, but it is known to overestimate the true value [9]. This occurs as a result of assuming no restriction on the times that offences occur. In reality, no offences occur while a chronic offender is in jail. We approximate \(\mu_j\) as the sum of the average time in remand and the average jail sentence. We are unable to get an estimate for \(\mu_{c,s}\) for chronic offenders, thus we approximate \(\mu_{c,s}\) from the average length of probation for the general population convicted of property crimes [18]. To approximate \(\chi_j\)
we double the estimate for the general population in [18] as chronic offenders are more likely to receive a jail sentence [19]. Since chronic offenders are unlikely to modify their behaviour while serving a community sentence [19], we estimate $\mu_{c,f}$ as $\mu_s$. Table 2.4 summarizes our choices for parameter estimates.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_s$</td>
<td>47</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>37</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_{c,s}$</td>
<td>443</td>
<td>days</td>
</tr>
<tr>
<td>$\mu_{c,f}$</td>
<td>47</td>
<td>days</td>
</tr>
<tr>
<td>$\chi_j$</td>
<td>0.8</td>
<td>none</td>
</tr>
</tbody>
</table>

Table 2.4: The model parameter estimates.

2.3 Summary

We formulated a single agent-based model describing the movement of chronic offenders between the street and serving a sentence. Sentences are served in the community or in jail. The agent-based model requires distributions for the time spent in each state. The choice of distributions leads to differing continuous approximations. When the times in each state are exponentially distributed, the continuous approximation is a system of linear ODEs (Equations 2.5, 2.6 and 2.7). Introducing deterministic jail sentences leads to a system of DDEs (Equations 2.8, 2.9 and 2.10) as the continuous approximation.
Chapter 3

Model Analysis

Chapter 3 presents the analysis of the mathematical models described in Chapter 2. We analyze the agent-based model with the two choices of distributions (deterministic and exponentially distributed jail sentences) and their corresponding continuous approximations. To understand the long time behaviour of the model, we solve for the steady state solution and determine its stability. Simulations of both versions of the agent-based model as well as solutions to the continuous models are presented. The system of ODEs is solved analytically and the system of DDEs is solved numerically.

3.1 Ordinary Differential Equation Model

To understand the dynamics of the ODE model, we first examine the steady states. Steady states give the long time behaviour of the model. Studying the stability of the steady states allows us to determine which initial values converge to a steady state for a given a set of model parameters. When multiple steady states exist, stability describes the relationship between the steady states and the initial data. We present the analytical solution to the system of ODEs and compare it to the agent-based model with exponentially distributed jail sentences.

3.1.1 Steady State Solution

Our original system of ODEs consists of three equations, but we are able to eliminate one of the equations using the conservation of chronic offenders \((s(t) + j(t) + c(t) = 1)\). We
notice that Equation 2.5 is a linear combination of Equations 2.6 and 2.7. By setting \( s(t) = 1 - j(t) - c(t) \) we eliminate Equation 2.5 and Equations 2.6 and 2.7 become the reduced system of equations. They are

\[
\frac{dj}{dt} = -\left( \frac{\chi_j}{\mu_s} + \frac{1}{\mu_j} \right) j(t) + \left( \frac{1}{\mu_{c,f}} - \frac{\chi_j}{\mu_s} \right) c(t) + \frac{\chi_j}{\mu_s}, \tag{3.1}
\]

\[
\frac{dc}{dt} = -\left( 1 - \frac{\chi_j}{\mu_s} \right) j(t) - \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} + \frac{1 - \chi_j}{\mu_s} \right) c(t) + \frac{1 - \chi_j}{\mu_s}. \tag{3.2}
\]

We denote \( \mathbf{y} = \langle j(t), c(t) \rangle^T \) and write the reduced system of ODEs as

\[
\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{b}, \tag{3.3}
\]

where

\[
A = \begin{bmatrix}
-\left( \frac{\chi_j}{\mu_s} + \frac{1}{\mu_j} \right) & -\left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} + \frac{1 - \chi_j}{\mu_s} \right) \\
-\frac{1 - \chi_j}{\mu_s} & -\left( \frac{1}{\mu_{c,s}} + \frac{1 - \chi_j}{\mu_s} \right)
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
\frac{\chi_j}{\mu_s} \\
\frac{1 - \chi_j}{\mu_s}
\end{bmatrix}.
\]

A steady state, \( \langle \bar{j}, \bar{c} \rangle \), is a solution that satisfies \( \frac{dj}{dt} = \frac{dc}{dt} = 0 \). This simply says that the rate of change is zero. Therefore, the steady state satisfies

\[
a_{1,1}\bar{j} + a_{1,2}\bar{c} + b_1 = 0,
\]

\[
a_{2,1}\bar{j} + a_{2,2}\bar{c} + b_2 = 0,
\]

where \( a_{i,j} \) is the element of \( A \) in the \( i^{th} \) row and \( j^{th} \) column and \( b_i \) is the \( i^{th} \) element of \( \mathbf{b} \). The solution is

\[
\bar{j} = \frac{a_{2,2}b_2 - a_{2,1}b_1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}, \tag{3.4}
\]

\[
\bar{c} = \frac{a_{2,1}b_1 - a_{1,1}b_2}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}, \tag{3.5}
\]

assuming \( \det(A) \neq 0 \). We compute the determinant of \( A \),

\[
\det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = \frac{\chi_j}{\mu_s\mu_{c,s}} + \frac{1}{\mu_j\mu_{c,s}} + \frac{1 - \chi_j}{\mu_j\mu_s} + \frac{1}{\mu_s\mu_{c,f}} > 0,
\]

showing the system (Equations 3.1 and 3.2) has a unique steady state. The steady state for chronic offenders on the street is simply

\[
\bar{s} = 1 - \bar{j} - \bar{c}. \tag{3.6}
\]
CHAPTER 3. MODEL ANALYSIS

We need to ensure that the steady state is a feasible solution, $\bar{s}, \bar{j}, \bar{c} \in [0,1]$. First, we observe

$$a_{1,2}b_2 - a_{2,2}b_1 = \frac{1}{\mu_s \mu_{c,f}} + \frac{\chi_j}{\mu_s \mu_{c,s}} > 0$$

since model parameters are assumed to be positive. This shows $\bar{j} > 0$. We note

$$a_{2,1}b_1 - a_{1,1}b_2 = \frac{1 - \chi_j}{\mu_j \mu_s} \geq 0$$

since $\chi_j \leq 1$ showing $\bar{c} \geq 0$. Having $\bar{j} > 0$ and $\bar{c} \geq 0$ implies $\bar{s} < 1$. For $\bar{s} \geq 0$, we need $\bar{j} + \bar{c} \leq 1$. We showed above that for $\bar{j}$ and $\bar{c}$ the numerators are non-negative and denominators are positive. Summing $\bar{j}$ and $\bar{c}$ gives

$$\bar{j} + \bar{c} = \frac{a_{1,2}b_2 - a_{2,2}b_1 + a_{2,1}b_1 - a_{1,1}b_2}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}.$$

When we subtract the numerator from the denominator and simplify, we obtain

$$a_{1,1}a_{2,2} - a_{1,1}a_{2,1} - a_{1,2}b_2 + a_{2,2}b_1 - a_{2,1}b_1 + a_{1,1}b_2 = \frac{1}{\mu_j \mu_{c,f}} + \frac{1}{\mu_j \mu_{c,s}} = \frac{\mu_{c,s} + \mu_{c,f}}{\mu_j \mu_{c,s} \mu_{c,f}} > 0,$$

giving $\bar{j} + \bar{c} < 1$ which implies $\bar{s} > 0$. Since $\bar{s} > 0$, it follows $\bar{j}, \bar{c} < 1$. To see this, without loss of generality, suppose $\bar{j} = 1$. Looking at $\bar{s}$,

$$\bar{s} = 1 - \bar{j} - \bar{c} = 1 - 1 - \bar{c} = -\bar{c} \leq 0,$$

a contradiction. This completes the proof, showing $(\bar{s}, \bar{j}, \bar{c})$ is a feasible solution. In fact, the steady states satisfy

$$0 < \bar{s} < 1, \ 0 < \bar{j} < 1, \ 0 \leq \bar{c} < 1.$$

This implies the only subpopulation which can become extinct are the chronic offenders serving a community sentence.

We have shown if the initial conditions satisfy the steady states, our solution remains constant. Since we are not guaranteed to be given initial conditions satisfying the steady state, we need to investigate the model behaviour for initial conditions away from the steady state solution. This is a question of stability which is the next step in understanding the ODE model.
3.1.2 Stability Analysis

We study the stability of the steady state by looking at the eigenvalues of \( A \), described in [1, 16]. The forcing term, \( \vec{b} \), does not play a role in the stability of the solution as the particular solution corresponding to the non-homogeneous system is constant, as described in Section 3.1.3. Since \( A \) is a \( 2 \times 2 \) matrix, it has two eigenvalues, \( \lambda_+, \lambda_- \). The \( \pm \) corresponds to the sign in the quadratic equation when we solve for the eigenvalues (Equation 3.8). If the real part of both eigenvalues, \( \Re(\lambda_-) \) and \( \Re(\lambda_+) \), is negative the steady state is stable [1, 16].

An eigenvalue, \( \lambda \), satisfies
\[
A \vec{v} = \lambda \vec{v}, \quad \vec{v} \in \mathbb{R}^2, \quad \lambda \in \mathbb{C},
\]
where \( \vec{v} \) is the eigenvector paired with \( \lambda \). To solve for the eigenvalue, we need \((A - I_2 \lambda) \vec{v} = 0\) to result in non-trivial solutions where \( I_2 \) is the \( 2 \times 2 \) identity matrix. For this to hold true, we need
\[
\det (A - I_2 \lambda) = 0,
\]
giving
\[
(\frac{\chi_j}{\mu_s} + 1 + \lambda) \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} + \frac{1 - \chi_j}{\mu_s} + \lambda \right) + \left( \frac{1 - \chi_j}{\mu_s} \right) \left( \frac{1}{\mu_{c,f}} - \frac{\chi_j}{\mu_s} \right) = 0,
\]
which expands to
\[
\lambda^2 + \left( \frac{1}{\mu_j} + \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} + \frac{1}{\mu_s} \right) \lambda + \left( \frac{1}{\mu_s \mu_{c,f}} + \frac{\chi_j}{\mu_s \mu_{c,s}} + \frac{1}{\mu_j \mu_{c,s}} + \frac{1}{\mu_j \mu_{c,f}} + \frac{1 - \chi_j}{\mu_j \mu_s} \right) = 0.
\]
In the general form, the eigenvalues satisfy,
\[
\det (A - I_2 \lambda) = \lambda^2 + C_1 \lambda + C_2 = 0, \quad C_1, C_2 \in \mathbb{R}. \quad \text{(3.7)}
\]
Using the quadratic formula, the eigenvalues are
\[
\lambda_{\pm} = \frac{-C_1 \pm \sqrt{C_1^2 - 4C_2}}{2}. \quad \text{(3.8)}
\]
Since our model parameters are assumed to be positive, we note \( C_1, C_2 > 0 \). From \( C_1 > 0 \) it follows that \( \Re(\lambda_-) < 0 \). If \( C_1^2 - 4C_2 \leq 0 \), then
\[
\Re(\lambda_+) = -\frac{C_1}{2} < 0.
\]
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We now suppose that $C_1^2 - 4C_2 > 0$. Using $C_2 > 0$, we note

$$C_1^2 - 4C_2 < C_1^2.$$ 

Both sides of the inequality are positive allowing us to take the square root,

$$\sqrt{C_1^2 - 4C_2} < C_1,$$

giving

$$\lambda_+ = \frac{-C_1 + \sqrt{C_1^2 - 4C_2}}{2} < 0.$$ 

This shows that $\Re(\lambda_+)$ and $\Re(\lambda_-)$ are always negative for realistic model parameters. That is, for positive model parameters and any choice of feasible initial data, the long time behaviour is given by the steady state.

3.1.3 Analytical Solution

To solve the system of ODEs analytically, we write $\vec{y}(t)$ as

$$\vec{y}(t) = \vec{y}_p(t) + \vec{y}_h(t)$$

where $\vec{y}_p(t)$ is the particular solution and $\vec{y}_h(t)$ is the general homogenous solution to Equation 3.3. Individually they satisfy

$$\frac{d\vec{y}_p}{dt} = A\vec{y}_p + \vec{b} \quad \text{and} \quad \frac{d\vec{y}_h}{dt} = A\vec{y}_h. \quad (3.9)$$

to find the particular solution we set $\vec{y}_p(t) \equiv \vec{y}_p \in \mathbb{R}^2$. This leads to,

$$Ay_p = -\vec{b} \Rightarrow y_p = -A^{-1}\vec{b}. \quad (3.10)$$

From Section 3.1.1, $\det(A) \neq 0$, thus $A^{-1}$ is guaranteed to exist. To find the general homogeneous solution, we assume $A$ is diagonalizable, which follows if $\lambda_+$, $\lambda_-$ are distinct. For our choice of parameters,

$$\lambda_+ \approx -0.029 \neq -0.043 \approx \lambda_-, $$

allowing for $A$ to be written as

$$A = PDP^{-1},$$
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where $P$ and $D$ are the following

$$P = \begin{bmatrix} \vec{v}_+ & \vec{v}_- \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}. \quad (3.11)$$

The homogeneous problem becomes

$$\frac{d\bar{y}_h}{dt} = PDP^{-1}\bar{y}_h \Rightarrow P^{-1}\frac{d\bar{y}_h}{dt} = DP^{-1}\bar{y}_h.$$ 

If we set $\vec{x} = P^{-1}\bar{y}_h$, the system is now decoupled in terms of $\vec{x}$,

$$\frac{d\vec{x}}{dt} = D\vec{x} \Rightarrow \frac{dx_1}{dt} = \lambda_+ x_1, \quad \frac{dx_2}{dt} = \lambda_- x_2.$$ 

Solving the first order decoupled system gives

$$x_1(t) = d_1 e^{\lambda_+ t} \quad \text{and} \quad x_2(t) = d_2 e^{\lambda_- t},$$

resulting in

$$\bar{y}_h = P\vec{x} = d_1 \vec{v}_+ e^{\lambda_+ t} + d_2 \vec{v}_- e^{\lambda_- t}, \quad d_1, d_2 \in \mathbb{R}.$$ 

Summing Equations 3.9 and 3.10, the solution to Equation 3.3 is

$$(j(t), c(t))^T = d_1 \vec{v}_+ e^{\lambda_+ t} + d_2 \vec{v}_- e^{\lambda_- t} - A^{-1}\vec{b}, \quad \text{with } s(t) = 1 - j(t) - c(t). \quad (3.12)$$

The initial conditions for $j(t), c(t)$ determine $d_1, d_2 \in \mathbb{R}$. For model parameters where $A$ is no longer diagonalizable, extra caution is required. Regardless of $A$ being diagonalizable, the steady state from the analytical solution is given by

$$\lim_{t \to \infty} \bar{y}(t) = -A^{-1}\vec{b}$$

since $\Re(\lambda_+)$ and $\Re(\lambda_-) < 0$. This matches Equations 3.4 and 3.5 as expected.

3.1.4 Numerical Results

We compare the analytical ODE solution with simulations of the agent-based model with exponentially distributed jail sentences to confirm our analysis. The agent-based model consists of 379 agents, the number of chronic offenders identified in Vancouver in 2006 [19], initially on the street. The values used for the remaining model parameters, the average time
in each state and the probability of receiving a jail sentence, are defined in Table 2.4. Each simulation has a final time of $T = 180$ days with a time step of $\delta t = 0.05$ days. Figure 3.1 shows the variance between simulations by plotting 50 simulations as well as the averaged output. We compute 500 agent-based model simulations to compare to the analytical ODE solution. The averaged output is compared to the ODE solution in Figure 3.3. We also compute the relative error between solutions,

$$E_s^{500} = \frac{|s_{\text{ODE}}(T) - s_{\text{agent}}^{500}(T)|}{s} = 1.2 \times 10^{-3},$$

$$E_j^{500} = \frac{|j_{\text{ODE}}(T) - j_{\text{agent}}^{500}(T)|}{j} = 4.1 \times 10^{-3},$$

$$E_c^{500} = \frac{|c_{\text{ODE}}(T) - c_{\text{agent}}^{500}(T)|}{c} = 6.8 \times 10^{-3},$$

and see that the ODE solution and the averaged agent solution match and converge as expected. The error $E_{xN}$ compares the agent-based solution, $x_{\text{agent}}^N(T)$, averaged over $N$ simulations against the continuous solution, $x_{\text{ODE}}(T)$, at time $T$ for state $x$.

Figure 3.1: Simulations of the agent-based model with all agents initially on the street. Time in each subpopulation is exponentially distributed. The single simulations (blue) show the variance between model runs with the average (red) converging to a steady state. The fraction of chronic offenders on the street is found by using the conservation of chronic offenders.
Figure 3.2: All chronic offenders initially on the street

Figure 3.3: The analytical solution to the system of ODEs and the agent-based model averaged over 500 simulations with all chronic offenders initially on the street. The time in each subpopulation is exponentially distributed. Both solutions match and converge to the steady state as expected. The fraction of chronic offenders on the street is calculated using the conservation of chronic offenders.

3.2 Delayed Differential Equation Model

The analysis of the DDE model (Equations 2.8, 2.9 and 2.10) is similar to the analysis performed on the ODE model in Section 3.1. We study the steady state solutions and their stability. The change from the ODE model to the DDE model greatly complicates the stability analysis as there no longer exists a single steady state for the DDE model. Numerics suggest that the DDE model has multiple stable steady states while the agent-based model only has one. A possible explanation for the discrepancy between the agent-based model and the DDE model is discussed. Finally, we restrict allowable initial data and present a modified DDE model. Numerical results for both DDE models along with the agent-based model are presented to confirm our findings.

3.2.1 Agent-Based Model Simulations

We average the agent-based model output over 500 simulations, each with a final time of \( T = 180 \) days and time step of \( \delta t = 0.05 \) days. The agent-based model consists of 379 agents, the number of chronic offenders identified in Vancouver in 2006 [19] initially on the
street. The values for the remaining model parameters, the average time in each state and the probability of receiving a jail sentence are defined in Table 2.4. Figure 3.4 shows the variation between 50 individual simulations as well as their averaged output. The agent-based model is converging to a steady state when all chronic offenders are initially on the street.

![Figure 3.4: Simulations of the agent-based model with all agents initially on the street. Jail sentences are deterministic. The single simulations (blue) show the variance between model runs with the average (red) converging to a steady state. The fraction of chronic offenders on the street is found by using the conservation of chronic offenders.](image)

To find the steady states of the system of DDEs, we assume the system approaches a constant, \((\bar{s}, \bar{j}, \bar{c})\) [2]. If we set \(s(t) = \bar{s}\), \(j(t) = \bar{j}\) and \(c(t) = \bar{c}\), Equations 2.8, 2.9 and 2.10 lead to

\[
-\frac{1 - \chi_j}{\mu_s} \bar{s} + \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right) \bar{c} = 0, \tag{3.13}
\]

\[
\frac{\chi_j}{\mu_s} \bar{s} - \frac{\chi_j}{\mu_s} \bar{s} + \frac{1}{\mu_{c,f}} \bar{c} - \frac{1}{\mu_{c,f}} \bar{c} = 0, \tag{3.14}
\]

\[
\frac{1 - \chi_j}{\mu_s} \bar{s} - \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right) \bar{c} = 0. \tag{3.15}
\]
Equation 3.14 is satisfied for any choice of $\bar{s}$, $\bar{c}$ and Equation 3.13 is equivalent to Equation 3.15. Even with the conservation of chronic offenders constraint, $\bar{s} + \bar{j} + \bar{c} = 1$, the DDE model has one free variable resulting in an infinite number of steady states. This corresponds to the linear system (Equations 3.13, 3.14 and 3.15) having a zero eigenvalue. Unlike the ODE model where we only have one steady state (non-zero eigenvalues), we need to check the stability for each steady state. Before attempting this analysis, we first look at the steady state of the agent-based model.

We derive the number of chronic offenders we expect in each state for the agent-based model, labeled $\tilde{s}$, $\tilde{j}$, and $\tilde{c}$. Using the conservation of chronic offenders we write the expected number of chronic offenders on the street, $\tilde{s}$, as

$$\tilde{s} = N - \tilde{j} - \tilde{c}. \quad (3.16)$$

The expected number of chronic offenders in jail is

$$\tilde{j} = \frac{\chi j}{\mu_s} \mu_j \tilde{s} + \frac{1}{\mu_{c,f}} \mu_j \tilde{c}. \quad (3.17)$$

It is the sum of two terms. The first term is the number of chronic offenders arrested for a crime on the street that receive a jail sentence. Chronic offenders flow from the street at a rate of $\frac{1}{\mu_s}$. With a probability of $\chi_j$, the chronic offender is given a jail sentence. Their expected length of stay is $\mu_j$ days. We multiply it by the number of chronic offenders on the street as the flow corresponds to a single chronic offender. The second term is the number of chronic offenders failing their community sentence. This term is constructed in a similar fashion. We have the rate out of the subpopulation of chronic offenders failing community sentences as $\frac{1}{\mu_{c,f}}$ which flow to jail and stay for an expected length of $\mu_j$ days. Again we multiply it by the number of chronic offenders serving a community sentence. The final term we need is the expected number of chronic offenders serving a community sentence and it equals

$$\tilde{c} = \frac{1 - \chi_j}{\mu_s} \left( \frac{\mu_{c,s} \mu_{c,f}}{\mu_{c,s} + \mu_{c,f}} \right) \tilde{s}. \quad (3.18)$$

The expected percentage of chronic offenders serving a community sentence flow from the street at a rate of $\frac{1 - \chi_j}{\mu_s}$. Since we assume community sentences and time to fail a community sentence are both independent Poisson processes, the rate at which the chronic offenders
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leave the community sentence pool is \( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \). This gives the expected length of stay as

\[
\left[ \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right]^{-1} = \frac{\mu_{c,s} \mu_{c,f}}{\mu_{c,s} + \mu_{c,f}}
\]

days. Again we multiply this term by the number of chronic offenders on the street. Equations 3.17 and 3.18 are both constructed from terms containing a flow rate multiplied by the expected length of stay multiplied by the subpopulation the flow initiated from.

Using Equation 3.16, Equations 3.17 and 3.18 become

\[
\tilde{j} - \frac{\chi_j}{\mu_s} \mu_j \left( N - \tilde{j} - \tilde{c} \right) - \frac{1}{\mu_{c,f}} \mu_j \tilde{c} = 0, \quad (3.19)
\]
\[
\tilde{c} - \frac{1 - \chi_j}{\mu_s} \left( \frac{\mu_{c,s} \mu_{c,f}}{\mu_{c,s} + \mu_{c,f}} \right) \left( N - \tilde{j} - \tilde{c} \right) = 0. \quad (3.20)
\]

We multiply Equation 3.19 by \(-\frac{1}{\mu_j}\) and Equation 3.20 by \(-\frac{\mu_{c,s} + \mu_{c,f}}{\mu_{c,s} \mu_{c,f}}\), leading to

\[
- \left( \frac{\chi_j}{\mu_s} + \frac{1}{\mu_j} \right) \tilde{j} + \left( \frac{1}{\mu_{c,f}} - \frac{\chi_j}{\mu_s} \right) \tilde{c} + \frac{N \chi_j}{\mu_s} = 0, \quad (3.21)
\]
\[
- \frac{1 - \chi_j}{\mu_s} \tilde{j} - \left( \frac{1}{\mu_{c,f}} + \frac{1}{\mu_{c,s}} + \frac{1 - \chi_j}{\mu_s} \right) \tilde{c} + \frac{N (1 - \chi_j)}{\mu_s} = 0. \quad (3.22)
\]

Unlike the ODE model, the steady state, \( \langle \tilde{s}, \tilde{j}, \tilde{c} \rangle \), is the number of chronic offenders in each subpopulation. By dividing Equations 3.21 and 3.22 by \( N \), the steady state satisfies

\[
- \left( \frac{\chi_j}{\mu_s} + \frac{1}{\mu_j} \right) \tilde{j} + \left( \frac{1}{\mu_{c,f}} - \frac{\chi_j}{\mu_s} \right) \tilde{c} + \frac{\chi_j}{\mu_s} = 0,
\]
\[
- \frac{1 - \chi_j}{\mu_s} \tilde{j} - \left( \frac{1}{\mu_{c,f}} + \frac{1}{\mu_{c,s}} + \frac{1 - \chi_j}{\mu_s} \right) \tilde{c} + \frac{(1 - \chi_j)}{\mu_s} = 0.
\]

This is the linear system we solved to obtain Equations 3.4 and 3.5. This implies that the agent-based model with deterministic jail sentences, the agent-based model with exponentially distributed jail sentences and the ODE model all have the same steady states. This steady state is one of the DDE model steady states. Unfortunately, the DDE model has steady states which are not present in the other models. We need to investigate the stability of each steady state to see if there exists initial data that results in discrepancies between the models.
3.2.3 Comparing Steady States

The system of DDEs has an infinite number of steady states and the agent-based model has one steady state. We numerically solve the system of DDEs, using the built-in MATLAB solver DDE23, to investigate when the system of DDEs converges to the agent steady state. To examine if the DDE model has multiple stable steady states, we numerically solve the system of Equations (Equations 2.8, 2.9 and 2.10) with different sets of uniformly distributed initial data. Each set of initial data satisfies

\[ s(t), j(t), c(t) \in [0, 1] \quad \text{and} \quad s(t) + j(t) + c(t) = 1, \quad \forall t \in [-\mu_j, 0]. \]

Figure 3.5 shows the solution at time \( T = 180 \) days for 10 sets of randomly chosen initial data. We also plot the solution at time \( T = 180 \) days for the agent-based model averaged over 500 simulations with \( \delta t = 0.05 \) days. The initial data for the agent-based model matches the DDE initial data at time \( t = 0 \). We have included the agent expected steady state (Equations 3.4 and 3.5) as a comparison. For most sets of initial data, the solution for the DDE model disagrees with the agent-based model steady state. The agent-based model converges to the same steady state for each set of initial data. There are two possibilities, either the DDE has multiple stable steady states or \( T \) is not large enough.

To show \( T \) is sufficiently large and there exists at least one other stable steady state, we choose constant initial data

\[ s(t) = 1, \quad j(t) = 0, \quad c(t) = 0, \quad \forall t \in [-\mu_j, 0]. \]

This is unrealistic, but simulates the case where police were not active with respect to the chronic offenders until time \( t = 0 \). We compare the solution to the agent-based model averaged over 500 iterations where the initial data is consistent with the DDE model.

Figure 3.6 shows that the solution of the system of DDEs is converging, but it is not converging to the agent steady state. This suggests that \( T \) is sufficiently large and that Figure 3.5 illustrates that multiple stable steady states exist for the DDE model while the agent-based model has a single stable steady state. A derived DDE model with steady states that do not match the corresponding agent-based model is a field of ongoing research and is discussed in [6].
Figure 3.5: The solution at time $T = 180$ days for both the agent-based model with deterministic jail sentences averaged over 500 runs as well as the system of DDEs for 10 different sets of random initial data. The expected steady state is included for a comparison.

We propose a possible explanation for the discrepancy between the DDE model and the agent-based model. During the construction of the DDE model, we assumed the number of chronic offenders leaving jail is the expected number of chronic offenders who entered jail $\mu_j$ days prior. This means the subpopulations are not independent from each other as agents are being forced to stay exactly $\mu_j$ days in jail. Initial data for $t \in \left[-\mu_j, 0\right]$ that does not satisfy this relationship is equivalent to having chronic offenders leaving jail early or late and possibly to an incorrect state. The discrepancy propagates throughout the solution shown by the following scenario. Suppose a chronic offender is released to the streets early at some $t_1 \leq 0$ but should have been released at some $t_2 > 0$. The DDE model will release another chronic offender at time $t_2$. Since the number of chronic offenders is fixed, it must underestimate the number of chronic offenders in jail and overestimate the number of chronic offenders on the street. The incorrect movement of chronic offenders in the initial data is believed to cause the solution to be perturbed towards an alternate steady state.

To test our theory, we solve the DDE model for $t > \mu_j$ using two sets of initial data. We use the averaged output over 500 simulations and over a single simulation as initial data for $t \in \left[0, \mu_j\right]$. A single simulation will satisfy the model properties while still including
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Figure 3.6: The numerical solution to the system of DDEs and the agent-based model with deterministic jail sentences averaged over 500 runs. All chronic offenders are initially on the street. The initial data for the system of DDEs is \( s(t) = 1, \ j(t) = 0, \ c(t) = 0, \ \forall t \in [-\mu_j, 0] \). The numerical solution converges to an alternate steady state. The fraction of chronic offenders on the street is calculated using the conservation of chronic offenders.

randomness. Figure 3.7 shows that the numerical solution from the system of DDEs appears to converge to the agent steady state. The relative error between solutions as defined in Section 3.1.4 is

\[
E^s_{500} = \frac{|s^{500}_{\text{DDE}}(T) - s^{500}_{\text{agent}}(T)|}{s} = 1.6 \times 10^{-3},
\]

\[
E^j_{500} = \frac{|j^{500}_{\text{DDE}}(T) - j^{500}_{\text{agent}}(T)|}{j} = 2.3 \times 10^{-3},
\]

\[
E^c_{500} = \frac{|c^{500}_{\text{DDE}}(T) - c^{500}_{\text{agent}}(T)|}{c} = 1.1 \times 10^{-3}.
\]

For the continuous solution, the superscript denotes the number of simulations the initial data is averaged over. Using a single simulation introduces a larger error as we are no longer giving the system of DDEs a smooth set of initial data, seen in the relative error

\[
E^s_{500} = \frac{|s^1_{\text{DDE}}(T) - s^{500}_{\text{agent}}(T)|}{s} = 1.8 \times 10^{-2},
\]

\[
E^j_{500} = \frac{|j^1_{\text{DDE}}(T) - j^{500}_{\text{agent}}(T)|}{j} = 2.6 \times 10^{-2},
\]

\[
E^c_{500} = \frac{|c^1_{\text{DDE}}(T) - c^{500}_{\text{agent}}(T)|}{c} = 1.3 \times 10^{-2}.
\]
We can reduce the error with a single agent simulation by decreasing the time step for simulating the agent-based model from $\delta t = 0.05$ days to $\delta t = 0.005$ days. Using a smaller time step results in a more accurate agent-based simulation. The relative error becomes

$$E_{500}^s = \frac{|s_{DDE}(T) - s_{agent}^{500}(T)|}{s_{agent}^{500}} = 7.4 \times 10^{-3},$$

$$E_{500}^j = \frac{|j_{DDE}(T) - j_{agent}^{500}(T)|}{j_{agent}^{500}} = 1.7 \times 10^{-3},$$

$$E_{500}^c = \frac{|c_{DDE}(T) - c_{agent}^{500}(T)|}{c_{agent}^{500}} = 4.8 \times 10^{-2}.$$

To avoid running the agent-based model while ensuring the initial data satisfies the agent-based model properties, we employ the strategy outlined in [6] and use the Heaviside function to only allow the lag terms to be in effect for $t > \mu_j$. We denote the following as the modified DDE model:

$$\frac{ds}{dt} = -\frac{1}{\mu_s} s(t) + H(t - \mu_j) \frac{\chi_j}{\mu_s} s(t - \mu_j) + \frac{1}{\mu_{c,s}} c(t) + H(t - \mu_j) \frac{1}{\mu_{c,f}} c(t - \mu_j),$$

$$\frac{dj}{dt} = \frac{\chi_j}{\mu_s} s(t) - H(t - \mu_j) \frac{\chi_j}{\mu_s} s(t - \mu_j) + \frac{1}{\mu_{c,f}} c(t) - H(t - \mu_j) \frac{1}{\mu_{c,f}} c(t - \mu_j),$$

$$\frac{dc}{dt} = 1 - \frac{\chi_j}{\mu_s} s(t) - \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right) c(t).$$

The Heaviside function is defined piecewise as

$$H(t) = \begin{cases} 
1 & \text{if } t \geq 0, \\
0 & \text{if } t < 0,
\end{cases}$$

which is a discontinuous step function. The modified DDE model only requires initial data for $t = 0$, but this is only justified when there are initially only chronic offenders located in locations with non-deterministic length of stays. The initial data must satisfy

$$j(0) = 0, \ s(0) + c(0) = 1.$$

The solution to the modified DDE is equivalent to solving the system of DDEs (Equations 2.8, 2.9 and 2.10), with initial data for $t \in [0, \mu_j]$ satisfying,

$$\frac{ds}{dt} = -\frac{1}{\mu_s} s(t) + \frac{1}{\mu_{c,s}} c(t),$$

$$\frac{dj}{dt} = \frac{\chi_j}{\mu_s} s(t) + \frac{1}{\mu_{c,f}} c(t),$$

$$\frac{dc}{dt} = \frac{1 - \chi_j}{\mu_s} s(t) - \left( \frac{1}{\mu_{c,s}} + \frac{1}{\mu_{c,f}} \right) c(t).$$
Figure 3.7: The numerical solution to the system of DDEs and the agent-based model with deterministic jail sentences. All chronic offenders are initially on the street. For the system of DDEs, the initial data \((t \in [0, \mu_j])\) is the averaged agent-based model solution. The fraction of chronic offenders on the street is calculated using the conservation of chronic offenders.
Figure 3.8: The numerical solution to the modified system of DDEs and the agent-based model averaged over 500 simulations with deterministic jail sentences. Both solutions match and converge to the steady state as expected. The fraction of chronic offenders on the street is found by using the conservation of chronic offenders.
We compare the numerical modified DDE solution to the agent-based model averaged over 500 simulations for two different sets of allowable initial data. Figure 3.8 shows that the two models are both converging to the same steady state. The relative errors between solutions when all offenders are initially on the street, defined in Section 3.1.4, is

\[
E_s^{500} = \frac{|s_{MDDE}(T) - s_{agent}^{500}(T)|}{s_{agent}} = 1.5 \times 10^{-3},
\]

\[
E_j^{500} = \frac{|j_{MDDE}(T) - j_{agent}^{500}(T)|}{j_{agent}} = 2.4 \times 10^{-3},
\]

\[
E_c^{500} = \frac{|c_{MDDE}(T) - c_{agent}^{500}(T)|}{c_{agent}} = 1.7 \times 10^{-3}.
\]

While understanding the discrepancies between models is of mathematical interest, we leave it for future research. From a modelling perspective, having chronic offenders all receive the exact same sentence is not realistic, thus understanding the end behaviour given feasible initial data is sufficient.

### 3.2.4 Numerical Results

Figure 3.9 has the solution for each model and summarizes the differences between the models with deterministic jail sentences. The DDE models are numerically solved using MATLAB’s solver DDE23. All solutions, except for the DDE model with constant initial data, converge to the agent steady state. We have chosen allowable initial data for the modified DDE model. If the initial data does not satisfy the constraint \(j(0) = 0\), the modified DDE is no longer justified and the agent-based model must be used as initial data. We also note that while a single agent-based model simulation as initial data converges near the steady state, with numerical error, it is best to average over several agent-based model simulations for initial data.
Figure 3.9: A summary of the numerical results for the agent-based model with deterministic jail sentences. The agent-based model results are averaged over 500 simulations. The fraction of chronic offenders on the street is found by using the conservation of chronic offenders.
Chapter 4

Sensitivity Analysis

When we approximate each parameter from either data or literature, we need to understand the sensitivity of the model solution as a function of its parameters. If the solution changes significantly by changing parameter $X_i$, then we say $X_i$ is significant. For any feasible set of parameter values we are able to solve for the long time behaviour using the steady states. This avoids the numerical cost of solving the continuous model or simulating the agent-based model. As a result, we can efficiently simulate the model for numerous choices of model parameters. Although there are many methods that are used, as outlined in [14], we focus on two methods to understand the sensitivity of each parameter without restricting the remaining parameters. This is referred to as global sensitivity analysis.

To study the global sensitivity of the model parameters, we first produce scatter plots to visualize which parameters appear the most influential for each state. We then employ a variance-based method which approximates the variance of the model output as a function of the parameters. Prior to performing the sensitivity analysis we choose a parameter space that includes all realistic parameter values. To avoid biases, each parameter is sampled from uniform distributions between the minimum value for the parameter and the maximum value. These are noted in Table 4.1.

4.1 Scatter Plots

If we expect parameter $X_i$ to influence variable $Y$, then we expect the scatter plot of $X_i$ versus $Y$ to have a visible structure [14]. This means that the scatter plot should not appear
### Parameter Distribution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_s$</td>
<td>$\mathcal{U}(1, 180)$</td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>$\mathcal{U}(1, 180)$</td>
</tr>
<tr>
<td>$\mu_{c,s}$</td>
<td>$\mathcal{U}(90, 540)$</td>
</tr>
<tr>
<td>$\mu_{c,f}$</td>
<td>$\mathcal{U}(1, 180)$</td>
</tr>
<tr>
<td>$\chi_j$</td>
<td>$\mathcal{U}(0, 1)$</td>
</tr>
</tbody>
</table>

Table 4.1: The parameter distributions used in the sensitivity analysis, where $\mathcal{U}(a, b)$ is the uniform distribution on $[a, b]$.

To be uniformly distributed noise. Averages over the output $Y$ with the parameter $X_i$ fixed will have a large variance when the scatter plot has a distinct shape. The large variance suggests the parameter influences the output. If the scatter plot appears to be uniform, the averages are nearly constant. The small variance implies $X_i$ is insignificant to $Y$.

To construct the scatter plots we take $N_s$ sets of sampled parameter values and calculate the $N_s$ corresponding steady states ($\bar{s}$, $\bar{j}$ and $\bar{c}$). The model parameters are

$$X = \langle \mu_s, \mu_j, \mu_{c,s}, \mu_{c,f}, \chi_j \rangle,$$

which results in 5 scatter plots for each variable. To construct the scatter plot for parameter $\mu_s$ and variable $\bar{s}$ we take the $N_s$ samples (each having all parameters randomly sampled) and plot $\bar{s}$ as a function of $\mu_s$. Specifically, for each sampled point in our parameter space, we plot the $\mu_s$ entry on the $x$-axis and the corresponding $\bar{s}$ on the $y$-axis. This process is repeated for each parameter and for each variable.

### 4.1.1 Preliminary Results

Figure 4.1 suggests that $\mu_s$ followed by $\mu_j$ are the most influential parameters for the expected number of chronic offenders on the street. It is reasonable to assume $\mu_s$ is most significant for $\bar{s}$ as $\mu_s$ is the average time a chronic offender spends on the street. We now look at the scatter plots corresponding to $\bar{j}$. Figure 4.2 indicates that $\mu_s$ and $\mu_j$ appear again to be the most influential parameters, but $\mu_j$ is now more significant than $\mu_s$. Again, this is not surprising for the same reason $\mu_s$ was the most significant for $\bar{s}$. From Figure 4.3 it is not clear which parameters are most significant for $\bar{c}$. Intuition leads us to believe $\mu_{c,s}$ should be significant for the expected number of chronic offenders serving a community sentence,
Figure 4.1: Scatter plots for the expected fraction of chronic offenders on the street, $\bar{s}$, versus each model parameter. The $y$-axis is $\bar{s}$ and the $x$-axis is the model parameter. Vertical slices represent fixing a single model parameter and allowing the remaining model parameters to change.
Figure 4.2: Scatter plots for the expected fraction of chronic offenders in jail, \( \tilde{j} \), versus each model parameter. The \( y \)-axis is \( \tilde{j} \) and the \( x \)-axis is the model parameter. Vertical slices represent fixing a single model parameter and allowing the remaining model parameters to change.
Figure 4.3: Scatter plots for the expected fraction of chronic offenders serving a community sentence, $\bar{c}$, versus each model parameter. The $y$-axis is $\bar{c}$ and the $x$-axis is the model parameter. Vertical slices represent fixing a single model parameter and allowing the remaining model parameters to change.
but this is not the case. It appears $\chi_j$ and $\mu_{c,f}$ are both significant. It is not clear from Figure 4.3 which parameter is more significant. The scatter plots (Figures 4.1, 4.2 and 4.3) show that the most significant parameters in our agent-based model are $\mu_s$, $\mu_j$ and $\chi_j$. Scatter plots are only used as a preliminary tool requiring an alternate method to support the results. We use a variance method to confirm the scatter plot findings.

### 4.2 A Variance Method

In the variance method, the first order measure of sensitivity is the conditional variance of the expected model output, $E[Y]$, conditioned on $X_i$, scaled by the total variance. The sensitivity index $S_i$ is written as

$$S_i = \frac{\text{Var}[E[Y|X_i]]}{\text{Var}[Y]}.$$  

The expected model output conditioned on $X_i$, the $i^{th}$ model parameter, is $E[Y|X_i]$, and $\text{Var}[E[Y|X_i]]$ is the variance of the expected model output as we change $X_i$. This is $X_i$’s contribution to the total variance. Therefore scaling by $\text{Var}[Y]$ gives a measure of sensitivity, $S_i \in [0,1]$. The measure, $S_i$, is classified as a first-order index and only represents the contribution of $X_i$ to the variance of the model output [14]. This measure does not include effects that correspond to interacting parameters. To measure the total effect, as in [14], we compute

$$S_{T_i} = \frac{\text{Var}[Y] - \text{Var}[E[Y|X_{\sim i}]]}{\text{Var}[Y]} = 1 - \frac{\text{Var}[E[Y|X_{\sim i}]]}{\text{Var}[Y]}.$$  

We calculate the total variance of the model and subtract the variance that is caused by changing all model parameters except $X_i$, denoted by $X_{\sim i}$ in the numerator. If

$$\text{Var}[E[Y|X_{\sim i}]] \ll \text{Var}[Y]$$

holds, then there is little change in the expected model output with changes in model parameters $X_{\sim i}$. This implies that most of the total variance must be from $X_i$ interacting with other model parameters. Scaling by $\text{Var}[Y]$ gives a measure of the interacting parameter significance, $S_{T_i} \in [0,1]$. For both $S_i$ and $S_{T_i}$, the closer the measure is to one, the more significant parameter $X_i$ is.

Our aim is to approximate $S_i$ and $S_{T_i}$ using a Monte Carlo framework. Typically Monte Carlo approximations are computationally expensive. Fortunately, we have a closed formula
for the long time behaviour for each variable, allowing for $N_v \gg 1$ sets of parameter values to be calculated efficiently.

### 4.2.1 Approximating Sensitivity Indices

To approximate $S_i$ and $S_{T_i}$, we use the Monte Carlo approximation outlined in [13, 14]. We explain the process for computing the sensitivity of a single model output, $Y = \bar{s}$. We populate the sample matrices $A$ and $B$,

$$
A = \begin{bmatrix}
\mu_{s,1}^{(1)} & \mu_{j,1}^{(1)} & \mu_{c,s,1}^{(1)} & \mu_{c,f,1}^{(1)} & \chi_{j,1}^{(1)} \\
\mu_{s,1}^{(2)} & \mu_{j,1}^{(2)} & \mu_{c,s,1}^{(2)} & \mu_{c,f,1}^{(2)} & \chi_{j,1}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_{s,v}^{(N_v)} & \mu_{j,1}^{(N_v)} & \mu_{c,s,1}^{(N_v)} & \mu_{c,f,1}^{(N_v)} & \chi_{j,1}^{(N_v)} 
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
\mu_{s,2}^{(1)} & \mu_{j,2}^{(1)} & \mu_{c,s,2}^{(1)} & \mu_{c,f,2}^{(1)} & \chi_{j,2}^{(1)} \\
\mu_{s,2}^{(2)} & \mu_{j,2}^{(2)} & \mu_{c,s,2}^{(2)} & \mu_{c,f,2}^{(2)} & \chi_{j,2}^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_{s,v}^{(N_v)} & \mu_{j,2}^{(N_v)} & \mu_{c,s,2}^{(N_v)} & \mu_{c,f,2}^{(N_v)} & \chi_{j,2}^{(N_v)} 
\end{bmatrix},
$$

where each parameter is randomly sampled from the uniform distributions in Table 4.1, and $N_v$ is the number of samples. We construct the sample matrix $C_i$, $i = 1 \ldots 5$, from $A$ and $B$. The $j \neq i$ columns of $C_i$ are the $j^{th}$ columns of $B$ and the $i^{th}$ column of $C_i$ is the $i^{th}$ column of $A$. We calculate the steady state for each set (row) of parameter values for each matrix and define the output vectors as

$$
\bar{Y}_A = \bar{s}(A), \quad \bar{Y}_B = \bar{s}(B), \quad \bar{Y}_{C_i} = \bar{s}(C_i),
$$

where $\bar{s}(X) \in \mathbb{R}^{N_v \times 1}$ computes the steady state for the parameter values in matrix $X$. The $i^{th}$ element of $\bar{s}(X)$ is the steady state calculated with the parameter values in the $i^{th}$ row of $X$. We also denote $s_0^2 = \left( \frac{1}{N_v} \sum_{j=1}^{N_v} Y_{A}^{(j)} \right)^2$. From [13, 14] we have the following approximations,

$$
S_i = \frac{\text{Var} \left[ E \left[ Y | X_i \right] \right]}{\text{Var} \left[ Y \right]} \approx \frac{1}{N_v} \sum_{j=1}^{N_v} \frac{Y_{A}^{(j)} Y_{C_i}^{(j)} - s_0^2}{\left( Y_{A}^{(j)} \right)^2 - s_0^2},
$$

$$
S_{T_i} = 1 - \frac{\text{Var} \left[ E \left[ Y | X_{\sim i} \right] \right]}{\text{Var} \left[ Y \right]} \approx 1 - \frac{1}{N_v} \sum_{j=1}^{N_v} \frac{Y_{B}^{(j)} Y_{C_i}^{(j)} - s_0^2}{\left( Y_{A}^{(j)} \right)^2 - s_0^2}.
$$

The derivation of $S_i$, $S_{T_i}$ is described in detail in [13], but we give a brief overview of how each term is estimated.
First we note
\[ E \left[ E \left[ Y \mid X_i = \tilde{X} \right]^2 \right] = E \left[ Y \left( X_1, X_2, \ldots, X_{i-1}, \tilde{X}, X_{i+1}, \ldots \right) \right] E \left[ Y \left( X'_1, X'_2, \ldots, X'_{i-1}, \tilde{X}, X'_{i+1}, \ldots \right) \right], \]
where \( X'_i \) is a new random variable with the same probabilities as \( X_i \). This leads to
\[ E \left[ E \left[ Y \mid X_i = \tilde{X} \right]^2 \right] = E \left[ \hat{Y} \left( \tilde{X}, \tilde{X}' \right) \right], \tag{4.1} \]
with the assumption
\[ \hat{Y} \left( \tilde{X}, \tilde{X}' \right) = Y \left( \tilde{X} \right) Y \left( \tilde{X}' \right) \quad \text{and} \quad X_i = X'_i. \]

We also note
\[ E \left[ E \left[ Y \mid X_i \right] \right] = E \left[ Y \right]. \tag{4.2} \]

Using the identity \( \text{Var} \left[ X \right] = E \left[ X^2 \right] - E \left[ X \right]^2 \) along with Equations 4.1 and 4.2 gives
\[ \frac{\text{Var} \left[ E \left[ Y \mid X_i \right] \right]}{\text{Var} \left[ Y \right]} = \frac{E \left[ \left( E \left[ Y \mid X_i = \tilde{X} \right]^2 \right) - (E \left[ E \left[ Y \mid X_i \right] \right])^2 \right]}{E \left[ Y^2 \right] - E \left[ Y \right]^2} \]
\[ = \frac{E \left[ \hat{Y} \left( \tilde{X}, \tilde{X}' \right) \right] - E \left[ Y \right]^2}{E \left[ Y^2 \right] - E \left[ Y \right]^2} \]
\[ \approx \frac{\frac{1}{N_v} \sum_{j=1}^{N_v} Y_A^{(j)} Y_C^{(j)} - \bar{s}_0^2}{\frac{1}{N_v} \sum_{j=1}^{N_v} \left( Y_A^{(j)} \right)^2 - \bar{s}_0^2}. \]

We have used the fact that the \( i \)th columns in \( Y_A \) and \( Y_C \) are identical, satisfying the \( X_i = X'_i \) constraint. The expected value is approximated using the sample mean. To approximate \( S_{T_i} \), we use the same strategy but instead use \( Y_B \) and \( Y_C \), as the \( i \)th column is the only column that differs between the two matrices representing samples conditioned on \( X_\sim \). We repeat the same process for the remaining variables.

### 4.2.2 Results

This variance method requires a parameter space and the number of samples. Table 4.2 defines the parameter space. We take \( N_v = 20,000 \) samples to construct each sample matrix.
CHAPTER 4. SENSITIVITY ANALYSIS

Table 4.2: The sensitivity analysis results for the variance based method.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( s(t) )</th>
<th>( j(t) )</th>
<th>( c(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_s \sim \mathcal{U}(1,180) )</td>
<td>0.64</td>
<td>0.61</td>
<td>0.29</td>
</tr>
<tr>
<td>( \mu_j \sim \mathcal{U}(1,180) )</td>
<td>0.29</td>
<td>0.3</td>
<td>0.55</td>
</tr>
<tr>
<td>( \mu_{c,s} \sim \mathcal{U}(90,540) )</td>
<td>0.03</td>
<td>-0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>( \mu_{c,f} \sim \mathcal{U}(1,180) )</td>
<td>0.04</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>( \chi_j \sim \mathcal{U}(0,1) )</td>
<td>0.05</td>
<td>0.01</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Since each measure should be non-negative with \( S_i \leq S_{T_i} \), error is present in Table 4.2. This is to be expected as a Monte Carlo estimate is exact only for \( N_v \to \infty \). We are only interested in the ordering of the parameters, thus the error does not effect the results. Overall, Table 4.2 is consistent with our findings in Section 4.1.

The significant parameter for the expected number of chronic offenders on the street is \( \mu_s \) followed by \( \mu_j \). The parameter \( \mu_s \) is significant as it directly impacts the number of chronic offenders on the street. The parameter \( \mu_j \) indirectly impacts \( s(t) \) because most chronic offenders spend some time in jail, either for a crime on the street or a crime while serving a community sentence, before returning back to the streets. The most significant parameter for the expected number of chronic offenders in jail is \( \mu_j \) followed by \( \mu_s \). As expected, the average jail sentence has a significant impact on the number of chronic offenders in jail. The parameter \( \mu_s \) is significant since it indirectly impacts the number of jailed chronic offenders in two ways. The first is from the chronic offenders that flow from the street directly to jail. The second is from the chronic offenders who flow from the street to the community and then to jail. Lastly, the most significant parameter for the expected number of chronic offenders serving a community sentence is \( \chi_j \) followed by \( \mu_{c,f} \). The parameter \( \chi_j \) is significant as it controls the number of chronic offenders who move from the street to the community subpopulation. Unlike chronic offenders in jail, the only way chronic offenders move to the community subpopulation is from the street. The parameter \( \mu_{c,f} \) is significant as most chronic offenders fail their community sentence. This results in \( \mu_{c,f} \) being the true expected time chronic offenders spend serving a community sentence.
To summarize, the parameters which are most significant are $\mu_s$, $\mu_j$ and $\chi_j$. The parameter that is not significant for any variable is $\mu_{c,s}$. This means chronic offenders are unlikely to modify their behaviour when given a community sentence. If the expected community sentence is longer than the expected time between arrests while serving a community sentence, it is unlikely for a chronic offender to successfully complete a community sentence. As a result, a typical chronic offender is likely to only spend $\mu_{c,f}$ days serving a community sentence resulting in $\mu_{c,s}$ being insignificant.
Chapter 5

Conclusion

The chapter summarizes the construction and analysis of the model. A hypothetical scenario is implemented to demonstrate how the model can be used to estimate the impact of policy changes. We discuss incomplete elements and limitations of the current model. Finally we outline the future work including possible extensions of the model.

5.1 Summary

Prior to constructing the model, we explored current research on agent-based modelling in criminology. The model is motivated by the lack of analysis of chronic offenders. We formulated an agent-based model with chronic offenders as the single agent type. In our model chronic offenders are either on the street, in jail or serving a community sentence. If a chronic offender is on the street or serving a community sentence they are free to commit crimes whereas chronic offenders in jail are unable to commit crimes. If a chronic offender is arrested for an offence on the street they are given either a community sentence or a jail sentence. When a chronic offender is arrested while serving a community sentence, their community sentence is replaced with a jail sentence. After a chronic offender successfully completes their sentence, they return to the offending population on the street. We assumed the time spent in each state follows exponential distributions and showed that the expected output of the agent-based model is a solution to a system of ODEs. Moreover, fixing the length of all jail sentences resulted in a system of DDEs as the continuous approximation.

Simulations of the agent-based model with exponentially distributed jail sentences showed
that the average over multiple agent-based model simulations matches the analytical solution to the system of ODEs. We then analyzed the system of ODEs and found that any set of feasible initial data converges to a single steady state. The steady state is a function of the model parameters. It determines the expected fraction of chronic offenders in each subpopulation in the long time limit. The analysis of the agent-based model with deterministic jail sentences and its continuous approximation led to inconsistent results. The agent-based model results in a single stable steady state while the continuous approximation has multiple stable steady states. Only initial data which satisfies the model properties leads to consistent results. We presented a possible explanation for the disagreement between the two models. For each model the numerical results, summarized in Figure 5.1, show our analysis matches the behaviour of the model. Each model converges to the expected steady state as long as the initial data satisfies the agent-based model properties.

![Figure 5.1](image)

Figure 5.1: A summary of the numerical results for each model presented and analyzed. All solutions, except when using constant initial data, converge as expected. The agent-based model results are averaged over 500 simulations. The fraction of chronic offenders on the street is found by using the conservation of chronic offenders.

Sensitivity analysis describes which model parameters are significant. This is required to understand which parameters need accurate estimates and which parameters can be approximated with rough estimates. We found that the average jail sentence and time between arrests are significant for the expected fraction of chronic offenders on the street and in jail.
We found that the average community sentence is insignificant for the expected fraction of chronic offenders serving a community sentence. The fraction of chronic offenders given a jail sentence and average time to fail a community sentence are significant to the community sentence subpopulation. This suggests that chronic offenders rarely complete a community sentence.

Overall, we discovered that using deterministic jail sentences is both inaccurate in reality as well as a poor choice for modelling. Deterministic jail sentences lead to inconsistencies between the agent-based model and the continuous approximation. Exponential distributions gave mathematically consistent results, but the results have not been validated.

5.2 Predicting the Impact of Policy Changes

We use the model to estimate how changing policing strategies and sentencing policies impact the total number of crimes committed. The total number of crimes committed is directly related to the fraction of offenders on the street. Typically, it is the long term effects that measure the success of a policy change. This implies that we should implement policies which reduce $\bar{s}$. From the sensitivity analysis, only $\mu_j$ and $\mu_s$ are significant to $\bar{s}$. This suggests we should focus on polices that impact $\mu_s$ and $\mu_j$. The first scenario incorporates a more vigilant police response. This can be approximated by studying the impact of changing the times between arrests $\mu_s$. The second scenario changes the lengths of jail sentences $\mu_j$. By comparing these two scenarios, we can see if a proactive approach (shortening times between arrests) or reactive approach (longer jail terms) is more likely to have a greater impact on the chronic offenders and reducing crime.

Preliminary results shown in Figure 5.2 suggest that a proactive approach yields better results than a reactive approach. Police tactics which reduce $\mu_s$ to less than 11.5 days are more effective than increasing $\mu_j$ to more than 180 days. The slope of the curve determines the effectiveness of the policy change. A flat section of the $\bar{s}$ curve suggests an ineffective policy change.
Figure 5.2: We modify a single parameter ($\mu_j$ or $\mu_s$) to predict whether a proactive or a reactive approach is more effective for reducing the fraction of offenders on the street. The remaining parameters are fixed to the estimated values described in Table 2.4.

5.3 Future Work

There are several avenues of future work for our model. We first need to validate our model to operational data. We would like to understand the impact our simplifying assumptions have on the solution. Finally, we discuss possible model extensions which reduce our simplifying assumptions.

5.3.1 Model Validation

We plan to validate the model using police data from the records management system of the VPD. Unfortunately, we were unable to get the exact data required to populate the current model. The Vancouver Police Department recently provided us with offender level offence data for 302 chronic offenders in Vancouver. The data consists of anonymized offender and offence data from 01/01/2008 to 04/13/2013. We have offender characteristics such as age, gender, ethnicity, number of police contacts, addiction status, mental health status and whether the offender is known to be violent. For each chronic offender, we have their history of police contacts in Vancouver. This includes, for each of their past known
crimes, the date and time, the role of the chronic offender in relation to the offence, status of the offence (charged, suspect chargeable, subject of complaint, etc.), the offence type, drug and alcohol involvement and the crime location type. We also have police street check data for the chronic offenders.

Since we were unable to get sentencing data, we plan on modifying our agent-based model in Section 2.1.1 by removing community sentences \((\chi_j = 1)\). Using the chronic offender data allows us to construct a distribution of the times between arrests. This is done by calculating the frequency of the times between charges. We will fit the frequencies to an exponential distribution to remain consistent with our prior analysis. Conceptually this is accomplished by choosing the average time between arrests which has the highest likelihood of producing the data. We will assume jail sentences are exponentially distributed and use the average sentence as the free parameter to tune the model. We will choose \(\mu_j\) which minimizes the differences between the agent-based model and the data. We simulate the agent-based model as in Section 2.1.1. To validate the model, we will compare the total number of arrests from the agent-based model to the data not used in building the distribution. We are unable to use agent states as the data does not differentiate when chronic offenders are incarcerated or free.

5.3.2 Model Improvements

A mathematical justification of the inconsistency between the agent-based model with deterministic jail sentences and the system of DDEs is of interest. In addition, we have only simulated the agent-based model with two different sets of distributions. More simulations need to be calculated to examine if there exists a choice of distributions where the agent-based model converges to an alternate steady state. Once the agent-based model is completely analyzed, we plan to expand the agent-based model. This can be done by changing the number of subpopulations and/or introducing more sophisticated agents.

To expand the number of subpopulations, we propose adding remand. Chronic offenders move from the street to remand prior to receiving their sentence. Chronic offenders in remand behave as the offenders in jail behave in the current model. Another option is to separate community sentences into conditional sentences and probation sentences. This allows us to study the impact of different community sentence types. Probation can be given
as either the lone punishment, or to be served at the completion of a conditional or jail sentence. If an offender is arrested during probation, they receive a new sentence. If an offender is arrested while serving a conditional sentence they will serve the remainder of their current sentence in jail.

Adding characteristics to the agents is a natural extension of our current model. Characteristics of interest are age, gender and criminal history. By introducing dynamic characteristics which depend on the agent’s history, we are no longer able to assume chronic offenders remain in the system. Resampling agents is required when we include dynamic attributes, such as offending frequency decreasing with age. For this scenario, failure to do this results in the population eventually consisting of only chronic offenders with low offending frequencies. In reality, we expect some agents to be young with high offending frequencies. We need to allow for chronic offenders to age out and for new chronic offenders to enter the system. How different agent characteristics affect their behaviour still requires further research. Once we have an agent-based model that accurately replicates reality, we can start simulating scenarios that are of interest to the VPD and the COU.
Bibliography


